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BASIC HOMOTOPY THEORY OF LOCALLY SEMIALGEBRAIC SPACES

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Dedicated to the memory of Gus Efroymson

We use the notations of our survey article [1] on locally semialgebraic spaces in this volume. We want to indicate that it is possible to develop a homotopy theory for locally semialgebraic spaces over an arbitrary real closed field R which works as efficiently as the homotopy theory for topological spaces. For this purpose we give the basic definitions and some results.

NOTATION. Let M, N be locally semialgebraic spaces over R and $A_1, \ldots, A_r$ resp. $B_1, \ldots, B_r$ be locally semialgebraic subsets of M resp. N. By $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]$ we denote the set of homotopy classes of locally semialgebraic maps $(M, A_1, \ldots, A_r) \to (N, B_1, \ldots, B_r)$ (cf. [1 Definition 1 in §4]). In the case $R = \mathbf{R}$ we denote by $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]_{top}$ the set of continuous homotopy classes of continuous maps between the associated systems of topological spaces $(M_{top}, A_{1,top}, \ldots, A_{r,top}), (N_{top}, B_{1,top}, \ldots, B_{r,top})$.

MAIN THEOREM 1. *Let $\tilde{R}$ be a real closed field extension of R. Let M be an affine semialgebraic space and N be a regular locally semialgebraic space over R. For closed semialgebraic subsets $A_1, \ldots, A_r$ of M and locally semialgebraic subsets $B_1, \ldots, B_r$ of N the natural map* $[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)] \to [(M(\tilde{R}), A_1(\tilde{R}), \ldots, A_r(\tilde{R})), (N(\tilde{R}), B_1(\tilde{R}), \ldots, B_r(\tilde{R}))]$ *which maps the class $[f]$ of a map f to the class $[f_{\tilde{R}}]$ of the base extension $f_{\tilde{R}}$ of f* (cf. [1] Example 1.7) *is bijective.*

Unfortunately we do not yet know whether Theorem 1 is also true if M is not affine semialgebraic. Our proof uses, besides the method of simplicial approximation, semialgebraic mapping spaces and Tarski's principle, and these techniques are restricted to the affine semialgebraic setting. The situation is slightly better when we compare semialgebraic homotopy sets with topological homotopy sets in the case $R = \mathbf{R}$.

MAIN THEOREM 2. *Let M, N be regular locally semialgebraic spaces over*

R *and* $A_1, \ldots, A_r$ *resp.* $B_1, \ldots, B_r$ *be closed locally semialgebraic subsets of* M *resp.* N. *Assume that either*

i) M *is an affine semialgebraic space or*

ii) M *and* N *are paracompact spaces.*

Then the natural map

$$[(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)] \to [(M, A_1, \ldots, A_r), (N, B_1, \ldots, B_r)]_{top}$$

is bijective.

These two theorems are sufficient to deal with homotopy groups which we define in precise analogy to the topological case.

NOTATION. I denotes the unit interval $[0, 1]$ in R. $I^n = \{(t_1, \ldots, t_n) \in R^n | 0 \leqq t_i \leqq 1,\ 1 \leqq i \leqq n\}$ is the n-fold cartesian product of I, $\mathring{I}^n = \{(t_1, \ldots, t_n) \in I^n \mid \text{all } t_i > 0\}$ its interior, $\partial I^n = I^n - \mathring{I}^n$ its boundary and J_n is the subset $(\partial I^{n+1} - (\mathring{I}^n \times \{0\}))$ of $\partial I^{n+1} (n \geqq 1)$. J_0 is simply the one point set $\{1\}$.

DEFINITION 1. Let M be a locally semialgebraic space over R and A be a locally semialgebraic subset of M. Then we define for every base point $x_0 \in A$ and $n \geqq 1$

$$\pi_n(M, A, x_0) := [(I^n, \partial I^n, J_{n-1}), (M, A, \{x_0\})]$$

and for every $x_0 \in M$ and $n \geqq 1$

$$\pi_n(M, x_0) := \pi_n(M, \{x_0\}, x_0).$$

Introducing on $\pi_n(M, A, x_0)$ for $n \geqq 2$ (resp. on $\pi_n(M, x_0)$ for $n \geqq 1$) the multiplication

$$[f] \cdot [g] = [f * g]$$

with

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) \text{ for } 0 \leqq t_1 \leqq \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) \text{ for } \frac{1}{2} \leqq t_1 \leqq 1, \end{cases}$$

these homotopy sets become groups, called the (semialgebraic) homotopy groups of (M, A, x_0) (resp. the (semialgebraic) homotopy groups of (M, x_0)).

