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Hans Delfs
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Locally Semialgebraic Spaces

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To Christl and Gisela

Preface

The primary occupation of real algebraic geometry, or better "semialgebraic geometry", is to study the set of solutions of a finite system of polynomial inequalities in a finite number of variables over the field $\mathbb{R}$ of real numbers. One wants to do this in a conceptual way, not always mentioning the polynomial data, similarly as in algebraic geometry, say over $\mathbb{C}$, where one most often avoids working explicitely with the systems of polynomial equalities (and non-equalities $f \neq 0$) involved.

But a semialgebraic geometry which deserves its name should be able to work - at least - over an arbitrary real closed field R instead of the field $\mathbb{R}$. Such fields are useful and even unavoidable in semialgebraic geometry for much the same reason as algebraically closed fields of characteristic zero - at least - are unavoidable in algebraic geometry over $\mathbb{C}$, as soon as one tries to avoid transcendental techniques or even then.

In order to illustrate this we give a somewhat typical example. Let $f: V \to W$ be an algebraic map between irreducible varieties over $\mathbb{R}$. This yields, by restriction, a continuous map $f_{\mathbb{R}}: V(\mathbb{R}) \to W(\mathbb{R})$ between the sets of real points. We assume that $W(\mathbb{R})$ is Zariski dense in W which means that $W(\mathbb{R})$ contains non singular points or, equivalently, that the function field $\mathbb{R}(W)$ is formally real. The generic fibre X of f, i.e. $X = f^{-1}(\eta)$ with η the generic point of W (regarding V and W as schemes), is an algebraic scheme over the function field $\mathbb{R}(W)$ of W, which contains a lot of information about f and $f_{\mathbb{R}}$. But it may be too difficult to study X, since the field $\mathbb{R}(W)$ is usually very complicated. In algebraic geometry one often replaces X by the algebraic variety $\overline{X}$ obtained from X by extension of the base field $\mathbb{R}(W)$ to the algebraic closure C of $\mathbb{R}(W)$. It is much easier to study the "geometric generic fibre" $\overline{X}$

instead of X, and still one may hope to extract relevant information about f from $\overline{X}$. But in semialgebraic geometry this procedure is not advisable, since most real phenomena in X will be destroyed in $\overline{X}$. Instead of $\overline{X}$ one should study the varieties X_α, obtained from X by base extension from $\mathbb{R}(W)$ to the real closures R_α of $\mathbb{R}(W)$ with respect to the various orderings α of the function field $\mathbb{R}(W)$, and the sets of rational points $X_\alpha(R_\alpha)$. For every such α we have $R_\alpha(\sqrt{-1}) = C$. Thus the R_α are "as near as possible" to C and nevertheless we may hope to detect some of the real phenomena of X, and ultimately of f, in the sets $X_\alpha(R_\alpha)$.

The variety X is the projective limit of the schemes $f^{-1}(U) = V \times_W U$ with U running through the Zariski-open subsets of W, since these U are the Zariski neighbourhoods of the generic point η in W. Similarly $\overline{X}$ is the projective limit of the fibre products $V \times_W U$, with respect to the etale morphisms $\varphi : U \to W$ from arbitrary varieties U over $\mathbb{R}$ to W ($U \neq \emptyset$, but $U(\mathbb{R})$ may be empty), since these morphisms φ are the etale neighbourhoods of η. How about the X_α? An ordering α of $\mathbb{R}(W)$ corresponds uniquely to an ultrafilter F in the Boolean lattice $\mathfrak{Y}(W(\mathbb{R}))$ of semialgebraic subsets of $W(\mathbb{R})$ such that every $A \in F$ has a non empty interior $\mathring{A}$ in the strong topology (= classical topology on $W(\mathbb{R})$), which means that A is Zariski dense in W, cf. [B, 8.11],[Br, §4]. (A rational function $h \in \mathbb{R}(W)$ is positive with respect to α if and only if h is defined and positive on some set $A \in F$). It turns out that X_α is the projective limit of the fibre products $V \times_W U$ with respect to those etale morphisms $\varphi : U \to W$ such that $\varphi(U(\mathbb{R})) \in F$. (N.B. $\varphi(U(\mathbb{R}))$ is semialgebraic.) This is due to the fact that R_α can be interpreted as the union of the rings of Nash functions $\mathcal{N}_W(U)$ on the various smooth open sets $U \in F$, cf. [Ry].

Much more can be said about a geometric interpretation over $\mathbb{R}$ of the fields R_α, the varieties X_α and the points in $X_\alpha(R_\alpha)$. But this would take us too far afield. We only mention that the real spectra of commutative rings invented by M. Coste and M.F. Coste-Roy provide exactly the right language to understand all this, cf. [CR], [Ry], and the literature cited there and, for an introduction to real spectra , also [L, §4, §7], [Br, §3, §4], [K], [BCR, Chap. 7].

We have been somewhat vague above. In particular we did not make precise the various direct systems which yield the projective limits $\overline{X}$ and X_α. We only wanted to indicate that in semialgebraic geometry over $\mathbb{R}$ real closed fields may come up in a natural and geometric way.

The present lecture notes give a contribution to a basic but rather modest aspect of semialgebraic geometry: the topological phenomena of semialgebraic sets in V(R) for V a variety over a real closed field R. There is a difficulty with the word "topological" here. Of course, V(R) is equipped with the strong topology coming from the topology of the ordered field R. But, except in the case $R = \mathbb{R}$, the topological space V(R) is totally disconnected.

These pathologies can be remedied by considering on V(R) a topology in the sense of Grothendieck, where only open <u>semialgebraic</u> subsets U of V(R) are admitted as "open sets", and for such a set U essentially only coverings by finitely many open semialgebraic subsets of U are admitted as "open coverings".

It seems that the <u>category of semialgebraic spaces and maps</u> over a real closed field R, which has been introduced in our paper [DK_2], provides the right framework for this "semialgebraic topology". Already in that paper and later in other ones ([D], [D_1], [DK_3], [DK_4], [DK_5]) we found analogues of many results in classical topology. Sometimes things are even nicer here. This is not astonishing since, in the case $R = \mathbb{R}$, the semialgebraic sets are rather tame from a topological viewpoint.

In the case $R = \mathbb{R}$ the category of semialgebraic spaces can be compared with the category of topological spaces, and this affords us a new perspective concerning the two branches of mathematics involved, semialgebraic geometry and algebraic topology, cf. the introduction of [B]. For example, a long journey along this road should give a thorough understanding of why so many spaces occuring in usual algebraic topology are semialgebraic sets.

Nevertheless the category of semialgebraic spaces is too restrictive for some purposes. A good instance where this can be seen is the theory of <u>semialgebraic coverings</u>. If M is a connected affine semialgebraic space over R, and x_o is some point in M, we can define the fundamental group $\pi_1(M,x_o)$ in the usual way as the set of semialgebraic homotopy classes of semialgebraic loops with base point x_o (cf. III,§6)*). This is an honest to goodness group, generated by finitely many elements satisfying finitely many relations. On the other hand we evidently have the notion of an (unramified) covering $p: N \to M$ of M, p being a locally trivial semialgebraic map with discrete (= zero-dimensional) fibres. Of course, one would like to classify the coverings of M by subgroups of $\pi_1(M,x_o)$. But a zero-dimensional semialgebraic space is necessarily a finite set. Thus every semialgebraic covering has finite degree. It can be shown that indeed the isomorphism classes of semialgebraic coverings of M correspond uniquely to the conjugacy classes of subgroups of finite index in $\pi_1(M,x_\Theta)$ in the usual way. But there should also exist coverings of a more general nature which correspond to the other subgroups of $\pi_1(M,x_o)$. In particular there should exist a universal covering of M. These more general coverings can be defined in

*) This means §6 in Chapter III of this book.

the <u>category of "locally semialgebraic" spaces and maps</u>.

After several years of experimenting with locally semialgebraic spaces we are convinced that these spaces exist "in nature". The coverings of affine semialgebraic spaces are <u>regular paracompact</u> locally semialgebraic spaces, to be defined in I, §4. Regular paracompact spaces seem to be the "good" locally semialgebraic spaces, analogous to the affine spaces in the semialgebraic category. For instance, for these spaces there exists a satisfactory cohomology theory of sheaves, based on flabby and soft sheaves, which parallels the classical theory for topological paracompact spaces. We will not deal with these matters here, except for some brief remarks in Appendix A, but they are quite important for defining homology and cohomology groups of various kinds for these spaces, cf. [D], [D_1], [D_2].

Although regular paracompact spaces are a very satisfying subclass of locally semialgebraic spaces one has to face the fact that there exist many locally semialgebraic spaces in nature which are not paracompact. (It seems that regularity may be assumed in most applications.) For example, studying open subsets of quite innocently looking real spectra may lead to regular spaces which are not paracompact, cf. Appendix A. Thus it is not just for fun or for systematic reasons that we study in Chapter I more general spaces. In the later chapters we are forced to restrict to paracompact spaces, since otherwise our deeper techniques break down.

There is one phenomenon in our theory which may seem somewhat unusual for a reader of our previous papers. In a semialgebraic space M it is strictly forbidden to work with subsets of M other than the semialgebraic subsets [DK_2, §7]. But in a locally semialgebraic space M there exist two natural classes of admissible subsets, the class $\mathcal{T}(M)$ of locally semialgebraic subsets of M and the smaller class $\gamma(M)$ of semialgebraic subsets of M. The interplay between $\gamma(M)$ and $\mathcal{T}(M)$ is a theme which recurs througout the whole theory.

The goal of the first volume of our lecture notes is to establish the category of locally semialgebraic spaces and maps over an arbitrary real closed field R on firm grounds, and to prove enough results about these spaces and maps, that the reader will feel well acquainted with them and will regard them as concrete and accessible objects. The next topics, to be covered in the second volume, are the theory of locally semialgebraic fibrations and fibre bundles (Chapter IV) and the theory of coverings (Chapter V).

As background material we assume our papers [DK_2], [DK_4], [DK_5], some sections of [DK_3], and Robson's paper [R]. Here you find nearly everything which we need about semialgebraic spaces, written up in a systematic way compatible with the spirit of these lecture notes. Of course, it would have been more comfortable for the reader if we had started the lecture notes with a review of the results of those papers. But this is not really necessary and would have made the lecture notes too long. Of course, the book [BCR] of Bochnak and the Costes - as soon as it has appeared - will contain most basic facts which are necessary for an understanding of these lecture notes and much more.

A survey on some basic results about semialgebraic spaces has been given in [DK]. Another survey on basic results about locally semialgebraic spaces, which, of course, all will be covered by the two volumes of these lecture notes, has been given in [DK_6] and [DK_7].

We hope that these lecture notes, designed in first place for the needs of semialgebraic geometry, are also of interest for topologists. The main results are usually non trivial also in the case $R = \mathbb{R}$ and not much easier to be proved in this special case. The category of locally semialgebraic spaces over $\mathbb{R}$ lies somewhat "in between" the category TOP of topological Hausdorff spaces and the category PL of piecewise linear spaces, being less rigid than PL and, in some respects, less pathological than TOP.

The central result of the whole volume seems to be Theorem 4.4 in Chapter II, §4, which states that every regular paracompact locally semialgebraic space M can be triangulated, and moreover a given locally finite family of locally semialgebraic subsets of M can be triangulated simultaneously. Thus we may regard every regular paracompact space as a locally finite polyhedron with some open faces missing (cf. the definition of strictly locally finite simplicial complexes, in I, §2, which is slightly different from the classical definition). But in contrast to PL-theory, we may subdivide simplices not only linearly but "semialgebraically". Nevertheless, in the special case that $R = \mathbb{R}$ and M is partially complete, Shiota and Yokoi have recently proved that any two PL structures on M which refine the given semialgebraic structure are isomorphic ([SY, Th. 4.1], they prove this more generally for suitable locally subanalytic spaces). This remarkable theorem can be extended to partially complete regular paracompact spaces over any R, as we hope to explain in the second volume.

If S is a real closed field containing R then, as a consequence of Tarski's principle, we can associate with every locally semialgebraic space M over R a locally semialgebraic space M(S) over S by "<u>extension of the base field R to S</u>", cf. I.2.10. This yields a very good natured functor $M \mapsto M(S)$ from the category of regular paracompact spaces over R to the category of regular paracompact spaces over S, which is of crucial importance for our whole theory. The homotopy groups (cf. III, §6), the homology groups (cf. III, §7) and also the various K-groups of M (orthogonal, unitary, symplectic, cf. Chapter IV in the second volume) are preserved under base field extension from R to S. These are examples of the main message of our whole theory, that over a complicated real closed field the locally semialgebraic spaces are in many respects not more complicated than over a simple field, as the field $\mathbb{R}$ or the field R_o of real algebraic numbers. We believe that this message is by no means trivial. It may be regarded as a vast generalization of Tarski's principle for topological statements. As soon as one leaves the cadre of semialgebraic topology and works, say with algebraic functions then the analogue of our message seems to hold only under severe restrictions. For example, it is well known that, in general, semialgebraic functions on the unit interval [0,1] in R cannot be approximated uniformly by polynomials, in contrast to the Stone-Weierstraß theorem for $R = \mathbb{R}$.

The book has two appendices. Appendix B (to Chapter I) contains some easy but fundamental results in the theory of base extension. They have not been included into Chapter I since some of the techniques needed to derive them seem to have their natural place in Chapter II. Appendix A is of different kind. Here we draw the connections between our theory and "abstract" semialgebraic geometry which, starting from the notion of the real spectrum, now is in a process of rapid development. Appendix A is not needed for our theory in a technical sense, but there we will find the occasion to explain some more points of our philosophy about the "raison d'être" of locally semialgebraic spaces.

We thank the members of the former Regensburger semialgebraic group, in particular Roland Huber and Robby Robson, for stimulating discussions and criticism about the contents of these lecture notes. Special thanks are due to Josê Manuel Gamboa and R. Huber for a penetrating (and very successful) search for mistakes in the final version of the manuscript.

We thank Marina Richter for her patience and excellence in typing the book and R. Robson for eliminating some of the most annoying grammatical mistakes. We are well aware that we could have written a better book

in our native language, but since the book is designed as a "topologie générale" for semialgebraic geometry which should be useful as a widely accepted reference, we have written in that language which will be understood by the most.

Regensburg, July 1985

Hans Delfs, Manfred Knebusch

TABLE OF CONTENTS

Chapter I. - The basic definitions

The goals of this chapter are modest. We present the basic definitions and some elementary observations needed by anyone who wants to work with locally semialgebraic spaces. The coverings mentioned in the preface will be a special class of the "partially proper maps" considered in §6.

§1 - Locally semialgebraic spaces and maps

Our idea is to define locally semialgebraic spaces as suitable ringed spaces in the sense of Grothendieck. Then the more delicate question how to define locally semialgebraic maps becomes trivial. These maps will be simply all the morphisms between the locally semialgebraic spaces in the category of ringed spaces. We used the same procedure already to define semialgebraic spaces in [DK_2, §7].

Definition 1. A generalized topological space *) is a set M together with a set $\mathring{\mathfrak{T}}(M)$ of subsets of M, called the "open subsets" of M, and a set Cov_M of families $(U_\alpha | \alpha \in I)$ in $\mathring{\mathfrak{T}}(M)$ **), called the "admissible coverings" such that the following properties hold:

i) $\emptyset \in \mathring{\mathfrak{T}}(M)$, $M \in \mathring{\mathfrak{T}}(M)$.

ii) If $U_1 \in \mathring{\mathfrak{T}}(M)$, $U_2 \in \mathring{\mathfrak{T}}(M)$, then $U_1 \cap U_2 \in \mathring{\mathfrak{T}}(M)$ and $U_1 \cup U_2 \in \mathring{\mathfrak{T}}(M)$.

iii) Every family $(U_\alpha | \alpha \in I)$ in $\mathring{\mathfrak{T}}(M)$ with I finite is an element of Cov_M.

iv) If $(U_\alpha | \alpha \in I)$ is an element of Cov_M, then the union $U := \cup (U_\alpha | \alpha \in I)$ of this family is an element of $\mathring{\mathfrak{T}}(M)$.

---

*) This term should be regarded as ad hoc.

**) In order to guarantee that Cov_M is really a set one should only allow subsets of some fixed large set as index sets I. We will ignore all set-theoretic difficulties here.

For any $U \in \mathring{\mathfrak{T}}(M)$ we denote the subset of all $(U_\alpha | \alpha \in I) \in \mathrm{Cov}_M$ with $U(U_\alpha|\alpha\in I) = U$ by $\mathrm{Cov}_M(U)$ and we call these coverings the "admissible coverings of U".

v) If $(U_\alpha|\alpha\in I)$ is an admissible covering of $U \in \mathring{\mathfrak{T}}(M)$ and if $V \in \mathring{\mathfrak{T}}(M)$ is a subset of U, then $(U_\alpha \cap V | \alpha \in I)$ is an admissible covering of V.

vi) If an admissible covering $(U_\alpha|\alpha\in I)$ of $U \in \mathring{\mathfrak{T}}(M)$ is given and for every $\alpha \in I$ an admissible covering $(V_{\alpha\beta}|\beta \in J_\alpha)$ of U_α is given, then $(V_{\alpha\beta}|\alpha \in I, \beta \in J_\alpha)$ is an admissible covering of U.

vii) If $(U_\alpha|\alpha\in I)$ is a family in $\mathring{\mathfrak{T}}(M)$ with $U(U_\alpha|\alpha\in I) = U \in \mathring{\mathfrak{T}}(M)$ and if $(V_\beta|\beta\in J) \in \mathrm{Cov}_M(U)$ is a refinement of $(U_\alpha|\alpha\in I)$ {i.e. there exists a map $\lambda : J \to I$ with $V_\beta \subset U_{\lambda(\beta)}$ for every $\beta \in J$}, then $(U_\alpha|\alpha\in I) \in \mathrm{Cov}_M(U)$.

viii) If $U \in \mathring{\mathfrak{T}}(M)$, $(U_\alpha|\alpha\in I) \in \mathrm{Cov}_M(U)$ and if V is a subset of U with $V \cap U_\alpha \in \mathring{\mathfrak{T}}(M)$ for every $\alpha \in I$, then $V \in \mathring{\mathfrak{T}}(M)$ [and thus, by (v), $(V \cap U_\alpha | \alpha \in I)$ is an admissible covering of V].

Comments.

a) We usually just write "M" for the triple $(M, \mathring{\mathfrak{T}}(M), \mathrm{Cov}_M)$. A generalized topological space M is a (rather special) example of a site in the sense of Grothendieck. Thus we have Grothendieck's theory of sheaves over such spaces M at our disposal, cf. [A], [SGA4, Exp. II]. Let us recall the notion of an (abelian) sheaf here in our special situation. A presheaf $\mathcal{F}$ on M is an assignment $U \mapsto \mathcal{F}(U)$ of an abelian group $\mathcal{F}(U)$ to every $U \in \mathring{\mathfrak{T}}(M)$ equipped with a restriction homomorphism $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for every pair of open sets U,V with $V \subset U$ such that $r^U_U = \mathrm{id}$ and $r^V_W \circ r^U_V = r^U_W$ for $U \supset V \supset W$. A presheaf $\mathcal{F}$ is a sheaf if in addition for any admissible covering $(U_\alpha|\alpha\in I)$ of any $U \in \mathring{\mathfrak{T}}(M)$, the usual sequence

$$0 \to \mathcal{F}(U) \to \prod_{\alpha\in I} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta)\in I\times I} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact. Thus a sheaf is very much the same notion as for usual topological spaces, except that now only sets in $\mathring{\mathfrak{T}}(M)$ are allowed as open sets and only admissible coverings are allowed as open coverings.
b) The last two axioms (vii) and (viii) in the definition of a generalized topological space are less substantial than the others. Axiom (vii) is just a technical device to make formal arguments smoother. Notice that if axioms (i) - (vi) are fulfilled, then by enriching the sets $Cov_M(U)$, $U \in \mathring{\mathfrak{T}}(M)$, by all families $(U_\alpha | \alpha \in I)$ in $\mathring{\mathfrak{T}}(M)$ which have the union U and which admit refinements lying in $Cov_M(U)$, we obtain a new site $(M, \mathring{\mathfrak{T}}(M), Cov_M')$ which fulfills (i) - (vii) and which has the same sheaves as the original site $(M, \mathring{\mathfrak{T}}(M), Cov_M)$. The rôle of axiom (viii) is more subtle and will be discussed at the end of this section.

From now on R denotes a fixed real closed field.

<u>Definition 2.</u> A <u>ringed</u> <u>space</u> <u>over</u> R is a pair $(M, \mathcal{O}_M)$ consisting of a generalized topological space M and a sheaf $\mathcal{O}_M$ of commutative R-algebras. A <u>morphism</u> $(\varphi, \vartheta): (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between ringed spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ over R is defined in the obvious way: φ is a continuous map from M to N, i.e. every open set V in N has an open preimage $\varphi^{-1}(V)$ and for every $(V_\alpha | \alpha \in I) \in Cov_N(V)$ the family $(\varphi^{-1}(V_\alpha) | \alpha \in I)$ is an admissible covering of $\varphi^{-1}(V)$. The second component ϑ is a homorphism from the sheaf $\mathcal{O}_N$ to the sheaf $\varphi_* \mathcal{O}_M$ respecting the R-algebra structures. In other words, for any open sets U in M and V in N with $\varphi(U) \subset V$ we have an R-algebra homomorphism

$$\vartheta_{U,V}: \mathcal{O}_N(V) \to \mathcal{O}_M(U)$$

with the usual compatibilities with respect to the restriction maps.

<u>Example.</u> Let M be a <u>semialgebraic space</u> over R as defined in [DK_2, §7]. Choose for $\mathring{\mathfrak{T}}(M)$ the set $\mathring{\mathfrak{r}}(M)$ of all open semialgebraic subsets of M, and for $U \in \mathring{\mathfrak{r}}(M)$ define the set $Cov_M(U)$ to be the set of all families $(U_i | i \in I)$ in $\mathring{\mathfrak{r}}(M)$ such that the union $\cup(U_i | i \in I) = U$ and such that finitely many $U_i, i \in I$, already cover U.
The axioms (i) - (viii) are clearly fulfilled. Thus M is a generalized topological space. (N.B. In this way every "restricted topological space", as defined in [DK_2, §7], can be regarded as a generalized topological space). The sheaves on this generalized topological space M are the same as the sheaves on the "restricted topological space" M considered in [DK_2,§7]. In particular on M there is a sheaf $\mathcal{O}_M$ of R-algebras defined as follows: If $U \in \mathring{\mathfrak{r}}(M)$, then $\mathcal{O}_M(U)$ is the R-algebra of semialgebraic functions $f: U \to R$. For open semialgebraic sets $V \subset U$ the restriction map r_V^U from $\mathcal{O}_M(U)$ to $\mathcal{O}_M(V)$ is the obvious restriction of functions $f \mapsto f|V$. This ringed space $(M, \mathcal{O}_M)$ over R is really the same object as the semialgebraic space M and will be identified with it.

If $(M, \mathcal{O}_M)$ is a ringed space over R then for any $U \in \mathring{\mathfrak{T}}(M)$ we obtain "by restriction" a ringed space $(U, \mathcal{O}_M|U)$ over R as follows: $\mathring{\mathfrak{T}}(U)$ consists of all $V \in \mathring{\mathfrak{T}}(M)$ with $V \subset U$. Cov_U consists of all families $(V_\alpha | \alpha \in I) \in Cov_M$ with $V_\alpha \subset U$ for every $\alpha \in I$, and $\mathcal{O}_M|U$ is the restriction of the sheaf $\mathcal{O}_M$ to U, i.e. $(\mathcal{O}_M|U)(V) = \mathcal{O}_M(V)$ for every $V \in \mathring{\mathfrak{T}}(U)$. These ringed spaces $(U, \mathcal{O}_M|U)$ are called the <u>open subspaces</u> of $(M, \mathcal{O}_M)$. An open subset U of M is called an <u>open semialgebraic subset</u> if $(U, \mathcal{O}_M|U)$ is a semialgebraic space over R, as defined in the example above.

<u>Definition 3.</u> A <u>locally semialgebraic space</u> over R is a ringed space $(M, \mathcal{O}_M)$ over R which possesses an admissible covering $(M_\alpha | \alpha \in I) \in Cov_M(M)$ such that all M_α are open semialgebraic subsets of M.

Let us look at these definitions more closely. Assume that $(M,\mathcal{O}_M)$ is a locally semialgebraic space and that $(M_\alpha|\alpha\in I)$ is an admissible covering of M by open semialgebraic subsets M_α. What are the other open semialgebraic subsets of M? Clearly, if $U\in \mathring{\mathfrak{T}}(M)$ is semialgebraic, then $(U\cap M_\alpha|\alpha\in I)$ is an admissible covering of the semialgebraic space U, and thus U is contained in the union of finitely many sets M_α. Conversely if $U\in \mathring{\mathfrak{T}}(M)$ and $U\subset M_{\alpha_1}\cup\ldots\cup M_{\alpha_r}$ for finitely many indices $\alpha_1, \ldots, \alpha_r \in I$ then the set $M_{\alpha_1}\cup\ldots\cup M_{\alpha_r} = W$ is open semialgebraic in M by the very definition of semialgebraic spaces. Moreover $U\in \mathring{\mathfrak{T}}(W)$. Thus $(U,\mathcal{O}_M|U)$ is also a semialgebraic space. The open semialgebraic subsets of M are precisely those sets $U\in \mathring{\mathfrak{T}}(M)$ which are contained in the union of finitely many sets M_α. We will henceforth denote the subset of $\mathring{\mathfrak{T}}(M)$ consisting of all open semialgebraic subsets of M by $\mathring{\gamma}(M)$. Notice that $\mathring{\gamma}(M) = \mathring{\mathfrak{T}}(M)$ if and only if M itself is semialgebraic.

We clearly have the following relations between the sets $\mathring{\gamma}(M)$ and $\mathring{\mathfrak{T}}(M)$.

a) A subset W of M belongs to $\mathring{\gamma}(M)$ if and only if for every $(U_\lambda|\lambda\in\Lambda)\in \mathrm{Cov}_M(U), U\in \mathring{\mathfrak{T}}(M)$, all the intersections $W\cap U_\lambda$ are elements of $\mathring{\mathfrak{T}}(M)$ and $W\cap U$ is covered by finitely many sets $W\cap U_\lambda$.

b) A subset U of M belongs to $\mathring{\mathfrak{T}}(M)$ if and only if $U\cap W\in \mathring{\gamma}(M)$ for every $W\in \mathring{\gamma}(M)$. Also $U\in \mathring{\mathfrak{T}}(M)$ if and only if $U\cap M_\alpha \in \mathring{\gamma}(M)$ for every $\alpha\in I$.

It is also an easy consequence of our definitions, in particular of the axioms vi), vii), viii) in Definition 1, that $\mathring{\gamma}(M)$ determines the set Cov_M in the following way:

c) A family $(U_\lambda|\lambda\in\Lambda)$ in $\mathring{\mathfrak{T}}(M)$ belongs to Cov_M if and only if for every $W\in \mathring{\gamma}(M)$ the intersection $W\cap U$ of W with the union U of the family is covered by finitely many sets $W\cap U_\lambda, \lambda\in\Lambda$. In fact, it suffices that for every $\alpha\in I$ the intersection $M_\alpha\cap U$ is covered by finitely many sets $M_\alpha\cap U_\lambda, \lambda\in\Lambda$.

<u>Definition 4.</u> A family $(X_\lambda | \lambda \in \Lambda)$ of subsets of M is called <u>locally finite</u> if any $W \in \mathring{\mathcal{T}}(M)$ meets only finitely many X_λ, in other words, if $W \cap X_\lambda \neq \emptyset$ for only finitely many $\lambda \in \Lambda$. Again it is only necessary to check that, for every $\alpha \in I$, the set M_α meets only finitely many X_λ.

As a special case of our observation (c), we have

<u>Proposition 1.1.</u> Every locally finite family in $\mathring{\mathcal{T}}(M)$ is an element of Cov_M. In particular, the union of this family is an element of $\mathring{\mathcal{T}}(M)$.

For every $x \in M$, the stalk $\mathcal{O}_{M,x}$ is a local ring and the natural map from R to the residue class field $\mathcal{O}_{M,x}/\mathfrak{m}_{M,x}$ of $\mathcal{O}_{M,x}$ is an isomorphism. Indeed, this is known to be true for all the semialgebraic spaces $(M_\alpha, \mathcal{O}_M | M_\alpha)$ and thus also holds for $(M, \mathcal{O}_M)$. We identify $\mathcal{O}_{M,x}/\mathfrak{m}_{M,x}$ with the field R. If $U \in \mathring{\mathcal{T}}(M)$ and $f \in \mathcal{O}_M(U)$ then f yields an R-valued function $\overline{f} : U \to R$, which maps every $x \in U$ to the natural image of f in $\mathcal{O}_{M,x}/\mathfrak{m}_{M,x}$. The element $f \in \mathcal{O}_M(U)$ is uniquely determined by this function $\overline{f}$, since the corresponding fact is known to be true for all the restrictions $r^U_{U \cap M_\alpha}(f) \in \mathcal{O}_M(U \cap M_\alpha)$ of f. We identify f with $\overline{f}$. Thus we regard $\mathcal{O}_M$ as a subsheaf of the sheaf of all R-valued functions on M. In particular the restriction maps $r^U_V : \mathcal{O}_M(U) \to \mathcal{O}_M(V)$ are now the naive restriction maps $f \mapsto f|V$ for functions. From now on we will call the elements of $\mathcal{O}_M(U)$ the <u>locally semialgebraic functions</u> on U (with respect to M). Notice that, in the special case where U is a semialgebraic open subset of M, these functions are just the semialgebraic functions on the semialgebraic space $(U, \mathcal{O}_M | U)$. Thus in this case the $f \in \mathcal{O}_M(U)$ will also be designated as the "semialgebraic functions on U".

Now let $(N, \mathcal{O}_N)$ be a second locally semialgebraic space over R.

<u>Definition 5.</u> A <u>locally semialgebraic map</u> from $(M,\mathcal{O}_M)$ to $(N,\mathcal{O}_N)$ is a morphism $(f,\mathfrak{F}):(M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ in the category of ringed spaces over R (cf. Def. 2 above).

The following theorem is known for semialgebraic spaces [DK_2,Th.7.2] and extends immediately to locally semialgebraic spaces.

<u>Theorem 1.2.</u> Let $(f,\mathfrak{F}):(M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ be a locally semialgebraic map. For any open sets U and V of M and N with $f(U) \subset V$ and any $h \in \mathcal{O}_N(V)$ we have

$$\mathfrak{F}_{U,V}(h)(x) = h(f(x))$$

for all $x \in U$.

Thus $(f,\mathfrak{F})$ is determined by its first component f and will henceforth be identified with the map f from the set M to the set N. Clearly a map $f:M \to N$ is locally semialgebraic if and only if f is continuous (cf. Def. 2 above), and if for every $U \in \mathring{\mathfrak{T}}(N)$ and every $h \in \mathcal{O}_N(U)$ the function $h \circ f$ is locally semialgebraic on $f^{-1}(U)$. Notice that, in case M and N are semialgebraic spaces, the locally semialgebraic maps from M to N are just the semialgebraic maps from M to N as defined in [DK_2,§7]. In general the following Proposition 1.3 gives a good hold on locally semialgebraic maps in terms of semialgebraic maps. Notice that for every continuous map $f:M \to N$ and every admissible covering $(N_\beta|\beta\in J)$ of N by open semialgebraic sets there certainly exists a covering $(M_\alpha|\alpha\in I)$ of M with the properties needed in the proposition.

<u>Proposition 1.3.</u> Let $f:M \to N$ be a (set theoretical) map. Let $(M_\alpha|\alpha\in I)$ and $(N_\beta|\beta\in J)$ be admissible coverings of M and N by open semialgebraic subsets. Assume that there is a map $\mu:I \to J$ such that $f(M_\alpha) \subset N_{\mu(\alpha)}$ for every $\alpha\in I$. Then f is locally semialgebraic if and only if the

restriction $f|M_\alpha : M_\alpha \to N_{\mu(\alpha)}$ of f to M_α is a semialgebraic map for every $\alpha \in I$.

Proof. The "only if" direction is obvious. So assume that $f|M_\alpha : M_\alpha \to N_{\mu(\alpha)}$ is semialgebraic for all $\alpha \in I$. Let $U \in \mathring{\mathfrak{T}}(N)$. Then for every $\alpha \in I$

$$f^{-1}(U) \cap M_\alpha = f^{-1}(N_{\mu(\alpha)} \cap U) \cap M_\alpha = (f|M_\alpha)^{-1}(N_{\mu(\alpha)} \cap U)$$

is an open semialgebraic subset of M_α. Hence, by axiom (viii) in Definition 1, $f^{-1}(U) \in \mathring{\mathfrak{T}}(M)$. Let now $(U_\lambda | \lambda \in \Lambda)$ be an admissible covering of $U \in \mathring{\mathfrak{T}}(N)$. Then for every $\alpha \in I$

$$(f^{-1}(U_\lambda) \cap M_\alpha | \lambda \in \Lambda) = ((f|M_\alpha)^{-1}(U_\lambda \cap N_{\mu(\alpha)}) | \lambda \in \Lambda)$$

is an admissible covering of $f^{-1}(U) \cap M_\alpha = (f|M_\alpha)^{-1}(U \cap N_{\mu(\alpha)})$, i.e. it possesses a finite refinement. We conclude that $(f^{-1}(U_\lambda) | \lambda \in \Lambda)$ is an admissible covering of $f^{-1}(U)$. Thus f is continuous. For a given function $h \in \mathcal{O}_N(U)$ all the restrictions $h|U \cap N_\beta$, $\beta \in J$, are semialgebraic functions. Since the maps $f|M_\alpha : M_\alpha \to N_{\mu(\alpha)}$ are semialgebraic we see that all the functions $h \circ f|f^{-1}(U) \cap M_\alpha$, $\alpha \in I$, are semialgebraic. Thus $h \circ f \in \mathcal{O}_M(f^{-1}U)$.

q.e.d.

Corollary 1.4. Let $(M, \mathcal{O}_M)$ be a locally semialgebraic space over R and U an open subset of M. Then the locally semialgebraic maps from U to the semialgebraic standard space $(R, \mathcal{O}_R)$ are just the functions $f \in \mathcal{O}_M(U)$.

This is evident from Proposition 1.3 and the corresponding fact for semialgebraic spaces. Similarly, the locally semialgebraic maps from $(M, \mathcal{O}_M)$ to the semialgebraic space $(R^n, \mathcal{O}_{R^n})$ $(n \geq 1)$ are the n-tuples $(f_1, \ldots, f_n)$ of locally semialgebraic functions $f_1, \ldots, f_n$ on M.

Corollary 1.5. Let $(M,\mathcal{O}_M)$ be a semialgebraic space and $(N,\mathcal{O}_N)$ a locally semialgebraic space over R. Let $(N_\beta \mid \beta \in J)$ be an admissible covering of N by open semialgebraic subsets. Then a map $f: M \to N$ is locally semialgebraic if and only if there exists a finite subset J' of J such that f(M) is contained in the union N' of the sets N_β with $\beta \in J'$ and the map f from M to the open semialgebraic subspace N' of N is semialgebraic.

Again this is evident from Proposition 1.3. We will often call the locally semialgebraic maps from a semialgebraic space $(M,\mathcal{O}_M)$ to a locally semialgebraic space $(N,\mathcal{O}_N)$ the "semialgebraic maps from $(M,\mathcal{O}_M)$ to $(N,\mathcal{O}_N)$".

We are ready for a discussion of axiom (viii) in the definition of a generalized topological space (Def. 1). We have seen in the proof of Proposition 1.3 that this axiom is important to obtain a slick description of locally semialgebraic maps in terms of semialgebraic maps and spaces. On the other hand one may verify the following observation concerning our definition. Assume we had defined locally semialgebraic spaces using axioms (i) - (vii) omitting (viii), and that $(M,\mathring{\mathcal{T}}(M),Cov_M,\mathcal{O}_M)$ were a locally semialgebraic space in this new sense. Then we could obtain a locally semialgebraic space $(M,\mathring{\mathcal{T}}'(M),Cov'_M,\mathcal{O}'_M)$ in the old sense as follows. Let $\mathring{\gamma}(M)$ be the set of all $U \in \mathring{\mathcal{T}}(M)$ such that every $(U_\alpha \mid \alpha \in I) \in Cov_M(U)$ has a finite refinement. Define $\mathring{\mathcal{T}}'(M)$ as the set of all $U \subset M$ with $U \cap W \in \mathring{\gamma}(M)$ for every $W \in \mathring{\gamma}(M)$, and define Cov'_M as the set of all families $(U_\alpha \mid \alpha \in I)$ in $\mathring{\mathcal{T}}'(M)$ such that the union U of the U_α is an element of $\mathring{\mathcal{T}}'(M)$, and such that for every $W \in \mathring{\gamma}(M)$ the set $U \cap W$ can be covered by finitely many U_α. Then $(M,\mathring{\mathcal{T}}'(M),Cov'_M)$ fulfills all the axioms (i) - (viii). Moreover, every sheaf $\mathcal{F}$ on the site $(M,\mathring{\mathcal{T}}(M),Cov_M)$ extends uniquely to a sheaf $\mathcal{F}'$ on the new site $(M,\mathring{\mathcal{T}}'(M),Cov'_M)$. Clearly $(M,\mathring{\mathcal{T}}'(M),Cov'_M,\mathcal{O}'_M)|W$

$= (M, \mathring{\mathfrak{T}}(M), \mathrm{Cov}_M, \mathcal{O}_M)|W$ for every $W \in \mathring{\gamma}(M)$, and these spaces are semialgebraic. In particular $(M, \mathring{\mathfrak{T}}'(M), \mathrm{Cov}'_M, \mathcal{O}'_M)$ is locally semialgebraic in the old sense.

Thus, despite its importance, the axiom (viii) should be regarded as an axiom which does not restrict the generality of our concept of locally semialgebraic spaces.

§2 - Inductive limits; some examples of locally semialgebraic spaces.

The category of locally semialgebraic spaces over R contains the category of semialgebraic spaces over R as a full subcategory. The new category may well be regarded as an enlargement of the category of semialgebraic spaces established to guarantee the existence of suitable inductive limits of semialgebraic spaces.

We first remark that inductive limits exist in the category of generalized topological spaces (with continuous maps as morphisms) in complete generality. Indeed, let $((M_\alpha, \mathring{\mathfrak{T}}_\alpha, \mathrm{Cov}_\alpha) \mid \alpha \in I)$ be a diagram of generalized topological spaces, i.e. $\alpha \mapsto (M_\alpha, \mathring{\mathfrak{T}}_\alpha, \mathrm{Cov}_\alpha)$ is a functor from a small category I into the category of generalized topological spaces. Let M denote the inductive limit $\varinjlim_{\alpha \in I} M_\alpha$ of the sets M_α. For every $\alpha \in I$ let φ_α denote the canonical map from M_α to M. We define $\mathring{\mathfrak{T}}(M)$ as the set of all subsets U of M such that $\varphi_\alpha^{-1}(U) \in \mathring{\mathfrak{T}}_\alpha$ for every $\alpha \in I$, and we define Cov_M as the set of all families $(U_\lambda \mid \lambda \in \Lambda)$ in $\mathring{\mathfrak{T}}(M)$ such that $(\varphi_\alpha^{-1}(U_\lambda) \mid \lambda \in \Lambda) \in \mathrm{Cov}_\alpha$ for every $\alpha \in I$. It is easy to check that the triple $(M, \mathring{\mathfrak{T}}(M), \mathrm{Cov}_M)$ fulfills the axioms (i) - (viii) of Definition 1 in §1 and thus is a generalized topological space. By construction every map $\varphi_\alpha : M_\alpha \to M$ is a continuous map from $(M_\alpha, \mathring{\mathfrak{T}}_\alpha, \mathrm{Cov}_\alpha)$ to $(M, \mathring{\mathfrak{T}}_M, \mathrm{Cov}_M)$, and it is easy to see that our new space $(M, \mathring{\mathfrak{T}}_M, \mathrm{Cov}_M)$ is the direct limit of the given diagram of generalized topological spaces $(M_\alpha, \mathring{\mathfrak{T}}_\alpha, \mathrm{Cov}_\alpha)$ via the maps φ_α. Next assume, that every generalized topological space $M_\alpha = (M_\alpha, \mathring{\mathfrak{T}}_\alpha, \mathrm{Cov}_\alpha)$ is equipped with a sheaf $\mathcal{O}_\alpha$ of R-valued functions such that $((M_\alpha, \mathcal{O}_\alpha) \mid \alpha \in I)$ is a diagram in the category of locally semialgebraic spaces. We introduce on M a sheaf $\mathcal{O}_M$ of R-valued functions as follows. Let $U \in \mathring{\mathfrak{T}}(M)$ be given. Then $\mathcal{O}_M(U)$ is the R-algebra of all functions $f : U \to R$ such that for every $\alpha \in I$ the function

$$f \circ (\varphi_\alpha | \varphi_\alpha^{-1} U) : \varphi_\alpha^{-1}(U) \to R$$

is an element of $\mathcal{O}_\alpha(\varphi_\alpha^{-1}(U))$. It is easy to check that $(M,\mathcal{O}_M)$ is the inductive limit of the diagram $((M_\alpha,\mathcal{O}_\alpha)|\alpha \in I)$ in the category of ringed spaces over R. The question arises whether or not $(M,\mathcal{O}_M)$ is a locally semialgebraic space. If this is the case, then, of course, $(M,\mathcal{O}_M)$ is the inductive limit of the diagram in the category of locally semialgebraic spaces. We present a special case in which $(M,\mathcal{O}_M)$ is locally semialgebraic. This case will suffice for our purposes in this chapter.

<u>Lemma 2.1.</u> Assume that for the diagram $((M_\alpha,\mathcal{O}_\alpha)|\alpha \in I)$ of locally semialgebraic spaces over R the following four conditions hold:

a) The map $\varphi_\alpha : M_\alpha \to M$ is injective for every $\alpha \in I$

b) For every $U \in \mathring{\mathcal{T}}_\alpha$, with $\alpha \in I$, and for every $\beta \in I$ the set $\varphi_\beta^{-1}\varphi_\alpha(U)$ is an element of $\mathring{\mathcal{T}}_\beta$.

c) For every $(U_\lambda|\lambda\in\Lambda) \in \mathrm{Cov}_\alpha, \alpha \in I$, and for every $\beta \in I$ the family $(\varphi_\beta^{-1}\varphi_\alpha U_\lambda|\lambda\in\Lambda)$ is an element of Cov_β.

d) For every $U \in \mathring{\mathcal{T}}_\alpha, \alpha \in I$, and every $f \in \mathcal{O}_\alpha(U)$ the function $f \circ \varphi_\alpha^{-1} \circ (\varphi_\beta | \varphi_\beta^{-1}\varphi_\alpha U)$ is an element of $\mathcal{O}_\beta(\varphi_\beta^{-1}\varphi_\alpha U)$ for $\beta \in I$.

Then $\varphi_\alpha(M_\alpha) \in \mathring{\mathcal{T}}(M)$ for every $\alpha \in I$ and φ_α is an isomorphism from the ringed space $(M_\alpha,\mathcal{O}_\alpha)$ to the open subspace $(\varphi_\alpha(M_\alpha),\mathcal{O}_M|\varphi_\alpha(M_\alpha))$ of M. Moreover $(\varphi_\alpha(M_\alpha)|\alpha \in I) \in \mathrm{Cov}_M(M)$. Thus $(M,\mathcal{O}_M)$ is locally semialgebraic.

This is completely trivial. Now assume that our small category I is a partially ordered set (i.e. between any two objects α,β of I there exists at most one morphism $\alpha \to \beta$, and we write $\alpha \leq \beta$ if a morphism exists). Further assume that for any $\alpha,\beta \in I$ there exists some $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. In other words, $((M_\alpha,\mathcal{O}_\alpha)|\alpha \in I)$ is a "directed system" of locally semialgebraic spaces. Finally assume that in the diagram $((M_\alpha,\mathcal{O}_\alpha)|\alpha \in I)$ the transition map $\varphi_{\beta\alpha} : M_\alpha \to M_\beta$ is an isomorphism of $(M_\alpha,\mathcal{O}_\alpha)$ with an open subspace $(\varphi_{\beta\alpha}(M_\alpha),\mathcal{O}_\beta|\varphi_{\beta\alpha}(M_\alpha))$ of $(M_\beta,\mathcal{O}_\beta)$ for every pair (α,β) with $\alpha \leq \beta$. Then the conditions a), b), c), d) of the preceding lemma are clearly fulfilled. Replacing the M_α by their images $\varphi_\alpha(M_\alpha)$ in M we arrive at the following special case of Lemma 2.1.

Lemma 2.2. Let M be a set and let $(M_\alpha|\alpha\in I)$ be a directed system of subsets of M with $\cup(M_\alpha|\alpha\in I) = M$. ($M_\alpha \subset M_\beta$ if $\alpha \leq \beta$). Assume that every set M_α is equipped with the structure of a locally semialgebraic space $(M_\alpha, \mathcal{O}_\alpha)$ over R, and that for $\beta < \alpha$ the space $(M_\beta, \mathcal{O}_\beta)$ is an open subspace of $(M_\alpha, \mathcal{O}_\alpha)$. Let $(M, \mathcal{O}_M)$ be the inductive limit of the $(M_\alpha, \mathcal{O}_\alpha)$ in the category of ringed spaces over R. Then $(M_\alpha|\alpha\in I) \in \mathrm{Cov}_M(M)$ and every space $(M_\alpha, \mathcal{O}_\alpha)$ is an open subspace of $(M, \mathcal{O}_M)$. Thus M is a locally semialgebraic space over R.

Example 2.3. Let $(M, \mathcal{O}_M)$ be a locally semialgebraic space and let $(M_\alpha|\alpha\in I)$ be an admissible covering of M. Adding to the family $(M_\alpha|\alpha\in I)$ all unions of finitely many M_α we obtain a directed system $(N_\beta|\beta\in J)$ of open subsets of M. The space $(M, \mathcal{O}_M)$ is the inductive limit of the subspaces $(N_\beta, \mathcal{O}_M|N_\beta)$. In particular, every locally semialgebraic space is the inductive limit of a directed system of semialgebraic spaces.

Example 2.4 (Existence of direct sums). Let $((M_\alpha, \mathcal{O}_\alpha)|\alpha \in I)$ be a family of locally semialgebraic spaces over R. Then the disjoint union $\bigsqcup(M_\alpha|\alpha\in I) = M$ of the sets M_α may be equipped in a unique fashion with the structure of a locally semialgebraic space over R such that $(M_\alpha|\alpha\in I)$ is an element of $\mathrm{Cov}_M(M)$ and every $(M_\alpha, \mathcal{O}_\alpha)$ is an open subspace of M. This is the special case of Lemma 2.1 where the category I is discrete, i.e. has no morphisms except the identities. A subset U of M is open in M if and only if $U \cap M_\alpha$ is open in M_α for every $\alpha \in I$, and a family $(U_\lambda|\lambda\in\Lambda)$ of open subsets of M belongs to Cov_M if and only if $(U_\lambda \cap M_\alpha|\lambda \in \Lambda) \in \mathrm{Cov}_\alpha$ for every $\alpha \in I$. Clearly $(M, \mathcal{O}_M)$ is the direct sum of the family $((M_\alpha,\mathcal{O}_\alpha)\,|\,\alpha\in I)$ in the category of locally semialgebraic spaces. Notice that, in the special case in which all $(M_\alpha,\mathcal{O}_\alpha)$ are semialgebraic, the space $(M,\mathcal{O}_M)$ is semialgebraic if and only if I is finite.

<u>Example 2.5</u> (Existence of direct products). Let $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$ be locally semialgebraic spaces over R. We equip the cartesian product $M\times N$ of the sets M,N with the structure of a locally semialgebraic space as follows. We first choose admissible coverings $(M_\alpha\,|\,\alpha\in I)$ and $(N_\beta\,|\,\beta\in J)$ of M and N by open semialgebraic subsets. By adding all unions of finitely many sets M_α to the family $(M_\alpha\,|\,\alpha\in I)$ we obtain an admissible covering $(M_\alpha\,|\,\alpha\in\tilde{I})$ of M which is a directed system of open semialgebraic subsets of M.

Similarly, we pass from $(N_\beta\,|\,\beta\in J)$ to an admissible covering $(N_\beta\,|\,\beta\in\tilde{J})$ of N which is a directed system of semialgebraic subsets of N. On every set $M_\alpha\times N_\beta$ with $(\alpha,\beta)\in\tilde{I}\times\tilde{J}$ we consider the direct product $(M_\alpha\times N_\beta,\mathcal{O}_{\alpha\beta})$ of the semialgebraic spaces $(M_\alpha,\mathcal{O}_M|M_\alpha)$ and $(N_\beta,\mathcal{O}_N|N_\beta)$ over R, as defined in [DK_2, Th. 7.3]. We then equip the set $M\times N$ with the structure of the ringed space over R which is the inductive limit of the directed system of semialgebraic spaces $((M_\alpha\times N_\beta,\mathcal{O}_{\alpha\beta})\,|\,(\alpha,\beta)\in\tilde{I}\times\tilde{J})$. According to Lemma 2.2, this space $(M\times N,\mathcal{O}_{M\times N})$ is locally semialgebraic. Moreover, $(M_\alpha\times N_\beta\,|\,(\alpha,\beta)\in I\times J)$ is an admissible covering of $M\times N$. Clearly this ringed space $(M\times N,\mathcal{O}_{M\times N})$ over R is the <u>unique</u> one for which $(M_\alpha\times N_\beta\,|\,(\alpha,\beta)\in I\times J)$ is an admissible covering of $M\times N$ and $(M_\alpha\times N_\beta,\mathcal{O}_{\alpha\beta})$ is an open subspace of $(M\times N,\mathcal{O}_{M\times N})$ for every $(\alpha,\beta)\in I\times J$. The natural projections $p: M\times N\to M$ and $q: M\times N\to N$ are locally semialgebraic maps. We claim that for any two locally semialgebraic maps $f: L\to M$ and $g: L\to N$ from a locally semialgebraic space $(L,\mathcal{O}_L)$ to $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$ the composite map $(f,g): L\to M\times N$ is locally semialgebraic from $(L,\mathcal{O}_L)$ to $(M\times N,\mathcal{O}_{M\times N})$. Indeed, the coverings $(f^{-1}(M_\alpha)\,|\,\alpha\in I)$, $(g^{-1}(N_\beta)\,|\,\beta\in J)$ of L are admissible. Thus by axioms v) and vi) in §1, Definition 1,

$(f^{-1}(M_\alpha) \cap g^{-1}(N_\beta) \mid (\alpha,\beta) \in I\times J)$ is also an admissible covering of L. Using axiom vi) we may find an admissible covering $(L_\gamma \mid \gamma \in K)$ of L by open semialgebraic sets L_γ refining this covering. If $L_\gamma \subset f^{-1}(M_\alpha) \cap g^{-1}(N_\beta)$ then $(f,g)\mid L_\gamma$ is a semialgebraic map from L_γ to the semialgebraic space $M_\alpha \times N_\beta$ since the components $f\mid L_\gamma : L_\gamma \to M_\alpha$ and $g\mid L_\gamma : L_\gamma \to N_\beta$ of this map are semialgebraic. Thus (f,g) is locally semialgebraic. All this means that $(M\times N, \mathcal{O}_{M\times N})$ is the direct product of the spaces $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$ in the category of locally semialgebraic spaces over R via the projection maps p and q. In particular the space structure $(M\times N, \mathcal{O}_{M\times N})$ on the set $M\times N$ does not depend on the choice of the coverings $(M_\alpha \mid \alpha \in I)$ and $(N_\beta \mid \beta \in J)$.

<u>Example 2.6.</u> Let $(M,\mathcal{O}_M)$ be an affine <u>locally complete</u> semialgebraic space over R. The latter condition means that every point of M has a semialgebraic neighbourhood which is a complete semialgebraic space*). If $(M,\mathcal{O}_M)$ is embedded in some R^n then M is a locally closed semialgebraic subset of R^n [DK_5, §3]. We choose an embedding $M \subset R^n$ with M bounded in R^n. Let $d : \overline{M} \to R$ denote the function on the closure $\overline{M}$ of M giving the distance from $\partial M := \overline{M} \setminus M$, i.e. $d(x) = \min(\| x-y \| \mid y \in \partial M)$ with $\| \ \|$ the euclidean norm on R^n. (N.B. The minimum exists, since ∂M is complete. d is a semialgebraic function on $\overline{M}$.) Let $\mathfrak{k}$ denote the directed system of complete semialgebraic subsets K of M. For every $K \in \mathfrak{k}$ let $(K,\mathcal{O}_K)$ denote the semialgebraic subspace of $(M,\mathcal{O}_M)$ corresponding to K, cf. [DK_2, p. 186]. Then $((K,\mathcal{O}_K) \mid K \in \mathfrak{k})$ is a directed system of semialgebraic spaces. Let $(M_{loc}, \mathcal{O}_M^{loc})$ denote the inductive limit of this system in the category of ringed spaces over R. Notice that this space has the same underlying set M as the original space $(M,\mathcal{O}_M)$ but that the topology of M_{loc} is different. We claim that $(M_{loc}, \mathcal{O}_M^{loc})$

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*) Actually every locally complete semialgebraic space M is affine, cf. §7.

is locally semialgebraic. In order to verify this we introduce the open semialgebraic subset

$$M(\varepsilon) = \{x \in M \mid d(x) > \varepsilon\}$$

of M for every $\varepsilon > 0$ in R. Notice that every set $K \in \mathfrak{k}$ is contained in some set $M(\varepsilon)$ and that every $M(\varepsilon)$ is contained in some $K \in \mathfrak{k}$, namely in the closure $\overline{M(\varepsilon)}$ of $M(\varepsilon)$ in M. (N.B. $d(x) > 0$ for every $x \in M$.) The directed system $((M(\varepsilon), \mathcal{O}_{M(\varepsilon)}) \mid \varepsilon > 0)$ of open subspaces of M is "equivalent" to the system $((K, \mathcal{O}_K) \mid K \in \mathfrak{k})$. Thus $(M_{loc}, \mathcal{O}_M{}^{loc})$ is also the inductive limit of the spaces $(M(\varepsilon), \mathcal{O}_{M(\varepsilon)})$. We infer from Lemma 2.2 that $(M_{loc}, \mathcal{O}_M{}^{loc})$ is indeed locally semialgebraic, and moreover, that $(M(\varepsilon) \mid \varepsilon > 0)$ is an admissible covering of this space by open semialgebraic subsets. Up to now we tacitly assumed that M is not complete since otherwise d is not defined. If M is complete we put $(M_{loc}, \mathcal{O}_M{}^{loc}) = (M, \mathcal{O}_M)$.

We observed in §1 that for any locally semialgebraic space $(N, \mathcal{O}_N)$ the locally semialgebraic maps from $(N, \mathcal{O}_N)$ to the semialgebraic space $(R, \mathcal{O}_R)$ are just the functions $f \in \mathcal{O}_N(N)$. What are the locally semialgebraic maps from $(N, \mathcal{O}_N)$ to $(R_{loc}, \mathcal{O}_R{}^{loc})$? R_{loc} has the admissible covering $(]-c,c[\mid c > 0)$ with c running through the positive elements of R. Here as usual $]-c,c[$ denotes the interval $\{x \in R \mid -c < x < c\}$. Thus it follows from Proposition 1.3 that a function $f: N \to R$ is a locally semialgebraic map from $(N, \mathcal{O}_N)$ to $(R_{loc}, \mathcal{O}_R{}^{loc})$ if and only if $f \in \mathcal{O}_N(N)$ and, in addition, f is bounded on every open semialgebraic subset of N.

Example 2.7. Let $(M, \mathcal{O}_M)$ be a semialgebraic space or more generally a locally semialgebraic space, and let X be a subset of M which is a union of open semialgebraic subsets of M. {In the terminology to be developed in §3 this means that X is an open subset of M in the strong topology}. We choose some family $(X_\alpha \mid \alpha \in I)$ in $\mathring{\gamma}(M)$ such that the following holds:

(*) Every X_α is contained in X and every $W \in \mathring{\gamma}(M)$ with $W \subset X$ is contained in the union of finitely many X_α.

For example we may take as family $(X_\alpha | \alpha \in I)$ the family of all $W \in \mathring{\gamma}(M)$ with $W \subset X$. As a consequence of (*) the set X is the union of all X_α. Let $(Y_\beta | \beta \in J)$ denote the directed system of all finite unions of sets X_α ($\beta \leq \gamma$ if $Y_\beta \subset Y_\gamma$). Let $(X, \mathcal{O}_X)$ denote the inductive limit of the directed system of semialgebraic spaces $((Y_\beta, \mathcal{O}_M | Y_\beta) | \beta \in J)$. By Lemma 2.2 the space $(X, \mathcal{O}_X)$ is locally semialgebraic. Clearly $(Y_\beta | \beta \in J)$ is an admissible covering of X and thus also $(X_\alpha | \alpha \in I)$ is an admissible covering of X with respect to this space structure.

If we replace $(X_\alpha | \alpha \in I)$ by the larger family of all $W \in \mathring{\gamma}(M)$ with $W \subset X$, then it follows easily from the condition (*) that the space structure on X obtained from the new family is the same as the old one. Thus the space $(X, \mathcal{O}_X)$ does not depend on the choice of the family $(X_\alpha | \alpha \in I)$. We call $(X, \mathcal{O}_X)$ the locally semialgebraic space <u>induced on</u> X by the space $(M, \mathcal{O}_M)$. Of course, if $X \in \mathring{\gamma}(M)$ then this space is just the open subspace $(X, \mathcal{O}_M | X)$ of $(M, \mathcal{O}_M)$.

<u>Subexample 2.8.</u> Let $(M, \mathcal{O}_M)$ be a semialgebraic space and let $f : M \to R$ be some semialgebraic function on M. For every $r > 0$ in R we introduce the open semialgebraic subset

$$M_r := \{x \in M | -r < f(x) < r\}$$

of M. Let $v : R^* \twoheadrightarrow \Gamma$ be some non-trivial valuation of R compatible with the unique ordering of R (additive notation; cf. [P, §7], [L, §5], or [KW]). We fix some $\lambda \in \Gamma$ and consider the subets

$$X := \{x \in M | v(f(x)) \geq \lambda\} \quad \text{and}$$
$$Y := \{x \in M | v(f(x)) > \lambda\}$$

of M. Then X is the union of the family $(M_r | r > 0, v(r) \geq \lambda)$ and Y is the union of the family $(M_r \mid r > 0, v(r) > \lambda)$. It is not difficult to verify that both these families fulfill the condition (*) in 2.7. (Notice that for every $W \in \mathring{\mathfrak{o}}(M)$ the set $f(W)$ is semialgebraic in R. If $W \subset X$ (resp. $W \subset Y$), then $f(W)$ is necessarily bounded in R, since $v(x) \geq \lambda$ (resp. $v(x) > \lambda$) for every $x \in f(W)$ and v is non-trivial.)

Example 2.9. The examples 2.6 and 2.7 can be regarded as special cases of a more general construction. Let $(M, \mathcal{O}_M)$ be a locally semialgebraic space and let $(X_\alpha \mid \alpha \in I)$ be an arbitrary family in $\mathring{\mathfrak{o}}(M)$. Then the unions of finitely many X_α constitute a directed system of open subspaces of M, which fulfill the conditions of Lemma 2.2. Thus the inductive limit of this system is a locally semialgebraic space $(X, \mathcal{O}_X)$. The space $(X, \mathcal{O}_X)$ has the following description: The set X is the union of the sets X_α. $\mathring{\mathfrak{o}}(X)$ is the set of all $U \in \mathring{\mathfrak{o}}(M)$ which can be covered by finitely many X_α, and $\mathring{\mathfrak{T}}(X)$ is the set of all subsets U of X such that $U \cap X_\alpha \in \mathring{\mathfrak{o}}(M)$ for every $\alpha \in I$. A family $(U_\lambda \mid \lambda \in \Lambda)$ in $\mathring{\mathfrak{T}}(X)$ with union U belongs to Cov_X if and only if $U \cap X_\alpha$ is covered by finitely many sets $U_\lambda \cap X_\alpha$ for every $\alpha \in I$. In particular $(X_\alpha \mid \alpha \in I)$ is an admissible covering of X. Finally $\mathcal{O}_X \mid X_\alpha = \mathcal{O}_M \mid X_\alpha$ for every $\alpha \in I$.

Example 2.7 is the special case that

$$\mathring{\mathfrak{o}}(X) = \{U \in \mathring{\mathfrak{o}}(M) \mid U \subset X\}.$$

In general the set $\mathring{\mathfrak{o}}(X)$ may be smaller than the set on the right hand side. For instance, let M be a locally complete - but not complete - semialgebraic space and $(X_\alpha \mid \alpha \in I)$ be the family of all open semialgebraic subsets U of M with $\overline{U}$ complete, indexed in some way. Then $X = M$, but $\mathring{\mathfrak{o}}(X)$ is only the set of all these subsets U. We obtain the space $(X, \mathcal{O}_X) = (M_{loc}, \mathcal{O}_M^{loc})$ considered in 2.6. The spaces in 2.6 and 2.7 are particularly nice since they are determined by the set X and the space structure on M.

Example 2.10 (Base field extension). Let $\tilde{R}$ be a real closed field extension of R. Then from any locally semialgebraic space $(M,\mathcal{O}_M)$ over R we obtain a locally semialgebraic space $(M(\tilde{R}),\tilde{\mathcal{O}}_M)$ over $\tilde{R}$ in the following way. We start by choosing an admissible covering $(M_\alpha|\alpha\in I)$ of M by open semialgebraic subsets which is a directed system ($M_\alpha \subset M_\beta$ if $\alpha < \beta$, cf. Example 2.3). Then we have the directed system $((M_\alpha,\mathcal{O}_\alpha)|\alpha\in I)$ of semialgebraic spaces over R with $\mathcal{O}_\alpha = \mathcal{O}_M|M_\alpha$. By base extension from R to $\tilde{R}$ ([D, §9], [DK_3, §4]), we obtain a directed system $((M_\alpha(\tilde{R}),\tilde{\mathcal{O}}_\alpha)|\alpha\in I)$ of semialgebraic spaces which fulfills the conditions of Lemma 2.1. We then define $(M(\tilde{R}),\tilde{\mathcal{O}}_M)$ as the inductive limit of this system. Having constructed this space we easily see that $(M(\tilde{R}),\tilde{\mathcal{O}}_M)$ is the inductive limit of the directed system of base extensions of all open semialgebraic subspaces of M. In particular $(M(\tilde{R}),\tilde{\mathcal{O}}_M)$ does not depend on the choice of the covering $(M_\alpha|\alpha\in I)$. We call this space "the space over $\tilde{R}$ obtained from $(M,\mathcal{O}_M)$ by base extension". For every $W \in \mathring{\mathfrak{T}}(M)$ we identify the set $W(\tilde{R})$ with its image in $M(\tilde{R})$, which is an open subset of $M(\tilde{R})$. Clearly every family $(U_\lambda|\lambda\in\Lambda)$ in Cov_M yields a family $(U_\lambda(\tilde{R})|\lambda\in\Lambda)$ in $\mathrm{Cov}_{M(\tilde{R})}$, but in general $\mathrm{Cov}_{M(\tilde{R})}$ also contains families which are not "defined over R". Every M_α can be regarded as a subset of $M_\alpha(\tilde{R})$ in the natural way (loc.cit.). Thus also M can - and will - be regarded as a subset of $M(\tilde{R})$. For every $W \in \mathring{\mathfrak{T}}(M)$ we have $W(\tilde{R}) \cap M = W$.

Starting with our analysis of locally semialgebraic maps in terms of semialgebraic maps (Proposition 1.3) and using the theory of base extension of semialgebraic maps [loc.cit.], it is evident how to obtain a locally semialgebraic map

$$f_{\tilde{R}} : (M(\tilde{R}),\mathcal{O}_{M(\tilde{R})}) \to (N(\tilde{R}),\mathcal{O}_{N(\tilde{R})})$$

over $\tilde{R}$ from any locally semialgebraic map $f : (M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ over R in a canonical way. Thus we have a functor "base field extension" from the category of locally semialgebraic spaces over R to the category of locally semialgebraic spaces over $\tilde{R}$. This functor preserves direct sums and finite direct products.

All this sounds very simple. Nevertheless some caution is indicated.

Example 2.11. Let M be a locally complete semialgebraic space over R. Then $M(\tilde{R})$ is a locally complete semialgebraic space over $\tilde{R}$. The base extension $M_{loc}(\tilde{R})$ of the space M_{loc} defined in Example 2.6 is usually different from $M(\tilde{R})_{loc}$. Indeed, $M_{loc}(\tilde{R})$ is the inductive limit of the spaces $M(\varepsilon)(\tilde{R}) = M(\tilde{R})(\varepsilon)$ with ε running through the positive elements of R (notation from 2.6). On the other hand $M(\tilde{R})_{loc}$ is the inductive limit of the spaces $M(\tilde{R})(\varepsilon)$ with ε running through the positive elements of $\tilde{R}$. The space $M_{loc}(\tilde{R})$ can be obtained from $M(\tilde{R})_{loc}$ by the process described in Example 2.9.

In order to explicate our last example we need some definitions of a combinatorial nature, cf. [DK_3, §2]. Recall that an open simplex in some vector space V over R is a set

$$\sigma = \{\sum_{i=0}^{n} t_i e_i \mid t_i \in R, t_i > 0, \sum_{i=0}^{n} t_i = 1\}$$

with affinely independent points $e_0,\ldots,e_n$ of V, which are called the vertices of σ. The closure $\bar{\sigma}$ of σ is defined as the convex hull of $e_0,\ldots,e_n$ in V. This notion of closure is purely combinatorial (since we do not equip V with a topology). But the vector space $V(\sigma)$ spanned by σ is finite dimensional and thus is an affine semialgebraic space in a canonical way. $\bar{\sigma}$ is the closure of the semialgebraic set σ in $V(\sigma)$.

Definition 1. A (geometric) simplicial complex over R is a pair $(X,\Sigma(X))$ consisting of a subset X of some vector space V over R and a family $\Sigma(X)$ of open simplices σ in V such that the following two properties hold:

i) X is the disjoint union of all $\sigma \in \Sigma(X)$.

ii) The intersection $\bar{\sigma} \cap \bar{\tau}$ of the closures of any two simplices $\sigma,\tau \in \Sigma(X)$ is either empty or a face of both $\bar{\sigma}$ and $\bar{\tau}$.

The closure of the complex $(X,\Sigma(X))$ is the pair $(\bar{X},\Sigma(\bar{X}))$, with $\Sigma(\bar{X})$ the set of all open faces of all $\sigma \in \Sigma(X)$ and $\bar{X}$ the union of all $\tau \in \Sigma(\bar{X})$. This is another simplicial complex. Of course, we call the complex $(X,\Sigma(X))$ closed if $X = \bar{X}$, or equivalently if $\Sigma(X) = \Sigma(\bar{X})$.

Remarks.

1) Our notion of a simplicial complex differs slightly from the classical one. (Classical simplicial complexes are what we call closed simplicial complexes.) The reason for this is that combinatorial semialgebraic topology should emphasize at least as much open simplices as closed simplices. In classical combinatorial topology an open simplex σ is usually regarded as a simplicial complex consisting of infinitely many closed simplices, but in our theory this would confuse the semialgebraic space σ with the locally semialgebraic space σ_{loc} introduced in Example 2.6.

2) The complete simplicial complexes, as defined in [DK_3, §2], are the closed and finite (cf. Def. 3 below) simplicial complexes in our present terminology.

We will often denote a simplicial complex $(X,\Sigma(X))$ simply by the letter X.

Definition 2. A simplicial complex Y is called a subcomplex of X if $Y \subset X$ and $\Sigma(Y) \subset \Sigma(X)$. For any two subcomplexes Y_1, Y_2 of X the intersection $Y_1 \cap Y_2$ is the complex $(Y_1 \cap Y_2, \Sigma(Y_1) \cap \Sigma(Y_2))$. A subcomplex Y of X is called closed in X, if Y is the intersection of the subcomplexes $\bar{Y}$ and X of $\bar{X}$. This means that for every $\sigma \in \Sigma(Y)$ every open face τ of σ which is an element of $\Sigma(X)$ is also an element of $\Sigma(Y)$. A subcomplex Y of X is called open in X if the complex $X \setminus Y$ is closed in X. Here $X \setminus Y$ is defined as the subcomplex Z with $\Sigma(Z) = \Sigma(X) \setminus \Sigma(Y)$, hence $Z = X \setminus Y$ as a set.

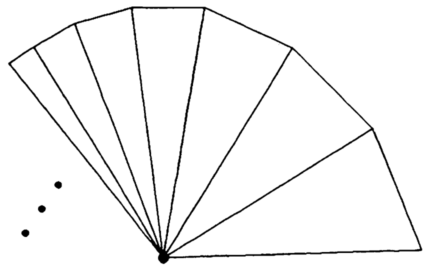
For any subset A of X the smallest open subcomplex of X containing A is called the star $St_X(A)$ of A in X. It consists of all $\sigma \in \Sigma(X)$ with $\bar{\sigma} \cap A \neq \emptyset$.

Definition 3. A simplicial complex X is called finite if $\Sigma(X)$ is a finite set. X is called locally finite, if every $\sigma \in \Sigma(X)$ is contained in a finite open subcomplex of X, i.e. if $St_X(\sigma)$ is finite for every $\sigma \in \Sigma(X)$.

Clearly X is finite if and only if $\bar{X}$ is finite. But it may happen that X is locally finite and $\bar{X}$ is not. When considering triangulations of locally semialgebraic spaces one usually has to avoid this slightly pathological situation. Thus we insert here yet another definition for later purpose.

Definition 4. X is called strictly locally finite if the complex $\bar{X}$ is locally finite. This means that every vertex of $\bar{X}$ is a vertex of only finitely many $\sigma \in \Sigma(X)$.

Here is an example of a simplicial complex X which is locally finite but not strictly locally finite:



$\overline{X}$ is an infinite collection of adjacent closed triangles with one common vertex p, and $X = \overline{X} \setminus \{p\}$.

We now consider "simplexwise affine" maps between simplicial complexes. There seem to be two natural definitions.

Definition 5. Let $(X, \Sigma(X))$ and $(Y, \Sigma(Y))$ be simplicial complexes in vector spaces V and W respectively. A weakly simplicial map f from $(X, \Sigma(X))$ to $(Y, \Sigma(Y))$ is a map f from the set X to the set Y with the following two properties:

i) For every $\sigma \in \Sigma(X)$ the image $f(\sigma)$ is an element τ of $\Sigma(Y)$ and $f|\sigma:\sigma \to \tau$ is the restriction of an affine map $\overline{f}_\sigma$ from $\overline{\sigma}$ to $\overline{\tau}$, which of course is uniquely determined by f.

ii) If $\tau \in \Sigma(X)$ is an open face of $\sigma \in \Sigma(X)$ then $\overline{f}_\sigma|\tau = f|\tau$.

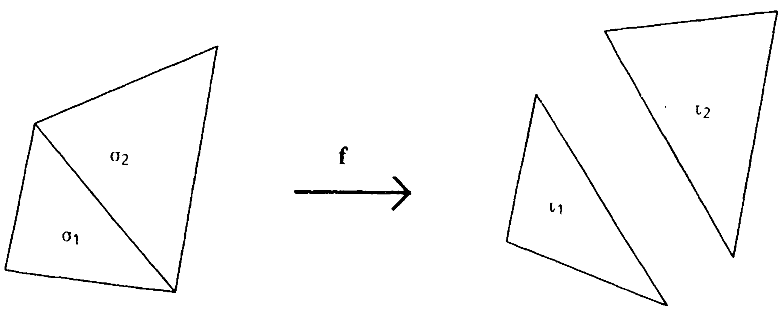
The first property can also be expressed as follows: If σ has the vertices $e_0, \ldots, e_n$ then

$$f\left(\sum_{i=0}^{n} t_i e_i\right) = \sum_{i=0}^{n} t_i e_i' \qquad \left(t_i > 0, \sum_{i=0}^{n} t_i = 1\right)$$

with e_i' running through all the vertices of some $\tau \in \Sigma(Y)$, possibly with repetitions.

Definition 6. A simplicial map f from the complex $(X,\Sigma(X))$ to $(Y,\Sigma(Y))$ is a map f from the set X to the set Y which extends to a weakly simplicial map $\overline{f}:\overline{X}\to\overline{Y}$ from $(\overline{X},\Sigma(\overline{X}))$ to $(\overline{Y},\Sigma(\overline{Y}))$. Of course, $\overline{f}$ is uniquely determined by f. We call $\overline{f}$ the closure of the simplicial map f.

If X is closed then the weakly simplicial maps and the simplicial maps from X to Y coincide. The image of such a map f is a subcomplex Z of Y which is closed, and $f:X\to Z$ is a simplicial map in the traditional sense. But in general not every weakly simplicial map is simplicial.



Here is a picture of a weakly simplicial map which is not simplicial. X consists of two open triangles σ_1 and σ_2 which have a common face (not belonging to X). Y consists of two open triangles τ_1 and τ_2 without a common face. f maps σ_1 affinely onto τ_1 and σ_2 affinely onto τ_2.

Remark. For most considerations concerning triangulations of locally semialgebraic spaces the useful maps are the simplicial maps and not the weakly simplicial maps.

We now are ready to present our last example of a locally semialgebraic space.

Example 2.12. Let X be a locally finite simplicial complex in some vector space V over R. The set of all finite open subcomplexes of X can be regarded as a directed system $(X_\lambda | \lambda \in \Lambda)$ of subsets of X with X the union of all X_λ ($\mu \leq \lambda$ if $X_\mu \subset X_\lambda$). Every set X_λ spans a finite dimensional vector space V_λ in V. Every V_λ is an affine semialgebraic space in a canonical way, and, as a semialgebraic subset of V_λ, X_λ also becomes an affine semialgebraic space in a canonical way. If $\mu \leq \lambda$ then V_μ is a (closed) semialgebraic subspace of V_λ (in the sense of [DK_2, §7]). Thus X_μ is a semialgebraic subspace of V_λ and hence also a semialgebraic subspace of X_λ, since $X_\mu \subset X_\lambda$. Now X_μ is an open subcomplex of X_λ. Thus the set X_μ is open in X_λ in the strong topology, which implies that X_μ is an open semialgebraic subspace of X_λ [DK_2, Prop. 7.5]. We equip X with the structure of the inductive limit of the directed system of semialgebraic spaces $(X_\lambda | \lambda \in \Lambda)$. By Lemma 2.2 X is locally semialgebraic.

