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Semialgebraic Topology over a Real Closed Field II: Basic Theory of Semialgebraic Spaces

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Our first goal in this paper is to develop a basic language for the theory of "semialgebraic spaces" over an arbitrary real closed field R (§6–§9). This language seems to be very convenient for "topological" considerations in the space $X(R)$ of rational points of an algebraic variety X over R. We then study paths and path components in semialgebraic spaces (§10–§13) expanding and completing the results of Part I of the paper [6].

We use the notations and terminology from part I without further explanation. We refer the reader to the introduction (§1) in part I to get an idea about the program pursued here.

We hope to continue our work in the near future by research on the homology, cohomology, and homotopy of semialgebraic spaces. An essential purpose of the present paper is to lay firm ground for all that. Thus we feel it necessary to proceed carefully and even pedantically in all foundational matters, especially in §§6 and 7.

Contents

§6. Semialgebraic Sets and Maps

Let X be an affine variety over R.

Definition 1. A subset A of the set $X(R)$ of real points of X is called *semialgebraic in* X, if there exist finitely many functions f_{ij}, g_{ik} in the affine ring $R[X]$

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$=\Gamma(X,\mathcal{O}_X)$ of X ($i=1,\dots,r; j=1,\dots,s_i;\ k=1,\dots,t_i$) such that

$$A=\bigcup_{i=1}^{r}\{x\in X(R)\,|\,f_{i1}(x)>0,\dots,f_{is_i}(x)>0,\\ g_{i1}(x)=0,\dots,g_{it_i}(x)=0\}.$$

Notice that in this definition the conditions $g_{i1}(x)=0,\dots,g_{it_i}(x)=0$ can be condensed to a single equation $g_i(x)=0$ with $g_i:=g_{i1}^2+\dots+g_{it_i}^2$.

We denote the family of all subsets A of $X(R)$ which are semialgebraic in X by $\mathfrak{S}(X)$. The following two lemmas are easily verified.

Lemma 6.1. *$\mathfrak{S}(X)$ is the smallest family $\mathfrak{S}$ of subsets of $X(R)$ which has the following properties:*

(i) *For every f in $R[X]$ the set $\{x\in X(R)\,|\,f(x)>0\}$ is an element of $\mathfrak{S}$.*

(ii) $A\in\mathfrak{S}\Rightarrow X(R)\setminus A\in\mathfrak{S}$.

(iii) $A\in\mathfrak{S},\ B\in\mathfrak{S}\Rightarrow A\cup B\in\mathfrak{S}$.

Lemma 6.2. *Let $\varphi\colon X\to Y$ be a morphism between affine varieties X, Y over R. Consider the induced map $\varphi_R\colon X(R)\to Y(R)$ on the rational points. For every $A\in\mathfrak{S}(Y)$ the preimage $\varphi_R^{-1}(A)$ is an element of $\mathfrak{S}(X)$.*

Example 6.3. Let Z be a locally closed affine subscheme of our affine variety X. Then for every set $A\in\mathfrak{S}(X)$ the intersection $A\cap Z(R)$ is semialgebraic in Z.

Lemma 6.4. *Let Z be an open affine subscheme of X. Then $\mathfrak{S}(Z)\subset\mathfrak{S}(X)$. Thus $\mathfrak{S}(Z)$ consists of all $A\in\mathfrak{S}(X)$ with $A\subset Z(R)$.*

Proof. For any function h in $R[X]$ we denote by X_h the open affine subscheme of X where h does not vanish. We have $Z=X_{h_1}\cup\dots\cup X_{h_r}$ with finitely many functions $h_1,\dots,h_r$ in $R[X]$. Then

$$A=(A\cap X_{h_1}(R))\cup\dots\cup(A\cup X_{h_r}(R)).$$

We know from above that every intersection $A\cap X_{h_i}(R)$ is semialgebraic in X_{h_i}. It suffices to prove that these sets are semialgebraic in X. Thus we may assume without loss of generality that $Z=X_h$ with some $h\in R[X]$. The family

$$\mathfrak{T}:=\{A\subset Z(R)\,|\,A\in\mathfrak{S}(X)\}$$

fulfills with respect to Z the properties (ii) and (iii) of Lemma 6.1. We have to show $\mathfrak{T}\supset\mathfrak{S}(Z)$. By Lemma 6.1 it suffices to verify that for any function $f\in R[Z]$ the set

$$A:=\{x\in Z(R)\,|\,f(x)>0\}$$

is semialgebraic in X. Now $R[Z]$ is the ring of fractions $R[X]_h$ (standard notation), in particular $f=h^{-2n}g$ with some $g\in R[X]$ and some $n\geq 1$. Clearly

$$A=\{x\in X(R)\,|\,g(x)\cdot h(x)^2>0\}.$$

Thus indeed A is semialgebraic in X. q.e.d.

We now define semialgebraic subsets of $X(R)$ for X an arbitrary variety over R. In contrast to part I of the paper *we do not assume that X is separated.* Thus a variety over R means here just a scheme of finite type over R. (Starting from § 9 we shall again consider only separated varieties.)

Definition 2. Let $(U_i | i \in I)$ be a covering of X by affine open subsets. A subset A of $X(R)$ is called *semialgebraic* in X, if $A \cap U_i = A \cap U_i(R)$ is semialgebraic in U_i for every $i \in I$. The family of all these sets A will again be denoted by $\mathfrak{S}(X)$.

We have to verify that the condition on A quoted in this definition does not depend on the choice of the affine open covering $(U_i | i \in I) = \mathfrak{U}$ of X. It suffices to show that a set A in $X(R)$ which is semialgebraic with respect to $\mathfrak{U}$ remains semialgebraic with respect to $\mathfrak{U} \cup \mathfrak{V}$ for any family $(V_j | j \in J) = \mathfrak{V}$ of open affine subsets of X. For every V_j there exist finitely many members $U_{i_1}, \ldots, U_{i_r}$ of $\mathfrak{U}$ with

$$V_j \subset U_{i_1} \cup \ldots \cup U_{i_r}.$$

We have

$$A \cap V_j = \bigcup_{k=1}^{r} (A \cap U_{i_k} \cap V_j).$$

We cover $U_{i_k} \cap V_j$ by finitely many affine open subsets W_l^{kj}; $l = 1, \ldots, s(k,j)$. Then

$$A \cap V_j = \bigcup_{k=1}^{r} \bigcup_{l=1}^{s(k,j)} (A \cap W_l^{kj}).$$

By Example 6.3 the set $A \cap W_l^{kj}$ is semialgebraic in W_l^{kj} and by Lemma 6.4 this set is also semialgebraic in V_j. Thus $A \cap V_j$ is semialgebraic in V_j for every $j \in J$, as we wanted to prove.

Proposition 6.5. *Let Y be a locally closed subscheme of the variety X.*

i) *If a subset A of $X(R)$ is semialgebraic in X then $A \cap Y = A \cap Y(R)$ is semialgebraic in Y.*

ii) *If a subset A of $Y(R)$ is semialgebraic in Y then A is semialgebraic in X. Thus $\mathfrak{S}(Y) = \{A \in \mathfrak{S}(X) | A \subset Y(R)\}$.*

Proof. Both assertions are evident from the consideration above if Y is open in X. Thus we may assume that Y is a closed subscheme of X. We easily retreat to the case that X is affine. Now the first assertion is trivial. The second assertion follows from the fact that every function $\bar{f}$ in $R[Y]$ extends to a function f in $R[X]$.

Proposition 6.6. *Let $\varphi: X \to Y$ be a morphism between varieties over R and let $\varphi_R: X(R) \to Y(R)$ denote the restiction of φ to the rational points. For every subset B of $Y(R)$ which is semialgebraic in Y the preimage $\varphi_R^{-1}(B)$ is semialgebraic in X.*

Proof. One easily retreats first to the case that Y is affine and then to the case that also X is affine. Now we are back to Lemma 6.2.

Corollary 6.7. *Let X and Y be algebraic varieties over R. Let M and N be subsets of $X(R)$ and $Y(R)$ which are semialgebraic in X and Y respectively. Then $M \times N$ is semialgebraic in the variety $X \times Y$.*

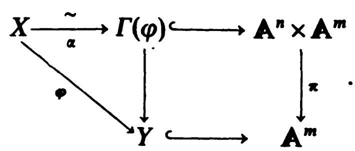
Proof. Let $p: X \times Y \to X$ and $q: X \times Y \to Y$ denote the projections from $X \times Y$ to X and Y. We have

$$M \times N = p_R^{-1}(M) \cap q_R^{-1}(N).$$

This is semialgebraic in $X \times Y$ by the preceding proposition.

Theorem 6.8 (Tarski). *Let $\varphi: X \to Y$ be a morphism between algebraic varieties over R and A be a subset of $X(R)$ which is semialgebraic in X. Then the set $\varphi(A) = \varphi_R(A)$ is semialgebraic in Y.*

Proof. We easily retreat to the case that X and Y are affine varieties, hence closed subvarieties of affine standard spaces $\mathbf{A}^n = \mathbf{A}_R^n$ and $\mathbf{A}^m$. We have a natural commutative diagram of morphisms



Here $\Gamma(\varphi)$ denotes the graph of φ which is a closed subscheme of $X \times Y$, hence of $\mathbf{A}^n \times \mathbf{A}^m$. The triangle is the usual factorization of the morphism φ by the graph. The horizontal arrows in the square are the inclusion morphisms, and π is the projection from $\mathbf{A}^n \times \mathbf{A}^m$ to $\mathbf{A}^m$. Since α is an isomorphism, the subset $B := \alpha(A)$ of $\Gamma(\varphi)(R)$ is semialgebraic in $\Gamma(\varphi)$, hence also in $\mathbf{A}^n \times \mathbf{A}^m$, and we have $\varphi(A) = \pi(B)$. Now B is a semialgebraic subset of R^{n+m} in the classical sense. By a well known theorem of Tarski ([13, 12, 3]) the projection $\pi(B)$ of B in R^m is semialgebraic in $\mathbf{A}^m$, hence also in Y.

Theorem 6.8 is closely related to the famous Tarski principle, which allows to transfer "elementary statements" from one real closed field to another, cf. [13, 12, 3] for the details. Here we mention just one application of Tarski's principle which we shall need later. For any point a of R^n and any $\varepsilon > 0$ in R we denote by $B_\varepsilon(a)$ the ball consisting of all $x \in R^n$ with $\|x - a\| < \varepsilon$, i.e. $\sum_{i=1}^{n} (x_i - a_i)^2 < \varepsilon^2$.

Theorem 6.9 (Implicit function theorem). *Let $f_1(X,T), \ldots, f_m(X,T)$ be polynomials in $R[X_1, \ldots, X_n, T_1, \ldots, T_m]$. Let (x_0, t_0) be a point of $R^n \times R^m$ at which all polynomials $f_i(X,T)$, $1 \leqq i \leqq m$, vanish and the matrix $(\partial f_i/\partial T_k)_{1 \leqq i,k \leqq m}$ has full rank m. Then there exist elements $\varepsilon > 0$ and $\delta > 0$ in R and a map $\varphi: B_\varepsilon(x_0) \to B_\delta(t_0)$ which is continuous in the strong topology and has the following property: For any point (x,t) in $B_\varepsilon(x_0) \times B_\delta(t_0)$ the polynomials $f_1, \ldots, f_m$ vanish at (x,t) if and only if $t = \varphi(x)$.*

Indeed, after fixing natural numbers $n, m, d_1, \ldots, d_m$ the theorem can be easily casted into an elementary statement over R involving all systems of m polynomials $f_1(X,T),\ldots,f_m(X,T)$ in $R[X_1,\ldots,X_n,T_1,\ldots,T_m]$ of degrees less or equal $d_1,\ldots,d_m$. This statement does not contain any free variables or constants from the field R. It is well known to be true over the field $\mathbb{R}$ of real numbers. By Tarski's principle it holds true over every real closed field R.

Exercise. Give a proof of the implicit function theorem without using Tarski's principle for the case $m=1$. Then prove this theorem for $m>1$ applying [9, Th. 7.6] (cf. the beginning of the proof of Theorem 9.1)[1].

Another proof of Theorem 6.9 without use of Tarski's principle has been given by Brumfiel [2, § 8.7].

For any variety X over R and any subset M of $X(R)$ which is semialgebraic in X we denote by $\mathring{\mathfrak{S}}_X(M)$ the family of all subsets U of M which are semialgebraic in X and open in M in the strong topology.

Definition 3. Let X and Y be varieties over the real closed field R, and let M and N be subsets of $X(R)$ and $Y(R)$ which are semialgebraic in X and Y respectively. A map $f\colon M\to N$ is called *semialgebraic with respect to X and Y*, if the following two conditions are fulfilled.

i) For every $V\in\mathring{\mathfrak{S}}_Y(N)$ the preimage $f^{-1}(V)$ is an element of $\mathring{\mathfrak{S}}_X(M)$.

ii) The graph $\Gamma(f)$ of f, which is a subset of $M\times N$, is semialgebraic in $X\times Y$.

Remarks. a) Under assumption of condition (ii) the condition (i) simply means that $f\colon M\to N$ is continuous in the strong topologies. b) The first condition (i) alone would define a class of maps which is far too broad for our purposes. Take for example $X=Y=\mathbf{A}^1$, M and N as intervals in R. Then every monotone bijective mapping $f\colon M\to N$ fulfills condition (i).

Examples. 1) For any morphism $\varphi\colon X\to Y$ the map $\varphi_R\colon X(R)\to Y(R)$ is semialgebraic with respect to X and Y. Indeed, $\Gamma(\varphi_R)$ is just the set of rational points $\Gamma(\varphi)(R)$ of the graph $\Gamma(\varphi)$ of φ, and thus is certainly semialgebraic in $X\times Y$. Moreover φ_R is continuous.

2) The map $\varphi\colon B_\delta(x_0)\to B_\varepsilon(y_0)$ occurring in the implicit function theorem 6.9 is semialgebraic with respect to $\mathbf{A}^n$ and $\mathbf{A}^m$.

Theorem 6.10. *Let M and N be subsets of the sets $X(R)$, $Y(R)$ of real points of varieties X and Y over R which are semialgebraic in X and Y respectively. Let $\varphi\colon M\to N$ be a semialgebraic map with respect to X and Y. Then for every subset A of M which is semialgebraic in X the image $\varphi(A)$ is semialgebraic in Y. Also for every subset B of N which is semialgebraic in Y the preimage $\varphi^{-1}(B)$ is semialgebraic in X.*

Proof. Let $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ denote the natural projections from $X\times Y$ to X and Y. The set $p_R^{-1}(A)\cap\Gamma(\varphi)$ is semialgebraic in $X\times Y$. Thus by Tarski's theorem 6.8

$$\varphi(A)=q_R(p_R^{-1}(A)\cap\Gamma(\varphi))$$

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[1] We thank G. Faltings for indicating to us this proof of the implicit function theorem

is semialgebraic in Y. Similarly the set $q_R^{-1}(B)\cap\Gamma(\varphi)$ is semialgebraic in $X\times Y$. Thus

$$\varphi^{-1}(B)=p_R(q_R^{-1}(B)\cap\Gamma(\varphi))$$

is semialgebraic in X. q.e.d.