In the usual way one sees that $\pi_n(M, A, x_0)$ for $n \geqq 3$ (resp. $\pi_n(M, x_0)$ for $n \geqq 2$) is an abelian group. $\pi_1(M, A, x_0)$ is considered as a set with a base point, represented by the constant map from I to x_0. By $\pi_0(M, x_0)$ we denote the set of components of M with base point the component of x_0.

Every path $\alpha: [0, 1] \to A$ (resp. $\alpha: [0, 1] \to M$) with $\alpha(0) = x_0$ and $\alpha(1) = x_1$ yields an isomorphism $\alpha_\#$ from the set with base point $\pi_n(M, A, x_0)$ to $\pi_n(M, A, x_1)$ for every $n \geqq 1$ (resp. from $\pi_n(M, x_0)$ to $\pi_n(M, x_1)$ for every $n \geqq 0$). $\alpha_\#$ is defined in the same way as in topology [**5**, p. 126ff.] and depends only on the homotopy class of α with fixed end points. For $n \geqq 2$ (resp. $n \geqq 1$) $\alpha_\#$ is an isomorphism of groups. In particular the fundamental group $\pi_1(A, x_0)$ operates on $\pi_n(M, A, x_0)$ and $\pi_1(M, x_0)$ operates on $\pi_n(M, x_0)$. In case $n = 1$ the latter operation is just by conjugation.

Again copying classical arguments [**5**, p. 115ff.)], we obtain for every pair (M, A) of locally semialgebraic spaces and every $x_0 \in A$ a long exact homotopy sequence, as in the topological theory

$$(3) \qquad \begin{aligned} &\to \pi_n(A, x_0) \to \pi_n(M, x_0) \to \pi_n(M, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \\ &\to \pi_1(M, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \to \pi_0(M, x_0). \end{aligned}$$

The fundamental group $\pi_1(A, x_0)$ operates on this sequence equivariantly.

THEOREM 4. *Let M be a regular locally semialgebraic space over* **R** *and A be a locally semialgebraic subset of M. Then for every $x_0 \in A$ (resp. $x_0 \in M$) the natural homomorphisms*

$$\pi_n(M, A, x_0) \to \pi_n(M, A, x_0)_{top} (n \geqq 2) \, (resp. \; \pi_n(M, x_0) \to \pi_n(M, x_0)_{top} (n \geqq 1))$$

from the semialgebraic homotopy groups to the topological homotopy groups are isomorphisms.

PROOF. The absolute case and, if A is closed in M, also the relative case are covered by Theorem 2. If A is not closed in M a comparison between the semialgebraic and the topological long exact homotopy sequence of the pair (M, A) yields the result.

THEOREM 5. ("Whitehead's theorem"). *Let f: $M \to N$ be a locally semialgebraic map between connected paracompact regular locally semialgebraic spaces over R. Assume that f is a weak homotopy equivalence, i.e., f induces for one point $x_0 \in M$ (and then for every $x_0 \in M$) isomorphisms*

$$f_*: \pi_n(M, x_0) \simeq \pi_n(N, f(x_0)) \; (n \geqq 1).$$

Then f is a locally semialgebraic homotopy equivalence.

EXAMPLE 6. Let M be a locally complete semialgebraic space. Assume that the base field R contains a sequence $(\varepsilon_n | n \in \mathbf{N})$ of positive elements converging to zero. Then the partially complete locally semialgebraic space M_{loc} is paracompact [**1**, Example 2.1]. Obviously M_{loc} has the same homotopy groups as M. Thus the map $g\colon M_{loc} \to M$, $x \mapsto x$, is a homotopy equivalence.

