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ISOALGEBRAIC GEOMETRY: FIRST STEPS

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In this talk I want to explain how it is possible to develop a "semialgebraic complex analysis" over an arbitrary algebraically closed field C of characteristic zero, which I call "isoalgebraic geometry." I shall omit nearly all proofs deferring a systematic exposition of the theory to the future.

Our theory emerges from the presence of a real closed field R in C with $R(\sqrt{-1}) = C$. A similar though more complicated theory should be possible using a p-adic field in C instead of R. The inequalities in the definition of semialgebraic sets in §1 below then have to be replaced by integrality conditions.

§ 1 Semialgebraic sets in an algebraic variety X over C.

We choose once and for all in C a subfield R with $R(\sqrt{-1}) = C$. This is always possible, usually in many different ways. R is real closed and has in particular a unique ordering. R is a topological field, a basis of open sets being given by the open intervals $\{x \in R \mid a < x < b\}$. Thus $C \cong R^2$ is also a topological field.

Let X be an algebraic variety over C. We assume for simplicity always that X is reduced, and we identify X with the set X(C) of C-rational points of X. The topology of C induces on X a "strong topology," finer than the Zariski topology, as follows: Cover X by (finitely many) affine Zariski open subsets X_i, $i \in I$. Embed each X_i into an affine standard space C^{n_i}. Take on each X_i the subspace topology of the cartesian product C^{n_i}. By definition a subset U of X is open in the strong topology if $U \cap X_i$ is open in X_i for every $i \in I$. It is easily seen that the strong topology does not depend on the choice of the X_i

nor on the choice of the embeddings into the C^{n_i}.

Unfortunately, if $R \neq \mathbb{R}$, this topology is always totally disconnected. I recapitulate from the paper [DK] how to remedy this pathology. X can be regarded as the set $Y(R)$ of R-rational points of a variety $Y = r_{C|R}(X)$ obtained from X by restriction of scalars from C to R. Intuitively Y can be described as follows: Choose as above a finite affine open covering $(X_i | i \in I)$ of X and embeddings $X_i \hookrightarrow C^{n_i}$. Identify C^{n_i} with R^{2n_i} in the usual way. Let Y_i denote the Zariski closure of X_i in C^{2n_i}. These varieties Y_i over R glue together in a rather obvious way and yield the desired variety $Y = \bigcup_{i \in I} Y_i$ over R.

More generally for <u>any</u> variety Y over R we can equip $Y(R)$ with a strong topology coming from the topology of R. We call a subset M of $Y(R)$ semialgebraic (in Y) if for a given (finite) affine Zariski open covering $(Y_i | i \in I)$ of Y, defined over R of course, every intersection $M \cap Y_i(R)$ is a finite union of sets

$$\{x \in Y_i(R) \mid f_1(x) > 0, \ldots, f_r(x) > 0, g_1(x) = 0, \ldots g_s(x) = 0\},$$

with functions $f_1, \ldots, f_r, g_1, \ldots, g_s$ in the affine ring $R[Y_i]$. This property does not depend on the choice of the covering $(Y_i | i \in I)$, cf. [DK, §6].

Let $f : M \to N$ be a map from a semialgebraic subset M of $Y(R)$ to a semialgebraic subset N of $Z(R)$ with Z another variety over R. We call the map f semialgebraic (with respect to Y and Z), if f is continuous (in the strong topologies, as always) and the graph $\Gamma(f) \subset M \times N$ of f is semialgebraic in $Y \times Z$.

A general principle of our theory is, that in all considerations only semialgebraic sets are allowed and that instead of continuous maps only semialgebraic maps are allowed. We have the following facts as a consequence of Tarski's theorem on the elimination of quantifiers in the elementary theory of real closed fields: For M a semialgebraic subset of $Y(R)$ the interior $\mathring{M}$ and the closure $\bar{M}$ are again semialgebraic. Under any semialgebraic map $f : M \to N$ images and preimages of semialgebraic sets are again semialgebraic. The composite $g \circ f$ of two semialgebraic maps f,g is again a semialgebraic map (cf. [DK, §6]).

A <u>semialgebraic path</u> in a semialgebraic set $M \subset Y(R)$ is a semialgebraic map from the unit interval $[0,1]$ in R to M. We compose semialgebraic paths in the usual way and thus have a partition of any semialgebraic set M into path components. The following fundamental theorem has been proved in [DK, §10 - 12].