Up to now at least our definitions could have been established over an arbitrary ordered base field instead of R. (Perhaps a more careful definition of the families $\mathring{\mathfrak{S}}_X(M)$ would be appropriate). But in the following crucial theorem our assumption that R is real closed enters in an essential way - as it does in the preceding theorem 6.10 - since we need for the proof Tarski's theorem 6.8. Without the assumption that R is real closed we do not know how to compose semialgebraic maps.

Theorem 6.11. *Let X, Y, Z be varieties over R and let M, N, S be subsets of $X(R)$, $Y(R)$, $Z(R)$ which are semialgebraic in X, Y, Z respectively. Let $f\colon M\to N$, $g\colon N\to S$ be maps which are semialgebraic with respect to X, Y and Z. Then the composite map $g\circ f\colon M\to S$ is semialgebraic with respect to X and Z.*

Proof. $g\circ f$ clearly fulfills the condition (i) in Definition 3. It remains to prove that the graph $\Gamma(g\circ f)$ of $g\circ f$ is semialgebraic in $X\times Z$. The subset

$$A:=(\Gamma(f)\times S)\cap(M\times\Gamma(g))$$

of $X(R)\times Y(R)\times Z(R)$ is semialgebraic in $X\times Y\times Z$. Let p denote the projection of $X\times Y\times Z$ onto $X\times Z$. The graph

$$\Gamma(g\circ f)=p_R(A)$$

is semialgebraic in $X\times Z$ by Theorem 6.8. q.e.d.

Proposition 6.12. *Let X_1, X_2, Y_1, Y_2 be varieties over R and M_1, M_2, N_1, N_2 be subsets of $X_1(R)$, $X_2(R)$, $Y_1(R)$, $Y_2(R)$ which are semialgebraic with respect to X_1, X_2, Y_1, Y_2. Let $f_1\colon M_1\to N_1$ and $f_2\colon M_2\to N_2$ be semialgebraic maps with respect to X_1, Y_1 and X_2, Y_2. Then $f_1\times f_2\colon M_1\times M_2\to N_1\times N_2$ is semialgebraic with respect to $X_1\times X_2$ and $Y_1\times Y_2$.*

Proof. Introducing the switch

$$\tau\colon X_1\times X_2\times Y_1\times Y_2\xrightarrow{\sim} X_1\times Y_1\times X_2\times Y_2$$

of the second with the third factor in $X_1\times X_2\times Y_1\times Y_2$ we have

$$\tau(\Gamma(f_1\times f_2))=\Gamma(f_1)\times\Gamma(f_2).$$

Thus $\Gamma(f_1\times f_2)$ is a semialgebraic subset of $X_1(R)\times X_2(R)\times Y_1(R)\times Y_2(R)$. Furthermore the map $f_1\times f_2$ is continuous in the strong topologies. Thus $f_1\times f_2$ is semialgebraic.

Corollary 6.13. *Let X, Y_1, Y_2 be varieties over R and M, N_1, N_2 be semialgebraic subsets of $X(R)$, $Y_1(R)$, $Y_2(R)$. Let $f_1\colon M\to N_1$ and $f_2\colon M\to N_2$ be semialgebraic maps with respect to X, Y_1, Y_2. Then the map $(f_1,f_2)\colon M\to N_1\times N_2$ is semialgebraic with respect to X and $Y_1\times Y_2$.*

This is clear since (f_1, f_2) is the composite of $f_1 \times f_2\colon M \times M \to N_1 \times N_2$ with the diagonal map from M to $M \times M$.

Definition 4. Let X be a variety over R and M a subset of $X(R)$ which is semialgebraic in X. A *semialgebraic function f on M with respect to X* is a map $f\colon M \to R$ which is semialgebraic with respect to X and $\mathbb{A}^1_R$.

Proposition 6.14. *If $f\colon M \to R$ and $g\colon M \to R$ are semialgebraic with respect to X then the same holds true for the functions $f+g$ and $f \cdot g$. If in addition f has no zeros on M then also the function $1/f$ on M is semialgebraic with respect to X.*

Proof. $f+g$ is the composite of the map $(f, g)\colon M \to R \times R$ and the map $(x, y) \to x+y$ from $R \times R$ to R. By Corollary 6.13 and Theorem 6.11 we conclude that $f+g$ is semialgebraic. In the same way we see that $f \cdot g$ is semialgebraic. Assume now that f has no zeros on M. Then f and $1/f$ may be regarded as maps from M to R^*. The map $f\colon M \to R^*$ is semialgebraic with respect to X and $\mathbb{A}^1\setminus\{0\}$ (use Proposition 6.5). The map $1/f\colon M \to R^*$ is the composite of f with the map $x \to x^{-1}$ from R^* to R^*. The last map comes from an automorphism of the scheme $\mathbb{A}^1\setminus\{0\}$. Thus also $1/f\colon M \to R^*$ is semialgebraic with respect to X and $\mathbb{A}^1\setminus\{0\}$. Then this map is also semialgebraic with respect to X and $\mathbb{A}^1$. q.e.d.

§ 7. The Category of Semialgebraic Spaces

Up to now we have been forced to consider a semialgebraic set M always together with a fixed embedding into some algebraic variety over R. The purpose of this section is to get rid of this inconvenience. We shall establish the "category of semialgebraic spaces" in a way which should leave no doubts that our definitions are the natural ones. Recall that our base field R is always assumed to be real closed.

Definition 1. A *restricted topological space M* is a set M equipped with a family $\mathring{\mathfrak{S}}(M)$ of subsets of X, called the "open subsets" of M, such that the following holds true:

i) $\emptyset \in \mathring{\mathfrak{S}}(M)$, $M \in \mathring{\mathfrak{S}}(M)$.

ii) If U_1 and U_2 are elements of $\mathring{\mathfrak{S}}(M)$ then also $U_1 \cup U_2$ and $U_1 \cap U_2$ are elements of $\mathring{\mathfrak{S}}(M)$.

Notice that the difference to usual topology is that only *finite* unions of open sets are again open.

Example 1. Take for M a subset of the set $X(R)$ of real points of a variety X over R which is semialgebraic in X and take for $\mathring{\mathfrak{S}}(M)$ the family $\mathring{\mathfrak{S}}_X(M)$ of all open subsets of M which are semialgebraic in X, as defined in § 6.

We regard a restricted topological space M as a (very special) site in the sense of Grothendieck. The category of the site has as objects the open sets of M and as morphisms the inclusion maps. The coverings $(U_i \to U)_{i \in I}$ are the *finite* systems of inclusions with $\bigcup_{i \in I} U_i = U$. A *sheaf F* on M is then an assignment $U \mapsto F(U)$ for every $U \in \mathfrak{S}(M)$ with sets, abelian groups, etc. as values $F(U)$ fulfilling the usual sheaf conditions, except that now only finite open coverings are admitted.

Definition 2. A *ringed space over R* is a pair $(M, \mathcal{O}_M)$ with M a restricted topological space and $\mathcal{O}_M$ a sheaf of R-algebras. A morphism (φ, ϑ): $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between ringed spaces over R is defined in the obvious way: φ is a continuous map from M to N, i.e. every open set V in N has an open preimage $\varphi^{-1}(V)$, and ϑ is a map from the sheaf $\mathcal{O}_N$ to $f_*\mathcal{O}_M$ respecting the R-algebra structures. In other words, for any open sets U in M and V in N with $\varphi(U) \subset V$ we have an R-algebra homomorphism

$$\vartheta_{U,V}\colon \mathcal{O}_N(V) \to \mathcal{O}_M(U)$$

with the usual compatibilities with respect to the restriction maps.

Example 2. Let X be an algebraic variety over R and M a subset of $X(R)$ which is semialgebraic in X. We equip M with the restricted topology as described in Example 1. For any open subset U of M we take for $\mathcal{O}_M(U)$ the R-algebra of semialgebraic functions on U as described at the end of § 6. Then $(M, \mathcal{O}_M)$ is a ringed space over R. We call such a ringed space a *semialgebraic subspace of the variety X*.

Notice that if X is a locally closed subscheme of another variety Y over R then according to § 6 the semialgebraic subspaces of X are just the semialgebraic subspaces $(M, \mathcal{O}_M)$ of Y with M contained in the subset $X(R)$ of $Y(R)$.

Definition 3. An *affine semialgebraic space over R* is a ringed space $(M, \mathcal{O}_M)$ over R which is isomorphic to a semialgebraic subspace of an affine variety X over R and hence of $\mathbb{A}_R^n$ for some $n \geq 0$. A *semialgebraic space over R* is a ringed space $(M, \mathcal{O}_M)$ over R which has a finite covering $(M_i \mid i \in I)$ by open sets M_i such that the ringed spaces $(M_i, \mathcal{O}_M \mid M_i)$ over R are affine semialgebraic spaces over R. A *morphism between semialgebraic spaces* is a morphism in the category of ringed spaces over R as described above.

Remark. Any semialgebraic subspace $(M, \mathcal{O}_M)$ of a variety X over R is clearly a semialgebraic space over R. If X is *quasiprojective* then $(M, \mathcal{O}_M)$ is even *affine*. This follows from the well known fact that every projective space $\mathbb{P}_R^n$ over R contains an affine open subscheme X with $\mathbb{P}^n(R) = X(R)$. Indeed, choose X as the complement of a closed hypersurface in $\mathbb{P}_R^n$ which has no real points.

From the affine case the following is clear: Let $(M, \mathcal{O}_M)$ be a semialgebraic space. Then for every point x of M the stalk $\mathcal{O}_{M,x}$, defined in the usual way, is a local ring. Moreover the residue class field $\mathcal{O}_{M,x}/\mathfrak{m}_{M,x}$ with respect to the maximal ideal $\mathfrak{m}_{M,x}$ of $\mathcal{O}_{M,x}$ is canonically isomorphic to R. Thus for any open subset U of M and any section $h \in \mathcal{O}_M(U)$ we can define the value $h(x) \in R$ at any point $x \in U$ as the image of h in $\mathcal{O}_{M,x}/\mathfrak{m}_{M,x}$. Any two sections h_1, h_2 of $\mathcal{O}_M$ over U with $h_1(x) = h_2(x)$ for every x in U are equal. Thus we regard since now always $\mathcal{O}_M$ as a subsheaf of the sheaf of all R-valued functions on M.

From § 6 the following assertion is evident.

Proposition 7.1. *Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be semialgebraic subspaces of the varieties X and Y over R and let $f\colon M \to N$ be a semialgebraic map with respect to*

X and Y in the sense of § 6. *For (semialgebraic) open subsets U of M and V of N with* $f(U)\subset V$ *we have well defined R-algebra homomorphisms*

$$f^*_{U,V}: \mathcal{O}_N(V)\to\mathcal{O}_M(U), \quad h\mapsto h\circ f.$$

f and these maps $f^*_{U,V}$ *together yield a morphism* (f, f^*) *from* $(M, \mathcal{O}_M)$ *to* $(N, \mathcal{O}_N)$.

We now establish an important converse of this statement.

Theorem 7.2. *Let* (f, ϑ): $(M, \mathcal{O}_M)\to(N, \mathcal{O}_N)$ *be a morphism between semialgebraic spaces. For any open subsets U and V of M and N with* $f(U)\subset V$ *and any* $h\in\mathcal{O}_N(V)$ *we have*

$$\vartheta_{U,V}(h)(x)=h(f(x))$$

for all $x\in U$. *If* $(M, \mathcal{O}_M)$ *and* $(N, \mathcal{O}_N)$ *are semialgebraic subspaces of varieties X and Y then moreover f is a semialgebraic map from M to N with respect to X and Y in the sense of* § 6.

Proof. One easily retreats to the case that $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ are affine. Then in particular $(N, \mathcal{O}_N)$ is isomorphic to a semialgebraic subspace of $\mathbf{A}^m$ for some m. We first prove the theorem in the case that $(N, \mathcal{O}_N)$ actually is a semialgebraic subspace of $\mathbf{A}^m$. We may assume $Y=\mathbf{A}^m$ in the second assertion of the theorem. N is a semialgebraic subset of R^m in the classical sense. Let $y_1, \ldots, y_m\in\mathcal{O}_N(N)$ denote the restrictions to N of the standard coordinate functions on R^m. We introduce the R-valued functions $f_i:=\vartheta_{M,N}(y_i)\in\mathcal{O}_M(M)$ on M. Our first claim is that for every point $a\in M$ the point $f(a)$ has the coordinates $f_1(a), \ldots, f_m(a)$. Indeed, the section $y_i-f_i(a)=z_i\in\mathcal{O}_N(N)$ has the image $\vartheta_{M,N}(z_i) =f_i-f_i(a)$ in $\mathcal{O}_M(M)$ which vanishes at a. Since $\vartheta_{M,N}(z_i)$ gives in the stalk $\mathcal{O}_{M,a}$ an element of the maximal ideal $\mathfrak{m}_{M,a}$ the section z_i cannot give a unit in the stalk $\mathcal{O}_{N,f(a)}$. Thus z_i vanishes at $f(a)$, and this means $y_i(f(a))=f_i(a)$. We now know that $f: M\to N$ coincides with the map $(f_1, \ldots, f_m)$ from M to R^n. If in particular M is a semialgebraic subspace of a variety X over R then clearly f is a semialgebraic map from M to N with respect to X and $\mathbf{A}^m$ in the sense of § 6.

Let U and V be relatively open semialgebraic subsets of M and N with $f(U)\subset V$ and let $h\in\mathcal{O}_N(V)$ be given. $h: V\to R$ is semialgebraic with respect to $\mathbf{A}^m$ and $\mathbf{A}^1$. We consider the graph G of h in $V\times R$ which is semialgebraic in $\mathbf{A}^{m+1}$. The map

$$g:=(\mathrm{id}, h): V\xrightarrow{\sim} G$$

is bijective and semialgebraic with respect to $\mathbf{A}^m$ and $\mathbf{A}^{m+1}$, and the inverse map g^{-1} is semialgebraic with respect to $\mathbf{A}^{m+1}$ and $\mathbf{A}^m$. Thus by Proposition 7.1

$$(g, g^*): (V, \mathcal{O}_V)\to(G, \mathcal{O}_G)$$

is an isomorphism between the semialgebraic subspace $(V, \mathcal{O}_V)$ of $\mathbf{A}^m$ and the semialgebraic subspace $(G, \mathcal{O}_G)$ of $\mathbf{A}^{m+1}$.

Let $(\tilde{f}, \tilde{\vartheta})\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ denote the restriction of (f, ϑ) to U and V. Let finally y_{m+1} denote the last coordinate function on G, i.e. the restriction of the projection $V \times R \to R$ to G. We apply what has been proved above to the morphism

$$(g, g^*) \circ (\tilde{f}, \tilde{\vartheta})\colon (U, \mathcal{O}_U) \to (G, \mathcal{O}_G)$$

and we learn:

$$(\tilde{\vartheta} \circ g^*)_{U,G}(y_{m+1}) = y_{m+1} \circ g \circ \tilde{f} = h \circ \tilde{f}.$$

But $g^*_{V,G}(y_{m+1}) = h$. Thus $\vartheta_{U,V}(h) = h \circ \tilde{f}$. This finishes the proof of the theorem in the case that $(N, \mathcal{O}_N)$ is a semialgebraic subspace of $\mathbf{A}^m$ for some m.