We also consider the directed system $(X_\kappa | \kappa \in K)$ of all finite subcomplexes $(K \supset \Lambda)$ in X. Again, every X_κ is an affine semialgebraic space and X_κ is a subspace of X_ρ if $\kappa \leq \rho$. Since the directed system $(X_\lambda | \lambda \in \Lambda)$ is cofinal in the new directed system $(X_\kappa | \kappa \in K)$, we see that our space X is also the inductive limit of the system of spaces $(X_\kappa | \kappa \in K)$.

Clearly $(St_X(\sigma) | \sigma \in \Sigma(X))$ is an admissible covering of X by open semialgebraic subsets. Using this fact we see that X is a semialgebraic space if and only if the complex X is finite.

If $f: X \to Y$ is a weakly simplicial map from X to a locally finite simplicial complex Y then it is easy to check that f is a locally semialgebraic map between the spaces X and Y. More precisely the weakly simplicial maps from X to Y are those locally semialgebraic maps from X to Y which map open simplices affinely onto open simplices.

Finite simplicial complexes are extremely important in the theory of semialgebraic spaces, since the following result holds which has been proved in [DK_3, §2] and will also appear in [BCR, Chap. 9] (cf. [Hi], [Lo], [Gi] for the case $R = \mathbb{R}$).

<u>Theorem 2.13.</u> For any affine semialgebraic space $(M, \mathcal{O}_M)$ and finitely many semialgebraic subsets $A_1, \dots, A_r$ of M there exists a finite simplicial complex X together with subcomplexes $Y_1, \dots, Y_r$ and a semialgebraic isomorphism $\varphi : (X, \mathcal{O}_X) \to (M, \mathcal{O}_M)$ with $\varphi(Y_i) = A_i$ for $i = 1, \dots, r$.

We call such an isomorphism φ a <u>simultaneous triangulation</u> of M, $A_1, \dots, A_r$. In the next chapter we will generalize this theorem to a triangulation theorem for a prominent class of locally semialgebraic spaces using strictly locally finite simplicial complexes.

§3 - Locally semialgebraic subsets

From now on a locally semialgebraic space $(M,\mathcal{O}_M)$ will often be denoted by just the letter M.

Let M be a locally semialgebraic space over R and let $(M_\alpha|\alpha\in I)$ be a fixed admissible covering of M by open semialgebraic subsets.

Definition 1. A subset X of M is called locally semialgebraic if, for every $W\in \mathring{\mathfrak{r}}(M)$, the set $X\cap W$ is semialgebraic in the semialgebraic space W. This already holds if $X\cap M_\alpha$ is semialgebraic in M_α for every $\alpha\in I$. The set of all locally semialgebraic subsets of M is denoted by $\mathcal{T}(M)$.

It is evident that $\mathcal{T}(M)$ has the following three properties:

1) $\mathring{\mathcal{T}}(M)\subset \mathcal{T}(M)$.

2) $X\in\mathcal{T}(M) \Rightarrow M\setminus X\in\mathcal{T}(M)$

3) The union and the intersection of any locally finite family $(X_\lambda|\lambda\in\Lambda)$ in $\mathcal{T}(M)$ (cf. Definition 4 in §1) is an element of $\mathcal{T}(M)$.

Proposition 3.1. Let $f: M\to N$ be a locally semialgebraic map between locally semialgebraic spaces. Then the preimage $f^{-1}(Y)$ of any locally semialgebraic subset Y of N is a locally semialgebraic subset of M.

Proof. Let $W\in\mathring{\mathfrak{r}}(M)$ be given. There exists some $V\in\mathring{\mathfrak{r}}(N)$ with $f(W)\subset V$. The map $g: W\to V$ obtained from f by restriction is semialgebraic. Since $W\cap f^{-1}(Y) = g^{-1}(V\cap Y)$ and $V\cap Y$ is semialgebraic we conclude that $W\cap f^{-1}(Y)$ is semialgebraic. q.e.d.

Caution. In contrast to the semialgebraic theory the image of a locally semialgebraic set under a locally semialgebraic map is not necessarily locally semialgebraic, cf. 6.3.a below.

We now endow a given set $X \in \mathcal{T}(M)$ with the structure of a locally semialgebraic space. We enlarge our admissible covering $(M_\alpha | \alpha \in I)$ of M to the directed system of open semialgebraic subsets $(N_\beta | \beta \in J)$ consisting of all finite unions of sets M_α. Notice that $(N_\beta | \beta \in J)$ is again an admissible covering of M. For every $\beta \in J$ the intersection $N_\beta \cap X$ will be regarded as a semialgebraic subspace of the semialgebraic space N_β [DK_2, §7]. Let $(X, \mathcal{O}_X)$ be the inductive limit of the directed system of semialgebraic spaces $(N_\beta \cap X | \beta \in J)$. By Lemma 2.2 $(X, \mathcal{O}_X)$ is locally semialgebraic. We call these spaces $(X, \mathcal{O}_X)$ the (locally semialgebraic) <u>subspaces of</u> $(M, \mathcal{O}_M)$. In the special case that $X \in \mathring{\mathcal{T}}(M)$ the space $(X, \mathcal{O}_X)$ is just the open subspace $(X, \mathcal{O}_M | X)$ of $(M, \mathcal{O}_M)$ as considered in §1.

Clearly the inclusion map $j: X \to M$ is a locally semialgebraic map from $(X, \mathcal{O}_X)$ to $(M, \mathcal{O}_M)$. Using our analysis of locally semialgebraic maps in terms of semialgebraic maps (Proposition 1.3) it is a trivial matter to prove

<u>Proposition 3.2.</u> Let $f: L \to M$ be a locally semialgebraic map from a locally semialgebraic space L to M. Assume that $f(L) \subset X$. Then the map $g: L \to X$ obtained from f by restriction of the image space is a locally semialgebraic map from L to the subspace X of M.

In particular, our space $(X, \mathcal{O}_X)$ does not depend on the choice of the admissible covering $(M_\alpha | \alpha \in I)$ of M.

<u>Example 3.3.</u> If Y is a subcomplex of a locally finite simplicial complex X, then Y is a subspace of X, where both X and Y have their canonical structure as locally semialgebraic spaces (Example 2.12).

In every one of the following four propositions the first statement is an immediate consequence of the analogous statement for semialgebraic spaces, and the second statement then follows from Proposition 3.2 in a purely formal way.

<u>Proposition 3.4.</u> Let X be a locally semialgebraic subset of M and let Y be a subset of X. Then $Y \in \mathcal{T}(X)$ if and only if $Y \in \mathcal{T}(M)$. In this case the subspace structure on Y with respect to X is the same as the subspace structure with respect to M.

<u>Proposition 3.5</u> (Existence of fibre products). Let $\varphi : M \to S$, $\psi : N \to S$ be locally semialgebraic maps over R. The subset $M \times_S N$ of $M \times N$ consisting of all pairs (x,y) in $M \times N$ with $\varphi(x) = \psi(y)$ is a locally semialgebraic subset of $M \times N$ and hence a locally semialgebraic space over R. The diagram

$$\begin{tikzcd} M \times_S N \arrow[r, "q"] \arrow[d, "p"'] & N \arrow[d, "\psi"] \\ M \arrow[r, "\varphi"'] & S \end{tikzcd}$$

where p and q are respectively the restrictions of the projections from $M \times N$ to M and N, is a pull back in the category of locally semialgebraic spaces.

<u>Proposition 3.6.</u> With notation as in Proposition 3.5, let X and Y be respectively locally semialgebraic subsets of M and N. Then the subset

$$X \times_S Y = \{(x,y) \in X \times Y \mid \varphi(x) = \psi(y)\}$$

of $M \times_S N$ is locally semialgebraic in the space $M \times_S N$. The subspace structure on this set with respect to $M \times_S N$ coincides with the structure as the fibre product of X and Y with respect to the locally semialgebraic maps $\varphi|X$ and $\psi|Y$.

Proposiion 3.7. Let $f: M \to N$ be a locally semialgebraic map. Then the graph $\Gamma f)$ is a locally semialgebraic subset of $M \times N$. The map $x \mapsto (x, f(x))$ from M o $\Gamma(f)$ is an isomorphism from the locally semialgebraic space M to th subspace $\Gamma(f)$ of $M \times N$.

We now ook for locally semialgebraic subsets which are "small".

Definiton 2. Let M be a locally semialgebraic space over R. A subset X of M s called semialgebraic, if X is locally semialgebraic and if the subpace $(X, \mathcal{O}_X)$ of M is a semialgebraic space. The set of all semialgbraic subsets of M is denoted by $\gamma(M)$.

Clearlyour previous set $\mathring{\gamma}(M)$ is contained in $\gamma(M)$.

Proposiion 3.8. A locally semialgebraic subset X of M is semialgebraic if and nly if X is contained in some $W \in \mathring{\gamma}(M)$.

Proof. f X is contained in a set $W \in \mathring{\gamma}(M)$, then by Proposition 3.4 the space $(, \mathcal{O}_X)$ is a subspace of the semialgebraic space $(W, \mathcal{O}_W)$ and thus $(X, \mathcal{O}_X)$ tself is semialgebraic. Assume now that $X \in \gamma(M)$. Let $(M_\alpha | \alpha \in I)$ be an amissible covering of M by open semialgebraic subsets. Then $(M_\alpha \cap X | \in I)$ is an admissible covering of the subspace X of M. Since this spce is semialgebraic it is already covered by finitely many sets $M_\alpha \cap X$. The union W of the corresponding sets M_α is an element of $\mathring{\gamma}(M)$ an contains X. q.e.d.

Using tis proposition one easily verifies

Corollay 3.9. $\gamma(M)$ is the smallest set of subsets of M with the followig three properties:

i) $\mathring{\gamma}(M) \subset \gamma$

ii) $X,Y \in \gamma \Rightarrow X \cup Y \in \gamma$

iii) $X,Y \in \gamma \Rightarrow X \setminus Y \in \gamma$

From the theory of semialgebraic spaces and from Corollary 1.5 we infer

<u>Proposition 3.10.</u> Let $f:M \to N$ be a locally semialgebraic map and X be a semialgebraic subset of M. Then $f(X)$ is a semialgebraic subset of N.

From the theory of semialgebraic spaces (cf. [DK_2, 7.8]) also the following is evident.

<u>Proposition 3.11.</u> Let X be a subspace of a locally semialgebraic space M. Then for every $U \in \mathring{\gamma}(X)$ there exists some $V \in \mathring{\gamma}(M)$ with $V \cap X = U$.

We now introduce the <u>strong topology</u> on a given locally semialgebraic space M over R. This is the topology, in the classical sense, on the set M which has $\mathring{\mathcal{T}}(M)$ as a basis of open sets. Thus the open sets in the strong topology are the unions of <u>arbitrary</u> families in $\mathring{\gamma}(M)$ (or in $\mathring{\gamma}(M)$).

Notice that the strong topology on the direct product $M \times N$ of two locally semialgebraic spaces M,N is just the usual direct product of the strong topologies on M and N. Also the strong topology on a subspace X of M is the usual subspace topology of the strong topology on M, as follows from Proposition 3.11.

<u>Proposition 3.12.</u> A set $X \in \mathcal{T}(M)$ is an element of $\mathring{\mathcal{T}}(M)$ if and only if X is open in the strong topology.

Proof. f course, if $X \in \mathring{\mathfrak{T}}(M)$ then X is open in the strong topology. Assume ow that $X \in \mathfrak{T}(M)$ is open in the strong topology. Let $(M_\alpha | \alpha \in I)$ be an amissible covering of M by sets $M_\alpha \in \mathring{\gamma}(M)$. Then $X \cap M_\alpha \in \gamma(M_\alpha)$ and $X \cap _x$ is open in M_α with respect to the strong topology of M_α for evey $\alpha \in I$. Thus by the semialgebraic theory [DK_2, Prop. 7.5] $X \cap M_\alpha \in (M_\alpha)$ for every $\alpha \in I$. This implies that $X \in \mathring{\mathfrak{T}}(M)$ by axiom viii) in §1, efinition 1. q.e.d.

Hencefoth we use the following terminology. The words open, closed, dense, tc. all refer to the strong topology on M. The sets $U \in \mathring{\mathfrak{T}}(M)$ are refred to as the "open locally semialgebraic" subsets of M in accordace with Proposition 3.12. The sets $U \in \mathring{\gamma}(M)$ are called the "open smialgebraic" subsets of M. A map $f:M \to N$ between locally smialgebraic spaces will be called "continuous", if f is continuous with respect to the strong topologies, and "strictly continuous", if f is a continuous map from the generalized topological space $(M, \mathring{\mathfrak{T}}(M), Cov_M)$ to $(N, \mathring{\mathfrak{T}}(N), Cov_N)$. By definition every locally semialgbraic map $f:M \to N$ is strictly continuous. Every strictly continuous map is of course continuous.

From Corollary 1.5, Proposition 3.10, and the semialgebraic theory [DK_2, 7.] we infer

Proposiion 3.13. A map $f:M \to N$ from a semialgebraic space M to a locally semialgebraic space N over R is semialgebraic (= locally semialgebaic, cf. §1) if and only if f is continuous and $\Gamma(f)$ is a semialgbraic subset of $M \times N$.

Caution A continuous map $f:M \to N$ between locally semialgebraic spaces ith a locally semialgebraic graph $\Gamma(f) \subset M \times N$ is not necessarily loally semialgebraic. Consider for example the identity map

$g: R_{loc} \to R$, $x \mapsto x$, (Example 2.6), which is locally semialgebraic. The map $f = g^{-1}$ is continuous, since the strong topology on R_{loc} is the same as the strong topology on R. The graph $\Gamma(f)$ is locally semialgebraic in $R \times R_{loc}$ since $\Gamma(g)$ is locally semialgebraic in $R_{loc} \times R$. But f is not locally semialgebraic. This example also shows that Proposition 3.13 is false if we only demand that $\Gamma(f)$ be locally semialgebraic.

<u>Proposition 3.14.</u>

a) For every locally semialgebraic subset X of a locally semialgebraic space M the interior $\mathring{X}$ and the closure $\overline{X}$ of X (in M, with respect to the strong topology) are locally semialgebraic subsets of M.
b) If U is an open locally semialgebraic subset of the subspace X of M then there exists some $V \in \mathring{\mathcal{T}}(M)$ with $V \cap X = U$.

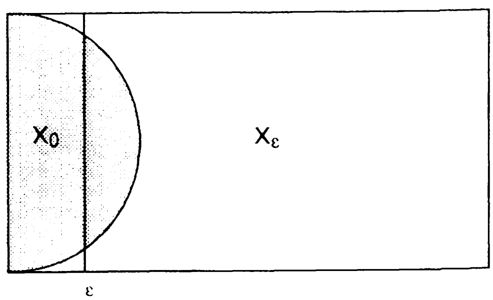
<u>Proof.</u> Statement (a) is evident from the semialgebraic theory [DK_2, 7.7]. In order to prove (b) consider the closure A of $X \setminus U$ in M. Then $A \in \mathcal{T}(M)$. Thus $V := M \setminus A \in \mathcal{T}(M)$. Since V is open in M, we have $V \in \mathring{\mathcal{T}}(M)$. Clearly $A \cap X = X \setminus U$. Thus $V \cap X = U$. q.e.d.

If X is semialgebraic, then, of course, also $\mathring{X}$ is semialgebraic. But it may happen that $\overline{X}$ is not semialgebraic, as the following example shows.

<u>Example 3.15.</u> We start with the semialgebraic space $M :=]0,1[\times]0,1[$ over R and consider the family $(X_\varepsilon \mid \varepsilon \in [0,1[)$ of open semialgebraic subsets of M defined as follows:

$$X_\varepsilon :=]\varepsilon,1[\times]0,1[\qquad \text{if } \varepsilon > 0,$$
$$X_0 := \{(x,y) \in]0,1[\times]0,1[\mid x^2 + (y-\tfrac{1}{2})^2 < \tfrac{1}{4}\}.$$

The union of these sets is M. We apply the procedure of Example 2.9 to this family and obtain on the set M a new structure of a locally semi-algebraic space. We call this new space M'. Notice that the strong topologies of M and M' coincide. The set X_0 is open semialgebraic in M' by construction. But its closure $\overline{X}_0$ in M' is not contained in the union of finitely many sets X . Thus $\overline{X}_0$ is not a semialgebraic subset of M'.



We now state a fact which is very useful for constructing locally semialgebraic maps. It stresses the importance of closed locally semialgebraic sets.

Proposition 3.16 (Gluing principle for locally semialgebraic maps). Let $(X_\lambda \mid \lambda \in \Lambda)$ be a family of closed locally semialgebraic subsets of M such that every $W \in \mathcal{T}(M)$ is contained in the union of finitely many X_λ. Let $f: M \to N$ be a map from M to a second locally semialgebraic space N. Assume that $f|X_\lambda$ is a locally semialgebraic map from X_λ to N for every $\lambda \in \Lambda$. Then f is locally semialgebraic.

Proof. Let $(M_\alpha \mid \alpha \in I)$ be an admissible covering of M by open semialgebraic subsets. It suffices to prove that $f|M_\alpha$ is locally semialgebraic for every $\alpha \in I$. There exists a finite subset J of I such that M_α is contained in $\cup(X_\lambda \mid \lambda \in J)$. $f(M_\alpha \cap X_\lambda)$ is a semialgebraic subset of N for every $\lambda \in J$. Thus $N_\alpha := f(M_\alpha)$ is a semialgebraic subset of N. It suffices to know that the map $g_\alpha : M_\alpha \to N_\alpha$ obtained from $f|M_\alpha$ by restriction of the range space is semialgebraic. But this is evident from the semialgebraic theory, since $g_\alpha | M_\alpha \cap X_\lambda$ is semialgebraic for every $\lambda \in J$. q.e.d.

<u>Example 3.17.</u> A map f from a locally finite simplicial complex X to a locally semialgebraic space N is locally semialgebraic if and only if for every $\sigma \in \Sigma(X)$ the restriction $f|\bar{\sigma} \cap X$ is a semialgebraic map from $\bar{\sigma} \cap X$ to N.

<u>Definition 3.</u> A <u>path</u> in a locally semialgebraic space M is a semialgebraic map from the unit intervall $[0,1]$ in R to M. The <u>path component</u> of a point $x \in M$ in M is the set of all $y \in M$ such that there exists a path γ in M with $\gamma(0) = x$ and $\gamma(1) = y$.

<u>Proposition 3.18.</u> Every path component M' of M is a closed and open locally semialgebraic subset of M. The family $(M_\lambda | \lambda \in \Lambda)$ of all path components of M is locally finite. In particular it is an admissible covering of M. Thus M is the direct sum (cf. Example 2.4) of the subspaces M_λ.

<u>Proof.</u> For any $W \in \mathring{\gamma}(M)$ the intersection $M' \cap W$ is a union of path components of the semialgebraic space W. Thus $M' \cap W$ is an open and closed semialgebraic subset of W, cf. [DK_2, §11 and §12]. This implies that M' is an open and closed locally semialgebraic subset of M. Since for any $W \in \mathring{\gamma}(M)$ and every $\lambda \in \Lambda$ the intersection $W \cap M_\lambda$ consists of full path components of W, there can be only finitely many λ with $W \cap M_\lambda \neq \emptyset$. q.e.d.

As in the semialgebraic theory, it is now clear that every path component M' of M is <u>connected</u>, i.e. there does not exist a partition of M' into two non empty disjoint open locally semialgebraic subsets. Thus we call the path components of M also the connected components of M. We denote the set of connected components of M by $\pi_o(M)$.

Example 3.19. We call a locally semialgebraic space M discrete, if $\{x\} \in \mathring{\mathfrak{T}}(M)$ for every $x \in M$ (hence $\{x\} \in \mathring{\gamma}(M)$), and if $(\{x\} \mid x \in M)$ is an admissible covering of M. This means that M is the direct sum of the one point spaces $\{x\}$, $x \in M$. Clearly a locally semialgebraic space M is discrete if and only if all its connected components are one point sets. Every set can be equipped with the structure of a discrete locally semialgebraic space over R.

Paths can be useful for testing properties of spaces and maps. We give an example. Other examples will come up later (Th. 6.7, Th. 6.9, II,§9, ...).

Proposition 3.20 (Path criterion for continuity). Let $f : M \to N$ be a map between locally semialgebraic spaces and let x be a point of M. Let U be a semialgebraic open neighbourhood of x in M. Assume that $f|U : U \to N$ has a semialgebraic graph. Assume further that, for every path $\gamma : [0,1] \to U$ with $\gamma(0) = x$, $\gamma(t) \neq x$ for $t > 0$, the map $f \circ \gamma : [0,1] \to N$ is continuous at $t = 0$. Then f is continuous at x.

Proof. We may assume U = M since continuity is a local property. Suppose that f is not continuous at x. Then there exists a semialgebraic neighbourhood W of f(x) such that every neighbourhood V of x contains a point z with $f(z) \notin W$. The set $A := f^{-1}(N \setminus W)$ is semialgebraic since f has a semialgebraic graph. The point x lies in $\bar{A} \setminus A$. By the semialgebraic curve selection lemma ([DK_2, §12], [DK_4, §2]) there exists a path $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(t) \in A$ for $t > 0$. We have $f \circ \gamma(0) = f(x)$ but $f \circ \gamma(t) \in N \setminus W$ for $t > 0$. Thus $f \circ \gamma$ is not continuous at $t=0$, contrary to our hypothesis. This proves that f is continuous at x. q.e.d.

This proposition should be regarded as an analogue of the criterion that a map $f: M \to N$ between metric spaces is continuous at a point $x \in M$ if and only if for every sequence of points $(x_n \mid n \in \mathbb{N})$ in $M \setminus \{x\}$ which converges to x the image sequence $(f(x_n) \mid n \in \mathbb{N})$ converges to $f(x)$. Notice that it may happen that our base field R does not contain a sequence of positive elements converging to zero. Over such a field it is, of course, impossible to test continuity by sequences.

<u>Definition 4.</u> The <u>dimension</u> $\dim M$ of a non empty locally semialgebraic space M is defined as the supremum (an integer ≥ 0 or ∞) of dimensions of all open semialgebraic subsets of M. For the empty space $\emptyset$ we define $\dim \emptyset = -1$. For $X \in \mathcal{T}(M)$ we understand by $\dim X$ the dimension of the subspace X of M. For every $x \in M$ the <u>local dimension</u> $\dim_x M$ is defined as the infimum of the dimensions of all $W \in \mathring{\gamma}(M)$ with $x \in W$. The space M is called <u>pure of dimension</u> n, if $\dim_x M = n$ for every $x \in M$. (Then necessarily $\dim M = n < \infty$, see statement 3.21.f below).

From the semialgebraic theory one easily derives the following facts about dimensions, cf. [DK_2, §8] for statements b) - f) and [CKLR, §1] for statement g). Statement a) is easily seen by considering paths in M.

<u>Proposition 3.21.</u> Let M be a non empty locally semialgebraic space.

a) $\dim M = 0$ if and only if M is discrete.

b) If $(X_\lambda \mid \lambda \in \Lambda)$ is a family in $\mathcal{T}(M)$ with union M and the property that every $W \in \mathring{\gamma}(M)$ is already contained in the union of finitely many X_λ then

$$\dim M = \sup(\dim X_\lambda \mid \lambda \in \Lambda) .$$

c) Assume that $X \in \mathcal{T}(M)$ has dimension $\dim X = \dim M < \infty$. Then the interior $\mathring{X}$ of X is not empty and $\dim \mathring{X} = \dim M$.

d) $\dim(\overline{X} \setminus X) < \dim X$ for every $X \in \mathcal{T}(M)$ with $\dim X < \infty$ and X not empty.

e) If there exists an injective locally semialgebraic map from M to a second locally semialgebraic space N, then $\dim M \leq \dim N$.

f) $\dim M$ is the supremum of all the local dimensions $\dim_x M, x \in M$.

g) For every integer $n \geq 0$ the set

$$\Sigma_n(M) := \{x \in M \mid \dim_x M \geq n\}$$

is a closed locally semialgebraic subset of M. If the set

$$\Sigma_n^o(M) := \Sigma_n(M) \setminus \Sigma_{n+1}(M) \in \mathcal{T}(M)$$

is not empty then this subspace of M is pure of dimension n. The family $(\Sigma_n^o(M) \mid n \geq 0)$ is locally finite.

<u>Caution.</u> If there exists a surjective locally semialgebraic map from M to N then it is not always true that $\dim M \geq \dim N$, in contrast to the semialgebraic case.

We finally discuss the behavior of subspaces of a locally semialgebraic space under extension of the base field R (cf. Example 2.10 for definitions).

<u>Proposition 3.22.</u> Let M and N be locally semialgebraic spaces over R and let $f: M \to N$ be a locally semialgebraic map. Let $\tilde{R}$ be a real closed overfield of R.

a) If $X \in \mathcal{T}(M)$, then the map $i_{\tilde{R}}: X(\tilde{R}) \to M(\tilde{R})$, obtained from the inclusion map $i: X \hookrightarrow M$ by base field extension is an isomorphism of the space $X(\tilde{R})$ onto some subspace of $M(\tilde{R})$. We identify henceforth $X(\tilde{R})$ with this subspace, thus $X(\tilde{R}) \in \mathcal{T}(M(\tilde{R}))$. We have $X(\tilde{R}) \cap M = X$.

b) If $X \in \gamma(M)$, then $X(\tilde{R}) \in \gamma(M(\tilde{R}))$.

c) If $X \in \mathcal{T}(M)$, then $\mathring{X}(\tilde{R})$ is the same as the interior $X(\tilde{R})^\circ$ of $X(\tilde{R})$ in $M(\tilde{R})$, and $\overline{X}(\tilde{R})$ is the same as the closure $\overline{X(\tilde{R})}$ of $X(\tilde{R})$ in $M(\tilde{R})$.

d) Let

$$\begin{array}{ccc} M\times_S N & \xrightarrow{q} & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow{\varphi} & S \end{array}$$

be a pull back in the category of locally semialgebraic spaces over R. Then

$$\begin{array}{ccc} (M\times_S N)(\tilde{R}) & \xrightarrow{q_{\tilde{R}}} & N(\tilde{R}) \\ {\scriptstyle p_{\tilde{R}}}\downarrow & & \downarrow{\scriptstyle \psi_{\tilde{R}}} \\ M(\tilde{R}) & \xrightarrow{\varphi_{\tilde{R}}} & S(\tilde{R}) \end{array}$$

is a pull back in the category of locally semialgebraic spaces over $\tilde{R}$. Henceforth we identify $(M\times_S N)(\tilde{R})$ with $M(\tilde{R})\times_{S(\tilde{R})} N(\tilde{R})$.

e) The subsets $\Gamma(f)(\tilde{R})$ and $\Gamma(f_{\tilde{R}})$ of $M(\tilde{R})\times N(\tilde{R}) = (M\times N)(\tilde{R})$ are equal.

f) Let X be a semialgebraic subset of M. Then $f(X)(\tilde{R}) = f_{\tilde{R}}(X(\tilde{R}))$.

g) Let Y be a locally semialgebraic subset of N. Then $f^{-1}(Y)(\tilde{R}) = f_{\tilde{R}}^{-1}(Y(\tilde{R}))$.

h) The map $f_{\tilde{R}}$ is injective iff the map f is injective.

i) If $(M_\alpha|\alpha\in I)$ is the family of connected components of M, then $(M_\alpha(\tilde{R})|\alpha\in I)$ is the family of connected components of $M(\tilde{R})$

j) $\dim M = \dim M(\tilde{R})$.

k) $\Sigma_n(M(\tilde{R})) = \Sigma_n(M)(\tilde{R})$ for every $n\geq 0$.

<u>Caution.</u> If the map f is surjective, then the map $f_{\tilde{R}}$ is not necessarily surjective. Consider e.g. the map $f:\mathbb{R}_{loc}\to\mathbb{R}$, $x\mapsto x$. Then for every real closed field extension $\tilde{R}$ of $\mathbb{R}$ the image of $f_{\tilde{R}}$ is the valuation ring $\mathfrak{o}$ of all elements of $\tilde{R}$ whose absolute values are not infinitely large with respect to $\mathbb{R}$. This example also shows that statement f) is false in general for locally semialgebraic subsets.

Proof. If M and N are affine semialgebraic the statements a) - h) are easily proved by use of Tarski's principle (cf. [DK_3, §9], [DK, §4], [BCR, Chap. 5]). The other statements i) - k) can be quickly deduced by use of Tarski's principle and some triangulation of M (Theorem 2.13). For statement i) it is convenient to use the combinatorial description of connectedness in simplicial complexes, cf. Chap. II, §7 below. {In [D] and [DK_3] we obtained i) by use of homology, a tool not really necessary for this simple statement.}

Starting from the semialgebraic case we obtain the general statement in a straightforward way. We indicate the proof of the first two statements leaving the rest to the reader. First assume that X is semialgebraic. We choose some $U \in \mathring{\gamma}(M)$ with $X \subset U$. Assertion a) is true for the inclusion map $X \hookrightarrow U$, since X and U are semialgebraic. By definition of the space $M(\tilde{R})$, this assertion is also true for the inclusion $U \hookrightarrow M$. Thus the assertion holds for $X \hookrightarrow M$. Since $U(\tilde{R}) \in \mathring{\gamma}(M(\tilde{R}))$ the assertion b) is now evident. To prove assertion a) for any $X \in \mathcal{T}(M)$ we choose an admissible covering $(U_\alpha | \alpha \in I)$ of M by open semialgebraic sets. Adding all unions of finitely many U_α to this family we assume that $(U_\alpha | \alpha \in I)$ is in addition a directed system of open semialgebraic subsets of M. Again by definition of the space $M(\tilde{R})$, we know that $(U_\alpha(\tilde{R}) | \alpha \in I)$ is an admissible covering of $M(\tilde{R})$ by open semialgebraics. Now assertion a) holds for all the inclusion maps $U_\alpha \cap X \hookrightarrow M$. Also $(U_\alpha \cap X | \alpha \in I) \in Cov_X$. Thus $X(\tilde{R})$ is the direct limit of the spaces $(U_\alpha \cap X)(\tilde{R})$. As a set, $X(\tilde{R})$ is the union of the directed system of $(U_\alpha \cap X)(\tilde{R})$. The map $i_{\tilde{R}}: X(\tilde{R}) \hookrightarrow M(\tilde{R})$ is injective on every subset $(U_\alpha \cap X)(\tilde{R})$. Thus $i_{\tilde{R}}$ is injective on the whole set $X(\tilde{R})$. We regard $X(\tilde{R})$ as a subset of $M(\tilde{R})$. For any $\alpha \in I$ we have $U_\alpha(\tilde{R}) \cap X(\tilde{R}) = (U_\alpha \cap X)(\tilde{R})$. Indeed, given any $x \in U_\alpha(\tilde{R}) \cap X(\tilde{R})$, we choose some $\beta \geq \alpha$ with $x \in (X \cap U_\beta)(\tilde{R})$ and conclude from the semialgebraic case that x is contained in $(X \cap U_\beta \cap U_\alpha)(\tilde{R}) = (X \cap U_\alpha)(\tilde{R})$. Thus $X(\tilde{R}) \in \mathcal{T}(M(\tilde{R}))$, and it is also clear that the given space structure on $X(\tilde{R})$ coincides with the subspace structure in $M(\tilde{R})$. In the same way we see that $X(\tilde{R}) \cap M = X$.

§4 - Regular and paracompact spaces

A locally semialgebraic space M is called <u>separated</u>, if M is Hausdorff in the strong topology, in other words, if the diagonal Δ of $M \times M$ is closed in $M \times M$. (N.B. Δ is locally semialgebraic by Prop. 3.7). Henceforth we tacitly assume that all our locally semialgebraic spaces are separated, and we will call them "<u>spaces</u>".

Nevertheless, the following stronger separation property appears to be the truly useful one.

<u>Definition 1.</u> A space M is called <u>regular</u>, if for every closed locally semialgebraic subset A of M and every point x in $M \setminus A$ there exist sets $U, V \in \mathring{\mathfrak{T}}(M)$ with $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. In other terms, every $x \in M$ has a fundamental system of closed semialgebraic neighbourhoods.

Robson's embedding theorem [R] states that a semialgebraic space X is affine if and only if X is regular. Since every subspace of a regular locally semialgebraic space is also regular this implies

<u>Proposition 4.1.</u> If M is regular then every semialgebraic subspace of M is affine.

We do not know wether the converse of this statement is true in general. We will prove the converse in a special case below (Prop. 4.7).

<u>Examples 4.2.</u> a) The spaces M_{loc} introduced in Example 2.6 are regular. More generally, the space X considered in Example 2.9 is regular if the ambient space M is regular. Indeed, every $x \in X$ is contained in some set X_α of the family $(X_\alpha | \alpha \in I)$ considered there. Every closed semialgebraic neighbourhood of x in M, which is contained in X_α, is

a closed semialgebraic neighbourhood of x in X since the strong topology of X is the subspace topology of the strong topology of M. Thus x has a fundamental system of closed semialgebraic neighbourhoods in X.

b) As already observed above, any locally semialgebraic subspace of a regular space is regular.

c) The direct product of finitely many regular spaces is regular. The fibre product $M_1 \times_N M_2$ of two regular spaces M_1 and M_2 with respect to arbitrary locally semialgebraic maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ is also regular, since it is a (closed) subspace of $M_1 \times M_2$.

Question 4.3. Is, for M a regular space over R, the extension $M(\tilde{R})$ with respect to any real closed overfield $\tilde{R}$ (cf. Example 2.10) again regular?

At first glance regular spaces might look only slightly more difficult to work with than affine semialgebraic spaces. But the ease with which we transfered results from semialgebraic to locally semialgebraic spaces is deceptive and disappears as soon as we study deeper topological - in particular cohomological - properties of such spaces. It seems to us that, at least at present, regular locally semialgebraic spaces are in general totally inaccessible objects. The best we can do is to study regular spaces which are also "paracompact". Fortunately, these spaces seem to suffice for many problems arising from the theory of affine semialgebraic spaces.

Definition 2. A locally semialgebraic space M is called paracompact, if there exists a locally finite (cf. Def. 4 in §1) covering $(M_\alpha | \alpha \in I)$ of M by open semialgebraic subsets. (Any such covering is admissible, cf. Prop. 1.1).

Example 4.4. a) Assume that in our base field R there exists a countable fundamental system of neighbourhoods of zero. This means there exists sequence $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$ of positive elements in R with $\lim_{n\to\infty} \varepsilon_n = 0$. Then for every locally complete semialgebraic space M over R the space M_{loc} introduced in Example 2.6 is paracompact. Indeed, the sets

$$M_o := \{x \in M \mid d(x) > \varepsilon_1\}$$
$$M_n := \{x \in M \mid \varepsilon_{n-1} > d(x) > \varepsilon_{n+1}\} \quad (n \geq 1)$$

are open semialgebraic in M_{loc}. They form a locally finite covering of M. Notice that nearly all real closed fields which occur in practice have a countable fundamental system of neighbourhoods of zero. For example every real closure R of a finitely generated field is even "microbial", i.e. R contains an element $\mathcal{J} > 0$ with $\lim_{n\to\infty} \mathcal{J}^n = 0$ [Du]. Every real closed field R is the union of a directed system of microbial real closed subfields.

b) Let M be a semialgebraic space and $(X_\alpha \mid \alpha \in I)$ a family of open semialgebraic subsets with union X such that every $W \in \mathring{\mathcal{T}}(M)$ with $W \subset X$ meets only finitely many sets X_α. Then the locally semialgebraic space introduced on this set X in Example 2.7 is paracompact. As an example, the smallest valuation ring σ of R which is compatible with the ordering of R namely,

$$\sigma = \{x \in R \mid \quad |x| \leq n \text{ for some } n \in \mathbb{N}\}$$

is a paracompact - and also regular - locally semialgebraic space.

c) More generally the space X considered in Example 2.9 is paracompact as soon as the given family $(X_\alpha \mid \alpha \in I)$ in $\mathring{\mathcal{T}}(M)$ has the following property: For any $\alpha \in I$ the set of all $\beta \in I$ with $X_\alpha \cap X_\beta \neq \emptyset$ is finite.

d) Every subspace of a paracompact space is paracompact.

e) Direct products of finitely many paracompact spaces are paracompact. More generally, a fibre product $M_1 \times_N M_2$ is paracompact if both M_1 and M_2 are paracompact.

f) Any extension $M(\tilde{R})$ of a paracompact space M over R with respect to a real closed overfield $\tilde{R} \supset R$ is paracompact.

We begin a basic study of paracompact spaces. So let M be a paracompact space and let $(M_\alpha | \alpha \in I)$ be a locally finite covering of M by open semialgebraic subsets.

Proposition 4.5. If $(U_\lambda | \lambda \in \Lambda)$ is an admissible covering of some $U \in \mathring{\mathfrak{T}}(M)$, then there exists a locally finite family $(V_\kappa | \kappa \in K)$ of open semialgebraic subsets V_κ of U which covers U and refines the covering $(U_\lambda | \lambda \in \Lambda)$.

Proof. For every $\alpha \in I$ we choose a finite subset Λ_α of Λ such that $(U_\lambda | \lambda \in \Lambda_\alpha)$ covers $M_\alpha \cap U$. Then $(M_\alpha \cap U_\lambda | \alpha \in I, \lambda \in \Lambda_\alpha)$ is a locally finite covering of U by open semialgebraic sets which refines the covering $(U_\lambda | \lambda \in \Lambda)$.

Proposition 4.6. For every semialgebraic subset X of M the closure $\overline{X}$ in M is again semialgebraic.

According to this proposition the regular space considered in Example 3.15 is certainly not paracompact.

Proof of Proposition 4.6. It suffices to verify that the sets $\overline{M}_\alpha$ are semialgebraic since this implies that the closure of the unions of finitely many M_α are semialgebraic, and every $\overline{X}$ is contained in such a set. Let some $\alpha \in I$ be given. We denote by $I(\alpha)$ the set of all $\beta \in I$ with $M_\alpha \cap M_\beta \neq \emptyset$. This set is finite. Clearly $\overline{M}_\alpha$ does not meet any M_γ with $\gamma \notin I(\alpha)$. Thus $\overline{M}_\alpha$ is contained in the open semialgebraic set $\cup(M_\beta | \beta \in I(\alpha))$, which implies that $\overline{M}_\alpha$ itself is semialgebraic. q.e.d.

We now discuss regular paracompact spaces. These are the "good" spaces in our theory.

Propostion 4.7. Again, let M be a paracompact space and $(M_\alpha | \alpha \in I)$ a locall finite covering of M by open semialgebraic sets. The following are eqivalent:

a) M i regular

b) Evey open semialgebraic subset of M is affine.

c) Evey finite union of sets M_α is affine.

Proof. a) ⇒ b) is contained in Proposition 4.1. b) ⇒ c) is trivial. Assume now that c) holds true. We want to prove that M is regular. Let x be a point of M and V some open semialgebraic neighbourhood of x. As explaied in the proof of Proposition 4.6, there exist finite subsets $J \subset K$ o I such that

$$V = \bigcup_{\alpha \in J} M_\alpha \subset \bigcup_{\alpha \in J} \overline{M}_\alpha \subset \bigcup_{\beta \in K} M_\beta .$$

The spce $\cup (M_\beta \,|\beta \in K) = N$ is affine, hence regular. Thus there exists some smialgebraic neighbourhood W of x in N with $\overline{W} \cap N \subset V$. But $\overline{W}$ is containedin $\cup(\overline{M}_\alpha | \alpha \in J)$, hence in N, and we conclude that $\overline{W} \subset V$. q.e.d.

Exampl 4.8. Every locally finite simplicial complex X (cf. Example 2.12) s a regular paracompact space. Indeed, the covering $(St_X(\sigma)\,|\,\sigma \in \Sigma())$ by all open stars is locally finite and fulfills property c) of the proposition.

If $(M_\alpha \alpha \in I)$ is a locally finite covering of M fulfilling property c) of Proosition 4.7, then for any real closed field $\tilde{R} \supset R$ the family $(M_\alpha(\tilde{R}) \alpha \in I)$ is a locally finite covering of $M(\tilde{R})$ with the same property. Ths we obtain a partial answer to Question 4.3.

Corollry 4.9. For any real closed field $\tilde{R} \supset R$ the base extension $M(\tilde{R})$ of a rgular paracompact space M over R is also regular paracompact.

Proposition 4.10 (Partition of 1, crude version). Let M be regular paracompact and let $(U_\lambda | \lambda \in \Lambda)$ be a locally finite covering of M by open semialgebraic subsets. Then there exists a family $(\varphi_\lambda | \lambda \in \Lambda)$ of locally semialgebraic functions $\varphi_\lambda : M \to [0,1]$ with $\varphi_\lambda^{-1}(]0,1]) = U_\lambda$ and

$$\sum_{\lambda \in \Lambda} \varphi_\lambda(x) = 1$$

for every $x \in M$.

Proof. According to Proposition 4.6 every closure $\bar{U}_\lambda$ is semialgebraic, hence affine semialgebraic. We choose for every λ a semialgebraic function $\chi_\lambda : \bar{U}_\lambda \to [0,1]$ with $\chi_\lambda^{-1}(0) = \bar{U}_\lambda \setminus U_\lambda$. (For example embed $\bar{U}_\lambda$ into some R^n as a bounded semialgebraic subset and let χ_λ be the euclidean distance from $\bar{U}_\lambda \setminus U_\lambda$ multiplied by some constant). By a mild application of Proposition 3.16 (gluing of locally semialgebraic maps) we know that the function $\psi_\lambda : M \to [0,1]$ defined by $\psi_\lambda | \bar{U}_\lambda = \chi_\lambda$ and $\psi_\lambda | M \setminus U_\lambda = 0$ is locally semialgebraic. Since the covering $(U_\lambda | \lambda \in \Lambda)$ is locally finite, the function $\psi = \sum_{\lambda \in \Lambda} \psi_\lambda$ on M is well defined and locally semialgebraic. This function has no zeros. The functions $\varphi_\lambda := \psi^{-1} \psi_\lambda$ fulfill the requirements of the proposition. q.e.d.

Theorem 4.11 (Shrinking of coverings). Assume that M is regular and that $(U_\lambda | \lambda \in \Lambda)$ is a locally finite covering of M by open semialgebraic sets. (In particular M is paracompact). Then there exists a covering $(V_\lambda | \lambda \in \Lambda)$ of M by open semialgebraic sets with $\bar{V}_\lambda \subset U_\lambda$.

Proof. For every $\lambda \in \Lambda$ we introduce the finite set

$$\Lambda(\lambda) := \{\mu \in \Lambda | U_\mu \cap U_\lambda \neq \emptyset\}$$

and the open semialgebraic set

$$W_\lambda := \cup (U_\mu | \mu \in \Lambda(\lambda)).$$

Then $\bar{U}_\lambda \subset W_\lambda$. We further introduce the finite set

$$I(\lambda) := \{\rho\in\Lambda \mid U_\rho \cap W_\lambda \neq \emptyset\} =$$
$$= \cup(\Lambda(\mu) \mid \mu\in\Lambda(\lambda)).$$

and the open semialgebraic set

$$M_\lambda := \cup(U_\rho \mid \rho\in I(\lambda)).$$

We have $U_\lambda \subset M_\lambda$. (Remark: $(M_\lambda \mid \lambda\in\Lambda)$ is still a locally finite covering of M). $(W_{\lambda\mu} \mid \mu\in I(\lambda))$ with $W_{\lambda\mu} := U_\mu$ is a finite covering of the affine semialgebraic space M_λ. By the shrinking lemma for affine semialgebraic spaces [DK_5, Lemma 1.4] there exists a covering $(V_{\lambda\mu} \mid \mu\in I(\lambda))$ of M_λ by open semialgebraic subsets $V_{\lambda\mu}$ of $W_{\lambda\mu}$ with $\overline{V}_{\lambda\mu}\cap M_\lambda \subset W_{\lambda\mu}$. We choose such a covering for every $\lambda\in\Lambda$. For every $\mu\in\Lambda$ we define the open semialgebraic set

$$V_\mu := \cup(V_{\lambda\mu} \mid \lambda\in\Lambda(\mu)).$$

(Notice that $\lambda\in\Lambda(\mu)$ if and only if $\mu\in\Lambda(\lambda)$, and that in this case $\mu\in I(\lambda)$).
For every $\lambda\in\Lambda(\mu)$ we have

$$U_\mu \subset W_\lambda \subset \overline{W}_\lambda \subset M_\lambda$$

hence

$$\overline{V}_{\lambda\mu} \subset \overline{W}_{\lambda\mu} = \overline{U}_\mu \subset \overline{W}_\lambda \subset M_\lambda,$$

and therefore

$$\overline{V}_{\lambda\mu} = \overline{V}_{\lambda\mu} \cap M_\lambda \subset W_{\lambda\mu} = U_\mu.$$

Thus $\overline{V}_\mu \subset U_\mu$. We claim that M is covered by the sets V_μ. Indeed, let $x \in M$ be given. Then $x \in U_\lambda$ for some $\lambda \in \Lambda$. Now

$$U_\lambda \subset M_\lambda = \cup(V_{\lambda\mu} \mid \mu\in I(\lambda)).$$

Thus $x \in V_{\lambda\mu}$ for some $\mu\in I(\lambda)$. Since $V_{\lambda\mu} \subset U_\mu$, we know that $U_\lambda \cap U_\mu$ is not empty, i.e., $\lambda \in \Lambda(\mu)$. Thus $x \in V_\mu$. q.e.d.

The reader should draw a schematic picture in order to understand the proof.

For every locally semialgebraic function $f:M \to R$ the set $\{x \in M \mid x \neq 0\}$ is open and locally semialgebraic in M. As usual we call the closure of this set the <u>support</u> $\mathrm{supp}(f)$ of f. It is a closed locally semialgebraic set. As a consequence of the last two results Prop. 4.1 and Theorem 4.11 we have

<u>Theorem 4.12</u> (Partition of 1, fine version). Let M be regular paracompact and let $(U_\lambda \mid \lambda \in \Lambda)$ be a locally finite covering of M by open semialgebraic sets. Then there exists a family $(\varphi_\lambda \mid \lambda \in \Lambda)$ of locally semialgebraic functions $\varphi_\lambda : M \to [0,1]$ with $\mathrm{supp}(\varphi_\lambda) \subset U_\lambda$ for every $\lambda \in \Lambda$ and

$$\sum_{\lambda \in \Lambda} \varphi_\lambda(x) = 1$$

for every $x \in M$.

As usual we call such a family $(\varphi_\lambda \mid \lambda \in \Lambda)$ a (locally semialgebraic) <u>partition of unity subordinate to the covering</u> $(U_\lambda \mid \lambda \in \Lambda)$.

<u>Theorem 4.13</u> ("Tietze's extension theorem"). Let M be paracompact and regular. Let $f:A \to K$ be a locally semialgebraic function on a closed locally semialgebraic subset A of M with values in some generalized interval of R (i.e. some convex semialgebraic subset K of R). Then there exists a locally semialgebraic function $g:M \to K$ with $g|A = f$.

<u>Proof.</u> We choose a locally finite covering $(M_\alpha \mid \alpha \in I)$ of M by open semialgebraic subsets and then a partition of unity $(\varphi_\alpha \mid \alpha \in I)$ subordinate to this covering. Since "Tietze's extension theorem" holds for affine semialgebraic spaces [DK$_5$, Th.4.5], there exists for every $\alpha \in I$ a semialgebraic function $g_\alpha : M_\alpha \to K$ which extends $f|A \cap M_\alpha$. Consequently $g := \sum_{\alpha \in I} \varphi_\alpha g_\alpha$ is a well defined locally semialgebraic function on M which extends f and has values in K.

Example 4.14. If A and B are two disjoint closed locally semialgebraic subspaces of a regular paracompact space M then there exists a locally semialgebraic function $f:M \to [0,1]$ which has constant values 0 and 1 on A,B respectively. In particular A and B can be separated by two disjoint open locally semialgebraic sets.

In fact a stronger statement is true.

Theorem 4.15. If A and B are disjoint closed locally semialgebraic subsets of a regular paracompact space M then there exists a locally semialgebraic function $f:M \to [0,1]$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

This is known to be true for M affine semialgebraic [DK_5, Th. 1.6], and thus follows for M regular paracompact by use of a partition of unity as in the proof of Theorem 4.13.

In classical topology it is often necessary to assume some axiom of "countability". In our context it turns out that such a countability condition is automatically satisfied by connected paracompact spaces.

Definition 3. A locally semialgebraic space M is called Lindelöf if M posesses an admissible open covering $(M_n \mid \in \mathbb{N})$ by countably many sets $M_n \in \mathring{\gamma}(M)$.

Here the word "Lindelöf" alludes to the Lindelöf-property for classical topological spaces [Ke, p. 50]. This is justified by the following proposition.

Proposition 4.16. A space M is Lindelöf if and only if for every admissible open covering $(U_\lambda \mid \lambda \in \Lambda)$ of M there exists a countable refinement

$(V_n|n\in\mathbb{N}) \in \mathrm{Cov}_M(M)$. In this case the V_n may always be chosen to be in $\mathring{\gamma}(M)$.

We leave the easy proof to the reader.

Theorem 4.17. Every connected paracompact space M is Lindelöf.

Proof. We choose a locally finite covering $(M_\alpha|\alpha\in I)$ of M by open semialgebraic sets M_α. We want to prove that I is countable. For every index $\gamma \in I$ we define finite subsets

$$J_0(\gamma) \subset J_1(\gamma) \subset J_2(\gamma) \subset \ldots$$

of I inductively as follows: $J_0(\gamma) := \{\gamma\}$,

$$J_{n+1}(\gamma) := \{\alpha\in I | M_\alpha \cap M_\beta \neq \emptyset \text{ for some } \beta \in J_n(\gamma)\}.$$

Let $J(\gamma)$ denote the union of the $J_n(\gamma)$ for all $n \geq 0$, and let $N(\gamma)$ denote the union of the sets M_α with $\alpha \in J(\gamma)$. Since the family $(M_\alpha|\alpha\in J(\gamma))$ is locally finite, $N(\gamma)$ is an open locally semialgebraic subset of M. Moreover, for any $\alpha\in I$ either $M_\alpha \subset N(\gamma)$ or $M_\alpha \cap N(\gamma)$ is empty. In the first case clearly $N(\alpha) = N(\gamma)$, while in the second case $N(\alpha) \cap N(\gamma) = \emptyset$. This means that either $J(\alpha) = J(\gamma)$ or $J(\alpha) \cap J(\gamma) = \emptyset$. We choose a subset K of I such that I is the disjoint union of the sets $J(\gamma)$ with $\gamma\in K$. Then $(N(\gamma)|\gamma \in K)$ is an admissible covering of M by pairwise disjoint open locally semialgebraic subsets. Since M is assumed to be connected, we conclude that K is a one point set. Thus $I = J(\gamma)$ for any $\gamma \in I$, and I is countable. q.e.d.

From this theorem it is clear that a paracompact space M is Lindelöf if and only if M has at most countably many connected components. One might ask whether every Lindelöf space is already paracompact. The space M' considered in Example 3.15 is a counterexample. We have already observed that this space is not paracompact since the closure $\bar{X}_0$ of the semialgebraic subset X_o of M' (cf. 3.15) is not semialgebraic. On the other hand, if R contains a sequence $(\varepsilon(n) \mid n \in \mathbb{N})$ of positive elements converging to zero, then M' has an admissible covering by the sets X_o and $X_{\varepsilon(n)}, n \in \mathbb{N}$. Thus in this case M' is Lindelöf.

This example is typical, since the following holds.

Proposition 4.18. Let M be a Lindelöf space. Assume that, for every $U \in \mathring{\gamma}(M)$ the closure $\bar{U}$ of U is semialgebraic. Then M is paracompact.

Proof. Let $(U_n \mid n \in \mathbb{N})$ be an admissible covering of M by open semialgebraic sets. Replacing every U_n by the set $U_1 \cup \cdots \cup U_n$, we may assume that $U_n \subset U_{n+1}$ for every $n \in \mathbb{N}$.
By assumption, every $\bar{U}_n$ is contained in some U_m with $m > n$. Choosing a suitable sequence

$$k(1) < k(2) < k(3) < \cdots$$

in $\mathbb{N}$, we obtain an admissible covering $(V_n \mid n \in \mathbb{N})$ of M, $V_n := U_{k(n)}$, such that $V_n \in \mathring{\gamma}(M)$ and $\bar{V}_n \subset V_{n+1}$ for every $n \in \mathbb{N}$. Define open semialgebraic sets $W_n, n \in \mathbb{N}$, as follows:

$$W_1 = V_1, \; W_2 = V_2, \; W_n = V_n \setminus \bar{V}_{n-2} \text{ for } n \geq 3.$$

Then $(W_n \mid n \in \mathbb{N})$ is a locally finite covering of M. Indeed, $W_n \cap V_m = \emptyset$ if $n \geq m+2$. q.e.d.

In the proof we constructed a particularly nice open covering of M. For later use we state a consequence of this proof, Theorem 4.17 and Proposition 4.6.

Corollary 4.19. Let M be a connected paracompact space. Then M has a locally finite covering $(M_n \mid n \in \mathbb{N})$ by open semialgebraic subsets M_n such that $M_n \cap M_m = \emptyset$ for $|n-m| \geq 2$.

Theorem 4.17 also tells us that without the assumption about R ^{n} Example 4.4 we cannot force the space M_{loc} considered to be par^{com-} pact.

<u>Remark 4.20.</u> Let M be an affine locally complete semialgebraic ^{pace} over R which is not complete. Assume that M_{loc} is paracompact. ^{hen} R contains a sequence $\varepsilon_1 > \varepsilon_2 > \ldots$ of positive elements with $\lim\limits_{n\to\infty} {}_n = 0$.

Indeed, we may assume that M is connected. Then M_{loc} is also co$^{nected.}$ By Theorem 4.17, M_{loc} is Lindelöf. We choose an admissible cove^{ing} $(U_n | n \in \mathbb{N})$ of M_{loc} by open semialgebraic subsets with $U_n \subset U_{n+1}$ ^{or} every $n \in \mathbb{N}$. We further choose an embedding of M into some R^m a ^{a} bounded semialgebraic subset, and we consider the distance func^{ion} d on M from the boundary $\bar{M} \setminus M$, cf. Example 2.6. Let $\varepsilon(n)$ denote t^{e} mini- mum of d on the complete semialgebraic subset $\bar{U}_n$ of M. The sequ^{nce} $(\varepsilon(n) | n \in \mathbb{N})$ must converge to zero, since d attains arbitrarily ^{small} positive values on M.

§5 - Semialgebraic maps and proper maps

The interplay between semialgebraic and locally semialgebraic sets in our theory becomes more transparent if we also introduce "semialgebraic maps" as a counterpart to locally semialgebraic maps. In particular this will help us to understand "proper maps" between paracompact spaces. (These maps will be defined in the obvious way - see Def. 2 below).

Definition 1. A map $f : M \to N$ between locally semialgebraic spaces is called semialgebraic (resp. affine semialgebraic) if f is locally semialgebraic and if the preimage $f^{-1}(X)$ of any semialgebraic (resp. affine semialgebraic) subset X of N is semialgebraic (resp. affine semialgebraic) in M.

We shall see that semialgebraic maps are pleasant and easy to be handled from a formal viewpoint. Affine semialgebraic maps are much more difficult, if no extra hypotheses are imposed on the spaces.

Examples 5.1.

a) Let X and Y be locally finite simplicial complexes and let $f : X \to Y$ be a weakly simplicial map. Then f is semialgebraic if and only if, for every open simplex τ of Y, the preimage $f^{-1}(\tau)$ is a finite subcomplex of X. In this case f is, in fact, affine semialgebraic.

b) Let M be a locally complete semialgebraic space, which is not complete. Then the natural map $M_{loc} \to M$, $x \mapsto x$, is not semialgebraic.

c) If M is a semialgebraic space (resp. affine semialgebraic space), then every locally semialgebraic map with domain M is semialgebraic (resp. affine semialgebraic). Thus our usage of the word "semialgebraic" for these maps in the preceding sections is in harmony with Definition 1.

We begin our short study of semialgebraic maps with some obvious remarks.

Remarks 5.2.

a) For every $X \in \mathcal{T}(M)$ the inclusion map $i : X \to M$ is affine semialgebraic.

b) Let $(N_\alpha | \alpha \in I)$ be an admissible covering of N by open semialgebraic sets N_α. Then a map $f : M \to N$ is semialgebraic if and only if the following two conditions are fulfilled:

i) $(f^{-1}(N_\alpha) | \alpha \in I)$ is an admissible covering of M by open semialgebraic sets.

ii) The restriction map

$$f | f^{-1}(N_\alpha) : f^{-1}(N_\alpha) \to N_\alpha$$

is a semialgebraic map between the semialgebraic spaces $f^{-1}(N_\alpha)$ and N_α for every $\alpha \in I$.

c) If $f : M \to N$ is semialgebraic then, for every locally semialgebraic map $g : N' \to N$, the map $f' : M \times_N N' \to N'$ obtained from f by base extension with respect to g is again semialgebraic.

d) If $f : M \to N$ is semialgebraic and N is paracompact, then M is also paracompact. Indeed, every locally finite covering $(N_\alpha | \alpha \in I)$ of N by sets $N_\alpha \in \mathring{\gamma}(N)$ yields a locally finite covering $(f^{-1}(N_\alpha) | \alpha \in I)$ of M by sets in $\mathring{\gamma}(M)$.

e) If $f : M \to N$ is affine semialgebraic and N is regular paracompact then M is regular paracompact (Use Proposition 4.7).

Proposition 5.3. Let $f : M \to N$ be a semialgebraic map. Then the image $f(X)$ of any locally semialgebraic subset X of M is again locally semialgebraic.

Proof. Let $U \in \mathring{\gamma}(N)$ be given and let $g : f^{-1}(U) \to U$ be the restriction of f to $f^{-1}(U)$. Then $f^{-1}(U)$ is a semialgebraic space and g is a semialgebraic map $X \cap f^{-1}(U)$ is a semialgebraic subset of $f^{-1}(U)$, and thus

$$g(X \cap f^{-1}(U)) = f(X) \cap U$$

is a semialgebraic subset of U. This means that $f(X) \in \mathcal{T}(N)$. q.e.d.

In the case that M is Lindelöf (e.g. M connected and paracompact, cf. Th. 4.17) we can prove a strong converse of this proposition.

Proposition 5.4. Let $f: M \to N$ be locally semialgebraic. Assume, that every component of M is Lindelöf. Assume, furthermore, that

a) all fibres of f are semialgebraic,

b) every (closed) zero dimensional locally semialgebraic subset A of M has a locally semialgebraic image f(A).

Then f is semialgebraic. If in addition M is regular, then f is affine semialgebraic.

Proof. We easily reduce to the case where N is affine semialgebraic and must then show that M is semialgebraic. If, in addition M is regular, then M is affine semialgebraic by Robson's embedding theorem [R]. Let $(M_i | i \in I)$ be the family of connected components of M. Then we choose admissible coverings $(M_{i,n} | n \in \mathbb{N})$ of all components M_i by open semialgebraic subsets. Replacing some of the sets $M_{i,n}$ by the empty set we may assume that either $M_{i,n}$ is not contained in $M_{i,1} \cup \ldots \cup M_{i,n-1}$ or $M_{i,n}$ is empty. We claim that all but finitely many sets $M_{i,n} (i \in I, n \in \mathbb{N})$ are empty. Then clearly M is semialgebraic and Proposition 5.4 is proved. Assume that $\Lambda := \{(i,n) \in I \times \mathbb{N} \mid M_{i,n} \neq \emptyset\}$ is infinite. For every $(i,n) \in \Lambda$ we choose an element $x_{i,n} \in M_{i,n} \setminus (M_{i,1} \cup \ldots \cup M_{i,n-1})$. Consider the set

$$A := \{x_{i,n} | (i,n) \in \Lambda\}.$$

Clearly A has a finite intersection with every $M_{i,n}$. Thus A is a (closed) locally semialgebraic subset of dimension zero in M. Of course, the same is true for every non-empty subset A' of A. By hypothesis b) the image $B = f(A)$ is a locally semialgebraic, hence semialgebraic, subset of N, and the same goes for every non-empty subset B' of B. If $\dim B > 0$, then B would contain a closed semialgebraic subspace isomorphic to $[0,1]$. But not every subset of $[0,1]$ is semialgebraic Thus $\dim B = 0$, i.e. B is finite. This implies that A contains an infinite subset C which is mapped by f to one point y. Now C is a closed zero-dimensional locally semialgebraic subset of the fibre $f^{-1}(y)$. Since $f^{-1}(y)$ is assumed to be semialgebraic C must be finite. This contradiction proves that indeed M is semialgebraic. q.e.d

Notation. Let M be a locally semialgebraic space. For convenience we denote the set of all closed semialgebraic subsets of M by $\mathfrak{F}(M)$ and the set of all closed locally semialgebraic subsets of M by $\overline{\mathfrak{T}}(M)$.

Definition 2. A locally semialgebraic map $f: M \to N$ is called proper, if for every locally semialgebraic map $g: N' \to N$ the following holds: The locally semialgebraic map $f': M \times_N N' \to N'$ obtained from f by base extension with respect to g maps every $X \in \overline{\mathfrak{T}}(M \times_N N')$ onto a set $f'(X) \in \overline{\mathfrak{T}}(N')$. A locally semialgebraic space M is called complete if the map from M to the one point space is proper.

Here, as elsewhere ([EGA II, §5.4], [DK_2, §9]), it is easy to verify that the class of proper maps has the following formal properties. (Recall that all our spaces are separated).

Remarks 5.5.

i) Every closed embedding $M \to N$ (i.e. locally semialgebraic isomorphism from M onto a closed subspace of N) is proper.

ii) The composition of proper maps is proper.

iii) Any map $f': M \times_N N' \to N'$ obtained from a proper map $f: M \to N$ by base extension is proper.

iv) If $f: M \to N, g: M' \to N'$ are proper maps "over" a space S (in the usual sense) then $f \times_S g : M \times_S M' \to N \times_S N'$ is proper.

Let $f: M \to N$ and $g: N \to L$ be locally semialgebraic maps.

v) If $g \circ f$ is proper then f is proper. In particular (L = point) every locally semialgebraic map whose domain is a complete space is proper.

vi) If $g \circ f$ is proper and f is surjective then g is proper. In particular the image of a complete space M under any locally semialgebraic map $f: M \to N$ is a complete subspace $f(M)$ of N. (N.B.. $f(M) \in \overline{\mathcal{T}}(N)$).

Proposition 5.6. Let f be a locally semialgebraic map from a semialgebraic space M to a space N. Then f is proper if and only if the semialgebraic set $f(M)$ is closed in N and if the map "f" : $M \to f(M)$ obtained from f by restriction of the image space is proper in the category of semialgebraic spaces, as defined in [DK_2, §9].

Proposition 5.7. Let $f: M \to N$ be a locally semialgebraic map and let $(U_\alpha | \alpha \in I)$ be an admissible open covering of N. Then f is proper if and only if all the maps $f^{-1}(U_\alpha) \to U_\alpha$, $\alpha \in I$, obtained from f by restriction are proper.

We omit the easy proofs of these two propositions.

Lemma 5.8. A complete locally semialgebraic space M has only finitely many connected components.

Proof. The family $(M_\alpha | \alpha \in I)$ of connected components of M is an admissible covering of M. In every set M_α we choose some point x_α. The set

$$A := \{x_\alpha | \alpha \in I\}$$

meets every M_α in one point. Thus A is closed and locally semialgebraic. The subspace A of M is complete and discrete. The same goes for every subset of A. Assume that I is infinite. We choose a sequence $(\alpha(n) | n \in \mathbb{N})$

of pairwise different indices in I and consider the complete and discrete subspace

$$B := \{x_{\alpha(n)} \mid n \in \mathbb{N}\}$$

of M. The function $f : B \to R$ defined by $f(x_{\alpha(n)}) = n$ is locally semialgebraic and proper. But its image $\mathbb{N}$ is not semialgebraic in R. This contradiction proves that I is finite. q.e.d.

We now come to the main result of this section.

<u>Theorem 5.9.</u> Let M be a space all whose connected components are Lindelöf (e.g. M paracompact, cf. Th. 4.17). Then every proper map $f : M \to N$ from M to an arbitrary space N is affine semialgebraic.

We first prove the theorem in the special case that N is a point. Then the theorem means the following.

<u>Corollary 5.10.</u> Let M be a complete space all whose connected components are Lindelöf (e.g. a complete paracompact space). Then M is affine semialgebraic.

In order to prove that M is semialgebraic we choose an admissible covering $(M_n \mid n \in \mathbb{N})$ of M by open semialgebraic subsets M_n which is possible by Lemma 5.8. Suppose M is not semialgebraic. Then M cannot be covered by finitely many M_n. As in the proof of Proposition 5.4 we assume that M_n is not contained in $M_1 \cup \ldots \cup M_{n-1}$ for every $n > 1$, and consider a set

$$A := \{x_n \mid n \in \mathbb{N}\}$$

with $x_n \in M_n \setminus (M_1 \cup \ldots \cup M_{n-1})$. Again A is closed and locally semialgebraic in M, and $\dim A = 0$. Thus A is a discrete complete space. This contradicts Lemma 5.8, and we see that M is indeed semialgebraic.

It remains to prove that M is affine*). By Robson's embedding theorem [R] it suffices to verify that M is regular. Let a point $x \in M$ and an affine open semialgebraic neighbourhood U of x be given. We claim that there exists a complete neighbourhood K of x with $K \subset U$. If we have proved this we are done, since K will be closed in M. We choose an isomorphism $\varphi : U \xrightarrow{\sim} Z$ from U to a bounded semialgebraic subset Z of some R^n. We apply the following general lemma, which will be proved below.

<u>Lemma 5.1.</u> Let X and Y be locally semialgebraic subsets of some spaces M and N, and let $\varphi : X \to Y$ be an isomorphism. Assume that M is complete and that X is open in its closure $\overline{X}$. Then Y is open in its closure $\overline{Y}$.

By this lemma Z is open in its closure $\overline{Z}$ in R^n. Let $p := \varphi(x)$ and $r := d(p, Z)$, the euclidean distance of p from $\partial Z = \overline{Z} \setminus Z$. We have $r > 0$. The set

$$L := \{z \in \overline{Z} \mid d(z, \partial Z) \geq \tfrac{r}{2}\}$$

is a complete neighbourhood of p in Z. Thus $K := \varphi^{-1}(L)$ is a complete neighbourhood of x in U.

It remains to prove Lemma 5.1. Replacing M and N by the closures $\overline{X}$ and $\overline{Y}$ of X and Y we assume that X is dense in M and Y is dense in N. Let $Z \subset X \times Y$ denote the graph of φ and $\overline{Z}$ its closure in $M \times N$. Let further p and q denote the projections of $\overline{Z}$ onto M and N. q is a proper surjective map. We have a commutative triangle

$$\begin{tikzcd}
Z \arrow[rr, hook, "j"] \arrow[dr, "q|Z"'] & & q^{-1}(Y) \arrow[dl, "q|q^{-1}(Y)"] \\
& Y &
\end{tikzcd}$$

with $q|Z$ an isomorphism and j the inclusion map from Z to $q^{-1}(Y)$. Now j is proper, since $q|Z$ is proper (cf. 5.5.v. We did not use yet that q

*) We thank R. Robson for communicating this proof to us. The first proof, to our knowledge, has been given by N. Schwartz [Sch, Th. 157].

is proper.) Since j is a dense embedding, we conclude that $Z = q^{-1}(Y)$. Replacing φ by φ^{-1} we see that $Z = p^{-1}(X)$. Assume now that X is open in M. Then $Z = p^{-1}(X)$ is open in $\bar{Z}$. Since $Z = q^{-1}(Y)$ we have

$$N \setminus Y = q(\bar{Z} \setminus Z),$$

and this set is closed, since q is proper. This finishes the proof of Lemma 5.11 and of Corollary 5.10.

We now come to the proof of Theorem 5.9 in general. From Corollary 5.10 we know that every fibre of f is (affine) semialgebraic. By Proposition 5.4 this implies that f is semialgebraic. Thus, for every affine semialgebraic subset X of N the preimage $f^{-1}(X)$ is semialgebraic. The restriction $f^{-1}(X) \to X$ of f is proper. We want to see that $f^{-1}(X)$ is affine. Thus we have to prove the following general fact.

Theorem 5.12. Let M be a semialgebraic space. Assume that there exists a proper map from M to an affine semialgebraic space N. Then M is affine.

A proof of this fact has been given by N. Schwartz using his theory of "real closed spaces" [Sch, Cor. 162]. (These are the "abstract locally semialgebraic spaces" in our terminology in Appendix A below.) A proof by our "geometric methods" (cf. App. A) would be desirable but seems to be difficult. A major point in Schwartz's proof is that he first verifies Corollary 5.10 for M semialgebraic and then applies this to the fibres of the abstraction $\tilde{f}: \tilde{M} \to \tilde{N}$ of f (cf. App. A), which can be interpreted as semialgebraic spaces over suitable real closed fields containing R.

Theorem 5.9 says in particular that, if M is not too wild, say paracompact, then a locally semialgebraic map $f: M \to N$ is proper if and only if, for a given admissible covering $(N_\alpha | \alpha \in I)$ of N by open semialgebraic sets, the preimages $f^{-1}(N_\alpha)$ are semialgebraic and the maps $f^{-1}(N_\alpha) \to N_\alpha$

obtained from f by restriction are proper maps in the category of semialgebraic spaces. This may be regarded as a negative result: Essentially nothing new happens if we study proper locally semialgebraic maps between paracompact spaces instead of proper semialgebraic maps between semialgebraic spaces. On the other hand properness is commonly regarded as a condition which is often necessary to obtain good results on "families of spaces". Should we admit that locally semialgebraic spaces are only a moderately useful generalization of semialgebraic spaces?

Fortunately there is a way out of this dilemma. In contrast to the classical space categories it is here possible to define "partially proper maps" which are more general than proper maps but behave nearly as well as proper maps. These maps will be discussed in the next section.

§6 - Partially proper maps

Recall that, for any space M, we denote the set of all closed semialgebraic subsets of M by $\overline{\gamma}(M)$ and the set of all closed locally semialgebraic subsets of M by $\overline{\mathfrak{T}}(M)$.

Definition 1.

i) A locally semialgebraic map $f : M \to N$ is called <u>partially proper</u> if, for every $A \in \overline{\gamma}(M)$, the restriction $f|A : A \to N$ is proper.*) According to Proposition 5.6 this means that we have $f(A) \in \overline{\gamma}(N)$ for every $A \in \overline{\gamma}(M)$, and the map $A \to f(A)$ obtained from f by restriction is proper semialgebraic.

ii) A locally semialgebraic space M is called <u>partially complete</u> if the map from M to the one point space is partially proper. This means that every closed semialgebraic subset of M is a complete semialgebraic space.

Of course every proper map is partially proper and every complete space is partially complete.

Examples 6.1. Here are examples of partially complete spaces:

i) Every discrete space (Ex. 3.19) is partially complete.

ii) The space M_{loc} considered in Example 2.6 is partially complete for every locally complete semialgebraic space M.

iii) Every closed locally finite simplicial complex (Ex. 2.12) is a partially complete space.

Relevant examples of partially proper maps will emerge naturally in the course of this section.

*) The term "proper in parts" would reflect this situation better, but is clumsier.

Some formal properties of partially proper maps follow easily from the corresponding properties of proper maps and/or directly from the definitions.

Remarks 6.2.

i) The composition of two partially proper maps is partially proper.

Let $f: M \to N$ and $g: N \to L$ be locally semialgebraic maps.

ii) If $g \circ f$ is partially proper, then f is partially proper. In particular every locally semialgebraic map with domain a partially complete space is partially proper.

iii) If $g \circ f$ is partially proper and f is surjective and semialgebraic, then g is partially proper.

iv) If $f: M \to N$ is partially proper then for every $X \in \bar{\mathfrak{I}}(N)$ the restriction $f^{-1}(X) \to X$ of f is partially proper. In particular, all fibres of a partially proper map are partially complete.

Partially proper maps warrant more cautious treatment than proper maps.

Counterexamples 6.3.

a) One can construct a partially proper "spiral map" h from $\mathbb{R}_{loc}$ into $\mathbb{R}^2$. (Infinite spiral with center O). $h(\mathbb{R}_{loc})$ is not even a semialgebraic subset of the semialgebraic space $\mathbb{R}^2$.

b) Let M be a locally complete semialgebraic space which is not complete. As stated above, the space M_{loc} constructed in Ex. 2.6 is partially complete. Let f be the locally semialgebraic map $x \mapsto x$ from M_{loc} to M, and let g be the map from M to the one point space. Then $g \circ f$ is partially proper and f is surjective. But g is not partially proper. Thus we see that in statement 6.2.iii we need some extra condition on f in addition to surjectivity. This also shows that the image of a partially complete space under a locally semialgebraic map is not necessarily partially complete - even if it is a locally semialgebraic set.

The question as to whether pull backs of partially proper maps remain partially proper is delicate. We want to prove that this is indeed the case, but this will only be feasible after some preparation.

Theorem 6.4. A locally semialgebraic map $f : M \to N$ is partially proper if and only if the following conditions hold:

i) $f(A) \in \overline{\gamma}(N)$ for every $A \in \overline{\gamma}(M)$.

ii) All fibres $f^{-1}(y)$, $y \in N$, are partially complete spaces.

Proof. Clearly conditions i) and ii) hold if f is partially proper. We now assume i) and ii) and want to prove that for a given $A \in \overline{\gamma}(M)$, the restriction $f|A : A \to N$ is proper. By assumption $f(A) \in \overline{\gamma}(N)$. It remains to be shown that the semialgebraic map $g : A \to f(A)$ obtained from f by restriction is proper. The image of every closed semialgebraic subset of A is closed in $f(A)$. For every $y \in f(A)$ the fibre $g^{-1}(y) = f^{-1}(y) \cap A$ is complete. Thus we know from [DK_2, Th. 12.5] that g is indeed proper.

q.e.d.

Corollary 6.5. Let $f : M \to N$ be a locally semialgebraic map. Assume that every point y of N has a locally semialgebraic neighbourhood L such that the restriction $f^{-1}(L) \to L$ of f is partially proper. Then f is partially proper.

Proof. All fibres of f are partially complete. Thus it suffices to verify that, for a given $A \in \overline{\gamma}(M)$, the image $f(A)$ is closed in N. Let y be a given point in the closure $\overline{f(A)}$. We choose a neighbourhood $L \in \mathcal{T}(M)$ of y such that $f^{-1}(L) \to L$ is partially proper. Then $f(A) \cap L = f(A \cap f^{-1}(L))$ is closed in L. Thus $y \in f(A) \cap L$ and a fortiori $y \in f(A)$. q.e.d.

In order to prove that pull backs of partially proper maps remain partially proper, we have to cope with the difficulty that in general the closure of a semialgebraic set may not be semialgebraic (Example 3.15). But there is a special case where this is true and that will suffice for our proof.

Definition 2. An incomplete path in a space M is a semialgebraic map $\alpha : [0,1[\to M$ from the half-open interval $[0,1[$ to M.

