Contemporary Mathematics
Volume 8, 1982

SEMIALGEBRAIC TOPOLOGY OVER A REAL CLOSED FIELD

Hans Delfs and Manfred Knebusch

We fix a real closed field R. By a variety over R we always mean a scheme of finite type over R. This paper gives a short survey about our theory of semialgebraic spaces over R. Semialgebraic spaces seem to be the adequate generalization of the classical notion of semialgebraic sets over R. Copying the classical definition we can consider also semialgebraic subsets of arbitrary varieties over R. Introducing the category of semialgebraic spaces we get rid of the inconvenience that every semialgebraic set is embedded in a variety. The basic definitions are given in §1. They can be found, as well as a lot of foundational material, in the paper [DK II]. The application to the theory of Witt rings outlined in §3 is contained in [DK I]. The results on triangulation and cohomology of affine semialgebraic spaces are contained in the thesis of the first author ([D]). We omit here nearly all proofs and refer the reader to these papers. But we emphasize that in all these proofs Tarski's principle is never used to transfer statements from the field $\mathbb{R}$ to other real closed base fields.

Contents

1980 Mathematics Subject Classification 14G30, 12D15, 58A07

§ 1 Basic definitions

For any variety V over R we denote by V(R) the set of R-rational points of V.

Definition 1: Let V = Spec(A) be an affine variety over R. A subset M of V(R) is called a semialgebraic subset of V, if M is a finite union of sets

$$\{x \in V(R) \mid f(x) = 0, g_j(x) > 0, j = 1,\dots,r\}$$

with elements f, $g_j \in A$.

The ordering of R induces a topology on the set V(R) of real points of every affine R-variety V, hence on every semialgebraic subset M of V. We call this topology the strong topology.

Definition 2: Let V, W be affine varieties over R and M, N be semialgebraic subsets of V resp. W.

A map f : M → N is called semialgebraic with respect to V and W, if f is continuous in strong topology and if the graph G(f) of f is a semialgebraic subset of $V \times_R W$. The semialgebraic maps from M to R with respect to V and A^1_R = Spec R[X] are called the semialgebraic functions on M with respect to V.

Definition 3: A restricted topological space M is a set M together with a family $\mathring{\mathfrak{S}}(M)$ of subsets of M, called the open subsets of M, such that the following conditions are satisfied:

i) $\emptyset \in \mathring{\mathfrak{S}}(M)$, $M \in \mathring{\mathfrak{S}}(M)$

ii) $U_1 \in \mathring{\mathfrak{S}}(M)$, $U_2 \in \mathring{\mathfrak{S}}(M) \Rightarrow U_1 \cup U_2 \in \mathring{\mathfrak{S}}(M)$.

Notice the essential difference to the usual topological spaces: Infinite unions of open subsets are in general not open.

We consider every restricted topological space M as a site in the following sense:
The category of the site has as objects the open subsets $U \in \mathring{\mathfrak{S}}(M)$ of M and as morphisms the inclusion maps between such subsets.

The coverings $(U_i \mid i \in I)$ of an open subset U of M are the finite systems of open subsets of M with $U = \bigcup_{i \in I} U_i$.

Example: Let V be an affine R-variety and M be a semialgebraic subset of V. Let $\mathring{\mathfrak{S}}(M)$ be the family of all subsets of M which are open in M in the strong topology and which are in addition semialgebraic in V. $(M, \mathring{\mathfrak{S}}(M))$ is a restricted topological space. We call this topology of M the semialgebraic topology (with respect to V) and denote this site by M_{sa}.

Definition 4: A ringed space over R is a pair $(M, \mathcal{O}_M)$ consisting of a restricted topological space M and a sheaf $\mathcal{O}_M$ of R-algebras on M. A morphism $(f, \vartheta) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between ringed spaces consists of a continuous map $f : M \to N$ (i.e. the preimages of all open subsets of N are open) and a family $(\vartheta_V)_{V \in \mathring{\mathfrak{S}}(N)}$ of R-algebra-homomorphisms which are compatible with restriction.