Theorem 1.1. Every semialgebraic set $M \subset Y(R)$ consists of only finitely many path components $M_1,\dots,M_r$. These are semialgebraic open subsets of M.

We call a semialgebraic set $N \subset Y(R)$ connected, if N is not the union of two non empty semialgebraic disjoint subsets open in N. It is immediately verified that the unit interval [0,1] is connected. Thus also the path components $M_1,\dots,M_r$ of M are connected. Since they are open in M they are certainly the right substitute of the topological components in the case $R = \mathbb{R}$. Already Brumfiel [B, p. 260 ff] and M.F. Coste-Roy [CR, part A, Theorem 7.3] both proved that every semialgebraic set is a union of finitely many open connected semialgebraic subsets.

For any semialgebraic subset M of Y(R) we define the semialgebraic dimension $\dim_R M$ as the dimension dim Z - in the sense of algebraic geometry - of the Zariski closure Z of M in Y. The next proposition implies in particular that $\dim_R M$ is a truly semialgebraic invariant of M.

Proposition 1.2. [DK, §8] For any semialgebraic map $f : M \to V(R)$ from M to the set of real points of a variety V over R the inequality $\dim_R M \geq \dim_R f(M)$ holds true. If f is injective then both dimensions are equal.

It is also shown in [DK, §8] that $\dim_R M$ is the maximal natural number n such that the unit ball in R^n, consisting of all points $(x_1,\dots,x_n) \in R^n$ with $x_1^2 + \dots x_n^2 < 1$, can be mapped onto some semialgebraic subset of M by a semialgebraic isomorphism. Thus without doubt we are in possession of a very satisfactory dimension theory of semialgebraic sets. Pathologies like Peano curves do not occur in the semialgebraic setting.

Let now X be an irreducible n-dimensional algebraic variety over C. We identify X with the set Y(R) of real points of the variety $Y = r_{C|R}(X)$ over R, as explained above. We have the following two basic facts about X.

Theorem 1.3. X is connected.

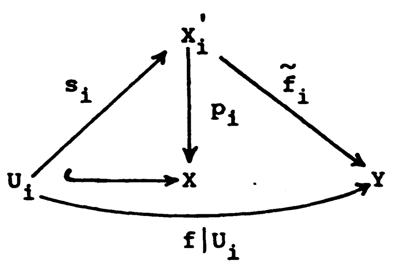
Theorem 1.4. X is pure of semialgebraic dimension $2n$, i.e. $\dim_R U = 2n$ for every non empty open semialgebraic subset U of X.

N.B. The analogues of both these theorems for the set $Y(R)$ of real points of an arbitrary irreducible variety Y over R are definitely wrong!

The last two theorems might give the impression that the semialgebraic structure of an irreducible variety X over C does not depend on the choice of R in an essential way. This is not true. Choosing a base point x_0 in X we define the fundamental group $\pi_1{}^R(X,x_0)$ in the usual way as the group of semialgebraic homotopy classes of semialgebraic paths $\alpha : [0,1] \to X$ with $\alpha(0) = \alpha(1) = x_0$. Examples by Serre [S] can be read as follows: There exists a smooth projective irreducible variety X over the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, defined over the absolute class field of a suitable imaginary quadratic number field, e.g. $\mathbb{Q}(\sqrt{-23})$, and real closures R,S of $\bar{\mathbb{Q}}$ such that the groups $\pi_1{}^R(X,x_0)$ and $\pi_1{}^S(X,x_0)$ are not isomorphic. On the other hand the profinite completions of both these groups are isomorphic since they both describe the finite etale coverings of X. Serre's examples are the more remarkable since both fields R,S are archimedian and in fact isomorphic.

§ 2 Isoalgebraic maps.

Let X and Y be varieties over C and U an open semialgebraic subset of X. A map $f : U \to Y$ is called isoalgebraic, if there exists a finite covering $(U_i | i \in I)$ of U by open semialgebraic subsets U_i such that every restriction $f|U_i$ has an "etale factorization." By this we mean that there exists a commutative diagram



with $p_i : X_i' \to X$ an etale morphism of varieties over C, s_i a semialgebraic section of p_i over U_i (in particular the Zariski open set $p_i(X_i')$ contains U_i), and $\tilde{f}_i$ a morphism of varieties over C.