Lemma 6.6. If $\alpha : [0,1[\to M$ is an incomplete path then the closure $\bar{A}$ of $A := \alpha([0,1[)$ in M is either A or $A \cup \{p\}$ with some point $p \in M$. Thus $\bar{A}$ is semialgebraic.

Proof. Assume that there exists some point $p \in \bar{A} \setminus A$. Let U be a semialgebraic neighbourhood of p. Then the semialgebraic subset $\alpha^{-1}(U)$ of $[0,1[$ contains some interval $]a,1[$ with $0 < a < 1$. Indeed, otherwise $U \cap \alpha([0,1[) \subset \alpha([0,a])$ for some $a \in]0,1[$. But $\alpha([0,a])$ is a closed subset of M which does not contain p, while p is in the closure of $U \cap \alpha([0,1[)$. Now if q were a point in $\bar{A} \setminus A$ different from p, then we could choose open semialgebraic neighbourhoods U and V of p,q respectively with $U \cap V = \emptyset$. But $\alpha^{-1}(U) \supset]a,1[$ and $\alpha^{-1}(V) \supset]b,1[$ with $a,b \in]0,1[$. Since $]a,1[\cap]b,1[$ is not empty, this is absurd. Thus $\bar{A} \setminus A$ contains at most one point p. q.e.d.

We shall also need the following fact about "lifting" incomplete paths.

Lemma 6.7. Let $f : M \to N$ be a semialgebraic map between arbitrary spaces and let $\alpha : [0,1[\to N$ be an incomplete path with $\alpha([0,1[) \subset f(M)$. Then there exists some $c \in [0,1[$ and a semialgebraic map $\beta : [c,1[\to M$ with $f \circ \beta = \alpha | [c,1[$.

Proof. We look at the cartesian square

$$\begin{tikzcd} L := [0,1[\times_N M \arrow[r, "q"] \arrow[d, "p"'] & M \arrow[d, "f"] \\ {[0,1[} \arrow[r, "\alpha"'] & N \end{tikzcd}$$

with p,q denoting the natural projections. The lemma means that there exists some $c \in [0,1[$ such that the restriction $p^{-1}([c,1[) \to [c,1[$ of p admits a semialgebraic section. The map p is surjective. It is also semialgebraic, hence the space L is semialgebraic. Let $L_1,\dots,L_r$ be finitely many (open) affine semialgebraic subsets of L which cover L. Then $[0,1[$ is the union of the semialgebraic sets $p(L_1),\dots,p(L_r)$. There exists some $a \in [0,1[$ such that one of these sets, say $p(L_1)$, contains the interval $[a,1[$. We now apply Hardt's theorem ([DK_3, Th. 6.4], [H] for $R = \mathbb{R}$, cf. II, Th. 6.3 below) to the restriction $\pi : L_1 \cap p^{-1}([a,1[) \to [a,1[$, which is a surjective semialgebraic map between affine spaces. By this theorem, $[a,1[$ contains a dense open semialgebraic subset U such that π is trivial over U, i.e. $\pi^{-1}(U)$ is isomorphic to a product $U\times F$ over U. Clearly π has a semialgebraic section over U, and U contains an interval $[c,1[$. q.e.d.

We now are ready to prove our main result about partially proper maps.

<u>Theorem 6.8</u> (Relative path completion criterion). For a locally semialgebraic map $f : M \to N$ the following are equivalent.

i) f is partially proper.

ii) For any path $\overline{\alpha} : [0,1] \to N$ and any incomplete path $\beta : [0,1[\to M$ with $f\circ\beta = \overline{\alpha}|[0,1[$ there exists a (unique) path $\overline{\beta} : [0,1] \to M$ such that $\overline{\beta}$ extends β (hence $f\circ\overline{\beta} = \overline{\alpha}$).

<u>Proof.</u> i) $\Rightarrow$ ii): Let $\overline{\alpha} : [0,1] \to N$ be a path and $\beta : [0,1[\to M$ an incomplete path with $f\circ\beta = \overline{\alpha}|[0,1[$. We want to extend β to a path in M. Let B denote the closure of $\beta([0,1[)$ in M. By Lemma 6.6 this set is semialgebraic. Thus $f(B) \in \overline{\gamma}(N)$ and the map $f|B : B \to f(B)$ is proper semialgebraic.

We can read $\overline{\alpha}$ as a path in $f(B)$ and β as an incomplete path in B. We now know from the semialgebraic theory [DK_4, Cor. 2.3] that β extends to a path $\overline{\beta} : [0,1] \to B$.

ii) $\Rightarrow$ i): Given some $L \in \overline{\gamma}(M)$ we have to prove that the restriction $f|L$ of f is proper. Condition ii) holds for $f|L$ since it holds for f. Thus, replacing M by L, we may assume from the beginning that the space M is semialgebraic. Now the map $f : M \to N$ is semialgebraic.. In order to prove that f is proper we have to consider a cartesian square

$$\begin{array}{ccc} M\times_N N' & \xrightarrow{q} & N' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[f]{} & N \end{array}$$

and some $A \in \overline{\mathcal{T}}(M\times_N N')$, and we have to verify that $q(A) \in \overline{\mathcal{T}}(N')$. Since q is semialgebraic (5.2.c) we know already that $q(A) \in \mathcal{T}(N')$. Let y be a point in the closure of $q(A)$. By the curve selection lemma ([DK_2, Th. 12.1], [DK_4, §2]) there exists a path $\overline{\alpha}$ in N' with $\overline{\alpha}([0,1[) \subset q(A)$ and $\overline{\alpha}(1) = y$. By Lemma 6.7 there exists some $c \in [0,1[$ and a semialgebraic map $\beta : [c,1[\to A$ such that $q\circ\beta = \overline{\alpha}|[c,1[$. Applying condition ii) to the map $p\circ\beta : [c,1[\to M$ and the map $(g\circ\overline{\alpha})|[c,1] : [c,1] \to N$ we see that $p\circ\beta$ extends semialgebraically (i.e. continuously) to $[c,1]$. Since also $q\circ\beta$ extends semialgebraically to $[c,1]$, the map β extends semialgebraically to a map $\overline{\beta} : [c,1] \to M\times_N N'$. Then $q\circ\overline{\beta} = \overline{\alpha}|[c,1]$. In particular, $q(\overline{\beta}(1)) = \overline{\alpha}(1) = y$. Since A is closed in $M\times_N N'$ we have $\overline{\beta}(1) \in A$, hence $y \in q(A)$, and we see that $q(A)$ is closed in N'. q.e.d.

In the special case that N is the one-point space the theorem means the following.

<u>Corollary 6.9</u> (Absolute path completion criterion). A space M is partially complete if and only if every incomplete path $\alpha : [0,1[\to M$ extends to a path $\overline{\alpha} : [0,1] \to M$.

For later use we state a consequence of (the easier part of) this corollary.

Corollary 6.10. If M is a partially complete subspace of some space N, then the set M is closed in N.

Proof. Let x be a point in the closure $\overline{M}$ of M. We choose a path $\alpha : [0,1] \to N$ with $\alpha([0,1[) \subset M$ and $\alpha(1) = x$. By Corollary 6.9 the map $\alpha | [0,1[$ from $[0,1[$ to M can be completed to a path in M. This means, that $x = \alpha(1) \in M$. q.e.d.

As another consequence of Theorem 6.8 we obtain a characterization of partially proper maps by a preimage condition. This characterization is remarkable since no conditions on the image space (e.g. "locally complete") is needed*), in contrast to the topological case, where one needs local compactness [Bo, Chap. I, §10, No. 3].

Corollary 6.11. For a locally semialgebraic map $f : M \to N$ the following are equivalent.

a) f is partially proper.
b) The preimage $f^{-1}(L)$ of every partially complete locally semialgebraic subset L of N is partially complete.
c) The preimage of every complete semialgebraic subset L of N with $\dim L \leq 1$ is partially complete.

Proof. a) $\Rightarrow$ b): The map $f^{-1}(L) \to L$ is partially proper by Corollary 6.10 and 6.2.iv. The space L is partially complete. Thus $f^{-1}(L)$ is partially complete.
The implication b) $\Rightarrow$ c) is trivial. The implication c) $\Rightarrow$ a) is evident by the path completion criteria 6.8 and 6.9.

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*) This fact has escaped out attention in [DK_5, §3].

Using the relative path completion criterion we are also able to verify that pull backs of partially proper maps are again partially proper.

Theorem 6.12. Let

$$\begin{tikzcd} M\times_N N' \arrow[r, "q"] \arrow[d, "p"'] & N' \arrow[d, "g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

be a cartesian square of locally semialgebraic maps. Assume that f is partially proper. Then q is also partially proper.

Proof. We verify condition (ii) of Theorem 6.8. Thus we are given semialgebraic maps $\bar{\alpha} : [0,1] \to N'$ and $\beta : [0,1[\to M\times_N N'$, with $q\circ\beta = \bar{\alpha}|[0,1[$ and we want to extend β to $[0,1]$. We introduce the maps $\bar{\gamma} := g\circ\bar{\alpha}$ from $[0,1]$ to N and $\delta := p\circ\beta$ from $[0,1[$ to M. Clearly $f\circ\delta = \bar{\gamma}|[0,1[$. Since f is partially proper, δ extends to a path $\bar{\delta} : [0,1] \to M$ with $f\circ\bar{\delta} = \bar{\gamma}$. Now

$$\bar{\beta}(t) = (\bar{\delta}(t), \bar{\alpha}(t)), \qquad 0 \le t \le 1,$$

is a path in $M\times_N N'$ with $\bar{\beta}|[0,1[= \beta$ and $q\circ\bar{\beta} = \bar{\alpha}$. q.e.d.

Proposition 6.13. Every partially proper semialgebraic map is proper.

Proof. Let $f : M \to N$ be partially proper and semialgebraic. Choose an admissible covering $(N_\alpha|\alpha\in I)$ of N by open semialgebraic subsets. The sets $f^{-1}(N_\alpha)$ are open semialgebraic in M. By Theorem 6.12 the maps $f^{-1}(N_\alpha) \to N_\alpha$ are partially proper, which in this case means they are proper. Thus f is proper. q.e.d.

If M is paracompact then it is now clear from Theorem 5.9 that a partially proper locally semialgebraic map $f : M \to N$ is proper if and only if f is semialgebraic.

We verify that an important class of locally semialgebraic maps consists of partially proper maps.

Definition 3. A locally semialgebraic map $f : M \to N$ is called weakly locally trivial if every point y of N has a semialgebraic neighbourhood L such that there exists a commutative triangle

$$\begin{tikzcd} f^{-1}(L) \arrow[rr, "\sim", "\varphi"'] \arrow[dr, "f|f^{-1}(L)"'] & & L \times f^{-1}(y) \arrow[dl, "pr_1"] \\ & L & \end{tikzcd} \qquad (*)$$

with φ a locally semialgebraic isomorphism and pr_1 the natural projection.

Remark. This notion is too weak for many - but not all - purposes. We call $f : M \to N$ locally trivial if N has an open admissible covering $(N_\alpha | \alpha \in I)$ such that every $f^{-1}(N_\alpha)$ is isomorphic to a product $N_\alpha \times F_\alpha$ over N_α.

Example 6.14. Every weakly locally trivial map $f : M \to N$ with partially complete fibres is partially proper.

Indeed, in the triangle (*) above the map pr_1 is partially proper by Theorem 6.12. Thus $f|f^{-1}(L)$ is partially proper for some neighbourhood L of any $y \in N$. By Corollary 6.5 the map f is partially proper.

The "locally semialgebraic covering" which we mentioned in the preface will be defined as the locally trivial maps with discrete (Ex. 3.19) fibres. These maps are certainly partially proper, a fact which is of good use to the theory of coverings, cf. Chapter V.

We turn to another example. Let $f : X \to Y$ be a simplicial map between strictly locally finite simplicial complexes (cf. definitions at the end of §2). Under which combinatorial conditions is f partially proper?

Let $\overline{f} : \overline{X} \to \overline{Y}$ denote the closure of f. This map is always partially proper since $\bar{X}$ is partially complete. Thus the simplicial map $g : \bar{f}^{-1}(Y) \to Y$ obtained from $\bar{f}$ by restriction is also partially proper (Th. 6.12). Looking at the commutative triangle

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & \bar{f}^{-1}(Y) \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

with j the inclusion map we see that f is partially proper if and only if j is partially proper (Remarks 6.2.i, ii). Now if j is partially proper then, for every open simplex σ of X, the semialgebraic set $\bar{\sigma} \cap X$ must be closed in $\bar{f}^{-1}(Y)$. This means $\bar{\sigma} \cap X = \bar{\sigma} \cap \bar{f}^{-1}(Y)$, and we see that f is partially proper if and only if $X = \bar{f}^{-1}(Y)$. Since X is paracompact the map f is proper if and only if in addition $\bar{f}$ is semialgebraic. This means that the preimage of every finite subcomplex of Y under f is a finite subcomplex of X. Thus we arrive at the following

Example 6.15. Let $f : X \to Y$ be a simplicial map between strictly locally finite complexes. Then f is partially proper if and only if the following holds for every open simplex $\sigma \in \Sigma(X)$: if τ is an open face of σ with $\bar{f}(\tau) \in \Sigma(Y)$, then $\tau \in \Sigma(X)$. f is proper if and only if, in addition, there exists only finitely many simplices $\sigma \in \Sigma(X)$ with $f(\sigma) = \rho$ for every $\rho \in \Sigma(Y)$.

A great amount of our work in these Lecture Notes will be spent on a rather special class of partially proper maps, the partially finite maps.

Definition 4. A locally semialgebraic map $f:M \to N$ is called finite, if f is proper and has discrete - hence finite (cf. Lemma 5.8) - fibres. f is called partially finite, if f is partially proper and has discrete fibres. As is easily seen f is partially finite if and only if the restriction $f|A : A \to N$ of f to every closed semialgebraic subset of M is finite. (Hint: Use the fact that a space which does not admit any non constant path is discrete.)

Caution. A partially proper map with finite fibres is not necessarily finite. For example, assume that R contains a sequence of positive elements converging to zero, and that M is an affine locally complete - but not complete - semialgebraic space. The space M_{loc} is partially complete. Thus the natural map $p : M_{loc} \to M$ is partially proper. But p is not proper, hence not finite. Indeed, M_{loc} is paracompact (Ex. 4.4.a). If p would be proper then, by Theorem 5.9, M_{loc} would be semialgebraic, which is not true.

Looking for examples of partially finite maps we state

Proposition 6.16. Let $f : X \to Y$ be a simplicial map between strictly locally finite simplicial complexes. The following are equivalent:

i) $\bar{f}$ is injective on the set $E(\sigma)$ of vertices of every $\sigma \in \Sigma(X)$.

ii) f has discrete fibres.

iii) $\bar{f} : \bar{X} \to \bar{Y}$ is partially finite.

Proof. i) $\Rightarrow$ iii): $\bar{f}$ maps the closure $\bar{\sigma}$ of every $\sigma \in \Sigma(X)$ isomorphically onto the closed simplex $\overline{f(\sigma)}$, hence also the closure $\bar{\sigma}$ of every $\sigma \in \Sigma(\bar{X})$ isomorphically onto the closed simplex $\bar{f}(\bar{\sigma})$. Thus every fibre of $\bar{f}$ meets every closed simplex in at most one point. A fortiori it meets every open simplex in at most one point. Since $\bar{X}$ is the union of its finite open subcomplexes, all fibres of f are discrete. The implication iii) $\Rightarrow$ ii) is trivial. ii) $\Rightarrow$ i): Suppose there exists an open simplex σ in X such that $\bar{f}$ is not injective on $E(\sigma)$. Then all fibres of the affine map $f|\sigma$ from σ to $f(\sigma)$ contain line segments and thus have at least dimension one. This contradicts the assumption that f has discrete fibres.

q.e.d.

Example 6.17. Let again $f : X \to Y$ be a simplicial map between strictly locally finite simplicial complexes. Then we see from Example 6.15 and the preceding Proposition 6.16 that f is partially finite iff $\bar{f}$ is injective on the set of vertices $E(\sigma)$ of every $\sigma \in \Sigma(X)$ and in addition $\tau \in \Sigma(X)$ for every open face τ of every $\sigma \in \Sigma(X)$ with $\bar{f}(\tau) \in \Sigma(Y)$. The map f is finite, iff, in addition, for every $\rho \in \Sigma(Y)$, there exist only finitely many $\sigma \in \Sigma(X)$ with $f(\sigma) = \rho$. Then f is also affine semialgebraic.

Remark 6.18. The statement in Examples 6.15 and 6.17 remain true if the complexes X and Y are only locally finite instead of strictly locally finite. This can be seen for the first statement in 6.15 by use of the path completion criterion for partial properness (Th. 6.8). Then the other statements can be deduced in much the same way as before.

We mention yet another class of examples of finite affine semialgebraic maps. Every finite morphism $\varphi : V \to W$ between algebraic varieties V and W over R (in the sense of algebraic geometry) yields a finite affine semialgebraic map $f = \varphi_R : V(R) \to W(R)$. As we shall explicate in the next section, the semialgebraic spaces V(R) and W(R) are locally complete (7.1.c and f may be also regarded as a locally semialgebraic map f_{loc} from the partially complete space $V(R)_{loc}$ to $W(R)_{loc}$. The semialgebraic map f is proper. This implies, by the definition of $V(R)_{loc}$ and $W(R)_{loc}$ that the locally semialgebraic map f_{loc} is affine semialgebraic. As a partially proper semialgebraic map f_{loc} is even proper. Thus f_{loc} is a finite affine semialgebraic map.

There remains the question, how our central notions "proper", "partially proper", "finite", "partially finite" behave under extension of the base field R to some real closed overfield $\tilde{R}$. This seems to be more difficult than the considerations about extension of the base field, say, at the end of §3. We shall deal with this question in Appendix B.

§7 - Locally complete spaces

As before the word "space", if used without further specification, means "locally semialgebraic space" (of course separated) over our fixed real closed base field R.

Definition 1. A space M is called locally complete, if every $x \in M$ has a semialgebraic neighbourhood which is a complete semialgebraic space.

By Corollary 5.10 every such neighbourhood K is an affine semialgebraic space. It is then clear from the regularity of K that x has a fundamental system of complete semialgebraic neighbourhoods in K, hence in M. In particular, every locally complete space M is regular. If M is also semialgebraic, then M is affine.

Locally complete semialgebraic spaces have been considered before in this chapter (Examples 2.6, 4.4; Remark 4.20). There the point was that such a space M leads naturally to a partially complete regular space M_{loc} which, under favourable circumstances, is also paracompact. We shall generalize the construction of M_{loc} below to an arbitrary locally complete space M.

Examples 7.1. a) Every partially complete regular space M is locally complete.

b) Every closed or open subspace of a locally complete space M is locally complete. Thus every locally closed subspace X of M ($X \in \mathcal{T}(M)$, X open in $\overline{X}$) is locally complete.

c) For any algebraic variety V over R the semialgebraic space V(R) is locally complete.

Indeed, if V is affine, V can be embedded as a closed subvariety into an affine standard variety $\mathbb{A}_R^n$. Then V(R) is a closed semialgebraic subspace of R^n. Since R^n is locally complete, $V(R)$ is also locally complete. In general, we cover V by finitely many Zariski-open subvarieties $V_1,\dots,V_r$, so $V(R)$ is covered by the finitely many open semialgebraic subsets $V_i(R)$ which are locally complete. Thus $V(R)$ is locally complete. $V(R)$ is also semialgebraic, hence affine semialgebraic. We shall explain in the second volume that for any variety W over $R(\sqrt{-1}) = C$ the set of geometric points $W(C)$ is also a semialgebraic space over R in a natural way (cf. [K_1, §1]). It then will be clear that $W(C)$ is locally complete. Additionally, the infinite coverings of $V(R)$ (resp. $W(C)$) - to be studied in Chapter V - are locally complete, since they are "locally isomorphic" to $V(R)$ (resp. $W(C)$). These spaces are no longer semialgebraic.

d) Direct products of finitely many locally complete spaces are locally complete. The fibre product $M_1 \times_N M_2$ of two locally complete spaces M_i with respect to any locally semialgebraic maps $M_i \to N$ $(i = 1,2)$ is locally complete since it is a closed subspace of $M_1 \times M_2$.

e) Going back to Example 2.9, assume that the space M there is locally complete. Then the space X constructed there is also locally complete.

f) We shall see in Appendix B that, for any locally complete space M over R and any real closed overfield $\tilde{R}$ of R, the space $M(\tilde{R})$ is locally complete.

Proposition 7.2. Let M be a dense locally semialgebraic subset of a space N and $x \in M$. Then every complete locally semialgebraic neighbourhood K of x in M is also a neighbourhood of x in N. In particular, if M is a locally complete subspace of a space L, then M is open in $\overline{M}$, the closure of M in L, i.e., M is locally closed in L.

Proof. Since K is complete, K is also closed in N. Hence N is the union of K and the closure $\overline{M \setminus K}$ of $M \setminus K$. Now $x \notin \overline{M \setminus K}$ and thus K is also a neighbourhood of x in N. q.e.d.

Let X be a locally finite simplicial complex over R (cf. §2). We look for a "combinatorial" criterion that X - regarded as a space - is locally complete. A small difficulty arises from the fact that $\overline{X}$ needs not be locally finite and therefore needs not have an interpretation as a space.

Proposition 7.3. The following are equivalent:

a) The space X is locally complete.

b) For every open simplex τ of X the stars $St_X(\tau)$ and $St_{\overline{X}}(\tau)$ are equal.

c) The complex $\overline{X} \setminus X$ is closed.

Proof. The equivalence b) $\Leftrightarrow$ c) is fairly obvious. X is the union of the subcomplexes $St_X(\tau)$, $\tau \in \Sigma(X)$, and these are open in X. If $\overline{X} \setminus X$ is closed, i.e. X is an open subcomplex of $\overline{X}$, then all stars $St_X(\tau)$ are open subcomplexes of $\overline{X}$. But $St_{\overline{X}}(\tau)$ is the smallest open subcomplex of $\overline{X}$ containing τ. Thus $St_{\overline{X}}(\tau) = St_X(\tau)$ for every $\tau \in \Sigma(X)$. Conversely, if this is true, then the complexes $St_X(\tau)$ are open in $\overline{X}$. Thus also X is open in $\overline{X}$.

a $\Rightarrow$ b): Fix some $\tau \in \Sigma(X)$ and let Y denote the finite open subcomplex $St_X(\tau)$ of X. Since X is locally complete, the open subspace Y of X is also locally complete. By proposition 7.2, Y is open in $\overline{Y}$. Now let σ be an open simplex of $St_{\overline{X}}(\tau)$, i.e. $\sigma \in \Sigma(\overline{X})$ and $\tau \leq \sigma$. Choose some $\rho \in \Sigma(X)$ with $\sigma \leq \rho$. Then $\rho \subset Y$ and $\sigma \subset \overline{Y}$. Since Y is open in $\overline{Y}$ and $\tau \leq \sigma$ we conclude that $\sigma \subset Y$. Thus $Y = St_{\overline{X}}(\tau)$.

b $\Rightarrow$ a): The family $(St_X(\tau), \tau \in \Sigma(X))$ is an (admissible) covering of the space X by open semialgebraic subets. It suffices to verify that, for any fixed $\tau \in \Sigma(X)$, the space $Y := St_X(\tau)$ is locally complete. By assumption, $Y = St_{\overline{X}}(\tau)$. Thus Y is an open subcomplex of $\overline{X}$. A fortiori, Y is an open subcomplex of the finite closed complex $\overline{Y}$. We conclude that Y, as a space, is open semialgebraic in the complete space $\overline{Y}$, and therefore Y is locally complete. q.e.d.

We shall see in Chapter II, §2 that every regular paracompact space is isomorphic to a subspace of some partially complete regular paracompact space. Once we know this, it is clear from Proposition 7.2 that the locally complete regular paracompact spaces are, up to isomorphism, just the open subspaces of the partially complete regular paracompact spaces.

In the case of a semialgebraic locally complete space M we can do better. We define a generalized topological space (cf. §1, Def. 1) $(M^+, \mathring{\gamma}(M^+), Cov_{M^+})$ as follows. The set M^+ is the disjoint union of M and one further point ∞. A subset U of M^+ is an element of $\mathring{\gamma}(M^+)$ if either $U \in \mathring{\gamma}(M)$ or $U = (M \setminus K) \cup \{\infty\}$ with K a complete semialgebraic subset of M. A family $(U_\alpha | \alpha \in I)$ in $\mathring{\gamma}(M^+)$ is an element of Cov_{M^+} if the union U of the U_α is an element of $\mathring{\gamma}(M^+)$ and U is actually the union of finitely many U_α. It is easily checked that the triple $(M^+, \mathring{\gamma}(M^+), Cov_{M^+})$ is indeed a generalized topological space. For every $U \in \mathring{\gamma}(M^+)$ we define a ring $\mathcal{O}_{M^+}(U)$ of R-valued functions on U as follows. If $U \in \mathring{\gamma}(M)$ then $\mathcal{O}_{M^+}(U) := \mathcal{O}_M(U)$. If $U = (M \setminus K) \cup \{\infty\}$, then a function $f: U \to R$ is an element of $\mathcal{O}_{M^+}(U)$ if the restriction $f|M \setminus K$ is an element of $\mathcal{O}_M(M \setminus K)$ and if f is also continuous at ∞ in the strong topology of U, i.e. for every $\varepsilon > 0$ in R there exists some complete semialgebraic set $L \supset K$ in M with $|f(x) - f(\infty)| < \varepsilon$ for all $x \in M \setminus L$. Clearly the assignment $U \mapsto \mathcal{O}_{M^+}(U)$ is a sheaf $\mathcal{O}_{M^+}$ of rings of functions on $(M^+, \mathring{\gamma}(M^+), Cov_{M^+})$. Henceforth we simply denote the ringed space $(M^+, \mathring{\gamma}(M^+), Cov_{M^+}, \mathcal{O}_{M^+})$ by M^+.

<u>Theorem 7.4.</u> M^+ is a complete semialgebraic space over R.

We call M^+ the <u>one-point completion</u> of M. If M is already complete, then M^+ is the direct sum of M and the one point space $\{\infty\}$. The theorem is trivial in this case.

In order to prove the theorem we start by considering the special case $M = R^n$. Let S^n denote the unit sphere in R^{n+1}, let e denote the north pole $(0,0,\ldots,1)$ of S^n, and let $p : S^n \setminus \{e\} \xrightarrow{\sim} R^n$ denote the stereographic projection, $p(x_1,\ldots,x_{n+1}) = (y_1,\ldots,y_n)$ with

$$y_i = x_i(1-x_{n+1})^{-1},\ 1 \leq i \leq n.$$

This is a semialgebraic isomorphism with the inverse map $(y_1,\ldots,y_n) \mapsto (x_1,\ldots,x_{n+1})$,

$$x_{n+1} = (r^2-1)(r^2+1)^{-1},\ x_i = 2y_i(r^2+1)^{-1}$$

for $1 \leq i \leq n$, where $r^2 = y_1^2 + \ldots + y_n^2$. We extend p to a bijection $\bar{p}$ from S^n to $(R^n)^+$ by $\bar{p}(e) = \infty$. It is easily checked that $\bar{p}$ is an isomorphism of the semialgebraic space S^n with the ringed space $(R^n)^+$. Henceforth we identify $(R^n)^+$ with S^n using this map $\bar{p}$.

Lemma 7.5. Any locally complete semialgebraic space M can be embedded in some R^n as a closed subspace.

Proof. Embed M somehow in a space R^m. Then M is open in its closure $\bar{M}$. Now embed M into $R^m \times R$ as the graph of the function $x \mapsto d(x,\bar{M} \setminus M)^{-1}$ with $d(-,\bar{M} \setminus M)$ the euclidean distance function from $\bar{M} \setminus M$. (This very classical proof, also contained in [DK_5, §3], is repeated here for the convenience of the reader).

Using this lemma, the proof of Theorem 7.4 is easy. We may assume that M is a closed subspace of R^n for some $n \geq 1$. We regard M^+ as a subset of $(R^n)^+$ identifying the point ∞ in M^+ with the point ∞ in $(R^n)^+$. This subset is closed and semialgebraic in $(R^n)^+$. It is now easily checked that the subspace structure on M^+ in $(R^n)^+$ coincides with $(\mathring{\mathcal{T}}(M^+), Cov_{M^+}, \mathcal{O}_{M^+})$.

Proposition 7.6. Let N be a partially complete space and let V be an open locally semialgebraic subset of N. Let $f : V \to M$ be a locally semialgebraic map from V to a locally complete semialgebraic space M. Extend f to a map $g : N \to M^+$ by $g(N \setminus V) = \{\infty\}$. Then the map g is locally semialgebraic (i.e. a morphism of ringed spaces) if and only if f is partially proper.

Proof. We have $g^{-1}(M) = V$. If g is a morphism, then g is partially proper, since N is partially complete. This implies that the restriction $f : g^{-1}(M) \to M$ of g is also partially proper. On the other hand, assume that f is partially proper. Once we know that g is continuous, then it is evident that g is locally semialgebraic. Indeed, for any semialgebraic subset W of N, the graph of $g|W$ is the union of the graph of $f|W \cap V$ and $g|W \setminus V$, and thus is semialgebraic. To verify continuity we have to check that for any $U \in \mathring{\gamma}(M^+)$ the preimage $g^{-1}(U)$ is open in N. This is obvious if $U \subset M$. Otherwise, $M^+ \setminus U$ is a complete semialgebraic subset K of M. Then $g^{-1}(K) = f^{-1}(K)$ is a partially complete locally semialgebraic subset of N. In particular, $g^{-1}(K)$ is closed in N, and $g^{-1}(U)$ is open in N.

Example 7.7. Any proper semialgebraic map $f : N \to M$ between locally complete semialgebraic spaces extends to a (proper) semialgebraic map $f^+ : N^+ \to M^+$ with $f^+(\infty) = \infty$.

It is also clear from Proposition 7.6 that, if a locally complete semialgebraic space M is embedded in a complete semialgebraic space $\bar{M}$ with $\bar{M} \setminus M$ a one point set, then $\bar{M}$ may be regarded as the one point completion M^+ of M. Indeed, this is trivial if M is complete. Otherwise, just apply the proposition to $N = \bar{M}$, $V = M$, f the identity map $M \to M$, and recall that a bijective proper map is an isomorphism.

Given a locally complete space M we now define a partially complete space M_{loc}, generalizing the construction in Example 2.6. This can be done in a straightforward way. Let $\gamma_c(M)$ denote the set of all complete semialgebraic subsets of M, and let $\mathring{\gamma}_c(M)$ denote the set of all open semialgebraics $U \in \mathring{\gamma}(M)$ with $\bar{U} \in \gamma_c(M)$. There is a tiny point here - if M is not paracompact, we do not know whether $\bar{U}$ is semialgebraic for every $U \in \mathring{\gamma}(M)$ (cf. Example 3.15). This subtlety will not disturb us. Both sets $\mathring{\gamma}_c(M)$ and $\gamma_c(M)$, ordered by inclusion, are directed systems of subspaces of $M^{*)}$.

We define M_{loc} as the inductive limit of the system $\mathring{\gamma}_c(M)$ in the category of ringed spaces over R with underlying set M. As observed in greater generality in Examples 2.9 and 4.2.a, M_{loc} is a regular locally semialgebraic space and $\mathring{\gamma}_c(M)$ is an admissible covering of M_{loc} by open semialgebraics. Furthermore, for each $U \in \mathring{\gamma}_c(M)$ the subspace structures on U with respect to M and M_{loc} are the same. From this we conclude that in fact $\mathring{\gamma}_c(M) = \mathring{\gamma}(M_{loc})$, and that M_{loc} is locally complete.

Every $U \in \mathring{\gamma}_c(M)$ is contained in the set $\bar{U} \in \gamma_c(M)$. Also, every $K \in \gamma_c(M)$ is contained in some $V \in \mathring{\gamma}_c(M)$. Indeed, K is contained in some $W \in \mathring{\gamma}(M)$, and the open subspace W of M is locally complete and semialgebraic. We have seen in Example 2.6 that K has a complete semialgebraic neighbourhood A in W. Thus $\mathring{A} \in \mathring{\gamma}_c(M)$ and $\mathring{A} \supset K$. Hence the directed systems of spaces $\mathring{\gamma}_c(M)$ and $\gamma_c(M)$ are equivalent, and we conlcude that

(7.8) $\qquad M_{loc} = \varinjlim \gamma_c(M)$.

Moreover, we see that every $K \in \gamma_c(M)$ is - in its given space structure - a complete subspace of M_{loc}, and that $\gamma_c(M_{loc}) = \gamma_c(M)$.

The inclusion maps $K \hookrightarrow M$ with K running through $\gamma_c(M)$ yield - by the very nature of the inductive limit (7.8) - a canonical locally semi-

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*) Every set of subsets of M will be regarded as a family of subsets of M, each member of the set being indexed by itself.

algebraic map

$$p_M : M_{loc} \to M.$$

We want to characterize the map p_M by a universal property. For this we need a new definition.

Definition 2. A space M is called taut, if the closure $\overline{X}$ of any semialgebraic subset X of M is again semialgebraic.

Examples 7.9. a) As observed in Proposition 4.6, every paracompact space is taut. The property "taut" should be considered as a weakening of the property "paracompact", needed for technical reasons.

b) If N is a taut space and $f : M \to N$ is a semialgebraic map, then M is taut. Indeed, given some $X \in \gamma(M)$, the set $Y := f(X)$ is semialgebraic in N. Thus $\overline{Y} \in \overline{\gamma}(N)$. Since f is semialgebraic, we conclude that $f^{-1}(\overline{Y}) \in \overline{\gamma}(M)$, and then that $\overline{X} \in \overline{\gamma}(M)$.

c) Every subspace of a taut space is taut.

d) The direct product $M_1 \times M_2$ of taut spaces M_1 and M_2 is taut. More generally, the fibre product $M_1 \times_N M_2$ with respect to arbitrary locally semialgebraic maps $M_1 \to N$ and $M_2 \to N$ is taut, since $M_1 \times_N M_2$ is a subspace of $M_1 \times M_2$.

e) If M is a taut space over R, then for any real closed field $\tilde{R} \supset R$ the space $M(\tilde{R})$ is taut. Indeed, for any semialgebraic subset X of $M(\tilde{R})$ there exists some open semialgebraic subset U of M with $X \subset U(\tilde{R})$. The closure $\overline{U}$ of U is semialgebraic. Its extension $\overline{U}(\tilde{R})$ is semialgebraic and contains $\overline{X}$. Hence $\overline{X} \in \gamma(M(\tilde{R}))$. Conversely, if $M(\tilde{R})$ is taut, then certainly M is taut (cf. Prop. 3.22.b) and c) and Th. B.1.i in App. B).

Remark 7.10. Every taut and partially complete space N is locally complete. Indeed, given some point x in N, we choose an open semialgebraic neighbourhood U of x. Then the closure $\overline{U}$ of U is a complete semialgebraic neighbourhood of x. In particular, N is regular.

We come back to the natural map $p_M : M_{loc} \to M$ for M an arbitrary locally complete space over R.

<u>Theorem 7.11.</u> The space M_{loc} is taut and partially complete. p_M has the following universal property. Given any locally semialgebraic map $f : N \to M$ with N a taut and partially complete space over R, there exists a unique locally semialgebraic map $g : N \to M_{loc}$ such that $f = p_M \circ g$.

$$\begin{tikzcd} & M_{loc} \arrow[d, "p_M"] \\ N \arrow[ur, dashed, "g"] \arrow[r, "f"'] & M \end{tikzcd}$$

<u>Proof.</u> Let U be an open semialgebraic subset of M_{loc}. Then $U \in \mathring{\gamma}_c(M)$. The closure $\bar{U}$ of U in M is also the closure of U in M_{loc}, since the strong topologies of M and M_{loc} are the same. We have $\bar{U} \in \gamma_c(M) = \gamma_c(M_{loc})$. In particular, $\bar{U}$ is semialgebraic in M_{loc}, and we see that M_{loc} is taut.

Let A be a closed semialgebraic subset of M_{loc}. Then A is closed in M, again since the strong topologies of M_{loc} and M are the same. Moreover, $A \in \gamma(U)$ for some $U \in \mathring{\gamma}_c(M)$. Thus $A \in \bar{\gamma}(K)$ for $K := \bar{U} \in \gamma_c(M)$. We conclude that $A \in \gamma_c(M) = \gamma_c(M_{loc})$. This proves that M_{loc} is partially complete.

Now consider a locally semialgebraic map $f : N \to M$ with N taut and partially complete. We are looking for a map $g : N \to M_{loc}$ with $f = p_M \circ g$ and are, of course, forced to define $g : N \to M_{loc}$ to be the same as f as a map between sets. We have to verify that this map g is locally semialgebraic. It suffices to check that, for any given $U \in \mathring{\gamma}(N)$, the map $g|U : U \to M_{loc}$ is semialgebraic. Since N is taut and partially complete, the closure $L := \bar{U}$ of U in N is semialgebraic and complete. If we know that $g|L$ is semialgebraic, then, of course, $g|U$ must also be semialgebraic. But this is trivial. $K := f(L)$ is an element of $\gamma_c(M)$, and $g|L$ is the composite of the semialgebraic map $h : L \to K$, obtained from f by restriction, and the inclusion map $K \hookrightarrow M_{loc}$ which is also semialgebraic. q.e.d.

It follows from Theorem 7.11 that any locally semialgebraic map $f : M \to N$ between locally complete spaces over R yields a unique locally semialgebraic map $f_{loc} : M_{loc} \to N_{loc}$ with $p_N \circ f_{loc} = f \circ p_M$. Moreover, the universal property described in the theorem is quite useful for analyzing the behavior of the functor $M \mapsto M_{loc}$ from the category of locally complete spaces over R to the full subcategory of taut partially complete spaces over R. We state two facts about this functor.

Let $f_i : M_i \to N$ $(i = 1,2)$ be locally semialgebraic maps between locally complete spaces over R and let $M_1 \times_N M_2 = M$ be the fibre product of M_1 and M_2 with respect to the f_i. Also let $P := M_{1,loc} \times_{N_{loc}} M_{2,loc}$ be the fibre product of $M_{1,loc}$ and $M_{2,loc}$ with respect to the maps $(f_i)_{loc}$. By Example 7.1.d the space M is locally complete and by Example 7.9.d and §6 the space P is taut and partially complete.

<u>Proposition 7.12.</u> The natural map $p : P \to M$ induced by the maps $p_{M_i} : (M_i)_{loc} \to M_i$ and $p_N : N_{loc} \to N$ has the universal property for M described in Theorem 7.11. In short,

$$(M_1 \times_N M_2)_{loc} = M_{1,loc} \times_{N_{loc}} M_{2,loc}$$

This is easily verified.

<u>Proposition 7.13.</u> Let $f : M \to N$ be a partially proper map between spaces over R. Assume that N is locally complete. Also assume that either M is taut or that f is semialgebraic. Then M is locally complete, and

the diagram

$$\begin{tikzcd} M_{loc} \arrow[r, "f_{loc}"] \arrow[d, "p_M"'] & N_{loc} \arrow[d, "p_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is a pullback in the category of spaces over R. In short

$$M_{loc} = M \times_N N_{loc}.$$

Proof. Let $x \in M$ be given. We choose a complete semialgebraic neighbourhood A of f(x) in N. Then $f^{-1}(A)$ is a closed neighbourhood of x in M, and the restriction $f^{-1}(A) \to A$ of f is partially proper. If f is semialgebraic, then $f^{-1}(A)$ is semialgebraic and complete. If M is taut, then x certainly has a closed semialgebraic neighbourhood D. Also, $B := D \cap f^{-1}(A)$ is a closed semialgebraic neighbourhood of x and the restriction $B \to A$ of f is proper. We conclude that B is complete. Thus, in both cases, M is locally complete. We consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "q"'] & N_{loc} \arrow[d, "p_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

for the maps f and p_N. The map g is partially proper. Thus P is partially complete. If f is semialgebraic, then g is semialgebraic, and we conclude from the tautness of N_{loc} that P is taut (Example 7.9.b). If M is taut, then P is also taut, since it is the fibre product of two taut spaces (Example 7.9.d). Now it is easily checked that q has the universal property described in Theorem 7.11 since p_N has this property. Thus q can be identified with p_M. q.e.d.

The constructions in this section enable us to strengthen one of the counterexamples in 6.3 for partially proper maps.

Counterexample 7.14. If $f : N \to M$ and $g : M \to S$ are locally semialgebraic maps with f surjective and $g \circ f$ partially finite then it may happen that g is not partially finite. Choose for example some semialgebraic locally complete space M which is not complete. Let f be the natural map $p_M : M_{loc} \to M$ and g be the inclusion map $M \hookrightarrow M^+$. Then f is surjective and $g \circ f$ is partially proper, since M_{loc} is partially complete. The fibres of $g \circ f$ are finite (empty or one-point sets). Thus $g \circ f$ is partially finite. But g is not partially finite.

We have seen that the notion of tautness is - besides partial completeness - the key to understand the spaces M_{loc} from a categorial view point (Th. 7.11). We shall later restrict our attention to paracompact spaces instead of taut spaces. The following proposition makes clear that this means to add a hypothesis of coutability.

Proposition 7.15. For a space M the following are equivalent:

i) M is paracompact and has countably many connected components.

ii) M is taut and Lindelöf (cf. §4, Def. 3).

Proof. The implication i) $\Rightarrow$ ii) is contained in Proposition 4.6 and Theorem 4.17. In order to prove ii $\Rightarrow$ i) we choose an admissible covering $(X_n \mid n \in \mathbb{N})$ of M by open semialgebraic subsets X_n. We then choose a family $(U_n \mid n \in \mathbb{N})$ in $\mathring{\gamma}(M)$ such that $U_1 \supset X_1$ and $U_n \supset X_n \cup \overline{U}_{n-1}$ for $n \geq 2$. We finally define open semialgebraic sets $V_1 := U_1$, $V_2 := U_2$, $V_n := U_n \setminus \overline{U}_{n-2}$ $(n \geq 3)$. The V_n cover the set M and every V_n meets at most three of the sets X_k. Thus the covering $(V_n \mid n \in \mathbb{N})$ of M is locally finite, which proves that M is paracompact. Of course, M has only countably many connected components, since every X_n meets only finitely many connected components. q.e.d.

Chapter II. - Completions and triangulations

§1 - Gluing paracompact spaces

Whereas in the first chapter we were mostly concerned with the analysis and explication of the fundamental notions, we now emphasize constructions. In this section we deal with the problem of gluing locally semialgebraic spaces along closed subspaces. Our central result, Theorem 1.3 below, will be useful for constructing "completions" of paracompact spaces, cf. §2 below. It is also interesting in its own right.

For technical reasons, we now do not assume that locally semialgebraic spaces are always understood to be separated. But if a space is called "paracompact" then we mean "separated and paracompact", as in Chapter I.

Ultimately we are only interested in separated spaces, or better, in regular spaces. We have to make sure that gluing separated (resp. regular) spaces always results in separated (resp. regular) spaces. This is guaranteed by

Lemma 1.1. Let M be a locally semialgebraic space over R, and let $(M_\alpha|\alpha\in I)$ be a locally finite family in $\overline{\mathfrak{T}}(M)$ with $M = \cup(M_\alpha|\alpha\in I)$. Assume that the subspaces M_α of M are all separated (resp. regular). Then M is also separated (resp. regular).

Proof. Assume first that all M_α are regular. Let x be a point in M and B be a closed locally semialgebraic subset of M with $x \notin B$. The family $(M_\alpha|\alpha\in I)$ is locally finite. Hence $I(x) := \{\alpha \in I | x \in M_\alpha\}$ is a finite set and $A_x := \cup(M_\alpha|\alpha\in I \setminus I(x))$ is a closed locally semialgebraic subset of M. We want to separate x and B by open locally semialgebraic sets. Replacing B by $B \cup A_x$ if necessary, we may assume that $A_x \subset B$. For every $\alpha \in I(x)$ there exists an $U_\alpha \in \mathring{\mathcal{T}}(M_\alpha)$ with $x \in U_\alpha$ and $\bar{U}_\alpha \cap B = \emptyset$. We choose sets $V_\alpha \in \mathring{\mathcal{T}}(M \setminus B)$ with $V_\alpha \cap M_\alpha = U_\alpha$ $(\alpha \in I(x))$, cf. I, Prop. 3.11. Then $V := \cap(V_\alpha | \alpha \in I(x))$ is an open semialgebraic neighbourhood of x. The closed semialgebraic set $Z := \cup(\bar{U}_\alpha | \alpha \in I(x))$ contains V and is disjoint from B. Thus x and B are separated by the open locally semialgebraic sets V and $M \setminus Z$. This proves that M is regular. Running through similar arguments, we see that M is separated if all sets M_α are separated spaces. q.e.d.

<u>Lemma 1.2.</u> Let M be a locally semialgebraic space over R and let $(M_\alpha | \alpha \in I)$ be a <u>locally finite</u> family in $\bar{\mathcal{T}}(M)$ with $\cup(M_\alpha | \alpha \in I) = M$. Then the structure of M as a locally ringed space is completely determined by the subspaces M_α in the following way:

a) A subset U of M belongs to $\mathring{\mathcal{T}}(M)$ if and only if $U \cap M_\alpha \in \mathring{\mathcal{T}}(M_\alpha)$ for every $\alpha \in I$.

b) A family $(U_\lambda | \lambda \in \Lambda)$ in $\mathring{\mathcal{T}}(M)$ belongs to Cov_M if and only if $(U_\lambda \cap M_\alpha | \lambda \in \Lambda) \in \mathrm{Cov}_{M_\alpha}$ for every $\alpha \in I$.

c) A function $f : U \to R$ on some $U \in \mathring{\mathcal{T}}(M)$ belongs to $\mathcal{O}_M(U)$ if and only if $f | U \cap M_\alpha \in \mathcal{O}_{M_\alpha}(U \cap M_\alpha)$ for every $\alpha \in I$.

Furthermore, if all subspaces M_α are paracompact, then M is also paracompact.

<u>Proof.</u> The statement (c) has already been proved in I, §3 (gluing of locally semialgebraic maps, cf. Prop. 3.16). Notice that also the "only if" parts in the statements (a) and (b) are trivial. To prove (a) consider a subset U of M with $U \cap M_\alpha \in \mathring{\mathcal{T}}(M_\alpha)$ for every $\alpha \in I$. Then $(M \setminus U) \cap M_\alpha \in \bar{\mathcal{T}}(M_\alpha)$ for every $\alpha \in I$. We conclude that $M \setminus U \in \bar{\mathcal{T}}(M)$ since it is the union of the <u>locally finite</u> family $((M \setminus U) \cap M_\alpha | \alpha \in I)$ of closed locally semialgebraic subsets. Thus U is indeed an element of $\mathring{\mathcal{T}}(M)$.

Now let $(U_\lambda|\lambda\in\Lambda)$ be a family in $\mathring{\mathfrak{T}}(M)$ with $(U_\lambda \cap M_\alpha|\lambda \in \Lambda) \in \mathrm{Cov}_{M_\alpha}$ for every $\alpha \in I$. Then, by (a), $U := \cup(U_\lambda|\lambda\in\Lambda)$ is an element of $\mathring{\mathfrak{T}}(M)$. Consider an open semialgebraic subset M' of M. Only finitely many sets M_α meet M'. Every $U \cap M_\alpha \cap M'$ is covered by finitely many sets U_λ. Thus $U \cap M'$ is also covered by finitely many sets U_λ. Hence $(U_\lambda|\lambda\in\Lambda)$ is an admissible covering of U. Statement (b) is proven.

Assume, finally, that all spaces M_α are paracompact. We want to show that M is also paracompact. M is separated by Lemma 1.1. Let $(M_{\alpha\beta}|\beta\in I_\alpha)$ be a locally finite covering of M_α by open semialgebraic subsets. Every closure $\overline{M}_{\alpha\beta}$ is contained in the union of the finitely many sets $M_{\alpha\gamma}$ with $M_{\alpha\gamma} \cap M_{\alpha\beta} \neq \emptyset$. Thus $\overline{M}_{\alpha\beta}$ is semialgebraic and $(\overline{M}_{\alpha\beta}|\alpha \in I, \beta \in I_\alpha)$ is a locally finite family in $\overline{\gamma}(M)$ with union M. Replacing $(M_\alpha|\alpha\in I)$ by $(\overline{M}_{\alpha\beta}|\alpha \in I, \beta \in I_\alpha)$ we may even assume that all spaces M_α are semialgebraic. For any $\alpha \in I$ we introduce the finite sets of indices

$$I(\alpha) := \{\beta \in I | M_\beta \cap M_\alpha \neq \emptyset\}$$

and the open locally semialgebraic neighbourhood

$$W_\alpha := M \setminus \cup(M_\beta | \beta \in I \setminus I(\alpha))$$

of M_α. Since W_α is covered by the finitely many semialgebraic sets M_β with $\beta \in I(\alpha)$, we see that W_α is actually open semialgebraic. The family $(W_\alpha|\alpha \in I)$ is locally finite and covers M. Thus M is indeed paracompact, and Lemma 1.2 is proven.

We are ready for the main result of this section.

<u>Theorem 1.3</u> (Gluing principle for regular paracompact spaces). Let M be a <u>set</u> and let $(M_\alpha|\alpha\in I)$ be a family of subsets of M with $M = \cup(M_\alpha|\alpha\in I)$. Assume that for every $\alpha \in I$ the set M_α is provided with

a structure $(\mathring{\mathfrak{T}}_\alpha, Cov_\alpha, \mathcal{O}_\alpha)$ of a regular paracompact locally semialgebraic space over R. Assume further that for any two indices $\alpha, \beta \in I$ the intersection $M_\alpha \cap M_\beta$ is a closed locally semialgebraic subset of M_α and M_β, i.e. in obvious notation, $M_\alpha \cap M_\beta \in \overline{\mathfrak{T}}_\alpha(M_\alpha)$ and $M_\alpha \cap M_\beta \in \overline{\mathfrak{T}}_\beta(M_\beta)$, and that the subspace structures on $M_\alpha \cap M_\beta$ with respect to $(\mathring{\mathfrak{T}}_\alpha, Cov_\alpha, \mathcal{O}_\alpha)$ and $(\mathring{\mathfrak{T}}_\beta, Cov_\beta, \mathcal{O}_\beta)$ are equal. Assume finally that for every $\alpha \in I$ the family $(M_\alpha \cap M_\beta \mid \beta \in I)$ in $\mathfrak{T}_\alpha(M_\alpha)$ is locally finite with respect to the structure $(\mathring{\mathfrak{T}}_\alpha, Cov_\alpha, \mathcal{O}_\alpha)$. Then there exists a unique structure $(\mathring{\mathfrak{T}}(M), Cov_M, \mathcal{O}_M)$ of a locally semialgebraic space over R on M such that:

i) $M_\alpha \in \overline{\mathfrak{T}}(M)$ for every $\alpha \in I$.

ii) The family $(M_\alpha \mid \alpha \in I)$ is locally finite.

iii) The subspace structure of M_α in M with respect to $(\mathring{\mathfrak{T}}(M), Cov_M, \mathcal{O}_M)$ coincides with $(\mathring{\mathfrak{T}}_\alpha, Cov_\alpha, \mathcal{O}_\alpha)$ for every $\alpha \in I$.

Moreover, the space M is regular and paracompact.

Proof. 1) We already know from Lemma 1.2 that there exists at most one structure $(\mathring{\mathfrak{T}}(M), Cov_M, \mathcal{O}_M)$ of a locally semialgebraic space on M (separated or not) which fulfills (i) - (iii) and how this structure may be described explicitly. We also know that this structure - if it exists - is regular and paracompact.

2) We prove existence first in the case that I is finite and every M_α with its given structure is affine semialgebraic. Using induction on the cardinality of I we retreat to the case that I is a two element set $\{1,2\}$. Let $A := M_1 \cap M_2$. The idea is to find a space structure on M fulfilling (i) - (iii) by injecting M suitably into some R^m. We choose semialgebraic embeddings $\varphi_\alpha : M_\alpha \hookrightarrow R^n$ ($\alpha = 1,2$) of M_α with respect to the given structure $(\mathring{\mathfrak{T}}_\alpha, Cov_\alpha, \mathcal{O}_\alpha)$. We extend $\varphi_\alpha | A$ to a semialgebraic map $\chi_\beta : M_\beta \to R^n$ ($(\alpha,\beta) = (1,2)$ or $= (2,1)$). This is possible by "Tietze's extension theorem" (I, Th. 4.13 or [DK_5, §1]). φ_α and χ_β yield a map $\psi_\alpha : M \to R^n$. We finally choose semialgebraic functions $h_1 : M_1 \to [-1,0]$ and $h_2 : M_2 \to [0,1]$ with $h_1^{-1}(0) = h_2^{-1}(0) = A$

(cf. I, 4.15). These functions are then glued to form a function $h: M \to [-1,1]$. We have $h^{-1}(0) = A$, $h^{-1}([-1,0]) = M_1$, and $h^{-1}([0,1]) = M_2$. The map

$$f = (\psi_1, \psi_2, h) : M \to R^n \times R^n \times R$$

is injective, and its restrictions to M_1 and M_2 are semialgebraic. In particular, $N_1 := f(M_1)$ and $N_2 := f(M_2)$ are semialgebraic subsets of R^m, $m = 2n+1$. Thus $N := N_1 \cup N_2$ is also semialgebraic. We equip N with the subspace structure of the semialgebraic standard space R^m. By construction, N_1 (resp. N_2) is the set of all points $(x_1,\ldots,x_m)$ in N with $x_m \leq 0$ (resp. $x_m \geq 0$). Thus N_1 and N_2 are closed semialgebraic subsets of N. The restriction $f|M_1 = (\varphi_1, \chi_1, h_1)$ of f is a semialgebraic isomorphism of M_1 onto N_1, since φ_1 is an embedding. (The inverse map is the composite of the natural projection from N_1 onto $\varphi_1(M_1)$ and $\varphi_1^{-1} : \varphi_1(M_1) \to M_1$). $f|M_2$ is an isomorphism of M_2 onto N_2. Pulling back the space structure of N to M by the map f we obtain a space structure on M with the desired properties (i) - (iii).

3) We consider the case that I is infinite but every M_α is still affine semialgebraic with its given space structure. Our assumption that the families $(M_\alpha \cap M_\beta | \beta \in I)$ are locally finite in M_α means in this case that every set M_α meets only finitely many sets M_β. We introduce the finite sets of indices

$$I(\alpha) := \{\beta \in I | M_\alpha \cap M_\beta \neq \emptyset\}$$

and the subsets of M

$$W_\alpha := M \setminus \cup (M_\gamma | \gamma \in I \setminus I(\alpha))$$

of M. Every W_α is the union of the finite family $(W_\alpha \cap M_\beta | \beta \in I(\alpha))$. We equip every $W_\alpha \cap M_\beta$ with its subspace structure in M_β. Applying what has been proved in step 2) to W_α and this family, we see that every W_α has a unique structure of a semialgebraic space such that the sets $W_\alpha \cap M_\beta, \beta \in I(\alpha)$, with their given space structures, are closed semialgebraic subspaces of W_α and that in this structure W_α is affine. More generally, we obtain for every finite subset K of I a unique semialgebraic space structure on the set $W_K := \cup(W_\alpha \mid \alpha \in K)$ such that for every $\beta \in \cup(I(\alpha) \mid \alpha \in K)$ the set $W_K \cap M_\beta$ with its subspace structure in M_β is a closed semialgebraic subspace of W_K, and W_K is affine. By use of Lemma 1.2 it is easily seen that for any two finite subsets $K \subset L$ of I the space W_K is an open semialgebraic subspace of W_L. We equip $M = \bigcup_K W_K$ with the space structure as the inductive limit of the directed system of these spaces W_K, cf. I, Lemma 2.2. Then every W_K - in its given structure - is an open semialgebraic subspace of M and $(W_\alpha \mid \alpha \in I)$ is a locally finite covering of M by open semialgebraic subsets. Every M_α meets only finitely many W_β and $M_\alpha \cap W_\beta \in \overline{\mathfrak{F}}(W_\beta)$ by construction. Thus every M_α is a closed semialgebraic subset of M. Moreover, M_α is - in its given structure - a closed semialgebraic subspace of M. Every W_β meets only finitely many M_α. Hence the family $(M_\alpha \mid \alpha \in I)$ is locally finite in M. The theorem is therefore proved in the given special case.

4) We finally come to the general case. We choose for every $\alpha \in I$ a locally finite family $(N_{\alpha\beta} \mid \beta \in J(\alpha))$ of affine closed semialgebraic subsets $N_{\alpha\beta}$ of M_α which covers M_α. This is possible since M_α is regular and paracompact (cf. the last step in the proof of Lemma 1.2 or I, Th. 4.11). Using the assumptions of the theorem we see the following: Every $N_{\alpha\beta}$ meets a set M_γ for only finitely many indices $\gamma \in I$. For any such index the intersection $N_{\alpha\beta} \cap M_\gamma$ is closed and semialgebraic in M_γ. Thus $N_{\alpha\beta}$ meets only finitely many sets of the family $(N_{\gamma\delta} \mid \gamma \in I, \delta \in J(\gamma))$. We equip every $N_{\alpha\beta}$ with its subspace structure in M_α and apply what has been proved in step 3 to the family $(N_{\alpha\beta} \mid \alpha \in I, \beta \in J(\alpha))$. We obtain a space structure on M which is easily checked to have all the desired properties by use of Lemma 1.2. This finishes the proof of the theorem.

Theorem 1.3 is somewhat more special than one would expect in view of Lemma 1.2. If we only assume that the spaces M_α are paracompact (instead of regular and paracompact), then we cannot guarantee that a locally semialgebraic space structure on M with the properties (i) - (iii) exists. But Theorem 1.3 suffices for all applications we have in mind.

Example 1.4. In I, §2 we equipped every locally finite simplicial complex X over R with the structure of a locally semialgebraic space such that for every open simplex $\sigma \in \Sigma(X)$ the set $\bar{\sigma} \cap X$ in its "simplex structure" is a closed semialgebraic subspace of X. Theorem 1.3 yields a new justification that this space structure exists on X, is unique and is regular and paracompact. This was stated in I, §4 (Example 4.8).

§2 - Existence of completions

In this whole section we assume that M is a regular paracompact space over R. Our goal is to prove the following

Theorem 2.1. There exists an embedding $i : M \hookrightarrow N$ of M into a partially complete regular paracompact space N (i.e. an isomorphism of M onto a locally semialgebraic subspace of N).

Replacing N by the closure of i(M) in N we then obtain an embedding i for which i(M) is also dense in N. Such an embedding will be called a (regular paracompact) completion of M. Notice that it does not make sense to look for an embedding of M into a complete space N, except in the trivial case that M is semialgebraic, since by I, §5 any complete paracompact space N is semialgebraic. "Partially complete" is the best we can hope for.

Our proof of Theorem 2.1 is based on §1 and the following easy lemma. We denote by S^n, more precisely by $S^n(R)$ if necessary, the standard n-sphere over R, i.e.

$$S^n = \{(x_o,\dots,x_n) \in R^{n+1} \mid x_o^2 + \dots + x_n^2 = 1\}.$$

We further denote the north pole $(0,\dots,0,1)$ of S^n by ∞.

Lemma 2.2. For any open semialgebraic subset U of M there exists, for some $n \in \mathbb{N}$, a locally semialgebraic map $\varphi : M \to S^n$ such that $\varphi|U$ is an embedding and $\varphi^{-1}(\infty) = M \setminus U$.

Proof. The closure $\overline{U}$ of U is again semialgebraic (I, Prop. 4.6). We choose an open semialgebraic neighbourhood M' of $\overline{U}$. It suffices to prove the assertion for M' and U instead of M and U. Thus we assume without loss of generality that the space M is affine semialgebraic. We choose an embedding $\psi : M \hookrightarrow R^n$ of M into an affine standard space R^n. We further choose a semialgebraic function $f : M \to R$ with $f^{-1}(0) = M \setminus U$ (e.g. the euclidean distance function from $M \setminus U$). Let $q : R^{n+1} \xrightarrow{\sim} S^{n+1} \setminus \{\infty\}$ be the inverse of the stereographic projection with center ∞ (cf. I, §7). Then

$$\varphi(x) := \begin{cases} q(\psi(x), f(x)^{-1}) & x \in U \\ \infty & x \in M \setminus U \end{cases}$$

is a semialgebraic map from M into S^{n+1} with $\varphi^{-1}(\infty) = M \setminus U$. The restriction $\varphi|U : U \to S^{n+1} \setminus \{\infty\}$ of φ is an embedding. q.e.d.

We now start proving the theorem. We choose a locally finite covering $(M_\alpha | \alpha \in I)$ of M by open semialgebraic subsets and a locally semialgebraic map $\varphi_\alpha : M \to S(\alpha)$ of M into a standard sphere $S(\alpha) = S^{n(\alpha)}$ for each $\alpha \in I$ such that the north pole ∞_α of $S(\alpha)$ has the preimage $M \setminus M_\alpha$ and $\varphi_\alpha | M_\alpha$ is an embedding. For every $\alpha \in I$ we introduce the finite sets of indices

$$J(\alpha) := \{\beta \in I | M_\alpha \cap M_\beta \neq \emptyset\}, \quad I(\alpha) := \cup(J(\beta) | \beta \in J(\alpha))$$
$$I(\alpha)' := I(\alpha) \setminus \{\alpha\}.$$

$U_\alpha := \cup(M_\beta | \beta \in J(\alpha))$ is an open semialgebraic neighbourhood of $\overline{M}_\alpha$. For each $\alpha \in I$ we choose a locally semialgebraic function $f_\alpha : M \to [0,1]$ with $f_\alpha^{-1}(1) = \overline{M}_\alpha$ and $f_\alpha^{-1}(0) = M \setminus U_\alpha$, cf. I, Th. 4.15. In the set

$$Z := \prod_{\beta \in I} (S(\beta) \times [0,1])$$

we consider the family of subsets

$$N(\alpha) := \prod_\beta N(\alpha)_\beta \qquad (\alpha \in I).$$

Here $N(\alpha)_\alpha = S(\alpha) \times \{1\}$, $N(\alpha)_\beta = \{(\infty_\beta, 0)\}$ - a one point set - for $\beta \in I \setminus I(\alpha)$, and $N(\alpha)_\beta = S(\beta) \times [0,1]$ for $\beta \in I(\alpha)'$. We regard every $N(\alpha)$ as a complete affine semialgebraic space isomorphic to the product

$$(S(\alpha) \times \{1\}) \times \prod_{\beta \in I(\alpha)'} (S(\beta) \times [0,1])$$

in the obvious way. We want to endow the subset

$$N := \cup(N(\alpha) \mid \alpha \in I)$$

of Z with the structure of a partially complete locally semialgebraic space over R. Clearly $N(\alpha) \cap N(\beta) \neq \emptyset$ if and only if $\beta \in I(\alpha)$, and in this case

$$N(\alpha) \cap N(\beta) = \prod_{\gamma \in I} N(\alpha,\beta)_\gamma$$

with $N(\alpha,\beta)_\gamma = S(\gamma) \times \{1\}$ for $\gamma = \alpha$ or $\gamma = \beta$, $N(\alpha,\beta)_\gamma = S(\gamma) \times [0,1]$ if $\gamma \in I(\alpha)' \cap I(\beta)'$, and $N(\alpha,\beta)_\gamma = \{(\infty_\gamma, 0)\}$ else. We see that a given $N(\alpha)$ meets $N(\beta)$ only for finitely many indices β, and that in this case $N(\alpha) \cap N(\beta)$ is closed semialgebraic in $N(\alpha)$ and in $N(\beta)$. Also, the subspace structures on $N(\alpha) \cap N(\beta)$ with respect to $N(\alpha)$ and $N(\beta)$ are identical. From Theorem 1.3 we know the following: There exists on N a unique locally semialgebraic space structure such that, for every $\alpha \in I$, the space $N(\alpha)$ - with its given structure - is a closed semialgebraic subspace of N, and the family $(N(\alpha) \mid \alpha \in I)$ is locally finite in N. We equip N with this structure. Then N is regular, paracompact, and partially complete. (The space structure on N can be described explicitly, cf. Lemma 1.2.)

We define for every point $x \in M$ a point $i(x) := (\varphi_\gamma(x), f_\gamma(x))_{\gamma \in I} \in Z$. If $x \in \overline{M}_\alpha$, then $(\varphi_\alpha(x), f_\alpha(x)) \in S(\alpha) \times \{1\}$ and $(\varphi_\beta(x), f_\beta(x)) = (\infty_\beta, 0)$ for every $\beta \in I \setminus I(\alpha)$. (Notice that $\beta \in I(\alpha)$ iff $\alpha \in I(\beta)$). Thus $i(x) \in N(\alpha) \subset N$. We claim that the map $i: M \to N$, $x \mapsto i(x)$, is a locally semialgebraic embedding. We prove this in five steps:

a) i is injective. Indeed, let x and y be two different points of M. If x and y both lie in some M_α, then $\varphi_\alpha(x) \neq \varphi_\alpha(y)$, since $\varphi_\alpha | M_\alpha$ is an embedding. If $x \in M_\alpha$, but $y \notin M_\alpha$, then $\varphi_\alpha(x) \neq \infty_\alpha$ and $\varphi_\alpha(y) = \infty_\alpha$. Thus in both cases $i(x) \neq i(y)$.

b) We prove that the map i is locally semialgebraic. For every finite subset K of I we introduce the semialgebraic sets

$$M_K := \cup(M_\alpha | \alpha \in K) \in \mathring{\gamma}(M),$$

$$N_K := \cup(N(\alpha) | \alpha \in K) \in \overline{\gamma}(N).$$

We have $i(M_K) \subset N_K$. We may regard N_K as a semialgebraic subset of the finite product $\prod_{\alpha \in L} (S(\alpha) \times [0,1])$, with $L := \cup(I(\alpha) | \alpha \in K)$. For every $\alpha \in K$ the subspace structure of $N(\alpha)$ in N coincides with the subspace structure of $N(\alpha)$ in this finite product. We conclude from Lemma 1.2 that the subspace structure of N_K in N coincides with the subspace structure in this finite product. We have

$$i|M_K = (\varphi_\beta, f_\beta)_{\beta \in L}.$$

Thus $i|M_K$ is a semialgebraic map from M_K to N_K, hence a semialgebraic map from M_K to N. But M is the inductive limit of the directed system of spaces $(M_K | K \subset I,\ K \text{ finite})$, cf. I, Ex. 2.3. Thus $i: M \to N$ is locally semialgebraic.

c) For every $\alpha \in I$ we have $i(M) \cap N(\alpha) = i(\overline{M}_\alpha)$. Thus the map i is semialgebraic (cf. I.5.2.b). Indeed, we know already that $i(\overline{M}_\alpha) \subset N(\alpha)$. On the other hand, if x is a point of M with $i(x) \in N(\alpha)$, then $f_\alpha(x) = 1$, hence $x \in \overline{M}_\alpha$.

d) We conclude from c) that $L := i(M)$ is a locally semialgebraic subset of N. We want to prove that, for any $\alpha \in I$, the semialgebraic map

$$i_\alpha = i|\overline{M}_\alpha : \overline{M}_\alpha \to N(\alpha)$$

is an isomorphism from $\overline{M}_\alpha$ to the subspace $L \cap N(\alpha)$ of N. For this it suffices to know that the image $i_\alpha(A) = i(A)$ of any closed semialgebraic subset A of $\overline{M}_\alpha$ is closed in $N(\alpha) \cap L$. Identifying $N(\alpha)$ with the space

$$(S(\alpha) \times \{1\}) \times \prod_{\beta \in I(\alpha)'} (S(\beta) \times [0,1])$$

(as above), we have

$$i_\alpha = (\varphi_\beta, f_\beta)_{\beta \in I(\alpha)} .$$

Let $x \in \overline{M}_\alpha$ be given with $i(x) \in \overline{i(A)}$. We have to verify that $x \in A$. Given any $\varepsilon > 0$ in R, there exists some $y \in A$ with

$$\| \varphi_\beta(x) - \varphi_\beta(y) \| < \varepsilon, \quad |f_\beta(x) - f_\beta(y)| < \varepsilon$$

for every $\beta \in I(\alpha)$. ($\| \ \|$ is the euclidean standard norm on the ambient space $R^{n(\beta)+1}$ of $S(\beta)$.) We choose an index $\gamma \in J(\alpha)$ with $x \in M_\gamma$. Then $\varphi_\gamma(x) \neq \infty$. For ε small enough we also have $\varphi_\gamma(y) \neq \infty_\gamma$ i.e. $y \in M_\gamma$. We conclude that $\varphi_\gamma(x)$ lies in the closure of $\varphi_\gamma(A \cap M_\gamma)$ in $\varphi_\gamma(M_\gamma)$. But $\varphi_\gamma | M_\gamma$ is a semialgebraic isomorphism from M_γ onto $\varphi_\gamma(M_\gamma)$, and $A \cap M_\gamma$ is closed in M_γ. Thus indeed $x \in A \cap M_\gamma \subset A$.

e) Pulling back the subspace structure of L in N to M by the bijective map $i : M \to L$, we obtain a space structure on M, which has the following properties according to d): Every $\overline{M}_\alpha$ is a closed semialgebraic subset of M and the subspace structure on $\overline{M}_\alpha$ coincides with the subspace structure of $\overline{M}_\alpha$ with respect to M in the old structure. The family $(N(\alpha) | \alpha \in I)$ is locally finite in N. Thus $(N(\alpha) \cap L | \alpha \in I)$ is locally finite in L, and $(\overline{M}_\alpha | \alpha \in I)$ is locally finite in M with respect to the new space structure. We conclude from Lemma 1.2 that the new space structure on M coincides with the old one. This means that i is indeed an embedding of M into N. The proof of Theorem 2.1 is finished.

Our next goal in this chapter is to prove that every regular paracompact space M can be triangulated, more precisely, that there exists a strictly locally finite simplicial complex X and a locally semialgebraic isomorphism $\psi : X \xrightarrow{\sim} M$. Notice that such a triangulation automatically gives a completion of M - namely the embedding of M into $\overline{X}$ by the map ψ^{-1}. Thus triangulations are a special sort of completions. Theorem 2.1 should be regarded as a first step towards the triangulation theorem.

§3 - Abstract simplicial complexes

Before we start our proof that regular paracompact spaces can be triangulated it seems wise to develop a combinatorial pattern for describing the simplicial complexes over R introduced in I, §2. From now on these complexes are called _geometric_ simplicial complexes whereas the combinatorial objects which correspond to them will be called _abstract_ simplicial complexes. Abstract simplicial complexes are widely used in classical topology. Notice, however, that our notion of a geometric simplicial complex is more general than the classical one (even for $R = \mathbb{R}$), and therefore our notion of an abstract simplicial complex also has to be more general than the classical one. In our theory the simplices of an abstract simplicial complex (see below) represent the _open_ simplices of the corresponding geometric simplicial complex, while in the classical theory they represent the _closed_ simplices. This deviation from the classical concepts seems to be inavoidable in semialgebraic geometry, cf. the first remark following Definition 1 in I, §2.

Definition 1. a) An _abstract simplicial complex_ K is a pair $(E(K),S(K))$ consisting of a set $E(K)$ and a set $S(K)$ of non empty finite subsets of $E(K)$ such that $E(K)$ is the union of all $s \in S(K)$ *). We call the elements of $E(K)$ the _vertices_ of K and the elements of $S(K)$ the _simplices_ of K. We do admit the empty complex $\emptyset = (\emptyset,\emptyset)$.
b) The _closure_ of an abstract simplicial complex K is the pair $\overline{K} = (E(\overline{K}),S(\overline{K}))$ with $E(\overline{K}) = E(K)$ and $S(\overline{K})$ the set of all non empty subsets t of all $s \in S(K)$. The complex K is called _closed_ if $K = \overline{K}$.
c) A _simplicial map_ $\alpha : K \to L$ between abstract simplicial complexes K,L is a map $\alpha : E(K) \to E(L)$ such that the image $\alpha(s)$ of every simplex s of

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*) Thus the datum $E(K)$ is determined by $S(K)$, but it seems convenient to maintain this redundancy.

K is a simplex of L.

d) A <u>subcomplex</u> of an abstract simplicial complex K is an abstract simplicial complex L with $E(L) \subset E(K)$ and $S(L) \subset S(K)$. Note that the closure $\bar{L}$ of L is a subcomplex of $\bar{K}$. For any two subcomplexes L_1 and L_2 of K we can form the <u>intersection</u> $L_1 \cap L_2$ and the <u>union</u> $L_1 \cup L_2$. The subcomplex $L_1 \cap L_2$ is determined by $S(L_1 \cap L_2) = S(L_1) \cap S(L_2)$ and the subcomplex $L_1 \cup L_2$ is determined by $S(L_1 \cup L_2) = S(L_1) \cup S(L_2)$. We have $E(L_1 \cup L_2) = E(L_1) \cup E(L_2)$ but could have $E(L_1 \cap L_2) \subsetneqq E(L_1) \cap E(L_2)$. Clearly $L_1 \cap L_2$ is a subcomplex of L_1 and L_2 and "contains" (i.e. has as subcomplex) every subcomplex of K contained in L_1 and L_2. Similarly $L_1 \cup L_2$ is the smallest subcomplex of K which contains L_1 and L_2. More generally, we may form the intersection and union of arbitrary families of subcomplexes of K.

e) The <u>closure</u> of a subcomplex L of K <u>in</u> K is the intersection $\bar{L} \cap K$ of the subcomplexes $\bar{L}$ and K of $\bar{K}$. In particular, L is called <u>closed in</u> K if $L = \bar{L} \cap K$.

f) A subcomplex L of K is called <u>full</u> <u>in</u> K, if

$$S(L) = \{s \in S(K) \mid s \subset E(L)\}$$

(In general S(L) may be smaller than the right hand set.) Of course every full subcomplex of K is closed in K.

g) The <u>complement</u> $K \setminus L$ of a subcomplex L <u>in</u> K is the subcomplex N of K with $S(N) = S(K) \setminus S(L)$. The subcomplex L is called <u>open</u> <u>in</u> K, if $K \setminus L$ is closed in K.

For the sake of brevity we will use the word "complex" instead of "simplicial complex" in the remainder of this section as well as in the whole entirety of the next section. Notice that for every simplex s of an abstract complex K the pair $(s,\{s\})$ is a subcomplex of K with a single simplex s. We usually identify s with this subcomplex of K. The <u>faces</u> of s are the simplices of the closure $\bar{s}$ of the complex s.

These are all the non-empty subsets t of the set s. We often write $t \leq s$ instead of $t \subset s$ if t is a face of s, and we write $t < s$ if $t \subset s$ but $t \neq s$ ("proper faces" of s). The simplices of the complex $\bar{s} \cap K$ are called the "<u>faces of s in K</u>".