It remains to prove the first assertion of the theorem in the case that there only exists an isomorphism

$$(g, \varepsilon)\colon (N, \mathcal{O}_N) \xrightarrow{\sim} (L, \mathcal{O}_L)$$

from $(N, \mathcal{O}_N)$ to a semialgebraic subspace $(L, \mathcal{O}_L)$ of $\mathbf{A}^m$. Now we can apply what has been proved to the isomorphism (g, ε) and to the morphism $(g, \varepsilon) \circ (f, \vartheta)$. Thus we know that ε is the "pulling back" g^* of functions by the map g, and

$$\vartheta \circ \varepsilon = (g \circ f)^* = f^* \circ g^*$$

Clearly $\varepsilon^{-1} = (g^{-1})^*$ and we conclude that indeed $\vartheta = f^*$. Theorem 7.2 is completely proved.

By this theorem a morphism $(f, \vartheta)\colon (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between semialgebraic spaces is completely determined by the map f and we write $\vartheta = f^*$.

Since now a semialgebraic space $(M, \mathcal{O}_M)$ will often be simply denoted by the letter M. An open subset U of M will again be regarded as a semialgebraic space with structure sheaf $\mathcal{O}_U = \mathcal{O}_M | U$. A morphism $(f, f^*)\colon (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ will be identified with the map $f\colon M \to N$. We usually call these maps $f\colon M \to N$ the *semialgebraic maps from the semialgebraic space M to the semialgebraic space N*. They are a generalization of the semialgebraic maps defined in § 6. It is pretty obvious that for any open subset U of a semialgebraic space M the functions in $\mathcal{O}_M(U)$ are just the semialgebraic maps from U to R. Here of course $R = \mathbf{A}^1(R)$ is considered as a semialgebraic subspace of the variety $\mathbf{A}^1$. Similarly we mean by R^n always the semialgebraic subspace $\mathbf{A}^n(R)$ of $\mathbf{A}^n$. The semialgebraic maps from U to R^n are just the n-tuples $(h_1, \ldots, h_n)$ of functions $h_i \in \mathcal{O}_M(U)$.

The following theorem is now easily established by use of § 6.

Theorem 7.3. 1) *Let M_1 and M_2 be semialgebraic spaces over R. Then there exists on the set $M_1 \times M_2$ a unique restricted topology and a unique sheaf $\mathcal{O}_{M_1 \times M_2}$ of R-valued functions such that $M_1 \times M_2$ becomes a semialgebraic space with the following properties:*

a) *The projection maps $pr_i\colon M_1 \times M_2 \to M_i (i = 1, 2)$ are semialgebraic.*

b) *The semialgebraic space $M_1 \times M_2$ equipped with these projection maps is the direct product of M_1 and M_2 in the category of semialgebraic spaces.*

2) *If U_1 and U_2 are open subsets of M_1 and M_2, hence again semialgebraic spaces, then the open semialgebraic subspace $U_1 \times U_2$ of $M_1 \times M_2$ coincides with the product of the space U_1 and U_2 as described above.*

3) *If M_i is a semialgebraic subspace of a variety X_i over $R (i=1,2)$ then the product space $M_1 \times M_2$ coincides with the semialgebraic subspace of $X_1 \times X_2$ which has as underlying point set the subset $M_1 \times M_2$ of $(X_1 \times X_2)(R) = X_1(R) \times X_2(R)$.*

Notice that the topology on $M_1 \times M_2$ is usually finer than the "product" of the restricted topologies of M_1 and M_2: Not every open subset of $M_1 \times M_2$ is a finite union of sets $U_1 \times U_2$ with U_i open in M_i.

For any semialgebraic space M we denote by $\mathfrak{S}(M)$ the smallest familiy of subsets of M which contains $\mathring{\mathfrak{S}}(M)$ and is closed with respect to taking finite unions and complements in M. If $(M_i)_{i\in I}$ is a finite open covering of M then clearly a subset A of M lies in $\mathfrak{S}(M)$ if and only if $A \cap M_i$ lies in $\mathfrak{S}(M_i)$ for every $i \in I$. Furthermore if M is a semialgebraic subspace of a variety X then $\mathfrak{S}(M)$ consists of all subsets A of M which are semialgebraic in X in the sense of § 6. Thus we see that the following holds true in any semialgebraic space M.

Proposition 7.4. *A subset A of M belongs to $\mathfrak{S}(M)$ if and only if A is the union of finitely many sets of the following shape:*

$$\{x \in U \mid f_1(x) > 0, \ldots, f_r(x) > 0, g(x) = 0\}$$

with U open in M and $f_1, \ldots, f_r, g$ functions in $\mathcal{O}_M(U)$. If M is affine U can be replaced by M in this statement.

We call the elements of $\mathfrak{S}(M)$ the *semialgebraic subsets* of M.

It is convenient to use on a semialgebraic space M also the *strong topology*, which is defined as follows: The open sets of this topology are the unions of arbitrary subfamilies of $\mathring{\mathfrak{S}}(M)$. The strong topology is a topology in the usual sense. If M is a semialgebraic subspace of a variety X over R, then the strong topology of M as now defined coincides with the topology induced by the strong topology of $X(R)$ as defined in §1. Henceforth we shall call the topology of a semialgebraic space M considered before "the restricted topology of M".

The following proposition is evident for an affine semialgebraic space M from the definitions and thus holds true for an arbitrary semialgebraic space M.

Proposition 7.5. *A subset U of M is open in the restricted topology if and only if U is semialgebraic in M and open in the strong topology.*

Since now we use the following terminology: The words open, closed, dense, etc. all refer to the strong topology of M. The open subsets of M in the restricted topology are called the "*open semialgebraic subsets*" of M. A map $f: M \to N$ between semialgebraic spaces will be called "*continuous*", if f is continuous in the strong topologies, and „*strictly continuous*", if f is continuous in the restricted topologies.

Theorem 7.6. *A map $\varphi: M \to N$ between semialgebraic spaces is semialgebraic if and only if φ is continuous and the graph $\Gamma(\varphi)$ is a semialgebraic subset of*

$M \times N$. In this case the preimage $\varphi^{-1}(B)$ of any semialgebraic subset B of N is semialgebraic in M and the image $\varphi(A)$ of any semialgebraic subset A of M is semialgebraic in N.

This theorem is clear from § 6 if M and N are affine. The proof in the general case is then only an exercise which can safely be left to the reader.

A further example for the usefulness of the strong topology is given by

Theorem 7.7. *Let A be a semialgebraic subset of a semialgebraic space M. Then also the closure $\bar{A}$ and the interior $\mathring{A}$ of A (in the strong topology of M) are semialgebraic subsets of M.*

Proof. It suffices to prove that $\bar{A}$ is semialgebraic, since $M \setminus \mathring{A} = (M \setminus A)^-$. We easily retreat to the case that M is a semialgebraic subspace of R^n for some n. The closure of A in M is the intersection of M with the closure of A in R^n. Thus it suffices to prove that the closure of A in R^n is semialgebraic, and we may assume without loss of generality that $M = R^n$. In this case the theorem is well known and in fact an easy consequence of Tarski's theorem on the "elimination of quantifiers", cf. [13, 12] or [3].

We turn any non empty semialgebraic subset A of a semialgebraic space M into a semialgebraic space in the following way: $\mathring{\mathfrak{S}}(A)$ consists of all subsets V of A which are open in A in the strong topology of M and semialgebraic in M. For any $V \in \mathring{\mathfrak{S}}(A)$ the R-algebra $\mathcal{O}_A(V)$ consists of all maps $f\colon V \to R$ which are continuous with respect to the strong topologies of M and R and have a graph $\Gamma(f)$ which is semialgebraic in $M \times R$. From the affine case studied in §6 it is clear that $(A, \mathcal{O}_A)$ is a semialgebraic space and that the inclusion map $A \to M$ yields a monomorphism $(A, \mathcal{O}_A) \to (M, \mathcal{O}_M)$ in the category of semialgebraic spaces. We call A equipped with this structure as a ringed spaces over R a *semialgebraic subspace* of M.

Remark 7.8. The strong topology of A is clearly the topology of A induced by the strong topology of M. The same holds true for the restricted topology: Every open semialgebraic subset V of A is the intersection $U \cap A$ of some open semialgebraic subset U of M with A. Indeed, for a proof we easily retreat to the case $M = R^n$. Consider the distance function

$$f(x) = \inf(\|x - y\| \mid y \in A \setminus V).$$

The infimum exists and is equal to $\min(\|x - y\| \mid y \in \overline{A \setminus V})$ (cf. § 9). f is a semialgebraic function on R^n, as is easily seen by use of Tarski's theorem on elimination of quantifiers (cf. also [2, 8.13.12]). V is the intersection of A with the open semialgebraic subset

$$U := \{x \in R^n \mid f(x) > 0\}$$

of R^n. We do not need these facts here.

The following proposition is obvious from Theorem 7.6.

Proposition 7.9. *Let $\varphi\colon M \to N$ be a semialgebraic map. If A is a semialgebraic subspace of N with $\varphi(M) \subset A$, then the map $\psi\colon M \to A$ obtained from φ by re-*

striction of the range N to A is again semialgebraic. In particular, since $\varphi(M)$ is a semialgebraic subspace of N, we have a canonical factorization of φ into a semialgebraic surjection $\tilde{\varphi}$ and a semialgebraic inclusion map

$$\begin{array}{ccc} & \varphi(M) & \\ {\scriptstyle\tilde{\varphi}}\nearrow & & \searrow \\ M & \xrightarrow{\varphi} & N. \end{array}$$

We now can prove that arbitrary pullbacks exist in our category.

Theorem 7.10. *Let $\varphi_1: M_1 \to N$, $\varphi_2: M_2 \to N$ be semialgebraic maps. The subset $M_1 \times_N M_2$ of $M_1 \times M_2$ consisting of all pairs (x_1, x_2) in $M_1 \times M_2$ with $\varphi_1(x_1) = \varphi_2(x_2)$ is a semialgebraic subset of $M_1 \times M_2$ and hence a semialgebraic space. The diagram*

$$\begin{array}{ccc} M_1 \times_N M_2 & \xrightarrow{p_2} & M_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ M_1 & \xrightarrow{\varphi_1} & N \end{array}$$

with p_1 and p_2 the restrictions of the projections of $M_1 \times M_2$ to M_1 and M_2 is a pullback in the category of semialgebraic spaces over R.

Proof. The diagonal Δ of $N \times N$ is a semialgebraic subset of $N \times N$ as is clear from the affine case. From φ_1 and φ_2 we obtain a semialgebraic map

$$\varphi_1 \times \varphi_2: M_1 \times M_2 \to N \times N.$$

$M_1 \times_N M_2$ is the preimage of Δ under this map, hence is semialgebraic in $M_1 \times M_2$. The pullback property of the square above follows now immediately from the fact that $M_1 \times M_2$ is the categorial product of M_1 and M_2 and from Proposition 7.9.

Similarly we obtain from Proposition 7.9 in a quite formal way

Proposition 7.11. *Let A_1 and A_2 be semialgebraic subspaces of semialgebraic spaces M_1 and M_2. Then $A_1 \times A_2$ is a semialgebraic subset of $M_1 \times M_2$. The structure of $A_1 \times A_2$ as a semialgebraic subspace of $M_1 \times M_2$ is the same as the product of the space A_1 and A_2.*

Definition 4. A semialgebraic space M is called *separated* if the usual Hausdorff property is fulfilled: For any two different points x and y of M there exist open sets U and V in M with $x \in U$, $y \in V$ and $U \cap V$ empty.

It is easily seen that M is separated if and only if the diagonal Δ of $M \times M$, which is a semialgebraic subset of $M \times M$, is closed in $M \times M$. Thus for a separated algebraic scheme X over R the semialgebraic space $X(R)$ is separated. Clearly also every semialgebraic subspace of a separated semialgebraic space is separated.

In our investigations only separated semialgebraic spaces will play a role; but as in the theory of schemes it sometimes is better not to exclude the other semialgebraic spaces. Moreover, since every semialgebraic subspace of a quasiprojective variety is affine, only affine semialgebraic spaces seem to be important in practice. In fact, we do not even know yet if there exist any other separated semialgebraic spaces. Nevertheless it certainly is important to have the notion of an arbitrary semialgebraic space. Remember for example differential topology where it is essential to have the general notion of a C^∞-manifold despite every manifold is isomorphic to a submanifold of some $\mathbb{R}^n$.

§ 8. Dimension

Our definition of the dimension of a semialgebraic space starts out from the following theorem.

Theorem 8.1. *Let M and N be semialgebraic subspaces of algebraic varieties X and Y over R. Let X_1 denote the Zariski closure of M in X (as always with the reduced subscheme structure) and Y_1 the Zariski closure of N in Y. Let $f\colon M\to N$ be a surjective map whose graph is semialgebraic in $M\times N$, hence in $X\times Y$ (e.g. f is a surjective semialgebraic map). Then* $\dim Y_1 \leqq \dim X_1$.

Proof. We easily retreat to the case that $X=\mathbf{A}^n$, $Y=\mathbf{A}^m$ are affine standard varieties. Thus M and N are semialgebraic subsets of R^n and R^m in the classical sense, and the graph $\Gamma(f)$ is semialgebraic in $R^n\times R^m$. Let V denote the Zariski closure of $\Gamma(f)$ in $\mathbf{A}^n\times\mathbf{A}^m$. The natural projection from $\mathbf{A}^n\times\mathbf{A}^m$ to $\mathbf{A}^m$ maps $\Gamma(f)$ onto N, hence V onto a Zariski dense subset of Y_1. Thus we have $\dim Y_1\leqq\dim V$ and it suffices to prove $\dim V\leqq d$ with d the dimension of X_1.

