



On the equivariant Tamagawa number  
conjecture in tame CM-extensions, II

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## ABSTRACT

We use the notion of non-commutative Fitting invariants to give a reformulation of the equivariant Iwasawa main conjecture (EIMC) attached to an extension  $F/K$  of totally real fields with Galois group  $\mathcal{G}$ , where  $K$  is a global number field and  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension 1 for an odd prime  $p$ . We attach to each finite Galois CM-extension  $L/K$  with Galois group  $G$  a module  $SKu(L/K)$  over the center of the group ring  $\mathbb{Z}G$  which coincides with the Sinnott-Kurihara ideal if  $G$  is abelian. We state a conjecture on the integrality of  $SKu(L/K)$  which follows from the equivariant Tamagawa number conjecture (ETNC) in many cases, and is a theorem for abelian  $G$ . Assuming the validity of the EIMC and the vanishing of the Iwasawa  $\mu$ -invariant, we compute Fitting invariants of certain Iwasawa modules, and we show that this implies the minus part of the ETNC at  $p$  for an infinite class of (non-abelian) Galois CM-extensions of number fields which are at most tamely ramified above  $p$ , provided that (an appropriate  $p$ -part of) the integrality conjecture holds.

## Introduction

Let  $L/K$  be a finite Galois extension of number fields with Galois group  $G$ . D. Burns [Bu01] used complexes arising from étale cohomology of the constant sheaf  $\mathbb{Z}$  to define a canonical element  $T\Omega(L/K)$  of the relative  $K$ -group  $K_0(\mathbb{Z}G, \mathbb{R})$ . This element relates the leading terms at zero of Artin  $L$ -functions attached to  $L/K$  to natural arithmetic invariants. It was shown that the vanishing of  $T\Omega(L/K)$  is equivalent to the ETNC for the pair  $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$  (cf. loc.cit., Th. 2.4.1).

The ETNC is known to be true if  $L$  is absolutely abelian as proved by D. Burns and C. Greither [BG03] with the exclusion of the 2-primary part; M. Flach [Fl02] extended the argument to cover the 2-primary part as well. If  $L$  is in addition totally real, the ETNC was independently proved in [RW02, RW03]. Some relatively abelian results are due to W. Bley [Bl06]; he showed that if  $L/K$  is a finite abelian extension, where  $K$  is an imaginary quadratic field which has class number one, then the ETNC holds for all intermediate extensions  $L/E$  such that  $[L : E]$  is odd and divisible only by primes which split completely in  $K/\mathbb{Q}$ . Finally, if  $L/K$  is a CM-extension and  $p$  is odd, the ETNC at  $p$  naturally decomposes into a plus and a minus part; it was shown by the author [Nia] that the minus part of the ETNC at  $p$  holds if  $L/K$  is abelian and at most tamely ramified above  $p$ , and the Iwasawa  $\mu$ -invariant vanishes if  $p$  divides  $|G|$  (and some additional technical condition is fulfilled). Note that the vanishing of  $\mu$  is a long standing conjecture of Iwasawa theory; the most general result is still due to B. Ferrero and L. Washington [FW79] and says that  $\mu = 0$  for absolutely abelian extensions.

These results make heavily use of the validity of the EIMC attached to the extension  $L_{\infty}^+/K$ , where  $L_{\infty}^+$  is the cyclotomic  $\mathbb{Z}_p$ -extension of  $L^+$  which is the maximal real subfield of  $L$ . Note that the

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EIMC is known for abelian extensions of totally real number fields with Galois group  $\mathcal{G}$  such that  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension 1 (cf. [Wi90a, RW02]).

In the abelian case, there is a natural formulation of the EIMC in terms of Fitting ideals. The theory of Fitting ideals also plays an important role within the descent methods used in [BG03, Bl06, Wi90b, Gr00, Ku03, Nia]. For not necessarily abelian  $\mathcal{G}$ , we will introduce a reformulation of the EIMC in terms of non-commutative Fitting invariants which have been introduced by the author [Nib]. This is the main purpose of section 2; we give some algebraic preparations on Iwasawa modules and their Fitting invariants in section 3.

Now let  $L/K$  be a Galois CM-extension with Galois group  $G$ . Assuming the validity of the EIMC attached to the extension  $L_{\infty}^+/K$  and the vanishing of  $\mu$ , we compute Fitting invariants of some natural Iwasawa modules in section 4; this generalizes results of C. Greither [Gr04]. In section 5, we introduce a module  $SKu(L/K)$  over the center of the group  $\mathbb{Z}G$  which is a non-commutative analogue of the Sinnott-Kurihara ideal (cf. [Si80], p. 193) and was already implicitly used in [Nic] and [BJ]. We formulate an integrality conjecture on  $SKu(L/K)$  which is implied by the ETNC in many cases and follows from the results in [Ba77], [Ca79], [DR80] if  $G$  is abelian. Assuming the validity of this integrality conjecture, we generalize a descent method due to A. Wiles [Wi90b] in the equivariant version of C. Greither [Gr00] to the non-abelian situation; this shows that the EIMC implies the minus part of the ETNC at  $p$  provided that  $\mu$  vanishes, the integrality conjecture holds and the ramification above  $p$  is at most tame (and, as in the abelian case, some technical extra assumption holds). For a special class of extensions, where no “trivial zeros” occur, the EIMC in fact implies the relevant part of the integrality conjecture. This generalizes [Nia], Th. 4 to the non-abelian situation. Moreover, it follows from the results in [Nic] that for the case at hand the EIMC implies the non-abelian analogues of Brumer’s conjecture, of the Brumer-Stark conjecture and of the strong Brumer-Stark property as formulated in loc.cit., provided that  $\mu = 0$  and the integrality conjecture holds.

## 1. Preliminaries

**1.0.1  $K$ -theory** Let  $\Lambda$  be a left noetherian ring with 1 and  $\text{PMod}(\Lambda)$  the category of all finitely generated projective  $\Lambda$ -modules. We write  $K_0(\Lambda)$  for the Grothendieck group of  $\text{PMod}(\Lambda)$ , and  $K_1(\Lambda)$  for the Whitehead group of  $\Lambda$  which is the abelianized infinite general linear group. If  $S$  is a multiplicatively closed subset of the center of  $\Lambda$  which contains no zero divisors,  $1 \in S$ ,  $0 \notin S$ , we denote the Grothendieck group of the category of all finitely generated  $S$ -torsion  $\Lambda$ -modules of finite projective dimension by  $K_0S(\Lambda)$ . Writing  $\Lambda_S$  for the ring of quotients of  $\Lambda$  with denominators in  $S$ , we have the following Localization Sequence (cf. [CR87], p. 65)

$$K_1(\Lambda) \rightarrow K_1(\Lambda_S) \xrightarrow{\partial} K_0S(\Lambda) \xrightarrow{\rho} K_0(\Lambda) \rightarrow K_0(\Lambda_S). \quad (1)$$

In the special case where  $\Lambda$  is an  $\mathfrak{o}$ -order over a commutative ring  $\mathfrak{o}$  and  $S$  is the set of all nonzerodivisors of  $\mathfrak{o}$ , we also write  $K_0T(\Lambda)$  instead of  $K_0S(\Lambda)$ . Moreover, we denote the relative  $K$ -group corresponding to a ring homomorphism  $\Lambda \rightarrow \Lambda'$  by  $K_0(\Lambda, \Lambda')$  (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87], p. 72)

$$K_1(\Lambda) \rightarrow K_1(\Lambda') \xrightarrow{\partial_{\Lambda, \Lambda'}} K_0(\Lambda, \Lambda') \rightarrow K_0(\Lambda) \rightarrow K_0(\Lambda').$$

It is also shown in [Sw68] that there is an isomorphism  $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$ . For any ring  $\Lambda$  we write  $\zeta(\Lambda)$  for the subring of all elements which are central in  $\Lambda$ . Let  $L$  be a subfield of either  $\mathbb{C}$  or  $\mathbb{C}_p$  for some prime  $p$  and let  $G$  be a finite group. In the case where  $\Lambda'$  is the group ring  $LG$  the reduced norm map  $\text{nr}_{LG} : K_1(LG) \rightarrow \zeta(LG)^{\times}$  is always injective. If in addition  $L = \mathbb{R}$ , there exists a canonical map  $\hat{\partial}_G : \zeta(\mathbb{R}G)^{\times} \rightarrow K_0(\mathbb{Z}G, \mathbb{R}G)$  such that the restriction of  $\hat{\partial}_G$  to the image of the

reduced norm equals  $\partial_{\mathbb{Z}G, \mathbb{R}G} \circ \text{nr}_{\mathbb{R}G}^{-1}$ . This map is called the extended boundary homomorphism and was introduced by Burns and Flach [BF01].

**1.0.2 Non-commutative Fitting invariants** For the following we refer the reader to [Nib]. We denote the set of all  $m \times n$  matrices with entries in a ring  $R$  by  $M_{m \times n}(R)$  and in the case  $m = n$  the group of all invertible elements of  $M_{n \times n}(R)$  by  $\text{Gl}_n(R)$ . Let  $A$  be a separable  $K$ -algebra and  $\Lambda$  be an  $\mathfrak{o}$ -order in  $A$ , finitely generated as  $\mathfrak{o}$ -module, where  $\mathfrak{o}$  is a complete commutative noetherian local ring with field of quotients  $K$ . Moreover, we will assume that the integral closure of  $\mathfrak{o}$  in  $K$  is finitely generated as  $\mathfrak{o}$ -module. The group ring  $\mathbb{Z}_p G$  of a finite group  $G$  will serve as a standard example. Let  $N$  and  $M$  be two  $\zeta(\Lambda)$ -submodules of an  $\mathfrak{o}$ -torsionfree  $\zeta(\Lambda)$ -module. Then  $N$  and  $M$  are called  $\text{nr}(\Lambda)$ -*equivalent* if there exists an integer  $n$  and a matrix  $U \in \text{Gl}_n(\Lambda)$  such that  $N = \text{nr}(U) \cdot M$ , where  $\text{nr} : A \rightarrow \zeta(A)$  denotes the reduced norm map which extends to matrix rings over  $A$  in the obvious way. We denote the corresponding equivalence class by  $[N]_{\text{nr}(\Lambda)}$ . We say that  $N$  is  $\text{nr}(\Lambda)$ -contained in  $M$  (and write  $[N]_{\text{nr}(\Lambda)} \subset [M]_{\text{nr}(\Lambda)}$ ) if for all  $N' \in [N]_{\text{nr}(\Lambda)}$  there exists  $M' \in [M]_{\text{nr}(\Lambda)}$  such that  $N' \subset M'$ . Note that it suffices to check this property for one  $N_0 \in [N]_{\text{nr}(\Lambda)}$ . We will say that  $x$  is contained in  $[N]_{\text{nr}(\Lambda)}$  (and write  $x \in [N]_{\text{nr}(\Lambda)}$ ) if there is  $N_0 \in [N]_{\text{nr}(\Lambda)}$  such that  $x \in N_0$ .

Now let  $M$  be a finitely presented (left)  $\Lambda$ -module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \quad (2)$$

be a finite presentation of  $M$ . We identify the homomorphism  $h$  with the corresponding matrix in  $M_{a \times b}(\Lambda)$  and define  $S(h) = S_b(h)$  to be the set of all  $b \times b$  submatrices of  $h$  if  $a \geq b$ . In the case  $a = b$  we call (2) a quadratic presentation. The Fitting invariant of  $h$  over  $\Lambda$  is defined to be

$$\text{Fitt}_\Lambda(h) = \begin{cases} [0]_{\text{nr}(\Lambda)} & \text{if } a < b \\ [\langle \text{nr}(H) | H \in S(h) \rangle_{\zeta(\Lambda)}]_{\text{nr}(\Lambda)} & \text{if } a \geq b. \end{cases}$$

We call  $\text{Fitt}_\Lambda(h)$  a Fitting invariant of  $M$  over  $\Lambda$ . One defines  $\text{Fitt}_\Lambda^{\max}(M)$  to be the unique Fitting invariant of  $M$  over  $\Lambda$  which is maximal among all Fitting invariants of  $M$  with respect to the partial order “ $\subset$ ”. If  $M$  admits a quadratic presentation  $h$ , one also puts  $\text{Fitt}_\Lambda(M) := \text{Fitt}_\Lambda(h)$  which is independent of the chosen quadratic presentation.

Now let  $C$  and  $C'$  be two finitely generated  $\mathfrak{o}$ -torsion  $\Lambda$ -modules of finite projective dimension and denote by  $[C]$  and  $[C']$  the corresponding classes in  $K_0 T(\Lambda)$ , respectively. If  $\rho([C] - [C']) = 0$ , we choose  $x \in K_1(A)$  such that  $\partial(x) = [C] - [C']$  and define (cf. [Nib], Def. 3.6)

$$\text{Fitt}_\Lambda(C : C') := [\langle \text{nr}_A(x) \rangle_{\zeta(\Lambda)}]_{\text{nr}(\Lambda)}.$$

**1.0.3 Equivariant  $L$ -values** Let us fix a finite Galois extension  $L/K$  of number fields with Galois group  $G$ . For any prime  $\mathfrak{p}$  of  $K$  we fix a prime  $\mathfrak{P}$  of  $L$  above  $\mathfrak{p}$  and write  $G_{\mathfrak{P}}$  resp.  $I_{\mathfrak{P}}$  for the decomposition group resp. inertia subgroup of  $L/K$  at  $\mathfrak{P}$ . Moreover, we denote the residual group at  $\mathfrak{P}$  by  $\overline{G}_{\mathfrak{P}} = G_{\mathfrak{P}}/I_{\mathfrak{P}}$  and choose a lift  $\phi_{\mathfrak{P}} \in G_{\mathfrak{P}}$  of the Frobenius automorphism at  $\mathfrak{P}$ .