Notice that in this situation s_i is a semialgebraic isomorphism from U_i to an open semialgebraic subset U_i' of X_i'. Notice also that an etale morphism $p : X' \to X$ over C has local semialgebraic sections at any point of $p'(X')$ by the implicit function theorem. (For elementary proofs of the implicit function theorem over an arbitrary real closed field R cf. [B, §8.7] or [DK, §6 Exercise].)

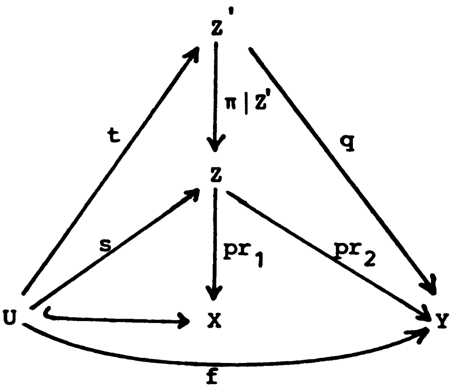
Morphisms between algebraic varieties over C will since now briefly be called "algebraic maps." Intuitively isoalgebraic maps serve the purpose "to make the inverse function theorem right for algebraic maps." Due to the presence of a real closed field in the base field C this can be done in a less formal way than has been done by M. Artin in his theory of algebraic spaces.

Since the covering $(U_i | i \in I)$ in the definition above is finite every isoalgebraic map is semialgebraic. It is easily seen that the composite $g \circ f$ of isoalgebraic maps $f : U \to Y$ and $g : V \to Z$ $(V \subset Y,\ f(U) \subset V)$ is again isoalgebraic. Also for two isoalgebraic maps $f_1 : U \to Y_1$, $f_2 : U \to Y_2$ the map $(f_1, f_2) : U \to Y_1 \times Y_2$ is isoalgebraic. We can state three useful theorems about isoalgebraic maps on normal varieties.

Theorem 2.1. (Global etale factorization) Assume that X is irreducible of algebraic dimension n, and that U is an open semialgebraic subset of X which is normal in X, i.e. every point of U is normal in X. Let $f : U \to Y$ be isoalgebraic. We denote by Z the Zariski closure of the graph $\Gamma(f) \subset U \times Y$ in $X \times Y$.

1. Claim: Z is irreducible and has algebraic dimension n.
Let $\pi : \bar{Z} \to Z$ be the normalization of Z and let Z' denote the set of all points of $\bar{Z}$ at which the map $pr_1 \circ \pi : \bar{Z} \to X$ is etale. (pr_1 = natural projection from Z to X; by standard algebraic geometry Z' is non empty and Zariski open in $\bar{Z}$.) Let $p : Z' \to X$ denote the etale map obtained from $pr_1 \circ \pi$ by restriction.

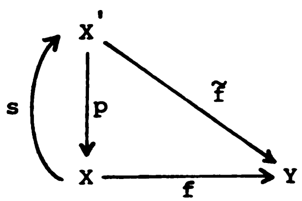
2. Claim: $U \subset p(Z')$. There exists a unique semialgebraic section $t : U \to Z'$ of p over U such that $\pi \circ t$ coincides with the evident section $s : x \to (x, f(x))$ of $pr_1 : Z \to X$ over U. Thus we have the factorization $f = q \circ t$ with $q = pr_2 \circ (\pi | Z')$.



All this can be proved in the same way as Artin and Mazur prove the analogous fact for Nash functions [AM, §2]. Artin and Mazur assume in their real setting that U is smooth in X, i.e. every point of U is regular in X. In our complex situation Theorem 1.4 above makes it possible to replace "smooth" by "normal."

Theorem 2.2. (*The case* $U = X$) *Let* $f : X \to Y$ *be an isoalgebraic map from a normal irreducible variety* X *to a variety* Y *over* C. *Then* f *is algebraic*.

This is a consequence of the preceeding theorem and Theorem 1.3: f has a global etale factorization



with X irreducible. By the implicit function theorem p is as a semialgebraic map locally trivial with finite fibres. Thus p is a "semialgebraic covering." This covering has a section s. But X' is connected by Theorem 1.3. Thus p must be a semialgebraic isomorphism and in particular bijective. We conclude by Zariski's main theorem that p is an algebraic isomorphism, since X is normal and we are in characteristic zero. Thus $f = \tilde{f} \circ p^{-1}$ is algebraic.