We proceed by induction on m. The case $m=0$ is trivial. Assume now $m>0$. We consider the projection $\pi\colon R^m\to R^{m-1}$, $\pi(y_1,\dots,y_m)=(y_1,\dots,y_{m-1})$. Let W denote the Zariski closure of the graph $\Gamma(\pi\circ f)$ in $\mathbf{A}^n\times\mathbf{A}^{m-1}$. By induction hypothesis $\dim W\leqq d$. Now $\Gamma(f)$ is semialgebraic in $\mathbf{A}^{n+m}$. Thus $\Gamma(f)=N_1\cup\dots\cup N_r$ with non empty sets

$$N_i=\{(w,t)\in R^{n+m-1}\times R \mid g_i(w,t)=0,\ f_{ij}(w,t)>0,\ j=1,\dots,s_i\}.$$

Here the g_i and f_{ij} are polynomials over R in $n+m$ variables $T_1,\dots,T_{n+m-1},T$. Let p denote the natural projection from $R^{n+m-1}\times R$ onto R^{n+m-1}, and let W_i denote the Zariski closure of $p(N_i)$ in $\mathbf{A}^{n+m-1}$. Then $W=W_1\cup\dots\cup W_r$. Clearly $V\subset V_1\cup\dots\cup V_r$ with V_i the closed reduced subvariety of zeros of $g_i(T_1,\dots,T_{n+m-1},T)$ of $W_i\times\mathbf{A}^1$. We now focus our attention on some arbitrarily chosen irreducible component W' of W_1. The set N_1 contains at least one point (w_1,t) with $w_1\in W'(R)$, since otherwise $p(N_1)$ would already be contained in the union of the other irreducible components of W_1. Over w_1 lies precisely one point of N_1, since quite generally over any point of $\Gamma(\pi\circ f)=p(\Gamma(f))$ lies precisely one point of $\Gamma(f)$. Thus the polynomial $g_1(w_1,T)\in R[T]$ is not zero. We conclude that the polynomial $g_1(T_1,\dots,T_{n+m-1},T)$ does not vanish identically on $W'\times\mathbf{A}^1$. Thus

$$\dim V_1 \leqq (\dim W_1 + 1) - 1 \leqq d.$$

For the same reason all V_i have at most the dimension d, hence $\dim V \leqq d$. q.e.d.

Definition 1. Let X be a variety over R and M a semialgebraic subspace of X. The *dimension* $\dim_X M$ *of* M *in* X is the dimension of the Zariski closure of the set M in X.

If Y is another variety over R and $f\colon M \to Y(R)$ is some semialgebraic map then we learn from Theorem 8.1 that

$$\dim_Y f(M) \leqq \dim_X M.$$

If f is also injective, then

$$\dim_Y f(M) = \dim_X M.$$

Indeed the map $f^{-1}\colon f(M) \to M$ has the graph

$$\Gamma(f^{-1}) = \tau(\Gamma(f))$$

with $\tau\colon M \times f(M) \to f(M) \times M$ the switching map. Thus $\Gamma(f^{-1})$ is semialgebraic in $f(M) \times M$, and we can apply Theorem 8.1 also to f^{-1}.

In the sequel M is always a semialgebraic space.

Definition 2. Let $(M_i | i \in I)$ be a finite covering of M by open semialgebraic subspaces M_i which are isomorphic to semialgebraic subspaces N_i of varieties X_i over R. We define the *dimension of* M as

$$\dim M = \sup_{i \in I} \dim_{X_i}(N_i).$$

From Theorem 8.1 and the subsequent discussion it is evident that this number $\dim M$ does not depend on the choice of the covering $(M_i | i \in I)$ and the N_i, X_i.

Proposition 8.2. a) *If A is a semialgebraic subspace of M then* $\dim A \leqq \dim M$.

b) *If $(A_j | j \in J)$ is a finite family of semialgebraic subspaces of M and if M is the union of the A_j then*

$$\dim M = \sup_{j \in J} \dim A_j.$$

Proof. We easily retreat to the case that M is a subspace of some variety X. Then both assertions are evident from the definitions.

Proposition 8.3. *If $f\colon M \to N$ is a surjective (resp. bijective) map between semialgebraic spaces whose graph is semialgebraic in $M \times N$ then* $\dim N \leqq \dim M$ *(resp.* $\dim N = \dim M$*).*

The proof is easy starting from the affine case (Theorem 8.1) and using Proposition 8.2.

Proposition 8.4. *Let A be a semialgebraic subspace of M and let $\bar{A}$ denote the closure of A in M in the strong topology, which by § 7 is again a semialgebraic subspace of M. The dimensions of A and $\bar{A}$ are equal.*

Proof. We easily retreat to the case that M is a subspace of a variety X. Let Z denote the Zariski closure of A in X. Then $A \subset Z(R)$ and $Z(R)$ is closed in the strong topology of $X(R)$. Thus $\bar{A} \subset Z(R)$, and we see that Z is also the Zariski closure of $\bar{A}$. We have $\dim \bar{A} = \dim Z = \dim A$.

Proposition 8.5. *For any semialgebraic spaces M and N*

$$\dim(M \times N) = \dim M + \dim N.$$

Proof. We easily retreat to the case that M and N are affine, hence subspaces of varieties $\mathbf{A}^n$ and $\mathbf{A}^m$. Let X_1 and Y_1 denote the Zariski closures of M and N in $\mathbf{A}^n$ and $\mathbf{A}^m$ respectively. It is easily verified that $M \times N$ has in $\mathbf{A}^{n+m}$ the Zariski closure $X_1 \times Y_1$. The assertion is now clear, since

$$\dim(X_1 \times Y_1) = \dim X_1 + \dim Y_1.$$

We now strive for a more intrinsic description of the dimension of a semialgebraic space M.

Proposition 8.6. *Let X be a regular irreducible variety over R of dimension n. Then every non empty open semialgebraic subset M of $X(R)$ has dimension n.*

Proof. Clearly $\dim M \leqq n$. We choose some point $p \in M$. There exists an etale morphism $\pi: U \to V$ from a Zariski open neighbourhood U of p in X onto a Zariski open subset V of $\mathbf{A}^n$ [9, Exp. 2]. By the implicit function Theorem 6.9 there exists an open semialgebraic neighbourhood M' of p in $U(R) \cap M$ and an open semialgebraic subset N of $V(R)$ such that the restriction $\pi | M'$ maps M' bijectively onto N and induces a semialgebraic isomorphism $M' \xrightarrow{\sim} N$. Thus $\dim M' = \dim N$. Since $\dim M' \leqq \dim M \leqq n$ it suffices to prove $\dim N = n$. Now N contains an open rectangle

$$B =]a_1, b_1[\times \ldots \times]a_n, b_n[$$

of R^n and we only have to verify that B is Zariski dense in $\mathbf{A}^n$. This is easy. We proceed by induction on n. The case $n=1$ is trivial since B is an infinite set. Assume $n>1$. Let $f(T_1, \ldots, T_n)$ be a polynomial over R in n variables $T_1, \ldots, T_n$ which vanishes on B. We have to verify that f is zero. Suppose that f is not zero and the variable T_n occurs in f. We write

$$f(T_1, \ldots, T_n) = \sum_{i=0}^{d} g_i(T_1, \ldots, T_{n-1}) T_n^i.$$

For every point w of $]a_1, b_1[\times \ldots \times]a_{n-1}, b_{n-1}[$ the polynomial $f(w, T_n)$ vanishes on $]a_n, b_n[$, hence is identically zero. Thus all $g_i(T_1, \ldots, T_{n-1})$ vanish on $]a_1, b_1[\times \ldots \times]a_{n-1}, b_{n-1}[$. By induction hypothesis the g_i are identically zero. This contradicts our assumption that $f \neq 0$, and the proposition is proved.

Proposition 8.6 implies the following "identity theorem" for morphisms between varieties. Notice that a special case of this theorem has been verified directly in the proof of Proposition 8.6.

Theorem 8.7. *Let $f_1: X \to Y$ and $f_2: X \to Y$ be two morphisms from an irreducible reduced variety X over R to a separated variety Y. Let X' denote the Zariski open subvariety of X consisting of the regular points of X. Assume that f_1 and f_2 coincide on some non empty open subset U of $X'(R)$. Then $f_1 = f_2$.*

Proof. We consider the morphism $\varphi = (f_1, f_2): X \to Y \times Y$ and the preimage $\varphi^{-1}(\Delta) = Z$ of the diagonal Δ of $Y \times Y$. Then Z is a closed subvariety of X. Let X have dimension n. We may choose the open set U above as semialgebraic in $X(R)$. By Proposition 8.6 $\dim U = n$. Now U is contained in $Z(R)$. Thus $\dim Z \geqq n$. This implies $Z = X$, hence $f_1 = f_2$.

Theorem 8.8. *Assume that M is a semialgebraic space of dimension n. Then M contains a semialgebraic open subset U which is semialgebraically isomorphic to an open semialgebraic subset of R^n.*

Proof. We may assume that M is a Zariski dense subspace of a reduced variety X over R. Then $\dim X = n$. Let S be the closed reduced subvariety of singular points of X. Then $S \neq X$. Thus M meets the set $X'(R)$ of real points of $X' := X \setminus S$, and $M \cap X'(R)$ is Zariski dense in X'. Replacing M by $M \cap X'(R)$ we assume that X is regular. We also assume without loss of generality that X is connected, hence irreducible, and affine.

We have $M = N_1 \cup \ldots \cup N_r$ with non empty sets

$$N_i := \{x \in X(R) \mid g_i(x) = 0, f_{ij}(x) > 0, j = 1, \ldots, s_i\}.$$

and functions g_i, f_{ij} in the affine ring $R[X]$. If all g_i were different from zero then the Zariski closure of M would be contained in the subvariety $\{g_1 = 0\} \cup \ldots \cup \{g_r = 0\}$ of X which is different from X. Thus say $g_1 = 0$, and N_1 is an open semialgebraic subset of $X(R)$. As explained in the proof of Proposition 8.6 the set N_1 certainly contains an open semialgebraic subset U which is isomorphic to an open semialgebraic subset of R^n. q.e.d.

From Proposition 8.6 and Theorem 8.8 we extract the following very satisfactory description of the dimension of M.

Corollary 8.9. *The dimension of a semialgebraic space M is the largest natural number n such that M contains a non empty semialgebraic subset A which is isomorphic to an open semialgebraic subset of R^n.*

In the proof of Theorem 8.8 we have seen that for M a Zariski dense semialgebraic subspace of an n-dimensional variety there exists a non empty semialgebraic subset U of M which is *open in* $X(R)$ and isomorphic to an open semialgebraic subset of R^n. From this observation we deduce immediately

Theorem 8.10. *Let M be a semialgebraic space of dimension n and let B be a semialgebraic subset of M which also has dimension n. Then the interior $\mathring{B}$ of B in M is non empty and has again dimension n. (Recall from § 7 that $\mathring{B}$ is semialgebraic.)*

Corollary 8.11. *Let A be a non empty semialgebraic subset of a semialgebraic space M. Let $\bar{A}$ denote the closure of A in M which is again semialgebraic. The semialgebraic subset $\bar{A} \setminus A$ is either empty or has strictly smaller dimension than A.*

Proof. Let $r := \dim A$. Then also $\bar{A}$ has the dimension r (Proposition 8.4) and the set $B := \bar{A} \setminus A$ has dimension $\dim B \leq r$. If $\dim B = r$ then by the preceding Theorem 8.10 the interior $\mathring{B}$ of B with respect to $\bar{A}$ would be non empty. This is absurd since A is dense in $\bar{A}$. Thus $\dim B < r$.

§ 9. Complete Semialgebraic Spaces

Starting from now we assume for convenience that *all semialgebraic spaces are separated.* Otherwise we would be forced in this section to impose on various semialgebraic maps $M \to N$ the condition that they are separated, i.e. that the corresponding diagonal map $M \to M \times_N M$ is an isomorphism from M to a closed semialgebraic subspace of $M \times_N M$. For separated spaces M, N this is automatically true, cf. the arguments in [8, I, § 5]. We also assume that all occurring varieties are separated.

We copy a definition from the theory of schemes.

Definition. A semialgebraic map f: $M \to N$ between semialgebraic spaces is called *proper*, if for the pullback

$$\begin{array}{ccc} M \times_N N' & \xrightarrow{f'} & N' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array} \qquad (*)$$

with an arbitrary semialgebraic map g the map f' is closed in the restricted topologies, i.e. f' maps a closed semialgebraic subset A' of $M \times_N N'$ onto *a* closed (semialgebraic) subset $f'(A')$ of N'. A semialgebraic space M is called *complete* if the map from M to the one point space is proper. This means that for any semialgebraic space N the projection $M \times N \to N$ is closed in the restricted topologies.

The following statements about proper maps are evident:

i) Every closed embedding $M \to N$ (i.e. semialgebraic isomorphism from M onto a closed subspace of N) is proper.

ii) The composition of proper maps is proper.

iii) For any pullback diagram of semialgebraic maps, as drawn above (*), the map f' is proper if f is proper.

From these facts we conclude in a well known purely formal way, cf. [8, II, § 5.4]:

iv) If f: $M \to N$, g: $M' \to N'$ are proper semialgebraic maps "over" a semialgebraic space S (in the usual sense) then $f \times_S g$: $M \times_S M' \to N \times_S N'$ is again proper.

Let f: $M \to N$ and g: $N \to L$ be semialgebraic maps.

v) If $g \circ f$ is proper then f is proper. In particular (L = point) every semialgebraic map starting at a complete semialgebraic space is proper.

vi) If $g \circ f$ is proper and f is surjective then g is proper. In particular the image of a complete space under any semialgebraic map is again complete.

We now strive for an insight when a semialgebraic space is complete. Our first goal is to prove that any bounded and closed semialgebraic subset of R^n for any $n \geqq 0$ is a complete semialgebraic space. The following remark gives a motivation for the direction of our considerations below.

Remark 9.1. Let M be a complete semialgebraic space. Then every semialgebraic function $f: M \to R$ attains a maximum and a minimum.

Proof. Consider f as a semialgebraic map into $\mathbb{P}^1(R) = R \cup \{\infty\}$. This map is proper (cf. statement (v) above). Thus $f(M)$ is a closed semialgebraic subset of $\mathbb{P}^1(R)$, which does not contain the point ∞. Thus $f(M)$ is contained in a closed interval $[-C, C]$. Now clearly the closed semialgebraic subsets of $[-C, C]$ are the unions of finitely many disjoint closed subintervals of $[-C, C]$ and of finitely many isolated points, (cf. Lemma 9.3 for a more general statement about curves). Clearly $f(M)$ contains a smallest and a largest element.

Proposition 9.2. *Let M be a closed and bounded semialgebraic subset of R^n for some $n \geqq 1$. Then every polynomial $f \in R[X_1, \ldots, X_n]$ attains on M a maximum and a minimum.*

This is well known to be true if R is the field of real numbers. The lemma can be transferred to an arbitrary real closed field by use of Tarski's principle. But we shall now give a direct proof of Proposition 9.2 without reference to the field of real numbers. Already Brumfiel has given such a proof [2, p. 207] using his theory of partially ordered rings. Our proof will be very different.

Of course it suffices to prove that f attains a maximum on M. We proceed by induction on the dimension d of M. The case $d = 0$ is trivial since then M is a finite set. Assume $d \geqq 1$. Let X be the Zariski closure of M in $\mathbf{A}^n$ and $X_1, \ldots, X_r$ the irreducible components of X. Every set $M_i := M \cap X_i(R)$ is Zariski dense in X_i and again closed and bounded in X. Thus we assume without loss of generality that X is irreducible.

We first treat the case $d = 1$. Here we need the following result which is clear from § 2 and in fact already contained in [11 II, § 7].