If  $S$  is a finite set of places of  $K$  containing the set  $S_\infty$  of all infinite places of  $K$ , and  $\chi$  is a (complex) character of  $G$ , we denote the  $S$ -truncated Artin  $L$ -function attached to  $\chi$  and  $S$  by  $L_S(s, \chi)$  and define  $L_S^*(0, \chi)$  to be the leading coefficient of the Taylor expansion of  $L_S(s, \chi)$  at  $s = 0$ . Recall that there is a canonical isomorphism  $\zeta(\mathbb{C}G) = \prod_{\chi \in \text{Irr}(G)} \mathbb{C}$ , where  $\text{Irr}(G)$  denotes the set of irreducible characters of  $G$ . We define the equivariant Artin  $L$ -function to be the meromorphic  $\zeta(\mathbb{C}G)$ -valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}(G)}.$$

We put  $L_S^*(0) = (L_S^*(0, \chi))_{\chi \in \text{Irr}(G)}$  and abbreviate  $L_{S_\infty}(s)$  by  $L(s)$ . If  $T$  is a second finite set of places of  $K$  such that  $S \cap T = \emptyset$ , we define  $\delta_T(s) := (\delta_T(s, \chi))_{\chi \in \text{Irr}(G)}$ , where  $\delta_T(s, \chi) = \prod_{\mathfrak{p} \in T} \det(1 - N(\mathfrak{p})^{1-s} \phi_{\mathfrak{p}}^{-1} | V_\chi^{I_{\mathfrak{p}}})$  and  $V_\chi$  is a  $G$ -module with character  $\chi$ . We put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp,$$

where we denote by  $^\sharp : \mathbb{C}G \rightarrow \mathbb{C}G$  the involution induced by  $g \mapsto g^{-1}$ . These functions are the so-called  $(S, T)$ -modified  $G$ -equivariant  $L$ -functions and we define Stickelberger elements

$$\theta_S^T := \Theta_{S,T}(0) \in \zeta(\mathbb{C}G).$$

If  $T$  is empty, we abbreviate  $\theta_S^T$  by  $\theta_S$ . Note that the  $\chi$ -part of  $\theta_S^T$  vanishes for a non-trivial character  $\chi$  if there is an (infinite) prime  $\mathfrak{p} \in S$  such that  $V_\chi^{G_{\mathfrak{p}}} \neq 0$ . Now let  $L/K$  be a Galois CM-extension, i.e.  $L$  is a CM-field,  $K$  is totally real and complex conjugation induces an unique automorphism  $j$  of  $L$  which lies in the center of  $G$ . If  $R$  is a subring of either  $\mathbb{C}$  or  $\mathbb{C}_p$  for a prime  $p$  such that 2 is invertible over  $R$ , we put  $RG_- := RG/(1+j)$  which is a ring, since the idempotent  $\frac{1-j}{2}$  lies in  $RG$ . For any  $RG$ -module  $M$  we define  $M^- = RG_- \otimes_{RG} M$  which is an exact functor since  $2 \in R^\times$ . Now Stark's conjecture (which is a theorem for odd characters, see [Ta84], Th. 1.2, p. 70) implies

$$\theta_S^T \in \zeta(\mathbb{Q}G_-). \quad (3)$$

Note that we actually have to exclude the special case  $|S_\infty(L)| = 1$  (cf. the proof of [Nia], Prop. 3, where (3) is shown in the relevant case  $S = S_\infty$  and  $T = \emptyset$ ), but in this situation the extension  $L/K$  is abelian. Here, we write  $S(L)$  for the set of places in  $L$  which lie above those in  $S$ , and  $S$  is any (finite) set of places of  $K$ . Let us fix an embedding  $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$ ; then the image of  $\theta_S^T$  in  $\zeta(\mathbb{Q}_p G_-)$  via the canonical embedding

$$\zeta(\mathbb{Q}G_-) \hookrightarrow \zeta(\mathbb{Q}_p G_-) = \bigoplus_{\substack{\chi \in \text{Irr}_{\mathbb{P}}(G)/\sim \\ \chi \text{ odd}}} \mathbb{Q}_p(\chi),$$

is given by  $\sum_{\chi} (\delta_T(0, \chi^{\iota^{-1}}) \cdot L_S(0, \check{\chi}^{\iota^{-1}}))^{\iota}$ , where we write  $\check{\chi}$  for the character contragredient to  $\chi$ . Here, the sum runs over all  $\mathbb{C}_p$ -valued irreducible odd characters of  $G$  modulo Galois action. Note that we will frequently drop  $\iota$  and  $\iota^{-1}$  from the notation.

**1.0.4 Ray class groups** Let  $T$  and  $S$  be as above. We write  $\text{cl}_L^T$  for the ray class group of  $L$  to the ray  $\mathfrak{M}_T := \prod_{\mathfrak{p} \in T(L)} \mathfrak{P}$  and  $\mathfrak{o}_S$  for the ring of  $S(L)$ -integers of  $L$ . Let  $S_f$  be the set of all finite primes in  $S(L)$ ; then there is a natural map  $\mathbb{Z}S_f \rightarrow \text{cl}_L^T$  which sends each prime  $\mathfrak{P} \in S_f$  to the corresponding class  $[\mathfrak{P}] \in \text{cl}_L^T$ . We denote the cokernel of this map by  $\text{cl}_S^T(L) =: \text{cl}_S^T$ . Further, we denote the  $S(L)$ -units of  $L$  by  $E_S$  and define  $E_S^T := \{x \in E_S : x \equiv 1 \pmod{\mathfrak{M}_T}\}$ . All these modules are equipped with a natural  $G$ -action and we have the following exact sequences of  $G$ -modules

$$E_{S_\infty}^T \hookrightarrow E_S^T \xrightarrow{v} \mathbb{Z}S_f \rightarrow \text{cl}_L^T \twoheadrightarrow \text{cl}_S^T, \quad (4)$$

where  $v(x) = \sum_{\mathfrak{p} \in S_f} v_{\mathfrak{p}}(x) \mathfrak{P}$  for  $x \in E_S^T$ , and

$$E_S^T \hookrightarrow E_S \rightarrow (\mathfrak{o}_S/\mathfrak{M}_T)^\times \xrightarrow{\nu} \text{cl}_S^T \twoheadrightarrow \text{cl}_S, \quad (5)$$

where the map  $\nu$  lifts an element  $\bar{x} \in (\mathfrak{o}_S/\mathfrak{M}_T)^\times$  to  $x \in \mathfrak{o}_S$  and sends it to the ideal class  $[(x)] \in \text{cl}_S^T$  of the principal ideal  $(x)$ . Note that the  $G$ -module  $(\mathfrak{o}_S/\mathfrak{M}_T)^\times$  is c.t. (short for cohomologically trivial) if no prime in  $T$  ramifies in  $L/K$ . If  $L/K$  is a CM-extension, we define

$$A_S^T := (\mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} \text{cl}_S^T)^-.$$

If  $S = S_\infty$ , we also write  $A_L^T$  and  $E_L^T$  instead of  $A_{S_\infty}^T$  and  $E_{S_\infty}^T$ . Finally, we suppress the superscript  $T$  from the notation if  $T$  is empty. If  $M$  is a finitely generated  $\mathbb{Z}$ -module and  $p$  is a prime, we put

$M(p) := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ . In particular,  $A_L(p)$  is the  $p$ -part of the minus class group if  $p$  is odd.

## 2. A reformulation of the equivariant Iwasawa main conjecture

Let  $p \neq 2$  be a prime and let  $F/K$  be a Galois extension of totally real fields with Galois group  $\mathcal{G}$ , where  $K$  is a global number field,  $F$  contains the cyclotomic  $\mathbb{Z}_p$ -extension  $K_\infty$  of  $K$  and  $[F : K_\infty]$  is finite. Hence  $\mathcal{G}$  is a  $p$ -adic Lie group of dimension 1 and there is a finite normal subgroup  $H$  of  $\mathcal{G}$  such that  $\mathcal{G}/H = \text{Gal}(K_\infty/K) =: \Gamma_K$ . Here,  $\Gamma_K$  is isomorphic to the  $p$ -adic integers  $\mathbb{Z}_p$  and we fix a topological generator  $\gamma_K$ . We denote the completed group algebra  $\mathbb{Z}_p[[\mathcal{G}]]$  by  $\Lambda(\mathcal{G})$  and the total ring of fractions of  $\Lambda(\mathcal{G})$  by  $Q(\mathcal{G})$ . If we pick a preimage  $\gamma$  of  $\gamma_K$  in  $\mathcal{G}$ , we can choose an integer  $m$  such that  $\gamma^{p^m}$  lies in the center of  $\mathcal{G}$ . Hence the ring  $R := \mathbb{Z}_p[[\Gamma^{p^m}]]$  belongs to the center of  $\Lambda(\mathcal{G})$ , and  $\Lambda(\mathcal{G})$  is an  $R$ -order in the separable  $\text{Quot}(R)$ -algebra  $Q(\mathcal{G})$ . Note that  $R$  is isomorphic to the power series ring  $\mathbb{Z}_p[[T]]$ . Let  $S$  be a finite set of places of  $K$  containing all the infinite places  $S_\infty$  and the set  $S_p$  of all places of  $K$  above  $p$ . Moreover, let  $M_S$  be the maximal abelian pro- $p$ -extension of  $F$  unramified outside  $S$ , and denote the Iwasawa module  $\text{Gal}(M_S/F)$  by  $X_S$ . There is a canonical complex

$$C^\bullet(F/K) : \dots \rightarrow 0 \rightarrow C^{-1} \rightarrow C^0 \rightarrow 0 \rightarrow \dots \quad (6)$$

of  $R$ -torsion  $\Lambda(\mathcal{G})$ -modules of projective dimension at most 1 such that  $H^{-1}(C^\bullet(F/K)) = X_S$  and  $H^0(C^\bullet(F/K)) = \mathbb{Z}_p$ . We put (cf. [RW04], §4)

$$\mathfrak{U}_S = \mathfrak{U}_S(F/K) := (C^{-1}) - (C^0) \in K_0T(\Lambda(\mathcal{G})).$$

Since  $\rho(\mathfrak{U}_S) = 0$ , there is a well defined Fitting invariant of  $\mathfrak{U}_S$ ; more precisely,

$$\text{Fitt}_{\Lambda(\mathcal{G})}(\mathfrak{U}_S) := \text{Fitt}_{\Lambda(\mathcal{G})}(C^{-1} : C^0).$$

Moreover, if  $\mathcal{F}$  is an exact functor from the category of  $R$ -torsion  $\Lambda(\mathcal{G})$ -modules of projective dimension at most 1 to itself, we also set

$$\text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(\mathfrak{U}_S)) := \begin{cases} \text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(C^{-1}) : \mathcal{F}(C^0)) & \text{if } \mathcal{F} \text{ is covariant} \\ \text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{F}(C^0) : \mathcal{F}(C^{-1})) & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

We recall some results concerning the algebra  $Q(\mathcal{G})$  due to Ritter and Weiss [RW04]. Let  $\mathbb{Q}_p^c$  be an algebraic closure of  $\mathbb{Q}_p$  and fix an irreducible ( $\mathbb{Q}_p^c$ -valued) character  $\chi$  of  $\mathcal{G}$  with open kernel. Choose a finite field extension  $E$  of  $\mathbb{Q}_p$  such that the character  $\chi$  has a realization  $V_\chi$  over  $E$ . Let  $\eta$  be an irreducible constituent of  $\text{res}_{H/H}^{\mathcal{G}} \chi$  and set

$$St(\eta) := \{g \in \mathcal{G} : \eta^g = \eta\}, \quad e_\eta = \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h, \quad e_\chi = \sum_{\eta | \text{res}_{H/H}^{\mathcal{G}} \chi} e_\eta.$$

For any finite field extension  $k$  of  $\mathbb{Q}_p$  with ring of integers  $\mathfrak{o}$ , we set  $Q^k(\mathcal{G}) := k \otimes_{\mathbb{Q}_p} Q(\mathcal{G})$  and  $\Lambda^{\mathfrak{o}}(\mathcal{G}) = \mathfrak{o}[[\mathcal{G}]]$ . By [RW04], corollary to Prop. 6,  $e_\chi$  is a primitive central idempotent of  $Q^E(\mathcal{G})$ . By loc.cit., Prop. 5 there is a distinguished element  $\gamma_\chi \in \zeta(Q^E(\mathcal{G})e_\chi)$  which generates a procyclic  $p$ -subgroup  $\Gamma_\chi$  of  $(Q^E(\mathcal{G})e_\chi)^\times$  and acts trivially on  $V_\chi$ . Moreover,  $\gamma_\chi$  induces an isomorphism  $Q^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(Q^E(\mathcal{G})e_\chi)$  by loc.cit., Prop. 6. For  $r \in \mathbb{N}_0$ , we define the following maps

$$j_\chi^r : \zeta(Q^E(\mathcal{G})) \rightarrow \zeta(Q^E(\mathcal{G})e_\chi) \simeq Q^E(\Gamma_\chi) \rightarrow Q^E(\Gamma_K),$$

where the last arrow is induced by mapping  $\gamma_\chi$  to  $\kappa^r(\gamma_\chi)\gamma_K^{w_\chi}$ , where  $w_\chi = [\mathcal{G} : St(\eta)]$  and  $\kappa$  denotes the cyclotomic character of  $\mathcal{G}$ . Note that  $j_\chi := j_\chi^0$  agrees with the corresponding map  $j_\chi$  in loc.cit. It is shown that for any matrix  $\Theta \in M_{n \times n}(Q(\mathcal{G}))$  we have

$$j_\chi(\text{nr}(\Theta)) = \det_{Q^E(\Gamma_K)}(\Theta | \text{Hom}_{EH}(V_\chi, Q^E(\mathcal{G})^n)). \quad (7)$$

Here,  $\Theta$  acts on  $f \in \text{Hom}_{EH}(V_\chi, Q^E(\mathcal{G})^n)$  via right multiplication, and  $\gamma_K$  acts on the left via  $(\gamma_K f)(v) = \gamma_K \cdot f(\gamma_K^{-1}v)$  for all  $v \in V_\chi$ . Hence the map

$$\begin{aligned} \text{Det}(\cdot)(\chi) : K_1(Q(\mathcal{G})) &\rightarrow Q^E(\Gamma_K)^\times \\ [P, \alpha] &\mapsto \det_{Q^E(\Gamma_K)}(\alpha | \text{Hom}_{EH}(V_\chi, E \otimes_{\mathbb{Q}_p} P)), \end{aligned}$$

where  $P$  is a projective  $Q(\mathcal{G})$ -module and  $\alpha$  a  $Q(\mathcal{G})$ -automorphism of  $P$ , is just  $j_\chi \circ \text{nr}$ . If  $\rho$  is a character of  $\mathcal{G}$  of type  $W$ , i.e.  $\text{res}_H^{\mathcal{G}} \rho = 1$ , then we denote by  $\rho^\sharp$  the automorphism of the field  $Q^c(\Gamma_K) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\Gamma_K)$  induced by  $\rho^\sharp(\gamma_K) = \rho(\gamma_K)\gamma_K$ . Moreover, we denote the additive group generated by all  $\mathbb{Q}_p^c$ -valued characters of  $\mathcal{G}$  with open kernel by  $R_p(\mathcal{G})$ ; finally,  $\text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$  is the group of all homomorphisms  $f : R_p(\mathcal{G}) \rightarrow Q^c(\Gamma_K)^\times$  satisfying

$$\begin{aligned} f(\chi \otimes \rho) &= \rho^\sharp(f(\chi)) && \text{for all characters } \rho \text{ of type } W \text{ and} \\ f(\chi^\sigma) &= f(\chi)^\sigma && \text{for all Galois automorphisms } \sigma \in \text{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p). \end{aligned}$$