Theorem 2.2 can be regarded as a "GAGA principle." The theorem is the more remarkable since no properness condition is needed for f.

Theorem 2.3. (Local nature of isoalgebraic maps) Let U be a semialgebraic open normal subset of X and let $f : U \to Y$ be a semialgebraic map into a variety Y over C. Suppose that every $x \in U$ has an open semialgebraic neighborhood $W_x \subset U$ such that the restriction $f|W_x$ is isoalgebraic. Then f is isoalgebraic.

This theorem is harder than it may look at first glance. One has to pay the debts for admitting only finite open coverings $(U_i | i \in I)$ in the definition of isoalgebraic maps. The main difficulty is in my opinion to understand why the Zariski closure of the graph of f has algebraic dimension n.*) On the other hand Theorem 2.3 is the result which makes isoalgebraic maps manageable allowing local considerations.

3 Differential quotients and Taylor series.

Let again X be a variety over C and U a semialgebraic open subset of X. By an isoalgebraic function on U we simply mean a semialgebraic map from U to the affine line $C = A^1$.

Example. (The function $\sqrt[d]{z}$) Take $X = A^1 = C$ and U the complement of the negative real axis $\{z \in R \mid z < 0\}$ in C. For every $d \in \mathbb{N}$ there exists a unique semialgebraic function $f_d : U \to C$ such that $f_d(z)^d = z$ for all $z \in U$ and $f_d(1) = 1$, and this function is isoalgebraic. Indeed, $\pi_1^R(U,1) = 1$. From this one can conclude that the finite etale covering $z \to z^d$, $C^* \to C^*$, has a unique semialgebraic section over U with $1 \to 1$.

We return to the general situation above. If f and g are isoalgebraic functions on U then f+g is the composite of $(f,g) : U \to C \times C$ and the addition map from $C \times C$ to C. In the same way we see that fg is isoalgebraic on U. Thus the set $A_X(U)$ of isoalgebraic functions on U is a commutative algebra over C. It follows from the definition of isoalgebraic functions that for every finite covering $(U_i, i \in I)$ of U by open semialgebraic subsets U_i the canonical sequence

$$0 \to A_X(U) \to \prod_{i \in I} A_X(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} A_X(U_i \cap U_j)$$

is exact. Thus $U \to A_X(U)$ is a sheaf A_X on X in the "semialgebraic topology" (cf. [DK, §7]).

If an isoalgebraic function $f : U \to C$ has no zeros on U then also

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* I thank Hans Delfs (Universität Regensburg) for much help in the proof of Theorem 2.3.

$1/f$ is isoalgebraic on U. Thus the stalks ($p \in X$)

$$A_{X,p} = \lim_{\substack{\rightarrow \\ U \ni p}} A_X(U)$$

of our sheaf A_X are local rings with residue class field C.

<u>Proposition 3.1</u>. $A_{X,p}$ <u>is the henselization</u> $\mathcal{O}^h_{X,p}$ <u>of the local ring</u> $\mathcal{O}_{X,p}$ <u>of algebraic functions on</u> X <u>at</u> p.

This is obvious: More or less by definition

$$\mathcal{O}^h_{X,p} = \underset{(X',p') \to (X,p)}{\underrightarrow{\lim}} \mathcal{O}_{X',p'}$$

with $(X',p') \to (X,p)$ running through the direct system of all pointed etale algebraic maps into (X,p), cf.[R, Chap. VIII]. Every etale map $(X',p') \to (X,p)$ has a semialgebraic section on a neighbourhood of p, unique up to restrictions, by the implicit function theorem. Using these sections we can identify the elements of the rings $\mathcal{O}_{X',p'}$ with germs of C-valued functions at p. These are just the germs of all isoalgebraic functions at p.

If p is a normal point of X then the ${}_{X,p}$-adic completion $\hat{\mathcal{O}}_{X,p}$ of $\mathcal{O}_{X,p}$ is an integral domain and henselian, hence

$$\mathcal{O}_{X,p} \subset A_{X,p} \subset \hat{\mathcal{O}}_{X,p} \ .$$

The following theorem of Nagata [N, Theorem 44.1] is often useful in considerations about isoalgebraic functions.