Example: Let M be a semialgebraic subset of an affine R-variety V equipped with its semialgebraic topology. For every open semialgebraic subset U of M let $\mathcal{O}_M(U)$ be the R-algebra of semialgebraic functions on U with respect to V. Then $(M, \mathcal{O}_M)$ is a ringed space over R. It is called a semialgebraic subspace of V.

Definition 5:

i) An affine semialgebraic space over R is a ringed space $(M, \mathcal{O}_M)$ which is isomorphic to a semialgebraic subspace of an affine R-variety V.

ii) A semialgebraic space over R is a ringed space $(M, \mathcal{O}_M)$ which has a (finite) covering $(M_i \mid i \in I)$ by open subsets M_i, such that $(M_i, \mathcal{O}_M \mid M_i)$ is for all $i \in I$ an affine semialgebraic space.

iii) A morphism between semialgebraic spaces is a morphism in the category of ringed spaces.

As is shown in [DK II, §7], a morphism $(f, \vartheta) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between semialgebraic spaces is completely determined by f : For $V \in \mathring{\mathfrak{S}}(N)$ and $g \in \mathcal{O}_N(V)$ we have $\vartheta_V(g) = g \circ f$. Hence we write simply f instead of (f,ϑ). These morphisms are also called semialgebraic maps. The morphisms $f : M \to N$ between semialgebraic subspaces M,N of affine varieties V,W are just the semialgebraic maps from M to N with respect to V and W ([DK II, §7]).

Let M be a semialgebraic space. We denote by $\mathfrak{S}(M)$ the smallest family of subsets of M which fulfills the following conditions:

i) $\mathring{\mathfrak{S}}(M) \subset \mathfrak{S}(M)$

ii) $A \in \mathfrak{S}(M) \Rightarrow M - A \in \mathfrak{S}(M)$

ii) $A,B \in \mathfrak{S}(M) \Rightarrow A \cup B \in \mathfrak{S}(M)$.

The elements of $\mathfrak{S}(M)$ are called the semialgebraic subsets of M. In the special case $M = V(R)$ this definition coincides with Definition 1.

The strong topology on M is the topology in the usual sense which has $\mathring{\mathfrak{S}}(M)$ as a basis of open sets. Thus the open sets in the strong topology are the unions of arbitrary families in $\mathring{\mathfrak{S}}(M)$. In the special case that M is a semialgebraic subspace of variety V this topology is of course the same as the strong topology in the previous sense.

Since now notions like "open", "closed", "dense", ... always refer to the strong topology. In this terminology the sets $U \in \mathring{\mathfrak{S}}(M)$ are the open semialgebraic subsets of M and their complements $M - U$ are the closed semialgebraic subsets of M.

It is easy to show that for any two semialgebraic maps $f : M \to N$ and $g : L \to N$ the fibre product $M \times_N L$ exists in the category of semialgebraic spaces. If N is the one point space we write simply $M \times L$ for this product.

Every semialgebraic subset $A \in \mathfrak{S}(M)$ of a semialgebraic space M is in a natural way equipped with the structure of a semialgebraic space:

The elements $U \in \mathring{\mathfrak{S}}(A)$ are those subsets of A which are open in A (in strong topology) and which are semialgebraic in M. A semialgebraic function f on $U \in \mathring{\mathfrak{S}}(A)$ is a function $f : U \to R$, which is continuous in strong topology and whose graph is a semialgebraic subset of $M \times R$.

A is then a subobject of M in the category of semialgebraic spaces.

§ 2 Some results in the theory of semialgebraic spaces

Definition 1: A semialgebraic space M over R is separated, if it fulfills the usual Hausdorff condition: Any two different points x and y of M can be separated by open disjoint semialgebraic neighbourhoods.