We have an isomorphism

$$\begin{aligned} \zeta(Q(\mathcal{G}))^\times &\simeq \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times) \\ x &\mapsto [\chi \mapsto j_\chi(x)]. \end{aligned}$$

By loc.cit., Th. 5 the map  $\Theta \mapsto [\chi \mapsto \text{Det}(\Theta)(\chi)]$  defines a homomorphism

$$\text{Det} : K_1(Q(\mathcal{G})) \rightarrow \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$$

such that we obtain a commutative triangle

$$\begin{array}{ccc} & K_1(Q(\mathcal{G})) & \\ \text{nr} \swarrow & & \searrow \text{Det} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{\sim} & \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times). \end{array} \tag{8}$$

We put  $u := \kappa(\gamma_K)$ . Each topological generator  $\gamma_K$  of  $\Gamma_K$  permits the definition of a power series  $G_{\chi,S}(T) \in \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \text{Quot}(\mathbb{Z}_p[[T]])$  by starting out from the Deligne-Ribet power series for abelian characters of open subgroups of  $\mathcal{G}$  (cf. [DR80]). One then has an equality

$$L_{p,S}(1-s, \chi) = \frac{G_{\chi,S}(u^s - 1)}{H_\chi(u^s - 1)},$$

where  $L_{p,S}(s, \chi)$  denotes the  $p$ -adic Artin  $L$ -function, and where, for irreducible  $\chi$ , one has

$$H_\chi(T) = \begin{cases} \chi(\gamma_K)(1+T) - 1 & \text{if } H \subset \ker(\chi) \\ 1 & \text{otherwise.} \end{cases}$$

Now [RW04], Prop. 11 implies that

$$L_{K,S} : \chi \mapsto \frac{G_{\chi,S}(\gamma_K - 1)}{H_\chi(\gamma_K - 1)}$$

is independent of the topological generator  $\gamma_K$  and lies in  $\text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_K)^\times)$ . Diagram (8) implies that there is a unique element  $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$  such that

$$j_\chi(\Phi_S) = L_{K,S}(\chi).$$

The EIMC as formulated in [RW04] now states that there is a unique  $\Theta_S \in K_1(Q(\mathcal{G}))$  such that  $\text{Det}(\Theta_S) = L_{K,S}$  and  $\partial(\Theta_S) = \mathcal{U}_S$ . By means of diagram (8), the following conjecture is equivalent to the EIMC without the uniqueness of  $\Theta_S$ .

CONJECTURE 2.1. The element  $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$  is a generator of  $\text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{U}_S)$ .

We also discuss Conjecture 2.1 within the framework of the theory of [CFKSV05], §3. For this, let

$$\pi : \mathcal{G} \rightarrow \mathrm{Gl}_n(\mathfrak{o}_E)$$

be a continuous homomorphism, where  $\mathfrak{o}_E$  denotes the ring of integers of  $E$  and  $n$  is some integer greater or equal to 1. There is a ring homomorphism

$$\Phi_\pi : \Lambda(\mathcal{G}) \rightarrow M_{n \times n}(\Lambda^{\circ E}(\Gamma_K)) \quad (9)$$

induced by the continuous group homomorphism

$$\begin{aligned} \mathcal{G} &\rightarrow (M_{n \times n}(\mathfrak{o}_E) \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_K))^\times = \mathrm{Gl}_n(\Lambda^{\circ E}(\Gamma_K)) \\ \sigma &\mapsto \pi(\sigma) \otimes \bar{\sigma}, \end{aligned}$$

where  $\bar{\sigma}$  denotes the image of  $\sigma$  in  $\mathcal{G}/H = \Gamma_K$ . By loc.cit., Lemma 3.3 the homomorphism (9) extends to a ring homomorphism

$$\Phi_\pi : Q(\mathcal{G}) \rightarrow M_{n \times n}(Q^E(\Gamma_K))$$

and this in turn induces a homomorphism

$$\Phi'_\pi : K_1(Q(\mathcal{G})) \rightarrow K_1(M_{n \times n}(Q^E(\Gamma_K))) = Q^E(\Gamma_K)^\times.$$

Let  $\mathrm{aug} : \Lambda^{\circ E}(\Gamma_K) \rightarrow \mathfrak{o}_E$  be the augmentation map and put  $\mathfrak{p} = \ker(\mathrm{aug})$ . Writing  $\Lambda^{\circ E}(\Gamma_K)_\mathfrak{p}$  for the localization of  $\Lambda^{\circ E}(\Gamma_K)$  at  $\mathfrak{p}$ , it is clear that  $\mathrm{aug}$  naturally extends to a homomorphism  $\mathrm{aug} : \Lambda^{\circ E}(\Gamma_K)_\mathfrak{p} \rightarrow E$ . One defines an evaluation map

$$\begin{aligned} \phi : Q^E(\Gamma_K) &\rightarrow E \cup \{\infty\} \\ x &\mapsto \begin{cases} \mathrm{aug}(x) & \text{if } x \in \Lambda^{\circ E}(\Gamma_K)_\mathfrak{p} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

If  $\Theta$  is an element of  $K_1(Q(\mathcal{G}))$ , we define  $\Theta(\pi)$  to be  $\phi(\Phi'_\pi(\Theta))$ . We need the following lemma.

LEMMA 2.2. *If  $\pi = \pi_\chi$  is a representation of  $\mathcal{G}$  with character  $\chi$  and  $r \in \mathbb{N}_0$ , then*

$$\begin{array}{ccc} K_1(Q(\mathcal{G})) & \xrightarrow{\Phi'_{\pi_\chi \kappa^r}} & K_1(M_{n \times n}(Q^E(\Gamma_K))) \\ \downarrow \mathrm{nr} & & \downarrow \simeq \mathrm{nr} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{j_\chi^r} & Q^E(\Gamma_K)^\times \end{array}$$

*commutes. In particular, we have  $\mathrm{nr} \circ \Phi'_{\pi_\chi} = \mathrm{Det}(\cdot)(\chi)$ .*

*Proof.* We recall that the map  $j_\chi$  induces a field extension  $Q^E(\Gamma_K)/Q^E(\Gamma_\chi)$ , where  $Q^E(\Gamma_\chi) = \zeta(Q^E(\mathcal{G})e_\chi)$ . The results in [RW04] imply that in fact  $Q^E(\Gamma_K)$  is a splitting field of  $Q^E(\mathcal{G})e_\chi$  and we thus have an isomorphism

$$Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})e_\chi \simeq M_{n \times n}(Q^E(\Gamma_K)). \quad (10)$$

Since  $1 \otimes \gamma_\chi = \gamma_K^{w_\chi} \otimes 1$  in  $Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})e_\chi$  and  $\pi_\chi(\gamma_\chi) \otimes \bar{\gamma}_\chi = 1 \otimes \gamma_K^{w_\chi}$  in  $M_{n \times n}(Q^E(\Gamma_K))$ , the homomorphism  $\Phi_{\pi_\chi}$  induces a realization of the above isomorphism (10). Hence  $\mathrm{nr} \circ \Phi'_{\pi_\chi}$  is just the reduced norm on  $Q^E(\mathcal{G})e_\chi$  which takes values in  $Q^E(\Gamma_\chi) \xrightarrow{j_\chi} Q^E(\Gamma_K)$ . This shows the lemma in the case  $r = 0$ . For arbitrary  $r$ , we similarly have  $j_\chi^r(\mathrm{nr}(\Theta)) = \det_{Q^E(\Gamma_K)}(\Theta | \mathcal{V}_\chi(r)) = \mathrm{nr}(\Phi'_{\pi_\chi \kappa^r}(\Theta))$ , where  $\Theta \in K_1(Q(\mathcal{G}))$  and  $\mathcal{V}_\chi(r)$  is the  $r$ -th Tate twist of the absolutely irreducible (right) module  $\mathcal{V}_\chi := \mathrm{Hom}_{EH}(V_\chi, Q^E(\mathcal{G}))$  over  $Q^E(\Gamma_K) \otimes_{Q^E(\Gamma_\chi)} Q^E(\mathcal{G})$ .  $\square$



Conjecture 2.1 now implies that there is an element  $\Theta_S \in K_1(Q(\mathcal{G}))$  such that  $\partial(\Theta_S) = \mathfrak{U}_S$  and for any  $r \geq 1$  divisible by  $p - 1$  we have

$$\Theta_S(\pi_\chi \kappa^r) = \phi(j_\chi^r(\Phi_S)) = L_S(1 - r, \chi).$$

### 3. Algebraic preparations

Let  $p \neq 2$  be a prime and let  $\mathcal{G}$  be a  $p$ -adic Lie group of dimension 1, i.e. there is a finite normal subgroup  $H$  of  $\mathcal{G}$  such that  $\Gamma := \mathcal{G}/H$  is isomorphic to  $\mathbb{Z}_p$ . For any ring  $\Lambda$  and any  $\Lambda$ -module  $M$ , we write  $\text{pd}_\Lambda(M)$  for the projective dimension of  $M$  over  $\Lambda$ . For any finitely generated  $\Lambda(\mathcal{G})$ -module  $M$ , we write  $\mu(M)$  for the Iwasawa  $\mu$ -invariant of  $M$ . As before, let  $\Gamma' \simeq \mathbb{Z}_p$  be a subgroup of  $\mathcal{G}$  which is central in  $\mathcal{G}$  and put  $R = \mathbb{Z}_p[[\Gamma']]$ .

**PROPOSITION 3.1.** *Let  $M$  be a finitely generated  $R$ -torsion  $\Lambda(\mathcal{G})$ -module which has no non-trivial finite submodule, has  $\mu(M) = 0$  and is cohomologically trivial as  $H$ -module. Then*

$$\text{pd}_{\Lambda(\mathcal{G})}(M) \leq 1.$$

*Proof.* For any topological ring  $\Lambda$ , we denote the category of compact  $\Lambda$ -modules by  $\mathcal{C}(\Lambda)$  and the category of discrete  $\Lambda$ -modules by  $\mathcal{D}(\Lambda)$ . We have a functor

$$\text{Hom}_{\Lambda(\mathcal{G})}(\ , \ ) : \mathcal{C}(\Lambda(\mathcal{G})) \times \mathcal{D}(\Lambda(\mathcal{G})) \longrightarrow \mathcal{D}(\mathbb{Z}_p)$$

and we can use either projective resolutions in  $\mathcal{C}(\Lambda(\mathcal{G}))$  or injective resolutions in  $\mathcal{D}(\Lambda(\mathcal{G}))$  to define functors

$$\mathcal{E}xt_{\Lambda(\mathcal{G})}^i(\ , \ ) : \mathcal{C}(\Lambda(\mathcal{G})) \times \mathcal{D}(\Lambda(\mathcal{G})) \longrightarrow \mathcal{D}(\mathbb{Z}_p), \ i \geq 0.$$

By [NSW00], Prop. (5.2.11), we have to show that  $\mathcal{E}xt_{\Lambda(\mathcal{G})}^2(M, N) = 0$  for all simple  $N$ . We consider the spectral sequence (cf. loc.cit., Ch. V, §2, Ex. 4):

$$E_2^{i,j} = H^i(\Gamma_K, \mathcal{E}xt_{\mathbb{Z}_p H}^j(M, N)) \implies E^{i+j} = \mathcal{E}xt_{\Lambda(\mathcal{G})}^{i+j}(M, N).$$

Since  $M$  has no non-trivial finite submodules and  $\mu(M) = 0$ , it is free and finitely generated as  $\mathbb{Z}_p$ -module. Moreover, it is c.t. as  $H$ -module by assumption and hence  $\mathbb{Z}_p H$ -projective. This implies  $E_2^{i,j} = 0$  for  $j > 0$ . Since  $N$  and hence  $\text{Hom}_{\mathbb{Z}_p H}(M, N)$  are  $p$ -torsion and the cohomological  $p$ -dimension of  $\Gamma_K$  is 1, we also have  $E_2^{i,j} = 0$  if  $i > 1$ . This implies  $\mathcal{E}xt_{\Lambda(\mathcal{G})}^2(M, N) = E^2 \simeq E_2^{2,0} = 0$ .  $\square$

**PROPOSITION 3.2.** *Let  $M$  be a finitely generated  $R$ -torsion  $\Lambda(\mathcal{G})$ -module such that  $\text{pd}_{\Lambda(\mathcal{G})}(M) \leq 1$  and  $\mu(M) = 0$ . Assume that  $\text{Fitt}_{\mathbb{Q}_p \Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes M)$  is generated by  $\text{nr}(\phi)$  over  $\mathbb{Q}_p \Lambda(\mathcal{G})$ , where  $\phi \in M_{n \times n}(\Lambda(\mathcal{G})) \cap \text{Gl}_n(Q(\mathcal{G}))$  such that  $\mu(\Lambda(\mathcal{G})^n / \phi(\Lambda(\mathcal{G})^n)) = 0$ . Then also*

$$\text{Fitt}_{\Lambda(\mathcal{G})}(M) = [\langle \text{nr}(\phi) \rangle_{\zeta(\Lambda(\mathcal{G}))}]_{\text{nr}(\Lambda(\mathcal{G}))}.$$

*Proof.* By [Nib], Lemma 6.2 the module  $M$  admits a quadratic presentation over  $\Lambda(\mathcal{G})$  such that  $\text{Fitt}_{\Lambda(\mathcal{G})}(M)$  exists and is generated by  $\text{nr}(\psi)$  for some  $\psi \in M_{n \times n}(\Lambda(\mathcal{G})) \cap \text{Gl}_n(Q(\mathcal{G}))$ . By assumption, we have

$$\langle \text{nr}(\phi) \rangle_{\zeta(\mathbb{Q}_p \Lambda(\mathcal{G}))} = \langle \text{nr}(\psi) \rangle_{\zeta(\mathbb{Q}_p \Lambda(\mathcal{G}))}$$

such that there is a unique  $x \in \zeta(\mathbb{Q}_p \Lambda(\mathcal{G}))^\times$  with  $\text{nr}(\psi) = x \cdot \text{nr}(\phi)$ . Let us denote the integral closure of  $\zeta(\Lambda(\mathcal{G}))$  in  $\zeta(Q(\mathcal{G}))$  by  $\mathfrak{Z}$ . Then the reduced norm maps  $K_1(\Lambda(\mathcal{G}))$  into  $\mathfrak{Z}^\times$  and  $K_1(\Lambda_{(p)}(\mathcal{G}))$  into  $\mathfrak{Z}_{(p)}^\times$ , where the subscript  $(p)$  means localization at the prime  $(p)$ . We have shown that there is a natural number  $N$  such that  $p^N \cdot x \in \mathfrak{Z}$ . Since the  $\mu$ -invariants vanish, the maps  $\phi$  and  $\psi$  become isomorphisms after localization at  $(p)$  and we find  $x \in \text{nr}(K_1(\Lambda_{(p)}(\mathcal{G}))) \subset \mathfrak{Z}_{(p)}^\times$ . Thus we can choose

a Weierstraß polynomial  $f$  such that  $f \cdot x \in \mathfrak{z}$  and hence  $x \in \mathfrak{z}$  by Lemma 3.3 below. Now [RW05], Th. B in conjunction with diagram (8) implies

$$x \in \mathfrak{z} \cap \text{nr}(K_1(\Lambda_{(p)}(\mathcal{G}))) \subset \text{nr}(K_1(\Lambda(\mathcal{G}))).$$

Hence the  $\zeta(\Lambda(\mathcal{G}))$ -modules generated by  $\text{nr}(\phi)$  and  $\text{nr}(\psi)$  are  $\text{nr}(\Lambda(\mathcal{G}))$ -equivalent.  $\square$

We have used the following easy lemma.