Clearly a subcomplex L of K is closed in K if for every simplex s of L all faces of s in K are simplices of L. Thus a subcomplex L of K is open in K if every simplex s of K posessing at least one face which is a simplex of L is itself a simplex of L.

A simplicial map $\alpha : K \to L$ between abstract complexes K, L can also be regarded as a simplicial map from $\bar{K}$ to $\bar{L}$. This will be denoted by $\bar{\alpha}$ and called the <u>closure of the simplicial map</u> α. If $\alpha : K \to L$ is a simplicial map and K_1 is a subcomplex of K, we have the image complex $\alpha(K_1) = L_1$ defined by

$$S(L_1) = \{\alpha(s) \mid s \in S(K_1)\}.$$

Note that $E(L_1) = \alpha(E(K_1))$. Furthermore, we have for every subcomplex L' of L the preimage complex $\alpha^{-1}(L') = K'$ defined by

$$S(K') = \{s \in S(K) \mid \alpha(s) \in S(L')\}.$$

The set $E(K')$ may be smaller than $\alpha^{-1}(E(L'))$. We have $\overline{\alpha(K_1)} = \bar{\alpha}(\bar{K}_1)$ but only $\overline{\alpha^{-1}(L')} \subset \bar{\alpha}^{-1}(\overline{L'})$.

Now let $X = (X, \Sigma(X))$ be a geometric complex over our fixed real closed field R (cf. I, §2). X yields an abstract complex $K(X) = (E(X), S(X))$ as follows: $E(X)$ is the set of all points in $\bar{X}$ which are vertices of open simplices σ of X (i.e. $\sigma \in \Sigma(X)$). In particular $E(X) = E(\bar{X})$. A subset s of $E(X)$ is an element of $S(X)$ if and only if s is the set of vertices of some $\sigma \in \Sigma(X)$. We call $K(X)$ the <u>abstraction</u> of the geometric complex X. Clearly the closure $\overline{K(X)}$ of $K(X)$ is the abstraction of the closure $\bar{X}$ of X. Also, the subcomplexes Y of X correspond uniquely to the subcomplexes L of K by abstraction. Our definition above of the words "closed in K", "union", "intersection", "complement", "open in K", for subcomplexes of K are compatible with the definitions of these words for subcomplexes of geometric complexes in I, §2. For example, a subcomplex Y of X is open in X if and only if $K(Y)$ is open in $K(X)$. We complete this pattern of definitions as follows.

Definitions 2. a) A subcomplex Y of a geometric complex X is called full in X, if $K(Y)$ is full in $K(X)$. This means that every open simplex σ of X whose vertices are all contained in $\overline{Y}$ is an open simplex of Y.
b) An abstract complex K is called finite if $E(K)$ is finite or, equivalently, if $S(K)$ is finite. K is called locally finite if every simplex s of K is a simplex of an open finite subcomplex of K. This means that every simplex of K is a face of only finitely many simplices of K (or of $\overline{K}$). Finally K is called strictly locally finite if $\overline{K}$ is locally finite. This means that every vertex p of $\overline{K}$ is an element of only finitely many simplices of K (or of $\overline{K}$).
c) The dimension $\dim s$ of an abstract simplex s is by definition the cardinality of s minus one, and the dimension $\dim K$ of an abstract complex K is the supremum of the dimensions of its simplices (a non negative integer or ∞). Similarly, the dimension $\dim X$ of a geometric complex X is defined as the supremum of the affine dimensions of its open simplices. The empty complex has dimension -1. Notice that for X locally finite this dimension coincides with the dimension of X as a locally semialgebraic space.

Any simplicial map $f: X \to Y$ between geometric complexes (cf. Definition 6 in I, §2) obviously yields a simplicial map $K(f): K(X) \to K(Y)$ between the abstractions of X and Y by restriction to the set of vertices. We call $K(f)$ the abstraction of the simplicial map f. The closure $\overline{f}: \overline{X} \to \overline{Y}$ of f has as abstraction the closure of $K(f)$, i.e. $K(\overline{f}) = \overline{K(f)}$.

We have established "the abstraction functor" K from the category of geometric complexes and geometric simplicial maps to the category of abstract complexes and abstract simplicial maps. The following proposition is easily verified.

Proposition 3.1. The abstraction functor K is fully faithful, i.e. for any two geometric complexes X and Y, the natural map $f \mapsto K(f)$ from the set of morphisms Hom(X,Y) to Hom(K(X),K(Y)) is bijective.

Definition 3. A realization over R of an abstract complex is a geometric complex X over R together with a simplicial isomorphism $\alpha : L \xrightarrow{\sim} K(X)$.

Proposition 3.2. Every abstract complex L admits a realization over R.

Indeed, we obtain a "canonical realization" of L in the same way as in the classical theory [Go, p. 39f]. Let $V = R^{(E(L))}$ be the free R-module with basis E(L), and let X denote the set of all open simplices in V, which are spanned by the finite sets $s \subset E(L)$, $s \in S(L)$. Then X is a geometric complex over R and the identity map from E(L) to E(X) = E(L) is an abstract simplicial isomorphism $\alpha : L \xrightarrow{\sim} K(X)$.

Notice that if X is a realization of L with the isomorphism $\alpha : L \xrightarrow{\sim} K(X)$ then $\overline{X}$ is a realization of $\overline{L}$ with the isomorphism $\overline{\alpha}$ from $\overline{L}$ to $K(\overline{X}) = \overline{K(X)}$. In the case $R = \mathbb{R}$ the geometric complex $\overline{X}$ may be regarded as a realization of $\overline{L}$ in the classical sense.

If X_1 and X_2 are two realizations of L with isomorphisms $\alpha_1 : L \xrightarrow{\sim} K(X_1)$ and $\alpha_2 : L \xrightarrow{\sim} K(X_2)$, then by Proposition 3.1 there exists a unique simplicial isomorphism $f : X_1 \xrightarrow{\sim} X_2$ such that $\alpha_2 = K(f) \circ \alpha_1$. In this sense any two realizations of L are canonically isomorphic. We usually choose for a given abstract complex L a fixed realization X over R and denote this geometric complex by $|L|_R$ (or simply $|L|$, if the field R is kept fixed). Usually we also identify $E(L)$ with the set $E(X)$ by the realization isomorphism $\alpha : L \xrightarrow{\sim} K(X)$. We call $|L|_R$ with this identification "the" realization of L over R. Of course, if L_1 is a subcomplex of L then we usually choose the subcomplex Y of X with $S(Y) = S(L_1)$ for the realization of L_1 over R. The isomorphism $\alpha_1 : L_1 \xrightarrow{\sim} K(Y)$ is obtained from α by restriction. In this sense we have formulas like

$$\begin{aligned} |L_1 \cap L_2|_R &= |L_1|_R \cap |L_2|_R, \\ \overline{|L|_R} &= |\bar{L}|_R \\ |L \setminus L_1|_R &= |L|_R \setminus |L_1|_R . \end{aligned}$$

In particular, for every $s \in S(L)$ we denote by $|s|_R$ the open simplex σ of $|L|_R$ whose set of vertices is s.

<u>Definition 4.</u> The <u>realization</u> of an abstract simplicial map $\gamma : L_1 \to L_2$ <u>over</u> R is the unique geometric simplicial map $g : |L_1|_R \to |L_2|_R$ with $K(g) = \gamma$. We denote this map g by $|\gamma|_R$.

Proposition 3.1 and 3.2 together mean that the functor K is an equivalence of the category of geometric complexes with the category of abstract complexes. We have just constructed a "quasi-inverse" functor to K, the realization functor $|\;|_R$, quite in the usual way.

The "canonical realization" of an abstract complex L, which we described in the proof of Proposition 3.2, is usually a geometric complex in a very large vector space. One can often do better.

<u>Proposition 3.3.</u> If L is a countable strictly locally finite complex with $\dim L \leq n$, then L has a realization as a geometric complex X in the vector space R^{2n+1} such that $\bar{X}$ is a closed locally semialgebraic subset of the locally semialgebraic space $(R^{2n+1})_{loc}$.

Proof. It suffices to realize $\bar{L}$ in R^{2n+1}. This can be done by the classical construction as described, for example, in [Sp, Chap. III, §2]. Then $\bar{X}$ becomes even a closed locally semialgebraic subset of M_{loc} for M a closed half-space in R^{2n+1}, e.g. $M = \{x_n \geq 0\}$.

We finally discuss extension of the base field. Let $\tilde{R}$ be a real closed overfield of R. Let X be a realization of an abstract complex L over R with the isomorphism $\alpha : L \xrightarrow{\sim} K(X)$. Then the geometric complex $X(\tilde{R})$ over $\tilde{R}$, obtained from X by base extension in the obvious way, has the same abstraction as X, i.e. $K(X(\tilde{R})) = K(X)$, and thus will be regarded as realization of L over $\tilde{R}$. In short,

(3.4) $$|L|_R(\tilde{R}) = |L|_{\tilde{R}}.$$

Also, for every abstract simplicial map $\gamma : L_1 \xrightarrow{\sim} L_2$ we obtain the realization $|\gamma|_{\tilde{R}}$ from $|\gamma|_R$ in an obvious way by base extension,

(3.5) $$|\gamma|_{\tilde{R}} = (|\gamma|_R)_{\tilde{R}}.$$

If L is locally finite, then (3.4) can also be read as a canonical isomorphism between the base extension of the locally semialgebraic space $|L|_R$ to $\tilde{R}$ and the locally semialgebraic space $|L|_{\tilde{R}}$. Similarly, if L_1 and L_2 are locally finite, then (3.5) can be read as an equality between locally semialgebraic maps.

§4 - Triangulation of regular paracompact spaces

Definition 1. a) Let M be a locally semialgebraic space over R. A triangulation of M is a locally semialgebraic isomorphism $\varphi : X \xrightarrow{\sim} M$ from a strictly locally finite geometric (simplicial) complex X onto M. (Recall from I, §2 that X has a natural structure as a locally semialgebraic space.)
b) Let $(A_\lambda | \lambda \in \Lambda)$ be a family of locally semialgebraic subsets of M. A simultaneous triangulation of M and the family $(A_\lambda | \lambda \in \Lambda)$ is a triangulation $\varphi : X \xrightarrow{\sim} M$ such that $\varphi^{-1}(A_\lambda)$ is a subcomplex Y_λ of X for every $\lambda \in \Lambda$. Then, of course, the restriction $\varphi | Y_\lambda : Y_\lambda \xrightarrow{\sim} A_\lambda$ is a triangulation of A_λ for every $\lambda \in \Lambda$.
c) If $\varphi : X \xrightarrow{\sim} M$ and $\psi : Y \xrightarrow{\sim} M$ are triangulations of M, we say that ψ refines φ, and write $\varphi \leq \psi$, if ψ is a simultaneous triangulation of M and the family $(\varphi(\sigma) | \sigma \in \Sigma(X))$ consisting of the images of all open simplices of X.
d) We say that two triangulations $\varphi : X \xrightarrow{\sim} M$ and $\psi : Y \xrightarrow{\sim} M$ are equivalent if $\psi^{-1} \circ \varphi$ is a simplicial isomorphism from X to Y. (N.B. This is a stronger condition than $\varphi \leq \psi$, $\psi \leq \varphi$.)
e) If two triangulations $\varphi : X \to M_1$ and $\psi : Y \to M_2$ of locally semialgebraic subsets M_1, M_2 of M are given, and if A is a locally semialgebraic subset of $M_1 \cap M_2$, then we say that φ refines ψ on A (resp. φ is equivalent to ψ on A), if $\varphi^{-1}(A)$ and $\psi^{-1}(A)$ are subcomplexes of X and Y and the restriction $\varphi^{-1}(A) \xrightarrow{\sim} A$ of φ refines (resp. is equivalent to) the restriction $\psi^{-1}(A) \xrightarrow{\sim} A$ of ψ.

Of course, if M has a triangulation, then M must be regular and paracompact. Our goal in this section is to prove the converse: Any regular and paracompact space M and any locally finite family $(A_\lambda | \lambda \in \Lambda)$ of locally semialgebraic subsets of M has a simultaneous triangulation. This is well-known to be true if M is affine semialgebraic (cf. I, Th. 2.13). The general case needs some preparation. Notice that it suffices to prove the theorem in the case that the regular paracompact space is partially complete, since by Theorem 2.1 any other regular paracompact space may be considered as a locally semialgebraic subset of such a space.

Lemma 4.1. (Gluing of triangulations for partially complete spaces). Let M be a partially complete space over R, and let $(M_\alpha | \alpha \in I)$ be a locally finite family in $\overline{\mathcal{T}}(M)$ with $M = \cup(M_\alpha | \alpha \in I)$. Assume that for every $\alpha \in I$ a triangulation $\varphi_\alpha : X_\alpha \xrightarrow{\sim} M_\alpha$ is given. (Thus, by Lemmas 1.1 and 1.2, M is regular and paracompact.) Assume further that for any two indices $\alpha, \beta \in I$ with $M_\alpha \cap M_\beta \neq \emptyset$ the triangulations φ_α and φ_β are equivalent on $M_\alpha \cap M_\beta$. Then there exists, up to equivalence, a unique triangulation $\varphi : X \xrightarrow{\sim} M$ such that φ is equivalent to φ_α on M_α for every $\alpha \in I$.

Proof. We prove the existence of φ, leaving the easy verification of uniqueness to the reader. For every $\alpha \in I$ the complex X_α is closed since the space M_α is partially complete. Let $K_\alpha = (E_\alpha, S_\alpha)$ denote the abstraction of X_α. We want to construct a closed abstract complex $K = (E,S)$ which is in some sense the "union" of the K_α. We define the set E of vertices as the quotient of the disjoint union $\bigsqcup(E_\alpha | \alpha \in I)$ by the following equivalence relation: Let $p \in E_\alpha$, $q \in E_\beta$. Then $p \sim q$ iff $\varphi_\alpha(p) = \varphi_\beta(q)$. (N.B. This makes sense since every complex X_α is closed. Transitivity of the relation is easily checked.) For every $\alpha \in I$ we have a natural injection $i_\alpha : E_\alpha \to E$. We define S as the union of the sets $\{i_\alpha(s) | s \in S_\alpha\}$ for all $\alpha \in I$. Clearly $K := (E,S)$ is a closed abstract complex. We claim that K is locally finite. Let $p \in E$ be given. We have to verify that p is a vertex of only finitely many $s \in S$. Choose an index $\alpha \in I$ such that $p = i_\alpha(p_\alpha)$ for some $p_\alpha \in E_\alpha$, uniquely determined by p. Let $s \in S$ be given with $p \in s$. We have $s = i_\beta(s_\beta)$ for some index $\beta \in I$ and some $s_\beta \in S_\beta$. Clearly $\varphi_\alpha(p_\alpha) \in M_\beta$. Since the family

$(M_\gamma | \gamma \in I)$ is locally finite, only finitely many indices β can occur in this manner. Also, $\varphi_\alpha(p_\alpha) = \varphi_\beta(p_\beta)$ with $p_\beta \in E_\beta$ uniquely determined by p_α. We have $p_\beta \in s_\beta$ for every simplex s_β as above. Thus, for β fixed, only finitely many simplices s_β can occur. We see that K is indeed locally finite.

We choose a realization $X = |K|$ of K. For every $\alpha \in I$ the injection $i_\alpha : E_\alpha \hookrightarrow E$ may be regarded as an abstract simplicial map from K_α to K. The realization $|i_\alpha| : X_\alpha \to X$ of i_α is an isomorphism from X_α onto a closed subcomplex Y_α of X. Also, the family $(Y_\alpha | \alpha \in I)$ of subcomplexes of X is locally finite and has X as its union. For every $\alpha \in I$ we have a locally semialgebraic isomorphism

$$\psi_\alpha = \varphi_\alpha \circ |i_\alpha|^{-1} : Y_\alpha \xrightarrow{\sim} M_\alpha .$$

Since the triangulations φ_α and φ_β are equivalent on $M_\alpha \cap M_\beta$, we have $\psi_\alpha(x) = \psi_\beta(x)$ for every $x \in Y_\alpha \cap Y_\beta$. Thus the ψ_α glue together to a locally semialgebraic isomorphism $\psi : X \xrightarrow{\sim} M$. Of course ψ is equivalent to φ_α on M_α for every $\alpha \in I$. q.e.d.

As the proof shows, Lemma 4.1 is a trivial statement. We did not do any serious work. Nevertheless the lemma leads to a sometimes very useful way to work with triangulations.

<u>Definition 2.</u> For any non negative integer n we denote by Δ_n (or more precisely by $\Delta_n(R)$ if necessary) the closed standard n-simplex over R, i.e. $\Delta_n(R) = [e_0, e_1, \ldots, e_n]$ with $e_0, \ldots, e_n$ the standard basis of R^{n+1}. A <u>simplicial atlas</u> of a locally semialgebraic space M is a family of semialgebraic maps $(\varphi_\alpha : \Delta_{n(\alpha)} \to M | \alpha \in I)$ with the following five properties:

a) Every φ_α is injective and hence an isomorphism from $\Delta_{n(\alpha)}$ onto a closed semialgebraic subset M_α of M.

b) The family $(M_\alpha | \alpha \in I)$ is locally finite and covers M.

c) For any two indices $\alpha, \beta \in I$ with $M_\alpha \cap M_\beta \neq \emptyset$ the sets $\varphi_\alpha^{-1}(M_\alpha \cap M_\beta)$ and $\varphi_\beta^{-1}(M_\alpha \cap M_\beta)$ are (closed) faces of $\Delta_{n(\alpha)}$ and $\Delta_{n(\beta)}$ respectively and

$$\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(M_\alpha \cap M_\beta) \to \varphi_\beta^{-1}(M_\alpha \cap M_\beta)$$

is an affine isomorphism.

d) $M_\alpha = M_\beta \Rightarrow \alpha = \beta$.

e) For every $\alpha \in I$ and every closed face τ of $\Delta_{n(\alpha)}$ we have $\varphi_\alpha(\tau) = M_\gamma$ for some $\gamma \in I$.

Notice that in this definition the last two conditions d), e) can always be met if a) - c) are fulfilled by simply omitting or adding suitable maps $\varphi_\alpha : \Delta_{n(\alpha)} \to M$. Notice also that, according to §1, the space structure on the set M is uniquely determined by a given atlas, and that M is necessarily regular paracompact and partially complete.

<u>Definition 3.</u> Two simplicial atlases $(\varphi_\alpha | \alpha \in I)$ and $(\psi_\lambda | \lambda \in \Lambda)$ of M are called <u>equivalent</u> if there exists a bijection $\kappa : I \xrightarrow{\sim} \Lambda$ such that, for every $\alpha \in I$, $\varphi_\alpha(\Delta_{n(\alpha)}) = \psi_{\kappa(\alpha)}(\Delta_{n(\kappa(\alpha))})$, in particular $n(\alpha) = n(\kappa(\alpha))$, and $\psi_{\kappa(\alpha)}^{-1} \circ \varphi_\alpha$ is an affine automorphism of $\Delta_{n(\alpha)}$.

This means that the index sets I and Λ may be identified in such a way that every ψ_α is gotten from the corresponding φ_α by composition with an affine automorphism of $\Delta_{n(\alpha)}$.

If a triangulation $\varphi : X \xrightarrow{\sim} M$ is given with X a closed locally finite geometric complex, we then obtain a simplicial atlas $(\varphi_\sigma : \Delta_{n(\sigma)} \to M | \sigma \in \Sigma(X))$ by choosing for every $\sigma \in \Sigma(X)$ an affine isomorphism $\psi_\sigma : \Delta_{n(\sigma)} \xrightarrow{\sim} \bar{\sigma}$, $n(\sigma) := \dim \sigma$, and setting $\varphi_\sigma = \varphi \circ \psi_\sigma$. This simplicial atlas is determined by the triangulation φ up to equivalence, and is called <u>an atlas</u>

of the triangulation φ.

Conversely, if a simplicial atlas $(\varphi_\alpha | \alpha \in I)$ of M is given then, applying Lemma 4.1 to M and the triangulations $\varphi_\alpha : \Delta_{n(\alpha)} \xrightarrow{\sim} \varphi_\alpha(\Delta_{n(\alpha)})$, we obtain

Corollary 4.2. Every simplicial atlas $(\varphi_\alpha | \alpha \in I)$ of a locally semialgebraic space M is equivalent to an atlas of some triangulation $\varphi : X \to M$ and φ is uniquely determined, up to equivalence, by the given atlas.

Thus, for partially complete spaces, simplicial atlases and triangulations are really the same thing. We shall switch back and forth between these concepts at will.

We now state a technical lemma which will help us to triangulate regular paracompact spaces.

Lemma 4.3 (Extension of a refinement of a triangulation). Let $\varphi : X \xrightarrow{\sim} M$ be a triangulation of a partially complete space M. Further, let M_o be a closed locally semialgebraic subset of M with $X_o := \varphi^{-1}(M_o)$ a (necessarily closed) subcomplex of X, and let $\varphi_o : X_o \xrightarrow{\sim} M_o$ denote the triangulation of M_o obtained from φ by restriction. Finally let $\psi_o : Y_o \xrightarrow{\sim} M_o$ be a triangulation of M_o which refines φ_o. Then there exists a triangulation $\psi : Y \xrightarrow{\sim} M$ which refines φ and is equivalent to ψ_o on M_o.

Proof. We assume without loss of generality that $X = M$, $\varphi = id_M$. Thus M is a closed locally finite simplicial complex and M_o is a closed subcomplex of M. For any $k \geq 0$ we denote by M^k the relative k-skeleton of (M, M_o) i.e. the union of M_o and all open (or closed) simplices of M of dimension $\leq k$. Let $(\chi_\alpha | \alpha \in J_o)$ be a simplicial atlas of the triangulation $\psi_o : Y_o \xrightarrow{\sim} M_o$ of M_o. We want to extend this atlas successively to simplicial atlases $(\chi_\alpha|\alpha\in J^k)$ for all skeleta M^k $(k\geq 0,\ J_o\subset J^o\subset J^1\subset\ldots)$ such that the resulting triangulation of M^k refines the tautological triangulation $\mathrm{id}: M^k \xrightarrow{\sim} M^k$ for every $k\geq 0$. Then $(\chi_\alpha|\alpha\in\cup(J^k|k\geq 0))$ becomes a simplicial atlas of M, which refines the tautological triangulation $\mathrm{id}: M\xrightarrow{\sim} M$.

The atlas $(\chi_\alpha|\alpha\in J^o)$ of M^o is easily found. M^o is the union of M_o and all vertices of M. Simply add the one point map $\chi_e: \Delta_o \xrightarrow{\sim} \{e\}$ to the family $(\chi_\alpha|\alpha\in J_o)$ for every $e\in E(M)\setminus E(M_o)$.

Suppose the family $(\chi_\alpha|\alpha\in J^k)$ has already been constructed for some $k\geq 0$. Let Σ_{k+1} denote the set of all open simplices in $M\setminus M_o$ of dimension $k+1$. For every $\sigma\in\Sigma_{k+1}$ the boundary $\partial\sigma = \bar{\sigma}\setminus\sigma$ is contained in M^k. Let $J^k(\sigma)$ denote the set of all indices $\alpha\in J^k$ with

$$M_\alpha := \chi_\alpha(\Delta_{n(\alpha)}) \subset \partial\sigma,$$

and let $\hat{\sigma}$ denote the barycenter of σ. For every pair (σ,α) with $\sigma\in\Sigma_{k+1}$, $\alpha\in J^k(\sigma)$, we introduce the injective semialgebraic map

$$\chi_{\sigma,\alpha}: \Delta_{n(\alpha)+1} \to M$$

defined by

$$\chi_{\sigma,\alpha}((1-t)u + te_{n(\alpha)+1}) = (1-t)\chi_\alpha(u) + t\hat{\sigma}.$$

In other words, we regard $\Delta_{n(\alpha)+1}$ as the cone over $\Delta_{n(\alpha)} = [e_o,\ldots,e_{n(\alpha)}]$ with vertex $e_{n(\alpha)+1}$ and extend the isomorphism $\chi_\alpha: \Delta_{n(\alpha)} \xrightarrow{\sim} M_\alpha$ in the obvious way to an isomorphism from $\Delta_{n(\alpha)+1}$ onto the cone over M_α with vertex $\hat{\sigma}$. As a simplicial atlas of M^{k+1} we choose the union of the three families $(\chi_\alpha|\alpha\in J^k)$, $(\Delta_o \xrightarrow{\sim} \{\hat{\sigma}\}|\sigma\in\Sigma_{k+1})$, $(\chi_{\sigma,\alpha}|\sigma\in\Sigma_{k+1},\alpha\in J^k(\sigma))$. It is easily checked that this union is indeed a simplicial atlas of M^{k+1}. q.e.d.

Now we are ready to prove our main theorem.

<u>Theorem 4.4</u> (Triangulation theorem). Let M be a regular paracompact space over R and $(A_\lambda \mid \lambda \in \Lambda)$ a locally finite family of locally semialgebraic subsets of M. Then there exists a simultaneous triangulation (cf. Def. 1) $\varphi : X \xrightarrow{\sim} M$ of M and this family.

<u>Proof.</u> Choosing a completion of M (Th. 2.1) we may assume in addition that M is partially complete. We may also assume that M is connected. We already know that the theorem is true if M is semialgebraic (I, Th. 2.13). In the general case we can cover M by a locally finite family $(U_n \mid n \in \mathbb{N})$ of open semialgebraic subsets with $U_n \cap U_m = \emptyset$ if $|n-m| \geq 2$, cf. I, Cor. 4.19. We shrink this covering to a covering $(V_n \mid n \in \mathbb{N})$ with $V_n \in \mathring{\gamma}(M)$, $\overline{V}_n \subset U_n$, cf. I, Th. 4.11. We shall work with the locally finite covering $(M_n \mid n \in \mathbb{N})$ of M by the closed semialgebraic subsets $M_n := \overline{V}_n$. Notice $M_n \cap M_m = \emptyset$ if $|n-m| \geq 2$. Also, for every $n \in \mathbb{N}$, the index set $\Lambda(n)$ consisting of all $\lambda \in \Lambda$ with $A_\lambda \cap M_n \neq \emptyset$ is finite.

Using the triangulation theorem for affine semialgebraic spaces we successively choose for $n = 1,2,3,\ldots$ simultaneous triangulations $\varphi_n : X_n \xrightarrow{\sim} M_n$ of M_n and the finitely many subsets $M_{n-1} \cap M_n$, $M_n \cap M_{n+1}$, $M_n \cap A_\lambda$ $(\lambda \in \Lambda(n))$ with the additional property that φ_{n+1} refines φ_n on $M_n \cap M_{n+1}$. (Read $M_o = \emptyset$). Then using Lemma 4.3 we refine φ_n to a triangulation $\psi_n : Y_n \xrightarrow{\sim} M_n$ such that ψ_n is equivalent to φ_n on $M_{n-1} \cap M_n$ and equivalent to φ_{n+1} on $M_n \cap M_{n+1}$. (Notice that $M_{n-1} \cap M_n$ and $M_n \cap M_{n+1}$ are disjoint closed semialgebraic subsets of M_n.) Now ψ_{n+1} is equivalent to ψ_n on $M_{n+1} \cap M_n$. By Lemma 4.1 the ψ_n can be glued to a triangulation $\psi : Y \xrightarrow{\sim} M$. By construction $\psi^{-1}(A_\lambda)$ is a subcomplex of Y for every $\lambda \in \Lambda$. q.e.d.

§5 - Triangulation of weakly simplicial maps, maximal complexes

Definition 1. Let $f: M \to N$ be a locally semialgebraic map.

a) A triangulation of f is a commutative square

$$\begin{tikzcd} X \arrow[r, "\sim", "\varphi"'] \arrow[d, "g"'] & M \arrow[d, "f"] \\ Y \arrow[r, "\sim", "\psi"'] & N \end{tikzcd} \qquad (1)$$

with X and Y strictly locally finite geometric (simplicial) complexes, φ and ψ locally semialgebraic isomorphisms, and g a simplicial map.

b) A completion of f is a commutative square

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] & \overline{M} \arrow[d, "\overline{f}"] \\ N \arrow[r, hook, "j"'] & \overline{N} \end{tikzcd} \qquad (2)$$

with i and j completions of the spaces M,N respectively (i.e. dense embeddings into partially complete spaces $\overline{M},\overline{N}$, cf. §2), and $\overline{f}$ a locally semialgebraic map.

Notice that the map $\overline{f}$ is partially proper. Notice also that every triangulation (1) of f yields a completion of f, namely

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[d, "f"'] & \overline{X} \arrow[d, "\overline{g}"] \\ N \arrow[r, hook, "j"] & \overline{Y} \end{tikzcd}$$

with i and j denoting the embeddings via φ^{-1} and ψ^{-1} into the closures $\overline{X},\overline{Y}$ of the complexes X, Y and $\overline{g}$ the closure of the simplicial map g.

We have the following general fact about completions of maps.

Proposition 5.1. Every locally semialgebraic map $f: M \to N$ between regular and paracompact spaces M,N can be completed. More precisely, given a completion $j : N \hookrightarrow Q$, we can find a completion (2) of f which involves j.

<u>Proof.</u> We choose some completion $\kappa : M \hookrightarrow T$. Let $\Gamma(f) \subset M \times N$ be the graph of f. We consider the natural factorization

$$\begin{tikzcd} M \arrow[r, "\sim", "\rho"'] \arrow[dr, "f"'] & \Gamma(f) \arrow[d, "p"] \\ & N \end{tikzcd}$$

of f via the graph. Let P be the closure of the image of $\Gamma(f)$ in $T \times Q$ under the embedding $\kappa \times j : M \times N \to T \times Q$, and let $i_1 : \Gamma(f) \hookrightarrow P$ denote the completion of $\Gamma(f)$ obtained from $\kappa \times j$ by restriction. Let finally $g : P \to Q$ be the restriction of the natural projection $pr_2 : T \times Q \to Q$ to P. We have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\sim", "\rho"'] \arrow[dr, "f"'] & \Gamma(f) \arrow[r, hook, "i_1"'] \arrow[d, "p"] & P \arrow[r, hook] \arrow[d, "g"] & T \times Q \arrow[dl, "pr_2"] \\ & N \arrow[r, hook, "j"'] & Q & \end{tikzcd}$$

The completion $i := i_1 \circ \rho$ of M and the map g fulfill the requirements of the proposition. q.e.d.

The question whether a given map can be triangulated - and how - is much more delicate. In the present section we deal with an easy case, namely a weakly simplicial map $f : X \to Y$ between strictly locally finite complexes X and Y (cf. I, §2, Def. 5). The problem here is, of course, that f does not extend continuously - hence simplicially - to a map from $\overline{X}$ to $\overline{Y}$. Our idea is to triangulate f by "modifying" the complexes X and Y in a somewhat canonical way, such that X and Y remain the same as locally semialgebraic spaces and f extends simplicially to the closures of the modified complexes.

Our procedure will be purely combinatorial. All constructions may be carried out in the category of strictly locally finite complexes with simplicial maps as morphisms. Actually we shall work in the larger category of all geometric simplicial complexes over R with the simplicial maps as morphisms. (So, in general, weakly simplicial maps are not regarded as morphisms.) This generalization will not cause any additional difficulties. On the other hand, a geometric simplicial complex X which is not locally finite can still be interpreted geometrically as a subspace of a "weak polytope", cf. end of III, §6. Thus our combinatorial considerations should be of additional interest later on.

When it seems to be convenient we switch over from the category of geometric simplicial complexes over R to the category of abstract simplicial complexes, with abstract simplicial maps as morphisms. These categories are equivalent via the abstraction and the realization functor, cf. §3. Thus it does not matter, in which category we work. In particular, the base field R is of no importance. As before, we speak simply of "complexes" instead of simplicial complexes.
In view of the examples 6.15, 6.17 and proposition 6.16 in Chapter I it is natural to introduce the following terminology.

Definition 2. Let $f : X \to Y$ be a simplicial map between geometric complexes X and Y.
a) f is called partially proper if, for every open simplex $\sigma \in \Sigma(X)$, the following holds: If τ is an open face of σ with $\overline{f}(\tau) \in \Sigma(Y)$, then $\tau \in \Sigma(X)$. The map f is called proper if, in addition, for every $\rho \in \Sigma(Y)$ there exist only finitely many $\sigma \in \Sigma(X)$ with $f(\sigma) = \rho$. In other words the preimage $f^{-1}(\rho)$ of any $\rho \in \Sigma(Y)$ is a finite subcomplex of X.
b) f is called partially finite if f is partially proper and $\overline{f}$ is injective on the set $E(\sigma)$ of vertices of every $\sigma \in \Sigma(X)$. The map f is

called <u>finite</u> if, in addition, the preimage $f^{-1}(\rho)$ of every $\rho \in \Sigma(Y)$ is a finite complex.

Clearly the closure $\bar{f}: \bar{X} \to \bar{Y}$ of any simplicial map $f: X \to Y$ is partially proper. Also, a partially proper map $f: X \to Y$ is finite if and only if all fibres of <u>f are finite sets</u>. If f is partially finite then also $\bar{f}$ is partially finite. But it may happen that f is finite and $\bar{f}$ is not finite. For example, let $\bar{X}$ be the disjoint union of countably many closed 1-simplices $\bar{\sigma}_n$ $(n \in \mathbb{N})$, and let $\bar{Y}$ be the union of countably many closed 1-simplices $\bar{\tau}_n (n \in \mathbb{N})$, all meeting in one common vertex p. We choose a vertex q_n in each $\bar{\sigma}_n$. Let $\bar{f}$ be the simplicial map from $\bar{X}$ onto $\bar{Y}$ which maps every $\bar{\sigma}_n$ onto $\bar{\tau}_n$ with $\bar{f}(q_n) = p$. Let X denote the subcomplex $\bar{X} \setminus \{q_n | n \in \mathbb{N}\}$ of $\bar{X}$ and Y denote the subcomplex $\bar{Y} \setminus \{p\}$ of $\bar{Y}$. The restriction $f: X \to Y$ of $\bar{f}$ is finite (in fact "dominant", see below). But $\bar{f}$ is not finite.

<u>Definition 2a.</u> A simplicial map $\alpha: K \to L$ between abstract complexes is called partially proper (resp. proper, resp. ...) if the realization $|\alpha|_R: |K|_R \to |L|_R$ is partially proper (resp. proper, resp. ...).

Notice that these properties of α do not depend on the choice of the base field R.

We now come to the main definition of this section.

<u>Definition 3.</u> A simplicial map $f: X \to Y$ between geometric complexes is called <u>dominant</u>, if f is partially proper and f is a bijection from the underlying set of X to the underlying set of Y. Then, of course, f is finite.

Notice that, in the case that X and Y are locally finite, this just means that f is an isomorphism between the locally semialgebraic spaces X and Y. Notice also that, for general complexes X and Y, the set theoretic inverse $f^{-1} : Y \to X$ of a dominant map $f : X \to Y$ is a weakly simplicial map. In some sense every weakly simplicial map can be built up from such maps f^{-1}, as we shall see later.

If X is closed, then every dominant simplicial map $f : X \to Y$ is a simplicial isomorphism and, in particular, Y is closed.

Definition 3a. A simplicial map $\alpha : K \to L$ between abstract complexes is called dominant if its realization $|\alpha|_R$ is dominant.

We state some formal properties of the notions introduced in the last two definitions, leaving the trivial proofs to the reader.

Remark 5.2. a) If X is a subcomplex of a geometric complex Y then the inclusion map $i : X \hookrightarrow Y$ is partially proper if and only if X is closed in Y. In this case i is finite.

b) Let $f : X \to Y$ and $g : Y \to Z$ be simplicial maps between geometric complexes. Let P be any one of the properties "partially proper", "partially finite", "finite".

i) If f and g both have property P then $g \circ f$ has property P.

ii) If $g \circ f$ has P then f has P.

iii) If $g \circ f$ has P and f is surjective (as map from the set X to the set Y), then g has P.

If P is the property "dominant", then the statements i) and iii) are also true. Statement ii) is only true if f is assumed to be surjective.

The reader should not be troubled by the fact that in the category of paracompact regular spaces statement iii) becomes false for P = "partially proper" (I, 6.3) or P = "partially finite" (I, 7.14). Such maps cannot always be triangulated.

In the category of abstract complexes, hence also in the category of geometric complexes, there exist arbitrary fibre products. Indeed, let $\alpha_1 : K_1 \to K$ and $\alpha_2 : K_2 \to K$ be abstract simplicial maps. We look at the set

$$F := E(K_1) \underset{E(K)}{\times} E(K_2) = \{(e_1,e_2) \in E(K_1) \times E(K_2) \mid \alpha_1(e_1) = \alpha_2(e_2)\},$$

with the natural projections $p_1 : F \to E(K_1)$, $p_2 : F \to E(K_2)$. We define a complex $L = (E(L),S(L))$ as follows. $S(L)$ is the set of all subsets s of F with $p_1(s) \in S(K_1)$ and $p_2(s) \in S(K_2)$, and $E(L)$ is the union of all the subsets $s \in S(L)$ of F. The restriction $\pi_i : E(L) \to E(K_i)$ of p_i is a simplicial map $\pi_i : L \to K_i$ $(i = 1,2)$ and the diagram

$$\begin{tikzcd} L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & K_2 \arrow[d, "\alpha_2"] \\ K_1 \arrow[r, "\alpha_1"'] & K \end{tikzcd} \qquad (*)$$

commutes. It is now easily checked that this diagram is a pull back in the category of abstract complexes. Thus L is the fibre product $K_1 \times_K K_2$.

The following statements are easily verified.

<u>Lemma 5.3.</u> Let P be any one of the properties "partially proper", "proper", "partially finite", "finite", "dominant". If in the pull back diagram (*) the map α_1 has property P then π_2 also has property P.

<u>Examples 5.4.</u> a) If both complexes K_1 and K_2 are closed then it is easily seen that $E(K_1 \times_K K_2) = E(K_1) \times_{E(K)} E(K_2)$.

b) (The case that K is the one point complex.) For any two abstract complexes K_1 and K_2 the direct product $K_1 \times K_2$ exists and may be described as follows: $E(K_1 \times K_2) = E(K_1) \times E(K_2)$. A subset s of $E(K_1) \times E(K_2)$

is a simplex of $K_1 \times K_2$ if and only if the projections $p_1(s)$ and $p_2(s)$ are simplices of K_1 and K_2 respectively. The reader is warned that the locally semialgebraic space $|K_1 \times K_2|_R$ usually is different from the product $|K_1|_R \times |K_2|_R$ of the locally semialgebraic spaces $|K_1|_R$ and $|K_2|_R$. In fact, this already occurs in the case where both K_1 and K_2 are closed and finite cf. [ES, p. 66ff]. The space $|K_1|_R \times |K_2|_R$ is then a strong deformation retract of $|K_1 \times K_2|_R$.

Definition 4. An abstract complex L is called maximal, if every dominant simplicial map $\alpha : L' \to L$ is a simplicial isomorphism. A maximal hull of an abstract complex K is a dominant map $\alpha : L \to K$ with L a maximal complex. Same terminology for geometric complexes.

Proposition 5.5. Let $\alpha : L \to K$ and $\beta : P \to K$ be simplicial maps. Assume that L is maximal and β is dominant. Then there exists a unique simplicial map $\eta : L \to P$ with $\beta \circ \eta = \alpha$. If α is dominant, i.e. α is a maximal hull of K, then η is also dominant. In particular, any two maximal hulls of K are isomorphic.

Proof. We consider the pull-back diagram

$$\begin{tikzcd} Q \arrow[r, "\delta"'] \arrow[d, "\gamma"'] & P \arrow[d, "\beta"] \\ L \arrow[r, "\alpha"'] & K \end{tikzcd}$$

of the maps α and β in the category of abstract complexes. Since β is dominant, γ is dominant (Lemma 5.3). Since L is maximal, γ is an isomorphism. Thus $\eta := \delta \circ \gamma^{-1}$ is a simplicial map with $\alpha = \beta \circ \eta$. Uniqueness of η is evident, say, from the pull-back property of the diagram. If α is dominant then δ is also dominant by Lemma 5.3. Thus η is dominant too. q.e.d.

We now start out to prove the existence of the maximal hull for a given abstract complex K. We construct a new complex $\hat{K}$ as follows. Let H(K) denote the set of all pairs $(p,s) \in E(K) \times S(K)$ with $p \in s$. We introduce on H(K) the coarsest equivalence relation $\sim$ such that $(p,t) \sim (p,s)$ whenever $p \in t \leq s$. Thus $(p_1,s_1) \sim (p_2,s_2)$ if and only if $p_1 = p_2$ and there exists a finite sequence.

$$s_1 = t_o, t_1, \dots, t_{2m} = s_2$$

with $p_1 \in t_i$ for $i = 1,\dots,2m$ and

$$t_o \leq t_1, t_1 \geq t_2, t_2 \leq t_3, \dots, t_{2m-1} \geq t_{2m}.$$

We denote the equivalence class of a pair $(p,s) \in H(K)$ by $[p,s]$, and we take as set of vertices $E(\hat{K})$ the set $H(K)/\sim$ of all these equivalence classes. We further introduce for every $s \in S(K)$ the finite subset

$$[s] := \{[p,s] \mid p \in s\}$$

of $E(\hat{K})$ and take as set of simplices of $\hat{K}$

$$S(\hat{K}) := \{[s] \mid s \in S(K)\}.$$

The pair $(E(\hat{K}),S(\hat{K}))$ is indeed an abstract complex $\hat{K}$, and the natural map $\pi = \pi_K : E(\hat{K}) \to E(K)$, $\pi([p,s]) = p$, is a simplicial map from $\hat{K}$ to K. More precisely, π maps every simplex $[s]$ of $\hat{K}$ bijectively onto the corresponding simplex s of K. If u is a face of some $[s] \in S(\hat{K})$ with $\pi(u) = t \in S(K)$, then $u = [t] \in S(\hat{K})$. Thus π is dominant. We shall see later that $\pi_K : \hat{K} \to K$ is indeed a maximal hull of K.

For any geometric complex X we denote by $\hat{X}$ the realization of $K(X)^\wedge$ and by $p_X : \hat{X} \to X$ the realization of $\pi_{K(X)} : K(X)^\wedge \to K(X)$. From now on a weakly simplicial map from X to a geometric complex Y will be denoted by a dotted arrow $X \dashrightarrow Y$ to keep in mind that it is not an arrow in the category of complexes.

<u>Theorem 5.6.</u> For any weakly simplicial map $f : X \dashrightarrow Y$ there exists a

unique simplicial map $\hat{f} : \hat{X} \to \hat{Y}$ such that the diagram

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "p_X"'] & \hat{Y} \arrow[d, "p_Y"] \\ X \arrow[r, dotted, "f"] & Y \end{tikzcd}$$

commutes.

Notice that, in the case that X and Y are strictly locally finite, $\hat{X}$ and $\hat{Y}$ are again strictly locally finite, and thus Theorem 5.6 yields a triangulation of the locally semialgebraic map f.

<u>Proof of Theorem 5.6.</u> Let $K := K(X)$, $L := K(Y)$. Thus $X = |K|$, $Y = |L|$. As s runs through $S(K)$, $|s|$ runs through the set $\Sigma(X)$ of open simplices of X. We have a map $\alpha : S(K) \to S(L)$, defined by $f(|s|) = |\alpha(s)|$. Notice that, in general, this map α is <u>not</u> induced by a map from $E(K)$ to $E(L)$. But if $t \in S(K)$ is a face of s then $\alpha(t)$ is a face of $\alpha(s)$. Moreover, f yields by restriction an affine map $f_s : |s| \twoheadrightarrow |\alpha(s)|$, which extends continuously to an affine map $\overline{f}_s : \overline{|s|} \to \overline{|\alpha(s)|}$, and $\overline{f}_s$ coincides with f_t on $|t|$ (cf. I, §2, Def. 5).

We now define a map β from $E(\hat{K})$ to $E(\hat{L})$ by

$$\beta([p,s]) = [\overline{f}_s(p), \alpha(s)] \quad (s \in S(K), p \in E(K), p \in s).$$

It is easily checked that β is well defined. Indeed, if $p \in t < s$, then $\overline{f}_t(p) = \overline{f}_s(p)$. Clearly $\beta([s]) = [\alpha(s)]$. Thus β is a simplicial map from $\hat{K}$ to $\hat{L}$. Let $\hat{f} : \hat{X} \to \hat{Y}$ denote the realization of β. For every $s \in S(K)$ the diagram

$$\begin{tikzcd} |[s]| \arrow[r, two heads, "\text{"}\hat{f}\text{"}"] \arrow[d, "\text{"}p_X\text{"}"', "\cong"] & |[\alpha(s)]| \arrow[d, "\text{"}p_Y\text{"}", "\cong"'] \\ |s| \arrow[r, two heads, "\text{"}f\text{"}"'] & |\alpha(s)| \end{tikzcd}$$

commutes. Thus indeed $f \bullet p_X = p_Y \circ \hat{f}$. The uniqueness statement in the theorem is evident, since p_Y is a bijection from the set $\hat{Y}$ to the set Y. q.e.d.

<u>Corollary 5.7.</u> For any geometric complex X the map $q := p_{\hat{X}} : \hat{\hat{X}} \to \hat{X}$ is a simplicial isomorphism.

<u>Proof.</u> Apply the theorem to the weakly simplicial map $g = p_X^{-1}$ from X to $\hat{X}$. We obtain a simplicial map $\hat{g}$ from $\hat{X}$ to $\hat{\hat{X}}$ with $q \bullet \hat{g} = g \bullet p_X = \mathrm{id}_{\hat{X}}$. Since q and $\hat{g}$ are both bijective, as maps between sets, also $\hat{g} \bullet q = \mathrm{id}_{\hat{\hat{X}}}$. q.e.d.

<u>Corollary 5.8.</u> For any geometrical complex X the complex $\hat{X}$ is maximal, and thus $p_X : \hat{X} \to X$ is a maximal hull of X.

<u>Proof.</u> Let $f : Y \to \hat{X}$ be a dominant map. Applying Theorem 5.6 to the weakly simplicial maps f and $g := f^{-1}$ we obtain a commutative diagram

$$\begin{tikzcd}
\hat{Y} \arrow[r, "\hat{f}"] \arrow[d, "p"'] & \hat{\hat{X}} \arrow[r, "\hat{g}"] \arrow[d, "q"] & \hat{Y} \arrow[d, "p"] \\
Y \arrow[r, "f"'] & \hat{X} \arrow[r, dashed, "g"'] & Y
\end{tikzcd}$$

with $p := p_Y$, $q := p_{\hat{X}}$. From the uniqueness statement in the same theorem we deduce that $\hat{g} \circ \hat{f} = \mathrm{id}_{\hat{Y}}$ and $\hat{f} \circ \hat{g} = \mathrm{id}_{\hat{\hat{X}}}$. Thus $\hat{f}$ and $\hat{g}$ are simplicial isomorphisms. By the preceding corollary also q is a simplicial isomorphism. Thus $h := p \circ \hat{f}^{-1} \circ q^{-1}$ is a right inverse of f. Since f is a bijection of sets, the simplicial map h is also a left inverse of f. Thus f is a simplicial isomorphism. q.e.d.

<u>Corollary 5.9.</u> Every weakly simplicial map $f : X \dashrightarrow Y$ from a maximal complex X to an arbitrary complex Y is simplicial.

<u>Proof.</u> Consider the commutative diagram

$$\begin{tikzcd}
\hat{X} \arrow[r, "\hat{f}"] \arrow[d, "p_X"'] & \hat{Y} \arrow[d, "p_Y"] \\
X \arrow[r, dotted, "f"'] & Y.
\end{tikzcd}$$

p_X is an isomorphism. Thus $f = p_Y \circ \hat{f} \circ p_X^{-1}$ is indeed simplicial.

q.e.d.

§6 - Triangulation of amenable partially finite maps

We will only consider regular and paracompact locally semialgebraic spaces until the end of this section. Thus from now on - unless something else is said - a "space" will always be a regular paracompact locally semialgebraic space.

We gather together some observations on partially finite simplicial maps which mostly appeared in §5. A few are already contained in I, §6.

Proposition 6.1. Let $f: Y \to X$ be a partially finite simplicial map between strictly locally finite geometric complexes. Then f and $\overline{f}: \overline{Y} \to \overline{X}$ - regarded as maps between spaces - have the following properties:

i) $(\overline{f})^{-1}(X) = Y$.

ii) $\overline{f}$ is partially finite.

iii) Every open simplex τ of Y is mapped under f isomorphically onto an open simplex σ of X. Also, the closure $\overline{\tau} \cap Y$ of τ in Y is mapped isomorphically onto $\overline{\sigma} \cap X$, and $\overline{\tau}$ is mapped under $\overline{f}$ isomorphically onto $\overline{\sigma}$.

iv) For every open simplex σ of X the connected components of the locally semialgebraic subset $f^{-1}(\sigma)$ of Y are precisely all open simplices τ of X with $f(\tau) = \sigma$. In particular, f is trivial over σ, i.e. $f^{-1}(\sigma)$ is isomorphic over σ to a product $\sigma \times F$, with F a discrete space in our case.

In this proposition only the last statement (iv) needs further explanation. Notice that the simplices $\tau \in \Sigma(Y)$ with $f(\tau) = \sigma$ are open connected subcomplexes of the complex $f^{-1}(\sigma)$ and thus are open semialgebraic subsets of $f^{-1}(\sigma)$. Moreover they are pairwise disjoint.

We draw the following consequence from the statements (iii) and (iv) in this proposition.

Corollary 6.2. Let X,Y,Z be strictly locally finite complexes and let

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

be a commutative diagram of locally semialgebraic maps with f and g partially finite. Hence h is also partially finite (cf. I.6.2.ii). Assume that f and g are simplicial. Then h is weakly simplicial. Thus, if in addition the complex Z is maximal (cf. §5), the map h is simplicial.

We want to triangulate partially finite maps. According to Proposition 6.1.iv a necessary condition for this is that the maps be "amenable" in the following sense.

Definition 1. A locally semialgebraic map $f: M \to N$ is called amenable if there exists a locally finite family $(N_\lambda | \lambda \in \Lambda)$ in $\mathcal{T}(N)$ such that N is the union of all N_λ and f is trivial over every N_λ, i.e. $f^{-1}(N_\lambda)$ is isomorphic over N_λ to a product $N_\lambda \times F_\lambda$ with some space F_λ.

Then, by the triangulation theorem 4.4, there exists a triangulation $\varphi: X \xrightarrow{\sim} N$ such that f is trivial over the image $\varphi(\sigma)$ of each open simplex $\sigma \in \Sigma(X)$.

We quote from [DK_3, Th. 6.4] (cf. also [BCR, Chap. 9]) the following important fact, which in the case $R = \mathbb{R}$ is due to R. Hardt [H], and therefore will often be cited as "Hardt's theorem".

Theorem 6.3. Every affine semialgebraic map $f: M \to N$ is amenable. Given a locally finite family $(M_i | i \in I)$ of locally semialgebraic subsets of M there even exists a locally finite partition $(N_\lambda | \lambda \in \Lambda)$ of N into semialgebraic sets such that f and all the restrictions $f|M_i : M_i \to N$ $(i \in I)$ are trivial over each N_λ.

Actually this has been proved in [DK_3] only in the category of semialgebraic spaces, but the generalization to the locally semialgebraic setting is immediate. Again, the sets N_λ can be chosen as the images of the open simplices of a suitable triangulation $Y \xrightarrow{\sim} N$.

Another important class of amenable maps are the coverings, to be studied later (Chap. V). There exist partially finite maps which are not amenable. For example, take $R = \mathbb{R}$ and construct an infinitely high Eiffel tower $T \subset [0,1] \times [0,1] \times \mathbb{R}_{loc}$ over $[0,1] \times [0,1]$ whose diameter tends to zero as the height increases. The projection $p : T \to [0,1] \times [0,1] \setminus \{(\frac{1}{2}, \frac{1}{2})\}$ is partially finite but not amenable. Even simpler, for any locally complete but not complete semialgebraic space M the identity map from M_{loc} to M is partially finite but not amenable.

To triangulate a given amenable partially finite map $f : M \to N$ we may assume that N is already triangulated in such a way that f is trivial over the image of every open simplex under the triangulation map $\varphi : X \xrightarrow{\sim} N$. We may as well assume that $N = X$ and φ is the identity. We thus meet the following

<u>Problem 6.4.</u> Let $f : M \to X$ be a partially finite map from a space M to a strictly locally finite complex X which is trivial over every open simplex of X. Under what further conditions on f can M be triangulated in such a way that f becomes simplicial? More precisely, when does there exist a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "\tilde{\psi}"] \arrow[dr, "g"'] & & M \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

with Y a strictly locally finite complex, g a simplicial map, and ψ a locally semialgebraic isomorphism?

If such a diagram exists then we may replace Y by its maximal hull $\hat{Y}$ (cf. §5) and assume as well that Y is maximal. According to Corollary 6.2 any other diagram of the same sort - with f fixed - is isomorphic to the given one in an obvious sense.

We stay with a map $f : M \to X$ as described in Problem 6.4. We denote by $\Sigma(M)$ the set of all connected components of all sets $f^{-1}(\sigma)$ with σ running through the set $\Sigma(X)$ of open simplices of X. The family $\Sigma(M)$ is clearly locally finite and is a partition of M into semialgebraic sets. For a given open simplex σ of X every connected component T of $f^{-1}(\sigma)$ is mapped isomorphically onto σ by f. We quote a general lemma, which will also be useful in later sections. Applying this lemma to the restriction $\overline{T} \to \overline{\sigma} \cap X$ of f, with $\overline{T}$ the closure of T in M, we see that f maps $\overline{T}$ isomorphically onto $\overline{\sigma} \cap X$.

<u>Lemma 6.5.</u> Let $\pi : L \to N$ be a proper map between spaces. Assume that S is a locally semialgebraic subset of N such that the fibres $\pi^{-1}(s)$ of all points $s \in S$ are connected and not empty. Assume further that S is dense in N and $\pi^{-1}(S)$ is dense in L. Assume finally that every point x in $N \setminus S$ has a fundamental system of open semialgebraic neighbourhoods U with $U \cap S$ connected. Then all fibres of π are connected and not empty. In particular if π is also finite, π is an isomorphism.

We postpone the proof of this lemma for a moment and continue the discussion of the partition $\Sigma(M)$ of M. If $\sigma_1 \in \Sigma(X)$ is an open face of

$\sigma \in \Sigma(X)$, and T is a connected component of $f^{-1}(\sigma)$, then $T_1 := \overline{T} \cap f^{-1}(\sigma_1)$ is a connected component of $f^{-1}(\sigma_1)$, since f maps T_1 isomorphically onto σ_1. In particular, the closure $\overline{T}$ of any $T \in \Sigma(M)$ is a union of sets $T' \in \Sigma(M)$. Thus we may regard $\Sigma(M)$ as a "semialgebraic stratification" of M, and we call the $T \in \Sigma(M)$ the "strata" of M.

We now start out to prove Lemma 6.5. We need the following general fact.

<u>Sublemma 6.6.</u> Let f be a proper surjective map from a space N to a space M. Assume that M as well as all fibres of f are connected. Then N is connected.

<u>Proof.</u> Suppose there exists a partition $N = A_1 \sqcup A_2$ of N into two disjoint open - hence also closed - non empty locally semialgebraic subsets. Since the fibres of f are connected, both A_1 and A_2 are unions of fibres. Since f is surjective M is the disjoint union of the sets $B_1 := f(A_1)$ and $B_2 := f(A_2)$. Since f is proper the sets B_i are closed and locally semialgebraic in M. But this contradicts our assumption that M is connected. Thus N is connected. q.e.d.

<u>Proof of Lemma 6.5.</u> $\pi(L)$ is closed and contains S. Thus $\pi(L) = N$. Let a point $x \in N \setminus S$ be given. Suppose $\pi^{-1}(x)$ is not connected. We choose a partition $\pi^{-1}(x) = A_1 \sqcup A_2$ of $\pi^{-1}(x)$ into two disjoint locally semialgebraic subsets, which are both closed in $\pi^{-1}(x)$ and non empty. By a standard argument for proper maps we find an open set $U \subset N$ in our given fundamental system of neighbourhoods of x such that $\pi^{-1}(U) = V_1 \cup V_2$ with V_i an open locally semialgebraic neighbourhood of A_i $(i=1,2)$ and $V_1 \cap V_2 = \emptyset$. We have

$$\pi^{-1}(U \cap S) = [V_1 \cap \pi^{-1}(S)] \sqcup [V_2 \cap \pi^{-1}(S)].$$

Both sets $V_i \cap \pi^{-1}(S)$ are non empty, locally semialgebraic, and open in $\pi^{-1}(U \cap S)$. But, by the sublemma, $\pi^{-1}(U \cap S)$ is connected. This contradiction proves that $\pi^{-1}(x)$ must be connected. q.e.d.

We continue studying the stratification $\Sigma(M)$ of M. Let $K(X) = (E(X),S(X))$ be the abstraction of the geometric complex X. We want to construct an abstract complex $K(X,f) = (E(X,f),S(X,f))$ which hopefully will serve to triangulate M in such a way that the strata become the images of the open simplices of $|K(X,f)|$.

Let $H(X,f)$ denote the set of pairs $(p,T) \in E(X)\times\Sigma(M)$ with $p \in \overline{f(T)}$. {$\overline{f(T)}$ means the closure of the open simplex $f(T)$ in the closed complex $\overline{X}$.} We introduce the coarsest equivalence relation $\sim$ in $H(X,f)$ such that $(p,T_1) \sim (p,T_2)$ whenever $\overline{T_1} \subset \overline{T_2}$. Two pairs (p_1,T_1) and (p_2,T_2) are equivalent if and only if $p_1 = p_2 =: p$ and there exists a sequence $U_o = T_1, U_1, \ldots, U_{2m} = T_2$ in $\Sigma(M)$ with $p \in \overline{f(U_i)}$ for $0 \le i \le 2m$ and

$$\overline{U}_o \subset \overline{U}_1,\ \overline{U}_1 \supset \overline{U}_2,\ \overline{U}_2 \subset \overline{U}_3, \ldots, \overline{U}_{2m-1} \supset \overline{U}_{2m}.$$

We denote the equivalence class of a pair $(p,T) \in H(X,f)$ by $[p,T]$, and we take the set of all these equivalence classes as set of vertices $E(X,f)$ of our complex $K(X,f)$, $E(X,f) := H(X,f)/\sim$. For every $T \in \Sigma(M)$ we define the finite subset

$$[T] := \{[p,T] \mid p \in \overline{f(T)} \cap E(X)\}$$

of $E(X,f)$, and we define the set of these subsets $[T]$, with T running through $\Sigma(M)$, to be the set of simplices $S(X,f)$ of $K(X,f)$. Then $K(X,f) = (E(X,f),S(X,f))$ is indeed an abstract simplicial complex. We have a well defined map

$$\pi = \pi_{X,f} : E(X,f) \to E(X),\ [p,T] \mapsto p,$$

which is a simplicial map from $K(X,f)$ to $K(X)$.

Lemma 6.7. π is partially finite (in the sense of Definitions 2 and 2a in §5).

Proof. Let $[T]$ be a simplex of $K(X,f)$. Clearly π is injective on $[T]$. In order to prove that π is partially finite we consider a proper face t_1 of $[T]$ with $\pi(t_1) \in S(X)$. We have to show that $t_1 \in S(X,f)$. Now $\pi([T])$ is the abstraction s of the open simplex $\sigma := f(T)$ of X, and $|\pi(t_1)|$ is a face σ_1 of σ with $\sigma_1 \in \Sigma(X)$. There exists a unique stratum $T_1 \subset \overline{T}$ of M with $f(T_1) = \sigma_1$ (see above). We have $[T_1] < [T]$, $\pi(t_1) = \pi([T_1])$, hence $t_1 = [T_1] \in S(X,f)$. Thus π is indeed partially finite.

We need the following general assumption on our stratification $\Sigma(M)$.
(*) If T_1 and T_2 are two different strata of M, then $[T_1] \neq [T_2]$.

Let us look for cases where (*) is fulfilled.

Lemma 6.8. Assume that T_1 and T_2 are two different strata of M with $f(T_1) = f(T_2) =: \sigma$ and that σ has at least one vertex p with $p \in X$ and $\overline{T}_1 \cap f^{-1}(p) \neq \overline{T}_2 \cap f^{-1}(p)$. Then $[T_1] \neq [T_2]$.

Proof. We claim that $[p,T_1] \neq [p,T_2]$. Otherwise there would exist a sequence of strata $U_o = T_1,\dots,U_{2m} = T_2$ with

$$\overline{U}_o \subset \overline{U}_1 \supset \overline{U}_2 \subset \overline{U}_3 \supset \dots \subset \overline{U}_{2m-1} \supset \overline{U}_{2m}$$

and $p \in \overline{f(U_i)}$ for $0 \leq i \leq 2m$. Let x_i be the point of $\overline{U}_j$ lying over p. We see successively that $x_o = x_1 = \dots = x_{2m}$. But $\{x_o\} = \overline{T}_1 \cap f^{-1}(p)$, $\{x_{2m}\} = \overline{T}_2 \cap f^{-1}(p)$. This contradicts the hypothesis of the lemma. Thus $[p,T_1] \neq [p,T_2]$. Since π maps both $[T_1]$ and $[T_2]$ bijectively onto the same simplex $s \in S(X)$ the vertex $[p,T_1]$ cannot be contained in $[T_2]$. q.e.d.

From this lemma we see immediately

Corollary 6.9. Condition (*) is certainly fulfilled in the following cases.

i) The complex X is closed and f is trivial over each closed simplex $\overline{\sigma}$ of X.

ii) f is a partially finite map from the space M to a strictly locally finite complex X_1, and f is trivial over each open simplex of X_1. The complex X is the barycentric subdivision of X_1.

Here the <u>barycentric subdivision</u> Y' of a geometric complex Y is defined in the obvious way: subdivide $\overline{Y}$ in the classical way, and collect all open simplices of $\overline{Y}'$ which are contained in the set Y.

A third case in which condition (*) is clearly fulfilled is the following: $M = X$, $f = \mathrm{id}_X$. Then the complex $K(X,f)$ is nothing other than the abstraction $K(\hat{X})$ of the maximal hull $\hat{X}$ of X defined in §5. This case leads back to the considerations in §5.

From now on <u>we always assume that condition</u> (*) <u>is fulfilled</u>.

<u>Lemma 6.10.</u> Let T be a stratum of M. Then the simplices $u \in S(X,f)$ having $[T]$ as a face are in one-to-one correspondence with the strata $U \in \Sigma(M)$ with $\overline{T} \subset \overline{U}$ via $u = [U]$. The complex $K(X,f)$ is locally finite.

<u>Proof.</u> Let $t := [T] \in S(X,f)$ and $\sigma := |\pi(t)| = f(T) \in \Sigma(X)$. Let $u \in S(X,f)$ be a simplex which has t as a proper face and $\rho := |\pi(u)| \in \Sigma(X)$. Then $\rho > \sigma$. We have $u = [U]$ with a stratum $U \in \Sigma(M)$, uniquely determined by u according to assumption (*). Since $f(U) = \rho$, there exists a unique stratum $T_1 \subset \overline{U}$ with $f(T_1) = \sigma$. Now $t_1 := [T_1]$ is a face of u with $|\pi(t_1)| = \sigma = |\pi(t)|$. Thus $t_1 = t$. By assumption (*) we have $T_1 = T$. The first assertion of the lemma is now clear. The second follows from the fact that the partition $\Sigma(M)$ of M is locally finite. Indeed, for a given $T \in \Sigma(M)$ and a given semialgebraic neighbourhood W of $\overline{T}$, there exist only finitely many strata U with $U \cap W \neq \emptyset$. A fortiori there

exist only finitely many strata U with $\overline{U} \cap T \neq \emptyset$, i.e. $T \subset \overline{U}$. Thus for a given $t \in S(X,f)$ there exist only finitely many $u \in S(X,f)$ with $u \geq t$.
q.e.d.

It may happen that $K(X,f)$ is not strictly locally finite.

Example 6.11. We consider the "infinite fan" M, described in I, §2 after Definition 4. This is a locally finite geometric complex which is not strictly locally finite. Let p be the point of $\overline{M}$ missing in M, and let $(\tau_i \mid i \in \mathbb{N})$ be the family of all open 2-simplices of M, labelled in such a way that $\overline{\tau}_i \cap \overline{\tau}_j \cap M \neq \emptyset$ iff $|i-j| < 2$. We introduce the obvious "folding map" $\overline{f}$ from $\overline{M}$ to the closed standard triangle $\Delta = [e_o, e_1, e_2]$, obtained by gluing together the affine maps $\overline{f_i} : \overline{\tau_i} \xrightarrow{\sim} \Delta$ with $\overline{f}_i(p) = e_o$ for all i, $\overline{f_i}(\overline{\tau}_{i-1} \cap \overline{\tau}_i) = [e_o, e_2]$ for i even, $\overline{f_i}(\overline{\tau}_{i-1} \cap \overline{\tau}_i) = [e_o, e_1]$ for i odd. Let $f : M \twoheadrightarrow X := \Delta \setminus \{e_o\}$ be the restriction of $\overline{f}$ to M. This map f is partially finite and is trivial over each open simplex of X. The complex $K(X,f)$ is naturally isomorphic to the abstraction of the geometric complex M. Notice that condition (*) is fulfilled for $f : M \to X$.

We denote the realization $|K(X,f)|$ of the abstract complex $K(X,f)$ by $X(f)$, and the realization of $\pi_{X,f} : K(X,f) \to K(X)$ by $p_{X,f}$ or briefly by p.

Theorem 6.12. (As before, we assume that condition (*) is fulfilled). There exists a unique locally semialgebraic map $\psi : X(f) \to M$, such that the diagram

$$\begin{tikzcd} X(f) \arrow[rr, "\psi"] \arrow[dr, "p_{X,f}"'] & & M \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

commutes, mapping every open simplex $|[T]|$ of $X(f)$ into (hence onto) the corresponding stratum T of M. This map ψ is a locally semialgebraic isomorphism.

<u>Proof.</u> For any stratum $T \in \Sigma(M)$, the map $p = p_{X,f}$ yields an affine isomorphism from the closure $\overline{\tau} \cap X(f)$ of the corresponding open simplex $\tau := |[T]|$ of $X(f)$ onto $f(\overline{T})$. On the other hand f yields also a semialgebraic isomorphism from $\overline{T}$ onto $f(\overline{T})$. Thus we have a unique semialgebraic isomorphism

$$\psi_T : \overline{\tau} \cap X(f) \xrightarrow{\sim} \overline{T}$$

with $(f|\overline{T}) \circ \psi_T = p|\overline{\tau} \cap X(f)$. These maps ψ_T glue to a locally semialgebraic map ψ from $X(f)$ to M, since for any stratum $U \subset \overline{T}$ and corresponding open simplex $\gamma := |[U]|$ of $X(f)$ we have $\gamma \leq \tau$ and $\psi_T|\gamma = \psi_U|\gamma$, and since these simplices γ are all the open faces of τ (Lemma 6.10). We have $f \circ \psi = p$, and clearly ψ is the only locally semialgebraic map from $X(f)$ to M with $f \circ \psi = p$ mapping every open simplex τ of $X(f)$ into the corresponding stratum T of M. In the same way we obtain a locally semialgebraic map $\chi : M \to X(f)$ with $p \circ \chi = f$ mapping every stratum T of M isomorphically onto the corresponding open simplex τ of $X(f)$. The map χ is inverse to ψ. Thus ψ is a locally semialgebraic isomorphism.

q.e.d.

If the map f is finite then it is pretty obvious that the complex $X(f)$ is <u>strictly</u> locally finite. Thus we obtain from Theorem 6.3, 6.12 and Corollary 6.9 the following final result for finite maps.

<u>Theorem 6.13.</u> Every finite map between spaces $f : M \to N$ can be triangulated. More precisely, if $\varphi : X \xrightarrow{\sim} N$ is a triangulation of N, such that f is trivial over $\varphi(\sigma)$ for every open simplex σ of X, then there exists a triangulation $\psi : Y \xrightarrow{\sim} M$ and a simplicial map g from Y to the barycentric subdivision X' of X such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\sim", "\psi"'] \arrow[d, "g"'] & M \arrow[d, "f"] \\ X' \arrow[r, "\sim", "\varphi"'] & N \end{tikzcd}$$

commutes. If N is partially complete and f is trivial over the image

$\varphi(\bar{\sigma})$ of each closed simplex $\bar{\sigma}$ of X the same is true for X instead of X'.

In the case $R = \mathbb{R}$ R. Hardt obtained a similar result. By use of subanalytic stratifications he proved that finite (= proper light) subanalytic maps may be triangulated $[H_1]$. Our approach is more elementary and more explicit.

Corollary 6.14. Every finite map $f : M \to N$ between spaces can be completed to a finite map $\bar{f} : \bar{M} \to \bar{N}$ between partially complete spaces (cf. Definition 1 in §5).

It is also obvious that the complex $X(f)$ is strictly locally finite if the complex X is closed. Thus we have

Theorem 6.15. Every amenable partially finite map $f : M \to N$ between partially complete spaces can be triangulated. More precisely, the statements in Theorem 6.13 remain true in this case.

In general, an amenable partially finite map $f : M \to N$ can not be completed to a partially finite map $\bar{f} : \bar{M} \to \bar{N}$ between partially complete spaces. We will not attempt to give a formal proof of this negative fact here. Just look at the "folding of the fan" $f : M \to \Delta \setminus \{e_o\}$ in Example 6.11. To obtain a completion in the category of spaces from the completion $\bar{f} : \bar{M} \to \Delta$ in the category of geometric complexes, one has to "blow up" the point p of $\bar{M}$. Then one gets a locally semialgebraic map $g : \tilde{M} \to \Delta$ between partially complete spaces which extends f. But the fibre $g^{-1}(e_o)$ is not discrete. It is intuitively clear that one cannot get around a blowing up process to complete f in the category of spaces.

Thus, for amenable partially finite maps, we have to be content with "weak triangulations" in general.

<u>Definition 2.</u> A <u>weak triangulation</u> of a space N is an isomorphism $\varphi : X \xrightarrow{\sim} N$ from a locally finite (but not necessarily strictly locally finite) geometric complex X to N. A weak triangulation of a locally semialgebraic map $f : M \to N$ between spaces is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow[\psi]{\sim} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\varphi]{\sim} & N \end{array}$$

with weak triangulations φ and ψ of N and M respectively and a simplicial map g.

Now observe that in our whole study in this section we do not need that X is <u>strictly</u> locally finite, as long as we don't demand that the complex X(f) becomes strictly locally finite. Thus we obtain

<u>Theorem 6.15a.</u> Every amenable partially finite map $f : M \to N$ between spaces can be weakly triangulated. More precisely, if $\varphi : X \xrightarrow{\sim} N$ is a weak triangulation of N, such that f is trivial over $\varphi(\sigma)$ for every open simplex σ of X, then there exists a weak triangulation $\psi : Y \xrightarrow{\sim} M$ and a simplicial map g from Y to the barycentric subdivision X' of X such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow[\psi]{\sim} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[\varphi]{\sim} & N \end{array}$$

commutes.

The following final proposition shows that condition (*) is indispensable for obtaining (even weak) triangulations of partially finite maps, and that our whole procedure is somewhat canonical.

<u>Proposition 6.16.</u> Again let $f : M \to X$ be a partially finite map of a

space M into a locally finite complex X. Suppose there exists a commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "\sim", "\chi"'] \arrow[dr, "g"'] & & M \arrow[dl, "f"] \\ & X & \end{tikzcd} \tag{**}$$

with Y a locally finite complex, g a simplicial map, and χ a locally semialgebraic isomorphism. Then the stratification fulfills condition (*). Thus we also have the commutative diagram described in Theorem 6.12. There exists a simplicial dominant map $h : X(f) \to Y$ with $\chi \circ h = \psi$ and $g \circ h = p_{X,f}$. In other words (cf. Cor. 6.2), the complex X(f) is maximal.

<u>N.B.</u> If we have a diagram (**) with g only weakly simplicial then, replacing Y by its maximal hull $\hat{Y}$, we may go ahead and assume that g is simplicial.

<u>Proof.</u> The open simplices τ of Y correspond bijectively with the strata T of M via $T = \chi(\tau)$, cf. Prop. 6.1, iii and iv. (As said before, this remains true under the condition that X and Y are locally finite instead of strictly locally finite.) We want to map the abstract complex K(X,f) simplicially onto K(Y) by a map η which sends a point $[p,T] \in E(X,f)$ to the unique point $q \in E(Y)$ with q a vertex of $\chi^{-1}(T) = \tau$ and $\overline{g}(q) = p$. To see that this map $\eta : E(X,f) \to E(Y)$ is well-defined we have to consider a sequence $T_o, T_1, \ldots, T_{2m}$ in $\Sigma(M)$ with $\overline{T}_{i-1} \subset \overline{T}_i$ for i odd and $\overline{T}_{i-1} \supset \overline{T}_i$ for i even, and a point $p \in E(X)$ with $p \in \overline{f(T_i)}$ for $0 \le i \le 2m$. Each simplex $\tau_i := \chi^{-1}(T_i)$ has a unique vertex q_i with $\overline{g}(q_i) = p$. We have to verify that $q_o = q_1 = \ldots = q_{2m}$. But this is obvious, since

$$\tau_o \le \tau_1 \ge \tau_2 \le \cdots \le \tau_{2m-1} \ge \tau_{2m},$$

and $\overline{g}$ is simplicial. For any $T \in \Sigma(M)$ the map η yields a bijection of the abstract simplex [T] of K(X,f) to the abstraction of the simplex

$\chi^{-1}(T)$ of Y. Thus η is a simplicial map from $K(X,f)$ to $K(Y)$. Also if $[T_1] = [T_2]$ for two strata T_1, T_2 of $\Sigma(M)$, then $\chi^{-1}(T_1) = \chi^{-1}(T_2)$, thus $T_1 = T_2$. So, condition (*) is fulfilled. Denoting the realization of η by h, we clearly have $\chi \cdot h = \psi$. Multiplying by f on the left we obtain $g \cdot h = p_{X,f}$. The map h is a bijection from the set $X(f)$ to Y. Thus h is dominant. q.e.d.

§7 - Stars and shells

In this section we study the "stars" and "shells" of a geometric (simplicial) complex X, and their behaviour under partially finite simplicial maps. Here we cannot rely on the classical literature where stars have been widely used since the early days of combinatorial topology since our complexes are more general than the classical ones. Our considerations in this section are usually purely combinatorial. Thus we work - as in §5 - in the category of all geometric complexes.

Let X be a geometric complex over R. We already defined the star $St_X(A)$ of any subset A of X in I, §2. We now generalize this definition to subsets A of $\overline{X}$.