Lemma 9.3. *Let Y be a complete regular curve over R. We choose an orientation on $Y(R)$. The closed semialgebraic subsets of $Y(R)$ are the unions of full components, finitely many pairwise disjoint closed intervals and finitely many isolated points.*

We return to our closed bounded semialgebraic set M in R^n which has dimension 1 and whose Zariski closure is an irreducible curve X in $\mathbf{A}^n$. Let $\bar{X}$ denote the projective completion of X in $\mathbb{P}^n$. Then M is a closed semialgebraic subset of $\bar{X}(R)$. Let $\pi: \tilde{X} \to \bar{X}$ be the normalization of $\bar{X}$ and $g: \tilde{X} \to \mathbb{P}^1$ the rational function obtained by composition of π with the rational function $f|X$ on $\bar{X}$. This function has no poles on the closed semialgebraic subset $\pi_R^{-1}(M) = \tilde{M}$ of $\tilde{X}(R)$ and we have $g(\tilde{M}) = f(M)$. Thus it suffices to prove that g attains a maximum on $\tilde{M}$. But this is clear from § 2 (or already [11 II, § 8]), since $\tilde{M}$ is a union of closed intervals and finitely many points by Lemma 9.3 above.

We now assume $d>1$. By Noether's normalization theorem there exists a finite surjective morphism $\pi_1: X \to \mathbf{A}^d$. Let $\pi_2: \mathbf{A}^d \to \mathbf{A}^{d-1}$ denote the standard projection by omission of the last coordinate and let $\pi = \pi_2 \circ \pi_1$ be the composite morphism from X onto $Y := \mathbf{A}^{d-1}$. All fibres $\pi^{-1}(y)$ of π at real points y of Y are – possibly not reduced or reducible – curves over R.

Let S denote the singular locus of X. There exists a non empty Zariski open subset V of Y such that the restriction

$$\varphi: U := (X \setminus S) \cap \pi^{-1}(V) \to V$$

of π is a smooth morphism [10, Chap. III, 10.7]. The reduced variety $T := X \setminus U$ has dimension at most $d-1$. We introduce the closed semialgebraic subset

$$N_1 := (M \cap T(R)) \cup (M \setminus \mathring{M})$$

of M with $\mathring{M}$ the interior of M in the space $X(R)$. The set N_1 has dimension at most $d-1$. Now consider the section $df \in \Gamma(U, \Omega_{U|V})$ of the invertible sheaf $\Omega_{U|V}$ of relative differentials of U over V. We consider first the case that df is not identically zero. Let Z' denote the zero locus of df on U and Z denote the Zariski closure of Z' in X. Z is either empty or a subvariety of X of dimension $d-1$. We introduce the closed semialgebraic subset

$$N := N_1 \cup (Z(R) \cap M)$$

of M, which again has dimension at most $d-1$. By induction hypothesis f attains a maximal value K on N. Let now x be a point of M not contained in N, hence in particular contained in U. Let C denote the scheme theoretic fibre of $\pi: X \to Y$ through x. The open subset $U \cap C$ of C is a regular curve. As we have seen in the case $d=1$ the function f attains a maximum on the closed semialgebraic subset $M \cap C(R)$ of $C(R)$ in some point x_0, since this set is a closed bounded semialgebraic set of dimension at most one in R^n. This point x_0 lies either in the complement of U, or x_0 lies on the boundary of the subset $M \cap C(R)$ of $C(R)$, hence in $M \setminus M^0$, or according to §2 we have $x_0 \in U$, $(d\bar{f})(x_0)=0$ for the regular function $\bar{f} = f|U \cap C$ on $U \cap C$, which means that x_0 lies in Z. Thus $x_0 \in N$, and we have $f(x) < f(x_0) \leqq K$. Thus f attains the maximum K also on M.

There remains the case that the section $df \in \Gamma(U, \Omega_{U|V})$ is zero on U. This means that f is constant on every Zariski-connected component of every fibre of the restriction $\pi^{-1}(V) \to V$ of π. We then choose another polynomial $h \in R[X_1, \dots, X_n]$ which is not constant on all these fibre-components and introduce the Zariski closure Z_1 of the set of zeros of $dh \in \Gamma(U, \Omega_{U|V})$ in X. The closed semialgebraic subset

$$A := N_1 \cup (Z_1(R) \cap M)$$

of M has again dimension at most $d-1$ and thus f attains on A a maximum K. Let x be a point of $M \setminus A$. Then $x \in U$. Let C denote the irreducible component of the fibre $\pi: X \to Y$ through x which contains x. The function $h|C$ attains on the set $M \cap C(R)$ a maximum in some point x_0. As shown above this point x_0 lies in A. Now f is constant on C. Thus

$$f(x) = f(x_0) \leqq K,$$

and Proposition 9.2 is completely proved.

We are ready to prove our first main result.

Theorem 9.4 (cf. [2, Proposition 8.13.5]). *Every bounded closed semialgebraic subset M of R^n, $n \geqq 1$ arbitrary, is a complete semialgebraic space.*

Proof. We have to verify that for any semialgebraic space N and any semialgebraic closed subset A of $M \times N$ the image $p(A)$ under the projection p: $M \times N \to N$ is closed in N. Suppose there exists a point $y_0 \in N$ which lies in the closure $\overline{p(A)}$ of $p(A)$ but not in $p(A)$. We choose a semialgebraic map φ: $L \to N$ from a bounded semialgebraic subset L of some R^m, $m \geqq 1$, into N which is a semialgebraic isomorphism from L onto an open semialgebraic neighbourhood $\varphi(L)$ of y_0 in N, and we consider the pullback

$$\begin{array}{ccc} M \times L & \xrightarrow{p'} & L \\ {\scriptstyle 1 \times \varphi} \big\downarrow & & \big\downarrow {\scriptstyle \varphi} \\ M \times N & \xrightarrow[p]{} & N. \end{array}$$

Let z_0 denote the preimage of y_0 under φ and let B denote the preimage $(1 \times \varphi)^{-1}(A)$ of A in $M \times L$, which is a closed semialgebraic subset of $M \times L$. The point z_0 lies in $\overline{p'(B)}$ but not in $p'(B)$. We now consider the canonical projection π: $R^n \times R^m \to R^m$ which extends p'. The set B is bounded and semialgebraic in $R^n \times R^m$, and z_0 lies in the closure $\overline{\pi(B)}$ of $\pi(B)$ in R^m, which is the same as $\pi(\bar{B})$ with $\bar{B}$ the closure of B in $R^n \times R^m$. But z_0 does not lie in $\pi(\bar{B})$. Indeed, B is closed in $R^n \times L$, since $M \times L$ is closed in $R^n \times L$. If there existed a point (x, z_0) in $\bar{B}$, then

$$(x, z_0) \in \bar{B} \cap (R^n \times L) = B,$$

and z_0 would lie in $\pi(B)$, which is not true. Now consider on $\bar{B}$ the polynomial function

$$p(x, z) = \|z - z_0\|^2$$

with $\| \; \|$ the euclidean norm on R^m. This function is nowhere zero on $\bar{B}$ but has values arbitrary near to zero. This contradicts Proposition 9.2. Our theorem is proved.

Proposition 9.5. *The projective space $\mathbb{P}^n(R)$ is complete for every $n \geqq 1$.*

Proof. We denote the projective standard coordinates on the variety $\mathbb{P}^n$ over R by $x_0, \ldots, x_n$. Let X denote the complement of the hypersurface $x_0^2 + \ldots + x_n^2 = 0$ in $\mathbb{P}^n$. We have the well known closed immersion

$$\varphi\colon X \to \mathbf{A}^N, \qquad N = \frac{(n+1)(n+2)}{2}$$
$$\varphi(x_0 : x_1 : \ldots : x_n) = (x_i x_j (x_0^2 + \ldots + x_n^2)^{-1} \mid i \leqq j).$$

Now $X(R) = \mathbb{P}^n(R)$ and φ_R is a semialgebraic isomorphism of $X(R)$ onto a closed algebraic subset M of the unit cube $[-1,1]^N$ in R^N. By the preceding theorem M is complete. Thus also $\mathbb{P}^n(R)$ is complete.

We now come to the second main result of this section. (A special case has been proved before: Theorem 4.2.)

Theorem 9.6. *Assume that $f\colon X \to Y$ is a proper morphism between algebraic varieties X and Y over R. Then the semialgebraic map $f_R\colon X(R) \to Y(R)$ is also proper.*

Proof. We first consider the case that f is projective. Then f has a factorization

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbb{P}^n \times Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

with p the natural projection and i a closed immersion. Since $\mathbb{P}^n(R)$ is complete the map p_R is proper. Also i_R is prover by the property (i) of proper maps stated at the beginning of this section. Thus f_R is proper.

In the general case we proceed by induction on the dimension d of X. If $d = 0$ there is nothing to prove. Assume $d \geqq 1$. By Chow's lemma [10, p. 107] there exists a commutative diagram of morphisms between varieties

$$\begin{array}{ccc} \tilde{X} & & \\ {\scriptstyle g}\downarrow & \searrow{\scriptstyle \tilde{f}} & \\ X & \xrightarrow[f]{} & Y \end{array}$$

with $\tilde{f}$ projective and g yielding an isomorphism from $g^{-1}(U)$ onto U for some open dense subscheme U of X. We have $X(R) = M \cup Z(R)$ with $M := g(\tilde{X}(R))$ and $Z := X \setminus U$. Now $\tilde{f}_R$ is proper as shown above. Consider the diagram

$$\begin{array}{ccc} \tilde{X}(R) & & \\ \downarrow & \searrow{\scriptstyle \tilde{f}_R} & \\ M & \xrightarrow[f_R|M]{} & Y \end{array}$$

with the vertical arrow obtained from g_R by restriction of the range. By property (vi) for proper maps we learn that $f_R|M$ is proper. On the other hand $\dim Z < d$ and the morphism $f|Z$ from Z to Y is proper. Thus by induction hypothesis $f_R|Z(R)$ is proper. The theorem now follows by an easy lemma whose proof will be left to the reader.

Lemma 9.7. *Let $f\colon M \to N$ be a semialgebraic map between semialgebraic spaces. Assume there exist semialgebraic subspaces $M_1, \ldots, M_r$ of M such that $M = M_1 \cup \ldots \cup M_r$ and all the restrictions $f|M_i\colon M_i \to N$ are proper. Then f is proper.*

We shall frequently use the following elementary fact about proper maps.

Proposition 9.8. *Let $f\colon M \to N$ be a proper semialgebraic map between semialgebraic spaces M and N. Assume that f is bijective. Then f is a semialgebraic isomorphism.*

Proof. f maps closed semialgebraic subsets of M onto closed semialgebraic subsets of N. Thus $f^{-1}\colon N \to M$ is strictly continuous. The graph of f^{-1} in $N \times M$ is obtained from the graph of f in $M \times N$ by applying the switch isomorphism $M \times N \xrightarrow{\sim} N \times M$. Thus the set $\Gamma(f^{-1})$ is semialgebraic in $N \times M$ and f^{-1} is a semialgebraic map.

We mention an application of our theory of proper maps.

Theorem 9.9. *Let $f\colon M \to N$ be a map from a semialgebraic space M to a complete semialgebraic space N. Assume that the graph of f is a closed semialgebraic subset of $M \times N$. Then f is semialgebraic.*

Proof. The problem is to prove that f is continuous at any given point x_0 of M. For this we may replace M by an arbitrary semialgebraic neighbourhood of x_0 in M. Thus we may assume that M is a bounded semialgebraic subset of R^n for some $n > 0$. Let $\bar{M}$ denote the closure of M in R^n and let $\bar{\Gamma}$ denote the closure of Γ in $R^n \times N$. Then $\bar{\Gamma}$ is a closed subspace of the complete space $\bar{M} \times N$. Thus $\bar{\Gamma}$ itself is a complete space. Let $\bar{p}$ denote the natural projection from $\bar{\Gamma}$ onto $\bar{M}$. This is a proper semialgebraic map. Since Γ is closed in $M \times N$ we have

$$\Gamma = \bar{\Gamma} \cap (M \times N) = \bar{p}^{-1}(M).$$

Thus the restriction $p\colon \Gamma \to M$ is again a proper semialgebraic map. p is the natural projection from Γ to M and thus bijective. By Proposition 9.8 p is a semialgebraic isomorphism. Introducing also the natural projection $q\colon \Gamma \to N$ we have $f = q \circ p^{-1}$, and we see that f is indeed a semialgebraic map.

Applying this Theorem 9.9 to the case $N = \mathbb{P}^m(R)$ we arrive at a result obtained by Brumfiel [2, Proposition 8.13.8].

Corollary 9.10. *Let $f\colon M \to R^m$ be a map which is locally bounded everywhere (i.e. every $x \in M$ has a neighbourhood U such that $f(U)$ is bounded). Assume that the graph of f is a closed semialgebraic subset of $M \times R^m$. Then f is semialgebraic.*

We now discuss proper maps in the case that R is the field $\mathbb{R}$ of real numbers. Let $f\colon M \to N$ be a semialgebraic map between semialgebraic spaces over $\mathbb{R}$. If f is a proper continuous map in the sense of topology [1, Chap. I, § 10], then clearly f is also a proper semialgebraic map. We want to prove the converse of this trivial fact.

Theorem 9.11. *Let $f\colon M \to N$ be a proper semialgebraic map over $\mathbb{R}$. Then f is a proper continuous map.*

We prove this first in the case that N is the one point space. Then M is a complete semialgebraic space, and we have to prove that M is a compact topological space. We need a lemma valid over any real closed field R and even for semialgebraic spaces which are not separated.

Lemma 9.12. *Let M be a semialgebraic space and V an open affine semialgebraic subset of M. Then there exists an open affine semialgebraic subset U of M which is dense in M and contains V.*

Proof. Let $(M_i | 1 \leqq i \leqq n)$ be a covering of M by finitely many open affine semialgebraic subsets with $M_1 = V$. Define $U_1 := M_1 = V$, $U_2 := M_2 \setminus \bar{U}_1$, $U_3 := M_3 \setminus (\bar{U}_1 \cup U_2), \ldots, U_n := M_n \setminus (U_1 \cup \ldots \cup U_{n-1})$. The U_i are pairwise disjoint affine open subsets of M. Thus the set

$$U := U_1 \cup \ldots \cup U_n$$

is again affine. Indeed, if every U_i can be semialgebraically embedded into R^m, then U can be embedded into R^{m+1} simply by choosing n distinct parallel hyperplanes $H_1, \ldots, H_n$ in R^{m+1} and embedding U_i into H_i. Clearly U is dense in M. q.e.d.

We shall use from Lemma 9.12 only the existence of a dense open affine semialgebraic subset U of M, but the lemma in the form stated above is more interesting.

We return to a complete semialgebraic space M over $\mathbb{R}$ and want to prove that M is compact. Assume first that M is affine, hence a semialgebraic subset of $\mathbb{R}^n$ for some n. Clearly M is closed in $\mathbb{R}^n$. The euclidean norm of $\mathbb{R}^n$ is a semialgebraic function on M, hence certainly bounded on M. Thus M is a closed bounded subset of $\mathbb{R}^n$, and we see that M is indeed compact.