**LEMMA 3.3.** *Let  $\Lambda$  be a ring,  $x \in \Lambda$  and  $y \in \zeta(\Lambda)$ . Assume that  $y$  is a nonzerodivisor and  $x$  is a nonzerodivisor modulo  $y$ . Let  $S$  be a multiplicatively closed subset of  $\zeta(\Lambda)$  which contains no zero divisors,  $1 \in S$ ,  $0 \notin S$  and let  $\Psi \in \Lambda_S$  such that  $x \cdot \Psi \in \Lambda$  and  $y \cdot \Psi \in \Lambda$ . Then also  $\Psi \in \Lambda$ .*

*Proof.* The equation  $x \cdot \Psi \cdot y = x \cdot y \cdot \Psi$  implies that  $y \cdot \Psi \equiv 0 \pmod{y}$ , since  $x$  is a nonzerodivisor modulo  $y$ . Hence there is  $\lambda \in \Lambda$  such that  $y \cdot \Psi = y \cdot \lambda$ . But  $y$  is a nonzerodivisor and thus  $\lambda = \Psi$ .  $\square$

If  $M$  is an Iwasawa torsion module, we write  $\alpha(M)$  for the Iwasawa adjoint of  $M$ . If  $H$  is a finite group and  $M$  is a  $\mathbb{Z}_p[H]$ -module, we denote the Pontryagin dual  $\text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$  of  $M$  by  $M^\vee$  which is equipped with the natural  $H$ -action  $(hf)(m) = f(h^{-1}m)$  for  $f \in M^\vee$ ,  $h \in H$  and  $m \in M$ .

**LEMMA 3.4.** *Let  $U$  be a subgroup of  $\mathcal{G}$  of finite index.*

- i) *For any  $\Lambda(U)$ -module  $N$ , we have an isomorphism  $\text{ind}_U^{\mathcal{G}}(N(1)) \simeq (\text{ind}_U^{\mathcal{G}}N)(1)$ .*
- ii) *If  $M = \text{ind}_U^{\mathcal{G}}\mathbb{Z}_p$ , then  $\alpha(M) \simeq M$ .*

*Proof.* Let us put  $N' := N(1)$ . Then  $\text{ind}_U^{\mathcal{G}}N' = \bigoplus_{\sigma} N'_{\sigma}$ , where  $\sigma$  runs through a set of (left) coset representatives, and where  $N'_{\sigma} = N'$  as sets and  $gn' = u_{\sigma}n'$  if  $g\sigma = \tilde{\sigma}u_{\sigma}$  for  $g \in \mathcal{G}$ ,  $u_{\sigma} \in U$ ,  $n' \in N'_{\sigma}$ ; similarly,  $\text{ind}_U^{\mathcal{G}}N = \bigoplus_{\sigma} N_{\sigma}$ . An easy computation shows that

$$\bigoplus_{\sigma} N'_{\sigma} \longrightarrow \left(\bigoplus_{\sigma} N_{\sigma}\right)(1), \quad \sum_{\sigma} n'_{\sigma} \mapsto \sum_{\sigma} \kappa(\sigma)n'_{\sigma}$$

is an isomorphism of  $\Lambda(\mathcal{G})$ -modules. This shows (i). For (ii) we compute

$$\begin{aligned} \alpha(M) &= \varprojlim_n \text{Hom}(M/p^n, \mathbb{Q}_p/\mathbb{Z}_p) \\ &= \varprojlim_n (\text{ind}_U^{\mathcal{G}}\mathbb{Z}_p/p^n)^{\vee} \\ &\simeq \varprojlim_n \text{ind}_U^{\mathcal{G}}\mathbb{Z}_p/p^n \\ &= M. \end{aligned}$$

$\square$

We point out that Lemma 3.4 and Proposition 3.2 are non-abelian generalizations of [Gr04], Lemma 1 and Lemma 2, respectively.

#### 4. Fitting invariants of Iwasawa modules

In this section we fix the following setting: let  $L/K$  be a Galois CM-extension of number fields with Galois group  $G$ , i.e.  $K$  is totally real and  $L$  is a totally imaginary quadratic extension of a totally real number field. This field is the maximal real subfield of  $L$  and will be denoted by  $L^+$ . Complex conjugation on  $\mathbb{C}$  induces an automorphism  $j$  on  $L$  which is independent of the embedding into  $\mathbb{C}$  and lies in the center of  $G$ . Let  $p \neq 2$  be a prime and assume that  $j$  lies in the decomposition group  $G_{\mathfrak{P}}$  for each prime  $\mathfrak{P}$  of  $L$  above  $p$  which is wildly ramified in  $L/K$  (we will call this condition *almost tame* above  $p$ ). In particular, we consider all Galois CM-extension which are at most tamely ramified

above  $p$ .

We choose a prime  $\mathfrak{p}_0 \nmid p$  of  $K$  which is unramified in  $L/K$  and define a set of places of  $K$  by

$$T = T_0 := \{\mathfrak{p}_0\} \cup S_{\text{ram}} \setminus (S_{\text{ram}} \cap S_p).$$

We may choose  $\mathfrak{p}_0$  such that  $E_S^T$  is torsionfree. Then  $A_L^T(p)$ , the  $p$ -part of the minus ray class group  $\text{cl}_L^{T,-}$ , is c.t. as  $G$ -module by [Nia], Th. 1.

Let  $L_\infty$  and  $K_\infty$  be the cyclotomic  $\mathbb{Z}_p$ -extensions of  $L$  and  $K$ , respectively. We denote the Galois group of  $K_\infty/K$  by  $\Gamma_K$ . Hence  $\Gamma_K$  is isomorphic to  $\mathbb{Z}_p$ , and we fix a topological generator  $\gamma_K$ . Furthermore, we denote the  $n$ -th layer in the cyclotomic extension  $K_\infty/K$  by  $K_n$  such that  $K_n/K$  is cyclic of order  $p^n$ . Accordingly, we set  $\Gamma_L = \text{Gal}(L_\infty/L)$  with a topological generator  $\gamma_L$  whose restriction to  $K_\infty$  is  $\gamma_K^a$  for an appropriate integer  $a$ . We enumerate the intermediate fields starting with  $L = L_a$  such that  $L_n/L$  is cyclic of order  $p^{n-a}$ . This is because in this case  $L_n$  is the smallest intermediate field of  $L_\infty/L$  which lies above  $K_n$ . It may also be convenient to define  $L_n = L$  if  $n \leq a$ . We put

$$\mathcal{X}_T^- := \varprojlim A_{L_n}^T(p).$$

We denote the Galois group of  $L_\infty/K$  by  $\mathcal{G}$ , hence  $\mathcal{G} = H \rtimes \Gamma$ , where  $H$  is a subgroup of  $G$  and  $\Gamma$  is topologically generated by a preimage  $\gamma$  of  $\gamma_K$  under the canonical epimorphism  $\mathcal{G} \twoheadrightarrow \mathcal{G}/H = \Gamma_K$ . Then  $\mathcal{X}_T^-$  is a finitely generated  $R$ -torsion  $\Lambda(\mathcal{G})_- := \Lambda(\mathcal{G})/(1+j)$ -module, where as before  $R = \mathbb{Z}_p[[\Gamma']]$  with  $\Gamma' \simeq \mathbb{Z}_p$  central in  $\mathcal{G}$ . Let  $L'$  be the maximal subfield of  $L_\infty$  fixed by  $\Gamma$ . Since  $L'$  is contained in  $L_n$  if  $n$  is sufficiently large, the layers of the cyclotomic extensions of  $L$  and  $L'$  agree for  $n \gg 0$  and  $A_{L_n}^T(p)$  is  $\text{Gal}(L_n/K_n)$ -c.t., since each of the extensions  $L_n/K_n$  inherits the required properties from the extension  $L/K$ . Hence  $\mathcal{X}_T^-$  is c.t. as  $H$ -module and has no nontrivial finite submodule (cf. [Gr04], first step of the proof of Prop. 6) such that Proposition 3.1 implies the following result.

**PROPOSITION 4.1.** *If  $L/K$  is almost tame above  $p$  and the Iwasawa  $\mu$ -invariant  $\mu(\mathcal{X}_T^-)$  vanishes, then the projective dimension of  $\mathcal{X}_T^-$  over  $\Lambda(\mathcal{G})_-$  is at most 1.*

Now let  $S$  be a finite set of places of  $K$  containing  $S_\infty$  (but not necessarily  $S_p$ ) and let  $M_S$  be the maximal abelian pro- $p$ -extension of  $L_\infty$  unramified outside  $S$ . Moreover, let  $M_\infty$  be the maximal abelian unramified extension of  $L_\infty$  and define  $\Lambda(\mathcal{G})$ -modules

$$X_S := \text{Gal}(M_S/L_\infty), \quad X_{\text{std}} := \text{Gal}(M_\infty/L_\infty).$$

Hence  $X_{\text{std}}$  is the “standard” Iwasawa module which is the projective limit of the  $p$ -parts of the class groups in the cyclotomic tower of  $L$ . If  $S = S_\infty \cup S_p$ , we also write  $X_{\{p\}}$  instead of  $X_{S_\infty \cup S_p}$ . Moreover, if  $S = T \cup S_\infty$ , there is an isomorphism  $X_{T \cup S_\infty}^- \simeq \mathcal{X}_T^-$ . Following Greither [Gr04], we will also define a “dual” Iwasawa module  $X_{\text{du}}$ : There is a minimal integer  $n_0$  such that all the  $p$ -adic places ramify in  $L_\infty/L_{n_0}$ . We denote the  $p$ -class field of  $L_{n_0}$  by  $M_{n_0}$  and put  $X_{\text{du}} := \text{Gal}(M_\infty/M_{n_0}L_\infty)$ . So  $X_{\text{du}}$  is a submodule of  $X_{\text{std}}$  of finite index and the subscript “du” is chosen because of the following description of  $X_{\text{du}}^-$  in the case  $\zeta_p \in L$ , where  $\zeta_p$  denotes a primitive  $p$ -th root of unity (cf. [Gr04], beginning of §2 - note that  $G$  is assumed to be abelian in loc.cit., but in all cases, where we will cite [Gr04], this assumption is not necessary):

$$X_{\text{du}}^- \simeq \alpha(X_{\{p\}}^+)(1).$$

If  $S$  contains all the  $p$ -adic places, we define an Iwasawa module  $Z_S = Z_{L,S}$  by

$$\begin{aligned} Z_S &= \alpha(X_S^+)(1) \text{ if } \zeta_p \in L, \\ Z_S &= (Z_{L(\zeta_p),S})_\Delta \text{ otherwise,} \end{aligned}$$

where  $\Delta = \text{Gal}(L(\zeta_p)/L)$ . Note that this definition slightly differs from the definition of the corresponding module in loc.cit. But since  $p \nmid |\Delta|$ , multiplication by  $N_\Delta := \sum_{\delta \in \Delta} \delta$  induces an isomorphism  $(Z_{L(\zeta_p), S})_\Delta \simeq (Z_{L(\zeta_p), S})^\Delta$ . For any prime  $\mathfrak{p}$  of  $K$ , we choose a prime  $\wp$  in  $L_\infty$  above  $\mathfrak{p}$  and put  $\mathfrak{P} = \wp \cap L$ . Setting  $Z_{\mathfrak{p}} := \text{ind}_{\mathcal{G}_{\wp}}^{\mathcal{G}} \mathbb{Z}_p$ , class field theory gives an exact sequence (cf. loc.cit., sequence (1); for the proof replace loc.cit., Lemma 1 (i) by Lemma 3.4 (i)):

$$\bigoplus_{\mathfrak{p} \in S' \setminus S_p} Z_{\mathfrak{p}}(1)^+ \hookrightarrow X_S^+ \twoheadrightarrow X_{\{p\}}^+ \quad (11)$$

We claim that this sequence induces an exact sequence

$$X_{\text{du}}^- \hookrightarrow Z_S \twoheadrightarrow \bigoplus_{\mathfrak{p} \in S' \setminus S_p} Z_{\mathfrak{p}}^-. \quad (12)$$

This is clear if  $\zeta_p \in L$ , since taking Iwasawa adjoints is exact on sequences of torsion Iwasawa modules without finite submodules and  $\alpha(Z_{\mathfrak{p}}(1))(1) = \alpha(Z_{\mathfrak{p}}) = Z_{\mathfrak{p}}$  by Lemma 3.4 (ii). If  $\zeta_p \notin L$ , we put  $L' = L(\zeta_p)$ ,  $L'_\infty = L_\infty(\zeta_p)$  etc. Since  $p \nmid |\Delta|$ , the  $p$ -class groups of the layers in the cyclotomic tower are c.t. as  $\Delta$ -modules and we have thus isomorphisms  $A_{L'_n}(p)_\Delta \simeq A_{L_n}(p)$  which combine to induce an isomorphism  $(X'_{\text{std}})_\Delta \simeq X_{\text{std}}^-$ . We have a commutative diagram

$$\begin{array}{ccccc} (X'_{\text{du}})_\Delta & \hookrightarrow & (X'_{\text{std}})_\Delta & \twoheadrightarrow & A_{L'_{n_0}}(p)_\Delta \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ X_{\text{du}}^- & \hookrightarrow & X_{\text{std}}^- & \twoheadrightarrow & A_{L_{n_0}}(p) \end{array}$$

Hence also the leftmost vertical arrow is an isomorphism and we obtain (12) in general, as we may adjoin  $\zeta_p$  first and then apply  $\Delta$ -coinvariants to sequence (12) for  $L'$ .

Let  $x \mapsto \dot{x}$  be the automorphism on  $\Lambda(\mathcal{G})$  induced by  $g \mapsto \kappa(g)g^{-1}$  for  $g \in \mathcal{G}$ . Let  $\mathcal{G}^+ := \mathcal{G}/\langle j \rangle = \text{Gal}(L_\infty^+/K)$  and let  $\Phi_S \in \zeta(Q(\mathcal{G}^+))^\times$  be the unique element satisfying  $j_\chi(\Phi_S) = L_{K,S}(\chi)$  for each even character of  $\mathcal{G}$  with open kernel. We define idempotents

$$e^- = \frac{1-j}{2}, \quad e^+ = \frac{1+j}{2}.$$

The following is a non-abelian generalization of [Gr04], Th. 2.