Definition 1. For any $x \in \overline{X}$ we define the star $St_X(x)$ of x in X as the union of all open simplices σ of X with $x \in \overline{\sigma}$. More generally we define for any subset A of $\overline{X}$ the star $St_X(A)$ of A in X to be the union of all $\sigma \in \Sigma(X)$ with $A \cap \overline{\sigma} \neq \emptyset$. Of course, $St_X(A)$ is the union of the stars $St_X(x)$ of all $x \in A$.

$St_X(A)$ is an open subcomplex of X and $X \setminus St_X(A)$ is the largest subcomplex Y of X with $\overline{Y} \cap A = \emptyset$. Also $St_X(A)$ is the union of the stars $St_X(\tau)$ with τ running through the open simplices of $\overline{X}$ which meet A. In particular, for any open simplex τ of $\overline{X}$, we have $St_X(\tau) = St_X(x)$ for every $x \in \tau$. The complex $St_X(\tau)$ is just the union of all $\sigma \in \Sigma(X)$ with $\tau \leq \sigma$.

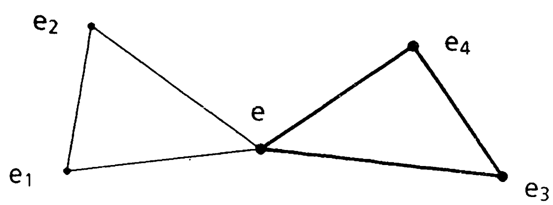
For any $x \in \overline{X}$ and $z \in St_X(x)$ the half-open line segment $]x,z]$ is contained in $St_X(x)$. Thus $St_X(x)$ is indeed a "star-shaped set" with center x, the center perhaps missing.

Suppose points $x \in \overline{X}$ and $z \in St_X(x)$ are given. Let σ denote the open simplex of X which contains z. Assume that $z \neq x$. There is a unique point $w \in \partial\sigma := \bar{\sigma} \setminus \sigma$ with $z \in]x,w[$. Let ρ denote the open simplex of $\bar{X}$ which contains w. Then $\rho < \sigma$. On the other hand, $x \notin \bar{\rho}$, since otherwise z would be contained in ρ, which is not true. Thus $w \notin St_X(x)$.

<u>Definition 2.</u> The <u>bordered star</u> $\overline{St}_X(x)$ of a point $x \in \bar{X}$ with respect to X is defined to be the set of all points $y \in \bar{X}$ with $(1-t)x + ty \in St_X(x)$ for every $t \in]0,1[$. The <u>shell</u> $Sh_X(x)$ of x is defined as the difference set $\overline{St}_X(x) \setminus St_X(x)$. This is the set of all points w arising as above.

$St_X(x) \setminus \{x\}$ is the set of all points $z = (1-t)x + tw$ with $w \in Sh_X(x)$ and $0 < t < 1$. The point w and the parameter t are uniquely determined by z. Notice that $x \in St_X(x)$ if and only if $x \in X$. Notice also that $Sh_X(x)$ is empty if and only if $St_X(x) = \{x\}$.

We give a simple example. Let X be the union of an open 2-simplex σ_1 and a closed 2-simplex $\bar{\sigma}_2$ such that $\bar{\sigma}_1$ and $\bar{\sigma}_2$ have precisely one vertex e in common.



Then $St_X(e) = X \setminus [e_3,e_4]$, and $Sh_X(e) =]e_1,e_2[\cup [e_3,e_4]$.

<u>Lemma 7.1.</u> $Sh_X(x)$ and $\overline{St}_X(x)$ are subcomplexes of $\bar{X}$. More precisely $Sh_X(x)$ is the union of all open simplices ρ of $\bar{X}$ with $x \notin \bar{\rho}$, and $\rho\tau \in \Sigma(X)$, where τ denotes the open simplex of $\bar{X}$ containing x, and $\rho\tau$ denotes the <u>open join</u> of ρ and τ, i.e. the set of all points $(1-t)u + tv$ with $u \in \rho$, $v \in \tau$ and $0 < t < 1$. (cf. also Def. 5 below. Notice that $\rho \cap \tau = \emptyset$. In case $\rho < \tau$ we have $\rho\tau = \tau$.)

The proof is very easy. As a consequence of this lemma we see that the shells $Sh_X(x_1)$ and $Sh_X(x_2)$ are equal for any two points x_1, x_2 in the same open simplex τ of $\overline{X}$; thus we also see that $\overline{St}_X(x_1) = \overline{St}_X(x_2)$. We are therefore justified in also denoting these subcomplexes of $\overline{X}$ by $Sh_X(\tau)$ and $\overline{St}_X(\tau)$ respectively and calling them the shell and the bordered star of τ with respect to X.

Lemma 7.2. The closure of the complex $St_X(x)$ - and also of the complex $\overline{St}_X(x)$ - is $St_{\overline{X}}(x)$. The closure of the complex $Sh_X(x)$ is $Sh_{\overline{X}}(x)$.

The proof is again easy. A star $St_X(x)$ is a finite complex if and only if $Sh_X(x)$ is a finite complex. By Lemma 7.2 we also know, that $St_X(x)$ is finite if and only if $St_{\overline{X}}(x)$ is finite. Notice that this is true for every $x \in X$ if and only if the complex X is locally finite, and for every $x \in \overline{X}$ if and only if X is strictly locally finite.

Definition 3. a) A geometric complex Y is called connected if Y is not the disjoint union of two subcomplexes Y_1, Y_2 which are both open (hence also closed) and non empty in Y.
b) If X is a geometric complex and x a point of X, then the connected component $C(x,X)$ of x in X is the intersection of all subcomplexes Y of X which are open and closed in X and contain x.

Notice that $C(x,X)$ is itself an open and closed subcomplex of X, and that this complex is connected. Thus $C(x,X)$ is the smallest open and closed subcomplex of X which contains x. If two points x_1 and x_2 of X lie in the same open simplex $\tau \in \Sigma(X)$ then $C(x_1,X) = C(x_2,X)$. Thus we also write $C(\tau,X)$ for this set and call $C(\tau,X)$ the connected component of τ in X. The complex X is the disjoint union of all its subcomplexes $C(\tau,X)$.

Definition 4. Two open simplices σ_1 and σ_2 of X are called connectable in X if there exists a finite sequence

$$\rho_o = \sigma_1, \rho_1, \ldots, \rho_{2m} = \sigma_2$$

in $\Sigma(X)$ with

$$\rho_o \leq \rho_1 \geq \rho_2 \leq \cdots \leq \rho_{2m-1} \geq \rho_{2m}.$$

"Connectable in X" is an equivalence relation on the set $\Sigma(X)$. For any $\tau \in \Sigma(X)$ the union of all simplices to which τ is connectable in X is an open, closed, and connected subcomplex of X. Thus this union is the connected component $C(\tau, X)$ of τ.

In the case that the complex X is locally finite, it is evident that the connected components of X as a complex are the same sets as the connected components of X as a locally semialgebraic space.

The equivalence relation needed in §5 to define the vertices of the maximal hull $\hat{X}$ of X is closely related to the relation "connectable in $St_X(x)$" for open simplices of $St_X(x)$, with x running through the vertices of X. Thus we arrive at the following statement.

Proposition 7.3. A geometric complex X is maximal if and only if the star $St_X(x)$ of every vertex x of X is connected.

For a maximal complex X it may still happen that the star of some point in $\overline{X} \setminus X$ which is not a vertex is not connected.

Example 7.4. Let $\overline{X}$ be the boundary $\partial\Delta$ of the standard three-dimensional closed simplex $\Delta = [e_o, e_1, e_2, e_3]$. Let X be the complex $\overline{X} \setminus]e_o, e_1[$. Then the stars $St_X(e_i)$, $0 \leq i \leq 3$, are all connected. Thus X is maximal. But for any $x \in]e_o, e_1[$ the star $St_X(x)$ has two connected components.

We denote the set of connected components of the complex X by $\pi_o(X)$. A simplicial map $f: X \to Y$ induces a map $\pi_o(f): \pi_o(X) \to \pi_o(Y)$ which sends any connected component X' of X to the connected component of Y which contains the (obviously) connected complex f(X'). We often write f_* instead of $\pi_o(f)$.

Proposition 7.5. Let X be any geometric complex and let x be a point in $\overline{X} \setminus X$. Then the inclusion maps $i: St_X(x) \hookrightarrow \overline{St}_X(x)$ and $j: Sh_X(x) \hookrightarrow \overline{St}_X(x)$ yield bijections $\pi_o(i)$ and $\pi_o(j)$. For any connected component A of $Sh_X(x)$ the connected component $B := j_*(A)$ of $\overline{St}_X(x)$ is the set of points $(1-t)x + tz$ with $z \in A$ and $0 < t \leq 1$, and the connected component $D = B \cap St_X(x)$ of $St_X(x)$ with $i_*(D) = B$ is the set of points $(1-t)x + tz$ with $z \in A$ and $0 < t < 1$.

Proof. This is intuitively obvious, since $\overline{St}_X(x)$ is the cone with base $Sh_X(x)$ and vertex x, deprived of its vertex, and since $St_X(x) = \overline{St}_X(x) \setminus Sh_X(x)$. In the case that X is strictly locally finite, we can regard $\overline{St}_X(x)$ as a semialgebraic space. Then the above argument is already a proof of the proposition. In general, let $(X_\alpha | \alpha \in I)$ be the directed system of all finite subcomplexes of X. We choose some index β with $x \in \overline{X}_\beta$. The complexes $St_X(x)$, $\overline{St}_X(x)$, $Sh_X(x)$ are respectively the unions of the directed systems of complexes $(St_{X_\alpha}(x) | \alpha \in I, \alpha \geq \beta)$, $(\overline{St}_{X_\alpha}(x) | \alpha \in I, \alpha \geq \beta)$ and $(Sh_{X_\alpha}(x) | \alpha \in I, \alpha \geq \beta)$ respectively. We know that the proposition is true for x and every $X_\alpha, \alpha \geq \beta$. From this the proposition follows easily for x and X.

If $f: X \to Y$ is a simplicial map between geometric complexes, then, for any $x \in \overline{X}$, we have

$$f(St_X(x)) \subset St_Y(\overline{f}(x)), f(\overline{St}_X(x)) \subset \overline{St}_Y(\overline{f}(x)),$$

as is easily verified. The following facts will be very useful for us in the next section, although it is easy to prove them.

Proposition 7.6. Assume that the simplicial map $f : X \to Y$ is partially finite (cf. §5, Def. 2). Let y be a point of $\overline{Y}$.
a) $f^{-1}(St_Y(y))$ is the union of the stars $St_X(x)$, with x running through $\overline{f}^{-1}(y)$. These stars are pairwise disjoint, open and closed in $f^{-1}(St_Y(y))$. In the cases where $y \in Y$, or X is maximal and y is a vertex of $\overline{Y}$, they are the connected components of the complex $f^{-1}(St_Y(y))$.
b) Restricting $\overline{f}$ to $\overline{St}_X(x)$, for any $x \in \overline{f}^{-1}(y)$, we obtain a partially finite map from $\overline{St}_X(x)$ to $\overline{St}_Y(y)$. The preimages of $St_Y(y)$ and $Sh_Y(y)$ under this map are $St_X(x)$ and $Sh_X(x)$ respectively.

Proof. a) Let σ be any open simplex in $f^{-1}(St_Y(y))$. Then $y \in \overline{f(\sigma)} = \overline{f}(\overline{\sigma})$, and hence $\overline{\sigma}$ contains some point $x \in \overline{f}^{-1}(y)$. This means that $\sigma \subset St_X(x)$, and we see that $f^{-1}(St_Y(y))$ is the union of the stars $St_X(x)$ with $x \in \overline{f}^{-1}(y)$. Suppose that for two points x and z in $\overline{f}^{-1}(y)$ the stars $St_X(x)$ and $St_X(z)$ are not disjoint. Let σ be an open simplex contained in both stars. Then $x \in \overline{\sigma}$, $z \in \overline{\sigma}$, and $\overline{f}(x) = \overline{f}(z) = y$. But $\overline{f}$ is injective on $\overline{\sigma}$ (cf. §5). Thus $x = z$. All the complexes $St_X(x)$, $x \in \overline{f}^{-1}(y)$, are open in $f^{-1}(St_Y(y))$, hence they are also closed in $f^{-1}(St_Y(y))$. If $y \in Y$, then $\overline{f}^{-1}(y) \subset X$, since $X = \overline{f}^{-1}(Y)$ by the partial properness of f. Thus all stars $St_X(x)$ with $x \in \overline{f}^{-1}(y)$ are connected. If y is a vertex of $\overline{Y}$, $y \notin Y$, then all $x \in \overline{f}^{-1}(y)$ are vertices of $\overline{X}$ not contained in X. If in addition X is maximal, then we know from Prop. 7.3 that every star $St_X(x)$ with $x \in \overline{f}^{-1}(y)$ is connected.
b) We fix some $x \in \overline{f}^{-1}(y)$ and consider the simplicial map g from $St_X(x)$ to $St_Y(y)$ obtained from f by restriction. This map is partially finite since $St_X(x)$ is closed in $f^{-1}(St_Y(y))$. The closure $\overline{g}$ of g is a partially finite map from $\overline{St}_{\overline{X}}(x)$ to $\overline{St}_{\overline{Y}}(y)$. Since g is partially proper, we know that $St_X(x)$ is the preimage of $St_Y(y)$ under $\overline{g}$. We shall verify

that $\overline{St}_X(x)$ is the preimage of $\overline{St}_Y(y)$ under $\bar{g}$. Then assertion b) of the proposition will be evident. For $z \in \overline{St}_X(x)$ we have $]z,x[\subset St_X(x)$, hence $]\bar{f}(z),y[\subset St_Y(y)$, which means that $\bar{f}(z) \in \overline{St}_Y(y)$. Conversely, if some $z \in \overline{St}_{\bar{X}}(x)$ with $\bar{f}(z) \in \overline{St}_Y(y)$ is given, then $]z,x[\subset St_{\bar{X}}(x)$ and the image of $]z,x[$ under $\bar{g}$ is contained in $St_Y(y)$. Thus $]z,x[\subset St_X(x)$, which means that $z \in \overline{St}_X(x)$. q.e.d.

<u>Corollary 7.7.</u> Let $f : X \to Y$ be a dominant simplicial map (cf. §5, Def. 3). Let $y \in \bar{Y}$.

i) By restriction, $\bar{f}$ yields, for every $x \in \bar{f}^{-1}(y)$, dominant maps from the complexes $St_X(x)$ $(\overline{St}_X(x), Sh_X(x))$ onto closed subcomplexes of $St_Y(y)$ $(\overline{St}_Y(y), Sh_Y(y))$.

ii) For different points $x_1, x_2 \in \bar{f}^{-1}(y)$ the complexes $\bar{f}(\overline{St}_X(x_1))$ and $\bar{f}(\overline{St}_X(x_2))$ are disjoint. Thus, for any $x \in \bar{f}^{-1}(y)$ the complexes $\bar{f}(\overline{St}_X(x))$ $(f(St_X(x)), \bar{f}(Sh_X(x)))$ are also open in $\overline{St}_Y(y)$ $(St_Y(y), Sh_Y(y))$.

iii) In the case that $y \in Y$ and x is the unique point in $f^{-1}(y)$ the complexes $St_X(x)$ $(\overline{St}_X(x), Sh_X(x))$ are mapped by $\bar{f}$ dominantly onto $St_Y(y)$, $(\overline{St}_Y(y), Sh_Y(y))$.

iv) If X is maximal and y is a vertex of $\bar{Y}$, then the complexes $f(St_X(x))$ $(\bar{f}(\overline{St}_X(x)), \bar{f}(Sh_X(x)))$ with x running through $\bar{f}^{-1}(y)$ are the connected components of the complexes $St_Y(y)$, $(\overline{St}_Y(y), Sh_Y(y))$.

<u>Proof.</u> Assertion i) is evident from the preceding proposition as soon as we know that $\bar{f}$ is injective on $\overline{St}_X(x)$. Let z_1 and z_2 be points in $\overline{St}_X(x)$ with $\bar{f}(z_1) = \bar{f}(z_2) = w$. For any $t \in]0,1[$ the two points $(1-t)z_i + tx$ $(i = 1,2)$ have the same image $(1-t)w + tx$. Since f is injective on the set X, these two points are equal. We conclude that $z_1 = z_2$. Assertion ii) can be proved in the same way, since we already know that $St_X(x_1)$ and $St_X(x_2)$ are disjoint, hence that $f(St_X(x_1))$ and $f(St_X(x_2))$

are disjoint. To prove assertion iii) observe that $\overline{St}_Y(y)$ is connected, hence $\overline{f}(\overline{St}_X(x)) = \overline{St}_Y(y)$ for the unique point x in $f^{-1}(y)$. Similarly we obtain assertion iv) from the fact that the complexes $St_X(x)$, $\overline{St}_X(x)$, $Sh_X(x)$ are connected by Propositions 7.3 and 7.5. q.e.d.

<u>A final remark.</u> The shell $Sh_X(\tau)$ of an open simplex $\tau \in \Sigma(\overline{X})$ (= $Sh_X(x)$ for any $x \in \tau$) is a somewhat unusual concept from the view point of classical combinatorial topology, even if the complex X is closed. Topologists prefer to work with the <u>link</u> $Lk_X(\tau)$ of τ instead of the shell (and, of course, they assume X to be closed). The right generalization of the classical notion of link for an arbitrary simplicial complex X and an arbitrary open simplex of $\overline{X}$ seems to be provided by the following definition.

<u>Definition 5.</u> a) Two open simplices $\tau =]u_o,\ldots,u_r[$ and $\rho =]v_o,\ldots,v_s[$ in a vector space over R are <u>joinable</u> if the vertices $u_o,\ldots,u_r,\ v_o,\ldots,v_s$ are affinely independent. In this case the <u>open join</u> $\tau\rho$ of τ and ρ is defined as the simplex $]u_o,\ldots,u_r,\ v_o,\ldots,v_s[$.
b) For any $\tau \in \Sigma(\overline{X})$ the <u>link</u> $Lk_X(\tau)$ <u>of</u> τ <u>in</u> X is the subcomplex of $\overline{X}$ whose open simplices are the $\rho \in \Sigma(\overline{X})$ such that τ and ρ are joinable and $\tau\rho \in \Sigma(X)$.

As is evident from Lemma 7.1,

$$Sh_X(\tau) = \begin{cases} Lk_X(\tau) * \partial\tau & \text{if } \tau \subset X, \\ [Lk_X(\tau) * \partial\tau] \setminus \partial\tau & \text{if } \tau \subset \overline{X} \setminus X, \end{cases}$$

with $Lk_X(\tau) * \partial\tau$ denoting the join of $Lk_X(\tau)$ and $\partial\tau$, i.e. the subcomplex of $\overline{X}$ which is the union of $Lk_X(\tau)$, $\partial\tau$, and all open joins $\rho\omega$ with ρ running through $\Sigma(Lk_X(\tau))$ and ω running through $\Sigma(\partial\tau)$. Thus $Sh_X(\tau)$ is determined by $Lk_X(\tau)$ and τ in both cases.

§8 - Pure hulls of dense pairs

As before a "space" is a regular paracompact locally semialgebraic space and a "complex" is a geometric simplicial complex.

Definition 1. a) A dense pair over R is a pair (P,M) consisting of a space P and a dense subspace M. A morphism $f:(P,M) \to (Q,N)$ between dense pairs is a locally semialgebraic map $f: P \to Q$ with $f(M) \subset N$. Notice that such a morphism f is uniquely determined by its restriction $f|M: M \to N$.

b) A completed space is a dense pair (P,M) with P partially complete.

In order to gain a better understandig of the various possibilities for completing a given space, we will be chiefly interested in the full subcategory of the category of dense pairs consisting of completed spaces. But in the following it is often more natural to work with arbitrary dense pairs instead of completed spaces.

Every strictly locally finite complex X may be considered as a completed space $(\overline{X},X)$, and every simplicial map $f: X \to Y$ between such complexes may be considered as a morphism $\overline{f}: (\overline{X},X) \to (\overline{Y},Y)$.

In the category of dense pairs over R there exist fibre products. Namely, if $f_i: (P_i,M_i) \to (Q,N)$ are morphisms $(i = 1,2)$, then define $M := M_1 \times_N M_2$, the fibre product with respect to the maps $f_i|M_i$, and define P as the closure of M in $P_1 \times_Q P_2$, the fibre product with respect to the maps f_i. We have natural projection maps $p_i: P \to P_i$ $(i = 1,2)$, which can be regarded as morphisms $p_i: (P,M) \to (P_i,M_i)$. The diagram

$$\begin{tikzcd}
(P,M) \arrow[r, "p_2"'] \arrow[d, "p_1"] & (P_2,M_2) \arrow[d, "f_2"] \\
(P_1,M_1) \arrow[r, "f_1"'] & (Q,N)
\end{tikzcd}$$

is a pull-back in the category of dense pairs. If P_1 and P_2 are partially complete, then P is also partially complete.

<u>Definition 2.</u> A morphism $f : (P,M) \to (Q,N)$ of dense pairs is <u>semialgebraic</u>, if the map $f : P \to Q$ is semialgebraic (hence $f|M : M \to N$ is also semialgebraic).

The following lemma will prove to be very useful.

<u>Lemma 8.1.</u> Let $f : (P,M) \to (Q,N)$ be a morphism of dense pairs such that $f|M : M \to N$ is semialgebraic. Then f is semialgebraic.

<u>Proof.</u> Let $U \in \mathring{\mathfrak{T}}(Q)$ be given. The set $(f|M)^{-1}(U \cap N) = f^{-1}(U \cap N) \cap M$ is semialgebraic. Consider a non-empty open subset V of $f^{-1}(U)$. Then $V \cap M \subset f^{-1}(U \cap N) \cap M$ and $V \cap M$ is not empty. We see that $f^{-1}(U \cap N) \cap M$ is dense in $f^{-1}(U)$. Therefore $f^{-1}(U)$ is also semialgebraic (cf. I, Prop. 4.6).

<u>Definition 3.</u> Let $f : (P,M) \to (Q,N)$ be a morphism between dense pairs. f is called <u>partially proper</u> (resp. <u>proper</u>), if the locally semialgebraic maps $f : P \to Q$ and $M \to N$ both are partially proper (resp. proper).

Notice that this forces $M = f^{-1}(N)$. If P is partially complete then $f : P \to Q$ is always partially proper. Thus, in this case, the morphism f is partially proper if and only if $f^{-1}(N) = M$.

<u>Definition 4.</u> Let $f : (P,M) \to (Q,N)$ be a morphism between dense pairs.
a) f is called <u>partially finite</u> (resp. <u>finite</u>) if the maps $f : P \to Q$ and

$f|M : M \to N$ both are partially finite (resp. finite).

b) f is called dominant if the map $f : P \to Q$ is partially proper and the map $f|M : M \to N$ is an isomorphism.

Remarks 8.2. a) If f is dominant, then we know from Lemma 8.1 that the map $f : P \to Q$ is semialgebraic. Thus the map $f : P \to Q$ is proper (cf. I, 6.13), and we conclude that the morphism f is proper. For the same reason we know that, if the map $f : P \to Q$ is partially finite and the map $f|M : M \to N$ is finite, then the morphism f is already finite.

b) All the properties of morphisms described in Definitions 2-4 are stable under composition and pull-backs. Also, if $f : (P,M) \to (Q,N)$ and $g : (Q,N) \to (S,L)$ are morphisms of dense pairs and if $g \circ f$ has one of these properties then f has the same property (cf. I, 5.5 for "proper" and I.6.2 for "partially proper"). Notice also that, if f is proper, then $f(P) = Q$ if and only if $f(M) = N$. If $f(P) = Q$ and $g \circ f$ is proper (resp. finite, resp. dominant) then g is proper (resp. finite, resp. dominant), cf. I.5.5. If Q is partially complete, and $f(P) = Q$, and $g \circ f$ is partially proper (resp. partially finite), then g is partially proper (resp. partially finite), since we easily check that $g^{-1}(L) = N$.

c) Any simplicial map $f : X \to Y$ between strictly locally finite complexes yields a morphism $\overline{f} : (\overline{X},X) \to (\overline{Y},Y)$ of completed spaces. The simplicial map f is partially proper (resp. partially finite, proper, finite, dominant) in the sense of §5 if and only if the morphism $\overline{f}$ of completed spaces is partially proper (resp. partially finite, proper, finite, dominant). In particular, if the morphism $\overline{f}$ is dominant, then it is also finite.

Definitions 5. a) A triangulation of a dense pair (P,M) is an isomorphism $\varphi : (Z,X) \xrightarrow{\sim} (P,M)$ of dense pairs with Z a strictly locally finite complex and X a subcomplex of Z. In other words, a triangulation of (P,M) is a simultaneous triangulation of P and M (cf. §4, Def. 1).

Notice that, if P is partially complete, we must have $Z = \overline{X}$ (cf. I, 6.10).

b) A <u>triangulation of a morphism</u> $f : (P,M) \to (Q,N)$ of dense pairs is a commutative square

$$\begin{tikzcd}
(Z,X) \arrow[r, "\sim", "\varphi"'] \arrow[d, "g"'] & (P,M) \arrow[d, "f"] \\
(T,Y) \arrow[r, "\sim", "\psi"'] & (Q,N)
\end{tikzcd}$$

with φ and ψ triangulations of (P,M) and (Q,N) and $g : Z \to T$ a simplicial map. (Of course, $g(X) \subset Y$.)

By §4 every dense pair (P,M) can be triangulated. By §6 every finite morphism $f : (P,M) \to (Q,N)$ can be triangulated (Th. 6.13). A dominant morphism can be triangulated if and only if it is finite (cf. Remark 8.2.c).

We want to study finite dominant morphisms along lines similar to those we used when studying simplicial dominant maps between complexes in §5. In fact the results of §5, as well as those of §6 and §7, will be quite useful for this.

<u>Definition 6</u>. A dense pair (Q,N) is called <u>pure</u> if every finite dominant morphism $(S,L) \to (Q,N)$ is an isomorphism. A <u>pure hull</u> of a dense pair (P,M) is a finite dominant morphism $(Q,N) \to (P,M)$ with (Q,N) pure.

The following analogue of Proposition 5.5 on maximal complexes can be proved by the same formal argument.

<u>Proposition 8.3.</u> Let $f : (Q,N) \to (P,M)$ and $g : (S,L) \to (P,M)$ be morphisms between dense pairs. Assume that (Q,N) is pure and g is finite and dominant. Then there exists a unique morphism $h : (Q,N) \to (S,L)$ with $g \circ h = f$. If f is finite and dominant, i.e. f is a pure hull of (P,M),

then h is also finite and dominant. In particular, any two pure hulls of (P,M) are isomorphic.

Our goal in this section is to prove the existence of pure hulls. This needs some preparation.

Definition 7. Let (P,M) be a dense pair and x a point of P. Then the set $\pi_0(M)_x$ is defined as the projective limit of the sets $\pi_0(U \cap M)$ of connected components of $U \cap M$, with U running through the directed system of open semialgebraic neighbourhoods of x in P. Of course, $\pi_0(M)_x$ is a one point set for $x \in M$.

Proposition 8.4. Let (P,M) be a dense pair with $|\pi_0(M)_x| = 1$ *) for every $x \in P$. If $f : (Q,N) \to (P,M)$ is a dominant morphism then all fibres of the proper map $f : Q \to P$ are connected. In particular, the pair (P,M) is pure.

This follows from Lemma 6.5 and the next evident Lemma 8.5 by use of a triangulation of (P,M).

Lemma 8.5. If X is a strictly locally finite complex, then the natural map $\pi_0(X)_x \to \pi_0(\mathrm{St}_X(x))$ is bijective for every $x \in \overline{X}$. More precisely, the sets $U_\lambda := \{(1-\lambda)x + \lambda z \mid z \in \mathrm{St}_{\overline{X}}(x)\}$, with λ running through $]0,1[$, are a fundamental system of open semialgebraic neighbourhoods of x in $\overline{X}$, and, for every $\lambda \in]0,1[$, the map $z \mapsto (1-\lambda)x + \lambda z$ is an isomorphism from $\mathrm{St}_X(x)$ onto $U_\lambda \cap X$.

Theorem 8.6. Let X_1 be a strictly locally finite complex. Let $X = X_1'$ be the (first) barycentric subdivision of X_1, and let $f : Y \to X$ be the maximal hull of the complex X (cf. §5). Then $|\pi_0(Y)_y| = 1$ for every $y \in \overline{Y}$. Thus the completed space $(\overline{Y},Y)$ is pure (cf. Prop. 8.4).

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*) For any set E we denote by $|E|$ the cardinality of E.

Proof. We fix some $x \in \overline{X} \setminus X$. We want to prove that $St_Y(y)$ is connected for every $y \in \overline{f}^{-1}(x)$. Then we are done by Lemma 8.5 above. Let τ denote the open simplex of $\overline{X}_1$ which contains x, and let T denote the open simplex of $\overline{X}$ which contains x. The barycenter u of τ is a vertex of T, hence $T \subset St_{\overline{X}}(u)$. Let $v_1,\ldots,v_r$ be the finitely many elements of $\overline{f}^{-1}(u)$. Since the complex Y is maximal, the stars $St_Y(v_i)$, $1 \le i \le r$, are connected (Prop. 7.3). The sets $V_i := f(St_Y(v_i))$, $1 \le i \le r$, are the connected components of $St_X(u)$, and f maps every star $St_Y(v_i)$ isomorphically onto V_i (Cor. 7.7). The inclusions $St_X(x) \hookrightarrow St_{X_1}(x)$ and $St_X(u) \hookrightarrow St_{X_1}(u)$ induce bijections between the sets of connected components of these stars (cf. Lemma 8.5). Now $St_X(x) \subset St_X(u)$ and $St_{X_1}(x) = St_{X_1}(u)$ and we conclude that the inclusion $St_X(x) \hookrightarrow St_X(u)$ induces a bijection between the sets of connected components. This means that the sets $V_i \cap St_X(x)$, $1 \le i \le r$, are the connected components of $St_X(x)$. In particular, we see that x lies in the closure $\overline{V}_i \cap St_{\overline{X}}(u)$ of V_i in $St_{\overline{X}}(u)$. The preimage $\overline{f}^{-1}(St_{\overline{X}}(u))$ of $St_{\overline{X}}(u)$ has as connected components $St_{\overline{Y}}(v_i)$, $1 \le i \le r$, and $\overline{f}$ is a finite map from this preimage to $St_{\overline{X}}(u)$. It maps every component $St_{\overline{Y}}(v_i)$ isomorphically onto the closure $\overline{V}_i \cap St_{\overline{X}}(u)$ of $V_i = f(St_Y(v_i))$ in $St_{\overline{X}}(u)$. Thus x has precisely r preimages $y_1,\ldots,y_r$ under $\overline{f}$ which can be numbered such that $y_i \in St_{\overline{Y}}(v_i)$. The map f sends

$$St_Y(y_i) = f^{-1}(St_X(x)) \cap St_{\overline{Y}}(v_i)$$

isomorphically onto its image, which is

$$St_X(x) \cap \overline{V}_i \cap St_{\overline{X}}(u) = St_X(x) \cap \overline{V}_i \cap St_X(u) = St_X(x) \cap V_i,$$

a connected set. Thus every star $St_Y(y_i)$, $1 \le i \le r$, is indeed connected.

q.e.d.

Theorem 8.7. Every dense pair (P,M) has a pure hull.

Proof. We first consider the case that P is partially complete. We choose a triangulation $\varphi: (\overline{X}_1, X_1) \xrightarrow{\sim} (P,M)$ of (P,M) (cf. §4). Let $X = X_1'$ be the barycentric subdivision of X_1 and $f: Y \to X$ its maximal hull. By

Theorem 8.6 the morphism $\bar{f} : (\bar{Y},Y) \to (\bar{X},X)$ is a pure hull of $(\bar{X},X)$. Thus $\varphi \circ \bar{f} : (\bar{Y},Y) \to (P,M)$ is a pure hull of (P,M). Also, by Theorem 8.6, $|\pi_0(Y)_y| = 1$ for every $y \in \bar{Y}$. Since any two pure hulls of (P,M) are isomorphic, we know the analogous fact for any other pure hull of (P,M).

In the general case we embed P as a dense subspace into some partially complete space $\bar{P}$ (cf. §2). The pair $(\bar{P},M)$ has a pure hull $g : (T,N) \to (\bar{P},M)$ and $|\pi_0(N)_z| = 1$ for every $z \in T$. Let Q be the preimage of P under g, and let $f : (Q,N) \to (P,M)$ be the morphism obtained from g by restriction. Clearly f is finite and dominant. Consider a point $x \in Q$. Then $U \cap Q$ runs through a fundamental system of neighbourhoods of x in Q as U runs through a fundamental system of neighbourhoods of x in T. Thus $\pi_0^Q(N)_x = \pi_0^T(N)_x$, where $\pi_0^Q(N)_x$ (resp. $\pi_0^T(N)_x$) is the set $\pi_0(N)_x$ defined with respect to Q (resp. T). We know that $|\pi_0^T(N)_x| = 1$ and conclude from Proposition 8.4 that f is a pure hull of (P,M). q.e.d.

In the course of the proof we have constructed a pure hull $f : (Q,N) \to (P,M)$ with $|\pi_0(N)_x| = 1$ for every $x \in Q$. Thus, recalling Proposition 8.4, we obtain the following corollary to Theorem 8.7.

<u>Corollary 8.8.</u> A dense pair (P,M) is pure if and only if $|\pi_0(M)_x| = 1$ for every $x \in P$.

Theorem 8.7 and its corollary 8.8 have an analogue in the classical theory of cuts in topology (cf. [M], [MV]). This will be apparent after a slight change of terminology.

<u>Definition 8.</u> a) A locally semialgebraic subset A of a space M is called <u>thin in</u> M if its interior is empty. If A is such a subset, then we say that A <u>does not cut</u> the space M <u>at a point</u> $x \in A$ if $|\pi_0(M \setminus A)_x| = 1$, i.e., if x has a fundamental system of neighbourhoods U in M with $U \setminus A$

connected.

b) Let A be a thin subset of M. A <u>cut of</u> M <u>along</u> A is a finite morphism $f : N \to M$ such that f maps $N \setminus f^{-1}(A)$ isomorphically onto $M \setminus A$ and $B := f^{-1}(A)$ is a thin subset of N which does not cut N at any point $y \in B$.

Our main results can be expressed in the following terms (cf. 8.3, 8.7, 8.8).

<u>Theorem 8.9.</u> Let A be a thin locally semialgebraic subset of a space M. Then there exists a cut $f : N \to M$ of M along A. If $g : L \to M$ is a locally semialgebraic map and B is a thin subset of L such that $g^{-1}(A) \subset B$ and B nowhere cuts L, then there exists a unique locally semialgebraic map $h : L \to N$ with $f \circ h = g$. In particular, if g is also a cut of M along A, then h is an isomorphism.

In the topological theory of cuts the "Stein factorisation" [MV] (= "monotone-light factorisation" in [M]) of a proper continuous map plays an important role. We shall meet the semialgebraic analogue of this in §12.

For later use we state a converse to Proposition 8.4.

<u>Proposition 8.10.</u> Let $f : (Q,N) \to (P,M)$ be a proper morphism between dense pairs. Assume that (Q,N) is pure. Assume further that the map $f : Q \to P$ is surjective and has connected fibres. Then (P,M) is pure.

<u>Proof.</u> Let $\pi : (\tilde{P},\tilde{M}) \to (P,M)$ be a pure hull of (P,M). By Proposition 8.3 we have a unique morphism $\tilde{f}$ from (Q,N) to $(\tilde{P},\tilde{M})$ with $\pi \circ \tilde{f} = f$. Now $\tilde{f}$ is proper, hence maps Q onto $\tilde{P}$. Since f has connected fibres it follows that π has connected fibres. This means that π is an isomorphism, i.e.

(P,M) is pure.

Example 8.11. Even if the morphism $f : (Q,N) \to (P,M)$ is dominant and has connected fibres, purity of (P,M) does not necessarily imply purity of (Q,N). For example, let Q be the closed unit disk in R^2 and let A be the closed thin subset $[-\frac{1}{2},\frac{1}{2}]\times\{0\}$ of Q. As we shall see in §10 (Prop. 10.4), there exists a proper semialgebraic map $f : Q \to P$ which collapses A into one point p and is an isomorphism from $N := Q\setminus A$ onto $M := P\setminus\{p\}$. The morphism $f : (Q,N) \to (P,M)$ is dominant and has connected fibres. (P,M) is pure, but (Q,N) is not pure.

We close this section with a very natural example of a dominant morphism between dense pairs.

Example 8.12 (Semialgebraic blowing up). Let (P,M) be a dense pair with $A:=P\setminus M$ closed in P. We choose $r\geq 2$ locally semialgebraic functions $f_1,\dots,f_r$ on P such that A is the set of common zeros of these functions. {Actually A can be written as the zero set of a single locally semialgebraic function [I, Th. 4.15], but for $r = 1$ the considerations to follow are trivial.} As usual, let $\mathbb{P}^n(R)$ denote the complete space of real points of the n-dimensional projective space $\mathbb{P}^n_R$ over R. Let Z be the closed locally semialgebraic subset of $P\times\mathbb{P}^{r-1}(R)$ consisting of all points $(x,y_1:\dots:y_r)$ with

$$f_i(x)y_j - f_j(x)y_i = 0$$

for $1\leq i\leq j\leq r$. The projection p from $P\times\mathbb{P}^{r-1}(R)$ to the first factor is a proper semialgebraic map. By restriction p yields an isomorphism from $N:=p^{-1}(M)\cap N$ to M, since for every $x\in M$ at least one of the functions f_i does not vanish. Let Q denote the closure of N in Z. Then the restriction $\pi : Q\to P$ is also a proper semialgebraic map, and π can be regarded as a dominant proper morphism from the dense pair (Q,N) to (P,M).

We know almost nothing about semialgebraic blowing up. The dimension of the fibre $\pi^{-1}(x)$ of a point $x \in A$ is a measure how much the functions $f_1,\dots,f_r$ are "independent" near x. Thus semialgebraic blowing up seems to be an interesting geometric device to study finite systems of locally semialgebraic functions.

<u>Question.</u> If (P,M) is pure, under which conditions on $(f_1,\dots,f_r)$ is (Q,N) again pure?

§9 - Ends of spaces, the LC-stratification

In this section we develop a theory of "ends" for regular paracompact spaces. It might be advisable for the reader to pursue the literature on the classical theory of ends in topology, e.g. Freudenthal's original papers [Fr], [Fr_1] or §1 of Hopf's paper [Ho]. This will give him a good feeling for the different "flavour" of the semialgebraic theory as opposed to the classical topological theory. In some sense our theory is much easier than the classical one, since from the view point of general topology our spaces and maps are rather special. So a lot of pathologies cannot occur. However, in other respects our theory is more complicated than the classical one. We shall meet new phenomena, in particular the "complexity" of a space (cf. Definitions 7 and 8 below), which are unfamiliar to topologists. The reason is that we do not restrict our attention to locally complete spaces. Thus the ends of a space M in some completion of M (cf. Def. 3 below) are not necessarily complete.

If M is semialgebraic and locally complete, then the ends of M can be regarded as points which are added to M to make M a complete space (cf. Th. 9.2 below). In this special case our theory of ends is more or less an adaption of Freudenthal's theory to the semialgebraic setting. But already if M is locally complete and not semialgebraic, our theory of ends is definitely distinct from the topological theory. For example, if N is a locally complete semialgebraic space, then the space N_{loc} has no ends at all in our theory, since N_{loc} is partially complete. But the strong topologies on N_{loc} and N are the same. So, in the case $R = \mathbb{R}$, N and N_{loc} have the same ends in the topological sense, and there may be many of these. To give yet another example, let X be a finite closed connected geometric simplicial complex over $\mathbb{R}$, and let $p : \tilde{X} \twoheadrightarrow X$ be the universal covering of X, cf. Chapter V. Then $\tilde{X}$ is partially complete, and thus has no ends in our theory. On the other hand, the number of topological ends of $\tilde{X}$ is a very interesting invariant $e(G)$ of the fundamental group G of X ($e(G) = 0,1,2$ or ∞, cf. [Ho], [SW, §5]).

Our theory of ends needs some preparation: As before a "space" is a regular paracompact locally semialgebraic space over a fixed real closed field R, and a "complex" is a geometric simplicial complex over R. We repeat a definition from §1.

Definition 1. A completion of a space M is a dense embedding $\varphi : M \hookrightarrow P$ of M in a partially complete space, i.e. an isomorphism from M onto a dense subspace $\varphi(M) \in \mathcal{T}(P)$ of P.

In this section it is sometimes more convenient to work with completions instead of completed spaces, as we did in the last section. Of course, these are equivalent notions. If $\varphi : M \hookrightarrow P$ is a completion then $(P,\varphi(M))$ is a completed space, and if $(\bar{M},M)$ is a completed space then the inclusion $M \hookrightarrow \bar{M}$ is a completion.

We are mainly interested in the completions of a fixed space M. In this context a morphism from a completion $\varphi : M \hookrightarrow P$ to a completion $\psi : M \hookrightarrow Q$ is a locally semialgebraic map $f : P \to Q$ with $f \circ \varphi = \psi$. We denote such a morphism by $f : P \xrightarrow[M]{} Q$. The locally semialgebraic map f is automatically proper, cf. Remark 8.2a, hence also surjective. The preimage of $Q \setminus \psi(M)$ is $P \setminus \varphi(M)$.

There exists at most one morphism from φ to ψ. Thus the set $\mathcal{D}(M)$ of isomorphism classes $[\varphi]$ of completions φ of M has a natural partial ordering: $[\varphi] \leq [\psi]$ if and only if there exists a morphism from ψ to φ.

If $\varphi_1 : M \hookrightarrow P_1$ and $\varphi_2 : M \hookrightarrow P_2$ are two completions of M, then there exists a smallest element $[\varphi]$ in $\mathcal{D}(M)$ with $[\varphi] \geq [\varphi_1]$, $[\varphi] \geq [\varphi_2]$. The completion $\varphi : M \hookrightarrow P$ is obtained as follows. Take the embedding $(\varphi_1, \varphi_2) : M \hookrightarrow P_1 \times P_2$. Let P be the closure of the image of M in $P_1 \times P_2$. Then φ is the restriction $M \hookrightarrow P$ of (φ_1, φ_2).

If M is already partially complete, then $\mathcal{D}(M)$ contains only one element, cf. I, Cor. 6.10. If M is semialgebraic and locally complete - but not complete - then $\mathcal{D}(M)$ has a smallest element, namely the one point completion $M \hookrightarrow M^+$, cf. I, Prop. 7.6.

In general, the set $\mathcal{D}(M)$ has neither a maximal nor a minimal element. This is a major difficulty in understanding the various completions of M. It seems to be wise to focus attention to the "pure completions" of M.

Definition 2. A morphism $f : Q \xrightarrow[M]{} P$ from a completion $\psi : M \hookrightarrow Q$ to a completion $\varphi : M \hookrightarrow P$ of M is called finite, if the fibres of the map $f : Q \to P$ are discrete, hence finite, i.e. if f is a finite semialgebraic map. A completion $\varphi : M \hookrightarrow P$ is called pure if every finite morphism from a completion of M to φ is an isomorphism. A pure hull of a completion $\varphi : M \hookrightarrow P$ is a finite morphism $Q \xrightarrow[M]{} P$ from a pure completion $\psi : M \hookrightarrow Q$ to φ.

These definitions are adaptions of some notions from §8 to the present needs. For example, a completion $\varphi : M \hookrightarrow P$ is pure if and only if the completed space $(P, \varphi(M))$ is pure. We know from §8 that every completion $\varphi : M \hookrightarrow P$ admits a pure hull (Th. 8.7). Any two pure hulls are isomorphic (Prop. 8.3). So, we often talk of "the" pure hull of φ. More generally we have the following proposition, which is a restatement of Proposition 8.3 in a special case.

Proposition 9.1. Let $f : Q \xrightarrow[M]{} P$ be a morphism from a pure completion

$\psi : M \hookrightarrow Q$ to a completion $\varphi : M \hookrightarrow P$. Further, let $g : T \xrightarrow[M]{} P$ be a finite morphism from a completion $\chi : M \hookrightarrow T$ to φ. Then there exists a unique morphism $h : Q \xrightarrow[M]{} T$ from ψ to χ with $g \circ h = f$.

Definition 3. If $\varphi : M \hookrightarrow P$ is a completion of the space M, then a connected component of the space $P \setminus \varphi(M)$ is called an end of M in P (with respect to φ). The set $\pi_0(P \setminus \varphi(M))$ of ends of M in P will usually be denoted by $\varepsilon(M,\varphi)$.

If M is semialgebraic, then P is semialgebraic, and $\varepsilon(M,\varphi)$ is a finite set. In general, $\varepsilon(M,\varphi)$ may be infinite, but is countable as long as M has only countably many connected components, cf. I, Th. 4.17.

If M is semialgebraic and locally complete then M has a pure completion such that the ends in this completion are points. More precisely,

Theorem 9.2. Let M be semialgebraic and locally complete. There exists - up to isomorphism - a unique pure completion $\varphi : M \hookrightarrow P$ such that the ends of M in P are one point sets. For any pure completion $\psi : M \hookrightarrow Q$ there exists a (unique) morphism $f : Q \xrightarrow[M]{} P$ from ψ to φ.

Proof. If M is already complete, there is nothing to prove. Assume now that M is not complete. Let $\varphi_0 : M \hookrightarrow M^+$ be the one point completion of M (I, §7). Let $\varphi : M \hookrightarrow P$, $p : P \xrightarrow[M]{} M^+$, be a pure hull of φ_0. The ends of M in P are finitely many points, namely the points of the fibre $p^{-1}(\infty)$. If $\psi : M \hookrightarrow Q$ is any completion of M then, by I, Prop. 7.6, we have a unique morphism $f_0 : Q \xrightarrow[M]{} M^+$ from ψ to φ_0. If ψ is pure then, by Prop. 9.1, we have a unique morphism $f : Q \xrightarrow[M]{} P$ from ψ to φ with $p \circ f = f_0$. By Proposition 8.4 the fibres of f are connected. If the ends of M in Q are also one point sets, then f must be bijective, hence an isomorphism (since f is proper). q.e.d.

We return to an arbitrary regular paracompact space M. Every morphism $f : Q \xrightarrow[M]{} P$ from a completion $\psi : M \hookrightarrow Q$ to a completion $\varphi : M \hookrightarrow P$ induces a surjection from the set $\varepsilon(M,\psi) = \pi_0(Q \setminus \psi(M))$ to the set $\varepsilon(M,\varphi) = \pi_0(P \setminus \varphi(M))$. We denote this surjection by $\kappa(\psi,\varphi)$. If B is an end of M in Q, then $\kappa(\psi,\varphi)(B)$ is the end A of M in P with $A \supset f(B)$.

<u>Proposition 9.3.</u> Assume that the completion φ is pure. Then $\kappa(\psi,\varphi)$ is bijective. If B is an end of M in Q, and A is the end $\kappa(\psi,\varphi)(B)$ of M in P, then $B = f^{-1}(A)$ and $A = f(B)$.

<u>Proof.</u> f gives by restriction a proper surjective map from $Q \setminus \psi(M)$ to $P \setminus \varphi(M)$. By Proposition 8.4 the fibres of this map are connected. Thus the connected components of $Q \setminus \psi(M)$ are the preimages of the connected components of $P \setminus \varphi(M)$, cf. Sublemma 6.6.

If $\varphi_1 : M \hookrightarrow P_1$ and $\varphi_2 : M \hookrightarrow P_2$ are two completions and if φ_1 is pure, then we obtain a surjection

$$\kappa(\varphi_1,\varphi_2) \; : \; \varepsilon(M,\varphi_1) \twoheadrightarrow \varepsilon(M,\varphi_2)$$

as follows. We choose a completion $\psi : M \hookrightarrow Q$ which admits morphisms $f_i : Q \xrightarrow[M]{} P_i$ $(i = 1,2)$ to both φ_1 and φ_2. Then $\kappa(\psi,\varphi_1)$ is a bijection. We define

$$\kappa(\varphi_1,\varphi_2) \; := \; \kappa(\psi,\varphi_2)\kappa(\psi,\varphi_1)^{-1} .$$

The following facts are now easily verified. (To prove the first statement, compare ψ with the "smallest" completion φ of M which admits morphisms to φ_1 and φ_2, as explained above.)

<u>Proposition 9.4.</u> The surjection $\kappa(\varphi_1,\varphi_2)$ does not depend on the choice of ψ. If φ_2 is also pure, then $\kappa(\varphi_1,\varphi_2)$ is bijective, the inverse map being $\kappa(\varphi_2,\varphi_1)$. If φ_3 is a third completion, and both φ_1 and φ_2 are pure, then

$$\kappa(\varphi_1,\varphi_3) = \kappa(\varphi_2,\varphi_3)\ \kappa(\varphi_1,\varphi_2).$$

For a description of the map $\kappa(\varphi_1,\varphi_2)$ which does not use an auxiliary completion ψ, see Cor. 9.12 below.

We now develop an "absolute" notion of ends which does not refer to any completion. Let $\mathcal{T}_c(M)$ be the set of all partially complete locally semi-algebraic subsets A of M. Every $K \in \mathcal{T}_c(M)$ is a closed subset of M (cf. I, Cor. 6.10). For any two elements K_1 and K_2 of $\mathcal{T}_c(M)$ the union $K_1 \cup K_2$ is again an element of $\mathcal{T}_c(M)$. In particular, the set $\mathcal{T}_c(M)$, ordered by inclusion, is a directed system of subsets of M.

<u>Definition 4</u> ("absolute" ends). An <u>end</u> λ of a space M is an assignment $K \mapsto (M\setminus K)_\lambda$ of a connected component $(M\setminus K)_\lambda$ of $M\setminus K$ to every $K \in \mathcal{T}_c(M)$ such that, whenever $K\subset L$, $(M\setminus K)_\lambda$ contains $(M\setminus L)_\lambda$. In other words, an end is an element of the set

$$\varepsilon(M) := \varprojlim_{K\in\mathcal{T}_c(M)} \pi_0(M\setminus K).$$

<u>Remark.</u> It is not difficult to verify that the set of ends of a locally compact and locally connected topological space M in the sense of Freudenthal [Fr] is the projective limit of the sets $\pi_0(M\setminus K)$ with K running through the compact subsets of M.

We want to establish for every pure completion $\varphi: M \hookrightarrow P$ a natural bijection $\kappa(\varphi)$ from $\varepsilon(M)$ to the set $\varepsilon(M,\varphi)$ of ends of M in P.

<u>Lemma 9.5.</u> Let $\varphi: M \hookrightarrow P$ be a completion of M. For every $K \in \mathcal{T}_c(M)$ the map φ_* from $\pi_0(M\setminus K)$ to $\pi_0(P\setminus\varphi(K))$, induced by φ, is surjective. If φ is pure the map φ_* is bijective. Moreover, in this case, the connected component $\varphi_*(A)$ of $P\setminus\varphi(K)$ is the closure of $\varphi(A)$ in $P\setminus\varphi(K)$ of any $A \in \pi_0(M\setminus K)$.

Proof. We assume without loss of generality that M is a subspace of P and φ is the inclusion map. φ_* is clearly surjective, since $M \setminus K$ is dense in $P \setminus K$. More precisely, every connected component C of $P \setminus K$ is the union of the closures $\tilde{A}_i := \bar{A}_i \cap (P \setminus K)$ in $P \setminus K$ of a family of components $(A_i \mid i \in I)$ of $M \setminus K$. (N.B. The family $(A_i \mid i \in I)$ is locally finite.)

Now assume that φ is pure. We have to verify that the closures $\tilde{A}_1$, $\tilde{A}_2$ in $P \setminus K$ of any two different components A_1 and A_2 of $M \setminus K$ are disjoint. Suppose to the contrary that there exists some point $x \in \tilde{A}_1 \cap \tilde{A}_2$. Since (P,M) is pure there exists an open neighbourhood U of x in $P \setminus K$ such that $U \cap M$ is connected (cf. Cor. 8.8). $U \cap M$ meets both A_1 and A_2. Thus $U \cap M \subset A_1$ and $U \cap M \subset A_2$, a contradiction. q.e.d.

Lemma 9.6. Let $(\bar{M},M)$ be a completed space. For any $K \in \mathcal{T}_c(M)$ there exists some $L \in \mathcal{T}_c(M)$ with $L \supset K$ such that the natural map from $\pi_0(\bar{M} \setminus M)$ to $\pi_0(\bar{M} \setminus L)$ is a bijection.

Proof. Choosing a simultaneous triangulation of $\bar{M}$, M, and K, we assume without loss of generality that M is a strictly locally finite complex with closure $\bar{M}$ and that K is a closed subcomplex of $\bar{M}$ contained in M.

In the following we argue in a purely combinatorial way. The assumption that $\bar{M}$ is locally finite will not be needed. Let $(A_\alpha \mid \alpha \in I)$ be the family of components of the complex $\bar{M} \setminus M$ and let U_α be the star $St_{\bar{X}}(A_\alpha)$ of A_α in the barycentric subdivision $\bar{X} = \bar{M}'$ of $\bar{M}$. The sets U_α are pairwise disjoint by the following general lemma, to be proved later.

Lemma 9.7. Let Y be a complex and let C_1 and C_2 be subcomplexes of Y with $\bar{C}_1 \cap C_2 = \emptyset$ and $C_1 \cap \bar{C}_2 = \emptyset$. Then the stars $St_{Y'}(C_1)$ and $St_{Y'}(C_2)$ of the C_i in the (first) barycentric subdivision Y' of Y are disjoint.

We continue proving Lemma 9.6. Every complex U_α is open in $\overline{X}$ and is connected, since A'_α is connected. U_α is contained in $\overline{X} \setminus K'$, since U_α is the smallest open subcomplex of $\overline{X}$ containing A'_α. Let L be the complement in $\overline{X}$ of the open subcomplex $\cup(U_\alpha \mid \alpha \in I)$. This is a closed subcomplex of $\overline{X}$ with $K' \subset L \subset X := M'$. Now $\overline{X} \setminus L$ is the disjoint union of the U_α. Thus $(U_\alpha \mid \alpha \in I)$ is the family of components of the complex $\overline{X} \setminus L$, while $(A'_\alpha \mid \alpha \in I)$ is the family of components of $\overline{X} \setminus X$. The map $\pi_0(\overline{X} \setminus X) \to \pi_0(\overline{X} \setminus L)$ is bijective. q.e.d.

We still have to prove Lemma 9.7. For that we may assume that the complex Y is closed. Let σ_i be an open simplex of Y contained in C_i $(i = 1,2)$. We verify that the stars $St_{Y'}(\sigma_1)$ and $St_{Y'}(\sigma_2)$ are disjoint. Assume on the contrary that there exists a simplex T of Y' which is contained in both stars $St_{Y'}(\sigma_1)$ and $St_{Y'}(\sigma_2)$. We have a chain $\tau_0 < \tau_1 < \ldots < \tau_r$ of open simplices τ_i of Y with $T =]\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_r[$, where $\hat{\tau}_i$ denotes the barycenter of τ_i. Then $\tau_0, \tau_1, \ldots, \tau_r$ are precisely all open simplices of Y which have non-empty intersection with $\overline{T}$. Thus $\sigma_1 = \tau_i$ and $\sigma_2 = \tau_j$ for suitable indices $i,j \in \{0,\ldots,r\}$. But, since $\overline{C}_1 \cap C_2$ and $C_1 \cap \overline{C}_2$ are both empty, neither is σ_1 a face of σ_2 nor is σ_2 a face of σ_1. This contradiction proves that indeed $St_{Y'}(\sigma_1)$ and $St_{Y'}(\sigma_2)$ are disjoint.

The lemmas 9.5 and 9.6 imply the following theorem.

<u>Theorem 9.8.</u> For any completion $\varphi : M \hookrightarrow P$ there exists a natural surjection

$$\kappa(\varphi) : \varepsilon(M) \twoheadrightarrow \varepsilon(M,\varphi),$$

to be described below explicitly, from the set $\varepsilon(M)$ of ends of M to the set $\varepsilon(M,\varphi)$ of ends of M in P. If φ is pure, then $\kappa(\varphi)$ is bijective. If there exists a morphism from φ to another completion ψ of M, then the triangle

$$\begin{tikzcd}
 & \varepsilon(M,\varepsilon) \arrow[dd, "\kappa(\varphi,\psi)"] \\
\varepsilon(M) \arrow[ur, "\kappa(\varphi)"] \arrow[dr, "\kappa(\psi)"'] & \\
 & \varepsilon(M,\psi)
\end{tikzcd}$$

commutes. If φ is a pure completion of M and ψ an arbitrary completion of M, then this triangle still commutes.

<u>Proof.</u> We obtain the map $\kappa(\varphi)$ as follows. We have a natural map

$$\varphi_* : \varepsilon(M) = \varprojlim_K \pi_0(M \setminus K) \to \varprojlim_K \pi_0(P \setminus \varphi(K))$$

with K running through $\mathcal{T}_c(M)$. By Lemma 9.5 this is a bijection if φ is pure. By Lemma 9.6 we have a natural bijection

$$\mu : \varepsilon(M,\varphi) = \pi_0(P \setminus \varphi(M)) \xrightarrow{\sim} \varprojlim_K \pi_0(P \setminus \varphi(K)) .$$

$\kappa(\varphi)$ is defined as the composite of φ_* with μ^{-1}. The commutativity of the triangle above is obvious in both cases from the definitions of $\kappa(\varphi)$, $\kappa(\psi)$, and $\kappa(\varphi,\psi)$. If ψ is an arbitrary completion of M, then there exists a morphism from a pure completion φ of M to ψ, and we learn from the commutative triangle above that $\kappa(\psi)$ is surjective.

The map $\kappa(\varphi)$ can be described neatly in terms of paths, as we shall now explain.

<u>Definition 5.</u> An incomplete path $\alpha : [0,1[\to M$ (cf. I, §6, Def. 2) is called <u>proper</u> if α is a proper semialgebraic map.

<u>Remark 9.9.</u> For any incomplete path α in M, the following properties are equivalent.

a) α is proper.

b) $\alpha([0,1[)$ is not contained in any set $K \in \mathcal{T}_c(M)$.

c) $\alpha([0,1[)$ is closed in M, but not complete.

<u>Proof.</u> c) ⇒ b): If $\alpha([0,1[)$ were contained in some $K \in \mathcal{T}_c(M)$, then $\alpha([0,1[)$ would be a closed semialgebraic subset of K, hence would be complete.

b) ⇒ a): It suffices to verify that, for a given complete semialgebraic subset L of M, the preimage $\alpha^{-1}(L)$ is also complete (I, Cor. 6.11). The set $\alpha^{-1}(L)$ is closed and semialgebraic in $[0,1[$. Thus it is a disjoint union of finitely many one point sets, closed intervals, and perhaps an interval $[c,1[$ with $c \in [0,1[$. Suppose we had $\alpha([c,1[) \subset L$ for some $c \in [0,1[$. Then $\alpha([0,1[)$ would be contained in the complete set $\alpha([0,c]) \cup L$, in contradiction to the assumption b). Thus $\alpha^{-1}(L)$ is a union of finitely many points and closed intervals. In particular, $\alpha^{-1}(L)$ is complete.

a) ⇒ c): Since α is proper, $L := \alpha([0,1[)$ is a closed semialgebraic subset of M. But L is not complete, because its preimage $\alpha^{-1}(L) = [0,1[$ is not complete.

<u>Definition 6.</u> Let α be a proper incomplete path in M. We define an end $\lambda \in \varepsilon(M)$ as follows. We assign to every $K \in \mathcal{T}_c(M)$ the connected component $(M \setminus K)_\lambda$ of $M \setminus K$ which contains $\alpha([c,1[)$ for some $c \in]0,1[$. We call λ <u>the end of</u> M <u>determined by</u> α, and denote this end by $\varepsilon(\alpha)$.

<u>Proposition 9.10.</u> If $\varphi : M \hookrightarrow P$ is a completion of M and α is a proper incomplete path in M, then the map $\kappa(\varphi) : \varepsilon(M) \twoheadrightarrow \varepsilon(M,\varphi)$ sends $\varepsilon(\alpha)$ to the connected component $\varepsilon(\alpha,\varphi)$ of $P \setminus \varphi(M)$ which contains the end point $\overline{\varphi\circ\alpha}(1)$ of the completion $\overline{\varphi\circ\alpha} : [0,1] \to P$ of the incomplete path $\varphi\circ\alpha$ in P (cf. I, Cor. 6.9).

This is pretty evident from the definition of $\kappa(\varphi)$. The proposition gives a vivid description of the map $\kappa(\varphi)$ as soon as we know that the following theorem holds.

Theorem 9.11. For every end λ of M there exists some proper incomplete path α in M with $\varepsilon(\alpha) = \lambda$.

Proof. We choose a pure completion $\varphi : M \hookrightarrow P$ and assume without loss of generality that M is a subspace of P and φ is the inclusion map. Let λ be an end of M and let $A \in \pi_0(P \setminus M)$ be the corresponding end $\kappa(\varphi)(\lambda)$ of M in P. We choose a point $x \in A$ and then a path $\beta : [0,1] \to P$ with $\beta([0,1[) \subset M$ and $\beta(1) = x$. Let α be the proper incomplete path $\beta | [0,1[$ in M. Then $\kappa(\varphi)(\varepsilon(\alpha)) = A$ by Proposition 9.10. But we also have $\kappa(\varphi)(\lambda) = A$. Since $\kappa(\varphi)$ is bijective (cf. Th. 9.8), we have $\lambda = \varepsilon(\alpha)$.

q.e.d.

From our description of the map $\kappa(\varphi)$ in terms of paths we immediately obtain a description of the maps $\kappa(\varphi,\psi)$, defined earlier, in terms of paths as follows.

Corollary 9.12. Let $M \underset{\varphi}{\hookrightarrow} P$ and $M \underset{\psi}{\hookrightarrow} Q$ be completions of M. Assume that either there exists a morphism from φ to ψ or that φ is pure. Let $A \in \varepsilon(M,\varphi) = \pi_0(P \setminus \varphi(M))$ be given. Then the component $B := \kappa(\varphi,\psi)(A)$ of $Q \setminus \psi(M)$ can be described as follows. Let $\alpha : [0,1] \to P$ be a path with $\alpha([0,1[) \subset M$ and $\alpha(1) \in A$. Let $\beta : [0,1] \to Q$ be the unique path in Q with $\beta(t) = \psi \circ \varphi^{-1} \circ \alpha(t)$ for $0 \leq t < 1$. (N.B. β exists by the path completion criterion I, Cor. 6.9). Then B is the component of $Q \setminus \psi(M)$ containing $\beta(1)$.

Remark. It is possible to verify this corollary directly, starting from the definition of $\kappa(\varphi,\psi)$, via an auxiliary pure completion χ with $[\chi] \geq [\varphi], [\chi] \geq [\psi]$, and not using the notion of an "absolute" end. We leave this as exercise to the reader.

We briefly discuss the functorial behavior of the set of ends under partially proper maps. Given a partially proper map $f : M \to N$ between spaces we obtain a map $f_* : \varepsilon(M) \to \varepsilon(N)$ as follows. Let $\lambda \in \varepsilon(M)$ be given. Then $f_*(\lambda)$ is the unique end μ of N such that, for every $L \in \mathfrak{T}_c(N)$, $(N \setminus L)_\mu$ is the component of $N \setminus L$ which contains the connected set $f((M \setminus f^{-1}(L))_\lambda)$. Notice that this makes sense, since $f^{-1}(L) \in \mathfrak{T}_c(M)$. In this way we obtain a functor ε from the category of spaces and partially proper maps over R to the category of sets. For example, if $\alpha : [0,1[\to M$ is a proper incomplete path, then the end $\varepsilon(\alpha)$ defined above is none other than the image of the unique end of $[0,1[$ under α_*.

Similarly we have an "end functor" ε from the category of completed spaces to the category of sets. {Now it seems more convenient to work with completed spaces $(\overline{M},M)$ than with completions $\varphi : M \hookrightarrow P$.} For any completed spaces $(\overline{M},M)$ the ends of $(\overline{M},M)$ are, of course, the ends of M in $\overline{M}$ with respect to the inclusion map $\varphi : M \hookrightarrow \overline{M}$, i.e. the connected components of $\overline{M} \setminus M$. We denote the set of ends of $(\overline{M},M)$ by $\varepsilon(\overline{M},M)$. If $f : (\overline{M},M) \to (\overline{N},N)$ is a partially proper morphism between completed spaces (cf. §8, Def. 3), then $f^{-1}(N) = M$, and the restriction $g : \overline{M} \setminus M \to \overline{N} \setminus N$ of f induces a map $\pi_0(g)$ from $\varepsilon(\overline{M},M) = \pi_0(\overline{M} \setminus M)$ to $\varepsilon(\overline{N},N) = \pi_0(\overline{N} \setminus N)$. We denote this map simply by f_*. We further denote the natural map $\kappa(\varphi)$ from $\varepsilon(M)$ to $\varepsilon(\overline{M},M)$ by $\kappa_{(\overline{M},M)}$. Recall that, in the case that $(\overline{M},M)$ is pure, $\kappa_{(\overline{M},M)}$ is bijective. We have a natural commutative diagram

$$(*) \qquad \begin{array}{ccc} \varepsilon(M) & \xrightarrow{\kappa_{(\overline{M},M)}} & \varepsilon(\overline{M},M) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \varepsilon(N) & \xrightarrow{\kappa_{(\overline{N},N)}} & \varepsilon(\overline{N},N) \end{array} .$$

The following proposition is an extension of our observations above on the map $\kappa(\varphi,\psi)$ from $\varepsilon(M,\varphi)$ to $\varepsilon(M,\psi)$ induced by a morphism $P \xrightarrow[M]{} Q$ between two completions $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$.

<u>Proposition 9.13.</u> a) If $f : M \to N$ is a proper surjective map, then

$f_* : \varepsilon(M) \to \varepsilon(N)$ is surjective. If, in addition, the fibres of f are connected, then f_* is bijective.

b) If $f : (\overline{M},M) \to (\overline{N},N)$ is a proper morphism from a completed space $(\overline{M},M)$ to a pure completed space $(\overline{N},N)$, such that $f(M) = N$ and the fibres of the restricted map $f|M : M \to N$ are connected, then $f(\overline{M}) = \overline{N}$ and the fibres of $f : \overline{M} \to \overline{N}$ are connected. In this case $f_* : \varepsilon(\overline{M},M) \to \varepsilon(\overline{N},N)$ is bijective.

Proof. If L runs through $\mathcal{T}_c(N)$, then $f^{-1}(L)$ runs through a cofinal subset of the directed set $\mathcal{T}_c(M)$, since f is proper. (N.B. Partial properness does not suffice for this.) f clearly yields a surjection $f_* : \pi_0(M \setminus f^{-1}(L)) \to \pi_0(N \setminus L)$ for every $L \in \mathcal{T}_c(N)$. If the fibres of f are connected, then all these maps f_* are bijective, cf. Sublemma 6.6. (N.B. Again partial properness does not suffice for this conclusion). Thus $f_* : \varepsilon(M) \to \varepsilon(N)$ is bijective. In order to prove the surjectivity of this map in the general case we extend f to a morphism $(\overline{M},M) \to (\overline{N},N)$ between completed pairs with $(\overline{N},N)$ pure (cf. Prop. 5.1 and Th. 8.7; $(\overline{M},M)$ could be chosen pure as well). We denote this morphism again by f. We are in the situation described above before Proposition 9.13. But now $f : \overline{M} \to \overline{N}$ is partially proper and semialgebraic (cf. Lemma 8.1), hence proper. This implies that the restriction $g : \overline{M} \setminus M \to \overline{N} \setminus N$ of f is surjective. Thus, in the diagram (*) above the right vertical arrow is surjective. $\kappa_{(\overline{N},N)}$ is bijective and $\kappa_{(\overline{M},M)}$ is surjective. This implies that $f_* : \varepsilon(M) \to \varepsilon(N)$ is surjective. If the fibres of $f : M \to N$ are connected then also the fibres of g are connected (Lemma 6.5). In this case the right vertical arrow in (*) is bijective.

We turn to a different type of question. Given two pure completions $M \xrightarrow[\varphi]{} P$ and $M \xrightarrow[\psi]{} Q$ of M, let $A \in \varepsilon(M,\varphi)$ and $B \in \varepsilon(M,\psi)$ be ends of M in P and Q with $\kappa(\varphi,\psi)(A) = B$. Are there any geometric properties which the spaces A and B have in common?

We shall give a positive answer for two very crude properties. The first one is semialgebraicity. It is easily seen that A is semialgebraic if and only if B is semialgebraic. Also, if $\lambda \in \varepsilon(M)$ is the absolute end of M with $\kappa(\varphi)(\lambda) = A$, then A is semialgebraic if and only if $(M \setminus K)_\lambda$ is semialgebraic for some $K \in \mathcal{T}_c(M)$. We call such an end λ semialgebraic.

The second property is "complexity". The complexity c(M) of a space M is a non negative integer or ∞, which we can associate to every space M, cf. Definitions 7 and 8 below, and which measures "how much" the space fails to be partially complete. It will turn out that A and B always have the same complexity (cf. Th. 9.25 below).

Definition 7. Let (P,M) be a completed space.

a) The derived sequence $((P_k, M_k) \mid k \geq 0)$ of (P,M) is the following inductively defined sequence of completed spaces. $(P_0, M_0) := (P,M)$. M_{k+1} is the subspace $P_k \setminus M_k$ of P_k and P_{k+1} is the closure of M_{k+1} in P_k.

b) If M is not empty then the complexity c(P,M) of (P,M) is defined as the supremum of all numbers $k \geq 0$ with $M_k \neq \emptyset$, i.e. c(P,M) is the highest index m with $M_m \neq \emptyset$, if it exists, and $c(P,M) = \infty$ otherwise. The complexity of the empty pair $(\emptyset,\emptyset)$ is defined as -1.

Let us take a rough look at the complexity c(P,M). Clearly

$$P_0 \supsetneqq P_1 \supsetneqq P_2 \supsetneqq \cdots\cdots \quad .$$

If $U \in \mathring{\mathcal{T}}(P)$ is an open locally semialgebraic subset of P, then obviously $M_k \cap U$ is dense in $P_k \cap U$. Hence we obtain

$$\dim P_k \cap U = \dim M_k \cap U > \dim M_{k+1} \cap U = \dim P_{k+1} \cap U,$$

provided $\dim M_k \cap U$ is finite and $M_k \cap U \neq \emptyset$ (cf. I, Prop. 3.21). Thus P_{k+1} is thin in P_k (cf. §8, Def. 8), and

$$\dim_x P_k \leq \dim_x P - k$$

for every $x \in P_k$. We see that $c(P,M) \leq \dim M$. Also

$$c(P_r, M_r) = 1 + c(P_{r+1}, M_{r+1}),$$

provided M_r is not empty.

What does it mean that (P,M) has complexity zero or one? We have $c(P,M) = 0$ if and only if $M = P$. Then M is partially complete. Conversely, if M is partially complete then M is closed in P (I, Cor. 6.10). hence $M = P$. Thus $c(P,M) = 0$ means that M is partially complete and not empty.

Assume now that M is not partially complete, i.e. $c(P,M) \geq 1$. We know from I, §7 (cf. Prop. 7.2) that M is locally complete if and only if $M_1 = P \setminus M$ is closed in P, i.e. $M_1 = P_1$. We conclude that $c(P,M) = 1$ if and only if M is locally complete but not partially complete.

In particular the properties "$c(P,M) = 0$" and "$c(P,M) = 1$" do not depend on the given completion $M \hookrightarrow P$ of M but only on the space M itself. This observation can be generalized, cf. Theorem 9.17 below.

We now take a closer look at the derived sequence $((P_k, M_k) \mid k \geq 0)$ of a given completed space (P,M). The set $P \setminus P_1$ is contained in M, and

$$M = (P \setminus P_1) \,\dot{\cup}\, (P_1 \setminus M_1) = (P \setminus P_1) \,\dot{\cup}\, M_2 \qquad *)$$

In particular $M_2 \subset M$, while M_1 is disjoint from M. Also $M \setminus M_2 = M \setminus P_1$ is open in M, and hence M_2 is closed in M. Applying all this to (P_k, M_k) instead of (P,M) we inductively see the following:

<u>Remark 9.14.</u> $M_0 \supsetneqq M_2 \supsetneqq M_4 \supsetneqq \cdots$, and $M_1 \supsetneqq M_3 \supsetneqq M_5 \supsetneqq \cdots$. Every set M_{2k} is closed in $M = M_0$, and every set M_{2k+1} is closed in M_1. The sets M_0 and M_1 are disjoint. We have, for every $k \geq 0$,

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*) $\dot{\cup}$ means "disjoint union" in the set theoretical sense.

$$M_k = M_{k+2} \cup (P_k \setminus P_{k+1}).$$

We introduce the subsets

$$S_k(P,M) := P_{2k} \setminus P_{2k+1} \quad (k \geq 0)$$

of M. Every $S_k(P,M)$ is open in P_{2k}, and hence is a locally complete space. We denote the closure of $S_k(P,M)$ in M by $\overline{S}_k(P,M)$.

<u>Lemma 9.15.</u> a) Every set $P_r \setminus P_{r+1}$ is dense in P_r.
b) $\overline{S}_k(P,M) = M_{2k}$.
c) M_{2k} is the disjoint union of all $S_l(P,M)$ with $l \geq k$.

<u>Proof.</u> We observed already that P_{r+1} is thin in P_r. This means that $P_r \setminus P_{r+1}$ is dense in P_r. The closure of $S_k(P,M) = P_{2k} \setminus P_{2k+1}$ in M is $M \cap P_{2k} = M \cap \overline{M}_{2k} = M_{2k}$, since M_{2k} is closed in M. The last assertion c) is evident from Remark 9.14, as soon as we know that the intersection of all the sets M_{2k} is empty. But this is evident since $\dim_x M_{2k} \leq \dim_x M-2k$ for every $x \in M_{2k}$, as we saw before. q.e.d.

According to this lemma the partition $(S_k(P,M) \mid k \geq 0)$ of M may be regarded as a stratification of M - in a weak sense. We have

$$S_k(P_1,M_1) = P_{2k+1} \setminus P_{2k+2}.$$

The closure $\overline{S_k(P,M)}$ of $S_k(P,M)$ in P is the set P_{2k} which is the union of all strata $S_l(P,M)$ of M with $l \geq k$ and all strata $S_l(P_1,M_1)$ of M_1 with $l \geq k$. Similarly $\overline{S_k(P_1,M_1)}$ is the union of the strata $S_l(P,M)$ of M with $l \geq k+1$ and the strata $S_l(P_1,M_1)$ of M_1 with $l \geq k$. Thus the stratifications $(S_k(P,M) \mid k \geq 0)$ and $(S_k(P_1,M_1) \mid k \geq 0)$ of M and M_1 fit together to a stratification of $P = M \cup M_1$.