Since in Theorem 1 the domain is required to be semialgebraic we cannot deduce Theorem 5 by transfer from the well known topological case. We will indicate a direct proof because an interesting phenomenon appears at this point.

Both spaces M, N can be triangulated, thus we can assume that they are simplicial complexes. But then, as in the affine semialgebraic case, (cf. [**3**, 2.5]) the biggest closed subcomplex M_0 (resp. N_0) of M (resp. N) is a strong deformation retract of M (resp. N), provided we have chosen a "good triangulation" [loc. cit.]. Hence replacing M by M_0, N by N_0 we can assume that M, N are partially complete spaces. Now if M is a locally semialgebraic subset of N and f is the inclusion map then Theorem 5 can be proved in exactly the same way as in the topological case (cf., e.g., [**6**, 7.5.2]).

To handle the general case we need the mapping cylinder $Z(f)$ of f (because then we could replace f by the inclusion $M \subset Z(f)$). Unfortunately this mapping cylinder in general is not a locally semialgebraic space. For example it is easily seen that the mapping cylinder $Z(g)$ of a locally semialgebraic map $g\colon M \to N$ between a partially complete space M and a complete semialgebraic space N cannot exist as a locally semialgebraic space if M is not a semialgebraic space. In particular, let us again consider Example 6. If M is not complete and $r\colon M \to M_0$ is a retraction from M onto a complete strong deformation retract of M, then $Z(r \circ g)$ is not a locally semialgebraic space.

At this point we are forced to leave the category of locally semialgebraic spaces, and to work with weak polytopes. A weak polytope X over R essentially is a ringed space over R (cf. [**1** §1]) which is an inductive limit $\varinjlim X_\alpha$ of complete semialgebraic spaces X_α over R in the category of ringed spaces over R. (It is assumed that the canonical maps $X_\alpha \to X$ are injective. Some more technical properties must be fulfilled). Every partially complete locally semialgebraic space is the inductive limit of its complete semialgebraic subsets, hence is a weak polytope. More precisely: The category of partially complete locally semialgebraic spaces over R is a full subcategory of the category of weak polytopes over R.

This new category has a great advantage: You can attach spaces as you like. Whenever $f\colon A \to Y$ is a morphism between weak polytopes and A is a closed "admissible" subset of a polytope X, then $X \cup_{A,f} Y$ is again a weak polytope and the map $X \cup Y \to X \cup_{A,f} Y$ is a quotient in the category of weak polytopes. In particular the mapping cylinder $Z(g)$ of an arbitrary morphism $g\colon X \to Y$ between weak polytopes exists in this category.

The proof of Theorem 5 is now easy. The mapping cylinder $Z(f)$ is a weak polytope which is homotopy equivalent to N by the obvious projection map $Z(f) \to N$. Thus it suffices to prove that the inclusion map $M \to Z(f)$ is a homotopy equivalence. But the proof of Theorem 5 for inclusion maps works equally well when the target space is merely a weak polytope instead of a locally semialgebraic space.

The importance of weak polytopes in semialgebraic homotopy theory is also indicated by the following example.

EXAMPLE 7. (Existence of Eilenberg-MacLane spaces) Let π be an abelian group and n be a natural number. Then there exists a weak polytope $K(\pi, n)$ over R such that $\pi_n(K(\pi, n)) \cong \pi$ and $\pi_k(K(\pi, n)) = 0$ for every $k \neq n$.

We close this section with a discussion of the "Huréwicz-homomorphisms" from semialgebraic homotopy to semialgebraic homology groups. For M a regular paracompact locally semialgebraic space, A a locally semialgebraic subset of M, G an abelian group, we can define homology groups $H_n(M, A, G)$ by the same procedure as described in [3, §3] in the affine semialgebraic case (cf. [2] for the details), using the triangulation theorem and sheaf cohomology. Every locally semialgebraic map $f: (M, A) \to (N, B)$ induces group homomorphisms $f_*: H_n(M, A, G) \to H_n(N, B, G)$, which only depend on the homotopy class of f. The analogues of our main theorems 1 and 2 remain true in homology: If $\tilde{R}$ is a real closed overfield of R, then the group $H_n(M(\tilde{R}), A(\tilde{R}), G)$ is "the same" as $H_n(M, A, G)$. If $R = \mathbf{R}$, the group $H_n(M, A, G)$ coincides with the singular homology group $H_n(M_{top}, A_{top}, G)$ of the pair (M_{top}, A_{top}) of topological spaces (cf. [3, Th. 3.7 and §4] for the affine semialgebraic case).