<u>Theorem 3.2</u>. $A_{X,p}$ <u>is the set of all elements in the integral domain</u> $\hat{\mathcal{O}}_{X,p}$ which are algebraic over the quotient field of $\mathcal{O}_{X,p}$.

If p is even regular in X and $z_1,\ldots,z_n$ is a regular system of parameters of $\mathcal{O}_{X,p}$ then

$$\hat{\mathcal{O}}_{X,p} = \hat{A}_{X,p} = C[[z_1,\ldots,z_n]],$$

the ring of formal power series in $z_1,\ldots,z_n$ with coefficients in C. Moreover there exists a semialgebraic open neighbourhood U of p such that $z_1,\ldots,z_n$ form an <u>isoalgebraic coordinate system</u> of X on U, i.e. the z_i are all defined on U and yield an isoalgebraic isomorphism

$$(z_1,\ldots,z_n) : U \xrightarrow{\sim} U'$$

onto an open semialgebraic subset U' of C^n. Indeed, the algebraic map

$(z_1,\ldots,z_n)$ induces an isomorphism form $\hat{O}_{C^n,o}$ onto $\hat{O}_{X,p}$ and thus is etale at p.

Let now $(U,z_1,\ldots,z_n)$ be any isoalgebraic coordinate system in X. (The z_i need not be algebraic as above but only isoalgebraic.) Then for every isoalgebraic function f on U the partial derivatives $\frac{\partial f}{\partial z_i} = D_i f$ exist in a literal sense. We identify as usual a point $a \in U$ with the sequence $(a_1,\ldots,a_n)$ of its coordinates.

Theorem 3.3. Let $f : U \to C$ be isoalgebraic. For every point $a = (a_1,\ldots a_n)$ of U and every $j = 1,\ldots,n$ there exists in C the limit

$$(D_jf)\ (a) := \lim_{z_j \to a_j} \frac{f(a_1,\ldots,z_j,\ldots,a_n) - f(a_1,\ldots,a_n)}{z_j - a_j}$$

The functions $D_jf : U \to C$ are again isoalgebraic. If f as an element of $\hat{A}_{X,a} = C[[z_1-a_1,\ldots,z_n-a_n]]$ is the power series

$$\sum_{\alpha\in\mathbb{N}_0^n} c_\alpha (z-a)^\alpha$$

(usual notation) then D_jf is the power series

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \alpha_j\geq 1}} \alpha_j c\ (z-a)^{\alpha-e_j},\quad e_j = (0,\ldots,1,\ldots 0).$$

From the last part of this theorem we obtain by iteration that for every multiindex $\alpha \in \mathbb{N}_0^n$

$$c_\alpha = \frac{(D^\alpha f)\ (a)}{\alpha!}$$

in the usual notation. Thus the power series of an isoalgebraic function f in $\hat{A}_{X,a}$ is the Taylor series of f at a.

Does the Taylor series converge to f in some neighbourhood of a? We assume without loss of generality that U is open and semialgebraic in C^n and $a = 0 \in U$. For any $z = (z_1,\ldots,z_n) \in C^n$ we denote by $|z|$ the standard norm of z, i.e. the positive square root of $z_1^2 + \ldots + z_n^2$.

Theorem 3.4. Let $f : U \to C$ be isoalgebraic and let

$$P_d(z) := \sum_{|\alpha|<d} \frac{(D^\alpha f)\ (0)}{\alpha!} z^\alpha$$

the d-th Taylor-polynomial of f. Let $r > 0$ in R be given such that the closed ball

$$\overline{B_r}(0) \quad := \quad \{z \in C^n \mid \, |z| < r\}$$

is contained in U. Let M(r) be the maximum of f on this ball (which exists, cf. [B, Prop. 8.13.5] or [DK, §9]). Then for every z in the open ball $B_r(0)$ of radius r around 0

$$|f(z) - P_d(z)| < \frac{|z|^{d+1}}{r^{d+1}} \frac{r}{r-|z|} M(r).$$

We obtain from this theorem the following

Corollary 3.5. The Taylor series of f converges to f uniformly on $B_{\nu r}(0)$ for every element $\nu > 0$ in R such that $(\nu^n \mid n \in \mathbb{N})$ is a null-sequence.