<u>Proposition 9.16.</u> Let (Q,N) be a second completed space with derived

sequence $((Q_k, N_k) \mid k \geq 0)$. Let $f : (P,M) \to (Q,N)$ be a proper morphism with $f(M) = N$. Then $c(P,M) = c(Q,N)$, and $f(M_k) = N_k$, $f(P_k) = Q_k$ for every $k \geq 0$.

Proof. $f(P)$ is a closed set which contains N, so $f(P) = Q$. We have $f^{-1}(N) = M$, and hence $f^{-1}(N_1) = M_1$. We conclude that $f(M_1) = N_1$, which implies that $f(P_1) = Q_1$. Thus f yields, by restriction, a proper morphism $f_1 : (P_1, M_1) \to (Q_1, N_1)$ with $f_1(M_1) = N_1$. Proceeding by induction on k, we obtain $f(M_k) = N_k$ and $f(P_k) = Q_k$ for every $k \geq 0$. In particular, $N_k \neq \emptyset$ if and only if $M_k \neq \emptyset$. Thus $c(Q,N) = c(P,M)$. q.e.d.

Theorem 9.17. If $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$ are two completions of a space M, then $c(P, \varphi(M)) = c(Q, \psi(M))$ and

$$\varphi^{-1} S_k(P, \varphi(M)) = \psi^{-1} S_k(Q, \psi(M))$$

for every $k \geq 0$.

Proof. We have a third completion $\chi : M \hookrightarrow T$ which admits morphisms $f : (T, \chi(M)) \to (P, \varphi(M))$ and $g : (T, \chi(M)) \to (Q, \psi(M))$ such that $f \circ \chi = \varphi$ and $g \circ \chi = \psi$. It suffices to prove the theorem for φ and χ instead of φ and ψ. The morphism f is proper and maps $\chi(M)$ isomorphically onto $\varphi(M)$. We conclude from the preceding proposition that indeed $c(P, \varphi(M)) = c(T, \chi(M))$ and $f(\overline{S}_k(T, \chi(M)) = \overline{S}_k(P, \varphi(M))$ for any $k \geq 0$. Since $f : \chi(M) \to \varphi(M)$ is bijective, this implies $f(S_k(T, \chi(M)) = S_k(P, \varphi(M))$, and then $S_k(T, \chi(M)) = f^{-1} S_k(P, \varphi(M))$. Taking preimages under χ, we obtain $\chi^{-1} S_k(T, \chi(M)) = \varphi^{-1} S_k(P, \varphi(M))$ as desired. q.e.d.

Definition 8. The complexity $c(M)$ of a space M is the number $c(P, \varphi(M))$ for any completion $\varphi : M \hookrightarrow P$. The LC-stratification of M is the partition $(S_k(M) \mid k \geq 0)$ of M into locally complete subsets

$$S_k(M) := \varphi^{-1} S_k(P, \varphi(M)).$$

(LC = "locally complete".) We call $S_k(M)$ the <u>k-th LC-stratum</u> of M. Its closure in M will be denoted by $\overline{S}_k(M)$.

The point is that, by the preceding theorem, the complexity and the LC-stratification of M do not depend on the choice of the completion φ of M.

Using the "absolute" notions of Definition 8 we re-formulate some of our observations on the derived sequence of a completed space.

<u>Resumée 9.18.</u> a) The subsets $S_k(M)$ of M are pairwise disjoint, and $\dim_x S_k(M) \leq \dim_x M-2k$ for every $x \in S_k(M)$.

b) $\overline{S}_k(M)$ is the union of the strata $S_l(M)$ with $l \geq k$.

c) $S_{k+1}(M) = S_0(\overline{S}_{k+1}(M)) = S_0(\overline{S}_k(M) \setminus S_k(M))$ for every $k \geq 0$.

d) $\overline{S}_k(\overline{S}_l(M)) = \overline{S}_{k+l}(M)$ for every $k \geq 0$, $l \geq 0$.

e) $c(M) \leq \dim M$.

f) If there exists a highest index r with $S_r(M) \neq \emptyset$, then

$$c(M) = \begin{cases} 2r & \text{if } \overline{S}_r(M) \text{ is partially complete} \\ 2r+1 & \text{else.} \end{cases}$$

If M is empty then $c(M) = -1$. If all $S_r(M) \neq 0$ (hence in particular $\dim M = \infty$), then $c(M) = \infty$.

g) If (P,M) is a non empty completed space, then $c(M) = 1 + c(P \setminus M)$. The closure of $S_k(M)$ in P is the union of all strata $S_l(M)$ with $l \geq k$ and all strata $S_l(P \setminus M)$ with $l \geq k$. The closure of $S_k(P \setminus M)$ in P is the union of all strata $S_l(M)$ with $l \geq k+1$ and all strata $S_l(P \setminus M)$ with $l \geq k$.

We also formulate Proposition 9.16 in absolute terms.

<u>Theorem 9.19.</u> Let $f : M \to N$ be a proper surjective map from a space M to a space N. Then $c(M) = c(N)$, and $f(\overline{S}_k(M)) = \overline{S}_k(N)$ for every $k \geq 0$.

Indeed, by Proposition 5.1 our map f can be extended to a morphism $(P,M) \to (Q,N)$ for suitable completed spaces (P,M) and (N,Q). This morphism is proper (cf. I.5.9, I.6.13, 8.1). Thus Proposition 9.16 gives the desired result.

We ask for an intrinsic description of the LC-strata $S_k(M)$ which does not involve any completion of M.

Proposition 9.20. $S_0(M)$ is the set of all $x \in M$ which have a complete neighbourhood in M.

Proof. We choose a completion $\varphi : M \hookrightarrow P$ of M. We assume without loss of generality that M is a subspace of P and φ is the inclusion map. $S_0(M)$ is, by definition, the interior $\dot{M}$ of the set M in P. If $x \in M$ has a complete neighbourhood in M, then $x \in \dot{M}$ by I, Proposition 7.2. On the other hand, $\dot{M}$ is a locally complete space. Thus every $x \in \dot{M}$ has a complete neighbourhood in $\dot{M}$, which is, of course, a complete neighbourhood in M. q.e.d.

By this proposition the LC-stratum $S_0(M)$ is the largest open locally complete subspace of M. Using 9.18.c we obtain an intrinsic - though complicated - description of every stratum $S_k(M)$. As another consequence of Proposition 9.20 we obtain

Proposition 9.21 (Local nature of the LC-stratification). Let U be an open subspace of M. Then $S_k(U) = S_k(M) \cap U$ for every $k \geq 0$.

Proof. This is evident for $k = 0$ from Proposition 9.20. We then have

$$\overline{S}_1(U) = U \setminus S_0(U) = (M \setminus S_0(M)) \cap U = \overline{S}_1(M) \cap U.$$

Thus $\overline{S}_1(U)$ is open in $\overline{S}_1(M)$. We conclude, with 9.18.c, that

$$S_1(U) = S_0(\overline{S}_1(U)) = S_0(\overline{S}_1(M)) \cap \overline{S}_1(U) =$$
$$= S_1(M) \cap \overline{S}_1(M) \cap U = S_1(M) \cap U.$$

Repeating this argument we obtain $S_k(U) = S_k(M) \cap U$ for every k. q.e.d.

From this proposition and 9.18.f we obtain immediately

Corollary 9.22. a) For every $U \in \mathring{\mathfrak{T}}(M)$ we have $c(U) \leq c(M)+1$ if $c(M)$ is even, and $c(U) \leq c(M)$ if $c(M)$ is odd.
b) If $(U_\alpha | \alpha \in I)$ is any admissible covering of M by open locally semialgebraic subsets then, for every $k \geq 0$,

$$S_k(M) = \cup(S_k(U_\alpha) | \alpha \in I),$$

and

$$c(M) = \sup(c(U_\alpha) | \alpha \in I) - \delta$$

with $\beta = 0$ if $c(M)$ is odd, and $\delta = 0$ or 1 if $c(M)$ is even.

Remark 9.23. Starting from 9.22.b we can establish an LC-stratification $(S_k(M) | k \geq 0)$ on any locally semialgebraic space M (not necessarily regular and paracompact) as follows. $S_k(M)$ is the union of all $S_k(U_\alpha)$ for any admissible covering $(U_\alpha | \alpha \in I)$ by open affine semialgebraic subsets. We further can define a complexity $c(M)$ as indicated in 9.18.f. It is now possible to generalize those facts on the LC-stratification and complexity which do not refer to a completion of M to arbitrary locally semialgebraic spaces. We leave the details to the reader.

Notation 9.24. If $\varphi : M \hookrightarrow P$ is a completion of a space M, then for any absolute end $\lambda \in \varepsilon(M)$ we denote the corresponding end $\kappa(\varphi)(\lambda)$ of M in P by $P(\lambda)$. Thus $(P(\lambda) | \lambda \in \varepsilon(M))$ is the family of all ends of M in P and $P \setminus \varphi(M)$ is the direct sum of the spaces $P(\lambda)$.

If M is not empty then clearly $c(M) = 1 + \sup(c(P(\lambda)) | \lambda \in \varepsilon(M))$. It turns out that, for φ pure, not only the supremum of the complexities $c(P(\lambda))$ but each number $c(P(\lambda))$ itself is independent of the choice of the completion φ of M.

<u>Theorem 9.25.</u> Let $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$ be pure completions of M. Then, for any $\lambda \in \varepsilon(M)$, the ends C of the space $P(\lambda)$ in its closure $\overline{P(\lambda)}$ in P correspond bijectively to the ends D of $Q(\lambda)$ in its closure $\overline{Q(\lambda)}$ in Q via the relation $\varphi^{-1}(C) = \psi^{-1}(D)$. (N.B. We have $C \subset \varphi(M)$ and $D \subset \psi(M)$.) In particular, $c(P(\lambda)) = c(Q(\lambda))$ and $|\varepsilon(P(\lambda))| = |\varepsilon(Q(\lambda))|$.

<u>Proof.</u> Let $A := P(\lambda)$ and $B := Q(\lambda)$. We may assume that there exists a morphism $f : Q \xrightarrow[M]{} P$ from ψ to φ (cf. proof of Th. 9.17). Thus f yields by restriction a proper surjective map $h : B \twoheadrightarrow A$ and also a proper surjective map $g : \overline{B} \twoheadrightarrow \overline{A}$. This already implies that $c(B) = c(A)$, cf. Theorem 9.19. We have $g^{-1}(A) = B$, and thus obtain a proper surjective map f_1 from $\overline{B} \setminus B$ to $\overline{A} \setminus A$ by restriction of f. But $\overline{B} \setminus B \subset \psi(M)$ and f maps $\psi(M)$ isomorphically onto $\varphi(M)$. Thus f_1 is an isomorphism from $\overline{B} \setminus B$ to $\overline{A} \setminus A$. The theorem is now obvious. q.e.d.

With the situation as in the proof, we know (cf. e.g. Proposition 8.4) that the fibres of the proper map $h : B \to A$ are connected. The fibres of the extended map $g : \overline{B} \to \overline{A}$ are, of course, also connected, since all fibres $g^{-1}(y)$ with $y \in \overline{A} \setminus A$ are one-point sets. Thus the morphism $g : (\overline{B}, B) \to (\overline{A}, A)$ is "shrinking" in the sense of the following definition

<u>Definition 9.</u> A morphism $g : (\overline{B}, B) \to (\overline{A}, A)$ between completed spaces is called <u>shrinking</u> if g is proper, $g(B) = A$, hence also $g(\overline{B}) = \overline{A}$, all fibres of $g : \overline{B} \to \overline{A}$ are connected, and all fibres $g^{-1}(y)$ with $y \in \overline{A} \setminus A$ are one-points sets. (N.B. Then g gives an isomorphism from $\overline{B} \setminus B$ onto $\overline{A} \setminus A$ by restriction).

We can add to Theorem 9.25 the following

<u>Corollary 9.26.</u> Let $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$ be completions of M which admit a morphism $f : Q \xrightarrow[M]{} P$. Assume that φ is pure. Then for every $\lambda \in \varepsilon(M)$ f yields by restriction a shrinking morphism $g_\lambda : (\overline{Q(\lambda)}, Q(\lambda)) \rightarrow (\overline{P(\lambda)}, P(\lambda))$ between the completed ends in Q and P corresponding to λ. The isomorphism from $\overline{Q(\lambda)} \setminus Q(\lambda)$ to $\overline{P(\lambda)} \setminus P(\lambda)$ induced by g_λ is the restriction of $\varphi \circ \psi^{-1}$ to the set $\overline{Q(\lambda)} \setminus Q(\lambda)$.

§10 - Some proper quotients

We would like to understand how the ends of a completed space can be "shrunk" or "expanded". Before we can attack this problem we have to know that suitable quotients (in particular mapping cylinders) exist in our category of spaces. These quotients deserve independent interest. Thus we try to prove their existence under the weakest possible assumptions. They will be slightly weaker than necessary for the applications in §11 and §12.

We work for a short time in the category of all (separated) locally semialgebraic spaces instead of regular paracompact spaces. So "space" now means "separated locally semialgebraic space".

Definition 1. A proper quotient of a space M by an equivalence relation $\mathcal{R}$ on the set M is a proper surjective semialgebraic*) map $p: M \to N$ whose fibres are the equivalence classes of $\mathcal{R}$. We then write $N = M/\mathcal{R}$ and call p the quotient projection from M to $M/\mathcal{R}$.

The definition is justified since, in this situation, N is indeed a quotient of M in the categorial sense, as stated by the following proposition.

Proposition 10.1. Let $p: M \to N$ be a proper surjective semialgebraic map. Let $f: M \to L$ be a locally semialgebraic map which sends every fibre of p to a point in L. Then the unique map $\bar{f}: N \to L$ with $\bar{f} \circ p = f$ is locally semialgebraic.

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*) Recall that, for M paracompact, every proper locally semialgebraic map $p: M \to N$ is semialgebraic (I, 5.9). Thus our condition "semialgebraic" here is not a serious restriction.

Proof. We choose an admissible covering $(U_\alpha|\alpha\in I)$ of N by open semialgebraic subsets. Then $(p^{-1}(U_\alpha)|\alpha\in I)$ is an admissible covering of M by open semialgebraic subsets. We have to verify that $\bar{f}|U_\alpha : U_\alpha \to L$ is semialgebraic for every $\alpha\in I$. Replacing N by U_α and M by $p^{-1}(U_\alpha)$ for some fixed α, we may assume that M and N are semialgebraic spaces. Replacing L by its semialgebraic subspace $f(M)$ we also assume that L is semialgebraic.

A subset B of N is closed and semialgebraic in N if and only if $p^{-1}(B)$ is closed and semialgebraic in M. From this fact one deduces immediately that $\bar{f}$ is continuous. The graph $\Gamma(\bar{f})\subset N\times L$ is the image of the graph $\Gamma(f)\subset M\times L$ under the (proper) semialgebraic map $p\times\mathrm{id} : M\times L \to N\times L$. Thus $\Gamma(\bar{f})$ is a semialgebraic subset of $N\times L$. This finishes the proof that $\bar{f}$ is semialgebraic. q.e.d.

Proposition 10.2. Let $p : M\to N$ be a proper surjective map. Assume that M is regular and paracompact. Then N is also regular and paracompact.

Proof. a) We first prove regularity. We have to separate a given closed locally semialgebraic set A in N from a given point x in $N\setminus A$. Since M is regular and paracompact there exist open locally semialgebraic sets U_1, U_2 in M with $p^{-1}(x)\subset U_1$, $p^{-1}(A)\subset U_2$ and $U_1\cap U_2 = \emptyset$ (I, 4.14). Since p is proper there exist open locally semialgebraic neighbourhoods V_1,V_2 of x and A respectively with $U_i\supset p^{-1}(V_i)$, namely $V_i := N\setminus p(M\setminus U_i)$. Since p is surjective $V_1\cap V_2 = \emptyset$. This proves that N is regular.

b) We choose a locally finite covering $(A_\alpha|\alpha\in I)$ of M by closed semialgebraic subsets (cf. I, Th. 4.11). Then $(p(A_\alpha)|\alpha\in I)$ is a covering of N by closed semialgebraic subsets. We claim that this family is locally finite. The map p is semialgebraic (I, 5.9). For any $U\in\mathring{\gamma}(N)$ the set $p^{-1}(U)$ is semialgebraic and hence meets A_α for only finitely many $\alpha\in I$. We conclude that U meets $p(A_\alpha)$ only for finitely many $\alpha\in I$.

We now know by Lemma 1.2 that N is indeed paracompact. q.e.d.

From now on a "space" will again mean a regular paracompact locally semialgebraic space. By Proposition 10.2 we know that a proper quotient of a space is again a space.

In order to prove the existence of proper quotients the following lemma will be useful.

<u>Lemma 10.3.</u> Let $(M_\alpha | \alpha \in I)$ be a locally finite family of locally semialgebraic subsets of a space M which covers M. Assume that either all M_α are closed or all M_α are open in M. Let $\mathcal{R}$ be an equivalence relation on the set M such that every M_α is a union of equivalence classes. Let $\mathcal{R}_\alpha$ denote the restriction of the equivalence relation to M_α. Assume that, for every $\alpha \in I$, the proper quotient $M_\alpha/\mathcal{R}_\alpha$ exists. Then the proper quotient $M/\mathcal{R}$ exists.

<u>N.B.</u> It is clear that, if the proper quotient $M/\mathcal{R}$ exists, all the sets $M_\alpha/\mathcal{R}_\alpha$ are closed, resp. open, locally semialgebraic subsets of $M/\mathcal{R}$, and that on $M_\alpha/\mathcal{R}_\alpha$ the subspace structure in $M/\mathcal{R}$ is the same as the quotient structure coming from M_α. Also, $(M_\alpha/\mathcal{R}_\alpha | \alpha \in I)$ is a locally finite covering of $M/\mathcal{R}$.

<u>Proof.</u> We denote the quotient $M_\alpha/\mathcal{R}_\alpha$ by N_α and the proper projection map from M_α to N_α by p_α. The subset $N_{\alpha\beta} := p_\alpha(M_\alpha \cap M_\beta)$ of N_α is locally semialgebraic and $p_\alpha^{-1}(N_{\alpha\beta}) = M_\alpha \cap M_\beta$. The restriction $p_{\alpha\beta} : M_\alpha \cap M_\beta \to N_{\alpha\beta}$ of p_α is again proper. This map shows the subspace $N_{\alpha\beta}$ of N_α to be the proper quotient of $M_\alpha \cap M_\beta$ by the equivalence relation $\mathcal{R}_{\alpha\beta}$ obtained from $\mathcal{R}$ by restriction. (Notice that $M_\alpha \cap M_\beta$ consists of full equivalence

classes of R.) Since the same holds for $N_{\beta\alpha}$ we have a canonical commutative triangle

$$\begin{tikzcd} & M_\alpha \cap M_\beta \arrow[dl, "p_{\alpha\beta}"'] \arrow[dr, "p_{\beta\alpha}"] & \\ N_{\alpha\beta} \arrow[rr, "\sim", "\chi_{\alpha\beta}"'] & & N_{\beta\alpha} \end{tikzcd}$$

with an isomorphism $\chi_{\alpha\beta}$.

We regard the spaces $N_\alpha = M_\alpha/R_\alpha$ as subsets of the set $N := M/R$. Then $N_{\alpha\beta} = N_{\beta\alpha}$ as sets, and $\chi_{\alpha\beta}$ becomes the identity map. Thus the subspace structures on $N_{\alpha\beta} = N_{\beta\alpha}$ with respect to N_α and N_β are equal. If every M_α is closed in M then every $N_{\alpha\beta}$ is closed in N_α and N_β. Then we know from Theorem 1.3 that there exists on N a unique space structure such that $(N_\alpha|\alpha\in I)$ is a locally finite family of closed locally semialgebraic subsets of N and the subspace structures on N_α are the given quotient structures of the spaces M_α. We equip N with this space structure. Clearly the natural projection $p: M \to N$ is a proper semialgebraic map, and we are done.

Assume now that every M_α is open in M. Then $p^{-1}(N_{\alpha\beta}) = M_\alpha \cap M_\beta$ is open in M_α and in M_β. It follows that $N_{\alpha\beta}$ is open in N_α and in N_β, since p_α and p_β are proper and surjective. On the set N we have a unique structure of a ringed space over R such that $(N_\alpha|\alpha\in I)$ is an admissible open covering and every N_α, in its given structure, is an open subspace of N. This space N is locally semialgebraic, but not yet known to be separated. Once we have verified this we are done: By Proposition I.5.7 and Theorem I.5.9 the map p is proper and semialgebraic.

Let x and y be different points of N. We have to separate them by open neighbourhoods. We choose indices $\alpha,\beta \in I$ with $x \in N_\alpha$ and $y \in N_\beta$. The fibres $p^{-1}(x)$ and $p^{-1}(y)$ are complete, hence closed in M. Since M is regular and paracompact we find open semialgebraic neighbourhoods $U_1 \subset M_\alpha$ and $U_2 \subset M_\beta$ of $p^{-1}(x)$ and $p^{-1}(y)$ with $U_1 \cap U_2 = \emptyset$. Since the maps p_α and p_β are proper we find open semialgebraic neighbourhoods $V_1 \subset N_\alpha$ and $V_2 \subset N_\beta$ of x and y with $p^{-1}(V_i) \subset U_i$. Clearly $V_1 \cap V_2 = \emptyset$. q.e.d.

We now prove the existence of proper quotients in a very special case.

Proposition 10.4. Let A be a complete - hence semialgebraic - subset of a space M. There exists the proper quotient M/A of M by the equivalence relation which has as equivalence classes the set A and the one point sets $\{x\}$ with $x \in M \setminus A$.

Proof. We have to find a proper surjective map $p: M \to N$ whose fibres are A and the sets $\{x\}$ with $x \in M \setminus A$. We choose an embedding $M \hookrightarrow P$ of M into a partially complete space P. It suffices to find a proper surjection $q: P \to L$ whose fibres are A and the sets $\{x\}$ with $x \in P \setminus A$. Then the restriction $p: M \to q(M)$ of q will be a proper quotient of M by the original equivalence relation. (N.B. q(M) is a subspace of L since q is semialgebraic). Thus we assume without loss of generality that the space M is partially complete.

We chosse an open semialgebraic neighbourhood U of A in M. The closure $B := \overline{U}$ of U in M is complete. It suffices to prove that the proper quotient B/A exists. Then, applying Lemma 10.3 to the covering of M by the two closed locally semialgebraic sets $M \setminus U$ and B, we see that the proper quotient M/A exists. Now we have reduced the proof to the case that the space M is complete and (hence) semialgebraic.

Let N denote the one point completion $(M \setminus A)^+$ of the locally complete semialgebraic space $M \setminus A$ (cf. I, §7). The identity map $M \setminus A \to M \setminus A$ extends to a proper map $p: M \to N$ with $p^{-1}(\infty) = A$ (I, Prop. 7.6), and we are done. q.e.d.

Given a locally semialgebraic map $f: X \to Y$ between spaces X and Y, we ask whether the mapping cylinder $Z(f)$ exists in the category of spaces. In analogy to the topological case we mean by $Z(f)$ the quotient of the direct sum

$$M := (X \times [0,1]) \sqcup Y$$

by the coarsest equivalence relation R with $(x,1) \sim f(x)$ for every $x \in X$.

Suppose that even the proper quotient $p: M \to Z(f)$ exists. Since Y is closed and locally semialgebraic in M, the set $p(Y)$ is closed and locally semialgebraic in $Z(f)$, and p maps Y isomorphically onto $p(Y)$. Identifying Y with $p(Y)$ we regard Y as a closed subspace of M and also of $Z(f)$. Now p maps $X \times \{1\}$ to Y and the restriction $X \times \{1\} \to Y$ of p is again proper. But this map is just a copy of $f: X \to Y$. Thus f must be proper. So, in the present framework, where we only look for proper quotients, we have to assume from the beginning that f is proper.

Theorem 10.5. Let $f: X \to Y$ be a proper map between spaces. Then the mapping cylinder $Z(f)$ exists as a proper quotient of $M := (X \times [0,1]) \sqcup Y$.

Proof. Let $(U_\alpha \mid \alpha \in J)$ be a locally finite covering of Y by open semialgebraic sets U_α. Suppose that for every $\alpha \in J$ the mapping cylinder of the restriction $f^{-1}(U_\alpha) \to U_\alpha$ of f exists as a proper quotient. Then it follows from Lemma 10.3 that $Z(f)$ exists as a proper quotient. Thus we may henceforth assume that X and Y are (affine) semialgebraic spaces.

Let $\bar{f}: \bar{X} \to \bar{Y}$ be a completion of f (cf. Prop. 5.1). Then $X = \bar{f}^{-1}(Y)$ since f is proper. Suppose that the proper quotient

$$\overline{p} : \overline{X}\times[0,1] \sqcup \overline{Y} \to Z(\overline{f})$$

exists. $L := \overline{p}(X\times[0,1] \sqcup Y)$ is a semialgebraic subset of $Z(\overline{f})$ with preimage $X\times[0,1] \sqcup Y$. Thus the restriction

$$p : X\times[0,1] \sqcup Y \to L$$

of $\overline{p}$ is a proper map, and this map shows L to be the mapping cylinder $Z(f)$ of f. These considerations allow us to retreat to the case that both X and Y are complete (affine) semialgebraic spaces.

In order to construct $Z(f)$ in this case we start with the proper projection map

$$\varphi : X\times I \twoheadrightarrow C(X) := X\times I/X\times\{1\}$$

of the "cone" $C(X)$ of X, which exists by the preceding proposition 10.4. Consider the proper surjective map

$$\psi : X\times Y\times I \xrightarrow[\sigma]{} X\times I\times Y \xrightarrow[\varphi\times id]{} C(X)\times Y,$$

with σ denoting the switch $(x,y,t) \mapsto (x,t,y)$. Notice that two different points (x_1,y_1,t_1) and (x_2,y_2,t_2) have the same image under ψ if and only if $t_1 = t_2 = 1$ and $y_1 = y_2$. We choose a point $x_o \in X$. Let Z be the image under ψ of the closed semialgebraic subset

$$N := (\Gamma(f)\times I) \cup (\{x_o\}\times Y\times\{1\})$$

of $X\times Y\times I$. {As usual, $\Gamma(f)$ denotes the graph of f.} Then Z is a complete space and the restriction $\chi : N \to Z$ of ψ is proper and surjective. We have another proper surjective map

$$\pi : (X\times I) \sqcup Y \to N,$$

defined by $\pi(x,t) = (x,f(x),t)$, $\pi(y) = (x_o,y,1)$. The composite $p := \chi\circ\pi$ is a proper surjective map from $(X\times I) \sqcup Y$ to Z which shows that Z is the mapping cylinder of f. q.e.d.

From Proposition 10.4 and Theorem 10.5 we obtain

<u>Corollary 10.6.</u> Let $f: X \to Y$ be a semialgebraic map from a complete space X to a space Y (hence f is proper). Then the <u>mapping cone</u>

$$C(f) := Z(f)/X \times \{0\}$$

exists as a proper quotient of $(X \times I) \sqcup Y$.

We now come to the main result of this section. Let A be a closed locally semialgebraic subset of a space X and let $f: A \to Y$ be a proper map to another space Y. We want to glue X to Y "along A", i.e. we ask for the quotient $X \cup_f Y$ of $X \sqcup Y$ by the coarsest equivalence relation with $a \sim f(a)$ for every $a \in A$.

<u>Theorem 10.7.</u> The proper quotient $X \cup_f Y$ of $X \sqcup Y$ exists.

Notice that this theorem contains our preceding results 10.4, 10.5 and 10.6 as special cases. Notice also that properness of f is certainly necessary for the existence of $X \cup_f Y$ as a proper quotient.

The proof of Theorem 10.7 is easy if $A = X$. In this case the proper map $\pi: X \sqcup Y \to Y$, $\pi(x) = f(x)$ $(x \in X)$, $\pi(y) = y$ $(y \in Y)$ shows that $Y = X \cup_f Y$.

We now proof Theorem 10.7 in the case that X is partially complete. Choosing a simultaneous triangulation of X and A we assume that X is a closed locally finite simplicial complex and A is a closed subcomplex of X.

We first consider the case that $X \setminus A$ contains only finitely many open simplices. We choose a filtration

$$A = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_r = X$$

of X by closed subcomplexes X_i such that $X_i = X_{i-1} \cup \sigma_i$ with a single open simplex σ_i whose boundary $\partial\sigma_i$ is contained in X_{i-1}.

Using induction on i, we prove that the proper quotient $X_i \cup_f Y$ exists for $i = 0,1,\dots,r$. We know this from above for $i = 0$. Assume that $0 \le i \le r-1$ and that the proper quotient map

$$p : X_i \sqcup Y \to X_i \cup_f Y =: Y_i$$

exists. We identify X_{i+1} with the mapping cone $C(\varphi)$ of the inclusion map $\varphi : \partial\sigma_{i+1} \hookrightarrow X_i$ in the obvious way such that, say, the barycenter of σ_{i+1} is the vertex of the cone. We also introduce the mapping cone $C(q\circ\varphi)$ of the composite of φ with the restriction $q : X_i \to Y_i$ of p. Let $\lambda : (\partial\sigma_{i+1}\times I) \sqcup X_i \to C(\varphi)$ and $\mu : (\partial\sigma_{i+1}\times I) \sqcup Y_i \to C(q\circ\varphi)$ be the proper quotient projections of these mapping cones. Then we have a unique map γ from the set $C(\varphi) \sqcup Y$ to the set $C(q\circ\varphi)$ such that the diagram

$$\begin{array}{ccc} (\partial\sigma_{i+1}\times I) \sqcup X_i \sqcup Y & \xrightarrow{\mathrm{id} \sqcup p} & (\partial\sigma_{i+1}\times I) \sqcup Y_i \\ \Big\downarrow{\scriptstyle \lambda \sqcup \mathrm{id}} & & \Big\downarrow{\scriptstyle \mu} \\ C(\varphi) \sqcup Y & \xrightarrow[\gamma]{} & C(q\circ\varphi) \end{array}$$

commutes. Since the other three maps are proper semialgebraic surjections we conclude that γ is also a proper semialgebraic surjection. Via this map γ, $C(q\circ\varphi)$ is the quotient $X_{i+1} \cup_f Y$ of $C(\varphi) \sqcup Y = X_{i+1} \sqcup Y$, as is easily checked. Thus our theorem is proved in the case that $X \setminus A$ contains only finitely many open simplices.

Assume now that $X \setminus A$ contains infinitely many open simplices. Let $(X_\lambda \mid \lambda\in\Lambda)$ be the family of all closed subcomplexes of X which contain A and contain only finitely many simplices outside of A. We order Λ in such a way that $\lambda \le \mu$ iff $X_\lambda \subset X_\mu$. Then $(X_\lambda \mid \lambda\in\Lambda)$ is a directed system of partially complete spaces. For every $\lambda \in \Lambda$ we have a proper quotient map

$$p_\lambda : X_\lambda \sqcup Y \to X_\lambda \cup_f Y =: M_\lambda .$$

If $\lambda \leq \mu$ then M_λ embeds into M_μ in a canonical way as a closed subspace. Thus $(M_\lambda | \lambda \in \Lambda)$ is again a directed system of partially complete spaces with closed embeddings as transition maps. It is now easily seen that the ringed space $M := \varinjlim_\lambda M_\lambda$ is a partially complete regular and paracompact locally semialgebraic space. For example, regard M as the union of the sets M_λ with $M_\lambda \subset M_\mu$ for $\lambda \leq \mu$, and apply Theorem 1.3 to the family consisting of the space Y and the spaces $p_\lambda(\bar{\sigma})$ with $\lambda \in \Lambda$, $\sigma \in \Sigma(X)$, $\sigma \subset X_\lambda$, $\sigma \not\subset X_\mu$ for $\mu < \lambda$, $\sigma \not\subset A$. The maps p_λ fit together to a proper surjective map $p : X \sqcup Y \to M$ (cf. I, Prop. 3.16), and by this map M is the desired quotient $X \cup_f Y$ of $X \sqcup Y$. This finishes the proof in the case that X is partially complete.

In order to prove the theorem in general we quote the following fact from [DK_5].

<u>Proposition 10.8</u> [DK_5, Prop. 2.2]. Let X be a locally finite simplicial complex and A a closed subcomplex of X. Let V denote the open star $St_{X'}(A)$ of A in the first barycentric subdivision X' of X. Then there exists a semialgebraic retraction map $r : V \to A$ such that, for every $x \in V \setminus A$, the open line segment $]x, r(x)[$ is contained in the same open simplex $S \subset V$ of X' as x.

This has been proved in [DK_5] in the case that the complex X is finite, by use of an explicit formula for r in terms of barycentric coordinates. The same formula and proof work if X is locally finite.

We return to the situation of Theorem 10.7. Choosing a simultaneous triangulation of X and A we learn from Proposition 10.8 that there exists an open locally semialgebraic neighbourhood U of A in X which admits a semialgebraic retraction r to A. (For this it suffices to know that Prop. 10.8 is true for strictly locally finite complexes). If we know that the proper quotient $U \cup_f Y$ exists, then we see by an application of Lemma 10.3 to the open subsets $X \setminus A$ and $U \sqcup Y$ of $X \sqcup Y$ that the proper quotient $X \cup_f Y$ exists. Thus we may replace X by U and henceforth assume that X admits a semialgebraic retraction R to A.

The function $g = f \circ r : X \to Y$ is semialgebraic (perhaps not proper) and extends $f : A \to Y$. We choose a completion $\overline{g} : \overline{X} \to \overline{Y}$ of g, cf. Proposition 5.1. We denote the closure of A in $\overline{X}$ by $\overline{A}$ and the restriction of $\overline{g}$ to $\overline{A}$ by $\overline{f} : \overline{A} \to \overline{Y}$. Now we forget the function g. The sole purpose of this function was to yield a completion $\overline{f} : \overline{A} \to \overline{Y}$ of f with $\overline{A}$ the closure of A in some completion $\overline{X}$ of X.

Since f is proper, we have $\overline{f}^{-1}(Y) = A$. As shown above the proper quotient map

$$p : \overline{X} \sqcup \overline{Y} \to \overline{X} \cup_{\overline{f}} \overline{Y}$$

exists. The locally semialgebraic subset $L := p(X \sqcup Y)$ of $\overline{X} \cup_{\overline{f}} \overline{Y}$ has the preimage $X \sqcup Y$ under p. Thus the restriction

$$p | X \sqcup Y : X \sqcup Y \to L$$

is also proper and shows L to be the quotient $X \cup_f Y$. This finishes the proof of Theorem 10.7.

§11 - Modification of pure ends

Definition 1. A shrinking (resp. an expansion) of a completed space $(\overline{A},A)$ is a shrinking morphism (cf. Def. 9 at the end of §9) $g : (\overline{A},A) \to (\overline{B},B)$ (resp. a shrinking morphism $h : (\overline{C},C) \to (\overline{A},A)$).

If $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$ are pure completions of a space M then there exists a third pure completion $\chi : M \hookrightarrow T$ which admits morphisms to φ and ψ. We know from Corollary 9.26 that, for any end $\lambda \in \varepsilon(M)$, we can go from the completed end $(\overline{P(\lambda)},P(\lambda))$ in P to the completed end $(\overline{T(\lambda)},T(\lambda))$ in T by an expension and then from $(\overline{T(\lambda)},T(\lambda))$ to $(\overline{Q(\lambda)},Q(\lambda))$ by a shrinking. The question arises: Which shrinkings and expansions of completed ends can be realized by morphisms between pure completions? For the shrinking problem we have a fully satisfactory answer.

Theorem 11.1. Let $\varphi : M \hookrightarrow P$ be a pure completion of M. Given a family of shrinkings $(g_\lambda \mid \lambda\in\varepsilon(M))$, $g_\lambda : (\overline{P(\lambda)},P(\lambda)) \to (\overline{B}_\lambda,B_\lambda)$, of the completed ends of M in P, there exists a morphism $f : P \xrightarrow[M]{} Q$ from the completion $\varphi : M \hookrightarrow P$ to another pure completion $\psi : M \hookrightarrow Q$ such that the shrinking morphisms $h_\lambda : (\overline{P(\lambda)},P(\lambda)) \to (\overline{Q(\lambda)},Q(\lambda))$ obtained from f are isomorphic to the g_λ under $(\overline{P(\lambda)},P(\lambda))$ {i.e. there exist isomorphisms $u_\lambda : (\overline{Q(\lambda)},Q(\lambda)) \xrightarrow{\sim} (\overline{B}_\lambda,B_\lambda)$ with $u_\lambda \circ h_\lambda = g_\lambda$}.

Proof. We regard M as a subspace of P and φ as the inclusion map. Let

$$A := P \setminus M = \bigsqcup (P(\lambda) \mid \lambda \in \varepsilon(M)).$$

Then

$$\overline{A} = \cup(\overline{P(\lambda)} \mid \lambda \in \varepsilon(M))$$

and the family $(\overline{P(\lambda)} \mid \lambda \in \varepsilon(M))$ is locally finite, since the family $(P(\lambda) \mid \lambda \in \varepsilon(M))$ is locally finite. Let $R(\lambda)$ denote the equivalence relation on $\overline{P(\lambda)}$ whose equivalence classes are the fibres of

$g_\lambda : \overline{P(\lambda)} \to \overline{B}_\lambda$. The set $\overline{P(\lambda)} \setminus P(\lambda)$ is a union of one-point equivalence classes. We conclude that, for $\lambda \neq \mu$, the set $\overline{P(\lambda)} \cap \overline{P(\mu)}$ is a union of one-point equivalence classes with respect to both relations $R(\lambda)$ and $R(\mu)$. Thus the equivalence relations $R(\lambda)$ fit together to an equivalence relation R on $\overline{A}$ to which Lemma 10.3 is applicable. By this lemma the proper quotient projection $h : \overline{A} \to \overline{A}/R =: L$ exists. For every $\lambda \in \varepsilon(M)$ we have a commutative triangle

$$\begin{tikzcd}
& \overline{P(\lambda)} \arrow[dl, "h_\lambda"'] \arrow[dr, "g_\lambda"] & \\
\overline{C}_\lambda \arrow[rr, "\sim", "u_\lambda"'] & & \overline{B}_\lambda
\end{tikzcd}$$

with $h_\lambda := h|\overline{P(\lambda)}$, $\overline{C}_\lambda := h(\overline{P(\lambda)})$, and u_λ an isomorphism. Since h is proper, the set $C_\lambda := h(P(\lambda))$ is locally semialgebraic, and the set $\overline{C}_\lambda$ is indeed the closure of C_λ in the partially complete space L. Also, by construction, $u_\lambda(C_\lambda) = B_\lambda$. Thus the proper projection h yields, for every λ, a morphism of completed spaces

$$h_\lambda : (\overline{P(\lambda)}, P(\lambda)) \to (\overline{C}_\lambda, C_\lambda),$$

which is isomorphic to g_λ under $(\overline{P(\lambda)}, P(\lambda))$. In particular, every h_λ is a shrinking morphism.

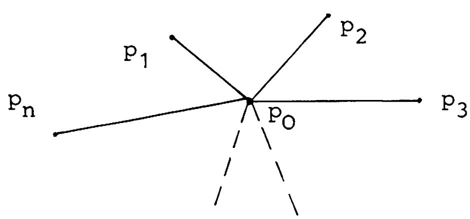
By the main result in the last section, Theorem 10.7, we have a proper projection map

$$p : P \sqcup L \to P \cup_h L =: Q$$

gluing the closed subspace $\overline{A}$ of P to L by the map h. {Here we use Theorem 10.7 only in the easy case where the space X in question is partially complete.} The restriction of p to P is still a proper surjection $f : P \to Q$. The set $N := f(M)$ is locally semialgebraic and dense in Q. Its preimage under f is M. Since all fibres $f^{-1}(y), y \in N$, are one-point sets, the restriction $f|M : M \to N$ is an isomorphism. Thus f yields a completion $\psi : M \to Q$ by restriction to M. Clearly f is a morphism from the completion $M \hookrightarrow P$ to ψ. By Proposition 8.10, ψ is pure. The shrinking maps of the morphism $f : P \xrightarrow[M]{} Q$ are the morphisms $h_\lambda : (\overline{P(\lambda)}, P(\lambda)) \to (\overline{C}_\lambda, C_\lambda)$ from above, which are isomorphic to the given morphisms $g_\lambda : (\overline{P(\lambda)}, P(\lambda)) \to (\overline{B}_\lambda, B_\lambda)$ under $(\overline{P(\lambda)}, P(\lambda))$. Theorem 11.1 is proved.

By Theorem 11.1 we may shrink the ends $P(\lambda)$ of M in a pure completion P in any way we want. In this way we can make them "simpler". For example, let $A = P(\lambda)$ be a semialgebraic end of M. We triangulate $(\overline{A}, A)$. Let K denote the maximal closed subcomplex of $\overline{A}$ contained in A with respect to the first barycentric subdivision of this triangulation. Then K is a strong deformation retract of A in the obvious semialgebraic sense, cf. Chapter III, §1 or [DK_3, p. 136]. In particular K is connected. Applying Theorem 11.1 to the projection $(\overline{A}, A) \to (\overline{A}/K, A/K)$ and the identity morphisms for the other ends of M, we shrink the end A to a contractible end A/K. If the end A was complete (i.e. $A = \overline{A}$), then the new end is actually a point, but otherwise there is no means for shrinking A to a point, since the complexity $c(A)$ has to be preserved (cf. Th. 9.25).

<u>Example 11.2.</u> Assume that $\dim M = 2$ and that the end $P(\lambda)$ of M in P is semialgebraic and not complete. Then $P(\lambda)$ is locally complete, since a priori $\dim P(\lambda) = 1$ and hence $c(P(\lambda)) \leq 1$. By the procedure just described, we can shrink $P(\lambda)$ in such a way that $(\overline{P(\lambda)}, P(\lambda))$ becomes isomorphic to the pair $(K_n, \dot{K}_n)$ with K_n the closed "star"



with $n \geq 1$ arms $[p_0,p_1],\dots,[p_0,p_n]$ and $\mathring{K}_n$ the open subcomplex where the points $p_1,p_2,\dots,p_n$ are removed. The number n is an invariant of λ, namely the number of ends of $P(\lambda)$ (cf. Th. 9.25).

We call an end $P(\lambda)$ of M in some completion P <u>pure</u> if the dense pair $(P,P\setminus P(\lambda))$ is pure. Clearly the completion is pure if and only if all ends are pure.

One may ask whether all stars $(K_n,\mathring{K}_n)$ occur as pure ends of 2-dimensional semialgebraic spaces. This is trivially true by the following general observation.

<u>Example 11.3.</u> Let $(\bar{A},A)$ be a completed space with A connected. Consider the completed space

$$(P,M) := (\bar{A}\times[0,1],\ (\bar{A}\times[0,1])\setminus(A\times\{0\})).$$

The pair (P,M) is pure and M has the unique end $A\times\{0\}$ in P. The completed end is isomorphic to $(\bar{A},A)$.

We now look for expansions of the ends of M in a pure completion $\varphi: M \hookrightarrow P$.

<u>Problem 11.4</u> (Expansion problem for ends). Given a family of expansions $(g_\lambda \mid \lambda \in \varepsilon(M))$,

$$g_\lambda : (\bar{C}_\lambda,C_\lambda) \to (\overline{P(\lambda)},P(\lambda)),$$

of the completed ends in P, does there exist a morphism $f: Q \xrightarrow[M]{} P$ from a pure completion $\psi: M \to Q$ to φ such that every shrinking morphism $h_\lambda : (\overline{Q(\lambda)},Q(\lambda)) \to (\overline{P(\lambda)},P(\lambda))$, obtained from f by restriction, is isomorphic to g_λ over $(\overline{P(\lambda)},P(\lambda))$ {i.e., such that there exists an isomorphism $v_\lambda : (\bar{C}_\lambda,C_\lambda) \xrightarrow{\sim} (\overline{Q(\lambda)},Q(\lambda))$ with $h_\lambda \circ v_\lambda = g_\lambda$}?

We call a morphism $f: Q \xrightarrow[M]{} P$ as described here a <u>solution of the expansion problem</u> for $\varphi: M \hookrightarrow P$ and the family $(g_\lambda \mid \lambda \in \varepsilon(M))$.

In order to study the expansion problem it suffices to consider spaces with only one end. This is a consequence of the following proposition.

<u>Proposition 11.5.</u> Let $\varphi: M \hookrightarrow P$ be a pure completion of the space M and let $(g_\lambda \mid \lambda \in \varepsilon(M))$ be a family of expansions of the completed ends of M in P, $g_\lambda : (\overline{C}_\lambda, C_\lambda) \to (\overline{P(\lambda)}, P(\lambda))$. We regard M as a subspace of P and φ as the inclusion map. The expansion problem is solvable for $M \hookrightarrow P$ and the family $(g_\lambda \mid \lambda \in \varepsilon(M))$ if and only if, for every $\lambda \in \varepsilon(M)$, the expansion problem is solvable for $P \smallsetminus P(\lambda) \hookrightarrow P$ and the expansion g_λ of the unique completed end of $P \smallsetminus P(\lambda)$ in P.

<u>Proof.</u> Let $f: Q \xrightarrow[M]{} P$ be a solution of the expansion problem for $M \hookrightarrow P$ and the family $(g_\lambda \mid \lambda \in \varepsilon(M))$. Fix an absolute end $\lambda \in \varepsilon(M)$. According to Theorem 11.1 we may shrink all ends $Q(\mu)$, $\mu \neq \lambda$, back to $P(\mu)$ leaving $Q(\lambda)$ unchanged, using the identity as shrinking map for $(\overline{Q(\lambda)}, Q(\lambda))$ and the restrictions of f as shrinking maps for $(\overline{Q(\mu)}, Q(\mu))$, $\mu \neq \lambda$. We then obtain a completion of $P \smallsetminus P(\lambda)$ which expands the unique end $P(\lambda)$ in P in the prescribed way. Assume conversely that, for every $\lambda \in \varepsilon(M)$, a proper surjective map $p_\lambda : Q^{(\lambda)} \to P$ is given which yields an isomorphism from $Q^{(\lambda)} \smallsetminus p_\lambda^{-1}(P(\lambda))$ to $P \smallsetminus P(\lambda)$ and whose restriction $p_\lambda^{-1}(\overline{P(\lambda)}) \to \overline{P(\lambda)}$ is isomorphic over $\overline{P(\lambda)}$ to the given map $g_\lambda : \overline{C}_\lambda \to \overline{P(\lambda)}$. Assume also that $Q^{(\lambda)} \smallsetminus p_\lambda^{-1}(P(\lambda)) \to Q^{(\lambda)}$ is a pure completion. For every finite set $S \subset \varepsilon(M)$ we introduce the fibre product $Q^{(S)}$ of the spaces $Q^{(\lambda)}$, $\lambda \in S$, over P which respect to the maps p_λ. The natural map $p_S : Q^{(S)} \to P$ is again a proper surjection. Moreover, p_S yields an isomorphism from $p_S^{-1}(P \smallsetminus \cup(P_\lambda \mid \lambda \in S))$ onto $P \smallsetminus \cup(P_\lambda \mid \lambda \in S)$. The inclusion map from $p_S^{-1}(P \smallsetminus \cup(P_\lambda \mid \lambda \in S))$ to $Q^{(S)}$ is a pure completion, as is easily seen. Over each $\overline{P}_\lambda$, $\lambda \in S$, p_S is isomorphic to p_λ, hence to g_λ. For finite subsets $S \subset T$ of $\varepsilon(M)$ we have a unique proper surjection $p_{T,S} : Q^{(T)} \to Q^{(S)}$ with $p_S \circ p_{T,S} = p_T$. We consider

the set (!)

$$Q := \varprojlim_{S} Q^{(S)},$$

with S running through the finite subsets of $\varepsilon(M)$. We want to equip Q with the structure of a partially complete space such that the natural surjections $q_S : Q \to Q^{(S)}$ are all proper.

This can be done easily. Let $q = q_\emptyset$ be the natural surjection from Q to P. We choose a simultaneous triangulation of P and M. Then every $\overline{P(\lambda)}$ is a closed subcomplex of P. For every closed simplex $\bar{\sigma}$ in P, the preimage $q^{-1}(\bar{\sigma})$ in Q is mapped bijectively to $p_S^{-1}(\bar{\sigma})$ by q_S for S large enough. We transfer the space structure of the complete semialgebraic space $p_S^{-1}(\bar{\sigma})$ to the set $q^{-1}(\bar{\sigma})$ by use of q_S^{-1}. For different simplices σ the complete spaces $q^{-1}(\bar{\sigma})$ fit together well. Applying Theorem 1.3, we obtain the structure of a partially complete space on Q such that every space $q^{-1}(\bar{\sigma})$, with its given space structure, is a subspace of Q, and the family of all $q^{-1}(\bar{\sigma})$ is locally finite. With this structure, the map $q : Q \to P$ is proper semialgebraic. It yields an isomorphism from $q^{-1}(M)$ to M and solves the expansion problem for the given family $(g_\lambda \mid \lambda \in \varepsilon(M))$. q.e.d.

The expansion problem seems to be much more difficult than the shrinking problem. With our present methods we can do almost nothing. Nevertheless, the discussion up to now makes it clear that, for all questions about modifications of ends, we can deal with each end of M separately. Thus, for the rest of this section, we usually assume that M has a single absolute end, and we speak of "the" end of M in a given completion.

The end A of M in P can be simplified by shrinking, but usually the way in which A is embedded in P, i.e. the relation of A to M, will become more complicated. Thus, for many purposes, it seems to be important to look for an expansion of A which makes the embedding as simple as possible. We focus attention on the case where the end A is locally complete, which means that M has complexity ≤ 2. Then $\partial A = \bar{A} \setminus A$ is closed in P. Let $\tilde{P} := P \setminus \partial A$.

<u>Definition 2.</u> The end A of M in P is called <u>collared</u> if there exists an open locally semialgebraic neighbourhood U of A in $\tilde{P}$ together with an isomorphism $\alpha : A \times [0,1] \xrightarrow{\sim} \bar{U} \cap \tilde{P}$ such that $\alpha(A \times [0,1[) = U$ and $\alpha(a,0) = a$ for every $a \in A$.

Of course, a collared end is pure and locally complete, hence has complexity ≤ 1.

<u>Theorem 11.6.</u> Assume that M has complexity $c(M) \leq 2$. Then, for any completion $M \hookrightarrow P$, there exists a completion $M \hookrightarrow Q$ and a proper morphism $f : Q \xrightarrow[M]{} P$ such that the end of M in Q is collared.

To prove this we need the following general fact which has been proved for locally complete semialgebraic spaces in [DK_5, §3]. The proof goes through for arbitrary locally complete (regular, paracompact) spaces without any serious modification.

<u>Theorem 11.7</u> [DK_5, Th. 2]. Let N be a locally complete space and B a closed locally semialgebraic subset of N. Then there exists an open locally semialgebraic neighbourhood U of B in N, a proper semialgebraic map g from $\partial U := \bar{U} \setminus U$ to B, and an isomorphism $\alpha : Z(g) \xrightarrow{\sim} \bar{U}$ from the mapping cylinder of g to $\bar{U}$ with $\alpha(u,0) = u$ for $u \in \partial U$ and $\alpha(b) = b$ for $b \in B$. (We regard $\partial U \times [0,1[$ and B as subspaces of $Z(g)$, as usual.)

The proof of Theorem 11.7 in [DK_5] gives an explicit description of U, g, and α. Choosing a simultaneous triangulation of N and B one takes for U the star of B in N with respect to the second barycentric subdivision of the triangulation. For g one takes the restriction to ∂U of the retraction r constructed in the proof of Proposition 10.8, cf. [DK_5, §2]. For α one takes the map described by the formulas

$$\alpha(p(u,t)) = (1-t)u + t\,r(u) \qquad (u \in \partial U, t \in [0,1]).$$

$$\alpha(b) = b \qquad (b \in B)$$

with $p : \partial U \times [0,1] \to Z(g)$ the canonical projection.

We shall use Theorem 11.7 only in the case where N is partially complete. We first prove Theorem 11.6 in the special case that the end $A := P \setminus M$ of M in P is partially complete. Applying Theorem 11.7 to the space $N := P$ and the subset $B := A$, we obtain a new completion Q of M by gluing $P \setminus U$ to $(\partial U) \times [0,1]$ along ∂U with the obvious isomorphism $\partial U \xrightarrow{\sim} \partial U \times \{0\}$ (cf. Theorem 1.3 or Theorem 10.7). Then gluing the identity of $P \setminus U$ with the map $\alpha \circ p : \partial U \times [0,1] \to \overline{U}$, where p is the projection from $\partial U \times [0,1]$ to $Z(g)$, we obtain a proper map $f : Q \to P$ (cf. I, Prop. 3.16). f is an morphism from the completion Q of M to the completion P of M, and the end of M in Q is collared.

Assume now that the end $A = P \setminus M$ of M in P is only locally complete. The set $\partial A = \overline{A} \setminus A$ is closed in P and contained in M. We consider the open subspace $L := M \setminus \partial A$ of M. The space P is a completion of L, and L has the end $\overline{A}$ in P. We know that the completion $L \hookrightarrow P$ of L is dominated by a completion $L \hookrightarrow Q'$ such that the end of L in Q' is collared. Let $f : Q' \xrightarrow[L]{} P$ be the morphism from the second completion of L to the first one. Let C denote the preimage of ∂A in Q' and let $h : C \twoheadrightarrow \partial A$ denote the restriction of f to C. Then

$$Q := Q' \cup_h \partial A$$

is a completion of M which dominates the completion P of M. The end of M in Q is collared, as is easily seen, and Theorem 11.6 is proved.

We shall prove in the next chapter (III, §8) that a collared end A of M is, up to homotopy equivalence, independent of the chosen completion P of M. Also, if M is an (open) locally semialgebraic manifold, then every collared end A of M is again a (closed) locally semialgebraic manifold. Thus M has a completion which is a locally semialgebraic manifold with boundary. We hope to explain this in the second volume of the lecture notes (cf. [SY, Th. 6.2] for $R = \mathbb{R}$ and M semialgebraic).

§12 - The Stein factorization of a semialgebraic map

We start with a proper (hence semialgebraic) map $f: M \to N$ between spaces. Let $\mathcal{R}(f)$ denote the equivalence relation whose equivalence classes are the connected components of the non empty fibres of f. We ask whether the proper quotient $M/\mathcal{R}(f)$ exists. Assume this is the case and $\varphi: M \to M/\mathcal{R}(f)$ is the quotient projection. Since f maps every equivalence class of $\mathcal{R}(f)$ to a point we have a factorization

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[r, "\varphi"] & M/\mathcal{R}(f) \arrow[dl, "\pi"] \\ N & \end{tikzcd} \qquad (*)$$

of f into proper maps which we call a <u>Stein factorization</u> of f (in analogy to the complex analytic Stein factorization). We shall often speak of "the" Stein factorization of f, but actually the quotient space $M/\mathcal{R}(f)$ is only determined up to isomorphism.

The second factor π in the Stein factorization (*) of f has finite fibres and is proper. Thus π is a finite map. The fibres of the first factor φ are connected, and φ is surjective. Assume that

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[r, "\psi"] & L \arrow[dl, "\rho"] \\ N & \end{tikzcd} \qquad (**)$$

is any factorization of f with ρ finite, ψ surjective, and the fibres of ψ connected. Then the fibres of ψ are the connected components of the fibres of f, and ψ is proper. Thus the map ψ shows L to be the quotient of M by $\mathcal{R}(f)$ and, up to isomorphism, the factorizations (*) and (**) are the same.

<u>Theorem 12.1.</u> Every proper map $f: M \to N$ between spaces has a Stein factorization.

Proof. The set $f(M)$ is closed and locally semialgebraic in N. Suppose we have found a Stein factorization $g = \rho\circ\varphi$ of the map $g: M \to f(M)$ obtained from f by restriction of the range space. Then $f = \pi\circ\varphi$, with π the composite of ρ and the inclusion $f(M) \to N$, is a Stein factorization of f. Thus, in order to prove the theorem, we may assume that f is surjective. Let

$$p: (M\times[0,1]) \sqcup N \to Z(f)$$

be the quotient projection onto the mapping cylinder of f. As usual, we identify the subspaces $M\times[0,1[$ and N of $(M\times[0,1])\sqcup N$ with their images under p, thus regarding them also as subspaces of $Z(f)$. Since f is surjective, the restriction $g: M\times[0,1] \to Z(f)$ of p is also surjective. Moreover g is proper. We now consider the dense pairs $(M\times[0,1], M\times[0,1[)$ and $(Z(f), M\times[0,1[)$. We may regard g as a morphism from the first pair to the second. Consequently we have a unique factorization of g

$$(***)\qquad \begin{tikzcd} M\times[0,1] \arrow[d, "g"'] \arrow[r, "\tilde{g}"] & \widetilde{Z(f)} \arrow[dl, "h"] \\ Z(f) & \end{tikzcd}$$

with h the pure hull of the dense pair $(Z(f), M\times[0,1[)$ (Prop. 8.3) The map h is finite and the map $\tilde{g}$ has connected fibres (Prop. 8.4). Moreover $\tilde{g}$ is surjective and proper. Let $\tilde{N} := h^{-1}(N)$. We have $g^{-1}(N) = \tilde{g}^{-1}(\tilde{N}) = M\times\{1\}$. Identifying $M\times\{1\}$ with M in the obvious way and restricting the diagram (***) to $M\times\{1\}$ and its images $\tilde{N}$ and N we obtain a Stein factorization of f. q.e.d.

As a consequence of this proof we have a side remark on mapping cylinders.

Corollary 12.2. Let $f: M \to N$ be a proper surjective map between spaces.

The dense pair $(Z(f), M\times[0,1[)$ is pure if and only if the fibres of f are connected.

The Stein factorization can also be characterized as a "maximal" factorization with the second factor a finite map.

Proposition 12.3. Assume $f : M \to N$ is a proper map and

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[r, "\varphi"] & S \arrow[dl, "\pi"] \\ N & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "f"'] \arrow[r, "\psi"] & T \arrow[dl, "\rho"] \\ N & \end{tikzcd}$$

are two factorizations of f into locally semialgebraic maps with π and ρ finite. Assume also that φ is surjective and has connected fibres. {i.e. the first factorization is Stein. Notice that φ and ψ are automatically proper.} Then there exists a unique locally semialgebraic map $h : S \to T$, such that $\rho\circ h = \pi$ and $h\circ\varphi = \psi$. {N.B. h is proper, hence semialgebraic

Proof. It is evident that the fibres of ψ are unions of connected components of the fibres of f. Thus there exists a unique locally semialgebraic map $h : S \to T$ with $h\circ\varphi = \psi$. Composing with ρ we obtain

$$\rho\circ h\circ\varphi = \rho\circ\psi = \pi\circ\varphi.$$

Since φ is surjective this implies $\rho\circ h = \pi$. q.e.d.

We seek a generalization of the Stein factorization of proper maps for an arbitrary semialgebraic map $f : M \to N$, in the sense that Proposition 12.3 remains true. It will turn out that such a factorization $f = \pi\circ\varphi$ does exist, and that the first factor φ is a "pure" semialgebraic map. We first explain what we mean by a pure map and state some easy observations about these maps.

Definition 1. A semialgebraic map $f : M \to N$ is called pure at a point

$y \in N$, if y has a fundamental system of open semialgebraic neighbourhoods U such that every preimage $f^{-1}(U)$ is non-empty and connected. The map f is called <u>pure</u> if f is pure at every point of N.

A pure map has a dense image.

<u>Example 12.4.</u> Let $f: M \to N$ be an embedding, i.e. an isomorphism from M onto a subspace of N. Then f is pure if and only if $f(M)$ is dense in N and the dense pair $(N, f(M))$ is pure in the sense of §8.

<u>Proposition 12.5.</u> For every semialgebraic map $f: M \to N$ the following are equivalent.

i) f is pure.

ii) For every non-empty connected open semialgebraic subset U of N the preimage $f^{-1}(U)$ is non-empty and connected.

iii) For every non-empty connected open locally semialgebraic subset U of N the preimage $f^{-1}(U)$ is non-empty and connected.

<u>Proof.</u> The implications iii) $\Rightarrow$ ii) $\Rightarrow$ i) are trivial. (Recall that every point of a locally semialgebraic space has a fundamental system of open connected semialgebraic neighbourhoods.) In order to prove i) $\Rightarrow$ iii), let a non-empty connected set $U \in \mathring{\mathfrak{T}}(N)$ be given. Certainly $f^{-1}(U) \neq \emptyset$, since f has dense image. Suppose $f^{-1}(U)$ is the disjoint union of two open non-empty locally semialgebraic subsets V_1, V_2. Then $f(f^{-1}(U)) = U \cap f(M)$ is the union of the two subsets $f(V_1), f(V_2)$. They are locally semialgebraic since f is semialgebraic. Since $U \cap f(M)$ is dense in U, we have

$$U = (U \cap \overline{f(V_1)}) \cup (U \cap \overline{f(V_2)}) .$$

Since U is connected, the sets $U \cap \overline{f(V_1)}$ and $U \cap \overline{f(V_2)}$ must meet in some point y. Let W be any open semialgebraic neighbourhood of y with $W \subset U$.

Then

$$W \cap f(M) = [W \cap f(V_1)] \cup [W \cap f(V_2)]$$

and neither set $W \cap f(V_1)$ nor $W \cap f(V_2)$ is empty. Taking preimages under f we obtain

$$f^{-1}(W) = [f^{-1}(W) \cap V_1] \cup [f^{-1}(W) \cap V_2].$$

The two sets on the right-hand side are open, semialgebraic, non-empty and disjoint. Thus $f^{-1}(W)$ is not connected, and we see that f is not pure at y. This contradiction proves that $f^{-1}(U)$ is connected.

q.e.d.

<u>Proposition 12.6.</u> A proper map $f: M \to N$ is pure if and only if $f^{-1}(y)$ is connected and non-empty for every $y \in N$.

<u>Proof.</u> If f is pure, then f has a dense image, hence f is surjective. Assume that a fibre $f^{-1}(y)$ is the union of two non-empty disjoint open semialgebraic subsets V_1, V_2. We may choose disjoint open semialgebraic subsets U_1, U_2 of M with $U_i \cap f^{-1}(y) = V_i$ $(i = 1,2)$. Since f is proper, there is an open connected semialgebraic neighbourhood U of y in N such that $f^{-1}(U)$ is contained in $U_1 \cup U_2$. Thus $f^{-1}(U)$ is not connected, in contradiction to Prop. 12.5.ii.

Conversely assume that $f^{-1}(y)$ is connected and non-empty for every $y \in N$. Then the preimage $f^{-1}(U)$ of any connected open locally semialgebraic subset U of N is connected by Sublemma 6.6.

<u>Lemma 12.7.</u> Let $f: M \to S$, $g: S \to N$ be semialgebraic maps.

i) If f and g are pure then $g \circ f$ is pure.

ii) If $g \circ f$ is pure and f has a dense image then g is pure.

iii) If $g \circ f$ is pure and g is an embedding, then f is pure.

Proof. i) The first statement is evident.

ii) Let U be a connected open semialgebraic set in N. The set $f^{-1}(g^{-1}(U))$ is connected, hence its image $g^{-1}(U) \cap f(M)$ under f is connected. Since $f(M)$ is dense, the closure of this set contains $g^{-1}(U)$. Thus $g^{-1}(U)$ is also connected.

iii) We regard g as an inclusion map. Let U be an open connected semialgebraic set in S. We choose an open semialgebraic set W in N with $W \cap S = U$ (cf. I, Prop. 3.14). Let V be the connected component of W containing U. Then also $V \cap S = U$. We have

$$f^{-1}(U) = (g \circ f)^{-1}(V).$$

Since $g \circ f$ is pure this set is connected. q.e.d.

Definition 2. A pure completion of a locally semialgebraic map $f : M \to N$ is a commutative diagram

$$\begin{tikzcd} M \arrow[r, hook, "\alpha"] \arrow[d, "f"'] & P \arrow[d, "h"] \\ N \arrow[r, hook, "\beta"'] & Q \end{tikzcd} \qquad (*)$$

with α and β pure completions of M and N.

Remark 12.8. Every locally semialgebraic map f has a pure completion (*) with prescribed pure completion β of N. If f is semialgebraic, then h is proper.

Indeed, given a pure completion β of N, we obtain a diagram (*) with α a completion of M by applying Proposition 5.1. Then we replace α by a pure hull $\tilde{\alpha} : M \hookrightarrow \tilde{P}$ of α and h by the composite $h \circ p$ with p the finite projection map from $\tilde{P}$ to P. The second sentence in the remark is evident from Lemma 8.1.

Lemma 12.7 has the following immediate consequence which gives us an understanding of purity of a semialgebraic map f in terms of a pure completion of f.

Proposition 12.9. Let

$$\begin{tikzcd} M \arrow[r, hook, "\alpha"] \arrow[d, "f"'] & P \arrow[d, "h"] \\ N \arrow[r, hook, "\beta"'] & Q \end{tikzcd} \qquad (*)$$

be a commutative diagram of semialgebraic maps with α and β dense embeddings.

i) If f and β are pure then h is pure.

ii) If h and α are pure then f is pure.

iii) In particular, if (*) is a pure completion of f, then f is pure if and only if h has connected fibres.

In the following we do not work with completions but with "proper extensions" of semialgebraic maps.

Definition 3. A proper extension of a semialgebraic map $f: M \to N$ is a commutative triangle

$$\begin{tikzcd} M \arrow[rr, hook, "\alpha"] \arrow[dr, "f"'] & & P \arrow[dl, "g"] \\ & N & \end{tikzcd} \qquad (**)$$

with α a dense embedding and g a proper map.

Remark 12.10. Every semialgebraic map $f: M \to N$ has a proper extension (**) with α pure. We obtain every such proper extension from a pure completion (*) of f by restricting h to the preimage of N under h (regarding α and β as inclusion maps).

Definition 4. A Stein factorization of a semialgebraic map $f: M \to N$ is a commutative triangle

$$\begin{tikzcd} M \arrow[dr, "\varphi"] \arrow[dd, "f"'] & \\ & S \arrow[dl, "\pi"] \\ N & \end{tikzcd} \qquad (***)$$

with φ pure and π finite.

Theorem 12.11. i) Every semialgebraic map $f: M \to N$ has a Stein factorization.

ii) If (***) is a Stein factorization and

$$\begin{tikzcd} M \arrow[dr, "\varphi_1"] \arrow[dd, "f"'] & \\ & S_1 \arrow[dl, "\pi_1"] \\ N & \end{tikzcd}$$

is any factorization of f with π_1 finite, then there exists a unique semialgebraic map $h: S \to S_1$ with $h \circ \varphi = \varphi_1$ and $\pi_1 \circ h = \pi$.

Proof. i) We choose a proper extension (**) of f with pure embedding α and a Stein factorization $g = \pi \circ \psi$ of g.

$$\begin{tikzcd} M \arrow[r, hook, "\alpha"] \arrow[dr, "f"'] & P \arrow[d, "g"] \arrow[r, "\psi"] & S \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

Then $\varphi := \psi \circ \alpha$ is a pure map and $f = \pi \circ \varphi$ is a Stein factorization of f.

ii) We choose a proper extension of the map $(\varphi, \varphi_1): M \to S \times S_1$.

$$\begin{tikzcd} M \arrow[rr, hook, "\alpha"] \arrow[dr, "{(\varphi,\varphi_1)}"'] & & P \arrow[dl, "{(\psi,\psi_1)}"] \\ & S \times S_1 & \end{tikzcd}$$

We then have a commutative diagram of maps

$$\begin{tikzcd}
 & P \arrow[dl, "\psi"'] \arrow[dr, "\psi_1"] & \\
S \arrow[dr, "\pi"'] & M \arrow[u, "\alpha"'] \arrow[l, "\varphi"] \arrow[r, "\varphi_1"'] \arrow[d, "f"] & S_1 \arrow[dl, "\pi_1"] \\
 & N &
\end{tikzcd}$$

with ψ pure (cf. Lemma 12.7.ii) and π and π_1 finite. Since α has dense image, $\pi \circ \psi = \pi_1 \circ \psi_1$. By Proposition 12.3 there exists a unique semialgebraic map $h : S \to S_1$ with $\pi_1 \circ h = \pi$ and $h \circ \psi = \psi_1$. Hence $h \circ \varphi = \varphi_1$. q.e.d.

According to the second part of this theorem any two Stein factorizations of a semialgebraic map f are isomorphic. Thus we are allowed to talk about "the" Stein factorization of f.

<u>Corollary 12.12</u> (Comparison of Stein factorizations). Let

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "f"'] & P \arrow[d, "g"] \\
N \arrow[r, "\beta"'] & Q
\end{tikzcd}$$

be a commutative square with α, β locally semialgebraic maps and f, g semialgebraic maps. Let

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "f"'] & S \arrow[dl, "\pi"] \\
N &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "g"'] & T \arrow[dl, "\chi"] \\
Q &
\end{tikzcd}$$

be the Stein factorizations of f and g. Then there exists a unique locally semialgebraic map $\gamma : S \to T$ with $\chi \circ \gamma = \beta \circ \pi$ and $\psi \circ \alpha = \gamma \circ \varphi$.

This can be easily seen by applying the last theorem to the Stein factorization of f and the finite map $N \times_Q T \to N$ obtained by pulling back $\chi : T \to Q$ along β.

We give an application of Corollary 12.12.

Theorem 12.13. Let

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "f"'] & P \arrow[d, "g"] \\ N \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & Q \end{tikzcd}$$

be a commutative square (solid arrows) with f a pure semialgebraic map, g a partially finite map, and α,β locally semialgebraic maps. Then there exists a unique locally semialgebraic map $\gamma : N \to P$ with $\gamma \circ f = \alpha$ and $g \circ \gamma = \beta$.

Proof. If M is semialgebraic then, replacing P by the semialgebraic subspace $\overline{\alpha(M)}$, we may assume that g is finite. In this case our claim is contained in Corollary 12.12. In the general case we choose an admissible covering $(N_i | i \in I)$ of N by open semialgebraic subsets. For every $i \in I$ the set $f^{-1}(N_i)$ is semialgebraic and the restriction $f_i : f^{-1}(N_i) \to N_i$ of f is again pure. Thus we have a unique semialgebraic map $\gamma_i : N_i \to P$ with $g \circ \gamma_i = \beta | N_i$ and $\gamma_i \circ f_i = \alpha | f^{-1}(N_i)$. By the uniqueness statement in Corollary 12.12 we have $\gamma_i | N_i \cap N_j = \gamma_j | N_i \cap N_j$. Thus the γ_i fit together to a locally semialgebraic map $\gamma : N \to P$ with the desired properties.

q.e.d.

In the special case where f is the inclusion map $L \times [0,1[\hookrightarrow L \times [0,1]$ for some space L this theorem means a relative path completion theorem with parameters for partially finite maps (cf. I, Th. 6.8 for relative path completion without parameters). It would be interesting to know something about relative path completion with parameters more generally for partially proper maps.

Using the Stein factorization $f = \pi \circ \varphi$ of a given semialgebraic map $f : M \to N$ we get a hold on the set of all points $y \in N$ with f pure at y (cf. Definition 1).

Theorem 12.14. For any $y \in N$, the map f is pure at y if and only if $\pi^{-1}(y)$ consists of one point. The set of these points y is locally semialgebraic.

Proof. Since φ is pure, f is pure at y if and only if π is pure at y. Thus we may replace f by π in the whole consideration and assume henceforth that f is finite. If $f^{-1}(y)$ is empty then there exists a neighbourhood of y which is disjoint from the closed set $f(M)$. Thus f is not pure at y. If $f^{-1}(y)$ contains several points $x_1,\dots,x_r$ then, by standard arguments, there exists an open semialgebraic neighbourhood U of y such that $f^{-1}(U)$ is the disjoint union of open semialgebraic neighbourhoods $V_1,\dots,V_r$ of $x_1,\dots,x_r$ respectively. The preimage $f^{-1}(U')$ of every semialgebraic neighbourhood $U' \subset U$ of y is disconnected. Thus f is not pure at y.

Assume now that $f^{-1}(y) = \{x\}$. There exists a triangulation of f such that y is a vertex of N, hence x is a vertex of M (Th. 6.13; we identify M and N with the corresponding simplicial complexes). Now f is a simplicial finite map, hence maps every open simplex of M isomorphically onto an open simplex of N. For purely combinatorial reasons the preimage of the open star U of y in N is the open star V of x in M. It then follows that for every $\lambda \in]0,1[$ the preimage of the set $(1-\lambda)U+\lambda\{y\}$ is $(1-\lambda)V+\lambda\{x\}$, and we see that f is pure at y. The last sentence in this theorem is now evident from Hardt's theorem (Th. 6.3). q.e.d.

We now want to describe the fibres of the pure factor φ of the Stein factorization $f = \pi\circ\varphi$ of a given semialgebraic map $f: M \to N$. Let $\mathrm{Con}(f)$ denote the set of all connected components of the preimages $f^{-1}(U)$ of all open semialgebraic subsets U of N with $U \cap f(M) \neq \emptyset$.

Lemma 12.15. Let

$$\begin{tikzcd} M \arrow[rr, "\chi"] \arrow[dr, "f"'] & & P \arrow[dl, "g"] \\ & N & \end{tikzcd}$$

be a commutative diagram of semialgebraic maps with χ pure. Then

$$\mathrm{Con}(f) = \{\chi^{-1}(W) \mid W \in \mathrm{Con}(g)\}.$$

Proof. For a given $U \in \dot{\gamma}(N)$, the preimage $f^{-1}(U) = \chi^{-1}g^{-1}(U)$ is not empty if and only if $g^{-1}(U)$ is not empty. Assume that we are in this case and let $W_1,\dots,W_r$ be the connected components of $g^{-1}(U)$. Then $f^{-1}(U)$ is the disjoint union of the non empty open sets $\chi^{-1}(W_1),\dots,\chi^{-1}(W_r)$, and these are connected (Prop. 12.5). Thus the $\chi^{-1}(W_i)$ are the connected components of $f^{-1}(U)$. This gives us the claim in the lemma.

As before, let $f : M \to N$ be a semialgebraic map. For any $x \in M$ we denote by $C(f,x)$ the connected component of x in the fibre $f^{-1}f(x)$ through x. We further denote by $D(f,x)$ the intersection of all sets $V \in \mathrm{Con}(f)$ which contain x. We have

$$C(f,x) \subset D(f,x) \subset f^{-1}f(x).$$

Lemma 12.15 has the following immediate consequence.

Lemma 12.16. In the situation of Lemma 12.15,

$$D(f,x) = \chi^{-1}(D(g,\chi(x)))$$

for every $x \in M$.

If f is proper then it is pretty obvious that $C(f,x) = D(f,x)$ for every $x \in M$. In particular, if f is finite then $D(f,x) = \{x\}$ for every $x \in M$. Applying this remark to the finite factor π of a Stein factorization $f = \pi \circ \varphi$ of an arbitrary semialgebraic map f we obtain the desired description of the fibres of φ from Lemma 12.16.