The group $H_n(I^n, \partial I^n, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$. As in the topological case we choose a standard generator σ_n of this group, cf., e.g., [7, Chap. 7, §4]. We choose a base point x_0 in A and define the Huréwicz map φ: $\pi_n(M, A, x_0) \to H_n(M, A, \mathbf{Z})$ for every $n \geqq 1$ by mapping the homotopy class $[f]$ of a semialgebraic map f from $(I^n, \partial I^n, J_{n-1})$ to (M, A, x_0) to the element $f_*(\sigma_n)$ in $H_n(M, A, \mathbf{Z})$. The map φ has the same properties as listed in [7, loc. cit.] in the topological case, and this can be seen by the same arguments as in topology (without using a transfer principle). In particular, φ is a group homomorphism for $n \geqq 2$, compatible with the action of the fundamental group $\pi_1(A, x_0)$. Here we decree that $\pi_1(A, x_0)$ acts trivially on $H_n(M, A, \mathbf{Z})$. The quotient $\pi_n'(M, A, x_0)$ of the group $\pi_n(M, A, x_0)$ with respect to the action of $\pi_1(A, x_0)$ is an abelian group also for $n = 2$, and φ induces a Huréwicz-homomorphism φ': $\pi'_n(M, A, x_0) \to H_n(M, A, \mathbf{Z})$ for every $n \geqq 2$. Clearly φ' is compatible with extension of the base field R to any real closed overfield $\tilde{R}$. Also, in case $R = \mathbf{R}$, φ' coincides with the topological Huréwicz homomorphism.

THEOREM 8. ("Relative Hurêwicz theorem"). *Let M be a connected regular paracompact locally semialgebraic space. Let A be a locally semialgebraic subset of M and x_0 be some point in A. Assume that $n \geqq 2$ and $\pi_q(M, A, x_0) = 0$ for $0 < q < n$. Then*

$$\varphi': \pi_n'(M, A, x_0) \to H_n(M, A, \mathbf{Z})$$

is an isomorphism.

PROOF. The theorem holds in the case $R = \mathbf{R}$ as a consequence of the topological Hurêwicz theorem. In general, by the triangulation theorem, (M, A) is isomorphic to a pair (X, Y) consisting of a strictly locally finite simplicial complex X and a subcomplex Y of X. Thus (M, A) is isomorphic to a pair $(M_0(R), A_0(R))$ with M_0 a regular paracompact space over the field R_0 of real algebraic numbers and A_0 a locally semialgebraic subset of M_0. We may choose the base point x_0 of A in A_0. Since the Hurêwicz homomorphisms behave well with respect to base extension it suffices to prove that the Hurêwicz homomorphism

$$\varphi_0': \pi_n'(M_0, A_0, x_0) \to H_n(M_0, A_0, \mathbf{Z})$$

is bijective. Now $\pi_q(M_0, A_0, x_0) = 0$ for $0 < q < n$. Thus also $\pi_q(M_0(\mathbf{R}), A_0(\mathbf{R}), x_0) = 0$, and we know from the case $R = \mathbf{R}$ that the Hurêwicz homomorphism

$$\varphi_{\mathbf{R}}': \pi_n'(M_0(\mathbf{R}), A_0(\mathbf{R}), x_0) \to H_n(M_0(\mathbf{R}), A_0(\mathbf{R}), \mathbf{Z})$$

is bijective. But $\varphi_{\mathbf{R}}'$ is essentially the same map as φ_0'. So also φ_0' is bijective.

By the same transfer method we obtain a semialgebraic version of the absolute Hurêwicz theorem. In particular, for any connected regular paracompact space M, the group $H_1(M, \mathbf{Z})$ is canonically isomorphic to the factor commutator group of $\pi_1(M, x_0)$.

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