We now prove that an arbitrary complete semialgebraic space M over $\mathbb{R}$ is compact. We proceed by induction on the dimension of M. If $\dim M = 0$ then M is a finite set, hence certainly compact. Assume now $\dim M > 0$. Let $\mathfrak{U}$ be a given covering of M by open sets. We have to show that there exists a finite open covering $\mathfrak{W}$ of M which refines $\mathfrak{U}$. We choose a covering $\mathfrak{B}$ of M which consists of open *semialgebraic* subsets of M and refines $\mathfrak{U}$. This is possible since the open semialgebraic sets are a basis of the topology of M. Let $(A_\alpha | \alpha \in I)$ be the family of the complements in M of all unions of finitely many members of $\mathfrak{B}$. All A_α are closed semialgebraic subsets of M. We have to show that $A_\alpha = \varnothing$ for some $\alpha \in I$. Then we know that already the finitely many members of $\mathfrak{B}$ used in the definition of A_α cover the whole space M.

Suppose that all A_α are non empty. By our Lemma 9.12 there exists a dense affine open semialgebraic subset U of M. The complement M' of U in M is a complete semialgebraic space of smaller dimension than M. By induction hypothesis M' is compact. If all A_α would meet M' then by the compactness of M' the intersection of all the sets $A_\alpha \cap M'$ would be non empty. But already the intersection of all A_α is empty. Thus there exists some $\gamma \in I$ with $A_\gamma \subset U$. Now A_γ is an affine complete semialgebraic space. As shown above A_γ is compact. All the sets $A_\alpha \cap A_\gamma$, $\alpha \in I$, are non empty. Thus the intersection of these sets is non empty. But this is the intersection of all A_α, $\alpha \in I$, which has to be empty.

This contradiction shows that indeed some A_α must be empty. We have proved that M is compact.

Let finally $f\colon M \to N$ be a proper semialgebraic map over $\mathbb{R}$. All fibres $f^{-1}(y)$, $y \in N$, are complete, hence compact. We want to prove that f is proper in the topological sense. For this it suffices to verify that the image $f(A)$ of any given closed subset A of M is closed in N [1, Chap. I § 10, Th. 1]. Suppose there exists some point y_0 in $\overline{f(A)}$ which is not contained in $f(A)$. The fibre $f^{-1}(y_0)$ is disjoint from the closed set A. For every point x of $f^{-1}(y_0)$ there exists an open semialgebraic neighbourhood $U(x)$ in M which does not meet the set A. Since $f^{-1}(y_0)$ is compact there exist finitely many points $x_1, \dots, x_n$ in $f^{-1}(y_0)$ with

$$f^{-1}(y_0) \subset U := U(x_1) \cup \dots \cup U(x_n).$$

The set $B := M \setminus U$ is closed and semialgebraic in M and contains A. Since f is proper semialgebraic the set $f(B)$ is closed in N. Thus $f(B)$ contains $\overline{f(A)}$. This is a contradiction since y_0 does not lie in $f(B)$. The map f must be proper in the topological sense, and Theorem 9.11 is proved.

A further result on proper semialgebraic maps will be proved in § 12 (Theorem 12.5).

§ 10. Semialgebraic Paths

The unit interval $[0,1]$ in R will be always considered as a semialgebraic subspace of the affine line $\mathbf{A}^1$.

Definition. *A semialgebraic path* α in a semialgebraic space M is a semialgebraic map α from $[0,1]$ to M.

Our goal in this section is to explain that for any variety X over R the semialgebraic paths in $X(R)$ are essentially the same objects as the combinatorial paths defined in § 3. The following theorem gives one half of this statement. It implies that every combinatorial path can be "parametrized" to become a semialgebraic path.

In contrast to the preceding sections § 6–§ 9 we now make strong use of the terminology and some results of part I of the paper.

Theorem 10.1. *Let γ be a non degenerate elementary path in the set $X(R)$ of real points of a variety X, as defined in § 3. Then there exists an order preserving semialgebraic isomorphism from the unit interval $[0,1]$ to the ordered closed semialgebraic subset γ of $X(R)$.*

Here we of course consider also $[0,1]$ as an elementary path, hence an ordered set, in the usual way.

Proof. We replace X by the Zariski closure of γ in X and assume that X is a reduced irreducible curve. Let $\tilde{X}$ denote the normalization of X and $\tilde{\gamma} \subset \tilde{X}(R)$ the normalization of γ. The canonical projection $\pi\colon \tilde{\gamma} \to \gamma$ is a bijective proper semialgebraic map, hence a semialgebraic isomorphism (cf. Proposition 9.8).

We now replace X and γ by $\tilde{X}$ and $\tilde{\gamma}$ and assume that X is also regular. We finally replace X by its regular completion. Thus we assume without loss of generality that X is a complete connected regular curve. We choose an orientation on $X(R)$ such that γ is an interval $[P,Q]$. We now choose an arbitrary function f in $R(X)^*$ which has no poles on γ. By §2 (or already [11 II, §8]) there exists a subdivision

$$P=P_0<P_1<P_2<\ldots<P_n=Q$$

of $[P,Q]$ such that f maps every subinterval $[P_{i-1},P_i]$ either strictly increasing or strictly decreasing onto a closed interval $[c_i,d_i]$ in R. As above we conclude that f yields a semialgebraic isomorphism from $[P_{i-1},P_i]$ onto $[c_i,d_i]$. We now define a strictly increasing semialgebraic isomorphism g_i: $[P_{i-1},P_i]\xrightarrow{\sim}[0,1]$ as follows:

$$g_i(x):=(d_i-c_i)^{-1}[f(x)-c_i]$$

if f is increasing on $[P_{i-1},P_i]$, and

$$g_i(x):=(d_i-c_i)^{-1}[d_i-f(x)]$$

if f is decreasing on $[P_{i-1},P_i]$. We finally define an order preserving semialgebraic isomorphism g: $[P,Q]\xrightarrow{\sim}[0,1]$ by

$$g(x):=\frac{i-1}{n}+\frac{g_i(x)}{n}$$

for x in $[P_{i-1},P_i]$. Then $\alpha:=g^{-1}$ is an order preserving isomorphism from $[0,1]$ to $\gamma=[P,Q]$ as desired.

Theorem 10.2. *Let f: $[a,b]\to X(R)$ be a semialgebraic map from a closed interval $[a,b]$ in R to the space of real points of an algebraic variety X over R. Then there exists a subdivision*

$$t_0=a<t_1<\ldots<t_n=b$$

of $[a,b]$ with the following properties: f maps every subinterval $[t_{i-1},t_i]$ onto an elementary path γ_i. If γ_i is non degenerate this map from $[t_{i-1},t_i]$ to the ordered semialgebraic set γ_i is an order preserving semialgebraic isomorphism.

Thus f is a "parametrization" of a combinatorial path $(\gamma_1,\ldots,\gamma_n)$.

Proof. Let Γ denote the graph of f in $[a,b]\times X(R)$ and let Z denote the Zariski closure of Γ in $\mathbb{P}^1\times X$. The semialgebraic space Γ is isomorphic to $[a,b]$. In particular $\dim\Gamma=1$, hence $\dim Z=1$. Let $Z_1,\ldots,Z_r$ denote the irreducible components of Z. Then $\Gamma_i:=\Gamma\cap Z_i(R)$ is Zariski dense in Z_i. Moreover Γ_i is a closed semialgebraic subset of Γ. Thus the image M_i of Γ_i in $[a,b]$ under the projection p: $\Gamma\xrightarrow{\sim}[a,b]$ is a disjoint union of closed subintervals I_{ij} of $[a,b]$ $(j=1,\ldots,s_i)$ and finitely many points (cf. Lemma 9.3). The Zariski closure of every set $p^{-1}(I_{ij})$ in $\mathbb{P}^1\times X$ must be the component Z_i. Thus certainly any two intervals I_{ij}, I_{kl} with $i\neq k$ have as intersection a finite set. This means that I_{ij}

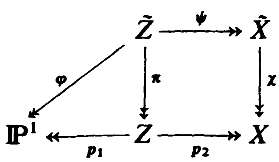
and I_{kl} are either disjoint or have one common endpoint. The I_{ij} altogether cover all of $[a,b]$ except perhaps finitely many points. Let

$$t_0 < t_1 < \ldots < t_n$$

be the endpoints of all the I_{ij}. Then we must have $t_0 = a$, $t_n = b$, and the I_{ij} are precisely all the intervals $[t_{k-1}, t_k]$, $k = 1, \ldots, n$. In particular the I_{ij} cover the whole of $[a,b]$.

It suffices now to prove the assertion of the theorem for the restrictions $f|[t_{k-1}, t_k]$ instead of f itself. Thus we assume without loss of generality that Z is irreducible. Let $p_1\colon Z \to \mathbb{P}^1$ and $p_2\colon Z \to X$ denote the natural projections from Z to $\mathbb{P}^1$ and X. We replace X by the Zariski closure of $p_2(Z)$ in X, hence assume that X is irreducible and reduced of dimension $\leqq 1$ and that p_2 is dominant. If X is a point there is nothing to prove. Since now we assume that X is a reduced irreducible curve. Replacing X by some projective completion $\bar{X}$ of X and Z by the Zariski closure $\bar{Z}$ of Z in $P^1 \times \bar{X}$ we also assume that X and Z are complete.

We have a commutative diagram of surjective morphisms



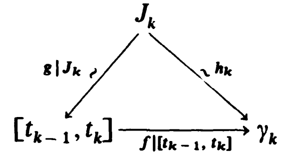
with π and χ the normalizations of Z and X. We choose orientations on $\tilde{Z}(R)$ and $\tilde{X}(R)$. The preimage $\pi_R^{-1}(\Gamma)$ of the graph Γ in $\tilde{Z}(R)$ is by Lemma 9.3 a union of finitely many closed intervals $J_1, \ldots, J_n$, any two of them having at most one end point in common, and finitely many points disjoint from the set

$$\tilde{\Gamma} := J_1 \cup \ldots \cup J_n.$$

The restriction $g\colon \tilde{\Gamma} \to [a,b]$ of φ_R has only finitely many fibres which contain more than one point. Subdividing the intervals J_k suitably we may assume that g is injective on every J_k, hence maps every J_k strictly increasing or decreasing onto a closed subinterval I_k of $[a,b]$ according to §2 (or already [11 II, §8]). Every map $g|J_k$ is a semialgebraic isomorphism from J_k onto I_k, cf. Proposition 9.8. We now see as above that there exists a subdivision

$$t_0 = a < t_1 < \ldots < t_n = b$$

of $[a,b]$ such that $I_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, n$ after a change of the enumeration of the intervals I_k, J_k. Again we have in particular $g(\tilde{\Gamma}) = [a,b]$. Subdividing the J_k further we achieve according to §2 that in addition ψ_R maps every interval J_k strictly monotonely and semialgebraically isomorphic onto an interval L_k in $\tilde{X}(R)$. Subdividing the J_k still further we also achieve that every L_k is mapped injectively into $X(R)$ under χ_R. Thus every map $\chi_R \circ \psi_R | J_k$ is a semialgebraic and order preserving isomorphism h_k from J_k onto an elementary non degenerate path γ_k in $X(R)$. We have commutative diagrams



Eventually reversing the order of γ_k we see that $f|[t_{k-1}, t_k]$ is an order preserving semialgebraic isomorphism from $[t_{k-1}, t_k]$ onto γ_k. Theorem 10.2 is now proved.

Remark. We could have given a shorter proof by use of Theorem 3.3 on the lifting of elementary paths. But we want to avoid Theorem 3.3 here since we shall strive in the next section for a second independent proof of the finiteness theorem in § 4.

In a way similar to the study of the graph Γ in this proof, but easier, one obtains from Lemma 9.3 also the following fact.

Proposition 10.3 (Triangulation of semialgebraic spaces of dimension $\leqq 1$). *Let M be a complete semialgebraic subspace of a variety with* $\dim M \leqq 1$. *Then*

$$M = \gamma_1 \cup \ldots \cup \gamma_r \cup \{P_1, \ldots, P_s\}$$

with finitely many elementary paths γ_i, such that for $i \neq j$ the sets γ_i and γ_j are either disjoint or have one common endpoint, and finitely many isolated points P_j.

In a later paper [5] we shall "triangulate" complete affine semialgebraic spaces of arbitrary dimension.

§ 11. Path Components Again

Starting from now we mean by a "path", if no further specification is given, always a semialgebraic path. We call two points P, Q of a semialgebraic space M *path connectable* (in M) if there exists a path γ: $[0,1] \to M$ with $\gamma(0) = P$ and $\gamma(1) = Q$. Clearly "path connectable" is an equivalence relation on M. We call the equivalence classes the *path components* of M and denote the set of all path components of M by $\pi_0(M)$. Any semialgebraic map φ: $M \to N$ between semialgebraic spaces M, N yields a map φ_*: $\pi_0(M) \to \pi_0(N)$ in the obvious way.

The following rather evident facts about a path connected semialgebraic space M, i.e. a space M with a single path component, will be frequently used.

Proposition 11.1. *Let M be path connected. Then M has no decomposition $M = U_1 \cup U_2$ into two disjoint non empty open semialgebraic subsets U_1, U_2. Every semialgebraic function f: $M \to R$ which has no zeros has constant sign on M.*

Proof. Suppose $M = U_1 \cup U_2$ with non empty disjoint open semialgebraic subsets U_1, U_2 of M. We choose two points x_1 in U_1 and x_2 in U_2. Since M is path connected there exists a path α: $[0,1] \to M$ with $\alpha(0) = x_1$ and $\alpha(1) = x_2$. Now U_1, U_2 are also closed in M. Thus $B_1 := \alpha^{-1}(U_1)$ and $B_2 := \alpha^{-1}(U_2)$ are disjoint closed semialgebraic subsets of $[0,1]$, which cover $[0,1]$ and are both non empty. B_1 and B_2 are both disjoint unions of finitely many closed intervals and one point sets (cf. Lemma 9.3). We obtain a decomposition

$$[0,1]=I_1 \cup \ldots \cup I_r$$

with $r \geqq 2$ and pairwise disjoint sets I_k which are closed intervals or contain only one point. But such a decomposition is clearly impossible. Thus our original decomposition $M=U_1 \cup U_2$ was impossible.

If $f\colon M \to R$ is a semialgebraic function on M without zeros then M is the union of the open semialgebraic subsets $f^{-1}(]\infty, 0[)$ and $f^{-1}(]0,\infty[)$. One of these sets must be empty. Thus f has constant sign on M. q.e.d.

Our goal in this section is a proof of the following generalization and improvement of Theorem 4.1.

Theorem 11.2. *For every semialgebraic space M the set $\pi_0(M)$ is finite. All path components of M are semialgebraic subsets of M.*

In § 12 we shall also see that the path components are open in M. This will generalize Theorem 4.3.

The proof of Theorem 11.2 needs some preparations. We recall an important lemma from Cohen's paper [3, § 1].