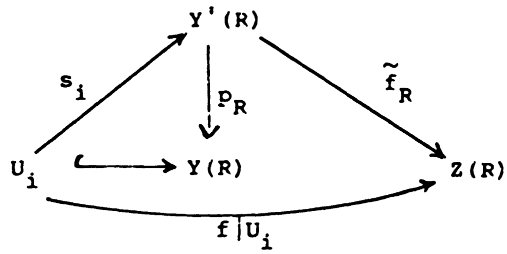
Such elements ν exist in most real closed fields R which occur in practice. For example any real closure of a finitely generated field contains such elements ν [D, §1].

Theorem 3.4 is in the case $R = \mathbb{R}$, $C = \mathbb{C}$ a consequence of Cauchy's formula, cf. [A, p. 101]. The only proof of Theorem 3.4 which I know is by transfer from the classical case $R = \mathbb{R}$ using Tarski's principle. But Corollary 3.5 I can also prove by elementary methods.

§ 4 Real and imaginary parts of an isoalgebraic function.

Isoalgebraic maps are our substitute for complex analytic maps and semialgebraic maps are our substitute for continuous maps if we leave the classical case $R = \mathbb{R}$, $C = C$. We also need a substitute for C^∞-maps and/or real analytic maps to make arguments of differential topological and differential geometric nature possible. A good substitute seems to be provided by the Nash maps.

Definition. Assume that Y,Z are varieties defined over R and that U is an open semialgebraic subset of Y(R) which is pure of dimension n=dim Y. A map $f : U \to Z(R)$ is called a Nash map if there exists a finite covering $(U_i \mid i \in I)$ of U by open semialgebraic subsets such that every restriction $f|U_i$ has an etale factorization. This means that we have commutative diagrams



with p_R the restriction to real points of an etale algebraic map $p : Y' \to Y$, defined over R, and $\tilde{f}_R$ the restriction to real points of an algebraic map $\tilde{f} : Y' \to Z$, defined over R, and s_i a semialgebraic section of p_R over U_i.

It has been explained by Artin and Mazur in [AM, §2] that in the case $R = \mathbb{R}$, U smooth in Y, this definition of Nash maps coincides with the more classical definition as "algebraic real analytic maps." The basic theorems for isoalgebraic maps stated in the last two sections have analogues for Nash maps with the notable exception of Theorem 2.2. In particular, if U is smooth in Y and $x_1,\ldots,x_n$ is a Nash coordinate system on U, then all partial derivatives $D^\alpha f$ of a Nash function $f : U \to R$ (here $Z = A^1_R$, $Z(R) = R$) exist and are again Nash functions, and the Taylor series of f at any point $a \in U$ converges to f in some ball around a. Of course, as in classical analysis, we do not have such a precise hold on the radius of the ball as Corollary 3.5 gives for isoalgebraic functions.

Let now X and Y be varieties over C and U an open semialgebraic subset of X. Assume for simplicity that X is pure of dimension n, e.g. X irreducible. Recall that we identify X and Y with the sets of real points of the varieties $r_{C|R}(X)$ and $r_{C|R}(Y)$ over R. Then it is fairly evident that every isoalgebraic map $f : U \to Y$ is a Nash map. In the case Y = C this means that the real part g and the imaginary part h of any isoalgebraic function $f : U \to C$ are Nash functions.

($f(z) = g(z) + \sqrt{-1}\, h(z)$ with a fixed choice of $\sqrt{-1}$ in C and $g(z), h(z) \in R$.)

Assume that U is smooth in X and has an isoalgebraic coordinate system $z_1,\ldots,z_n$. Then clearly the real and imaginary parts x_j, y_j of the functions z_j form a Nash coordinate system. Local considerations — legitimated by Theorem 2.3 — yield the following characterization of the isoalgebraic functions on U among the C-valued Nash functions.

Theorem 4.1. Let g and h be R-valued Nash functions on U. Then the C-valued function $f = g + \sqrt{-1}\, h$ is isoalgebraic if and only if the Cauchy-Riemann equations

$$\frac{\partial g}{\partial x_j} = \frac{\partial h}{\partial y_j}, \quad \frac{\partial g}{\partial y_j} = -\frac{\partial h}{\partial x_j}$$

$(j = 1,\ldots,n)$ hold true on U.

§ 5 Some one variable theory.