<u>Theorem 12.17.</u> For every $x \in M$ the fibre $\varphi^{-1}\varphi(x)$ through x of the pure factor φ of a Stein factorization of f coincides with the set $D(f,x)$.

<u>N.B.</u> In particular, the sets $D(f,x)$ are unions of components of $f^{-1}f(x)$.

Another application of Lemma 12.16 gives us

<u>Proposition 12.18.</u> Let

$$\begin{tikzcd} M \arrow[rr, hook, "\alpha"] \arrow[dr, "f"'] & & P \arrow[dl, "g"] \\ & N & \end{tikzcd}$$

be a proper extension of f with α pure. We regard α as an inclusion map. Then, for every $x \in M$

$$D(f,x) = C(g,x) \cap M.$$

It may well happen that $D(f,x) \neq C(f,x)$, if f is not proper. For example let $g : P \to N$ be a proper map with connected fibres and let $f : M \to N$ be the restriction of g to a pure dense locally semialgebraic subset M of P. Then, by Proposition 12.18, the sets $D(f,x)$ are just the fibres of f. But some fibres may be not connected.

Most of the results of this section have well known analogues in topology, cf. [M], [MV].

§13 - Semialgebraic spreads

We now make a more detailed study of the case where the pure factor of the Stein factorization of a given semialgebraic map f is an embedding (i.e. an isomorphism onto its image).

Definition 1. A locally semialgebraic map $f : M \to N$ is called a spread if the following holds for every $x \in M$: If U runs through the open semialgebraic neighbourhoods of $y := f(x)$ then the connected component of x in $f^{-1}(U)$, which we denote by $f^{-1}(U)_x$, runs through a fundamental system of neighbourhoods of x. Of course, a spread has discrete fibres.

We refer the reader to [F], [M], [M_1] for the theory of spreads in topology.

Examples 13.1. i) Every finite map is a spread.

ii) Every embedding is a spread.

iii) The composite $g \circ f$ of two spreads $f : M \to N$ and $g : N \to L$ is a spread. This is obvious, since for any point $x \in M$ and any open semialgebraic neighbourhood U of $g \circ f(x)$ we have, with $y := f(x)$,

$$(*) \qquad (g \circ f)^{-1}(U)_x = f^{-1}(g^{-1}(U)_y)_x.$$

iv) A map $f : M \to N$ between spaces is called a local isomorphism if every $x \in M$ has an open semialgebraic neighbourhood V such that f(V) is open and semialgebraic in N and f yields an isomorphism from V onto f(V). {Notice that here we use the word "local" in a weak sense, but cf. Theorem 13.8 below.} Clearly every local isomorphism is a spread. In particular, coverings, to be discussed in Chapter V , are spreads. Also, for every locally complete space M, the identity map from M_{loc} to M (cf. I, §7) is a local isomorphism and hence a spread.

v) More generally let $f : M \to N$ be a map between spaces. Assume that every $x \in M$ has a neighbourhood U in M, such that the restriction

$f|U: U \to N$ is a spread. Then f is a spread.

To prove this fact, let $x \in M$ and U be a neighbourhood of x in M such that $f|U$ is a spread. Since M is regular, the set $\mathcal{W} := \{V \in \mathring{\mathfrak{r}}(M) \mid x \in V, \overline{V} \subset U\}$ is a fundamental system of neighbourhoods of x in M. For every $V \in \mathcal{W}$ we find an open semialgebraic neighbourhood W of $f(x)$ such that $(f|U)^{-1}(W)_x \subset V$. Since $\overline{V} \subset U$, the set $(f|U)^{-1}(W)_x$ is not only open and closed in $U \cap f^{-1}(W)$, but also in $f^{-1}(W)$. Hence $f^{-1}(W)_x = (f|U)^{-1}(W)_x$. We see that f is indeed a spread.

vi) A partially finite map between spaces is a spread. This follows immediately from i) and v).

vii) If M is locally complete, then every map $f: M \to N$ with discrete fibres is a spread. Indeed, every $x \in M$ has a complete semialgebraic neighbourhood K. The map $f|K: K \to N$ is proper and has finite fibres, hence is finite. Thus $f|K$ is a spread. By v) f is also a spread.

Let $f: X \to Y$ be a simplicial map between locally finite complexes X, Y. Then f is a spread if and only if $f|\sigma$ is injective for every open simplex σ of X. Indeed, this condition is certainly necessary since a spread has discrete fibres. Assume now that $f|\sigma$ is injective for every $\sigma \in \Sigma(X)$. As always, we denote the completion of f to a simplicial map from $\overline{X}$ to $\overline{Y}$ by $\overline{f}$. Clearly $\overline{f}|\overline{\sigma}$ is injective for every $\sigma \in \Sigma(\overline{X})$. Let $x \in X$ be given and $y := f(x)$. It is pretty obvious that $f^{-1}(\mathrm{St}_Y(y))$ is the disjoint union of the stars $\mathrm{St}_X(z)$ with z running through $\overline{f}^{-1}(y)$ (cf. beginning of the proof of Prop. 7.6). Thus $\mathrm{St}_X(x)$ is the connected component of $f^{-1}(\mathrm{St}_Y(y))$ containing x. This implies that, for every $\lambda \in]0,1[$ the "shrunken star" $(1-\lambda)x + \lambda \mathrm{St}_X(x)$ is a connected component of $f^{-1}((1-\lambda)y + \lambda \mathrm{St}_Y(y))$. We see that f is a spread.

In the following we focus attention on semialgebraic spreads, i.e. spreads which are semialgebraic maps. But notice that $f: M \to N$ is a spread if and only if the restriction $f|X$ of f to every semialgebraic subset X of M is a spread (cf. 13.1.ii, iii, v). Thus our results will, to some extent, yield an understanding of arbitrary spreads.

Theorem 13.2. Let $f: M \to N$ be a semialgebraic map. f is a spread if and only if the pure factor φ in the Stein factorization of f is an embedding.

Proof. Let

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & S \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

be a Stein factorization of f. If φ is an embedding then both π and φ are spreads, hence f is also a spread.

Assume now that f is a spread. Then, for every $x \in M$, the set $D(f,x)$ introduced in §12 is the one point set $\{x\}$. Thus, by Theorem 12.17, the map φ is injective. In order to prove that φ is an embedding it suffices to verify that the bijective map $\psi: M \to \varphi(M)$, obtained from φ by restriction of the range, is open. Let $x \in M$, $y := \varphi(x)$. Since φ is pure, $\varphi^{-1}(\pi^{-1}(U)_y)$ is connected for every open semialgebraic neighbourhood U of $f(x)$. Thus by formula (*) in 13.1.iii, we have

$$\varphi^{-1}(\pi^{-1}(U)_y) = f^{-1}(U)_x.$$

Applying ψ to this equality we obtain

$$\psi(f^{-1}(U)_x) = \pi^{-1}(U)_y \cap \varphi(M).$$

Since f is a spread, $f^{-1}(U)_x$ runs through a fundamental system of neighbourhoods of x if U runs through the open semialgebraic neighbourhoods of $f(x)$. Thus ψ is indeed open. q.e.d.

We look for a criterion saying that a given semialgebraic map $f: M \to N$ has a <u>finite extension</u> (in the sense of §12, Def. 3), i.e. a factorization

$$(**) \qquad \begin{tikzcd} M \arrow[rr, hook, "\alpha"] \arrow[dr, "f"'] & & P \arrow[dl, "\pi"] \\ & N & \end{tikzcd}$$

with α an embedding and π finite. If such a factorization (**) exists then, replacing the embedding α by its pure hull $\tilde{\alpha}: M \to \tilde{P}$ and π by $\pi \circ p$, with p the finite projection from $\tilde{P}$ onto P, we obtain a factorization (**) with α a pure embedding and π finite. This is a Stein factorization of f. Thus a necessary (and obviously sufficient) condition for the existence of a finite extension of f is that the pure factor φ in the Stein factorization is an embedding. In view of Theorem 13.2 we may express this result in the following way.

<u>Theorem 13.3.</u> A semialgebraic map $f: M \to N$ admits a finite extension if and only if f is a spread. In this case we have, up to isomorphism, a unique finite extension (**) where the embedding α is pure, namely the Stein factorization of f.

In §6 we have seen that every finite map has a completion which is again finite (Cor. 6.14). Theorem 13.3 provides us with a new proof of this result.

<u>Corollary 13.4.</u> Let $f: M \to N$ be a finite map and let $\psi: N \hookrightarrow Q$ be a completion of the space N. Then there exists a unique completion

$$\begin{tikzcd} M \arrow[r, hook, "\varphi"] \arrow[d, "f"'] & P \arrow[d, "\bar{f}"] \\ N \arrow[r, hook, "\psi"] & Q \end{tikzcd}$$

of f with $\bar{f}$ finite and φ pure.

Indeed, $\psi \circ f$ is a spread and $\bar{f}, \varphi$ are the factors of the Stein factorization of $\psi \circ f$.

Semialgebraic spreads can be triangulated. Even the following holds.

<u>Theorem 13.5.</u> Let $f : M \to N$ be a semialgebraic spread. Let $(M_i \mid i \in I)$ and $(N_j \mid j \in J)$ be locally finite families of locally semialgebraic subsets of M and N respectively. Then there exists a triangulation

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\psi]{\sim} & N \end{array} \qquad (T)$$

of f such that every $\psi^{-1}(N_j)$ is a subcomplex of Y and every $\varphi^{-1}(M_i)$ is a subcomplex of X.

<u>Proof.</u> We choose a finite extension of f,

$$\begin{array}{ccc} M & \hookrightarrow & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & N & \end{array}$$

and regard M as a subspace of P. If we prove the claim for the finite map π and the families $(M, (M_i \mid i \in I))$ and $(N_j \mid j \in J)$ of subsets of P and N, then the claim is also proved in the original setting. Thus we may assume without loss of generality that f is finite. Applying Hardt's theorem (= Th. 6.3) we choose a simultaneous triangulation $\psi : Y \to N$ of N and the sets N_j such that the map f and also every restriction $f|M_i : M_i \to N$ is trivial over $\psi(\sigma)$ for every open simplex σ of Y. Replacing ψ by its first barycentric subdivision we know in addition that there exists a (unique) triangulation (T) of f which involves the given triangulation ψ of Y. (cf. Th. 6.13). We want to verify that, for every $i \in I$, the set $X_i := \varphi^{-1}(M_i)$ is a subcomplex of X. But this is obvious: The semialgebraic map $g|X_i : X_i \to Y$ is trivial over every $\sigma \in \Sigma(Y)$. Since g has discrete fibres this means that g maps every connected component of $g^{-1}(\sigma) \cap X_i$ isomorphically onto σ. Now $g^{-1}(\sigma)$ has as connected components the open simplices τ of X with $g(\tau) = \sigma$, and g maps every τ isomorphically onto σ. (This is a general fact for finite simplicial maps, cf. §6). Thus the connected components of $g^{-1}(\sigma) \cap X_i$ must be some of these simplices τ. Running through all $\sigma \in \Sigma(Y)$ we see that X_i is a union of open simplices of X. q.e.d.

This theorem implies a comparison statement for open coverings of M and N if a semialgebraic spread $f: M \to N$ is given.

<u>Corollary 13.6.</u> Again, let $f: M \to N$ be a semialgebraic spread. Let $(M_i \mid i \in I)$ and $(N_j \mid j \in J)$ be locally finite coverings of M and N by open semialgebraic sets. Then there exists a locally finite covering $(U_k \mid k \in K)$ of N by open semialgebraic sets and for every $k \in K$, a family $(V_{k\lambda} \mid \lambda \in \Lambda(k))$ consisting of connected components of $f^{-1}(U_k)$ such that the following holds:

a) Every U_k is contained in some N_j and every N_j is the union of finitely many sets U_k.

b) Every $V_{k\lambda}$ is contained in some M_i and every M_i is the union of finitely many sets $V_{k\lambda}$.

In the special case that f is finite the locally finite covering $(U_k \mid k \in K)$ by open semialgebraic sets can be chosen in such a way that a) and b) hold with $(V_{k\lambda} \mid \lambda \in \Lambda(k))$ the family of <u>all</u> connected components of $f^{-1}(U_k)$.

<u>Proof.</u> We choose a triangulation of f as indicated in Theorem 13.5. Replacing the triangulations of M and N by their barycentric subdivisions we may assume the following: $M = X$, $N = Y$ with strictly locally finite complexes X and Y. The map f is simplicial and its closure $\bar{f}$ is finite. Every M_i is an open subcomplex X_i of X, and every N_j is an open subcomplex Y_j of Y. Furthermore, every open simplex $\sigma \subset X$ (resp. $\sigma \subset X_i$) has at least one vertex lying in X (resp. in X_i), and every open simplex $\tau \subset Y$

(resp. $\tau \subset Y_j$) has at least one vertex lying in Y (resp. Y_j). In this situation it is easily verified that the covering $(U_k | k \in K)$ with U_k running through the open stars $St_Y(w)$ with $w \in E(Y) \cap Y$ and the families $(V_{k\lambda} | \lambda \in \Lambda(k))$ with $V_{k\lambda}$ running through the open stars $St_X(v)$ with $v \in f^{-1}(w)$, where $U_k = St_Y(w)$ have the desired properties. q.e.d.

We defined a spread by a local condition in the weak sense (cf. Def. 1 above). As a consequence of Theorem 13.5, or its Corollary 13.6, we now see that semialgebraic spreads can be characterized by a local condition in the strong sense.

Definition 2. A base of a space N is a subset $\mathcal{L}$ of $\mathring{\gamma}(N)$ such that every $U \in \mathring{\gamma}(N)$ is the union of a finite family in $\mathcal{L}$.

Proposition 13.7. Let $f: M \to N$ be a semialgebraic map and let $\mathcal{L}$ be a base of N. Let $Con(f,\mathcal{L})$ denote the set of all connected components of all sets $f^{-1}(B)$ with B running through $\mathcal{L}$. The map f is a spread if and only if $Con(f,\mathcal{L})$ is a base of M.

Proof. If $Con(f,\mathcal{L})$ is a base of M then it is evident directly from Definition 1 that f is a spread. Assume now that f is a spread, and let $X \in \mathring{\gamma}(M)$ be given. We have to verify that X is the union of finitely many open sets in $Con(f,\mathcal{L})$. Choosing an open semialgebraic set $U \supset f(X)$ in N we may replace N by U, M by $f^{-1}(U)$, f by its restriction from $f^{-1}(U)$ to U and $\mathcal{L}$ by $\{W \in \mathcal{L} | W \subset U\}$. Thus we may asssume without loss of generality that M and N are semialgebraic. Applying Corollary 13.6 to the covering $\{N\}$ of N and the covering $\{M,X\}$ of M, we obtain a finite family $\{U_1,\dots,U_r\}$ in $\mathring{\gamma}(N)$ and for every U_i a (finite) family $(V_{ij} | j \in J(i))$ of connected components of $f^{-1}(U_i)$ such that X is the union of all V_{ij} with $i \in \{1,\dots,r\}$, $j \in J(i)$. Choosing for every U_i a finite family $(B_{i\lambda} | \lambda \in \Lambda(i))$ in $\mathcal{L}$ with union U_i it is evident that

X is the union of some of the components of the preimages $f^{-1}(B_{i\lambda})$, and these are finitely many elements of $\mathrm{Con}(f,\mathcal{L})$. q.e.d.

Using the method in the proof of Theorem 13.5 and Corollary 13.6 we can also verify that a local isomorphism, as defined in 13.1.iv, i.e. "in the weak sense", is actually a local isomorphism "in the strong sense".

Theorem 13.8. Let $f: M \to N$ be a local isomorphism between spaces. Then there exists a locally finite covering $(U_\lambda | \lambda \in \Lambda)$ of M by open semialgebraic sets such that, for every $\lambda \in \Lambda$, f yields an isomorphism of U_λ onto an open semialgebraic set $f(U_\lambda)$ in N. If f is semialgebraic we can find such a covering $(U_\lambda | \lambda \in \Lambda)$ with $(f(U_\lambda) | \lambda \in \Lambda)$ also locally finite.

Proof. We may assume that M and N are semialgebraic. Now run through the proof of Corollary 13.6 again with $(M_i | i \in I) = \{M\}$, $(N_j | j \in J) = \{N\}$. We choose as the sets U_λ the open stars $St_X(v)$ with $v \in E(X) \cap X$. Since f maps a small neighbourhood of v isomorphically onto a small neighbourhood of $w := f(v)$ it is evident that f maps $St_X(v)$ isomorphically onto $St_Y(w)$. Thus the covering $(St_X(v) | v \in E(X) \cap X)$ of X has the desired properties. q.e.d.

Theorem 13.8 has first been proved in full generality by Roland Huber at Regensburg. He actually proved a more general theorem by an elementary direct method, which is independently interesting. We shall present a jazzed-up version of his theorem and proof in the next section (Huber worked only in the semialgebraic setting).

§14 - Huber's theorem on open mappings

A locally semialgebraic map $f: M \to N$ is called <u>open</u> if the image of any open semialgebraic set is open. This means that f is open in the topological sense with respect to the strong topologies on M and N.

<u>Theorem 14.1</u> (Huber). Let $f: M \to N$ be an amenable map (cf. §6) between spaces. Assume further that f is open and every point $x \in M$ has an open semialgebraic neighbourhood V such that $f|V: V \to N$ has connected fibres. Then there exists an admissible covering $(M_i | i \in I)$ of M by open locally semialgebraic sets such that, for every $i \in I$, the fibres of $f|M_i : M_i \to N$ are connected components of fibres of f. If the components of all fibres of f are semialgebraic then $(M_i | i \in I)$ can be chosen as a locally finite covering with semialgebraic sets M_i.

<u>Examples 14.2.</u> The assumptions of the theorem are fulfilled in the following cases.

a) f is a semialgebraic local isomorphism. In this case the theorem gives our previous result Theorem 13.8 anew.

b) M is an open locally semialgebraic subset of the product $L \times N$ of two spaces and f is the restriction to M of the canonical projection $L \times N \to N$.

In order to prove Theorem 14.1 we choose a suitable triangulation of N and then assume that N is a locally finite simplicial complex Y and f is trivial over each open simplex σ of Y.

Let $\Sigma(M)$ denote the set of all connected components of all inverse images $f^{-1}(\sigma)$ of all open simplices $\sigma \in \Sigma(Y)$. Clearly $\Sigma(M)$ is a locally finite partition of M into locally semialgebraic sets. We use the notation $\Sigma(M)$ since - as we shall see - $\Sigma(M)$ is a "stratification" of M (in a mild sense) similar to the stratification $\Sigma(Y)$ of Y into open simplices. The proof will be similar in spirit to the considerations in §6 about triangulations of partially finite amenable maps, where such a "stratification" was also studied.

We already start to call the sets $S \in \Sigma(M)$ the <u>strata</u> of M. Since f is trivial over each $\sigma \in \Sigma(Y)$, f maps every stratum S onto a whole open simplex σ of Y and, for every $y \in \sigma$, the intersection $S \cap f^{-1}(y)$ is a connected component of $f^{-1}(y)$. We divide the proof of Theorem 14.2 into several steps.

<u>First (and most important) step</u>. If S_1 and S_2 are different strata with $f(S_1) = f(S_2) = \sigma$, then $\overline{S_1} \cap \overline{S_2} = \emptyset$.

<u>Proof.</u> Suppose there exists a point $x \in \overline{S_1} \cap \overline{S_2}$. Then $y := f(x)$ is a point in $\partial\sigma$. We choose an open locally semialgebraic neighbourhood V of x such that $f|V$ has connected fibres. Then we choose some $U \in \mathring{\gamma}(M)$ with $x \in U \subset V$. Finally we choose some open semialgebraic neighbourhood W of f(x) in N with $W \subset f(U)$ and $W \cap \sigma$ connected. This is possible since f is open. Let $\Sigma(M,\sigma)$ denote the set of all strata $S \in \Sigma(M)$ with $f(S) = \sigma$, and let Λ denote the subset of all $S \in \Sigma(M,\sigma)$ with $U \cap f^{-1}(W) \cap S \neq \emptyset$. The set Λ is finite since $\Sigma(M)$ is locally finite. Clearly both S_1 and S_2 are elements of Λ. For every $S \in \Lambda$ the set $f(U \cap f^{-1}(W) \cap S)$ is open semialgebraic in $W \cap \sigma$ and

$$(*) \qquad W \cap \sigma = \bigcup_{S \in \Lambda} f(U \cap f^{-1}(W) \cap S).$$

This is clear since f is trivial over σ, $\Sigma(M,\sigma)$ is the set of connected components of $f^{-1}(\sigma)$, and f is open. For every $y \in W \cap \sigma$ we have

$$f^{-1}(y) \cap U \subset f^{-1}(y) \cap V \subset S$$

with a <u>unique</u> $S \in \Lambda$. Thus the right hand side of (*) is a <u>disjoint</u> union of non empty open semialgebraic subsets of $W \cap \sigma$, and there are at least two such sets. This contradicts the fact that $W \cap \sigma$ is connected. We conclude that $\overline{S_1} \cap \overline{S_2} = \emptyset$.

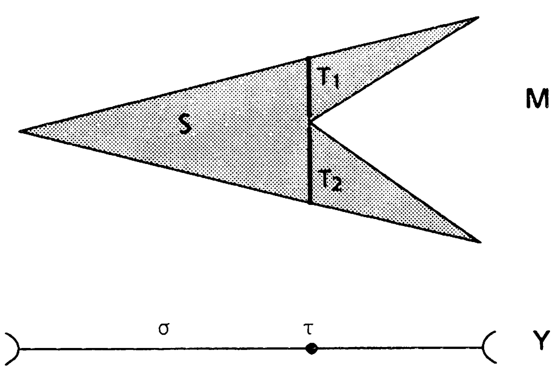
<u>2nd Step.</u> For every $\sigma \in \Sigma(Y)$ we have $\overline{f^{-1}(\sigma)} = f^{-1}(\bar{\sigma})$.
Indeed, $\overline{f^{-1}(\sigma)} \subset f^{-1}(\bar{\sigma})$. Given a point $x \in f^{-1}(\bar{\sigma})$ and an open neighbourhood V of x we have $f(V) \cap \sigma \neq \emptyset$ since $f(V)$ is open. Thus $V \cap f^{-1}(\sigma) \neq \emptyset$, and we see that $x \in \overline{f^{-1}(\sigma)}$.

<u>3rd Step.</u> Let σ and τ be open simplices of Y with $\tau \subset \bar{\sigma}$. Then, for every stratum $S \subset f^{-1}(\sigma)$, the intersection $\bar{S} \cap f^{-1}(\tau)$ is a union of strata $T \subset f^{-1}(\tau)$. Every stratum $T \subset f^{-1}(\tau)$ lies in the closure $\bar{S}$ of a unique stratum $S \subset f^{-1}(\sigma)$.

<u>Proof.</u> $f^{-1}(\tau) \subset f^{-1}(\bar{\sigma}) = \overline{f^{-1}(\sigma)}$, and $\overline{f^{-1}(\sigma)}$ is the disjoint union (cf. step 1) of the closures $\bar{S}$ of the connected components S of $f^{-1}(\sigma)$. Thus $f^{-1}(\tau)$ is the disjoint union of the closed semialgebraic sets $\bar{S} \cap f^{-1}(\tau)$ with S running through the strata $S \subset f^{-1}(\sigma)$. This gives both results.

q.e.d.

<u>Remark.</u> $\bar{S} \cap f^{-1}(\tau)$ may consist of several strata as the following picture shows.



Here M is an open subset in $\mathbb{R}^2$, f is the projection onto the first coordinate axis, and Y is an open interval subdivided by one point.

For every $T \in \Sigma(M)$ we define the <u>star</u> $St_M(T)$ as the union of all $S \in \Sigma(M)$ with $\overline{S} \supset T$. Since the star $St_Y(\tau)$ of $\tau := f(T)$ consists of finitely many open simplices, it is evident from the third step that $St_M(T)$ consists of finitely many strata. In particular, $St_M(T)$ is locally semialgebraic.

<u>4th Step.</u> $St_M(T)$ is open for every $T \in \Sigma(M)$.

<u>Proof.</u> If S is a stratum of M with $\overline{S} \not\supset T$ then $\overline{S}$ is disjoint from $St_M(T)$. Thus the complement of $St_M(T)$ in M is the union of the family $(\overline{S} \mid S \in \Sigma(M),\ \overline{S} \not\supset T)$. The family $(\overline{S} \mid S \in \Sigma(M))$ is locally finite since $\Sigma(M)$ is locally finite. Thus this union is closed. q.e.d.

<u>5th Step.</u> For every $T \in \Sigma(M)$ the fibres of $f \mid St_M(T)$ are connected components of the fibres of f.

<u>Proof.</u> Let $\tau := f(T)$. The image $f(St_M(T))$ is a subcomplex of $St_Y(\tau)$ which contains τ. Let $y \in f(St_M(T))$ and let σ denote the unique open simplex of $St_Y(\tau)$ containing y. By Step 3 there exists a unique stratum $S \subset f^{-1}(\sigma)$ with $\overline{S} \supset T$. Thus $f^{-1}(y) \cap St_M(T) = f^{-1}(y) \cap S$, and this is a connected component of $f^{-1}(y)$.

<u>6th Step.</u> The covering $(St_M(T) \mid T \in \Sigma(M))$ of M is admissible. Indeed, every $U \in \mathring{\gamma}(M)$ can be covered by finitely many strata. A fortiori, U can be covered by finitely many stars.

If the connected components of the fibres of f are semialgebraic, then the strata of M are semialgebraic, hence also their stars are semialgebraic. Now the closure $\overline{S}$ of a stratum S is the union of finitely many strata, and these are precisely all $T \in \Sigma(M)$ such that $S \cap St_M(T) \neq \emptyset$. Thus the family $(St_M(T) \mid T \in \Sigma(M))$ is locally finite.

Altogether we see that the theorem holds with the covering

$$(M_i \mid i \in I) := (St_M(T) \mid T \in \Sigma(M)) .$$

We ask for the relations between our stratification of M and analogous stratifications of suitable subspaces of M. This leads to a refinement of Theorem 14.1.

Theorem 14.3. Let $f : M \to N$ be a locally semialgebraic map between spaces. Assume that $(X(\lambda) \mid \lambda \in \lambda)$ is a family of locally semialgebraic subsets of M such that, for every $\lambda \in \Lambda$, the restriction $f \mid X(\lambda) : X(\lambda) \to N$ fulfills the assumptions made about f in Theorem 14.1. Assume further that the family of sets $(f(X(\lambda)) \mid \lambda \in \Lambda)$ is locally finite in N. (N.B. Every set $f(X(\lambda))$ is locally semialgebraic since $f \mid X(\lambda)$ is amenable.) Then there exists, for every $\lambda \in \Lambda$, a locally finite covering $(X(\lambda,i) \mid i \in I(\lambda))$ of $X(\lambda)$ by open locally semialgebraic subsets of $X(\lambda)$, and, for any two indices $\lambda, \mu \in \Lambda$ with $X(\lambda) \subset X(\mu)$, a map $\varphi_{\mu\lambda} : I(\lambda) \to I(\mu)$ such that the following holds.

a) The fibres of $f \mid X(\lambda,i)$ are connected components of fibres of $f \mid X(\lambda)$.

b) If $X(\lambda) \subset X(\mu)$ then every set $X(\lambda,i)$ is contained in $X(\mu,\varphi_{\mu\lambda}(i))$ and every set $X(\mu,j) \cap X(\lambda)$ is the union of all $X(\lambda,i)$ with $X(\mu,\varphi_{\mu\lambda}(i)) \subset$ $\subset X(\mu,j)$.

c) If $X(\lambda) \subset X(\mu) \subset X(\nu)$ then $\varphi_{\nu\mu} \circ \varphi_{\mu\lambda} = \varphi_{\nu\lambda}$.

Example 14.4. Let $f : M \to N$ be an open locally semialgebraic map between spaces. Assume that every $x \in M$ has a fundamental system of open semialgebraic neighbourhoods V such that $f \mid V$ has connected fibres. Let $(Z_\kappa \mid \kappa \in K)$ be any family of open semialgebraic subsets of M such that the family $(f(Z_\kappa) \mid \kappa \in K)$ is locally finite in N. Then the assumptions of the theorem hold for f and the family $(X(\lambda) \mid \lambda \in \Lambda)$ of all intersections of finitely many sets Z_κ. Indeed, the family $(f(X(\lambda)) \mid \lambda \in \Lambda)$ is again locally finite. Every $X(\lambda)$ is open and semialgebraic in M. By Hardt's theorem (Th. 6.3) every restriction $f|X(\lambda)$ is amenable.

<u>Proof of Theorem 14.3.</u> For every $\lambda\in\Lambda$ we choose a locally finite partition $(A_{\lambda\kappa}|\kappa\in K(\lambda))$ of $f(X(\lambda))$ into semialgebraic sets such that $f|X(\lambda)$ is trivial over each $A_{\lambda\kappa}$. The family $(A_{\lambda\kappa}|\lambda\in\Lambda,\kappa\in K(\lambda))$ is locally finite in N. We choose a simultaneous triangulation of N and this family. Then we may assume that N is a locally finite simplicial complex Y such that, for every $\lambda\in\Lambda$ and every $\sigma\in\Sigma(Y)$, the map $f|X(\lambda)$ is trivial over σ. Thus we have, for every $\lambda\in\Lambda$, the stratification $\Sigma(X(\lambda))$ of $X(\lambda)$ constructed above, whose strata are the connected components of the sets $X(\lambda)\cap f^{-1}(\sigma)$. We define $I(\lambda) := \Sigma(X(\lambda))$ and, for every $T\in I(\lambda)$, we define $X(\lambda,T) := \mathrm{St}_{X(\lambda)}(T)$. If $X(\lambda)\subset X(\mu)$, then every stratum $T\in I(\lambda)$ is contained in a unique stratum $T'\in I(\mu)$ and we define $\varphi_{\mu\lambda}(T) = T'$. In this setting the claims a), b), c) in the theorem are evident. q.e.d.

We now state a consequence of Theorem 14.1 for the theory of Stein factorizations.

<u>Theorem 14.5.</u> We consider the Stein factorization

$$\begin{tikzcd} M \arrow[d, "f"'] \arrow[dr, "\varphi"] & \\ N & S \arrow[l, "\pi"] \end{tikzcd}$$

of a proper map f. The following are equivalent.

i) φ is open and π is a local isomorphism.

ii) f is open and every point $x\in M$ has an open semialgebraic neighbourhood V such that the fibres of $f|V$ are connected.

<u>Proof.</u> i) $\Rightarrow$ ii): This is the trivial direction. Of course, $f = \pi\circ\varphi$ is open. Given a point $x\in M$ we choose an open semialgebraic neighbourhood

W of $\varphi(x)$ which is mapped isomorphically onto an open semialgebraic neighbourhood U of $f(x)$ under π. Then $V := \varphi^{-1}(W)$ is an open semialgebraic neighbourhood of x, and the fibres of $f|V$ are connected components of fibres of f.

ii) $\Rightarrow$ i): By Theorem 14.1 we have a locally finite covering $(M_i | i\in I)$ of M by open semialgebraic subsets M_i such that, for every $i \in I$, the fibres of $f|M_i$ are connected components of fibres of f. Since f is proper this implies that every M_i consists of full fibres of φ. The image $S_i :=$ $\varphi(M_i)$ is a semialgebraic subset of S, and $M_i = \varphi^{-1}(S_i)$. Again using that φ is proper, we conclude that $S \setminus S_i = \varphi(M \setminus M_i)$ is closed in S, hence S_i is open in S. The family $(S_i | i\in I)$ is a locally finite covering of S. Every S_i is mapped under π onto the open semialgebraic subset $N_i :=$ $f(M_i)$ of N. We consider the restrictions $\varphi_i : M_i \twoheadrightarrow S_i$, $\varphi_i : S_i \twoheadrightarrow N_i$, and $f_i = \pi_i \circ \varphi_i : M_i \twoheadrightarrow N_i$ of φ, π, f respectively. The fibres of φ_i coincide with the fibres of f_i. Thus π_i is bijective. Since f_i is open, π_i is also open. We see that π_i is an isomorphism from S_i onto N_i and that $\varphi_i = \pi_i^{-1} \circ f_i$ is open. Thus π is a local isomorphism and φ is open.

q.e.d.

The finite maps which are local isomorphisms are just the finite coverings to be studied in Chapter V. Theorem 14.5 indicates a source of examples of finite coverings.

Chapter III - Homotopies

§1 - Some strong deformation retracts

In this whole chapter a space means a regular paracompact locally semialgebraic space over some fixed real closed field R. In particular, a semialgebraic space is always assumed to be affine. A map between spaces M and N is implicitly assumed to be locally semialgebraic. If we do not mean this we call the map "set theoretic" or "map between the sets M and N".

In this section we recall some theorems about the existence of strong deformation retractions which were proved for semialgebraic spaces in the papers [DK_3] and [DK_5]. The proofs there generalize word for word to the locally semialgebraic setting. We start with some obvious definitions.

Definitions 1. a) Let f and g be two maps from a space M to a space N. A homotopy from f to g is a map $H: M \times [0,1] \to N$ such that $H_0 = f$ and $H_1 = g$. Here H_t denotes the map $x \mapsto H(x,t)$ from M to N ($t \in [0,1]$). As before the unit interval [0,1] in R will often be denoted by I.
b) A subspace A of M (= locally semialgebraic subset of M) is called a retract of M if there exists a map $r: M \to A$ with $r|A = id_A$. Notice that then A must be closed in M. Any such map r is called a retraction from M to A.
c) A subspace A of M is called a strong deformation retract of M if there exists a homotopy $H: M \times I \to M$ such that H_0 is the identity of M, H_1 is a retraction from M to A and $H(a,t) = a$ for every $a \in A$, $t \in I$. We then call H a strong deformation retraction from M to A.

Theorem 1.1 (cf. Theorem 2.1 and 2.7 in [DK_5]). Let A be a closed subspace of a space M. Then there exists a locally semialgebraic open neighbourhood U of A in M and a semialgebraic strong deformation retraction $H: \bar{U}\times[0,1] \to \bar{U}$ from the closure $\bar{U}$ of U in M to A such that restricting H to $U\times[0,1]$ yields a strong deformation retraction from U to A.

Remark. The map H is obtained from the retraction r used in II, Prop. 10.8 in an easy way: Choose a simultaneous triangulation of M and A. Then take for U the open star of A in M with respect to the second barycentric subdivision and put

$$H(x,t) = (1-t)x + tr(x),$$

cf. [DK_5, §2].

Proposition 1.2 (Extension of maps to a neighbourhood, [DK_5, Prop. 4.1]). Again let A be a closed subspace of a space M and let U be a neighbourhood of A with the properties claimed in Theorem 1.1. Any map $f: A \to Z$ into some space Z extends to a map $g: \bar{U} \to Z$. If g_1 and g_2 are two extensions of f to $\bar{U}$, then there exists a homotopy $G: \bar{U}\times I \to Z$ with $G_0 = g_1$, $G_1 = g_2$ and $G_t|A = f$ for every $t \in I$. (The same is true with $\bar{U}$ replaced by U.)

This is an easy consequence of Theorem 1.1. Indeed, let $H: \bar{U}\times I \to \bar{U}$ be a strong deformation retraction from $\bar{U}$ to A. Define $g := f\circ H_1$ and define $G:\bar{U}\times I \to Z$ as follows:

$$G(x,t) = \begin{cases} g_1\circ H(x,2t), & 0 \le t \le \frac{1}{2} \\ g_2\circ H(x,2(1-t)), & \frac{1}{2} \le t \le 1. \end{cases}$$

Theorem 1.3 (cf. Th. 5.1 in [DK_5]). If A is a closed subspace of a space M then $(A\times I) \cup (M\times\{0\})$ is a strong deformation retract of $M\times I$.

The fact that $(A\times I) \cup (M\times\{0\})$ is a retract of $M\times I$ means that the pair

(M,A) has the "homotopy extension property".

<u>Corollary 1.4</u> (Homotopy extension theorem). Let A be a closed subspace of a space M. Given a map $g : M \to Z$ into some space Z and a homotopy $F : A\times I \to Z$ with $F_o = g|A$ there exists a homotopy $G : M\times I \to Z$ with $G_o = g$ and $G|A\times I = F$.

$$\begin{tikzcd}
M\times I \arrow[drr, dashed, "G"] & & \\
(A\times I) \cup (M\times\{0\}) \arrow[u, hook] \arrow[rr, "F\cup g"'] & & Z
\end{tikzcd}$$

Indeed, we obtain G by composing $F \cup g$ with a retraction from $M\times I$ to $(A\times I) \cup (M\times\{0\})$.

As before, a <u>complex</u> means a geometric simplicial complex. In the following we shall often work with weak triangulations (cf. II, §6) of spaces instead of triangulations, i.e. we shall use locally finite instead of strictly locally finite complexes. This is a minor point in the theory since we usually have enough triangulations at our disposal, but we feel that the methods become clearer if we use weak triangulations. By the way, in the proof of Theorem 1.1 it was already sufficient to take a weak simultaneous triangulation of M and A.

<u>Definitions 2.</u> a) A subcomplex Y of a complex X is called <u>tame in</u> X if, for every open simplex σ of X, the following two conditions are fulfilled.

i) If $\sigma \subset Y$ then at least one vertex of σ is a point of Y.

ii) If every vertex of σ lies in Y then $\sigma \subset Y$ (thus also $\overline{\sigma} \cap X \subset Y$).

b) A simultaneous weak triangulation $\varphi : X \xrightarrow{\sim} M$ of a space M and a subspace A of M (i.e. φ is a weak triangulation of M and $\varphi^{-1}(A)$ is a subcomplex of X) is called <u>good on</u> A if the complex $\varphi^{-1}(A)$ is tame in $\overline{X}$.

It is an easy combinatorial exercise to verify

Remark 1.5. For every subcomplex Y of a complex X the barycentric subdivision Y' is tame in X'. Thus, if $\varphi : X \xrightarrow{\sim} M$ is a simultaneous triangulation of a space M and a subspace A, then the barycentric subdivision $\varphi' : X' \xrightarrow{\sim} M$ of φ (which as a map between spaces is the same as φ) is good on A and on M.

Definitions 3. a) The core of a complex X is the subcomplex Z of X consisting of all open simplices $\sigma \in \Sigma(X)$ with $\overline{\sigma} \subset X$. In other words, Z is the unique maximal subcomplex of X which is closed. We denote the core of X by $\operatorname{co} X$.

b) The core of a space M with respect to a weak triangulation $\varphi : X \xrightarrow{\sim} M$ is the partially complete subspace $\varphi(\operatorname{co} X)$ of M. We denote this subset of M by $\operatorname{co}(M,\varphi)$. If A is a subset of M such that $\varphi^{-1}(A)$ is a subcomplex Y of X, i.e. φ is a simultaneous triangulation of X and A, then by the core of A with respect to φ we mean the set $\operatorname{co}(A,\varphi|Y)$. We will write $\operatorname{co}(A,\varphi)$ instead of $\operatorname{co}(A,\varphi|Y)$.

Proposition 1.6 (cf. Prop. 2.5 in [DK_3] and its proof). Let $\varphi : X \xrightarrow{\sim} M$ be a simultaneous triangulation of M and A which is good on A. Let U denote the image under φ of the open star $St_X(\operatorname{co} Y)$ of the core of $Y := \varphi^{-1}(A)$. Notice that $U \supset A$. There exists a semialgebraic strong deformation retraction $H : U \times I \to U$ from U to $\operatorname{co}(A,\varphi)$ whose restriction to $A \times I$ is a strong deformation retraction from A to $\operatorname{co}(A,\varphi)$.

The map H is given by an explicit formula [DK_3, p. 26] which is so simple that we have space here to write it down. We identify X with M assuming $\varphi = id_X$. Then $H(x,t) = (1-t)x + tr(x)$ $(x \in U,\ t \in I)$ with a retraction $r : U \to \operatorname{co} Y$ defined as follows. If σ is an open simplex in U, $\sigma =]e_0,\dots,e_n[$ with $e_i \in Y$ for $0 \le i \le m$ and $e_i \notin Y$ for $m < i \le n$, then

$$(*) \qquad r\left(\sum_{i=0}^{n} t_i e_i\right) = \left(\sum_{i=0}^{m} t_i\right)^{-1} \left(\sum_{i=0}^{m} t_i e_i\right)$$

(all $t_i > 0$, $t_0 + \ldots + t_n = 1$). We call r the canonical retraction from U to co Y. Incidentally, the retraction used in II, Prop. 10.8 and in Theorem 1.1 is defined by a generalization of the formula (*), cf. [DK_5, §2].

Corollary 1.7. Let M be any space and K be a partially complete subspace of M. Then there exists a partially complete subspace $L \supset K$ in M such that L is a strong deformation retract of M.

Indeed, choose a simultaneous triangulation $\varphi : X \xrightarrow{\sim} M$ of M and K. Then the core L = co(M,φ') of M with respect to the barycentric subdivision φ' of φ has the desired properties (apply Remark 1.5 and Prop. 1.6 with A = M).

For any complex X which is tame in $\overline{X}$ we denote the canonical retraction from X to co X by r_X. (Caution: r_X is almost never a simplicial map!) We state two observations on canonical retractions for later use.

Proposition 1.8. Let X be a complex which is tame in $\overline{X}$ and let Y be a subcomplex of X with $\overline{Y} \cap \mathrm{co}\, X \subset Y$ (for example, Y closed in X). Then Y is tame in $\overline{Y}$ and $\mathrm{co}\, Y = Y \cap \mathrm{co}\, X$. The canonical retraction $r_Y : Y \to \mathrm{co}\, Y$ is the restriction of $r_X : X \to \mathrm{co}\, X$ to Y.

We leave the easy proof to the reader.

Proposition 1.9. Again let X be a complex which is tame in $\overline{X}$ and let Y be a subcomplex of X. Assume that every $\sigma \in \Sigma(Y)$ has at least one vertex contained in Y, and further that $Y \cap \mathrm{co}\, X$ is open in co X (for example, Y open in X). Then Y is tame in $\overline{X}$ and the restriction of r_X to

Y is a retraction from Y to $Y \cap \mathrm{co}\, X$, which is linearly homotopic to id_Y.

<u>Proof.</u> Let $\sigma = \,]e_o,\dots,e_n[$ be an open simplex of $\overline{X}$ all whose vertices e_i are points in Y. Since X is tame in $\overline{X}$ we know that $\sigma \subset \mathrm{co}\, X$. Since $e_o \in Y \cap \mathrm{co}\, X$ and $Y \cap \mathrm{co}\, X$ is open in $\mathrm{co}\, X$ we conclude that $\sigma \subset Y \cap \mathrm{co}\, X$ and a fortiori $\sigma \subset Y$. Thus Y is tame in $\overline{X}$.

Now let $\sigma = \,]e_o,\dots,e_n[$ be an open simplex in Y. We may assume that $e_o \in Y$, $e_i \in X$ for $0 \le i \le m$, and $e_i \notin X$ for $m < i \le n$, for some $m \in \{0,\dots,n\}$. Then $r_X(\sigma) = \,]e_o,\dots,e_m[$. This is an open simplex of $\mathrm{co}\, X$ contained in $\mathrm{St}_{\mathrm{co}\, X}(e_o)$ and e_o is a vertex of $Y \cap \mathrm{co}\, X$. Since $Y \cap \mathrm{co}\, X$ is open in $\mathrm{co}\, X$ we conclude that $r_X(\sigma) \subset Y \cap \mathrm{co}\, X$. The remaining assertions of the proposition are now obvious. q.e.d.

§2 - Simplicial approximations

We consider $(r+1)$-tupels $(M, M_1, \ldots, M_r)$ consisting of a space M and subspaces $M_1, \ldots, M_r$ of M $(r \geq 0)$. Such a tupel is called a system of spaces. If M is a locally finite simplicial complex X and M_i is a subcomplex X_i of X $(1 \leq i \leq r)$, then it is called a system of complexes.

A map

$$f: (M, M_1, \ldots, M_r) \to (N, N_1, \ldots, N_r)$$

between such systems of spaces is, of course, a (locally semialgebraic) map $f: M \to N$ with $f(M_i) \subset N_i$. Such a map f is called an isomorphism if $f: M \to N$ is an isomorphism of spaces and $f(M_i) = N_i$ for $i = 1, \ldots, r$. Then f induces isomorphisms $M_i \xrightarrow{\sim} N_i$.

A homotopy between two maps f,g from $(M, M_1, \ldots, M_r)$ to $(N, N_1, \ldots, N_r)$ is a map

$$H: (M \times I,\ M_1 \times I, \ldots, M_r \times I) \to (N, N_1, \ldots, N_r)$$

with $H_0 = f$ and $H_1 = g$. If such a homotopy exists we call f and g homotopic and write $f \simeq g$. Similarly we transfer all the usual terminology from topological homotopy theory to the category of spaces over R.

We denote the homotopy class of a map f from $(M, M_1, \ldots, M_r)$ to $(N, N_1, \ldots, N_r)$ by $[f]$ and the set of all these homotopy classes by $[(M, M_1, \ldots, M_r),\ (N, N_1, \ldots, N_r)]$. The goal of the next three sections will be to obtain some insight into the general nature of these homotopy sets. (We shall prove two "main theorems"). A major tool for this purpose are simplicial approximations, to be explained now. We have to be a little more careful than in the topological theory since our simplicial complexes are not necessarily closed.

By a <u>weak triangulation</u> (resp. <u>triangulation</u>) <u>of the system</u> $(M,M_1,\dots,M_r)$ we mean a simultaneous weak triangulation (resp. triangulation) of $M,M_1,\dots,M_r$. This is nothing more than an isomorphism

$$\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$$

with X a locally finite (resp. strictly locally finite) complex and $X_1,\dots,X_r$ subcomplexes of X.

<u>Definitions 1.</u> a) A system of complexes $(X,X_1,\dots,X_r)$ is called <u>tame</u> if X is tame in $\overline{X}$ (cf. §1) and every X_i is tame in $\overline{X}_i$. In particular $(r = 0)$ we call a complex X tame if X is tame in $\overline{X}$.
b) A weak triangulation $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$ of a system of spaces is called <u>good</u>, if $(X,X_1,\dots,X_r)$ is tame.

Notice that a weak triangulation $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$ is good if and only if all the triangulations $\varphi : X \xrightarrow{\sim} M$, $\varphi|X_i : X_i \xrightarrow{\sim} M_i$ $(1 \le i \le r)$ are good.

By Remark 1.5 the first barycentric subdivision $\varphi' : (X',X_1',\dots,X_r') \xrightarrow{\sim} (M,M_1,\dots,M_r)$ of any weak triangulation $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$ is good. Also, if the subspaces M_i are closed in M, then a weak triangulation $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$ is good if and only if the weak triangulation $\varphi : X \xrightarrow{\sim} M$ is good (cf. Prop. 1.8).

<u>Definition 2.</u> The <u>core</u> of a system of complexes $(X,X_1,\dots,X_r)$ is the system $(\operatorname{co} X, X_1 \cap \operatorname{co} X, \dots, X_r \cap \operatorname{co} X)$. It will be denoted by $\operatorname{co}(X,X_1,\dots,X_r)$. The inclusion map from $\operatorname{co}(X,X_1,\dots,X_r)$ to $(X,X_1,\dots,X_r)$ will be denoted by j_X {or more precisely by $j_{(X,X_1,\dots,X_r)}$}.

Usually only the cores of tame systems of complexes will be considered, since otherwise $\operatorname{co}(X,X_1,\dots,X_r)$ may have little in common with $(X,X_1,\dots,X_r)$. We often will need an even stronger condition than tameness which guarantees that j_X is a homotopy equivalence, so that we can replace $(X,X_1,\dots,X_r)$ by its core in homotopy considerations.

Definition 3. A system $(X,X_1,\dots,X_r)$ of complexes is called well cored if the system is tame and if the canonical retraction $r_X : X \to \operatorname{co} X$ (cf. §1) maps every X_i into itself, hence onto $X_i \cap \operatorname{co} X$. We also denote the induced map from $(X,X_1,\dots,X_r)$ to $\operatorname{co}(X,X_1,\dots,X_r)$ by r_X (or more precisely by $r_{(X,X_1,\dots,X_r)}$) and call it the canonical retraction of the system $(X,X_1,\dots,X_r)$.

If $(X,X_1,\dots,X_r)$ is well cored then the canonical retraction $r_X : (X,X_1,\dots,X_r) \to \operatorname{co}(X,X_1,\dots,X_r)$ is a homotopy inverse of the inclusion map j_X. More precisely, $r_X \circ j_X$ is the identity of $\operatorname{co}(X,X_1,\dots,X_r)$ and $j_X \circ r_X$ is linearly homotopic to the identity of $(X,X_1,\dots,X_r)$.

By Propositions 1.8 and 1.9 a tame system $(X,X_1,\dots,X_r)$ is well cored if, for every $i \in \{1,\dots,r\}$, either $\overline{X}_i \cap \operatorname{co} X \subset X_i$ or $X_i \cap \operatorname{co} X$ is open in $\operatorname{co} X$.

Proposition 2.1. Let $(X,X_1,\dots,X_r)$ be a tame system of complexes. Assume that every subcomplex X_i is locally closed in X (i.e. X_i is open in $\overline{X}_i \cap X$ or, what means the same, X_i is closed in $\mathrm{St}_X(X_i)$). Then $(X,X_1,\dots,X_r)$ is well-cored.

Indeed, we know that $r_X(Z) \subset Z$ for every subcomplex Z of X which is closed or open in X. The same holds for an intersection $Z = Z_1 \cap Z_2$ of subcomplexes with Z_1 open and Z_2 closed in X.

Definition 4. a) Let $f : (M,M_1,\dots,M_r) \to (N,N_1,\dots,N_r)$ be a map between systems of spaces. A simplicial approximation to f is a triple (φ,ψ,g) consisting of good weak triangulations $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$,

$\psi : (Y,Y_1,\dots,Y_r) \to (N,N_1,\dots,N_r)$, and a simplicial map

$$g : \mathrm{co}(X,X_1,\dots,X_r) \to \mathrm{co}(Y,Y_1,\dots,Y_r)$$

such that, for every vertex e of co(X),

$$\psi^{-1}\circ f\circ\varphi(St_X(e)) \subset St_Y(g(e)).$$

b) If $f : (X,X_1,\dots,X_r) \to (Y,Y_1,\dots,Y_r)$ is a map between tame systems of locally finite complexes, then by a simplicial approximation to f we usually mean just a simplicial map g from $\mathrm{co}(X,X_1,\dots,X_r)$ to $\mathrm{co}(Y,Y_1,\dots,Y_r)$ such that the triple (φ,ψ,g), with φ and ψ the identity maps of $(X,X_1,\dots,X_r)$ and $(Y,Y_1,\dots,Y_r)$, is a simplicial approximation to f in the sense above.

Notice that a simplicial map $g : \mathrm{co}(X,X_1,\dots,X_r) \to \mathrm{co}(Y,Y_1,\dots,Y_r)$ is a simplicial approximation to $f : (X,X_1,\dots,X_r) \to (Y,Y_1,\dots,Y_r)$ if and only if $g : \mathrm{co}\,X \to \mathrm{co}\,Y$ is a simplicial approximation to $f : X \to Y$. In this case g maps $\mathrm{co}\,X_i$ into $\mathrm{co}\,Y_i$ and the restriction $g|\mathrm{co}\,X_i : \mathrm{co}\,X_i \to \mathrm{co}\,Y_i$ is a simplicial approximation to $f|X_i : X_i \to Y_i$ $(1 \le i \le r)$. Also, if $k : \mathrm{co}(Y,Y_1,\dots,Y_r) \to \mathrm{co}(Z,Z_1,\dots,Z_r)$ is a simplicial approximation to a second map $h : (Y,Y_1,\dots,Y_r) \to (Z,Z_1,\dots,Z_r)$ between tame systems of locally finite complexes, then $k\circ g$ is a simplicial approximation to $h\circ f$.

<u>Lemma 2.2.</u> Let $f : X \to Y$ be a map between tame locally finite complexes, and let $g : \mathrm{co}\,X \to \mathrm{co}\,Y$ be a simplicial map. Then g is a simplicial approximation to f if and only if, for every $x \in X$, the following condition holds:

$$(*) \qquad f(x) \in \rho \in \Sigma(Y) \Rightarrow g\circ r_X(x) \in \overline{\rho}.$$

(Recall that r_X is the canonical retraction from X to co X).

<u>Proof.</u> Let $x \in X$ be given. Let $\sigma :=]e_o,\dots,e_n[$ be the open simplex of X containing x, and let ρ be the open simplex of Y containing f(x). We assume that $e_i \in X$ for $0 \le i \le m$ and $e_i \notin X$ for $m < i \le n$ $(m \ge 0$, recall that

X is tame). Then $r_X(\sigma)$ is the simplex $]e_o,\dots,e_m[$ in $\operatorname{co} X$, and $g\circ r_X(\sigma)$ is the open simplex τ spanned by the vertices $g(e_o),\dots,g(e_m)$. On the other hand, $f(x)$ is contained in the intersection of the sets $f(St_X(e_i))$ with $0 \leq i \leq m$.

Assume that g is a simplicial approximation to f. Then $f(x)$ is contained in the intersection of the sets $St_Y(g(e_i))$ with $0 \leq i \leq m$, which is the star $St_Y(\tau)$ of τ in Y. Thus we also have $\rho \subset St_Y(\tau)$, i.e. $\tau \subset \overline{\rho}$. Since $g\circ r_X(x) \in \tau$, the condition (*) is verified.

Assume now that condition (*) holds for every $x \in X$ and assume that our point x lies in $St_X(e)$ for some $e \in X \cap E(X)$. Then e is one of the vertices $e_o,\dots,e_m$. We conclude from (*) that τ is a face of ρ, which implies that $g(e)$ is a vertex of ρ, i.e. $\rho \subset St_Y(g(e))$. Thus $f(St_X(e)) \subset St_Y(g(e))$. q.e.d.

<u>Proposition 2.3.</u> Let $f : (X,X_1,\dots,X_r) \to (Y,Y_1,\dots,Y_r)$ be a map from a well cored system to a tame system of locally finite complexes. Let $g : \operatorname{co}(X,X_1,\dots,X_r) \to \operatorname{co}(Y,Y_1,\dots,Y_r)$ be a simplicial approximation to f. Then the maps f and $j_Y\circ g\circ r_X$ from $(X,X_1,\dots,X_r)$ to $(Y,Y_1,\dots,Y_r)$ are linearly homotopic.

<u>Proof.</u> We put $X_o := X$, $Y_o := Y$. Let a point $x \in X_k$ be given for some $k \in \{0,\dots,r\}$. Let ρ be the open simplex of Y_k containing $f(x)$. By condition (*) in Lemma 2.2, the half open line segment $[f(x), g\circ r_X(x)[$ is contained in ρ, hence in Y_k. Since r_X maps X_k to $X_k \cap \operatorname{co} X$, the point $g\circ r_X(x)$ lies in $Y_k \cap \operatorname{co} Y \subset Y_k$. Thus the closed line segment $[f(x), g\circ r_X(x)]$ is contained in Y_k, more precisely, in $Y_k \cap \overline{\rho}$. This implies the proposition. q.e.d.

This proposition gives the reason why simplicial approximations are interesting to us in homotopy theory. Under favourable conditions it tells us that, up to homotopy, we may replace a map between triangulated systems of spaces by the composition of a canonical retraction, a simplicial approximation and an inclusion. This map is a much simpler map than the original one.

But we still have to show that simplicial approximations exist under fairly general assumptions. We first state another obvious consequence of Lemma 2.2 which gives us a hint as to how to construct simplicial approximations.

Remark 2.4. Let $g : \mathrm{co}(X,X_1,\dots,X_r) \to \mathrm{co}(Y,Y_1,\dots,Y_r)$ be a simplicial approximation to a map $f : (X,X_1,\dots,X_r) \to (Y,Y_1,\dots,Y_r)$ between tame systems of complexes. For every vertex $e \in E(X) \cap X$ the point $g(e)$ is a vertex of the open simplex ρ of Y which contains $f(e)$. In particular, if the restriction $f|X_i : X_i \to Y_i$ is simplicial for some $i \in \{1,\dots,r\}$, then the maps $f|\mathrm{co}\,X_i$ and $g|\mathrm{co}\,X_i$ from $\mathrm{co}\,X_i$ to $\mathrm{co}\,Y_i$ are equal.

Theorem 2.5 (Existence of simplicial approximations). Let $f : (M,M_1,\dots,M_r) \to (N,N_1,\dots,N_r)$ be a map between systems of spaces. For every index $i \in \{1,\dots,r\}$ assume that either M_i is closed in M or N_i is locally closed in N. Let $\varphi : (X,X_1,\dots,X_r) \xrightarrow{\sim} (M,M_1,\dots,M_r)$ and $\psi : (Y,Y_1,\dots,Y_r) \xrightarrow{\sim} (N,N_1,\dots,N_r)$ be weak triangulations of $(M,M_1,\dots,M_r)$ and $(N,N_1,\dots,N_r)$. Assume that $(X,X_1,\dots,X_r)$ is tame and that the system $(Y,Y_1,\dots,Y_r)$ is the first barycentric subdivision of another system of complexes. Assume finally that $\psi^{-1}\circ f\circ\varphi$ maps every open simplex of X into an open simplex of Y. Then there exists a simplicial map

$$g : \mathrm{co}(X,X_1,\dots,X_r) \to \mathrm{co}(Y,Y_1,\dots,Y_r)$$

such that (φ,ψ,g) is a simplicial approximation to f.

Proof. We assume without loss of generality that $(M,M_1,\dots,M_r) = (X,X_1,\dots,X_r)$, $(N,N_1,\dots,N_r) = (Y,Y_1,\dots,Y_r)$ and that φ,ψ are the identity maps. We put $X_o := X$, $Y_o := Y$. For any $\tau \in \Sigma(Y)$ we denote the intersection of all Y_i, $0 \leq i \leq r$, with $\tau \subset Y_i$ by $Y(\tau)$. Since $(Y,Y_1,\dots,Y_r)$ is the barycentric subdivision of another system of complexes the complex $Y(\tau)$ is tame in $\overline{Y(\tau)}$.

Recall that, for any complex Z, we denote the set of vertices of Z by $E(Z)$ and the abstraction of Z by $K(Z) = (E(Z),S(Z))$ (II, §3). We set out to define an abstract simplicial map $\mu : K(\mathrm{co}\,X) \to K(\mathrm{co}\,Y)$. For any $\sigma \in \Sigma(X)$ the support $\mathrm{supp}\,f(\sigma)$ of $f(\sigma)$ is defined as the unique simplex $\tau \in \Sigma(Y)$ which contains $f(\sigma)$ (by the last assumption in the theorem). For every vertex e of $\mathrm{co}\,X$, i.e. $e \in X \cap E(X)$, we choose a vertex $\mu(e)$ of $\rho := \mathrm{supp}\,f(e)$ which lies in $Y(\rho)$. This is possible since $Y(\rho)$ is tame in $\overline{Y(\rho)}$. In this way we obtain a map $\mu : E(\mathrm{co}\,X) \to E(\mathrm{co}\,Y)$.

We claim that μ is a simplicial map from the abstract complex $K(\mathrm{co}\,X)$ to $K(\mathrm{co}\,Y)$. Let $\{e_o,\dots,e_n\}$ be the set of vertices of some open simplex $\sigma \subset \mathrm{co}\,X$. We want to verify that $\mu(e_o),\dots,\mu(e_n)$ are the vertices (possibly with repetitions) of some $\tau \in \Sigma(Y)$. Then $\tau \subset \mathrm{co}(Y)$ and we are done.

Let ρ denote the support of $f(\sigma)$. We have $f(e_i) \in f(\overline{\sigma}) \subset \overline{\rho}$ for every $i = 0,\dots,n$. Thus the support ρ_i of $f(e_i)$ is a face of ρ and $\mu(e_i)$ is a vertex of ρ in Y. The points $\mu(e_o),\dots,\mu(e_n)$ span a simplex τ of $\overline{Y}$. Since Y is tame in $\overline{Y}$ this simplex τ lies in Y and our claim is proved.

If e is a vertex of $\mathrm{co}(X_k)$ for some $k \in \{1,\dots,r\}$ then $f(e) \in Y_k$ and hence $\mu(e) \in E(Y_k) \cap Y_k = E(\mathrm{co}\,Y_k)$ by definition of μ. This implies that

μ maps the subcomplex $K(\mathrm{co}\,X_k)$ of $K(\mathrm{co}\,X)$ into $K(\mathrm{co}\,Y_k)$.

The realization $g = |\mu|$ of μ is a simplicial map from $\mathrm{co}\,X$ to $\mathrm{co}\,Y$ which maps $\mathrm{co}\,X_k$ into $\mathrm{co}\,Y_k$ for $1 \le k \le r$. We claim that $g : \mathrm{co}\,X \to \mathrm{co}\,Y$ is a simplicial approximation to $f : X \to Y$. We have to verify that, for a given vertex $e \in X \cap E(X)$, the set $f(St_X(e))$ is contained in $St_Y(g(e))$. Let σ be an open simplex in $St_X(e)$, and let ρ denote the support of $f(\sigma)$. Since e is a vertex of σ we have $f(e) \in \overline{f(\sigma)} \subset \overline{\rho}$. Thus $\mathrm{supp}\, f(e)$ is a face of ρ and $g(e) = \mu(e)$ is a vertex of ρ. This means $\rho \subset St_Y(g(e))$ and implies that $f(\sigma) \subset St_Y(g(e))$.

We finally have to verify, for a given index $k \in \{1,\dots,r\}$, that g maps $X_k \cap \mathrm{co}\,X$ into $Y_k \cap \mathrm{co}\,Y$. If X_k is closed in X then $X_k \cap \mathrm{co}\,X = \mathrm{co}\,X_k$ (cf. Prop. 1.8), and we already know that $g(\mathrm{co}\,X_k)$ is contained in $\mathrm{co}\,Y_k \subset Y_k \cap \mathrm{co}\,Y$. If X_k is not closed in X then, by assumption, $Y_k \cap \mathrm{co}\,Y$ is locally closed in $\mathrm{co}\,Y$. This means that $Y_k \cap \mathrm{co}\,Y = Z_k \cap U_k$ with $Z_k = \overline{Y}_k \cap \mathrm{co}\,Y$ and $U_k = St_{\mathrm{co}\,Y}(Y_k \cap \mathrm{co}\,Y) = St_{\mathrm{co}\,Y}(\mathrm{co}\,Y_k)$. Since g is simplicial and maps $\mathrm{co}\,X_k$ into $\mathrm{co}\,Y_k$ we have $g(X_k \cap \mathrm{co}\,X) \subset g(St_{\mathrm{co}\,X}(\mathrm{co}\,X_k)) \subset St_{\mathrm{co}\,Y}(\mathrm{co}\,Y_k) = U_k$. It remains to verify that $g(X_k \cap \mathrm{co}\,X) \subset Z_k$. Since f is continuous, $f(\overline{X}_k \cap \mathrm{co}\,X) \subset Z_k$. In particular, for every open simplex $\sigma =]e_o,\dots,e_n[$ in $X_k \cap \mathrm{co}\,X$, the point $f(e_i)$ lie in Z_k. Thus $\mathrm{supp}\, f(e_i) \subset Z_k$, which implies that $g(e_i) = \mu(e_i) \in Z_k$. Since Z_k is tame in $\mathrm{co}\,Y$ the simplex $]g(e_o),\dots,g(e_n)[= g(\sigma)$ is contained in Z_k. Thus indeed $g(X_k \cap \mathrm{co}\,X) \subset Y_k \cap \mathrm{co}\,Y$, and Theorem 2.5 is proved.

Notice that, in view of Remark 2.4, our procedure for constructing the simplicial approximation g was the most general one. It seems difficult to imagine a natural hypothesis more general than the last one in the theorem under which this construction works. We did not use the assumption that $(X,X_1,\dots,X_r)$ is tame in the proof. But since good triangulations exist in abundance and tameness of the system $(X,X_1,\dots,X_r)$ is essential for a simplicial approximation to be homotopic to the original map (Prop. 2.3), it seems to be reasonable to demand tameness in the definition of simplicial approximations, as we did.

We now ask to what extent a simplicial approximation of a given map between systems of locally finite complexes is determined by f.

Definition 5 (cf. [Spa, p. 130])*). Let $g_1, g_2 : (X_o, X_1, \ldots, X_r) \rightrightarrows (Y_o, Y_1, \ldots, Y_r)$ be two simplicial maps between systems of complexes. We say that g_1 and g_2 are strictly contiguous if, for every $\sigma \in \Sigma(X_i)$, the vertices of $g_1(\sigma)$ and $g_2(\sigma)$ span an open simplex of Y_i $(0 \le i \le r)$. We say that g_1 and g_2 are contiguous if there exists a finite sequence

$$h_o = g_1, h_1, h_2, \ldots, h_t = g_2$$

of simplicial maps from $(X_o, \ldots, X_r)$ to $(Y_o, \ldots, Y_r)$ with h_i strictly contiguous to h_{i+1} for $i = 0, \ldots, t-1$. (Contiguity is the equivalence relation generated by strict contiguity. In this section only strict contiguity will play a role.)

Notice that strictly contiguous maps are linearly homotopic. Thus contiguous maps are homotopic.

Proposition 2.6. Any two simplicial approximations

$$g_1, g_2 : \mathrm{co}(X_o, \ldots, X_r) \rightrightarrows \mathrm{co}(Y_o, \ldots, Y_r)$$

of a given map $f : (X_o, \ldots, X_r) \to (Y_o, \ldots, Y_r)$ between tame locally finite complexes are strictly contiguous, provided one of the following two conditions holds:

i) For every index $k \in \{1, \ldots, r\}$ either X_k is closed in X_o or Y_k is locally closed in Y_o.

*) Spanier uses "contiguous" for our "strictly contiguous" and has no word for our "contiguous".

ii) $(Y_o,\dots,Y_r)$ is the barycentric subdivision of another system of complexes.

Proof. We put $X := X_o$, $Y = Y_o$. Let σ be an open simplex in $X_k \cap \operatorname{co} X$ for some $k \in \{0,1,\dots,k\}$. We choose a point x in σ and denote the open simplex of Y_k containing $f(x)$ by ρ. By Lemma 2.2 the points $g_1(x)$ and $g_2(x)$ are both contained in $\overline{\rho}$. This implies that $g_1(\sigma) \cup g_2(\sigma) \subset \overline{\rho} \cap Y_k \cap \operatorname{co} Y$. Let τ denote the open simplex of Y spanned by the vertices of $g_1(\sigma)$ and $g_2(\sigma)$. Of course, $\tau \subset \operatorname{co} Y$ and $g_i(\sigma) \leq \tau \leq \rho$ for $i = 1,2$. We will verify that, under either of the assumptions i), ii), $\tau \subset Y_k$, and then will be done. If Y_k is locally closed in Y_o, then $\tau \subset Y_k$ since $g_1(\sigma) \subset Y_k$ and $\rho \subset Y_k$. If X_k is closed in X, then $X_k \cap \operatorname{co} X = \operatorname{co} X_k$ (cf. Prop. 1.8, but this is trivial). Since $\sigma \subset \operatorname{co} X_k$ and the g_i are simplicial we have $g_1(\sigma) \cup g_2(\sigma) \subset \operatorname{co} Y_k$ and then $\tau \subset \operatorname{co} Y_k \subset Y_k$. Finally if $(Y_o,\dots,Y_r) = (Z_o',\dots,Z_r')$ for a system of complexes $(Z_o,\dots,Z_r)$ then we have open simplices $S_o < \dots < S_t$ of Z_o whose barycenters $\hat{S}_i$ run through the set $E(g_1(\sigma)) \cup E(g_2(\sigma))$ of vertices of $g_1(\sigma)$ and $g_2(\sigma)$. Assume, without loss of generality that $\hat{S}_t$ is a vertex of $g_1(\sigma)$. Then $g_1(\sigma) \subset S_t$ and $g_1(\sigma) \subset Y_k$, hence $S_t \subset Y_k$. This implies that $\tau =]\hat{S}_o,\dots,\hat{S}_t[\subset Y_k$. Thus $\tau \subset Y_k \cap \operatorname{co} Y$ in both cases. q.e.d.

We now have established a rather satisfactory theory of simplicial approximations from the view point of homotopy theory. Let $f : (M,M_1,\dots,M_r) \to (N,N_1,\dots,N_r)$ be a map between systems of spaces and assume that, for every index $i \in \{1,\dots,r\}$, either M_i is closed in M or N_i is locally closed in N. Then Theorem 2.5 provides us with many simplicial approximations to f. Indeed, choose some triangulation $\tilde{\psi} : (\tilde{Y},\tilde{Y}_1,\dots,\tilde{Y}_r) \to (N,N_1,\dots,N_r)$, which is possible by the triangulation theorem (II, Th. 4.4). Let $\psi : (Y,Y_1,\dots,Y_r) \to (N,N_1,\dots,N_r)$ be the first barycentric subdivision of $\tilde{\psi}$. The family $(f^{-1}\psi(\tau) \mid \tau \in \Sigma(Y))$ is locally finite in M. Thus, again by the triangulation theorem, there exists a simultaneous triangulation $\varphi : (X, X_1, \ldots, X_r) \xrightarrow{\sim} (M, M_1, \ldots, M_r)$ of $M, M_1, \ldots, M_r$ and this family. Replacing φ by its barycentric subdivision, if necessary, we may assume that $(X, X_1, \ldots, X_r)$ is tame. According to Theorem 2.5 there exists a simplicial map

$$g : \mathrm{co}(X, X_1, \ldots, X_r) \to \mathrm{co}(Y, Y_1, \ldots, Y_r)$$

such that (φ, ψ, g) is a simplicial approximation to f and, as the proof of the theorem shows, g can be found by a canonical procedure. By Proposition 2.6 any other simplicial map g_1 of this kind is strictly contiguous to g. Finally, if every M_i is locally closed in M, then, by Propositions 2.1 and 2.3, the map $\psi \circ j_Y \circ g \circ r_X \circ \varphi^{-1}$ is homotopic to f.

§3 - The first main theorem on homotopy sets; mapping spaces.

We consider two systems of spaces $(M,A_1,\dots,A_r)$, $(N,B_1,\dots,B_r)$ over R and a real closed field $S \supset R$. Every map (= locally semialgebraic map, cf. §1) $f:(M,A_1,\dots,A_r) \to (N,B_1,\dots,B_r)$ over R yields by base extension (cf. I, 2.10 and I, 4.9) a map

$$f_S : (M(S),A_1(S),\dots,A_r(S)) \to (N(S),B_1(S),\dots,B_r(S))$$

between systems of spaces over S. If g is a second map from $(M,A_1,\dots,A_r)$ to $(N,B_1,\dots,B_r)$ and $H:(M\times I,A_1\times I,\dots,A_r\times I) \to (N,B_1,\dots,B_r)$ is a homotopy from f to g then

$$H_S : (M(S)\times I(S),\dots,A_r(S)\times I(S)) \to (N(S),B_1(S),\dots,B_r(S))$$

is a homotopy from f_S to g_S. (Notice that I(S) is just the unit interval in S.) Thus we have a canonical map $\kappa : [f] \mapsto [f_S]$ from the homotopy set (cf. beginning of §2) $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]$ to the homotopy set $[(M(S),A_1(S),\dots,A_r(S)),(N(S),B_1(S),\dots,B_r(S))]$. The latter set will be briefly denoted by $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)](S)$.

The purpose of this section is to prove the following theorem.

<u>Theorem 3.1</u> (First main theorem). Assume that the space M is semialgebraic.

a) The canonical map $\kappa : [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)] \to [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)](S)$ is always injective.

b) Assume that every A_i is locally closed in M. Assume further that, for every $i \in \{1,\dots,r\}$, either A_i is closed in M or B_i is locally closed in N. Then κ is also surjective.

In order to simplify the notation we assume from now on that $r = 1$. The reader will see easily that all arguments remain valid for any $r \geq 0$. We write A,B instead of A_1,B_1.

In both parts a) and b) of the theorem we may assume without loss of generality that the space N is also semialgebraic. Indeed, to prove surjectivity, we have to consider a map $f : (M(S), A(S)) \to (N(S), B(S))$ and have to find a map $g : (M,A) \to (N,B)$ such that $f \simeq g_S$. The image $f(M(S))$ is a semialgebraic subset of $N(S)$. Now, every semialgebraic subset of $N(S)$ is contained in the base extension $D(S)$ of some (closed) semialgebraic subset D of N. Choosing such a set D with $f(M(S)) \subset D(S)$ we may consider f as a map from $(M(S), A(S))$ to $(D(S), (B \cap D)(S))$. It suffices to prove the claim for this new map. In order to prove injectivity we have to consider two maps f,g from (M,A) to (N,B) and a homotopy $\widetilde{H}$ from f_S to g_S, and we have to find a homotopy H from f to g. Now $\widetilde{H}$ takes values in D(S) for some semialgebraic subset D of N and again we may replace (N,B) by $(D, B \cap D)$.

We choose fixed embeddings of M and N into standard spaces R^n, R^m. Now M is a semialgebraic subset of R^n, N is a semialgebraic subset of R^m, and M(S), N(S) are semialgebraic subsets of S^n, S^m, defined by the same polynomial inequalities (read over S) as M and N. The proof will be based on the following study of "semialgebraic mapping spaces".

We fix natural numbers d,r,s. Let G_i and F_{ij}, $1 \le i \le r$, $1 \le j \le s$ be copies of the vector space P(n,m,d) of polynomials over R of total degree $\le d$ in the variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$. We introduce the R-vector space

$$F := F(n,m,d,r,s) := \prod_{i=1}^{r} (G_i \times \prod_{j=1}^{s} F_{ij})$$

and the following semialgebraic subset L of $M \times N \times F$.

$L := L(M,N,d,r,s) := \{(x,y,(g_i,(f_{ij}))) \in M \times N \times F \mid$ there exists some $i \in \{1,\ldots,r\}$ with $g_i(x,y) = 0$ and $f_{ij}(x,y) > 0$ for $1 \le j \le s\}$.

Let $\pi: L \to F$ be the restriction to L of the canonical projection $M \times N \times F \to F$. For every element $\xi \in F$ we have a semialgebraic subset $\pi^{-1}(\xi)$ of $M \times N$ (identifying $M \times N \times \{\xi\}$ with $M \times N$, as usual). Notice that we obtain all semialgebraic subsets of $M \times N$ in this way, if we vary d,r,s.

We are interested in the subset $Map(M,N,d,r,s)$ of F which consists of all $\xi \in F$ such that $\pi^{-1}(\xi)$ is the graph $\Gamma(f)$ of a semialgebraic map $f : M \to N$. An element $\xi = (g_i,(f_{ij}))$ of F lies in $Map(M,N,d,r,s)$ if and only if the following two conditions are satisfied:

i) For every $x \in M$ there exists a unique $y \in N$ with $(x,y,\xi) \in L$.

ii) For every $(x_o,y_o) \in M \times N$ with $(x_o,y_o,\xi) \in L$ and every $\varepsilon > 0$ in R there exists a $\delta > 0$ in R such that, for every $x \in M$ with $\| x-x_o \| < \delta$ there exists a point $y \in N$ with $\| y-y_o \| < \varepsilon$ and $(x,y,\xi) \in L$.

