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Abstract. Let \((M, g)\) be a compact Riemannian spin manifold. The Atiyah-Singer index theorem yields a lower bound for the dimension of the kernel of the Dirac operator. We prove that this bound can be attained by changing the Riemannian metric \(g\) on an arbitrarily small open set.

1. Introduction and statement of results

Let \(M\) be a spin manifold, we assume that all spin manifolds come equipped with a choice of orientation and spin structure. We denote by \(-M\) the same manifold \(M\) equipped with the opposite orientation. For a Riemannian manifold \((M, g)\) we denote by \(U_p(r)\) the set of points for which the distance to the point \(p\) is strictly less than \(r\).

The Dirac operator \(D^g\) of \((M, g)\) is a first order differential operator acting on sections of the spinor bundle associated to the spin structure on \(M\). This is an elliptic, formally self-adjoint operator. If \(M\) is compact, then the spectrum of \(D^g\) is real, discrete, and the eigenvalues tend to plus and minus infinity. In this case the operator \(D^g\) is invertible if and only if 0 is not an eigenvalue, which is the same as vanishing of the kernel.

The Atiyah-Singer Index Theorem states that the index of the Dirac operator is equal to a topological invariant of the manifold,

\[ \text{ind}(D^g) = \alpha(M). \]

Depending on the dimension \(n\) of \(M\) this formula has slightly different interpretations. If \(n\) is even there is a \(\pm\)-grading of the spinor bundle and the Dirac operator \(D^g\) has a part \((D^g)^+\) which maps from positive to negative spinors. If \(n \equiv 0, 4 \mod 8\) the index is integer valued and computed as the dimension of the kernel minus the dimension of the cokernel of \((D^g)^+\). If \(n \equiv 1, 2 \mod 8\) the index is \(\Z/2\Z\)-valued and given by the dimension modulo 2 of the kernel of \(D^g\) (if \(n \equiv 1 \mod 8\)) resp. \((D^g)^+\) (if \(n \equiv 2 \mod 8\)). In other dimensions the index is zero. In all dimensions \(\alpha(M)\) is a topological invariant depending only on the spin bordism class of \(M\). In particular, \(\alpha(M)\) does not depend on the metric, but it depends on the spin structure in dimension \(n \equiv 1, 2 \mod 8\). For further details see [9, Chapter II, §7].

The index theorem implies a lower bound on the dimension of the kernel of \(D^g\) which we can write succinctly as

\[ \dim \ker D^g \geq a(M), \quad (1) \]

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Let $M$ be a compact connected Riemannian spin manifold of dimension $n \geq 2$. Let $p \in M$ and $r > 0$. Then there is a $D$-minimal metric $\bar{g}$ on $M$ with $\bar{g} = g$ on $M \setminus U_p(r)$.

The new ingredient in the proof of this theorem is the use of the “invertible double” construction which gives a $D$-minimal metric on any spin manifold of the type $(-M)\#M$ where $\#$ denotes connected sum. For dimension $n \geq 5$ we can then use the surgery method from [3] with surgeries of codimension $\geq 3$. For $n = 3, 4$ we need the stronger surgery result of [1] preserving $D$-minimality under surgeries of codimension $\geq 2$. The case $n = 2$ follows from [1] and classical facts about Riemann surfaces.

**1.1. Generic metrics.** We denote by $\mathcal{R}(M, U_p(r), g)$ the set of all smooth Riemannian metrics on $M$ which coincide with the metric $g$ outside $U_p(r)$ and by $\mathcal{R}_{\min}(M, U_p(r), g)$ the subset of $D$-minimal metrics. From Theorem 1.1 it follows that a generic metric from $\mathcal{R}(M, U_p(r), g)$ is actually an element of $\mathcal{R}_{\min}(M, U_p(r), g)$, as made precise in the following corollary.

**Corollary 1.2.** Let $(M, g)$ be a compact connected Riemannian spin manifold of dimension $\geq 3$. Let $p \in M$ and $r > 0$. Then $\mathcal{R}_{\min}(M, U_p(r), g)$ is open in the $C^1$-topology on $\mathcal{R}(M, U_p(r), g)$ and it is dense in all $C^k$-topologies, $k \geq 1$.

The proof follows [2, Theorem 1.2] or [10, Proposition 3.1]. The first observation of the argument is that the eigenvalues of $D^y$ are continuous functions of $g$ in the $C^1$-topology, from which the property of being open follows. The second observation is that spectral data of $D^y$ for a linear family of metrics $g_t = (1 - t)g_0 + tg_1$ depends real analytically on the parameter $t$. If $g_0 \in \mathcal{R}_{\min}(M, U_p(r), g)$ it follows that metrics arbitrarily close to $g_1$ are also in this set, from which we conclude the property of being dense.