Definition 2: A semialgebraic space M over R is called complete, if M is separated and if for all semialgebraic spaces N over R the projection

$$M \times N \to N$$

is closed, i.e., every closed semialgebraic subset A of $M \times N$ is mapped onto a closed semialgebraic subset of N.

Since the closed semialgebraic subsets of the projective line $P^1_R(R) = R \cup \{\infty\}$ which do not contain the point ∞ are finite unions of closed bounded intervals of R, we obtain immediately

Proposition 2.1. Let M be a complete semialgebraic space over R. Then every semialgebraic function $f : M \to R$ attains a minimum and a maximum on M.

Complete semialgebraic spaces are the substitute for compact spaces in usual topology.

Theorem 2.2 ([DK II, 9.4]).
Let M be a semialgebraic space over the field $\mathbb{R}$ of real numbers. Then M is complete if and only if M is a compact topological space.

Theorem 2.3 ([DK II, 9.6]).
Let V be a complete R-variety. Then the semialgebraic space V(R) is also complete.

We consider every semialgebraic subset $M \subset R^n$ of $A^n_R = \operatorname{Spec} R[X_1,\ldots,X_n]$ as a semialgebraic subspace of A^n_R.

Theorem 2.4 ([DK II, 9.4])
A closed and bounded semialgebraic subset of R^n is a complete semialgebraic space.

If R is not the field of real numbers, the spaces R^n and hence all spaces V(R), where V is an affine R-variety, are totally disconnected. To get reasonable connected components, one must find another notion of connectedness.

Definition 3: Let M be a semialgebraic space. A (semialgebraic) path in M is a semialgebraic map $\alpha : [0,1] \to M$ from the unit interval in R to M. Two points P,Q of M are connectable, if there is a path α in M with $\alpha(0) = P$ and $\alpha(1) = Q$.

Every semialgebraic space M splits under the equivalence relation "connectable" into path components.

Theorem 2.5 ([DK II, 11.2]):
Let M be a semialgebraic space. Then M has a finite number of path components. Each of these components is a semialgebraic subset of M.

It follows from a well known theorem of Tarski that the closure $\overline{N}$ of a semialgebraic subset N of a semialgebraic space M in strong topology is again a semialgebraic subset of M.

Theorem 2.6 (Curve Selection Lemma; [DK II, 12.1]):
Let N be a semialgebraic subset of a semialgebraic space M and let P be a point in the closure $\overline{N}$ of N in M (in the strong topology). Then there exists a path $\alpha : [0,1] \to M$ with $\alpha(0) = P$ and $\alpha(]0,1]) \subset N$.

An immediate consequence of the last two theorems is

Corollary 2.7 The finitely many path components of a semialgebraic space M are closed and hence open semialgebraic subsets of M.

Theorems 2.5 and 2.6 can be easily derived from the triangulation theorem (4.1), but it is not necessary to use such a strong result. The proofs in [DK II] are rather easy and elementary.

Definition 4 ([B], p.249): A semialgebraic space M is called connected if it is not the union of two non empty open semialgebraic subsets.

The images of (path-)connected semialgebraic spaces are obviously again (path-)connected. Since the unit interval in R is connected, we see that any path connected space is connected. Corollary 2.7 implies that also the converse is true.

Corollary 2.8: A semialgebraic space is connected if and only if it is path connected.

From now on we say simply "component" instead of "path component". The number of components is in the following sense a birational invariant.

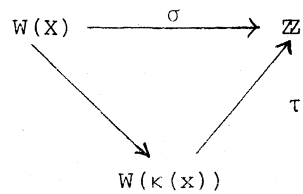
Theorem 2.9 ([DK II, 13.3]):
Let V and W be birationally equivalent smooth complete varieties over R. Then V(R) and W(R) have the same number of components.