**THEOREM 4.2.** *Assume that the EIMC holds for  $L_\infty^+/K$ .*

i) *If  $\zeta_p \in L$ , then*

$$\text{Fitt}_{\Lambda(\mathcal{G})}^{\max}(Z_S) = \text{Fitt}_{\Lambda(\mathcal{G})}^{\max}(Z_p(1))^\# [\langle \dot{\Phi}_S e^- + e^+ \rangle]_{\text{nr}(\Lambda(\mathcal{G}))}.$$

ii) *If  $\zeta_p \notin L$ , then  $\text{pd}_{\Lambda(\mathcal{G})}(Z_S) \leq 1$  and*

$$\text{Fitt}_{\Lambda(\mathcal{G})}(Z_S) = [\langle \dot{\Phi}_S e^- + e^+ \rangle]_{\text{nr}(\Lambda(\mathcal{G}))}.$$

*Proof.* Assume that  $\zeta_p \in L$ . The canonical complex (6) for the extension  $L_\infty^+/K$  gives an exact sequence

$$X_S^+ \hookrightarrow C^{-1} \rightarrow C^0 \twoheadrightarrow \mathbb{Z}_p.$$

Applying the functor  $\alpha(\cdot)(1)$  to this sequence yields

$$\mathbb{Z}_p(1) \hookrightarrow \alpha(C^0)(1) \rightarrow \alpha(C^{-1})(1) \twoheadrightarrow Z_S. \quad (13)$$

Now it follows from [Nib], Prop. 6.3 (ii) that

$$\begin{aligned} \text{Fitt}_{\Lambda(\mathcal{G})}^{\max}(Z_S) &= \text{Fitt}_{\Lambda(\mathcal{G})}^{\max}(Z_p(1))^\# \cdot \text{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mathcal{U}_S)(1)) \\ &= \text{Fitt}_{\Lambda(\mathcal{G})}^{\max}(Z_p(1))^\# \cdot [\langle \dot{\Phi}_S e^- + e^+ \rangle]_{\text{nr}(\Lambda(\mathcal{G}))}, \end{aligned}$$

where the second equality holds by the EIMC and loc.cit., Prop. 6.3 (i).

If  $\zeta_p \notin L$ , we again put  $L' = L(\zeta_p)$ . We apply  $\Delta = \text{Gal}(L'/L)$ -coinvariants to sequence (13) (for  $L'$ ) and obtain an exact sequence

$$\alpha(C^0)(1)_\Delta \hookrightarrow \alpha(C^{-1})(1)_\Delta \twoheadrightarrow Z_S.$$

Hence  $Z_S$  has projective dimension at most 1 and

$$\begin{aligned} \text{Fitt}_{\Lambda(\mathcal{G})}(Z_S) &= \text{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mathcal{U}_S((L'_\infty)^+/K))(1)_\Delta) \\ &= \text{Fitt}_{\Lambda(\mathcal{G})}(\alpha(\mathcal{U}_S(L_\infty^+/K))(1)) \\ &= [\langle \dot{\Phi}_S e^- + e^+ \rangle]_{\text{nr}(\Lambda(\mathcal{G}))}, \end{aligned}$$

where the second equality follows from [RW04], Prop. 12, whereas the last equality is the EIMC.  $\square$

As in [Gr04], Prop. 6 we have an exact sequence

$$\mathbb{Z}_p(1) \hookrightarrow \bigoplus_{\mathfrak{p} \in T} \mathbb{Z}_{\mathfrak{p}}(1)^- \rightarrow \mathcal{X}_T^- \twoheadrightarrow X_{\text{std}}^- \quad (14)$$

if  $\zeta_p \in L$ , and without the leftmost term if  $\zeta_p \notin L$ . For  $\mathfrak{p} \notin S_p$  we put

$$\Xi_{\mathfrak{p}} := \varepsilon_{\mathfrak{p}} \frac{\kappa(\phi_{\wp}) - \phi_{\wp}}{1 - \phi_{\wp}} + 1 - \varepsilon_{\mathfrak{p}} \in \mathbb{Q}_p \Lambda(\mathcal{G}_{\wp}), \text{ where } \varepsilon_{\mathfrak{p}} = |I_{\mathfrak{p}}|^{-1} N_{I_{\mathfrak{p}}} \in \mathbb{Q}_p H,$$

$$\xi_{\mathfrak{p}} := \text{nr}(1 \otimes \Xi_{\mathfrak{p}}).$$

Here,  $\phi_{\wp} \in \mathcal{G}$  and  $I_{\mathfrak{p}}$  are the Frobenius and the inertia subgroup at a chosen prime  $\wp$  in  $L_\infty$  above  $\mathfrak{p}$ , respectively; note that the inertia subgroup depends only on the prime  $\mathfrak{p}$  in  $L$  above  $p$ , since  $\mathfrak{p}$  lies not above  $p$  and is thus unramified in the cyclotomic extension. The element  $1 \otimes \Xi_{\mathfrak{p}}$  belongs to  $\mathbb{Q}_p \Lambda(\mathcal{G}) = \text{ind}_{\mathcal{G}_{\wp}}^{\mathcal{G}} \mathbb{Q}_p \Lambda(\mathcal{G}_{\wp})$ . Note that  $\phi_{\wp}$  and  $I_{\mathfrak{p}}$  depend on the choice of  $\wp$ , but  $\xi_{\mathfrak{p}}$  does not. If  $S$  is a finite set of places of  $K$  containing  $S_p \cup S_\infty$ , we put

$$\Psi_S := \prod_{\mathfrak{p} \in S \setminus S_p} \xi_{\mathfrak{p}} \cdot \dot{\Phi}_S e^- \in \zeta(Q(\mathcal{G})_-).$$

**PROPOSITION 4.3.** *The Fitting invariant  $\text{Fitt}_{\mathbb{Q}_p \Lambda(\mathcal{G})_-}(\mathbb{Q}_p \mathcal{X}_T^-)$  is generated by  $\Psi_{T \cup S_p}$ . In particular,  $\Psi_{T \cup S_p} \in \zeta(\mathbb{Q}_p \Lambda(\mathcal{G})_-)$ .*

*Proof.* We first observe that  $\mathbb{Q}_p \Lambda(\mathcal{G})$  is a maximal  $\mathbb{Q}_p \otimes R$ -order in  $Q(\mathcal{G})$ . In this case every finitely generated  $\mathbb{Q}_p \Lambda(\mathcal{G})$ -module has a quadratic presentation, and taking Fitting invariants is multiplicative on short exact sequences of  $\mathbb{Q}_p \otimes R$ -torsion  $\mathbb{Q}_p \Lambda(\mathcal{G})$ -modules. It suffices to assume the EIMC in the “maximal order case” which is a theorem ([RW04], Th. 16; cf. also loc.cit., remark H), and we may use Theorem 4.2 over  $\mathbb{Q}_p \Lambda(\mathcal{G})$  without assuming the EIMC. We put  $i = -1$  if  $\zeta_p \in L$  and  $i = 0$  otherwise. Since  $\mathbb{Q}_p X_{\text{du}} = \mathbb{Q}_p X_{\text{std}}$ , the exact sequences (12) and (14) imply that

$$\begin{aligned} \text{Fitt}(\mathbb{Q}_p \mathcal{X}_T^-) &= \text{Fitt}(\mathbb{Q}_p Z_{T \cup S_p}^-) \cdot \text{Fitt}(\mathbb{Q}_p(1))^i \cdot \prod_{\mathfrak{p} \in T} \text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}^-)^{-1} \cdot \text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}(1)^-) \\ &= \langle \dot{\Phi}_{T \cup S_p} e^- \rangle \cdot \prod_{\mathfrak{p} \in T} \text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}^-)^{-1} \cdot \text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}(1)^-), \end{aligned}$$

where all Fitting invariants are taken over  $\mathbb{Q}_p \Lambda(\mathcal{G})_-$  and the second equality holds by Theorem 4.2. The Fitting invariant of  $\mathbb{Q}_p Z_{\mathfrak{p}}^-$  is generated by  $\text{nr}(1 \otimes x_{\mathfrak{p}})$  with  $x_{\mathfrak{p}} = 1 - \varepsilon_{\mathfrak{p}} + (1 - \phi_{\wp})\varepsilon_{\mathfrak{p}}$ , since  $\mathbb{Q}_p Z_{\mathfrak{p}} = \text{ind}_{\mathcal{G}_{\wp}}^{\mathcal{G}} \mathbb{Q}_p$  and  $\mathbb{Q}_p$  is isomorphic to  $\mathbb{Q}_p \Lambda(\mathcal{G}_{\wp})/x_{\mathfrak{p}}$  as  $\mathbb{Q}_p \Lambda(\mathcal{G}_{\wp})$ -module. Likewise, the Fitting invariant of  $\mathbb{Q}_p Z_{\mathfrak{p}}(1)^-$  is generated by  $\text{nr}(1 \otimes \dot{x}_{\mathfrak{p}})$ . We obtain

$$\text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}^-)^{-1} \cdot \text{Fitt}(\mathbb{Q}_p Z_{\mathfrak{p}}(1)^-) = \langle \text{nr}(1 \otimes (x_{\mathfrak{p}} \dot{x}_{\mathfrak{p}}^{-1})) \rangle = \langle \xi_{\mathfrak{p}} \rangle.$$

$\square$

We now prove the non-abelian analogue of [Gr04], Th. 6.

**THEOREM 4.4.** *Let  $L/K$  be almost tame above  $p$ . Assume that the EIMC holds for the extension  $L_\infty^+/K$  and  $\mu(\mathcal{X}_T^-) = 0$ . Then  $\Psi_{T \cup S_p}$  generates the Fitting invariant  $\text{Fitt}_{\Lambda(\mathcal{G})_-}(\mathcal{X}_T^-)$ .*

*Proof.* By Proposition 4.3, we know that  $\Psi_{T \cup S_p}$  is a generator of  $\text{Fitt}_{\mathbb{Q}_p \Lambda(\mathcal{G})_-}(\mathbb{Q}_p \mathcal{X}_T^-)$ . Let us write  $\Psi_{T \cup S_p} = \text{nr}(B)$ , where  $B = \Theta \cdot \prod_{\mathfrak{p} \in T} (1 \otimes \Xi_{\mathfrak{p}})$  and  $\text{nr}(\Theta) = \dot{\Phi}_{T \cup S_p}$ . Since  $\mathcal{X}_T^-$  is of projective dimension at most 1 by Proposition 4.1, it suffices to show that  $B \in M_{n \times n}(\Lambda(\mathcal{G})_-)$  for some  $n$  by Proposition 3.2.

The validity of the EIMC implies that  $\Theta$  is in the image of  $K_1(\Lambda_{(p)}(\mathcal{G}))$ ; in fact, this is equivalent to the EIMC by [RW05], Th. A, but it is clearly necessary, since  $\mathfrak{U}_S$  vanishes if we localize at  $(p)$ . Hence we can choose a Weierstraß polynomial  $f$  such that  $f \cdot \Theta$  lies in  $M_{n \times n}(\Lambda(\mathcal{G})_-)$ . We claim that  $(1 - \phi_{\wp})\Xi_{\mathfrak{p}}$  belongs to  $\Lambda(\mathcal{G}_{\wp})$ . Taking this for granted for the moment, we see that  $f \cdot \prod_{\mathfrak{p} \in T} 1 \otimes (1 - \phi_{\wp}^N) \cdot B$  belongs to  $M_{n \times n}(\Lambda(\mathcal{G})_-)$ , where  $N$  is a large integer such that  $\phi_{\wp}^N$  is central in  $\Lambda(\mathcal{G})$  for all  $\mathfrak{p} \in T$ . Since we already know that  $p^b \cdot B \in M_{n \times n}(\Lambda(\mathcal{G})_-)$  for sufficiently large  $b$ , Lemma 3.3 implies that also  $B$  belongs to  $M_{n \times n}(\Lambda(\mathcal{G})_-)$ .

We are left with the claim. Setting  $q_{\mathfrak{p}} = \kappa(\phi_{\wp})$  and  $|I_{\mathfrak{p}}| = p^r \cdot e'$  with  $p \nmid e'$ , we find that

$$(1 - \phi_{\wp})\Xi_{\mathfrak{p}} = 1 - \phi_{\wp} - \varepsilon_{\mathfrak{p}}(q_{\mathfrak{p}} - 1)$$

and we have to show that  $p^r$  divides  $(q_{\mathfrak{p}} - 1)$ . We may assume  $r > 0$ . Let  $q \neq p$  be the rational prime below  $\mathfrak{p}$  and denote the  $q$ -Sylow subgroup of  $I_{\mathfrak{p}}$  by  $R_{\mathfrak{p}}$ . There is a natural inclusion  $I_{\mathfrak{p}}/R_{\mathfrak{p}} \hookrightarrow (\mathfrak{o}_L/\mathfrak{p})^\times$  which preserves the group action of  $G_{\mathfrak{p}}/I_{\mathfrak{p}}$ , where  $\mathfrak{p}$  is a place in  $L$  above  $\mathfrak{p}$ . Let  $\mathcal{I}$  be the subgroup of  $G_{\mathfrak{p}}/I_{\mathfrak{p}}$ -invariants of  $I_{\mathfrak{p}}/R_{\mathfrak{p}}$ . Then  $\mathcal{I}$  maps into  $(\mathfrak{o}_K/\mathfrak{p})^\times$  and hence  $|\mathcal{I}|$  divides  $(q_{\mathfrak{p}} - 1)$ . We have to show that the  $p$ -parts of  $|\mathcal{I}|$  and  $|I_{\mathfrak{p}}|$  coincide. Let  $a$  be a generator of  $I_{\mathfrak{p}}/R_{\mathfrak{p}}$  and choose a lift  $b \in G_{\mathfrak{p}}/R_{\mathfrak{p}}$  of  $\phi_{\mathfrak{p}}^{-1}$  which is of maximal order of all such elements; here,  $\phi_{\mathfrak{p}} = \phi_{\wp} \bmod \Gamma_L$ . By [Ch85], Lemma p. 369 we have  $b^{-1}ab = a^{q_{\mathfrak{p}}}$  and hence  $a^i$  belongs to  $\mathcal{I}$  if and only if  $|I_{\mathfrak{p}}/R_{\mathfrak{p}}|$  divides  $(q_{\mathfrak{p}}^i - i)$ . Since  $r > 0$ , this in particular implies that  $q_{\mathfrak{p}}^i - i \equiv 0 \bmod p$  if  $a^i \in \mathcal{I}$ . But  $p \neq q$  such that  $p \nmid i$ . It follows that  $\mathcal{I}$  is generated by  $a^m$  for some divisor  $m$  of  $|I_{\mathfrak{p}}/R_{\mathfrak{p}}|$  which is not divisible by  $p$ . Hence  $p^r$  is the exact power of  $p$  which divides  $|\mathcal{I}| = |I_{\mathfrak{p}}| \cdot (|R_{\mathfrak{p}}| \cdot m)^{-1}$ .  $\square$