**1.2. The invertible double.** Let $N$ be a compact connected spin manifold with boundary. The double of $N$ is formed by gluing $N$ and $-N$ along the common boundary $\partial N$ and is denoted by $(-N) \cup_{\partial N} N$. If $N$ is equipped with a Riemannian metric which has product structure near the boundary, then this metric naturally gives a metric on $(-N) \cup_{\partial N} N$. The spin structures can be glued together to obtain a spin structure on $(-N) \cup_{\partial N} N$. The spinor bundle $(-N) \cup_{\partial N} N$ is obtained by
gluing the spinor bundle of \( N \) with the spinor bundle of \(-N\) along their common boundary \( \partial N \). It is straightforward to check that the appropriate gluing map is the map used in [6, Chapter 9].

If a spinor field is in the kernel of the Dirac operator on \((-N) \cup_{\partial N} N\), then it restricts to a spinor field which is in the kernel of the Dirac operator on \( N \) and vanishes on \( \partial N \). By the weak unique continuation property of the Dirac operator it follows that such a spinor field must vanish everywhere, and we conclude that the Dirac operator on \((-N) \cup_{\partial N} N\) is invertible. For more details on this argument see [6, Chapter 9] and [5, Proposition 1.4].

**Proposition 1.3.** Let \((M, g)\) be a compact connected Riemannian spin manifold. Let \( p \in M \) and \( r > 0 \). Let \((-M) \# M\) be the connected sum formed at the points \( p \in M \) and \( p \in -M \). Then there is a metric on \((-M) \# M\) with invertible Dirac operator which coincides with \( g \) outside \( U_p(r) \)

This Proposition is proved by applying the double construction to the manifold with boundary \( N = M \setminus U_p(r/2) \), where \( N \) is equipped with a metric we get by deforming the metric \( g \) on \( U_p(r) \setminus U_p(r/2) \) to become product near the boundary.

Metrics with invertible Dirac operator are obviously \( D \)-minimal, so the metric provided by Proposition 1.3 is \( D \)-minimal.

2. **Proof of Theorem 1.1**

Let \( M \) and \( N \) be compact spin manifolds of dimension \( n \). Recall that a spin bordism from \( M \) to \( N \) is a manifold with boundary \( W \) of dimension \( n + 1 \) together with a spin preserving diffeomorphism from \( N \# (-M) \) to the boundary of \( W \). The manifolds \( M \) and \( N \) are said to be spin bordant if such a bordism exists.

For the proof of Theorem 1.1 we have to distinguish several cases.

2.1. **Dimension** \( n \geq 5 \).

*Proof of Theorem 1.1 in the case \( n \geq 5 \).* To prove the Gromov-Lawson conjecture, Stolz [11] showed that any compact spin manifold with vanishing index is spin bordant to a manifold of positive scalar curvature. Using this we see that \( M \) is spin bordant to a manifold \( N \) which has a \( D \)-minimal metric \( h \), where the manifold \( N \) is not necessarily connected. For details see [3, Proposition 3.9].

By removing an open ball from the interior of a spin bordism from \( M \) to \( N \) we get that \( N \# (-M) \) is spin bordant to the sphere \( S^n \).
Since $S^n$ is simply connected and $n \geq 5$ it follows from [9, Proof of Theorem 4.4, page 300] that $S^n$ can be obtained from $N \amalg (-M)$ by a sequence of surgeries of codimension at least 3. By making $r$ smaller and possibly move the surgery spheres slightly we may assume that no surgery hits $U_p(r) \subset M$. We obtain a sequence of manifolds $N_0, N_1, \ldots, N_k$, where $N_0 = N \amalg (-M)$, $N_k = S^n$, and $N_{i+1}$ is obtained from $N_i$ by a surgery of codimension at least 3.

Since the surgeries do not hit $U_p(r) \subset M \subset N \amalg (-M) = N_0$ we can consider $U_p(r)$ as a subset of every $N_i$. We define the sequence of manifolds $N'_0, N'_1, \ldots, N'_k$ by forming the connected sum $N'_i = M \# N_i$ at the points $p$. Then $N'_0 = N \amalg (-M) \# M$, $N'_k = S^n \# M = M$, and $N'_{i+1}$ is obtained from $N'_i$ by a surgery of codimension at least 3 which does not hit $M \setminus U_p(r)$. 
We now equip $N_0'$ with a Riemannian metric. On $N$ we choose a $D$-minimal metric. The manifold $(-M)\# M$ has vanishing index, so a $D$-minimal metric is a metric with invertible Dirac operator. From Proposition 1.3 we know that there exists such a metric on $(-M)\# M$ which coincides with $g$ outside $U_p(r)$. Note that we here use the assumption that $M$ is connected. Together we get a $D$-minimal metric $g_0'$ on $N_0'$.