The proof of Theorem 2.9 in [DK II] is a straightforward adaption of classical arguments and illustrates that it is possible by our theory to transfer quite a lot of geometric ideas, familiar in the case $R = \mathbb{R}$, to arbitrary real closed base fields. A quite different proof of Theorem 2.9 has been given by M.F. Coste-Roy ([C]).

§ 3 An application to Witt rings

Let X be a divisorial variety over an arbitrary (not necessarily real closed) field k. (Notice that all regular and all quasiprojective varieties over k are divisorial). We consider the signatures of X, i.e. the ring homomorphisms from the Witt ring W(X) of bilinear spaces over X to the ring of integers $\mathbb{Z}$ (cf. [K] for the general theory and meaning of signatures).

It is shown in [K, $\overline{\underline{V}}$.1] that any signature factors through some point x of X, i.e., there exists a commutative diagram



with $W(X) \to W(\kappa(x))$ the natural map from W(X) to the Witt ring of the residue class field $\kappa(x)$. Using our theory we are able to prove

Theorem 3.1 ([DK I,5.1]):
Every signature σ of X factors through a closed point x of X.

Theorem 3.1 is first proved in the case that k = R is a real closed field. The proof runs essentially along the same lines as the proof in the special case that R = $\mathbb{R}$ [K, Chap $\overline{\underline{V}}$]. It is based on Theorem 2.5 which states that X(R) has only a finite number of components. Using the theory of real closures of schemes ([K_1]), it is possible to extend Theorem 3.1 to arbitrary base fields.

§ 4 Triangulation of affine semialgebraic spaces

To avoid confusion what is meant by a triangulation we first give two definitions.

Definition 1: An open non degenerate n-simplex S over R is the interior (in strong topology) of the convex closure of n + 1 affine independent points $e_0,\dots,e_n$ in some space R^m, called the vertices of S, i.e.

$$S = \left\{ \sum_{i=0}^{n} t_i e_i \mid t_i \in R,\ t_i > 0,\ \sum_{i=0}^{n} t_i = 1 \right\}.$$

Definition 2. A simplicial complex over R is a pair $(X, X = \bigcup_{k=1}^{r} S_k)$ consisting of a semialgebraic subset X of some affine space

R^m and a decomposition $X = \bigcup_{k=1} S_k$ of X into disjoint open nondegenerate simplices S_k, such that the following condition is fulfilled:
The intersection $\overline{S_k} \cap \overline{S_l}$ of the closures of any two simplices S_k, S_l is either empty or is a face (defined as usual) of $\overline{S_k}$ as well as of $\overline{S_l}$.

We can now state the triangulation theorem which says that finitely many semialgebraic subsets of an affine semialgebraic space can be triangulated simultaneously.

Theorem 4.1 ([D, 2.2]):
Let $\{M_j\}_{j \in J}$ be a finite family of semialgebraic subsets of an affine semialgebraic space M. Then there exists a semialgebraic isomorphism

$$\phi : X = \bigcup_{k=1}^{r} S_k \xrightarrow{\sim} \bigcup_{j \in J} M_j ,$$

such that each set M_j is a union of certain images $\phi(S_k)$ of simplices S_k of X.

In the proof of Theorem 3 one easily retreats to the case that the sets M_j are bounded semialgebraic subsets of some R^n. Then induction on n is used, the case n = 1 being trivial. The main problem is to find a substitute for the analytic tools used in the classical proofs for $R = \mathbb{R}$ (cf. [H]).

§ 5 Cohomological dimension

The following "inequality of Lojasciewicz" can be proved with help of Theorem 2.4.