We close this section with a few preparations for the Galois descent. If  $\chi$  is a character of  $\mathcal{G}$  with open kernel, we define

$$S_{\chi} := \{\mathfrak{p} \subset K \mid I_{\mathfrak{p}} \not\subset \ker(\chi)\}.$$

**LEMMA 4.5.** *Let  $S$  be a finite set of primes of  $K$  containing  $S_{\infty}$ . Let  $\chi$  be an even character of  $\mathcal{G}$  with open kernel and put  $\Sigma := S \cup S_p$  and  $\Sigma_{\chi} := (S \cap S_{\chi}) \cup S_p$ .*

i) *If  $\chi$  is of type  $S$  (i.e.  $\Gamma \subset \ker(\chi)$ ), we have an equality*

$$L_{p,\Sigma}(s, \chi) = L_{p,\Sigma_{\chi}}(s, \chi) \prod_{\mathfrak{p} \in \Sigma \setminus \Sigma_{\chi}} \det(1 - \sigma_{\wp} u^{-sc_{\mathfrak{p}}} |V_{\chi \omega^{-1}}^{I_{\mathfrak{p}}}|), \quad (15)$$

where we write  $\phi_{\wp} = \sigma_{\wp} \cdot \gamma^{c_{\mathfrak{p}}}$  with  $\sigma_{\wp} \in H$ ,  $c_{\mathfrak{p}} \in \mathbb{Z}_p$ , and where  $\omega$  denotes the Teichmüller character.

ii) *We have an equality*

$$G_{\chi,\Sigma}(T) = G_{\chi,\Sigma_{\chi}}(T) \prod_{\mathfrak{p} \in \Sigma \setminus \Sigma_{\chi}} g_{\mathfrak{p},\chi \omega^{-1}}(T),$$

where  $g_{\mathfrak{p},\chi}(T) := \det_{Q^c(\Gamma_K)}(1 - \phi_{\wp}^{-1} \varepsilon_{\mathfrak{p}} |V_{\chi^{-1}}|)$ .

*Proof.* For (i), we have to evaluate both sides at  $s = 1 - r$ , where  $r \geq 1$  is divisible by  $(p - 1)$ . We observe that

$$u^{(r-1)c_{\mathfrak{p}}} = \kappa(\phi_{\wp})^{r-1} \kappa(\sigma_{\wp})^{1-r} = N(\mathfrak{p})^{r-1} \omega(\sigma_{\wp}).$$

Now we compute that the right hand side of equation (15) at  $s = 1 - r$  equals

$$\begin{aligned} L_{p, \Sigma_\chi}(1 - r, \chi) \prod_{\mathfrak{p} \in \Sigma \setminus \Sigma_\chi} \det(1 - \sigma_\varphi \omega(\sigma_\varphi) N(\mathfrak{p})^{r-1} | V_{\chi \omega^{-1}}^{I_\varphi}) &= L_{\Sigma_\chi}(1 - r, \chi) \prod_{\mathfrak{p} \in \Sigma \setminus \Sigma_\chi} \det(1 - \sigma_\varphi N(\mathfrak{p})^{r-1} | V_\chi^{I_\varphi}) \\ &= L_{\Sigma}(1 - r, \chi) \\ &= L_{p, \Sigma}(1 - r, \chi). \end{aligned}$$

This proves (i). For (ii) we observe that  $g_{\mathfrak{p}, \chi}(u^s - 1) = \det(1 - \sigma_\varphi u^{-sc_\varphi} \varepsilon_\mathfrak{p} | V_\chi)$  if  $\chi$  is of type  $S$ . Hence (i) implies (ii) in this case. If  $\chi = \psi \otimes \rho$ , where  $\psi$  is of type  $S$  and  $\rho$  is of type  $W$ , then we have an equality

$$g_{\mathfrak{p}, \psi \otimes \rho} = g_{\mathfrak{p}, \chi}(\rho(\gamma_K)(1 + T) - 1).$$

Since similar equalities hold for  $G_{\chi, \Sigma}$  and  $G_{\chi, \Sigma_\chi}$ , we get (ii) in general.  $\square$

**COROLLARY 4.6.** *Keep the notation of Lemma 4.5, but assume that  $\chi$  is an odd character and  $\Sigma$  contains  $S_{\text{ram}}$ . Then*

$$j_\chi(\dot{\Phi}_\Sigma) = L_{K, \Sigma_\chi}(\chi^{-1} \omega) \prod_{\mathfrak{p} \in \Sigma \setminus \Sigma_\chi} g_{\mathfrak{p}, \chi^{-1}}(\gamma_K - 1).$$

The following proposition is still contained in the author's dissertation [Ni08], Prop. 3.2.7, but it was not yet published in a peer reviewed journal:

**PROPOSITION 4.7.** *Let  $L/K$  be a Galois CM-extension with Galois group  $G$ ,  $p \neq 2$  a rational prime and  $T$  a finite  $G$ -invariant set of places of  $L$  such that  $T \cap S_p = \emptyset$ . If  $\mathcal{X}_T^-$  denotes the projective limit of the minus  $p$ -ray class groups  $A_{L_n}^T(p)$ , there is an exact sequence of  $\mathbb{Z}_p G_-$ -modules*

$$\bigoplus_{\mathfrak{p} \in S_p} (\text{ind}_{G_\mathfrak{p}}^G \mathbb{Z}_p)^- \rightarrow (\mathcal{X}_T^-)_{\Gamma_L} \rightarrow A_L^T(p).$$

*Proof.* The canonical restriction map  $X_T \rightarrow \text{cl}_L^T(p)$  is surjective on minus parts, since the cokernel is a quotient of  $\Gamma_L$  on which  $j$  acts trivially. It clearly factors through  $(\mathcal{X}_T^-)_{\Gamma_L}$ .

Recall that  $M_T$  is the maximal abelian pro- $p$ -extension of  $L_\infty$  unramified outside  $T$ . We put  $\mathcal{Y}_T = \text{Gal}(M_T/L)$ . Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_s$  be the primes in  $L$  above  $p$ . Exactly these primes ramify in  $L_\infty/L$ , and we denote the finitely many primes in  $L_\infty$ , which lie above  $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ , by  $\mathfrak{P}_{ik}^\infty$ ,  $1 \leq i \leq s$ . Moreover, we choose above each  $\mathfrak{P}_{ik}^\infty$  a prime  $\mathfrak{P}_{ik}$  in  $M_T$ , and denote its inertia group in  $\mathcal{Y}_T$  by  $I_{ik}$ .

We obviously have an isomorphism  $\mathcal{Y}_T/X_T \simeq \Gamma_L$ . So we can pick a preimage  $\gamma \in \mathcal{Y}_T$  of  $\gamma_L$ , and thus

$$\mathcal{Y}_T = X_T \cdot \overline{\langle \gamma \rangle}. \quad (16)$$

Let  $\mathcal{Y}'_T$  be the closure of the commutator subgroup of  $\mathcal{Y}_T$ . Then  $G$  acts on  $\mathcal{Y}_T/\mathcal{Y}'_T$  via conjugation, and we may assume that  $\gamma^j \equiv \gamma \pmod{\mathcal{Y}'_T}$ , as we may choose a lift  $\tilde{j} \in \text{Gal}(M_T/K)$  of  $j$  and replace  $\gamma$  by  $\gamma^{(1+\tilde{j})/2}$ . The condition on the set  $T$  forces that the extension  $M_T/L_\infty$  does not ramify above  $p$ . Therefore  $I_{ik} \cap X_T = 1$ , and we get inclusions

$$I_{ik} \hookrightarrow \mathcal{Y}_T/X_T = \Gamma_L.$$

Hence, each  $I_{ik}$  is isomorphic to  $\Gamma_L^{p^{n_{ik}}}$  for an appropriate integer  $n_{ik}$ . We fix a topological generator  $\sigma_{ik}$  of  $I_{ik}$  which maps to  $\gamma_L^{p^{n_{ik}}}$  via the above inclusion. But for fixed  $i$ , each two of these inertia groups are conjugate, and hence  $n_i := n_{ik}$  does not depend on  $k$ . Corresponding to (16) we write  $\sigma_{ik} = a_{ik} \gamma^{p^{n_i}}$  with  $a_{ik} \in X_T$ .

Let  $M_0$  be the  $p$ -ray class field of  $L$  to the ray  $\mathfrak{M}_T$  such that  $\text{Gal}(M_0/L) \simeq \text{cl}_L^T(p)$ . Because of the obvious exact sequence

$$\text{Gal}(M_T/M_0) \hookrightarrow \mathcal{Y}_T \twoheadrightarrow \text{cl}_L^T(p)$$

we are interested in the Galois group  $\text{Gal}(M_T/M_0)$ . We claim that it equals the subgroup  $\mathcal{N}$  of  $\mathcal{Y}_T$  generated by  $\mathcal{Y}'_T$  and the inertia groups  $I_{ik}$ . For this, let  $N$  be the intermediate field of the extension  $M_T/L$  fixed by  $\mathcal{N}$ . Then  $N$  is the largest subfield of  $M_T$  which is abelian over  $L$  and unramified above  $p$ . Thus  $M_0 \subset N$ . If we assume that  $M_0 \neq N$ , we find an intermediate field  $N_0$  of finite degree over  $L$  such that  $M_0 \subsetneq N_0 \subset N$ . Let  $\mathfrak{N}$  be the conductor of  $N_0/L$ . Then the primes which divide  $\mathfrak{N}$  are exactly the primes in  $T$ . The commutative diagram

$$\begin{array}{ccccccc} \mathfrak{o}_L & \longrightarrow & (\mathfrak{o}_L/\mathfrak{N})^\times & \longrightarrow & \text{cl}_L^\mathfrak{N} & \twoheadrightarrow & \text{cl}_L \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ \mathfrak{o}_L & \longrightarrow & (\mathfrak{o}_L/\mathfrak{M}_T)^\times & \longrightarrow & \text{cl}_L^T & \twoheadrightarrow & \text{cl}_L \end{array}$$

now implies that the order  $m$  of the kernel of the surjection  $\text{cl}_L^\mathfrak{N} \twoheadrightarrow \text{cl}_L^T$  is prime to  $p$ , since the primes dividing  $m$  are below the primes in  $T$ . What we have shown is  $N_0 = M_0$ , in contradiction to our assumption.

LEMMA 4.8. *Let  $\mathcal{Y}'_T$  be the closure of the commutator subgroup of  $\mathcal{Y}_T$ . Then*

$$\mathcal{Y}'_T = X_T^{\gamma_L-1}.$$

*Proof.* The proof of [Wa82], Lemma 13.14 nearly remains unchanged. We only have to replace the inertia subgroup  $I_1$  in loc.cit. by  $\langle \overline{\gamma} \rangle$ .  $\square$

Since  $\gamma^j = \gamma \bmod \mathcal{Y}'_T$ , the above Lemma implies that we obtain an isomorphism

$$A_L^T(p) \simeq \mathcal{X}_T^- / \langle (\mathcal{X}_T^-)^{\gamma_L-1}, a_{ik}^{e-} \rangle.$$

As already mentioned, the inertia groups  $I_{ik}$  are conjugate for fixed  $i$ , hence  $\sigma_{ik} \equiv \sigma_{i1} \bmod \mathcal{Y}'_T$  and likewise  $a_{ik} \equiv a_{i1} \bmod \mathcal{Y}'_T$  for all  $k$ . Hence

$$A_L^T(p) \simeq \mathcal{X}_T^- / \langle (\mathcal{X}_T^-)^{\gamma_L-1}, a_1, \dots, a_s \rangle,$$

where we have defined  $a_i := a_{i1}^{e-}$ . Since  $\mathcal{X}_T^- / (\mathcal{X}_T^-)^{\gamma_L-1} = (\mathcal{X}_T^-)_{\Gamma_L}$ , Proposition 4.7 follows from the following lemma.  $\square$

LEMMA 4.9. *If  $\mathfrak{P}_j = \mathfrak{P}_i^g$  for an element  $g \in G$ , then  $a_j \equiv a_i^g \bmod (\mathcal{X}_T^-)^{\gamma_L-1}$ .*

*Proof.* Let  $\tau \in \text{Gal}(M_T/K)$  be a lift of  $g$ . Then  $g$  acts on  $(\mathcal{X}_T^-)_{\Gamma_L}$  via conjugation by  $\tau$ .  $\tilde{\mathfrak{P}}_{i1}^\tau$  is a prime in  $M_T$  above  $\mathfrak{P}_j$ , hence there exists an  $x \in \mathcal{Y}_T$  such that  $\tilde{\mathfrak{P}}_{i1}^\tau = \tilde{\mathfrak{P}}_{j1}^x$ . Replacing  $\tau$  by  $x^{-1}\tau$  we may assume that  $x = 1$ . Hence

$$\langle \overline{\sigma_{j1}} \rangle = I_{j1} = I_{i1}^\tau = \langle \overline{\sigma_{i1}^\tau} \rangle.$$

Since the restriction to  $L_\infty$  induces an isomorphism  $I_{j1} \simeq \Gamma_L^{p^{n_j}}$  and

$$\sigma_{i1}^\tau|_{L_\infty} = (\gamma_L^{p^{n_i}})^\tau = (\gamma_L^{p^{n_i}})^g = \gamma_L^{p^{n_i}},$$

we have  $n_i = n_j$  and  $\sigma_{j1} = \sigma_{i1}^\tau$ , i.e.

$$a_{j1} = (a_{i1}\gamma_L^{p^{n_j}})^\tau \cdot \gamma_L^{-p^{n_j}}.$$

But  $\gamma^\tau|_{L_\infty} = \gamma_L$  implies that  $\gamma^\tau = x_\tau \cdot \gamma$  for an element  $x_\tau \in X_T$ . Hence, the assertion follows from the above equation, since  $x_\tau^{e-}$  vanishes in  $(\mathcal{X}_T^-)_{\Gamma_L}$ , as  $j$  trivially acts on  $\gamma \bmod \mathcal{Y}'_T$  and commutes with  $\tau$ .  $\square$



## 5. An integrality conjecture

Let  $L/K$  be a Galois CM-extension with Galois group  $G$ . Let  $S$  and  $T$  be two finite sets of places of  $K$  such that

- $S$  contains all the infinite places of  $K$  and all the places which ramify in  $L/K$ , i.e.  $S \supset S_{\text{ram}} \cup S_{\infty}$ .
- $S \cap T = \emptyset$ .
- $E_S^T$  is torsionfree.