From [3, Proposition 3.6] we know that the property of being $D$-minimal is preserved under surgery of codimension at least 3. We apply the surgery procedure to $g_0'$ to produce a sequence of $D$-minimal metrics $g_i'$ on $N_i'$. Since the surgery procedure of [3, Theorem 1.2] does not affect the Riemannian metrics outside arbitrarily small neighborhoods of the surgery spheres we may assume that all $g_i'$ coincide with $g$ on $M \setminus U_p(r)$. The Theorem is proved by choosing $\tilde{g} = g_k'$ on $N_k' = M$. □

2.2. Dimensions $n = 3$ and $n = 4$.

Proof of Theorem 1.1 in the case $n \in \{3, 4\}$. In these cases the argument works almost the same, except that we can only conclude that $S^n$ is obtained from $N\setminus (-M)$ by surgeries of codimension at least 2, see [7, VII, Theorem 3] for $n = 3$ and [8, VIII, Proposition 3.1] for $n = 4$. To take care of surgeries of codimension 2 we use [1, Theorem 1.2]. Since this surgery construction affects the Riemannian metric only in a small neighborhood of the surgery sphere we can finish the proof as described in the case $n \geq 5$. □

Alternatively, it is straight-forward to adapt the perturbation proof by Maier [10] to prove Theorem 1.1 in dimensions 3 and 4.

2.3. Dimension $n = 2$.

Proof of Theorem 1.1 in the case $n = 2$. The argument in the case $n = 2$ is different. Assume that a metric $g$ on a compact surface with chosen spin structure is given. In [1, Theorem 1.1] it is shown that for any $\varepsilon > 0$ there is a $D$-minimal metric $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$. Using the following Lemma 2.1, we see that for $\varepsilon > 0$ sufficiently small, there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to $g$ on $M \setminus U_p(r)$. As the dimension of the kernel of the
Dirac operator is preserved under spin-preserving conformal diffeomorphisms, $\tilde{g}$ is $D$-minimal as well.

**Lemma 2.1.** Let $M$ be a compact surface with a Riemannian metric $g$ and a spin structure. Then for any $r > 0$ there is an $\varepsilon > 0$ with the following property: For any $\tilde{g}$ with $\|g - \tilde{g}\|_{C^1} < \varepsilon$ there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^*\tilde{g}$ is conformal to $g$ on $M \setminus U_p(r)$.

To prove the lemma one has to show that a certain differential is surjective. This proof can be carried out in different mathematical languages. One alternative is via Teichmüller theory and quadratic differentials. We will follow a different way of presentation and notation.

**Sketch of Proof of Lemma 2.1.** If $g_1$ and $g_2$ are metrics on $M$, then we say that $g_1$ is spin-conformal to $g_2$ if there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\psi^*g_2 = g_1$. This is an equivalence relation on the set of metrics on $M$, and the equivalence class of $g_1$ is denoted by $\Phi(g_1)$. Let $\mathcal{M}$ be the set of equivalence classes. Showing the lemma is equivalent to showing that $\Phi(R(M, U_p(r), g))$ is a neighborhood of $g$ in $\mathcal{M}$.

Variations of metrics are given by symmetric $(2,0)$-tensors, that is by sections of $S^2T^*M$. The tangent space of $\mathcal{M}$ can be identified with the space of transverse (= divergence free) traceless sections,

$$S^{TT} := \{h \in \Gamma(S^2T^*M) \mid \text{div}^g h = 0, \text{tr}^g h = 0\},$$

see for example [4, Lemma 4.57] and [12].

The two-dimensional manifold $M$ has a complex structure which is denoted by $J$. The map $H : T^*M \to S^2T^*M$ defined by $H(\alpha) := \alpha \otimes \alpha - \alpha \circ J \otimes \alpha \circ J$ is quadratic, it is 2-to-1 outside the zero section, and its image are the trace free symmetric tensors. Furthermore $H(\alpha \circ J) = -H(\alpha)$. Hence by polarization we obtain an isomorphism of real vector bundles from $T^*M \otimes \mathbb{C} T^*M$ to the trace free part of $S^2T^*M$. Here the complex tensor product is used when $T^*M$ is considered as a complex line bundle using $J$. A trace free section of $S^2T^*M$ is divergence free if and only if the corresponding section $T^*M \otimes \mathbb{C} T^*M$ is holomorphic, see [12, pages 45-46]. We get that $S^{TT}$ is finite-dimensional, and it follows that $\mathcal{M}$ is finite dimensional.

In order to show that $\Phi(R(M, U_p(r), g))$ is a neighborhood of $g$ in $\mathcal{M}$ we show that the differential $d\Phi : TR(M, U_p(r), g) \to T\mathcal{M}$ is surjective at $g$. Using the above identification $T\mathcal{M} = S^{TT}$, $d\Phi$ is just orthogonal projection from $\Gamma(S^2T^*M)$ to $S^{TT}$.

Assume that $h_0 \in S^{TT}$ is orthogonal to $d\Phi(TR(M, U_p(r), g))$. Then $h_0$ is $L^2$-orthogonal to $TR(M, U_p(r), g)$. As $TR(M, U_p(r), g)$ consists of all sections of $S^2T^*M$ with support in $U_p(r)$ we conclude that $h_0$ vanishes on $U_p(r)$. Since $h_0$ can be identified with a holomorphic section of $T^*M \otimes \mathbb{C} T^*M$ we see that $h_0$ vanishes everywhere on $M$. The surjectivity of $d\Phi$ and the lemma follow.

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**References**