Lemma 5.1 ([D, 3.2]): Let M be a closed and bounded semialgebraic subset of R^n and f,g be semialgebraic functions on M. Assume that for all $x \in M$ $f(x) = 0$ implies $g(x) = 0$. Then there is a constant $C > 0$, $C \in R$, and a natural number m, such that for all $x \in M$

$$|f(x)| \geq C \ |g(x)|^m .$$

From Lemma 4.1 one can derive in a similar way as it is done in a special case for $R = \mathbb{R}$ in [BE]

Theorem 5.2 ([D, 3.3]):
Let M be a semialgebraic subset of an affine R-variety V = Spec A and let U be an open semialgebraic subset of M. Then U is a finite union of sets

$$\{x \in M \mid f_i(x) > 0,\ i = 1,\ldots,r\}$$

with $f_i \in A$, $i = 1,\ldots,r$.
Other proofs of this fact have been given by Delzell [De, Chap. II] and M.F. Coste-Roy ([C]).

A sheaf F on a semialgebraic space M assigns to every open semialgebraic subset U of M an abelian group F(U) such that the usual compatibilities with respect to restriction and the sheaf condition are fulfilled. Since we admit only finite coverings in the semialgebraic topology M_{sa} of M (cf. §1), the sheaf condition must hold only for finite coverings.

It is clear from Grothendieck's definition what the cohomology groups $H^q(M,F)$ of M with coefficients in an abelian sheaf F on M are ([G],[A]):
Choose a resolution

$$0 \to F \to J^0 \to J^1 \to J^2 \to \ldots\ldots$$

of F by injective sheaves J^k, apply the global section functor and take the cohomology groups of the arising complex:

$$H^q(M,F) = \mathrm{Ker}(J^q(M) \to J^{q+1}(M)) \,/\, \mathrm{Im}(J^{q-1}(M) \to J^q(M)).$$

It follows from Theorem 5.2 that an affine semialgebraic space has similar separating properties as a usual paracompact topological space. Using this fact we can prove

Theorem 5.3 ([D, 5.2]):
Let M be an affine semialgebraic space and F be an abelian sheaf on M. Then the canonical homomorphism

$$\check{H}^p(M,F) \to H^p(M,F)$$

from Čech- to Grothendieck cohomology is for all $p \geq 0$ an isomorphism.

Every semialgebraic space M has a certain well defined dimension ([DK II, §8]). If M is a semialgebraic subset of an R-variety V, dim M is simply the dimension of the Zariski-closure of M in V. Since every affine semialgebraic space can be triangulated (Theorem 4.1) and Grothendieck-cohomology coincides with Čech-cohomology, similar arguments as in [Go, II.5.12] show that the cohomological dimension does not exceed the topological dimension.

Theorem 5.4 ([D, 5.9]):
Let M be an affine semialgebraic space of dimension n. Then for all abelian sheaves F on M

$$H^q(M,F) = 0$$

for all $q > n$.

§ 6 The homotopy axiom in semialgebraic cohomology

Let M be a semialgebraic space and G be an abelian group. G yields the constant sheaf G_M on M: For an open semialgebraic subset U of M

$$G_M(U) = \prod_{\pi_0(U)} G ,$$

where $\pi_0(U)$ is the finite set of components of U.

We denote by $[0,1]$ the unit interval in R.

In the classical theory homotopic maps induce the same homomorphisms in cohomology. This is also true in semialgebraic topology over an arbitrary real closed field, at least in the affine case.

Theorem 6.1 ([D, 7.1]): Let $f_0, f_1 : M \rightrightarrows N$ be homotopic semialgebraic maps between affine semialgebraic spaces, i.e. there exists a semialgebraic map $H : M \times [0,1] \to N$, such that $H(-,0) = f_0$,

$H(-,1) = f_1$. Then f_0 and f_1 induce the same homomorphisms in cohomology:

$$f_0^* = f_1^* : H^q(N,G_N) \to H^q(M,G_M).$$

We proof Theorem 6.1 by use of Alexander-Spanier-cohomology which is defined in a similar manner as in the classical case (cf. [S]). The sheaves of Alexander-Spanier-cochains yield a resolution of G_M by flask sheaves (defined as in the topological case). Flask resolutions can be used to determine cohomology. Thus Alexander-Spanier-cohomology is the same as Grothendieck-cohomology. The existence of infinitely small elements in a non archimedian field rises many difficulties in the proof of Theorem 6.1, compared with the classical case. For example, we cannot make intervals and triangles "arbitrarily small" by barycentric or even "linear" subdivision. Our proof is based on a careful investigation of the roots of a system of polynomials.