We refer to the above hypotheses as  $\text{Hyp}(S, T)$ . For a fixed set  $S$  we define  $\mathfrak{A}_S$  to be the  $\zeta(\mathbb{Z}G)$ -submodule of  $\zeta(\mathbb{Q}G)$  generated by the elements  $\delta_T(0)$ , where  $T$  runs through the finite sets of places of  $K$  such that  $\text{Hyp}(S, T)$  is satisfied. Note that  $\mathfrak{A}_S$  equals the  $\mathbb{Z}G$ -annihilator of the roots of unity of  $L$  if  $G$  is abelian by [Ta84], Lemma 1.1, p. 82.

For each finite prime  $\mathfrak{p}$  of  $K$ , we define a  $\mathbb{Z}G_{\mathfrak{p}}$ -module  $U_{\mathfrak{p}}$  by

$$U_{\mathfrak{p}} := \langle N_{I_{\mathfrak{p}}}, 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1} \rangle_{\mathbb{Z}G_{\mathfrak{p}}} \subset \mathbb{Q}G_{\mathfrak{p}},$$

where we recall that  $\varepsilon_{\mathfrak{p}} = |I_{\mathfrak{p}}|^{-1} N_{I_{\mathfrak{p}}}$ . Note that  $U_{\mathfrak{p}} = \mathbb{Z}G_{\mathfrak{p}}$  if  $\mathfrak{p}$  is unramified in  $L/K$  such that the definition of the following  $\zeta(\mathbb{Z}G)$ -module is indeed independent of the set  $S$  as long as  $S$  contains the ramified primes:

$$U := \langle \prod_{\mathfrak{p} \in S \setminus S_{\infty}} \text{nr}(u_{\mathfrak{p}}) | u_{\mathfrak{p}} \in U_{\mathfrak{p}} \rangle_{\zeta(\mathbb{Z}G)} \subset \zeta(\mathbb{Q}G).$$

DEFINITION 5.1. Let  $S$  be a finite set of primes which contains  $S_{\text{ram}} \cup S_{\infty}$ . We define a  $\zeta(\mathbb{Z}G)$ -module by

$$SKu(L/K, S) := \mathfrak{A}_S \cdot U \cdot L(0)^{\#} \subset \zeta(\mathbb{Q}G).$$

We call  $SKu(L/K) := SKu(L/K, S_{\text{ram}} \cup S_{\infty})$  the (fractional) *Sinnott-Kurihara ideal*.

For abelian  $G$ , this definition coincides with the Sinnott-Kurihara ideal  $SKu(L/K)$  in [Gr07] (see also [Si80], p. 193).

Let  $\mathcal{I}(G)$  be the  $\zeta(\mathbb{Z}G)$ -module generated by the elements  $\text{nr}(H)$ ,  $H \in M_{n \times n}(\mathbb{Z}G)$ ,  $n \in \mathbb{N}$ . Actually,  $\mathcal{I}(G)$  is a commutative ring and we have inclusions

$$\zeta(\mathbb{Z}G) \subset \mathcal{I}(G) \subset \zeta(\mathfrak{M}(G)),$$

where  $\mathfrak{M}(G)$  is a maximal order in  $\mathbb{Q}G$ . We now state the following integrality conjecture:

CONJECTURE 5.2. The Sinnott-Kurihara ideal  $SKu(L/K)$  is contained in  $\mathcal{I}(G)$ .

Remark 1. i) Since clearly  $SKu(L/K, S) \subset SKu(L/K, S')$  if  $S' \subset S$ , Conjecture 5.2 implies  $SKu(L/K, S) \subset \mathcal{I}(G)$  for all admissible sets  $S$ .

ii) If the sets  $S$  and  $T$  satisfy  $\text{Hyp}(S, T)$ , the Stickelberger element  $\theta_S^T$  is contained in  $SKu(L/K, S)$ . Hence Conjecture 5.2 predicts that  $\theta_S^T \in \mathcal{I}(G)$  which is part of [Nic], Conjecture 2.1.

iii) In the above definitions, we may replace  $\mathbb{Z}$  and  $\mathbb{Q}$  by  $\mathbb{Z}_p$  and  $\mathbb{Q}_p$ , respectively. We obtain a local Sinnott-Kurihara ideal  $SKu_p(L/K)$  contained in  $\zeta(\mathbb{Q}_p G)$  and a  $\zeta(\mathbb{Z}_p G)$ -module  $\mathcal{I}_p(G)$ . Since we have an equality

$$\mathcal{I}(G) = \bigcap_p \mathcal{I}_p(G) \cap \zeta(\mathbb{Q}G),$$

we have an equivalence

$$SKu(L/K) \subset \mathcal{I}(G) \iff SKu_p(L/K) \subset \mathcal{I}_p(G) \quad \forall p.$$

If  $G$  is abelian, we obviously have  $\mathcal{I}(G) = \zeta(\mathbb{Z}G) = \mathbb{Z}G$  and the results in [Ba77], [Ca79], [DR80] each imply the following theorem (cf. [Gr07], §2).

THEOREM 5.3. *Conjecture 5.2 holds if  $L/K$  is an abelian CM-extension.*

## 6. The ETNC in almost tame extensions

Let us fix a finite Galois extension  $L/K$  of number fields with Galois group  $G$  and a finite set  $S$  of places of  $K$  which contains  $S_{\text{ram}} \cup S_{\infty}$ . In [Bu01] the author defines the following element of  $K_0(\mathbb{Z}G, \mathbb{R})$ :

$$T\Omega(L/K, 0) := \psi_G^*(\chi_{G, \mathbb{R}}(\tau_S, \lambda_S^{-1}) + \hat{\partial}_G(L_S^*(0)^\sharp)).$$

Here,  $\psi_G^*$  is a certain involution on  $K_0(\mathbb{Z}G, \mathbb{R})$  which is not important for our purposes, since we will be only interested in the nullity of  $T\Omega(L/K, 0)$ . Furthermore,  $\tau_S \in \text{Ext}_G^2(E_S, \Delta S)$  is Tate's canonical class (cf. [Ta66]), where  $\Delta S$  is the kernel of the augmentation map  $\mathbb{Z}S(L) \rightarrow \mathbb{Z}$  which maps each  $\mathfrak{P} \in S(L)$  to 1. Finally,  $\lambda_S$  denotes the negative of the usual Dirichlet map, so  $\lambda_S : \mathbb{R} \otimes E_S \rightarrow \mathbb{R} \otimes \Delta S$ ,  $u \mapsto -\sum_{\mathfrak{P} \in S(L)} \log |u|_{\mathfrak{P}} \mathfrak{P}$ , and  $\chi_{G, \mathbb{R}}(\tau_S, \lambda_S^{-1})$  is the refined Euler characteristic associated to the perfect 2-extension whose extension class is  $\tau_S$ , metrised by  $\lambda_S^{-1}$ . For more precise definitions we refer the reader to [Bu01]. The ETNC for the motive  $h^0(L)$  with coefficients in  $\mathbb{Z}G$  in this context asserts that the element  $T\Omega(L/K, 0)$  is zero. Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [GRW99] (cf. [Bu01], Th. 2.3.3).

It is also proven in loc.cit. that  $T\Omega(L/K, 0)$  lies in  $K_0(\mathbb{Z}G, \mathbb{Q})$  if and only if Stark's conjecture holds. In this case the ETNC decomposes into local conjectures at each prime  $p$  by means of the isomorphism

$$K_0(\mathbb{Z}G, \mathbb{Q}) \simeq \bigoplus_{p \nmid \infty} K_0(\mathbb{Z}_p G, \mathbb{Q}_p).$$

Now let  $L/K$  be a Galois CM-extension. Since Stark's conjecture is known for odd characters (cf. [Ta84], Th. 1.2, p. 70),  $T\Omega(L/K, 0)$  has a well defined image  $T\Omega(L/K, 0)_p^-$  in  $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$ . Recall that  $T$  consists of a prime  $\mathfrak{p}_0 \nmid p$  and all finite places of  $K$  which ramify in  $L/K$  and do not lie above  $p$ , and we have chosen  $\mathfrak{p}_0$  such that  $E_S^T$  is torsionfree. We have the following reformulation of [Nia], Th. 2.

**THEOREM 6.1.** *Let  $p$  be an odd prime and  $L/K$  a Galois CM-extension which is almost tame above  $p$ . Then*

$$T\Omega(L/K, 0)_p^- = 0 \iff \text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)) = [\langle \theta_{S_1}^T \rangle]_{\text{nr}(\mathbb{Z}_p G_-)},$$

where  $S_1$  denotes the set of all wildly ramified primes above  $p$ .

We have the following connection to the integrality conjecture 5.2 (cf. [Nic], proof of Th. 5.1 and Cor. 5.6):

**THEOREM 6.2.** *Let  $p$  be an odd prime and  $L/K$  a Galois CM-extension and assume that  $T\Omega(L/K, 0)_p^-$  vanishes. If the  $p$ -part of the roots of unity of  $L$  is a c.t.  $G$ -module or if  $L/K$  is almost tame above  $p$ , then the  $p$ -part of Conjecture 5.2 holds, i.e.  $SKu_p(L/K) \subset \mathcal{I}_p(G)$ .*

The aim of this section is to prove a partial reverse of this theorem for almost tame extensions.

**LEMMA 6.3.** *Let  $p$  be an odd prime and  $L/K$  a Galois CM-extension which is almost tame above  $p$ . Assume that the Iwasawa  $\mu$ -invariant vanishes and that the EIMC holds for the extension  $L_{\infty}^+/K$ . Then*

$$\text{Fitt}_{\mathbb{Z}_p G_-}(\mathcal{X}_T^-/(\gamma_L - 1)) = [\langle \theta_{S_p}^T \rangle]_{\text{nr}(\mathbb{Z}_p G_-)}.$$

*Proof.* By Theorem 4.4, the Fitting invariant of  $\mathcal{X}_T^-$  over  $\Lambda(\mathcal{G})_-$  is generated by  $\Psi_{\Sigma}$ , where we put  $\Sigma = T \cup S_p$ . Now [Nib], Th. 6.4 implies that  $\text{Fitt}_{\mathbb{Z}_p G_-}(\mathcal{X}_T^-/(\gamma_L - 1))$  is generated by

$$\sum_{\chi \in \text{Irr}(G)} \text{aug}_{\Gamma_L}(j_{\chi}(\Psi_{\Sigma}))e_{\chi}. \quad (17)$$

But using Corollary 4.6 we compute

$$\begin{aligned}
 j_\chi(\Psi_\Sigma) &= \left( \prod_{\mathfrak{p} \in T} j_\chi(\xi_{\mathfrak{p}}) \right) \cdot j_\chi(\dot{\Phi}_\Sigma) \\
 &= \left( \prod_{\mathfrak{p} \in T} j_\chi(\text{nr}(\Xi_{\mathfrak{p}} \cdot (1 - \phi_{\mathfrak{p}}^{-1} \varepsilon_{\mathfrak{p}}))) \right) L_{K, \Sigma_\chi}(\chi^{-1} \omega) \\
 &= \left( \prod_{\mathfrak{p} \in T} j_\chi(\text{nr}(\varepsilon_{\mathfrak{p}}(1 - N(\mathfrak{p})\phi_{\mathfrak{p}}^{-1}) + 1 - \varepsilon_{\mathfrak{p}})) \right) L_{K, \Sigma_\chi}(\chi^{-1} \omega).
 \end{aligned}$$

Hence (17) equals

$$\sum_{\chi \in \text{Irr}(G)} \left( \prod_{\mathfrak{p} \in T} \det(1 - N(\mathfrak{p})\phi_{\mathfrak{p}}^{-1} | V_\chi^{I_{\mathfrak{p}}}) \right) L_{\Sigma_\chi}(0, \chi^{-1}) = \theta_{S_p}^T.$$

□

We define an element  $\alpha_p \in \zeta(\mathbb{Q}_p G_-)$  by

$$\alpha_p = \prod_{\mathfrak{p} \in S_p \setminus S_1} \text{nr}(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{p}}^{-1})$$

such that we have an equality  $\theta_{S_1}^T \cdot \alpha_p = \theta_{S_p}^T$ . We start with the following special case, where we get Conjecture 5.2 for free.

**PROPOSITION 6.4.** *Let  $p$  be an odd prime and  $L/K$  a Galois CM-extension such that  $j \in G_{\mathfrak{p}}$  for all  $\mathfrak{p}$  above  $p$ . Assume that the Iwasawa  $\mu$ -invariant vanishes and that the EIMC holds for the extension  $L_\infty^+/K$ . Then  $T\Omega(L/K, 0)_p^- = 0$  and the  $p$ -part of Conjecture 5.2 holds.*

*Proof.* As before, the canonical restriction map  $\mathcal{X}_T^- \rightarrow A_L^T(p)$  is surjective. By [Nib], Prop. 3.5 (i) this implies

$$\text{Fitt}_{\mathbb{Z}_p G_-}(\mathcal{X}_T^- / (\gamma_L - 1)) \subset \text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p)).$$

Since we have  $j \in G_{\mathfrak{p}}$  for all  $\mathfrak{p}$  above  $p$  by assumption, the element  $\alpha_p$  lies in  $\text{nr}(K_1(\mathbb{Z}_p G_-))$  and thus Lemma 6.3 implies  $\theta_{S_1}^T \in \text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p))$ . In particular, we have  $\theta_{S_1}^T \in \mathcal{I}_p(G)$ . Let  $E$  be a splitting field of  $\mathbb{Q}_p G$ . Since  $\theta_{S_1}^T = (\delta_T(0, \chi) L_{S_1}(0, \chi^{-1}))_\chi$  and

$$|A_L^T(p)| = x \cdot \prod_{\substack{\chi \in \text{Irr}(G) \\ \chi \text{ odd}}} (\delta_T(0, \chi) L_{S_1}(0, \chi^{-1}))^{\chi(1)}$$

with an appropriate unit  $x \in \mathfrak{o}_E^\times$  by [Nia], Prop. 4, the Stickelberger element  $\theta_{S_1}^T$  is actually a generator of  $\text{Fitt}_{\mathbb{Z}_p G_-}(A_L^T(p))$  by [Nib], Prop. 5.4. Now Theorem 6.1 implies the vanishing of  $T\Omega(L/K, 0)_p^-$  which also implies Conjecture 5.2 by Theorem 6.2. □

Let us denote the normal closure of  $L$  over  $\mathbb{Q}$  by  $L^{\text{cl}}$  which is again a CM-field. We will henceforth make the following additional assumption:

$$L^{\text{cl}} \not\subset (L^{\text{cl}})^+(\zeta_p).$$

Note that this assumption fails only for finitely many primes  $p$ , since such a  $p$  has to ramify in  $L^{\text{cl}}/\mathbb{Q}$ .