Let now X and Y be smooth curves over C, U a connected open semialgebraic subset of X, and $f : U \to Y$ an isoalgebraic map which does not map U to a single point. The following lemma can be proved by the same power series argument as in the complex analytic case. (Recall Nagata's theorem 3.2.)

Lemma 5.1. If w is a local isoalgebraic coordinate of Y at the point $b = f(a)$, then there exists a local isoalgebraic coordinate z of X at a such that $w \circ f = z^e$ with some natural number e.

Since f looks locally like the map $z \to z^e$ in C near the origin, we obtain immediately

Theorem 5.2. ("Invariance of domain") f is open. If f is injective then f is an isoalgebraic isomorphism from U onto $f(U)$.

The first part of this theorem can be generalized to several variables, and the hypothesis that U is smooth can be eliminated.

Theorem 5.3. Let U be a connected semialgebraic subset of a variety X over C and let $f : U \to Y$ be a non constant isoalgebraic map into a smooth curve Y. Then f is open.

For a proof we may restrict the function f to a curve Z through a given point a of U on which f is not constant near a. This is possible. Then we can apply Theorem 5.2 to the normalization of Z.

In the case $Y = A^1$ this theorem 5.3 has the following immediate consequence.

Corollary 5.4. (Maximum principles) In the situation of the preceding theorem 5.3 the absolute value $|f|$ does not attain a maximum on U. The same holds true for the real part and for the imaginary part of f.

In particular any isoalgebraic function on a connected open semialgebraic set U is uniquely determined by its real part up to an additive constant in $\sqrt{-1}$ R.

Returning to one variable let us consider an isoalgebraic map f from the unit disc

$$D = \{z \in C \mid |z| < 1\}$$

into itself with f(0) = 0. It is easily seen that the function g on D defined by

$$g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

is again isoalgebraic. Applying the maximum principle to the absolute value of g we obtain by the classical argument [A, p. 110]

Theorem 5.4. (Schwarz's lemma) $|f(z)| \leq |z|$ for every $z \in D$ and $|f'(o)| \leq 1$. If $|f(z)| = |z|$ for some z, or if $|f'(o)| = 1$, then f is the multiplication with a constant c of absolute value 1.

This theorem yields in particular as in the classical analytic theory a description of all isoalgebraic automorphisms of the unit disc D. Switching from D to the upper half plane

$$H = \{x + \sqrt{-1}\, y \in C \mid y > 0\}$$

by a standard map, e.g.

$$z \to \sqrt{-1}(1+z)(1-z)^{-1},$$

we obtain

Corollary 5.5. The isoalgebraic automorphisms of the upper half plane H are precisely all maps

$$z \to (az+b)(cz+d)^{-1}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$$

Here a serious obstacle in our theory comes into sight. There is an obvious way to define (reduced) "isoalgebraic spaces" by glueing finitely many open semialgebraic subsets of varieties with isoalgebraic glueing maps, cf. the definition of "semialgebraic spaces" in [DK, §7]. In view of Corollary 5.5 one would like to construct isoalgebraic spaces H/Γ as quotients of H by suitable "discrete" subgroups Γ of SL(2,R) and would like to prove that these spaces "are" algebraic curves. (N.B.: By

Theorem 2.2 there exists at most one algebraic structure on H/Γ inducing the given isoalgebraic structure.) Then the projection map $p : H \to H/\Gamma$ would be isoalgebraic and in particular semialgebraic. The fibres of p would be zero dimensional semialgebraic subsets of H, hence finite. But the fibres of p are not finite.

To overcome this obstacle we have to change the isoalgebraic structure on H. We have to make H a "locally isoalgebraic space," i.e. a limit of a suitable direct system of isoalgebraic spaces with open isoalgebraic immersions as transition maps. The fibres of the projection p above then will be only locally isoalgebraic zero dimensional spaces, which are allowed to be infinite.

Such a locally isoalgebraic structure does exist on H. Switching back to the unit disk D one takes on D the structure as a direct limit of the standard isoalgebraic subspaces

$$D_r = \{z \in C \mid |z| < r\}, \quad 0 < r < 1$$

of the affine line C with the inclusions as transition maps. Corollary 5.5 remains true for the corresponding isoalgebraic structure on H, and many quotients H/Γ become indeed algebraic curves.

I refrain from entering into this subject here. It would need another talk.

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