We use Theorem 6.1 to identify the semialgebraic cohomology groups $H^q(M,G_M)$ with certain simplicial cohomology groups. For the rest of this section let M be an affine semialgebraic space and G be a fixed abelian group. Consider a triangulation

$$\phi : X = \bigcup_{i=1}^{r} S_i \xrightarrow{\sim} M$$

of M (§4). For technical reasons we have to assume that ϕ is a barycentric subdivision of another triangulation of M.

We then associate to ϕ the following abstract simplicial complex K: The set V(K) of vertices of K consists of all vertices of X lying in X. (Notice that X is not necessarily closed).
A subset $e_0,\dots,e_q$ of V(K) is a simplex of K, if $e_0,\dots,e_q$ are the vertices of a simplex S_i, $i \in \{1,\dots,r\}$.

In the usual way we form the simplicial cohomology groups $H^q(K,G)$.

<u>Proposition 6.2</u> ([D,8.4]):

$$H^q(M,G_M) = H^q(K,G) \quad \text{for all } q \geq 0.$$

In particular, the simplicial cohomology groups $H^q(K,G)$ do not depend on the chosen triangulation of M.

The proof of Proposition 6.2 uses the description of $H^q(M,G_M)$ as Čech-cohomology and Theorem 6.1 which implies that the covering $\{St(e)\}_{e\in V(k)}$ of M, consisting of the star neighbourhoods of the vertices of M with respect to ϕ, is a Leray-covering for the sheaf G_M.

It is now possible to define also homology groups.

<u>Definition 1</u>: $H_q(M,G) := H_q(K,G)$ is called the q-th <u>homology group</u> of M with coefficients in G.

This definition does not depend on the chosen triangulation of M as follows from the corresponding fact for cohomology (Proposition 6.2). The homology groups $H_q(M,G)$ are functorial in M since, according to Theorem 4.1, every semialgebraic map between affine semialgebraic spaces can be "approximated" by a simplicial map (cf. [D, §8]).

Proposition 6.2 implies also that for $R = \mathbb{R}$ the semialgebraic cohomology groups coincide with the usual (singular) cohomology groups determined with respect to strong topology. For homology this is true by definition.

Another immediate consequence of the simplicial interpretation of homology and cohomology is that the groups are invariant under change of the base field. We will illustrate this in a special case.

Let L be a real closed field containing R. Assume, M is a semialgebraic subset of R^n. We choose a description of M by finitely many polynomial inequalities and equalities. Let M_L denote the semialgebraic subset of L^n defined by the same inequalities and equalities. By use of Tarski's principle - here clearly legitimate and unavoidable - we see that the set M_L is independent of the choice of the description of M.
It also follows from Tarski's principle that the triangulation

$$\phi : X = \bigcup_{i=1}^{r} S_i \xrightarrow{\sim} M$$

yields a triangulation

$$\phi_L : X_L = \bigcup_{i=1}^{r} S_{iL} \xrightarrow{\sim} M_L$$

of M_L. The associated abstract complex of ϕ_L is also K and we get with Prop. 6.2 and Def. 1:

$$H^q(M,G) = H^q(M_L,G),$$

$$H_q(M,G) = H_q(M_L,G).$$

Example: The n-sphere $S_R^n = \{(x_0,\ldots,x_n) \in R^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\}$ has the same homology and cohomology as the n-sphere $S_{\mathbb{R}}^n$ over $\mathbb{R}$, since S_R^n and $S_{\mathbb{R}}^n$ both can be obtained from S_{R_0} by base extension, with R_0 the real closure of $\mathbb{Q}$. Thus