**LEMMA 6.5.** *Let  $N > 0$  be a natural number. Then there are infinitely many primes  $r \in \mathbb{Z}$  such that*

$$-r \equiv 1 \pmod{p^N}.$$

- $j \in G_{\mathfrak{R}}$  for all primes  $\mathfrak{R}$  in  $L$  above  $r$ .
- The Frobenius automorphism  $\text{Frob}_p$  at  $p$  in  $\text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$  generates  $\text{Gal}(k_r/\mathbb{Q})$ , where  $k_r$  denotes the unique subfield of  $\mathbb{Q}(\zeta_r)$  of degree  $p^N$  over  $\mathbb{Q}$ .

*Proof.* The proof of [Gr00], Prop. 4.1 carries over unchanged to the present situation.  $\square$

Let  $N \in \mathbb{N}$  be large and choose a prime  $r$  as in Lemma 6.5 which does not ramify in  $L^{\text{cl}}/\mathbb{Q}$ . We put  $L' := Lk_r$ ,  $K' = Kk_r$  and  $G' = \text{Gal}(L'/K) = G \times C_N$ , where  $C_N \simeq \text{Gal}(k_r/\mathbb{Q})$  is cyclic of order  $p^N$ , generated by  $\text{Frob}_p$ . Note that  $L'/K$  is again almost tame above  $p$ . Moreover, we define  $T' := T \cup S_r$ , where  $S_r$  denotes the set of places in  $K$  above  $r$ . Using the same arguments as in [Nia] following Prop. 9 we have an isomorphism

$$A_{L'}^{T'}(p) \simeq A_{L'}^T(p)$$

and hence  $A_{L'}^T(p)$  is  $G'$ -c.t. by loc.cit., Th. 1. As in loc.cit. the restriction map induces an isomorphism

$$(A_{L'}^T(p))_{C_N} \simeq A_L^T(p). \quad (18)$$

We will need the following lemma.

LEMMA 6.6. *Assume that  $G'$  is a direct product of a group  $G$  and an abelian group  $C$ . Then we have  $|G| \cdot \mathcal{I}_p(G') \subset \zeta(\mathbb{Z}_p G')$  for all primes  $p$ .*

*Proof.* Choose a maximal order  $\mathfrak{M}(G)$  containing  $\mathbb{Z}_p G$ . Then  $\mathfrak{M}(G)$  is a direct sum of matrix rings of type  $M_{n \times n}(\mathfrak{o}_D)$ , where  $\mathfrak{o}_D$  denotes the valuation ring of a skew field  $D$ . We have

$$\zeta(M_{n \times n}(\mathfrak{o}_D)) = \zeta(\mathfrak{o}_D) = \mathfrak{o}_F,$$

where  $\mathfrak{o}_F$  is the ring of integers of the field  $F = \zeta(D)$  which is finite over  $\mathbb{Q}_p$ . Since the reduced norm maps  $\mathfrak{M}(G)$  into its center and  $|G| \cdot \zeta(\mathfrak{M}(G)) \subset \zeta(\mathbb{Z}_p G)$ , it suffices to show that the reduced norm maps  $M_{m \times m}(M_{n \times n}(\mathfrak{o}_D)[C])$  into  $\mathfrak{o}_F[C]$ . Let us at first assume that  $D = F$ . Then the map

$$\begin{aligned} \sigma : M_{n \times n}(F)[C] &\longrightarrow M_{n \times n}(F[C]) \\ \sum_{c \in C} M_c c &\mapsto \left( \sum_{c \in C} \alpha_{ij}(c) c \right)_{i,j} \end{aligned}$$

is an isomorphism of rings, where  $M_c = (\alpha_{ij}(c))_{i,j}$  lies in  $M_{n \times n}(F)$ . Likewise,  $\sigma$  induces an isomorphism

$$\sigma : M_{n \times n}(\mathfrak{o}_F)[C] \simeq M_{n \times n}(\mathfrak{o}_F[C]).$$

Therefore, we have

$$\text{nr}(M_{m \times m}(M_{n \times n}(\mathfrak{o}_F)[C])) = \text{nr}(M_{nm \times nm}(\mathfrak{o}_F[C])) = \text{nr}(\mathfrak{o}_F[C]) = \mathfrak{o}_F[C].$$

For arbitrary  $D$ , there is a field  $E$ , Galois over  $F$  such that  $E \otimes_F D \simeq M_{s \times s}(E)$  for some integer  $s$ . We have just proven that the reduced norm maps  $M_{m \times m}(M_{n \times n}(\mathfrak{o}_D)[C])$  into  $\mathfrak{o}_E[C]$ . But the image is invariant under the action of  $\text{Gal}(E/F)$  and is therefore contained in  $\mathfrak{o}_F[C]$ .  $\square$

Let  $\alpha'_p \in \zeta(\mathbb{Q}_p G')$  be defined analogously to  $\alpha_p$  such that  $\theta_{S_1}^{T'} \cdot \alpha'_p = \theta_{S_p}^{T'}$ . Now choose a second natural number  $M \leq N$  and put

$$\nu := \sum_{i=0}^{p^M-1} \text{Frob}_p^{ip^{N-M}} \in \mathbb{Z}_p C_N \subset \zeta(\mathbb{Z}_p G').$$

LEMMA 6.7. *Let  $f$  be the least common multiple of the residual degrees  $f_{\mathfrak{p}}(K/\mathbb{Q})$  of all  $\mathfrak{p} \in S_p$ . If  $N - M \geq v_p(|G| \cdot f)$ , then  $|G| \cdot \alpha'_p$  is a nonzerodivisor in  $\zeta(\mathbb{Z}_p G')/\nu$ .*

*Proof.* We first observe that Lemma 6.6 implies that  $|G| \cdot \alpha'_p$  lies in  $\zeta(\mathbb{Z}_p G')$ . Since  $\mathbb{Z}_p C_N / \nu$  and likewise  $\zeta(\mathbb{Z}_p G') / \nu$  are reduced rings, we have to show that no minimal prime of  $\zeta(\mathbb{Z}_p G')$  contains both,  $|G| \cdot \alpha'_p$  and  $\nu$ . The minimal primes are given by

$$\mathfrak{p}_{\chi'} := \{x \in \zeta(\mathbb{Z}_p G') \mid \chi'(x) = 0\}, \quad \chi' \in \text{Irr}(G').$$

We may write  $\chi'$  as a product  $\chi \cdot \chi_N$  of irreducible characters  $\chi$  of  $G$  and  $\chi_N$  of  $C_N$ ; then  $\chi'(\text{Frob}_p) = \chi(1) \cdot \zeta_{p^s}$  for some  $s \leq N$ . Assume that  $\nu \in \mathfrak{p}_{\chi'}$ ; hence  $0 = \chi'(\nu) = \chi(1) \sum_{i=0}^{p^M-1} \zeta_{p^s}^{ip^{N-M}}$ . But since  $\chi(1) \neq 0$ , this implies  $s > N - M$ . If also  $|G| \cdot \alpha'_p \in \mathfrak{p}_{\chi'}$ , there is a prime  $\mathfrak{p} \in S_p$  and a prime  $\mathfrak{P}'$  in  $L'$  above  $\mathfrak{p}$  such that the inertia group at  $\mathfrak{P}'$  acts trivially on  $V_{\chi'}$  and  $\det(1 - \phi_{\mathfrak{P}'}^{-1} | V_{\chi'}^{f_{\mathfrak{P}'}} )$  vanishes. But this determinant is a product of some  $1 - \zeta \cdot \zeta_{p^s}^{-f_{\mathfrak{p}}}$ , where  $\zeta$  is a root of unity of order dividing  $|G|$  and, by assumption, we have  $v_p(\text{ord}(\zeta_{p^s}^{f_{\mathfrak{p}}})) = \frac{s}{v_p(f_{\mathfrak{p}})} > \frac{N-M}{v_p(f_{\mathfrak{p}})} \geq v_p(|G|)$ . This is a contradiction.  $\square$

We are ready to prove the main result of this section which generalizes [Nia], Th. 4.

**THEOREM 6.8.** *Let  $p$  be an odd prime and  $L/K$  a Galois CM-extension which is almost tame above  $p$ . Assume that the Iwasawa  $\mu$ -invariant vanishes and that  $L^{\text{cl}} \not\subset (L^{\text{cl}})^+(\zeta_p)$ . Moreover assume that for each integer  $M$  there is an integer  $N \geq M$  such that there is a prime  $r = r(N)$  as in Lemma 6.5, unramified in  $L^{\text{cl}}/\mathbb{Q}$  such that the  $p$ -part of Conjecture 5.2 is true for  $L'/K$  and the EIMC holds for the extension  $(L'_{\infty})^+/K$ . Then  $T\Omega(L/K, 0)_p^- = 0$ . In particular, the  $p$ -parts of the following conjectures hold:*

- i) the strong Stark conjecture for odd characters as formulated by T. Chinburg [Ch83], Conj. 2.2.
- ii) the (weak) non-abelian Brumer conjecture of [Nic], Conj. 2.1 and 2.3.
- iii) the (weak) non-abelian Brumer-Stark conjecture of [Nic], Conj. 2.6 and 2.7.
- iv) the weak non-abelian strong Brumer-Stark conjecture of [Nic], Conj. 3.6.

Moreover,  $L/K$  fulfills the non-abelian strong Brumer-Stark property at  $p$  (cf. [Nic], Def. 3.5).

*Remark 2.* i) Since the EIMC as well as Conjecture 5.2 are known for abelian Galois groups, it seems to be likely that we can prove these conjectures attached to the extensions  $L'/K$  if so for  $L/K$ . For example, the EIMC is known to be true for Galois groups  $\mathcal{G}$  which are pro- $p$  and have an abelian subgroup of index  $p$  (cf. [RW08]). But if this is true for  $\text{Gal}(L_{\infty}^+/K)$ , then also for  $\text{Gal}((L'_{\infty})^+/K)$ .

- ii) Since the strong Stark conjecture at  $p$  is a theorem for odd characters in the case at hand (cf. [Nia], Cor. 2), it follows from the results in [Nic] that the weak variants of the above conjectures are true unconditionally (cf. loc.cit., Cor. 4.2).

*Proof of Theorem 6.8.* We first observe that the Iwasawa  $\mu$ -invariant does not change if we enlarge  $L$  to  $L'$  by [NSW00], Th. 11.3.8. Now choose natural numbers  $M \leq N$  such that  $r = r(N)$  fulfills the above conditions and  $N - M \geq v_p(|G| \cdot f)$ , where  $f$  was defined in Lemma 6.7. Let  $\mathcal{G}' = \text{Gal}(L'_{\infty}/K)$  and let  $\mathcal{X}_{T'}^-$  be the projective limit of the minus  $p$ -ray class groups  $A_{L'_n}^{T'}(p)$ . Then  $\mathcal{X}_{T'}^-$  has projective dimension at most one and the EIMC for the extension  $(L'_{\infty})^+/K$  implies

$$\text{Fitt}_{\Lambda(\mathcal{G}')_-}(\mathcal{X}_{T'}^-) = [\langle \Psi_{T' \cup S_p} \rangle]_{\text{nr}(\Lambda(\mathcal{G}')_-)}.$$

For each prime  $\mathfrak{p}$  of  $K$  let  $\mathfrak{P}' \subset L'$  be a prime above  $\mathfrak{p}$ . By Proposition 4.7, we have a right exact sequence

$$\bigoplus_{\mathfrak{p} \in S_p} \text{ind}_{G_{\mathfrak{p}}}^G \mathbb{Z}_p \rightarrow (\mathcal{X}_{T'}^-)_{\Gamma_{L'}} \rightarrow A_{L'}^{T'}(p).$$

The Fitting invariant of the leftmost term is generated by  $\alpha'_p$ , whereas  $\theta_{S_p}^{T'} = \theta_{S_p}^{T'}(L'/K)$  is a generator of  $\text{Fitt}_{\mathbb{Z}_p G'_-}((\mathcal{X}_{T'}^-)_{\Gamma_{L'}})$  by Lemma 6.3. Since  $j \in G_{\mathfrak{P}'}$  for all primes above  $r$ , we may replace  $\theta_{S_p}^{T'}$  by

$\theta_{S_p}^T$ . The above sequence gives rise to the following inclusion of Fitting invariants (cf. [Nib], Prop. 3.5 (iii)):

$$\text{Fitt}_{\mathbb{Z}_p G'_-} \left( \bigoplus_{\mathfrak{p} \in S_p} \text{ind}_{G_{\mathfrak{p}}}^G \mathbb{Z}_p \right) \cdot \text{Fitt}_{\mathbb{Z}_p G'_-} (A_{L'}^{T'}(p)) \subset \text{Fitt}_{\mathbb{Z}_p G'_-} ((\mathcal{X}_{T'}^-)_{\Gamma_{L'}}).$$

If we choose a generator  $\xi'$  of  $\text{Fitt}_{\mathbb{Z}_p G'_-} (A_{L'}^{T'}(p))$ , there exists  $x \in \zeta(\mathbb{Z}_p G')$  such that

$$\alpha'_p \xi' = x \cdot \theta_{S_p}^T = x \cdot \alpha'_p \theta_{S_1}^T.$$

It follows from Lemma 6.6 that multiplication by  $|G|^2$  yields an equality in  $\zeta(\mathbb{Z}_p G')$  (since Conjecture 5.2 holds by assumption) such that Lemma 6.7 gives

$$|G| \cdot \xi' \equiv |G| \cdot x \cdot \theta_{S_1}^T \pmod{\nu}. \quad (19)$$

Let  $\text{aug} : \mathbb{Z}_p G' \rightarrow \mathbb{Z}_p G$  be the natural augmentation map. Since Fitting invariants behave well under base change (cf. [Nib], Lemma 5.5), the element  $\xi := \text{aug}(\xi')$  generates the Fitting invariant of  $A_L^T(p)$  by (18). But since  $\text{aug}(\theta_{S_p}^T(L'/K)) = \theta_{S_p}^T(L/K)$  and  $\text{aug}(\nu) = p^M$ , equation (19) implies

$$\xi \equiv \text{aug}(x) \cdot \theta_{S_1}^T(L/K) \pmod{p^{M-m} \mathcal{I}_p(G)},$$

where  $p^m$  is the exact power dividing  $|G|$ . This gives an inclusion

$$\text{Fitt}_{\mathbb{Z}_p G} (A_L^T(p)) \subset [\langle \theta_{S_1}^T \rangle]_{\text{nr}(\mathbb{Z}_p G)},$$

as we may choose  $M$  arbitrarily large. Now we can conclude as in Proposition 6.4 that  $\theta_{S_1}^T$  is in fact a generator of  $\text{Fitt}_{\mathbb{Z}_p G} (A_L^T(p))$  and we are done via Theorem 6.1.  $\square$

*Remark 3.* Note that we have not used the whole statement of Conjecture 5.2. It suffices to assume that the denominators of the elements  $\theta_{S_1}^T(L'/K)$  for varying  $r = r(N)$  are bounded, independently of  $N$ .

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