Lemma 11.3. *Let $f(X_1,\ldots,X_n,Y)$ be a polynomial in $n+1$ variables $X_1,\ldots,X_n,Y$ over R which for the variable Y has degree $\leqq d$. Then there exist R-valued functions $k,\xi_1,\ldots,\xi_d$ on R^n with the following properties:*

i) *k has its values in the finite set $\{-1,0,1,\ldots,d\}$.*

ii) *For every $a \in R^n$ with $k(a)>0$ we have*

$$\xi_1(a)<\xi_2(a)<\ldots<\xi_{k(a)}(a)$$

and these values $\xi_i(a)$ are all the real roots of $f(a,Y)$. If $k(a)=0$ then $f(a,Y)\neq 0$ but $f(a,Y)$ has no real roots. If $k(a)=-1$ then $f(a,Y)=0$.

iii) *All the functions k, $\xi_1,\ldots,\xi_d$ have semialgebraic graphs. (They are even "effective" in the sense of Cohen, but we do not need this here.)*

Lemma 11.4. *Let $f \in R[X_1,\ldots,X_n,Y]$ be a polynomial in $n+1$ variables over R and let M be a semialgebraic subset of R^n. Assume that the following holds true:*

i) *For every $k>0$ either $\dfrac{\partial^k f}{\partial Y^k}(x,Y)=0$ for all $x \in M$ or the polynomial $\dfrac{\partial^k f}{\partial Y^k}(x,Y)$ has for every $x \in M$ precisely r_k distinct real roots*

$$\xi_1^k(x)<\ldots<\xi_{r_k}^k(x)$$

with r_k independent of x.

ii) *Any two of these roots either coincide or they do not meet on M, that means: Either $\xi_i^k(x)=\xi_j^l(x)$ for all $x \in M$ or $\xi_i^k(x)\neq\xi_j^l(x)$ for all $x \in M$.*

Then all these roots ξ_i^k are semialgebraic functions on M. If M is path connected then for any two roots ξ_i^k and ξ_j^l which do not coincide either $\xi_i^k(x)>\xi_j^l(x)$ for all $x \in M$ or $\xi_i^k(x)<\xi_j^l(x)$ for all $x \in M$.

Proof. By Lemma 11.3 the ξ_i^k have semialgebraic graphs. Every ξ_i^k is a simple root everywhere on M of some polynomial $\frac{\partial^l f}{\partial Y^l}$. By the implicit function theorem 6.9 the function ξ_i^k is continuous on M, hence semialgebraic. The last assertion is now clear from Proposition 11.1

We enter the proof of Theorem 11.2. Since M can be covered by finitely many (open) affine semialgebraic subspaces, we may assume that already M itself is affine, hence that M is a semialgebraic subset of R^n for some n. We proceed by induction on n. The case $n=0$ is trivial. We now assume that the assertion of Theorem 11.2 holds true for all semialgebraic subsets of R^n for some $n \geqq 0$, and we want to prove the assertion for a given semialgebraic subset M of R^{n+1}. Let G_0 be a finite set of polynomials in $R[X_1, \ldots, X_n, Y]$ defining M, that is, M is a (finite) union of sets of the form

$$\{(x,y) \in R^n \times R \mid f(x,y)=0,\ g_i(x,y)>0 \text{ for } i=1,\ldots,r\}$$

with $f, g_1, \ldots, g_r$ in G_0. Let G_1 be the set of all derivatives $\frac{\partial^k g}{\partial Y^k}$ of the polynomials g in G_0 of arbitrary order $k \geqq 1$ with respect to the last variable Y, and let G denote the union of the finite sets G_0 and G_1. We regard every element of G as a polynomial in Y, whose coefficients are functions on R^n. Applying Lemma 11.3, Tarski's theorem 6.8, Lemma 11.4 and our induction hypothesis we see that there exists a decomposition of R^n into finitely many disjoint path connected semialgebraic sets $A_1, \ldots, A_s$, such that for each set A_i the following two properties hold true:

a) Every $g \in G$ either vanishes everywhere on A_i or has a constant number of real roots. These roots are - ordered by their magnitude - semialgebraic functions on A_i.

b) Any two roots of different polynomials $g, h \in G$ either coincide everywhere on A_i or are different everywhere on A_i.

Property b) implies that the distinct roots of all the polynomials in G_0 over A_i can be ordered by their magnitude. Let

$$\xi_1^i < \xi_2^i < \ldots < \xi_{m_i}^i$$

be these roots. Now $A_i \times R$ is the disjoint union of the following finitely many semialgebraic sets.

If $m_i = 0$:

(1) $A_i \times R$.

If $m_i > 0$:

(2) $\{(x,y) \in A_i \times R \mid y = \xi_j^i(x)\}, j=1, \ldots, m_i$.

(3) $\{(x,y) \in A_i \times R \mid \xi_j^i(x) < y < \xi_{j+1}^i(x)\}, \quad 1 \leqq j \leqq m_i - 1$.

(4) $\{(x,y) \in A_i \times R \mid y < \xi_1^i(x)\}$.

(5) $\{(x,y) \in A_i \times R \mid y > \xi_{m_i}^i(x)\}$.

All these sets are path connected. This is trivial for the sets of type (1). It is also clear for the other sets since any path α: $[0,1]\to A_i$ can be lifted to a path in the sets (2), (3), (4), (5) by

$$t\mapsto(\alpha(t),\xi^i_j(\alpha(t)))$$
$$t\mapsto(\alpha(t),\tfrac{1}{2}[\xi^i_j(\alpha(t))+\xi^i_{j+1}(\alpha(t))])$$
$$t\mapsto(\alpha(t),\xi^i_1(\alpha(t))-1)$$
$$t\mapsto(\alpha(t),\xi^i_{m_i}(\alpha(t))+1)$$

respectively. A given polynomial $g\in G$ is on each of these sets (1)–(5) either identically zero or nowhere zero. In the second case g has constant sign on this set by Proposition 11.1. Thus M is the union of finitely many of the path connected semialgebraic sets (1)–(5), and Theorem 11.2 is evident.

As the referee kindly pointed out to us, a similar argument had already been used by G. Efroymson in the case $R=\mathbb{R}$ to prove that a semialgebraic set has only finitely many connected components, cf. G. Efroymson, A Nullstellensatz for Nash rings, Pac. J. Math. 54 (1974), p. 104f. We feel that the last paragraph in Efroymson's proof (p. 105) is correct but needs further explanation.

§ 12. The Curve Selection Lemma

We shall prove the following theorem and draw some consequences.

Theorem 12.1 ("Curve selection lemma"). *Let M be a semialgebraic subset of a semialgebraic space L, and let P be a point in the closure $\bar{M}$ of M. Then there exists a path α: $[0,1]\to L$ running from P to M, i.e. with $\alpha(0)=P$ and $\alpha(]0,1])\subset M$.*

This theorem is essentially already contained in Brumfiel's book [2, Proposition 8.13.6]. Our proof will be more naive and rather different from Brumfiel's proof. We need a lemma which has some relation to Theorem 3.3 in part I.

Lemma 12.2. *Let p: $R^{n+1}\to R^n$ denote the natural projection from R^{n+1} to R^n by omission of the last coordinate. Let β: $[0,1]\to R^n$ be a path in R^n such that $\beta(0)\neq\beta(t)$ for all $t\in]0,1]$. Let α: $]0,1]\to R^{n+1}$ be a lifting of $\beta|]0,1]$, i.e. a semialgebraic map with $p\circ\alpha=\beta|]0,1]$. Assume that the set $A:=\alpha(]0,1])$ is bounded. Then this set A is not closed. Assume further that $\bar{A}\setminus A$ contains only one point P. Then the map γ: $[0,1]\to R^{n+1}$ defined by $\gamma(t)=\alpha(t)$ for $0<t\leq 1$ and $\gamma(0)=P$ is a path in R^{n+1} with $p\circ\gamma=\beta$.*

Proof. The space $\bar{A}$ is complete. Thus $p(\bar{A})$ is closed in R^n. But $p(A)=\beta(]0,1])$ is not closed since $\beta(0)$ does not lie in this set. Thus $A\neq\bar{A}$. Since $\beta([0,1])$ is closed we have $p(\bar{A})=\beta([0,1])$. Assume now that $\bar{A}=A\cup P$. Then $p(P)=\beta(0)$. By Theorem 10.2 it is evident that for some ε in $]0,1]$ the map β is injective on the interval $[0,\varepsilon]$. For the proof of the last assertion of the theorem we may replace β by $\beta|[0,\varepsilon]$ and α by $\alpha|]0,\varepsilon]$ and then assume without loss of generality that β is injective on $[0,1]$. The restriction π: $\bar{A}\to\beta([0,1])$ of p is now a bijective semialgebraic map between complete spaces, hence a semialgebraic isomorphism (cf. Proposition 9.8). The map γ defined in the lemma is the composite $\pi^{-1} \circ \beta$ and thus is indeed semialgebraic. q.e.d.

We now enter the proof of the curve selection lemma. We clearly may assume that $L=R^n$ for some $n \geqq 1$, and we proceed by induction on n. The case $n=1$ is trivial, since then M is a disjoint union of finitely many intervals and finitely many isolated points. Assume now $n \geqq 1$, and that the theorem holds true up to the number n, and that $L=R^{n+1}$. We may assume that $P=0$. We easily retreat to the case that

$$M=\{(x,y)\in R^n \times R \mid f(x,y)=0,\ g_1(x,y)>0,\dots,g_r(x,y)>0\}.$$

with some polynomials f, $g_1,\dots,g_r$ in $R[X_1,\dots,X_n,Y]$. We subject the coordinates of R^{n+1} to a suitable linear transformation and then assume that $g_j(0,Y)\not\equiv 0$ for $j=1,\dots,r$ and in the case $f\not\equiv 0$ also that $f(0,Y)\not\equiv 0$. Let p: $R^{n+1}\to R^n$ be the projection by omission of the last coordinate. The set $N:=p(M)$ is semialgebraic in R^n and contains the point 0 in its closure. Applying Lemma 11.3 and the main result Theorem 11.2 of the last section we see that N has a decomposition $N=N_1\cup\dots\cup N_m$ into pairwise disjoint path connected semialgebraic sets with the following properties:

i) If $0\in N$, then $N_1=\{0\}$

ii) $f, g_1,\dots,g_r$ and all their higher partial derivatives with respect to Y either vanish over N_i or they have a constant number of real roots over N_i.

We order the roots of any of these polynomials by increasing magnitude. Then they are well defined functions on N_i.

iii) Any two of these roots of different polynomials either coincide everywhere on N_i or they are different everywhere on N_i.

We then know from Lemma 11.4 and Proposition 11.1 that all the roots are smialgebraic functions on N_i and that for two roots ξ,η which do not coincide either $\xi(x)<\eta(x)$ for every $x\in N_i$ or $\xi(x)>\eta(x)$ for every $x\in N_i$.

We fix some N_i, such that 0 is contained in the closure of $M\cap(N_i\times R)$. Then we choose a path component M_0 of $M\cap(N_i\times R)$ which contains 0 in its closure. The semialgebraic set $N_0:=p(M_0)$ is then also path connected and contained in N_i, and 0 is in the closure of N_0. We look for a path running from 0 to M_0.

If $0\in N_0$, then $N_0=\{0\}$, and we are back to the case $n=1$ which is settled. Since now we assume that $0\notin N_0$. We distinguish two cases.

Case 1: $f\not\equiv 0$.
Then $f(0,Y)\not\equiv 0$, hence $f(x,Y)\not\equiv 0$ for all $x\in N_0$. Let $0,\eta_1,\dots,\eta_s$ be the different roots of $f(0,Y)$. We choose some $\varepsilon>0$ in R such that $|\eta_i|>2\varepsilon$ for $1\leqq i\leqq s$. Let $\xi_1(x)<\dots<\xi_m(x)$ be the roots of $f(x,Y)$ for $x\in N_0$. The ξ_i are semialgebraic functions on N_0 and M_0 is the union of the disjoint semialgebraic subsets

$$M_j:=\{(x,u)\in M_0 \mid u=\xi_j(x)\},\quad j=1,\dots,m.$$

which are closed in M_0. Since M_0 is path connected, we have

$$M_0=\{(x,u)\in N_0\times R \mid u=\xi_j(x)\}$$

with some fixed $j\in\{1,\dots,m\}$. Now 0 is already in the closure of the semialgebraic subset

$$M':=M_0\cap(R^n\times[-\varepsilon,\varepsilon])$$

of M, and 0 is also contained in the closure of the semialgebraic set $N':=p(M')$. By induction hypothesis there exists a path β: $[0,1]\to R^n$ running from 0 to N'. The map

$$\alpha\colon\,]0,1]\to M', \alpha(t):=(\beta(t),\xi_j\circ\beta(t))$$

is a semialgebraic lifting of $\beta|]0,1]$. By Lemma 12.2 the set $A:=\alpha(]0,1])$ is not closed. Every point in $\bar{A}\setminus A$ has coordinates $(0,\eta)$ with η a root of $f(0,Y)$. But $A\subset R^n\times[-\varepsilon,\varepsilon]$. Thus by our choice of ε we must have $\eta=0$, and we learn that $\bar{A}=A\cup\{0\}$. According to Lemma 12.1 the map γ: $[0,1]\to R^{n+1}$ with $\gamma(0)=0$, $\gamma(t)=\alpha(t)$ for $0<t\leqq 1$, is a semialgebraic path. This path runs from 0 to $M'\subset M_0$.

Case 2: $f\equiv 0$.
Since M_0 is a path component of $M\cap(N_i\times R)$ we have

$$M_0=\{(x,u)\in N_0\times R \mid \eta_1(x)<u<\eta_2(x)\}$$

with η_i a root of a polynomial g_{k_i} or $\eta_i\equiv\mp\infty$. Let h_i be the polynomial g_{k_i} in the first case and $h_i=1$ in the second case $(i=1,2)$. We choose some $\varepsilon>0$ in R such that $2\varepsilon<|\lambda|$ for every root λ of $h_1h_2(0,Y)$ which is different from zero. N_0 is the union of the following four semialgebraic subsets A_1,A_2,A_3,A_4:

$$\begin{aligned}
A_1&:=\{x\in N_0 \mid \eta_1(x)<-\varepsilon,\ \eta_2(x)>\varepsilon\},\\
A_2&:=\{x\in N_0 \mid \eta_1(x)\geqq-\varepsilon,\ \eta_2(x)\leqq\varepsilon\},\\
A_3&:=\{x\in N_0 \mid \eta_1(x)<-\varepsilon,\ \eta_2(x)\leqq\varepsilon\},\\
A_4&:=\{x\in N_0 \mid \eta_1(x)\geqq-\varepsilon,\ \eta_2(x)>\varepsilon\}.
\end{aligned}$$

Thus M_0 is the union of the semialgebraic subsets

$$M_i:=M_0\cap(A_i\times R),\qquad 1\leqq i\leqq 4.$$

Now $0\in\overline{M}_i$ for at least one of these four sets, and then also $0\in\bar{A}_i$. We treat the four cases separately.

a) $0\in\overline{M}_1$. By induction hypothesis there exists a path β in R^n from 0 to A_1. Then $\alpha(t):=(\beta(t),0)$, $0\leqq t\leqq 1$, is a path in R^{n+1} from 0 to M_1.

b) $0\in\overline{M}_2$. We choose a path β in R^n from 0 to A_2. The semialgebraic maps $(i=1,2)$

$$\alpha_i\colon\,]0,1]\to R^{n+1},\qquad \alpha_i(t)=(\beta(t),\eta_i\circ\beta(t)),$$

both have all values in $A_2 \times [-\varepsilon, \varepsilon]$, and every boundary point of $\alpha_i(]0,1])$ over $\beta(0)=0$ is a point $(0,\tau)$ with $(h_1 h_2)(0,\tau)=0$. We see as above with Lemma 12.2 that the maps α_i can be completed by the values $\alpha_i(0)=0$ to semialgebraic paths $\alpha_i\colon [0,1] \to R^{n+1}$. Now

$$\alpha(t)=\tfrac{1}{2}(\alpha_1(t)+\alpha_2(t)), \qquad 0 \leqq t \leqq 1,$$

is a path in R^{n+1} from 0 to M_2.

c) $0 \in \overline{M}_3$. The set A_3 is the union of two subsets

$$A_{31} := \{x \in A_3 \mid \eta_2(x) > -\frac{\varepsilon}{2}\},$$

$$A_{32} := \left\{x \in A_3 \mid \eta_2(x) \leqq -\frac{\varepsilon}{2}\right\},$$

and M_3 is the union of the two sets

$$M_{3i} := M_3 \cap (A_{3i} \times R); \qquad i=1,2.$$

Now every point z of M_{32} has euclidean distance $\|z\| \geqq \frac{\varepsilon}{2}$ from 0. Thus 0 lies in the closure of M_{31} and then also in the closure of A_{31}. We choose a path β in R^n from 0 to A_{31}. We then see as above that

$$\alpha(t)=(\beta(t), \eta_2 \circ \beta(t)), \qquad 0 < t \leqq 1, \ \alpha(0)=0,$$

is a semialgebraic path in R^{n+1}.