$$H_0(S_R^n,G) \cong H_n(S_R^n,G) \cong H^0(S_R^n,G) \cong H^n(S_R^n,G) \cong G,$$

$$H_q(S_R^n,G) = 0,\ H^q(S_R^n,G) = 0 \quad \text{for } q \neq 0,n.$$

§ 7 The duality theorems

As an example of our theory we want to explain that in a certain sense the classical duality theorems for manifolds remain true over an arbitrary real closed field. We consider a semialgebraic space M over R.

Definition 1. M is an n-dimensional semialgebraic manifold if every point $x \in M$ has an open semialgebraic neighbourhood which is isomorphic to an open semialgebraic subset of R^n.

Example 1: It follows from the implicit function theorem for polynomials (cf. [DK II, 6.9]) that every open semialgebraic subset U of the set V(R) of real points of an n-dimensional smooth R-variety V is an n-dimensional semialgebraic manifold ([DK II, §13]).

From now on we assume that M is affine and complete.
We choose a triangulation

$$\phi : X = \bigcup_{i=1}^{r} S_i \xrightarrow{\sim} M$$

and associate to ϕ an abstract simplicial complex K as in §6. Since M is complete, X is closed and bounded in its embedding space R^m.

We follow in our notation the book of Maunder on algebraic topology ([M]). For any point $x \in M$ we denote by $N_\phi(x)$ the union $\bigcup_{\phi^{-1}(x)\in\overline{S}_i} \phi(\overline{S}_i)$ of all closed "simplices" of M with respect to ϕ containing x. The union $Lk_\phi(x) := N_\phi(x) \setminus \left(\bigcup_{\phi^{-1}(x)\in\overline{S}_i} \phi(S_i)\right)$ of those "simplices" of $N_\phi(x)$ which do not contain x is called the link of x.

Definition 2: M is called a homology-n-manifold if for all $x \in M$

$$H_q(Lk_\phi(x),\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0,n \\ 0 & q \neq 0,n \end{cases}$$

Definition 2 does not depend on the chosen triangulation ϕ (cf. [M]).

Example 2: Every affine complete n-dimensional semialgebraic manifold is a homology-n-manifold ([D, §10]).

Subexample 2 a. The Zariski-open subset U of the m-dimensional projective space $\mathbb{P}^m_R$ over R, obtained by removing the hypersurface $X_0^2 + X_1^2 + \ldots + X_m^2 = 0$ from $\mathbb{P}^m_R$, is affine and has the same real points as $\mathbb{P}^m_R$. Hence the space V(R) of real points of a projective R-variety V is affine. It is also complete by Theorem 2.3. Thus if V is projective, smooth and has dimension n the space V(R) is a homology-n-manifold.

We return to our affine and complete semialgebraic space M and assume in addition that M is a homology-n-manifold.

Definition 3: If M is connected, we call M orientable if $H_n(M,\mathbb{Z}) = \mathbb{Z}$. In general M is called orientable, if each component of M is orientable.

Example: The n-sphere S^n_R over R is an orientable homology-n-manifold (cf. §6).

Theorem 7.1 (Poincaré-duality):
Assume M is orientable. Then there are canonical isomorphisms

$$H^q(M,\mathbb{Z}) \xrightarrow{\sim} H_{n-q}(M,\mathbb{Z}).$$

If M is not orientable, there are still isomorphisms

$$H^q(M,\mathbb{Z}/2) \xrightarrow{\sim} H_{n-q}(M,\mathbb{Z}/2)\ .$$

The proof is very easy: Consider a realization $|K|_{\mathbb{R}}$ of the abstract complex K over $\mathbb{R}$, i.e., a closed simplicial complex over $\mathbb{R}$ with associated abstract complex K. Then $|K|_{\mathbb{R}}$ is an (orientable) homology-n-manifold over $\mathbb{R}$ and the classical Poincaré-Duality applies to $|K|_{\mathbb{R}}$. But semialgebraic (co-)homology of M and singular (co-) homology of $|K|_{\mathbb{R}}$ coincide both with the (co-)homology of the abstract complex K.

In a similar way other duality theorems can also be transferred to an arbitrary real closed field.

Using relative homology- and cohomology groups, one derives from Poincaré-duality

Theorem 6. (Alexander-duality)
Let A be a semialgebraic subset of the n-sphere S^n_R. Then there are isomorphisms

$$\tilde{H}^q(A) \xrightarrow{\sim} \tilde{H}_{n-q-1}(S^n_R - A),$$

where $\tilde{H}^q$ (resp. $\tilde{H}_q$) denotes the reduced cohomology (resp.homology) group.

As an application we get the generalized Jordan curve theorem over any real closed field.

Corollary 7. Let M be a semialgebraic subset of S^{n+1}_R $(n \geq 1)$. Assume M is a homology-n-manifold with k components. Then $S^{n+1}_R - M$ has k+1 connected components. In particular: If M is semialgebraically isomorphic to S^n_R then $S^{n+1}_R - M$ has 2 components. In this case M is the common boundary of these two components.

Proof: From Alexander-duality we obtain

$$\tilde{H}_o(S^{n+1}_R - M,\mathbb{Z}) \cong \tilde{H}^n(M,\mathbb{Z}) \cong H^n(M,\mathbb{Z})$$

and by Poincaré-duality

$$H^n(M,\mathbb{Z}/2) \cong H_o(M,\mathbb{Z}/2) \cong \prod_{\pi_o(M)} \mathbb{Z}/2.$$

The statement concerning the boundary is proved by similar arguments.

References

[A] M. Artin, "Grothendieck topologies", Harvard University, 1962.

[B] G.W. Brumfiel, "Partially ordered rings and semialgebraic geometry", Cambridge University Press (1979).

[BE] J. Bochnak, G. Efroymson, Real algebraic geometry and the 17th Hilbert Problem, Math. Ann. 251, 213-241 (1980).

[C] M.F. Coste-Roy, Spectre réel d'un anneau et topos étale réel, Thèse, Université Paris Nord (1980).

[D] H. Delfs, Kohomologie affiner semialgebraischer Räume, Thesis, Regensburg (1980).

[DK I] H. Delfs, M. Knebusch, Semialgebraic topology over a real closed field I: Paths and components in the set of rational points of an algebraic variety, to appear in Math. Z.

[DK II] H. Delfs, M. Knebusch, Semialgebraic topology over a real closed field II: Basic theory of semialgebraic spaces, preprint, Regensburg (1980).

[De] C.N. Delzell, A constructive, continuous solution to Hilbert's 17th Problem, and other results in semialgebraic geometry, Thesis, Stanford University, 1980.

[G_o] R. Godement, "Théorie des faisceaux", Hermann, Paris (1958).

[G] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. Journ. A. IX (1957), 119-221.

[H] H. Hironaka, Triangulation of algebraic sets, Proc. Amer. Math. Soc., Symp. in Pure Math. 29 (1975), 165-185.

[K] M. Knebusch, "Symmetric bilinear forms over algebraic varieties". In: Conference on quadratic forms (Kingston 1976), 103-283. Queen's papers in Pure Appl. Math. 46 (1977).

[K_1] M. Knebusch, Real closures of algebraic varieties, ibid., 548-568.

[M] C.R.F. Maunder, "Algebraic topology", Van Nostrand Reinhold Company, London (1970).

[S] E.H. Spanier, "Algebraic topology", McGraw Hill Book Company, New York (1966).

FACHBEREICH MATHEMATIK DER UNIVERSITÄT
REGENSBURG
WEST GERMANY