Now

$$\gamma(t) := \alpha(t) + (0, -\tfrac{1}{2}\varepsilon t), \qquad 0 \leqq t \leqq 1,$$

is a path in R^{n+1} running from 0 to M_3.

d) $0 \in \overline{M}_4$. Using the automorphism $(x,u) \to (x,-u)$ of the space R^{n+1} we are back to the preceding case.

The proof of Theorem 12.1 is finished. This theorem has the following important consequence.

Theorem 12.3. *The finitely many path components $M_1, \ldots, M_r$ of a semialgebraic space M are closed, hence also open semialgebraic subsets of M.*

Indeed, if $P \in M$ lies in the closure of M_i, then there exists a path in M running from P to M_i. Thus $P \in M_i$.

Definition. A semialgebraic space M is *connected*, if M is not the union of two non empty semialgebraic open subsets of M.

As stated in Proposition 11.1 every path connected semialgebraic space is connected. On the other hand it is clear from Theorem 12.3 that a space which is not path connected is not connected. Thus we can state

Corollary 12.4. *A semialgebraic space M is connected if and only if M is path connected.*

Since now we justly call the path components of a semialgebraic space M also the "connected components" of M or simply "components".

We mention that already Brumfiel and Coste-Roy both have proved that every semialgebraic set is a union of finitely many connected open semialgebraic subsets ([2, p. 260ff], [4, part A, Theorem 7.3]). Coste-Roy makes strong use of Tarski's principle. Brumfiel's proof is in close relation to Whitney's proof in the classical case $R=\mathbb{R}$ [14]. Both authors do not use paths. Our method yields the additional result that any two points in a connected component can be joined by a path.

The curve selection lemma is a very useful tool also beyond connectivity questions. We give an application to the theory of proper semialgebraic maps.

Theorem 12.5 (cf. [1, Chap. I § 10 Th. 1] for the analogous theorem in topology). *Let $f: M \to N$ be a semialgebraic map with the following two properties:*

i) *The image $f(A)$ of every closed semialgebraic subset A of M is closed in N.*

ii) *Every fibre $f^{-1}(y)$, $y \in N$, is complete.*

Then f is proper.

Proof. For any pullback

$$\begin{array}{ccc} M \times_N N' & \xrightarrow{f'} & N' \\ \downarrow & & \downarrow g \\ M & \xrightarrow{f} & N \end{array}$$

of f and any given closed semialgebraic subset A of $M \times_N N'$ we have to verify that $f'(A)$ is closed in N'. Suppose there exists a point z_0 in the closure of $f'(A)$ which does not belong to the semialgebraic set $f'(A)$. By the curve selection lemma there exists a path $\alpha: [0,1] \to N'$ running from z_0 to $f'(A)$. Consider the pullback q of f' by the map α which coincides with the pullback of f by the map $\gamma := g \circ \alpha$,

$$\begin{array}{ccc} M \times_N [0,1] & \xrightarrow{q} & [0,1] \\ p \downarrow & & \downarrow \gamma \\ M & \xrightarrow{f} & N. \end{array}$$

Let B denote the preimage of A in $M \times_N [0,1]$, i.e. $B = A \times_{N'} [0,1]$. Then B is a closed semialgebraic subset of $M \times_N [0,1]$ and $q(B) =]0,1]$. Now the map p is proper since $[0,1]$ is complete. Thus $p(B)$ is closed and semialgebraic in M. By hypothesis (i) the set $f \circ p(B) = \gamma(]0,1])$ is closed in N. Since $\gamma(]0,1])$ is dense in $\gamma([0,1])$ this means that

$$\gamma(]0,1]) = \gamma([0,1]).$$

Applying the same consideration to the closed subset

$$B_c := B \cap q^{-1}[0,c] = A \times_{N'} [0,c]$$

of $M \times_N [0,c]$ with arbitrarily chosen c in $]0,1]$ we see more generally that

$$\gamma(]0,c]) = \gamma([0,c]).$$

Thus there exist values $t \in]0,1]$ arbitrarily near to 0 with $\gamma(t) = \gamma(0) = g(z_0)$. Let us denote this point $g(z_0)$ by y_0. The set $\gamma^{-1}(y_0)$ is closed and semialgebraic in $[0,1]$, hence is a disjoint union of finitely many closed intervals and finitely many points. Thus $\gamma^{-1}(y_0)$ must contain an interval $[0,\varepsilon]$ with $\varepsilon > 0$. Since γ maps $[0,\varepsilon]$ to one point y_0 we have

$$M \times_N [0,\varepsilon] = f^{-1}(y_0) \times [0,\varepsilon].$$

B_ε is a closed semialgebraic subset of this space which under the natural projection to the second factor has the image $]0,\varepsilon]$. But by hypothesis (ii) the space $f^{-1}(y_0)$ is complete. Thus B_ε should have a closed image. This contradiction finishes the proof of the theorem.

§ 13. Birational Invariance of $\pi_0(X(R))$

We prove the theorem - well known in the case $R = \mathbb{R}$ — that for any two birationally equivalent complete smooth varieties X and Y over R the semialgebraic spaces $X(R)$ and $Y(R)$ have the same number of components. The proof is a straightforward adaption of classical arguments and is meant as an illustration that it is possible by our theory to transfer quite a lot of geometric ideas, familiar in the case $R = \mathbb{R}$, to arbitrary real closed base fields. Again a quite different proof has been given before by Coste-Roy [4, part A § 8].

Definition 1. The *local dimension* $\dim_x M$ of a semialgebraic space M at a point x of M is the infimum of the dimensions of all open semialgebraic neighbourhoods of x in M. We call M *pure of dimension n*, if $\dim_x M = n$ for all $x \in M$. (Then certainly $\dim M = n$, cf. Theorem 8.8.)

Definition 2. An *n-dimensional semialgebraic manifold M over R* is a semialgebraic space M over R in which every point x of M has an open semialgebraic neighbourhood isomorphic to some open semialgebraic subset of R^n. (Of course M is then pure of dimension n.)

Notice that by the implicit function theorem for every regular variety X of dimension n over R the space $X(R)$ is a semialgebraic n-dimensional manifold over R (cf. proof of Proposition 8.6).

Lemma 13.1. *Assume that M is a semialgebraic space which is pure of dimension n, and that N is a semialgebraic subset of M of dimension $\leqq n-1$. Then $M \setminus N$ is dense in M, and the natural map from $\pi_0(M \setminus N)$ to $\pi_0(M)$ is surjective.*

Proof. Let x be a given point of M and U an arbitrary open semialgebraic neighbourhood of x. Since $\dim N \leqq n-1$, but $\dim U = n$, the set U cannot be contained in N. Thus U meets the set $M \setminus N$, and we see that $M \setminus N$ is dense in M. By the curve selection lemma every point of M can be connected by some path with some point of $M \setminus N$. This is the last assertion of the lemma.

Theorem 13.2. *Let M be a semialgebraic n-dimensional manifold over R and N a semialgebraic subset of M with $\dim N \leqq n-2$. Then the natural map from $\pi_0(M \setminus N)$ to $\pi_0(M)$ is bijective.*

Proof. We proceed by induction on n. The case $n \leqq 2$ is trivial. Assume $n \geqq 3$. According to the preceding lemma there only remains to prove the injectivity of the map from $\pi_0(M \setminus N)$ *to* $\pi_0(M)$. We have to verify for any two given points in $M \setminus N$ which are path connectable in M that they are path connectable in $M \setminus N$. Replacing M by the component of these points in M we may assume from the beginning that M is connected. Let $M_1, \ldots, M_r$ denote the components of $M \setminus N$. We have to show that $r=1$. Suppose $r>1$. By the preceeding lemma $M \setminus N$ is dense in M, hence

$$M = \overline{M}_1 \cup \ldots \cup \overline{M}_r.$$

The intersection of $\overline{M}_1$ with $\overline{M}_2 \cup \ldots \cup \overline{M}_r$ is not empty, since M cannot be decomposed into two disjoint non empty closed semialgebraic subsets. After a change of the enumeration of the M_i, $i \geqq 2$, we assume that $\overline{M}_1 \cap \overline{M}_2 \neq \emptyset$.

We choose a point x_0 in $\overline{M}_1 \cap \overline{M}_2$. Of course $x_0 \in N$. We further choose an open semialgebraic neighbourhood D of x_0 in M which admits an isomorphism $\varphi: D \xrightarrow{\sim} B$ onto the unit ball B in R^n with $\varphi(x_0)=0$. By the curve selection lemma there exist paths $\alpha_i: [0,1] \to D$, $i=1,2$, with $\alpha_i(0)=x_0$ and $\alpha_i(]0,1]) \subset D \cap M_i$. We shall connect $\alpha_1(1) \in M_1 \cap D$ with $\alpha_2(1) \in M_2 \cap D$ by a path in $D \setminus N$. This will be the desired contradiction and will finish the proof.

We transfer all objects to the unit ball B in R^n. Let $N' := \varphi(D \cap N)$, $M'_1 := \varphi(D \cap M_1)$, $M'_2 := \varphi(D \cap M_2)$, $\beta_i := \varphi \circ \alpha_i$. We consider some $r \in R$ with

$$0 < r < r_0 := \operatorname{Min}(\|\beta_1(1)\|, \|\beta_2(1)\|).$$

The sphere S_r of radius r around the origin has in $\mathbf{A}^n$ as Zariski closure the zero locus Σ_r of the irreducible polynomial $X_1^2 + \ldots + X_n^2 - r^2$. (This is clear since $\dim S_r = n-1$.) Let $Z_1, \ldots, Z_t$ be the irreducible components of the Zariski closure of N' *in* $\mathbf{A}^n$. If a variety Z_i is contained in some hypersurface Σ_r then of course r is uniquely determined by Z_i. Thus there can exist at most t values r in $]0, r_0[$ such that Σ_r contains some Z_i. We choose a value ρ in $]0, r_0[$ with $Z_i \not\subset \Sigma_\rho$ for $i=1, \ldots, t$. Since all the varieties Z_i are irreducible and have dimension at most $n-2$ the intersections $Z_i \cap \Sigma_\rho$ have dimension at most $n-3$. Thus $\dim(N' \cap S_\rho) \leqq n-3$.

Applying the induction hypothesis to the $(n-1)$-dimensional connected manifold S_ρ and the semialgebraic subset $N' \cap S_\rho$ of S_ρ we see that $S_\rho \setminus N'$ is path connected. Now by the mean value theorem, applied to the functions $\|\beta_1\|$ and $\|\beta_2\|$ on $[0,1]$, there exist elements t_1 and t_2 in $]0,1[$ such that the points $\beta_1(t_1)$ and $\beta_2(t_2)$ lie on S_ρ, and hence in $S_\rho \setminus N'$. They can be connected by a path in $S_\rho \setminus N'$. Thus $\beta_1(1)$ and $\beta_2(1)$ can be connected by a path in $B \setminus N'$. The proof of Theorem 13.2 is finished.

Definition 3. A rational map $f: X \to Y$ from a regular variety X to a complete variety Y over R induces a natural map $f_*: \pi_0(X(R)) \to \pi_0(Y(R))$ as follows: The domain of definition U of f (= maximal open subset of X on which f is regular) has in X a complement Z of codimension at least two ([8, 8.2.12]; it suffices that X is normal instead of regular). The morphism $g: U \to Y$ obtained from f by restriction induces a map

$$(g_R)_*: \pi_0(U(R)) \to \pi_0(Y(R)).$$

The inclusion $j: U \to X$ induces by the preceding theorem a bijective map

$$(j_R)_*: \pi_0(U(R)) \xrightarrow{\sim} \pi_0(X(R)).$$

We define

$$f_* = (g_R)_* \circ (j_R)_*^{-1}: \pi_0(X(R)) \to \pi_0(Y(R)).$$

Theorem 13.3. *Assume that $f: X \to Y$ is a birational map between regular complete varieties over R. Then the map f_* from $\pi_0(X(R))$ to $\pi_0(Y(R))$ is bijective.*

Proof. Again we denote by U the domain of definition of f and by $g: U \to Y$ the restriction of f to U. There exist open subschemes U_1 of U and V_1 of Y such that g gives be restriction an isomorphism $h: U_1 \xrightarrow{\sim} V_1$. We have the following commutative diagram:

$$\begin{array}{ccc}
\pi_0(X(R)) & & \\
\uparrow \cong & \searrow^{f_*} & \\
\pi_0(U(R)) & \xrightarrow{(g_R)_*} & \pi_0(Y(R)) \\
\uparrow & & \uparrow \\
\pi_0(U_1(R)) & \xrightarrow[(h_R)_*]{\sim} & \pi_0(V_1(R))
\end{array}$$

The vertical maps $\pi_0(U_1(R)) \to \pi_0(U(R))$ and $\pi_0(V_1(R)) \to \pi_0(Y(R))$ are surjective by Lemma 13.1. Thus f_* is surjective. In particular we have for the cardinalities of $\pi_0(X(R))$ and $\pi_0(Y(R))$ the inequality

$$|\pi_0(Y(R))| \leqq |\pi_0(X(R))|.$$

Interchanging the role of X and Y we see that

$$|\pi_0(Y(R))| = |\pi_0(X(R))|.$$

Thus $f_*: \pi_0(X(R)) \to \pi_0(Y(R))$ is bijective.[2]

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