Thus, by Tarski's theorem on the elimination of quantifiers, $Map(M,N,d,r,s)$ is a semialgebraic subset of F.

<u>Definition 1.</u> We denote, for any point $\xi \in Map(M,N,d,r,s)$, the semialgebraic map $f : M \to N$ with graph $\Gamma(f) = \pi^{-1}(\xi)$ by $<\xi>$, or more precisely by $<\xi>_R$, and we call ξ a <u>parameter</u> of f. The value $<\xi>(x)$ of $<\xi>$ at a point $x \in M$ will usually simply be written $\xi(x)$. We call these maps $<\xi>$ the semialgebraic maps from M to N <u>of type</u> (d,r,s), and we call $Map(M,N,d,r,s)$ the <u>parameter space of maps from</u> M <u>to</u> N <u>of type</u> (d,r,s).

The base extension $Map(M,N,d,r,s)(S)$ of the space $Map(M,N,d,r,s)$ is the parameter space of maps from $M(S)$ to $N(S)$ of type (d,r,s). If ξ is a point of $Map(M,N,d,r,s)$ then ξ is also a point of $Map(M(S),N(S),d,r,s)$, and the map $<\xi>_S : M(S) \to N(S)$ is the base extension of the map $<\xi> : M \to N$.

We now adapt things to the situation in the theorem. We first prove the injectivity of κ. We are given two maps f and g from (M,A) to (N,B) with $f_S \simeq g_S$, and we have to show that $f \simeq g$. For any triple (d,r,s) of natural numbers we look at the parameter space $H(f,g,d,r,s)$ of "homotopies of type (d,r,s)" from f to g, i.e. the set

$$H(f,g,d,r,s) := \{\xi \in Map(M\times I,N,d,r,s) \mid \langle\xi\rangle(A\times I) \subset B,$$
$$\xi(x,0) = f(x),\ \xi(x,1) = g(x) \text{ for every } x \in M\}.$$

This set is semialgebraic in $Map(M\times I, N, d,r,s)$ as follows again from Tarski's theorem. Clearly

$$H(f,g,d,r,s)(S) = H(f_S,g_S,d,r,s).$$

By our assumption on f and g there exists a triple (d,r,s) such that $H(f,g,d,r,s)(S)$ is not empty. But then $H(f,g,d,r,s)$ is also not empty. (N.B. Our theory of base extension, which has this consequence, is also an application of Tarski's theorem). This means that there exists a homotopy (of type (d,r,s)) from f to g, and the injectivity of κ is proved.

The proof of the surjectivity of κ will be harder. It uses the theory of simplicial approximations from §2. We need a lemma. For later use in §4 we state a more general version of the lemma than needed now. The application in the present section uses only the case where the set C below is empty, i.e. the map h can be omitted.

Lemma 3.2. Let $(M,A_1,\dots,A_r)$ and $(N,B_1,\dots,B_r)$ be systems of semialgebraic spaces over R and let $h: C \to N$ be a map on a subspace C of M. Assume there exists an isomorphism $f: M(S) \to N(S)$ such that $f|C(S) = h_S$, $f(A_i(S)) = B_i(S)$ for $1 \le i \le k$ with some $k \in \{0,\dots,r\}$, and $f(A_i(S)) \subset B_i(S)$ for $k < i \le r$. Then there exists an isomorphism $g: M \xrightarrow{\sim} N$ such that $g|C = h$, $g(A_i) = B_i$ for $1 \le i \le k$, and $g(A_i) \subset B_i$ for $k < i \le r$.

Proof. Again we choose fixed embeddings $M \subset R^n$, $N \subset R^m$. For any triple (d,r,s) of natural numbers we consider the subset $Q(d,r,s)$ of $Map(M,N,d,r,s)$ consisting of all points $\xi \in Map(M,N,d,r,s)$ such that $\langle\xi\rangle$ extends h and is an isomorphism from M onto N which maps A_i onto

B_i for $1 \leq i \leq k$ and into B_i for $k < i \leq r$. All these conditions can be expressed by an elementary polynomial formula in the parameter ξ over R by use of Tarski's theorem on elimination of quantifiers. Thus $Q(d,r,s)$ is semialgebraic in $Map(M,N,d,r,s)$. By the assumption of the lemma there exists a triple (d,r,s) such that $Q(d,r,s)(S)$ is not empty. We conclude that $Q(d,r,s)$ is not empty, which means that we have an isomorphism over R (of type d,r,s) with the desired properties. q.e.d.

We assume that either A is closed in M or both A and B are locally closed in M and N respectively. We are given a map $f : (M(S),A(S)) \to (N(S),B(S))$ and we have to find a map $g : (M,A) \to (N,B)$ such that g_S is homotopic to f.

Choosing triangulations of (M,A) and (N,B) and passing over to the first barycentric subdivisions for safety, we assume without loss of generality that both (M,A) and (N,B) are pairs of finite complexes over R which are barycentric subdivisions of other pairs of complexes. We then choose a simultaneous triangulation $\tilde{\varphi} : (\tilde{X},\tilde{X}_1) \xrightarrow{\sim} (M(S),A(S))$ of $(M(S),A(S))$ and the finitely many subsets $\rho(S)$, with $\rho \in \Sigma(M)$, and $f^{-1}(\tau(S))$, with $\tau \in \Sigma(N)$, of $M(S)$.

According to §2 there exists a simplicial approximation $\tilde{u} : co(\tilde{X},\tilde{X}_1) \to co(N(S),B(S))$ to $f \circ \tilde{\varphi}$ (cf. Th. 2.5), and the maps $f \circ \tilde{\varphi}$ and $j_{N(S)} \circ \tilde{u} \circ r_{\tilde{X}}$ from $(\tilde{X},\tilde{X}_1)$ to $(N(S),B(S))$ are (linearly) homotopic (Prop. 2.1 and 2.3). Now $(\tilde{X},\tilde{X}_1)$ is the realization $(|K|_S,|K_1|_S)$ over S of a pair (K,K_1) of finite abstract complexes (II, §3). Thus $(\tilde{X},\tilde{X}_1)$ is the base extension $(X(S),X_1(S))$ of the pair of finite complexes $(X,X_1) := (|K|_R,|K_1|_R)$. Also $\tilde{u}$ is the base extension u_S of a simplicial map $u : co(X,X_1) \to co(N,B)$ and, of course, $r_{\tilde{X}} = (r_X)_S$, $j_{N(S)} = (j_N)_S$. Thus $f \circ \tilde{\varphi} \simeq v_S$ with $v := j_N \circ u \circ r_X$.

Applying Lemma 3.2 to the isomorphism $\tilde{\varphi} : (X(S),X_1(S)) \xrightarrow{\sim} (M(S),A(S))$ we see that there exists a semialgebraic isomorphism $\chi : (X,X_1) \xrightarrow{\sim} (M,A)$ which maps every open simplex σ of X into that open simplex ρ of M whose base extension $\rho(S)$ contains $\tilde{\varphi}(\sigma(S))$. The maps $\tilde{\varphi}$ and χ_S map every open simplex $\sigma(S)$ of $X(S)$ into the same open simplex $\rho(S)$ of $M(S)$. Thus the maps $\tilde{\varphi}$ and χ_S from $(X(S),X_1(S))$ to $(M(S),A(S))$ are (linearly) homotopic, and we have

$$f\circ\chi_S \simeq f\circ\tilde{\varphi} \simeq v_S.$$

Multiplying by χ_S^{-1} on the right we obtain

$$f \simeq (v\circ\chi^{-1})_S.$$

This finishes the proof of Theorem 3.1.

It would be desirable to prove a version of Theorem 3.1 where our assumption that the space M is semialgebraic is eliminated. We may assume that M is connected. Then we have an exhaustion of M by a family of closed semialgebraic subsets $(M_n \mid n\in\mathbb{N})$ with, say, $M_n \subset M_{n+1}$ and the covering $(M_{n+2} \setminus M_n \mid n \geq 0)$ of M locally finite $(M_0 = \emptyset)$. It is tempting to try to prove such a version by applying Theorem 3.1 to the systems $(M_n, A_1 \cap M_n, \dots, A_r \cap M_n)$ and $(N,B_1,\dots,B_r)$ and somehow using the homotopy extension theorem Corollary 1.4.

An application of Corollary 1.4 would be difficult if some of the A_i are not closed in M. But, also in the case that every A_i is closed in M, our Corollary 1.4 seems to be too weak to admit such a proof. One needs a stronger version of Corollary 1.4 where only homotopies are considered which do not move points on C or $C(S)$ for a given closed semialgebraic subset C of M. This leads to a study of "relative" homotopy sets, cf. the next section §4. We shall prove an extension of Theorem 3.1 to relative homotopy sets in §4, provided the A_i are closed in M. We use essentially the same ideas as in the proof of Theorem 3.1 but apply them with more technical effort. We will then be able to generalize the theorem to the case where M is no longer semialgebraic.

§4 - Relative homotopy sets

Again we consider two systems of spaces $(M,A_1,\ldots,A_r)$, $(N,B_1,\ldots,B_r)$ over R. We fix a map $h: C \to N$ on a closed subspace C of M with $h(C \cap A_i) \subset B_i$ $(1 \leq i \leq r)$, and we only consider maps from $(M,A_1,\ldots,A_r)$ to $(N,B_1,\ldots,B_r)$ which extend h. We call any two such maps f,g homotopic relative C, and write $f \simeq g \text{ rel. } C$, if there exists a homotopy $H: (M\times I, A_1\times I,\ldots,A_r\times I) \to (N,B_1,\ldots,B_r)$ with $H_0 = f$, $H_1 = g$, and $H(x,t) = h(x)$ for every $x \in C$, $t \in I$. Such a homotopy H is called constant on C. The set of all $g: (M,A_1,\ldots,A_r) \to (N,B_1,\ldots,B_r)$ which are homotopic to f relative C will be denoted by $[f]^C$ and the set of all these "relative homotopy classes" will be denoted by $[(M,A_1,\ldots,A_r),(N,B_1,\ldots,B_r)]^h$.

Again let S be a real closed overfield of R. Then h yields a map $h_S: C(S) \to N(S)$ which maps every set $A_i(S) \cap C(S)$ into $B_i(S)$ $(1 \leq i \leq r)$. We denote the relative homotopy set $[(M(S),A_1(S),\ldots,A_r(S)),(N(S),B_1(S),\ldots,B_r(S))]^{h_S}$ briefly by $[(M,A_1,\ldots,A_r),(N,B_1,\ldots,B_r)]^h(S)$. As in §3 we have a canonical map $\kappa: [f]^C \mapsto [f_S]^{C(S)}$ from $[(M,A_1,\ldots,A_r),(N,B_1,\ldots,B_r)]^h$ into this set.

Proposition 4.1. If M is semialgebraic, then
$$\kappa: [(M,A_1,\ldots,A_r),(N,B_1,\ldots,B_r)]^h \to [(M,A_1,\ldots,A_r),(N,B_1,\ldots,B_r)]^h(S)$$
is injective.

This is clear by an obvious variation of the proof of the corresponding fact in §3. Instead of the spaces $H(f,g,d,r,s)$ used there, one uses the parameter spaces $H(f,g,C,d,r,s)$ of homotopies relative C between two given maps $f,g: (M,A) \rightrightarrows (N,B)$ which extend h.

Our goal in this section is to prove

Theorem 4.2 (First main theorem, second version). Assume that every A_i

is closed in M. Then the canonical map κ from $[(M,A_1,\dots,A_r),$ $(N,B_1,\dots,B_r)]^h$ to $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]^h(S)$ is bijective.

As in §3 we assume henceforth, for simplicity, that $r = 1$ and write $A_1 = A$, $B_1 = B$. Suppose we have already proved the surjectivity of κ. Then we see that κ is injective by applying the surjectivity result to the systems $(M\times I, A\times I)$, (N,B) and the closed subset $\tilde{C} := (C\times I) \cup (M\times\partial I)$ of $M\times I$ instead of (M,A), (N,B), C. Indeed, let $f,g : (M,A) \rightrightarrows (N,B)$ be two maps with $f|C = g|C = h$ and let $F : (M\times I, A\times I)(S) \to (N,B)(S)$ be a homotopy with $F(-,0) = f_S$, $F(-,1) = g_S$, $F(x,t) = h_S(x)$ for $x \in C(S)$, $t \in I(S)$. {We write $(M,A)(S)$ instead of $(M(S),A(S))$ etc.} We have a map

$$H : \tilde{C} := (C\times I) \cup (M\times 0) \cup (M\times 1) \to N,$$

defined by $H(x,t) = h(x)$ for $(x,t) \in C\times I$, $H(x,0) = f(x)$, $H(x,1) = g(x)$, such that F extends H_S. By the surjectivity result there exists a map

$$G : (M\times I, A\times I) \to (N,B)$$

with G_S homotopic to F relative $\tilde{C}(S)$. Clearly $G(-,0) = f$, $G(-,1) = g$, and $G(x,t) = h(x)$ for $(x,t) \in C\times I$.

Thus it suffices to prove the surjectivity of κ. Suppose this is already done in the case $r = 0$. Then the surjectivity is also clear for $r = 1$ (and more generally for every $r > 0$) by the following argument. Let a map $f : (M,A)(S) \to (N,B)(S)$ be given with $f|C(S) = h_S$. By the surjectivity for $r = 0$ there exists a homotopy $G : (A\times I)(S) \to B(S)$ with $G(-,0) = f|A(S)$, $G(x,t) = h_S(x)$ for $x \in (C\cap A)(S)$, $t \in I(S)$, and $G(-,1) = k_S$ for some map $k : A \to B$. We have $k|A\cap C = h|A\cap C$. Let $C_1 := C \cup A$, and let $h_1 : C_1 \to N$ be the map with restrictions h and k to C and A respectively. Gluing G with the constant homotopy $(x,t) \mapsto h_S(x)$ on $(C\times I)(S)$ we obtain a homotopy $H : (C_1\times I)(S) \to N(S)$. We have $H(-,0) = f|C_1(S)$ and $H(x,1) = (h_1)_S(x)$. Now C_1 is closed in M. By the homotopy extension theorem Cor. 1.4 there exists a homotopy $F : (M\times I)(S) \to N(S)$

with $F(-,0) = f$ and $F|(C_1 \times I)(S) = H$. F is constant on $C(S)$, maps $(A \times I)(S)$ into $B(S)$, and ends up with a map $f_1 : (M,A)(S) \to (N,B)(S)$ such that $f_1|C_1(S) = (h_1)_S$. Applying again the surjectivity of κ for $r = 0$ we obtain a homotopy $F_1 : (M \times I)(S) \to N(S)$ which is constant on $C_1(S)$, starts with f_1 and ends with g_S for some $g : M \to N$. The composite homotopy $\widetilde{F} := F * F_1$ is constant on $C(S)$, starts with f and ends with g_S. This proves the surjectivity of κ for $r = 1$ provided we know the surjectivity for $r = 0$. Notice that we used in an essential way that A is closed in M.

Henceforth we assume that $r = 0$. We now explain how the surjectivity of κ can be proved in general once we know that it holds for M semialgebraic. We may assume that M is connected. We choose a locally finite covering $(X_n | n \in \mathbb{N})$ of M by closed semialgebraic subsets. (We can even choose a covering with $X_n \cap X_m = \emptyset$ if $|n-m| > 1$, cf. I.4.19 and I.4.11.) We define $M_n := X_1 \cup \ldots \cup X_n$ and $C_n := M_n \cup C$.

Let a map $f : M(S) \to N(S)$ be given with $f|C(S) = h_S$. We have to find a map $g : M \to N$ with $f \simeq g_S$ rel. $C(S)$. Since we assume that the theorem holds in the semialgebraic case, there exists a homotopy $H_1' : (M_1 \times I)(S) \to N(S)$ with $H_1'(-,0) = f|M_1(S)$, H_1' constant on $(C \cap M_1)(S)$, and $H_1'(-,1)$ the base extension to S of some map from M_1 to N. By the homotopy extension theorem 1.4 we then obtain a homotopy

$$H_1 : (M \times I)(S) \to N(S)$$

with $H_1(-,0) = f$, $H_1(x,t) = h_S(x)$ for all $x \in C(S)$, $t \in I(S)$, and a final map $f_1 := H_1(-,1)$ from $M(S)$ to $N(S)$ such that $f_1|C_1 = (h_1)_S$ for some map $h_1 : C_1 \to N$ which extends h. (Recall that $C_1 := C \cup M_1$.) Now we repeat the same construction with f, h, M_1 replaced by f_1, h_1, M_2 and obtain a homotopy

$$H_2 : (M \times I)(S) \to N(S)$$

with $H_2(-,0) = f_1$, $H_2(-,1) =: f_2$, H_2 constant on $C_1(S)$ and $f_2|C_2 = (h_2)_S$ for some $h_2 : C_2 \to N$ extending h_1. By iteration we obtain a sequence of maps $(h_n|n\in\mathbb{N})$, $h_n : C_n \to N$, with $h_n|C_{n-1} = h_{n-1}$, a sequence of maps $(f_n|n\in\mathbb{N})$, $f_n : M(S) \to N(S)$, with $f_n|C_n(S) = (h_n)_S$, and a sequence of homotopies $(H_n|n\in\mathbb{N})$, $H_n : (M\times I)(S) \to N(S)$, with H_n constant on $C_{n-1}(S)$, $H_n(-,0) = f_{n-1}$ (let $C_0 = C$, $f_0 = f$) and $H_n(-,1) = f_n$. The maps h_n fit together to a map $g : M \to N$. We choose in $]0,1[\subset R$ a sequence $(s_n|n\in\mathbb{N})$ with $s_n < s_{n+1}$ for all n (say $s_n = 1-2^{-n}$). Then we define a set theoretic map $G : (M\times I)(S) \to N(S)$ by the following formulas: $G(x,t) = g_S(x)$ if $x \in M(S)$ and $t > s_n$ for every $n \in \mathbb{N}$, and

$$G(x,t) = H_n(x,(s_n-s_{n-1})^{-1}(t-s_{n-1}))$$

if $x\in M(S)$, $s_{n-1} \leq t \leq s_n$, with $s_0 := 0$. We claim that G is locally semialgebraic. By the gluing principle for maps (I, 3.16) it suffices to check for every $m \in \mathbb{N}$ that the restriction $G|(M_m\times I)(S)$ is semialgebraic. Now, for $x \in M_m(S)$, $t \in I(S)$, and $n > m$ we have $H_n(x,t) = f_{n-1}(x) = (h_m)_S(x) = g_S(x)$. Thus G is semialgebraic on $(M_m\times[\ s_m\ ,1])(S)$. But clearly G is also semialgebraic on $(M_m\times[0,\ s_m\ ])(S)$. Thus G is semialgebraic on $(M_m\times[0,1])(S)$. By construction $G(-,0) = f$, $G(-,1) = g_S$ and $G(x,t) = h_S(x)$ for $x \in C(S)$, $t \in I(S)$, as desired.

It remains to prove the surjectivity of κ in the case where M is semialgebraic. We then may also assume that N is semialgebraic, cf. the corresponding argument in §3. Let a map f from $M(S)$ to $N(S)$ be given with $f|C(S) = h_S$. We have to find a map g from M to N with $f \simeq g_S$ rel. $C(S)$

Since things are more complicated here than in the proof of surjectivity of κ in §3 we are less audacious than in §3 and first reduce to the case where M is complete. So assume that the theorem holds in the case that M is complete. We may then prove the theorem for M semialgebraic as follows.

Choosing a simultaneous triangulation of (M,C) and passing to its first barycentric subdivision, we assume without loss of generality that (M,C) is a system of finite complexes which is the barycentric subdivision of another system of complexes. Let $(M_o,C_o) = co(M,C)$ be the core of this system and let r denote the canonical retraction from (M,C) to (M_o,C_o). Since we assume that the theorem holds in the complete case we have a homotopy $F : (M_o\times I)(S) \to N(S)$ and a map $u : M_o \to N$ such that $F(-,0) = f|M_o(S)$, $F(-,1) = u_S$, $F(x,t) = h_S(x)$ for all $(x,t) \in (C_o\times I)(S)$. We now introduce the maps $F' := F\circ(r\times id_I)_S : (M\times I)(S) \to N(S)$, and $u\circ r : M \to N$. We have $F'(-,0) = f\circ r_S$, $F'(-,1) = u_S\circ r_S$, and $F'(x,t) = h_S\circ r_S(x)$ for all $(x,t) \in (C\times I)(S)$. We have obvious homotopies $G : (M\times I)(S) \to N(S)$ from f to $f\circ r_S$, and $K_o : (C\times I) \to N$ from $h\circ r$ to h, defined by $G(x,t) := f((1-t)x + t\,r_S(x))$, $K_o(x,t) := h(t\,x + (1-t)\,r(x))$. We extend K_o to a homotopy $K : (M\times I) \to N$ with $K(-,0) = u\circ r$. We define $g := K(-,1)$. Now we compose the three homotopies G,F',K_S, say at $t = \frac{1}{3}$ and $t = \frac{2}{3}$, to form a single homotopy

$$H : (M\times I)(S) \to N(S).$$

We draw a schematic picture of this homotopy.

$C(S)$		const	
$M(S)$	G	F'	K_S

f $\quad f\circ r_S$ $\quad (u\circ r)_S$ $\quad g_S$

Then $H|(C\times I)(S) = (H_o)_S$ with a homotopy $H_o : C\times I \to N$ which may be depicted as follows:

C	G_o	const	K_o

h $\quad h\cdot r$ $\quad h\cdot r$ $\quad h$

Here G_o is the inverse homotopy to K_o. Our homotopy H goes from f to g_S but is not yet constant on $C(S)$. This can now be easily remedied. We have an obvious homotopy

$$\Phi : C\times I\times I \to N$$

with $\Phi(-,0) = H_o$, $\Phi(x,t,1) = h(x)$ for all $(x,t) \in C\times I$, and Φ constant on $(C\times 0) \cup (C\times 1)$. By the homotopy extension theorem Φ_S can be extended to a homotopy

$$\Psi : (M\times I\times I)(S) \to N(S)$$

with $\Psi(-,0) = H$ and Ψ constant on $[(M\times 0) \cup (M\times 1)](S)$. Consider the map $\tilde{H} := \Psi(-,1)$ from $(M\times I)(S)$ to $N(S)$. We have $\tilde{H}(-,0) = f$, $\tilde{H}(-,1) = g_S$ and $\tilde{H}(x,t) = h_S(x)$ for all $(x,t) \in (C\times I)(S)$. Thus $\tilde{H}$ solves our problem.

From now on we assume that M is complete. We choose a closed semialgebraic neighbourhood D of C in M such that C is a strong deformation retract of D (cf. Th. 1.1). Then we choose a map $h_1 : D \to N$ which extends h and a homotopy $G : (D\times I)(S) \to N(S)$ from $f|D(S)$ to $(h_1)_S$ which is constant on $C(S)$ (cf. Prop. 1.2). We extend G to a homotopy $F : (M\times I)(S) \to N(S)$ with $F(-,0) = f$. Then $F(-,1)$ is a map f_1 from $M(S)$ to $N(S)$ with $f_1|D(S) = (h_1)_S$. We now replace f by f_1. Changing notation we assume again that f is a map from $M(S)$ to $N(S)$, but that the map h is defined on the closed neighbourhood D of C, $h : D \to N$, and that $f|D(S) = h_S$. As before we want to deform f relative $C(S)$ into a map which is the base extension g_S of a map $g : M \to N$. Loosely speaking, we have built a barrier $D \setminus C$ around C, which will help us to protect f from being disturbed within C. We choose a closed semialgebraic neighbourhood E of C in M which is contained in the interior $\mathring{D}$ of D (cf. I.4.14).

Using suitable triangulations of (M,C,D,E) and N we assume that (M,C,D,E) is a system of finite complexes, that N is a finite complex which is the barycentric subdivision of another complex, and that $h : D \to N$ maps every open simplex of D into an open simplex of N. Let L denote the closed subcomplex $M \setminus \mathring{D}$ of M. We choose a simultaneous triangulation of $L(S)$ and the finitely many subsets $\rho(S)$, with $\rho \in \Sigma(L)$, and $L(S) \cap f^{-1}(\tau(S))$, with $\tau \in \Sigma(N)$. Applying II, Lemma 4.3 (choosing $E(S) \sqcup L(S)$ as the subspace M_o there) we then obtain a simultaneous triangulation $\tilde{\varphi}: \tilde{X} \to M(S)$ of $M(S)$, and the subsets $\rho(S)$, $f^{-1}(\tau(S))$ of $M(S)$ with ρ running through $\Sigma(M)$ and τ running through $\Sigma(N)$, such that $\tilde{\varphi}$ is equivalent on $E(S)$ to the tautological triangulation $id_{E(S)}$. At this point our assumption that M is complete bears fruits.

Let $\tilde{X}_1 := \tilde{\varphi}^{-1}(C(S))$ and $\tilde{X}_2 := \tilde{\varphi}^{-1}(E(S))$. Then $\tilde{\varphi}|\tilde{X}_2$ is a simplicial isomorphism $\tilde{\psi}$ from $\tilde{X}_2$ to $E(S)$ which maps the subcomplex $\tilde{X}_1$ of $\tilde{X}_2$ onto $C(S)$. According to §2 (i.e. essentially by classical theory [Spa, Chap. 3] since M is complete) there exists a simplicial approximation $\tilde{u}: \tilde{X}_1 \to N(S)$ to $f \circ \tilde{\varphi}$, and $\tilde{u}$ is linearly homotopic to $f \circ \tilde{\varphi}$. As in §3 we write $(\tilde{X}, \tilde{X}_1, \tilde{X}_2) = (X, X_1, X_2)(S)$ with a system of complexes (X, X_1, X_2) over R. We have $\tilde{\psi} = \psi_S$ with a simplicial isomorphism $\psi: (X_2, X_1) \xrightarrow{\sim} (E, C)$, and $\tilde{u} = u_S$ with a simplicial map $u: X \to N$. By Lemma 3.2 there exists a semialgebraic isomorphism $\chi: X \xrightarrow{\sim} M$ which extends $\psi: X_2 \xrightarrow{\sim} E$ and maps every open simplex σ of X into the open simplex ρ of M whose base extension $\rho(S)$ contains $\tilde{\varphi}(\sigma(S))$. The maps χ_S and $\tilde{\varphi}$ from $X(S)$ to $M(S)$ are (linearly) homotopic relative $X_2(S)$. Thus $f \circ \chi_S$ is homotopic to $f \circ \tilde{\varphi}$ relative $X_2(S)$, and a fortiori

$$f \circ \chi_S \simeq f \circ \tilde{\varphi} \quad \text{rel } X_1(S)$$

On the other hand, we have a linear homotopy from $f \circ \tilde{\varphi}$ to u_S. But this homotopy moves points in $X_1(S)$. To remedy this, we choose a semialgebraic function $\lambda: X \to [0,1]$ with $\lambda^{-1}(0) = X_1$ and $\lambda^{-1}(1) = X \setminus \dot{X}_2$ (cf. I. Th. 4.15). Then we define a homotopy

$$H: (X \times I)(S) \to N(S)$$

by the formula $(x \in X(S), t \in I(S))$

$$H(x,t) = (1-t\lambda_S(x))(f \circ \tilde{\varphi})(x) + t\lambda_S(x)\, u_S(x) .$$

This homotopy is constant on $X_1(S)$ and starts with $H(-,0) = f \circ \tilde{\varphi}$. We have $H(-,1) = v_S$ with the following semialgebraic map $v: X \to N$:

$$v(x) = \begin{cases} u(x), & x \in X \setminus X_2, \\ (1-\lambda(x))(h \circ \psi)(x) + \lambda(x)\,u(x), & x \in X_2. \end{cases}$$

Thus $f \cdot \tilde{\varphi} \simeq v_S$ rel. $X_1(S)$. Altogether we have

$$f \circ \chi_S \simeq v_S \quad \text{rel } X_1(S)$$

Multiplying by the isomorphism χ_S^{-1} from $M(S)$ to $X(S)$ we obtain

$$f \simeq (v \cdot \chi^{-1})_S \quad \text{rel } C(S)$$

as desired. This completes the proof of Theorem 4.2.

<u>Discussion 4.3.</u> For a given base field R we denote by $LSA(R)$ the category of (locally semialgebraic regular paracompact) spaces and (locally semialgebraic) maps over R. We denote by $HLSA(R)$ the "homotopy category" of spaces over R. This category has the same objects as $LSA(R)$ but its morphisms are the homotopy classes of maps over R. If S is a real closed overfield of R then we have the base extension functor $LSA(R) \to LSA(S)$. Every object $\tilde{M}$ of $LSA(S)$ is isomorphic to the image $M(S)$ of an object M of $LSA(R)$. Indeed, every triangulation $|K|_S \xrightarrow{\sim} \tilde{M}$ yields such an object, namely $M = |K|_R$. Our functor induces a functor $\kappa : HLSA(R) \to HLSA(S)$. The main message of Theorem 4.2 is that κ <u>is an equivalence of categories</u>. More generally, Theorems 3.1 and 4.2 imply similar statements about suitable homotopy categories of systems of spaces.

<u>Final remark 4.4.</u> Theorem 4.2 remains true for <u>locally</u> finite systems $(A_\alpha | \alpha \in I)$, $(B_\alpha | \alpha \in I)$ of subspaces of M and N instead of the systems $(A_1, \ldots, A_r)$, $(B_1, \ldots, B_r)$, with every A_α closed in M, of course. The proof runs the same way with more notational effort.

§5 - The second main theorem; contiguity classes

We assume now that our base field R is the field $\mathbb{R}$ of real numbers. Let two systems of spaces $(M,A_1,\dots,A_r)$, $(N,B_1,\dots,B_r)$ over $\mathbb{R}$ be given. Beside locally semialgebraic maps and homotopies from the first system to the second we also consider continuous homotopy classes of continuous maps from $(M,A_1,\dots,A_r)$ to $(N,B_1,\dots,B_r)$ in the usual topological sense. We denote the set of these classes by $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]_{top}$. We have an obvious map

$$\lambda : [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)] \to [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]_{top},$$

which sends the locally semialgebraic homotopy class $[f]$ of a locally semialgebraic map f to the topological homotopy class $[f]_{top}$ of f.

More generally, given a locally semialgebraic map

$$h : (C,C\cap A_1,\dots,C\cap A_r) \to (N,B_1,\dots,B_r)$$

on a closed subspace C of A we have a map λ from the set $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]^h$ of locally semialgebraic homotopy classes relative C of locally semialgebraic maps extending h to the analogous set of topological relative homotopy classes $[(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]^h_{top}$. We want to prove

<u>Theorem 5.1</u> (Second main theorem). Assume that every A_i is closed in M. Then the map $\lambda : [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]^h \to [(M,A_1,\dots,A_r),(N,B_1,\dots,B_r)]^h_{top}$ is bijective.

To a great extent the proof will follow the same pattern as the proof of the first main theorem 4.2 in the preceding section. By four reduction steps which are fully analogous to the corresponding reduction steps in §4 we first see that it suffices to prove the surjectivity of λ and then that we may assume $r = 0$ and that M is semialgebraic and complete. The role of maps over R in §4 is played here by the locally semialgebraic maps over $\mathbb{R}$ and the role of maps over S in §4 is played by the continuous maps. Instead of the homotopy extension theorem 1.4 one uses the analogous topological fact, which is true as well. (Recall the proof of Corollary 1.4). We leave the details to the reader.

Now let $f : M \to N$ be a continuous map extending h. Since we assume that M is complete, M is a compact topological space. Thus $f(M)$ is compact as well. Choosing a covering of N by open semialgebraic subsets we can cover $f(M)$ by finitely many of them. Thus $f(M)$ is contained in a semialgebraic subspace of N. Replacing N by this subspace we assume henceforth that N is also semialgebraic.

As in §4 we construct a "barrier" around C, i.e. a closed subspace D of M with $\mathring{D} \supset C$ such that f is homotopic relative C to a map $f_1 : M \to N$ whose restriction to D is semialgebraic. We replace f by f_1 and we denote the restriction $f|D$ - instead of $f|C$ - by h. Thus h is now a semialgebraic map from D to N. As in §4 we choose a closed semialgebraic neighbourhood E of C in M with $E \subset \mathring{D}$.

Using triangulations we assume that (M,C,D,E) is a system of finite complexes and that N is a finite complex which is the first barycentric subdivision of another complex. Now we meet the big difference to §4. We cannot refine the tautological triangulation of (M,C,D,E) in such a way that f maps every open simplex of M into an open simplex of N, since now the preimages of the open simplices of N under f have no reason to be semialgebraic. Thus our existence theorem 2.5 for simplicial approximations is of no use here. But since the field $\mathbb{R}$ is archimedian and the topological space M is compact there exists a natural number n such that every open star of the n-th barycentric subdivision $M^{(n)}$ of M is contained in the preimage $f^{-1}(\mathrm{St}_N(p))$ of an open star $\mathrm{St}_N(p)$ for some vertex $p \in N \cap E(N)$, cf. [Spa, Chap. 3, §3, Th. 14]. The family of these stars is a (finite) open covering of N since N is tame . We replace (M,C,D,E) by its n-th barycentric subdivision. Then we choose for every vertex $e \in E(M)$ a vertex $\mu(e) \in N \cap E(N)$ with $f(\mathrm{St}_M(e)) \subset \mathrm{St}_N(\mu(e))$. The map $\mu : E(M) \to N \cap E(N)$ is an abstract simplicial map from $K(M)$ to $K(\mathrm{co}\, N)$, cf. [Spa, Chap. 3, §4]. Evidently the realization $u = |\mu| : M \to \mathrm{co}(N)$ is a simplicial approximation to $f : M \to N$. We have a linear homotopy H from f to u. This homotopy moves points in C. But since we have built a barrier $D \setminus C$ around C we can modify H to a homotopy which starts with f, is constant on C, and ends up with a semialgebraic map v by the same device as in §4. This completes the proof of the theorem.

Remark 5.2. Theorem 5.1 remains true for locally finite systems of subspaces instead of finite systems, as does the first main theorem 4.2.

Discussion 5.3. Let TOP denote the category of topological (Hausdorff) spaces and let HTOP denote the homotopy category of these spaces. We have forgetful functors $LSA(\mathbb{R}) \to TOP$ and $\lambda : HLSA(\mathbb{R}) \to HTOP$. The main message of Theorem 5.1 is that λ is an equivalence of $HLSA(\mathbb{R})$ with a full subcategory of HTOP. As objects of this subcategory we may take those topological spaces which are homeomorphic to locally finite simplicial complexes or those spaces which are homotopy equivalent to such spaces.

In §3 we gave a version of the first main theorem which only applies to semialgebraic spaces and "absolute" homotopy classes but there is much stronger than the second version Theorem 4.2, since the subspaces of M and N are allowed to be more general, for example open. We do not know whether an analogous version of the second main theorem holds. We pose a subquestion of this problem.

Question 5.4. Given - say - complete semialgebraic spaces M,N over $\mathbb{R}$, open subspaces U,V of M,N, and a continuous map $f:(M,U) \to (N,V)$, does there always exist a semialgebraic map $g:(M,U) \to (N,V)$ which is (continuously) homotopic to f?

The first main theorem, in version 4.2, and the second main theorem 5.1 together form a powerful tool to transfer results from topological (= classical) homotopy theory to locally semialgebraic spaces over any real closed base field. As a first example we give a combinatorial description of the homotopy set [X,Y] for X a finite complex and Y a locally finite complex. Other examples will come up later (cf. in particular §6).

From now on the base field R may again be an arbitrary real closed field. Let K and L be abstract locally finite complexes.

Definition (cf. [Spa, p. 130]). Let $\varphi,\psi : K \rightrightarrows L$ be two simplicial maps.

a) φ and ψ are strictly contiguous if for every simplex $s \in S(K)$ the union $\varphi(s) \cup \psi(s)$ is a simplex of L.
b) φ and ψ are contiguous if there exists a sequence $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_r = \psi$ of simplicial maps from K to L such that φ_{i-1} and φ_i are strictly contiguous for $i = 1,\ldots,r$. The contiguity class of φ is denoted by $[\varphi]$. The set of contiguity classes of simplicial maps from K to L is denoted by [K,L].

Notice that φ and ψ are strictly contiguous (resp. contiguous) precisely if the realizations $|\varphi|_R, |\psi|_R$ are strictly contiguous (resp. contiguous) maps from $|K|_R$ to $|L|_R$ in the sense of Definition 5 in §2.

By sd K we denote the first barycentric subdivision of K. This is a purely combinatorial notion: The vertices of sd K are the simplices of K and the simplices of sd K are the sets $\{s_0, s_1, \dots, s_r\}$ with $s_i \in S(K)$ and $s_0 \subset s_1 \subset \dots \subset s_r$. As realization $|\mathrm{sd}\,K|_R$ we can and will choose the barycentric subdivision $|K|_R'$ of $|K|_R$ in the geometric sense, a vertex s of sd K being realized by the barycenter of the simplex $|s|_R$.

From now on we assume that the complex K is <u>closed</u>. We will use the classical theory of simplicial approximations, as developed in [Spa, 3.4], transfered to an arbitrary real closed base field R in the obvious way. The more complicated theory of simplicial approximations from §2 may be ignored here since K is closed. We will consider simplicial maps from iterated barycentric subdivisions $\mathrm{sd}^r K$ of K to L. Any such map has its image in the core of L. Thus we may always replace L by co L . We assume henceforth that L is also closed.

Let $h : \mathrm{sd}\,K \to K$ be the abstraction of a simplicial approximation to the identity map of the space $|K|_R = |\mathrm{sd}\,K|_R$. We obtain every such map h by mapping every vertex s of sd K to an arbitrarily chosen vertex of s. Notice that, for any other real closed field $\tilde{R}$, the map $|h|_{\tilde{R}}$ is also a simplicial approximation to the identity of $|K|_{\tilde{R}}$.

h induces a map $\eta : [K,L] \to [\mathrm{sd}\,K,L]$, $[\varphi] \mapsto [\varphi \circ h]$. This map η does not depend on the choice of h. Indeed, if h_1 is the abstraction of another simplicial approximation to $\mathrm{id}_{|K|}$ then h and h_1 are strictly contiguous. Also $\eta([\varphi]) = [\psi]$ with $\psi : \mathrm{sd}\,K \to L$ the abstraction of <u>any</u> simplicial approximation $|\mathrm{sd}\,K| \to |L|$ of $|\varphi| : |K| \to |L|$.

Similarly, given two iterated barycentric subdivisions $\mathrm{sd}^r K$ and $\mathrm{sd}^s K$ with $r < s$, we obtain a canonical map

$$\eta_{rs} : [\mathrm{sd}^r K, L] \to [\mathrm{sd}^s K, L]$$

from the abstraction $h_{rs} : sd^sK \to sd^rK$ of any simplicial approximation $|sd^sK| \to |sd^rK|$ to the map $x \mapsto x$. If $r < s < t$ then $\eta_{st} \circ \eta_{rs} = \eta_{rt}$. Thus we have a direct system of sets $([sd^rK,L], r \geq 0)$.

If $\varphi, \psi : sd^rK \rightrightarrows L$ are two strictly contiguous maps then their realizations $|\varphi|, |\psi| : |K| \rightrightarrows |L|$ are linearly homotopic. Thus, if φ and ψ are contiguous then $|\varphi|$ and $|\psi|$ are homotopic. For that reason we have a well defined map

$$\rho_r : [sd^rK,L] \to [|K|,|L|],\ [\varphi] \mapsto [|\varphi|].$$

If $r < s$ then $\rho_s \circ \eta_{rs} = \rho_r$ since the map $|h_{rs}| : |K| \to |K|$ is homotopic to the identity. Thus the maps ρ_r, $r \in \mathbb{N}_0$, yield a map

$$\rho : \varinjlim_r\, [sd^rK,L] \to [|K|_R, |L|_R].$$

<u>Theorem 5.5.</u> If the complexes K and L are closed and K is finite then ρ is bijective.

<u>Proof.</u> We denote the map ρ more precisely by ρ^R. First assume that $R = \mathbb{R}$. The composite of $\rho^{\mathbb{R}}$ with the canonical bijection λ from $[|K|_{\mathbb{R}}, |L|_{\mathbb{R}}]$ to $[|K|_{\mathbb{R}}, |L|_{\mathbb{R}}]_{top}$ (cf. Th. 5.1) is the map from $\varinjlim\, [sd^rK,L]$ to $[|K|_{\mathbb{R}}, |L|_{\mathbb{R}}]_{top}$ discussed in [Spa 3.5]. By Theorem 8 in [Spa 3.5] this map is a bijection. Thus $\rho^{\mathbb{R}}$ is a bijection.

Now we assume that R is the field R_o of real algebraic numbers. The triangle

$$\begin{tikzcd}
 & {[|K|_{\mathbb{R}}, |L|_{\mathbb{R}}]} \\
\varinjlim\, [sd^rK,L] \arrow[ur, "\rho^{\mathbb{R}}"] \arrow[dr, "\rho^{R_o}"'] & \\
 & {[|K|_{R_o}, |L|_{R_o}]} \arrow[uu, "\kappa"', "\cong"]
\end{tikzcd}$$

with κ the canonical bijection (Th. 3.1 or 4.2) commutes. Since $\rho^{\mathbb{R}}$ is a bijection ρ^{R_o} is also a bijection. If R is an arbitrary real closed field then $R_o \subset R$. We compare ρ^{R_o} with ρ^R in the same way and conclude that ρ^R is bijective. q.e.d.

Remark 5.6. More generally we obtain in the same way a description by contiguity classes of the homotopy set $[(|K|,|K|_1,\ldots,|K_r|),(|L|,|L_1|,\ldots,|L_r|)]$ for K a finite and closed complex and subcomplexes K_i, L_i which are closed in K and L respectively. Spanier [loc.cit.] states and proves the corresponding topological theorem for $r = 1$. If K is not finite then the theorem becomes wrong for $r = 0$, cf. [Spa 3.5.7].

Theorem 5.5 gives a purely combinatorial - albeit complicated - description of the semialgebraic homotopy set $[|K|,|L|]$. Notice that this implies a description of $[M,N]$ for a semialgebraic space M and an arbitrary space N over R, as soon as good triangulations $\alpha : X \xrightarrow{\sim} M$ and $\beta : Y \xrightarrow{\sim} N$ are given. Simply take for K and L the abstractions of co X and co Y and notice that we have a bijection $[f] \mapsto [r_Y \circ f \circ j_X]$ from $[X,Y]$ to $[\mathrm{co}\,X, \mathrm{co}\,Y]$, with j_X the inclusion $\mathrm{co}\,X \hookrightarrow X$ and r_Y the canonical retraction $Y \to \mathrm{co}\,Y$, cf. §1.

Despite its easy proof Theorem 5.5 is quite astonishing if the base field R is non archimedian. The surjectivity of ρ means that for every semialgebraic map $f : |K| \to |L|$ we can find a natural number n and a simplicial map $\varphi : sd^n K \to L$ such that $|\varphi|$ is homotopic to f. In fact, using compactness arguments and the Lebesgue Lemma one proves for $R = \mathbb{R}$ an even stronger version [Spa, p. 128]: φ can be chosen in such a way that $|\varphi|$ is a simplicial approximation to f, hence is linearly homotopic to f. But this last statement becomes false if the base field is non archimedian as is shown by the following simple example.

Counterexample 5.7. Assume that R is non archimedian. We choose a positive element ε in R which is smaller than every positive rational number. Let X be the boundary of a 2-simplex $[P_0,P_1,P_2]$ in R^2. For any $\lambda \in [0,1]$ we denote by $P_{ij}(\lambda)$ the point $(1-\lambda)P_i + \lambda P_j$. We consider the map $f : X \to X$ with $f(P_0) = P_0$, $f(P_{01}(2\varepsilon)) = P_1$, $f(P_1) = P_{12}(1/2)$, $f(P_2) = P_2$ and f linear (= affine) on each interval $[P_0,P_{01}(2\varepsilon)]$, $[P_{01}(2\varepsilon),P_1]$, $[P_1,P_2]$, $[P_2,P_0]$. Assume that there exists a natural number n and a simplicial map $g : X^{(n)} \to X$ from the n-th barycentric subdivision $X^{(n)}$ of X to X with g linearly homotopic to f. Let $Q := P_{01}(2^{-n})$. The points $P_{01}(\varepsilon)$ and $P_{01}(3\varepsilon)$ in the simplex $]P_0,Q[$ of $X^{(n)}$ are mapped by f into $]P_0,P_1[$ and $]P_1,P_2[$ respectively, while g maps the whole simplex $]P_0,Q[$ into (in fact onto) one open simplex σ of X. Since the images under f of both points $P_{01}(\varepsilon)$, $P_{01}(3\varepsilon)$ are "linearly connectable" in X to their images under g, σ must be the 0-simplex $\{P_1\}$. This forces $g(P_0) = g(Q) = P_1$. A point T on $]P_2,P_0[$ near P_0 is mapped to a point near P_1 by g, hence to a point in $]P_0,P_1[\cup [P_1,P_2[$. But then $f(T) = T$ is not linearly connectable to $g(T)$. This contradiction proves that such a simplicial map g does not exist.

§6 - Homotopy groups

For every pointed space (M,x_o) over R we define homotopy groups $\pi_n(M,x_o)$, $n \geq 1$, in the classical way [Hu, IV, §2]:

$$\pi_n(M,x_o) := [(I^n,\partial I^n),\ (M,x_o)].$$

(We write (M,x_o) instead of $(M,\{x_o\})$ for simplicity.) The multiplication is given by

$$[f]\cdot[g] := [f*g],$$

the product $f*g$ of two maps $f,g : (I^n,\partial I^n) \rightrightarrows (M,x_o)$ being defined by

$$(f*g)(t_1,\ldots,t_n) = \begin{cases} f(2t_1,t_2,\ldots,t_n), & 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1-1,t_2,\ldots,t_n), & \frac{1}{2} \leq t_1 \leq 1 \end{cases}.$$

The inverse $[f]^{-1}$ of a class $[f]$ is the class $[h]$ with

$$h(t_1,\ldots,t_n) := f(1-t_1,t_2,\ldots,t_n).$$

For $n = 0$ we still define a pointed set $\pi_n(M,x_o)$ as above (I^o = one point space, $\partial I^o = \emptyset$). $\pi_o(M,x_o)$ turns out to be the set of connected components of the space M, the base point being the component of x_o. But, in general, there is no reasonable multiplication on $\pi_o(M,x_o)$.

Using a standard isomorphism from the pointed space $(I^n/\partial I^n,\partial I^n/\partial I^n)$ (cf. II, §10) to the space (S^n,∞), with S^n the unit sphere in R^{n+1} and ∞ the north pole of S^n, we may interpret $\pi_n(M,x_o)$ as the set of homotopy classes $[(S^n,\infty),(M,x_o)]$.

For every $n \geq 1$ we identify I^{n-1} with the subspace $I^{n-1}\times\{0\}$ of I^n, and we denote the closure of $\partial I^n \setminus I^{n-1}$ in ∂I^n by J_{n-1}. Then we define for any $n \geq 1$ and any triple (M,A,x_o), with M a space, A a subspace (= locally semialgebraic subset) of M, x_o a point of A, a pointed set

$$\pi_n(M,A,x_o) := [(I^n,\partial I^n,J_{n-1}),(M,A,x_o)],$$

the base point being the class of the constant map with value x_o. For $n \geq 2$ this set turns out to be a group with multiplication and inverse given by the same formulas as above, cf. [Hu, IV, §3]. On the other hand, $\pi_1(M,A,x_o)$ is the set $[(I,\{0,1\},1),(M,A,x_o)]$ of homotopy classes of semialgebraic paths in M which start in A and end at x_o. In general we apparently have no reasonable multiplication on this set. Notice that, for every $n \geq 1$, $\pi_n(M,\{x_o\},x_o) = \pi_n(M,x_o)$. We define $\pi_o(M,\{x_o\},x_o) := \pi_o(M,x_o)$.

Using a standard isomorphism $(I^n/J_{n-1},\partial I^n/J_{n-1},J_{n-1}/J_{n-1}) \xrightarrow{\sim} (D^n,S^{n-1},\infty)$ with D^n the closed unit ball in R^n, we may identify $\pi_n(M,A,x_o)$ with the set $[(D^n,S^{n-1},\infty),(M,A,x_o)]$. Choosing some point z_o in J_{n-1} we may also identify $\pi_n(M,A,x_o)$ with the set $[(I^n,\partial I^n,z_o),(M,A,x_o)]$ since the inclusion map from $(I^n,\partial I^n,z_o)$ to $(I^n,\partial I^n,J_{n-1})$ is a homotopy equivalence of triples, cf. [DKP, p. 201].

Every map $\varphi: (M,A,x_o) \to (N,B,y_o)$ between pointed pairs of spaces over R induces a map between pointed sets

$$\pi_n(\varphi) : \pi_n(M,A,x_o) \to \pi_n(N,B,y_o)$$

in the obvious way, $[f] \mapsto [\varphi\circ f]$, for every $n \geq 1$, and, in case $A=\{x_o\}$, $B=\{y_o\}$, also for $n = 0$. The map $\pi_n(\varphi)$ is a group homomorphism if $n \geq 2$, and also if $n=1$, $A=\{x_o\}$, $B=\{y_o\}$. As usual, we often write φ_* instead of $\pi_n(\varphi)$.

Having defined the absolute and the relative homotopy groups completely along classical lines, we can transfer several basic results about them from classical homotopy theory to the present locally semialgebraic setting just by copying classical proofs. But we have to avoid those proofs which use path spaces, since such spaces do not exist in our setting. For example, we see that the group $\pi_n(M,x_o)$ is abelian for

$n \geq 2$ by looking at the picture on p. 18 of Whitehead's book [W_1] (or p. 125 in [W]), rather than using the general argument there which involves the loop space of (M,x_o). The same picture can be used to prove that $\pi_n(M,A,x_o)$ is abelian for $n \geq 3$. By this picture it is also evident that we obtain the same multiplication on $\pi_n(M,A,x_o)$ as before if we compose homotopies by using a coordinate t_i, with $1 < i < n$ instead of t_1. The coordinate t_n may also be used in the case $A = \{x_o\}$.

For any pointed pair of spaces (M,A,x_o) and $n \geq 1$ we define a boundary map

$$\partial : \pi_n(M,A,x_o) \to \pi_{n-1}(A,x_o)$$

in the usual way, $\partial[f] := [f|I^{n-1}]$. Clearly ∂ maps the neutral element of $\pi_n(M,A,x_o)$ to the neutral element of $\pi_{n-1}(A,x_o)$. If $n \geq 2$ then ∂ is a group homomorphism. ∂ depends functorially on the triple (M,A,x_o).

The maps ∂ and the inclusions $i : (A,x_o) \hookrightarrow (M,x_o)$, $j : (M,x_o,x_o) \hookrightarrow (M,A,x_o)$ yield a long sequence

$$(6.1)\qquad \begin{array}{l} \xrightarrow{\partial} \pi_n(A,x_o) \xrightarrow{i*} \pi_n(M,x_o) \xrightarrow{j*} \pi_n(M,A,x_o) \xrightarrow{\partial} \pi_{n-1}(A,x_o) \\ \to \dots \to \pi_1(M,A,x_o) \xrightarrow{\partial} \pi_o(A,x_o) \xrightarrow{i*} \pi_o(M,x_o). \end{array}$$

Looking at the arguments in [Hu, IV, §7], which are entirely semialgebraic, we see that this sequence is exact.

We now discuss - among other things - the dependence of the absolute and the relative homotopy groups on the base point. Given a space M and two points x_o and x_1 we denote the set $[(I,0,1),(M,x_o,x_1)]$ of homotopy classes of paths in M from x_o to x_1 by $\pi_1(M,x_o,x_1)$. If x_2 is a third point in M we have a multiplication

$$\pi_1(M,x_o,x_1) \times \pi_1(M,x_1,x_2) \to \pi_1(M,x_o,x_2),$$

$[u][v] := [u*v]$, by composing paths in the usual way,

$$(u*v)(t) = \begin{cases} u(2t) & 0 \le t \le \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} .$$

We see, as in the topological theory, that the disjoint union $\Pi(M)$ of the sets $\pi_1(M,x_o,x_1)$, with (x_o,x_1) running through $M\times M$, is a groupoid, called the (semialgebraic) <u>fundamental groupoid of</u> M.

Let (E,D,C) be a triple of spaces with $D \supset C$ and both D and C closed in E (for example, $E = I^n$, $D = \partial I^n$, $C = J_{n-1}$). Let (M,A) be a pair of spaces and let x_o,x_1 be two points in A. Then every path u in A running from x_o to x_1 yields a bijection

$$u_{\#} : [(E,D,C),(M,A,x_1)] \to [(E,D,C),(M,A,x_o)]$$

of pointed sets as follows: Given a map $f : (E,D,C) \to (M,A,x_1)$ we extend the homotopy $C\times I \to A$, $(x,t) \mapsto u(t)$, to a homotopy $F : (E\times I,D\times I,C\times I) \to (M,A,A)$ with $F(-,1) = f$, and we define $u_{\#}[f] := [f_o]$ with $f_o := F(-,0)$. Applying the homotopy extension theorem 1.4 several times, we see, as in the topological theory [W, II, §1], that the map $u_{\#}$ is indeed well-defined and bijective, and that $u_{\#}$ depends only on the class $[u] \in \pi_1(M,x_o,x_1)$.

If v is a path in A from x_1 to a third point x_2 then clearly $(u*v)_{\#} = u_{\#}\circ v_{\#}$. Thus the family $([(E,D,C),(M,A,x)] \mid x \in A)$ is a local system of pointed sets on the space A. In other words, if we regard $\Pi(A)$ as a category, whose objects are the points of A and whose morphisms are the homotopy classes of paths, then we have a functor $x \mapsto [(E,D,C),(M,A,x)]$, $[u] \mapsto u_{\#}$, from $\Pi(A)$ to the category of pointed sets. In particular, the group $\pi_1(A,x)$ operates on the set $[(E,D,C),(M,A,x)]$ from the left.

As in topology one easily verifies the following useful fact (cf. [W, p. 101]).

Proposition 6.2. If C is contractible and A is connected then the forgetful map $[(E,D,C),(M,A,x)] \to [(E,D),(M,A)]$ induces a bijection from the set of orbits of $\pi_1(A,x)$ in $[(E,D,C),(M,A,x)]$ to the set $[(E,D),(M,A)]$.

Notice that, if C is complete, we can replace the triple (E,D,C) everywhere above by $(E/C,D/C,C/C)$, as is usually done in topology. But if C is only closed in E, then the spaces E/C, D/C do not necessarily exist in our setting. Nevertheless the considerations above go through in this generality.

If $(E,D,C) = (I^n,\partial I^n,J_{n-1})$ and $n \geq 2$ then the maps $u_{\#}$ above are group homomorphisms. The same holds if $(E,D,C) = (I^n,I^n,\partial I^n)$, $n \geq 1$, $M = A$. Thus we have local systems of groups $(\pi_n(M,A,x) \mid x \in A)$ for $n \geq 2$ and $(\pi_n(M,x) \mid x \in M)$ for $n \geq 1$.

Given a map $f : (M,A) \to (N,B)$ the induced homomorphisms $f_* : \Pi(A) \to \Pi(B)$ and $f_* : \pi_n(M,A,x) \to \pi_n(N,B,f(x))$ together yield a morphism from the local system $(\pi_n(M,A,x) \mid x \in A)$ to the local system $(\pi_n(N,B,y) \mid y \in B)$ for every $n \geq 1$. In other words, for every path u in A from a point x_o to a point x_1 the diagram

$$\begin{tikzcd}
\pi_n(M,A,x_1) \arrow[r, "u_{\#}"] \arrow[d, "f_*"] & \pi_n(M,A,x_o) \arrow[d, "f_*"] \\
\pi_n(N,B,f(x_1)) \arrow[r, "(f \cdot u)_{\#}"'] & \pi_n(N,B,f(x_o))
\end{tikzcd}$$

commutes. Similarly a map $f : M \to N$ yields a morphism from the local system $(\pi_n(M,x) \mid x \in M)$ to $(\pi_n(N,y) \mid y \in N)$ for every $n \geq 0$.

Finally, for every pair (M,A) of spaces the long exact homotopy sequences (6.1) of the triples (M,A,x) with x running through A fit together to a "local system of exact sequences" on A, in the obvious sense.

Up to now we transfered results from topological homotopy theory to the locally semialgebraic theory simply by copying suitable classical proofs. This method alone has a rather limited range. A much more efficient method is based on the following two theorems, which, up to trivial considerations, are special cases of the first main theorem (in one of its versions 3.1 or 4.2) and the second main theorem 5.1.

Theorem 6.3 (First main theorem for homotopy groups). Let $\tilde{R}$ be a real closed field containing the base field R.

i) For every pointed pair of spaces (M,A,x_o) over R and for $n \geq 2$ the natural map

$$\kappa : \pi_n(M,A,x_o) \to \pi_n(M(\tilde{R}),A(\tilde{R}),x_o),$$

defined by $\kappa[f] := [f_{\tilde{R}}]$, is an isomorphism of groups. If $A = \{x_o\}$, this holds for $n = 1$. In the remaining cases $n = 1$, $A \neq \{x_o\}$, and $n = 0$, $A = \{x_o\}$, κ is an isomorphism of pointed sets.

ii) Let M be a space over R. The natural map $[u] \mapsto [u_{\tilde{R}}]$ from $\Pi(M)$ to $\Pi(M(\tilde{R}))$ is an injective groupoid homomorphism. The image is the union of all sets $\pi_1(M(\tilde{R}),x_o,x_1)$ with $x_o,x_1 \in M$. Given a path u in M from a point x_o to a point x_1 the diagram

$$\begin{tikzcd} \pi_n(M,x_1) \arrow[r, "\sim"', "u_{\#}"] \arrow[d, "\kappa"', "\cong"] & \pi_n(M,x_o) \arrow[d, "\kappa"', "\cong"] \\ \pi_n(M(\tilde{R}),x_1) \arrow[r, "\sim"', "(u_{\tilde{R}})_{\#}"] & \pi_n(M(\tilde{R}),x_o) \end{tikzcd}$$

commutes for every $n \geq 0$.

iii) Similarly, given a pair of spaces (M,A) over R and a path u in A from a point x_o to a point x_1 the diagram

$$\begin{tikzcd}
\pi_n(M,A,x_1) \arrow[r, "\sim", "u_{\#}"'] \arrow[d, "\kappa"', "\cong"] & \pi_n(M,A,x_o) \arrow[d, "\kappa"', "\cong"] \\
\pi_n(M(\tilde{R}),A(\tilde{R}),x_1) \arrow[r, "\sim", "(u_{\tilde{R}})_{\#}"'] & \pi_n(M(\tilde{R}),A(\tilde{R}),x_o)
\end{tikzcd}$$

commutes for every $n \geq 1$.

<u>Theorem 6.4</u> (Second main theorem for homotopy groups). Assume that the base field R is the field $\mathbb{R}$ of real numbers.

i) Given a pair of pointed spaces (M,A,x_o) then, for every $n \geq 2$, the obvious map

$$\lambda : \pi_n(M,A,x_o) \to \pi_n(M,A,x_o)_{top}$$

from the n-th semialgebraic homotopy group to the n-th topological (= classical) homotopy group is an isomorphism of groups. This is also true if $n = 1$ and $A = \{x_o\}$. In the remaining cases $n = 1$, $A \neq \{x_o\}$, and $n = 0$, $A = \{x_o\}$, λ is an isomorphism of pointed sets.

ii) For every space M over $\mathbb{R}$ the obvious map $\lambda : \Pi(M) \to \Pi(M)_{top}$ from the semialgebraic to the topological fundamental groupoid is an isomorphism of groupoids. This map, together with the maps $\lambda : \pi_n(M,x) \xrightarrow{\sim} \pi_n(M,x)_{top}$ for $x \in M$ form an isomorphism from the semialgebraic local system $(\pi_n(M,x) \mid x \in M)$ to the topological local system $(\pi_n(M,x)_{top} \mid x \in M)$ on M $(n \geq 0)$.

iii) Similarly, given a pair of spaces (M,A) over $\mathbb{R}$, we have a natural isomorphism from the semialgebraic local system $(\pi_n(M,A,x) \mid x \in A)$ to the topological local system $(\pi_n(M,A,x)_{top} \mid x \in A)$ on A $(n \geq 1)$.

Theorem 6.4 tells us that the absolute semialgebraic homotopy groups of a space M over $\mathbb{R}$ and the relative semialgebraic homotopy groups of a pair of spaces (M,A) over $\mathbb{R}$ are really the same objects as the corresponding topological homotopy groups. The message of Theorem 6.3 is bound to be more complicated since, in general, $M(\tilde{R})$ and $A(\tilde{R})$ contain many more points than M and A. Anyway, every group $\pi_n(M(\tilde{R}),x)$

is <u>isomorphic</u> to a group $\pi_n(M,y)$ since there exists a path in $M(\tilde{R})$ from x to some point $y \in M$. Similarly every group $\pi_n(M(\tilde{R}),A(\tilde{R}),x)$ is isomorphic to a group $\pi_n(M,A,y)$ with $y \in A$.

By the main theorems 6.3 and 6.4 the homotopy groups of any triple (M,A,x_o) which is the base extension to R of some triple (M_o,A_o,x_o) over the field R_o of real algebraic numbers are identical with the topological homotopy groups of $(M_o(\mathbb{R}),A_o(\mathbb{R}),x_o)$. Indeed, $\pi_n(M_o(R),A_o(R),x_o) = \pi_n(M_o,A_o,x_o) = \pi_n(M_o(\mathbb{R}),A_o(\mathbb{R}),x_o) = \pi_n(M_o(\mathbb{R}),A_o(\mathbb{R}),x_o)_{top}$. (We write "=" since the isomorphisms are entirely canonical.) In particular, if V is an algebraic variety over R_o and x_o is a point in $V(R_o)$, then

$$\pi_n(V(R),x_o) = \pi_n(V(\mathbb{R}),x_o)_{top}.$$

If x is a point in $V(R)$ then $\pi_n(V(R),x)$ is isomorphic to $\pi_n(V(R),x_o)$ for any point $x_o \in V(R_o)$ which lies in the connected component of x in $V(R)$. Thus we know the semialgebraic homotopy groups of spheres, Grassmannians, Stiefel varieties, etc. over any real closed field to the same extent as we know the topological groups of these varieties over $\mathbb{R}$.

Now observe that, <u>up to isomorphism</u>, <u>every</u> triple (M,A,x_o) over R is the base extension of a triple over R_o. Indeed, choosing a triangulation of (M,A,x_o), we have a pair of abstract locally finite simplicial complexes (K,L) and a vertex e of L together with an isomorphism from $(|K|_R,|L|_R,e)$ to (M,A,x_o). Then

$$\pi_n(M,A,x_o) \cong \pi_n(|K|_R,|L|_R,e) = \pi_n(|K|_{\mathbb{R}},|L|_{\mathbb{R}},e)_{top}.$$

For example, if M is semialgebraic, then it is evident that the fundamental group $\pi_1(M,x_o)$ with respect to any base point is finitely presentable, since this is known for finite closed simplicial complexes in the topological theory ([Spa, 3.7]). (Choose a good triangulation

$(X,a) \xrightarrow{\sim} (M,x_o)$ and replace X by its core!)

More generally every system of spaces $(M,A_1,\ldots,A_r)$ over R is isomorphic to the base extension to R of a system over R_o, since $(M,A_1,\ldots,A_r)$ can be triangulated. Using this observation and our main theorems - either in the general versions in §3 - §5 or the special versions in this section - we are able to transfer a large part of the results of classical homotopy theory to the locally semialgebraic setting. We illustrate this here with two examples. Later we will leave such matters to the reader.

Our first example is the homotopy excision theorem of Blakers and Massey. We choose a version which (in the topological setting) can be found in [DKP, §15].

Theorem 6.5. Let A_1 and A_2 be open locally semialgebraic subsets of a space M over R with $M = A_1 \cup A_2$. Let $A_o := A_1 \cap A_2$. Assume that p and q are natural numbers with $p+q \geq 3$ and $\pi_r(A_1,A_o,x) = 0$ for $1 \leq r < p$, $\pi_r(A_2,A_o,x) = 0$ for $1 \leq r < q$, for every point $x \in A_o$. Then the homomorphism

$$j_* : \pi_n(A_2,A_o,x) \to \pi_n(M,A_1,x)$$

induced by the inclusion $j : (A_2,A_o) \hookrightarrow (M,A_1)$ is an isomorphism in dimensions $1 \leq n < p+q-2$ and an epimorphism in dimension $n = p+q-2$ for every $x \in A_o$.

Proof. Choosing a triangulation of (M,A_1,A_2) we may assume that

$$(M,A_1,A_2) = (X(R),Y_1(R),Y_2(R))$$

for some triple (X,Y_1,Y_2) over R_o with Y_1 and Y_2 open in X. Let $Y_o := Y_1 \cap Y_2$ and let i denote the inclusion map from (Y_2,Y_o) to (X,Y_1). We have $X = Y_1 \cup Y_2$ and $j = i_R$. We conclude from Theorem 6.3 that, for every $x \in Y_o$, the groups $\pi_r(Y_1,Y_o,x)$ and $\pi_r(Y_2,Y_o,x)$ vanish for $1 \leq r < p$ and $1 \leq r < q$ respectively. The sets $Y_1(\mathbb{R})$, $Y_2(\mathbb{R})$ are open in $X(\mathbb{R})$ and cover $X(\mathbb{R})$. Again by Theorem 6.3, the groups $\pi_r(Y_1(\mathbb{R}), Y_o(\mathbb{R}), x)$ and $\pi_r(Y_2(\mathbb{R}), Y_o(\mathbb{R}), x)$ vanish for $1 \leq r < p$ and $1 \leq r < q$ respectively and every $x \in Y_o(\mathbb{R})$ (not just $x \in Y_o$). Since the topological analogue of Theorem 6.5 is true [DKP, §15] we conclude from Theorem 6.4, that, for every $x \in Y_o(\mathbb{R})$,

$$(i_{\mathbb{R}})_* : \pi_n(Y_2(\mathbb{R}), Y_o(\mathbb{R}), x) \to \pi_n(X(\mathbb{R}), Y_1(\mathbb{R}), x)$$

is an isomorphism in dimensions $1 \leq n < p+q-2$ and an epimorphism in dimension $n = p+q-2$. Now apply Theorem 6.3 twice. We first see that the analogous statement is true over R_o and then that it is true over R, which is what we want. In the last argument special cases of the following obvious general fact have been used: If $f : (M,A,x_o) \to (N,B,y_o)$ is a map between pairs of pointed spaces over R, then for every $n \geq 1$ and every real closed overfield $\tilde{R}$ of R the diagram

$$\begin{array}{ccc}
\pi_n(M,A,x_o) & \xrightarrow{f_*} & \pi_n(N,B,y_o) \\
\kappa \downarrow \cong & & \kappa \downarrow \cong \\
\pi_n(M(\tilde{R}),A(\tilde{R}),x_o) & \xrightarrow[(f_{\tilde{R}})_*]{} & \pi_n(N(\tilde{R}),B(\tilde{R}),y_o)
\end{array}$$

commutes. q.e.d.

We are not aware of any proof of Blakers-Massey's excision theorem in topology which could be transfered to the general locally semialgebraic setting. The proof in [DKP] makes heavy use of the fact that $\mathbb{R}$ is archimedian.

Our second example of knowledge transfer from topological to semialgebraic homotopy theory is the following fundamental theorem which, in the setting of CW-complexes, is due to J.H.C. Whitehead, cf. e.g. [Spa, p. 405].

Theorem 6.6. Let $f : M \to N$ be a map between connected spaces over R. Let x_o be a point of M and $y_o := f(x_o)$.

i) Assume that, for every $r > 0$, the map $\pi_r(f)$ from $\pi_r(M,x_o)$ to $\pi_r(N,y_o)$ is an isomorphism. Then f is a homotopy equivalence.

ii) Given some natural number n, assume that $\pi_r(f) : \pi_r(M,x_o) \to \pi_r(N,y_o)$ is an isomorphism for $1 \leq r < n$ and an epimorphism for $r = n$. Then, for every space P over R with $\dim P < n$, the map $f_* : [P,M] \to [P,N]$ induced by f is bijective, while for every n-dimensional space P this map is surjective.

We give the proof of part i) in detail leaving part ii) to the reader. Choosing good triangulations of (M,x_o) and (N,y_o) we assume that there are given tame complexes X and Y over R_o with $M = X(R)$, $N = Y(R)$ and x_o, y_o vertices of X and Y lying in X and Y respectively. Now X and Y are homotopy equivalent to their cores, cf. §1. Replacing X and Y by their cores we assume that the complexes X and Y are closed. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are certainly CW-complexes. By Theorem 4.2 (= second version of the first main theorem) there exists a map $g : (X,x_o) \to (Y,y_o)$ such that g_R is homotopic to $f : (M,x_o) \to (N,y_o)$. We may replace f by g_R and assume henceforth that $f = g_R$. Applying Theorem 6.3 (i.e. again the main theorem) twice we see first that the map $\pi_r(g)$ is an isomorphism and then that $\pi_r(g_{\mathbb{R}})$ is an isomorphism for every $r > 0$. Now we know by the second main theorem 6.4 and by the topological Whitehead theorem [Spa, p. 403] that $g_{\mathbb{R}} : X(\mathbb{R}) \to Y(\mathbb{R})$ is a topological homotopy equivalence. By the second main theorem, in its general version 5.1, we know that $g_{\mathbb{R}}$ is a locally semialgebraic homotopy equivalence. Then, applying the first main theorem twice, we see that $f = g_R$ is a homotopy equivalence.

Example 6.7. Let M be a locally complete space. Then the canonical map $x \mapsto x$ from M_{loc} to M (cf. I, §7) is a homotopy equivalence. Indeed the hypothesis of part i) of the theorem is satisfied for trivial reasons.

How about a <u>proof</u> of Theorem 6.6 by semialgebraic methods? Let us again confine attention to part i). This suffices to understand the difficulties. As indicated in the proof above we may assume that M and N are partially complete. If the space M is semialgebraic then we can argue in a very classical way. We form the mapping cylinder $Z(f)$ of $f: M \to N$ (cf. II, §10; f is now proper). The map f is the composite $p \circ j$ of the inclusion map $j: M \hookrightarrow Z(f)$ and the natural projection $p: Z(f) \to N$. Now p is a homotopy equivalence. Thus we may replace f by the map j. Henceforth we assume that M is a closed subspace of the partially complete space N and f is the inclusion map. The assumption in (i) means that the relative homotopy groups $\pi_r(N,M,x_o)$ all vanish.

Since M is connected this means that every map $(D^r,S^{r-1}) \to (N,M)$ is homotopic to a constant map, cf. Prop. 6.2. From this one concludes in the usual way (e.g. [DKP, p. 212]) that every map $\varphi: D^r \to N$ with $\varphi(S^{r-1}) \subset M$ is homotopic <u>relative</u> S^{r-1} to a map ψ with $\psi(D^r) \subset M$ (φ can be "compressed" to M). Choosing a triangulation of (N,M) it is now easy to construct a strong deformation retraction from N to M working "simplex by simplex".

This is an entirely classical proof of part (i) of Whitehead's theorem for M semialgebraic. Unfortunately the proof does not work if M is only locally semialgebraic, since then we do not have a mapping cylinder $Z(f)$ at our disposal.

Here a serious handicap of our homotopy theory comes into sight. Some of the most basic constructions in topology are impossible in the category of - say - partially complete spaces. Even the cone $CX = X \times I / X \times \{1\}$ over a partially complete space X (= mapping cylinder of the map from X to the one point space) does not exist. Indeed, choosing a triangulation, we may assume that X is a closed locally finite complex. Then

CX exists in the category of simplicial complexes. But CX is not locally finite if the complex X is not finite.

All the more we cannot construct the suspension of a partially complete space which is not semialgebraic. Thus stable homotopy theory is out of bounds. We also do not have the analogues of many important infinite dimensional CW-complexes, as for example the Eilenberg-MacLane spaces $K(\pi,n)$, at our disposal.

This makes it desirable to admit suitable inductive limits of complete semialgebraic spaces over R which are more general than partially complete locally semialgebraic spaces. The authors developed a full fledged theory of such limits which we call "weak polytopes".

In the category of weak polytopes mapping cylinders, suspensions, Eilenberg-MacLance spaces, and all that exist. This leads, in particular, to a satisfactory stable homotopy theory over any real closed field R, opening a door to general cohomology theories over R. And, of course, with weak polytopes a direct proof of Theorem 6.6 can be given.

In developing the theory of weak polytopes one has to be careful that the limits of complete spaces do not become too "wild" and the geometry is destroyed. This needs more space and time than is available now. We have to delay the publication of this theory to a later occasion.

§7 - Homology, the Hurêwicz theorems

Having developed homotopy theory this far, it is natural to ask whether there exists an analogue of singular homology for locally semialgebraic spaces and a Hurêwicz homomorphism from the homotopy groups to the homology groups. We certainly can define singular chains using semialgebraic (instead of just continuous) maps from the standard simplices Δ^n, $n \geq 0$, to the given space but the authors do not know whether the homology groups defined in this way have reasonable properties.

Nevertheless a satisfactory homology theory of semialgebraic spaces has been developed avoiding the singular approach, in [D], [DK_3], [D_1]. The proofs there readily generalize to locally semialgebraic spaces. Thus we give here only a short rêsumê of the basics of this homology theory, condensed into two theorems. We refer the reader to the papers above for the details. [DK_3] gives mainly an overview on homology and cohomology. The missing proofs can be found in [D] and [D_1], the main difference between these two articles being the way the homotopy invariance of cohomology is proved. (Cohomology is established first, homology later, in contrast to the singular theory.) For a first reading we recommend [DK_3, §3 - §5].

All we need to know here from the theory developed in these papers is the following Theorem 7.1. Let HLSA(2,R) denote the category whose objects are the pairs of spaces (M,A) over a given real closed field R and whose morphisms are the homotopy classes of (locally semialgebraic) maps between such pairs. Let E denote the endomorphism $(M,A) \to (A,\emptyset)$ of this category. Let G be an abelian group.

Theorem 7.1. There exists a family $(h_n \mid n\in\mathbb{Z})$ of covariant functors from HLSA(2,R) to the category of abelian groups and a family $(\partial_n \mid n\in\mathbb{Z})$ of natural transformations $\partial_n : h_n \to h_{n-1} \circ E$ with the following properties.

1) For every pair of spaces (M,A) the long sequence
$$\ldots \to h_n(A,\emptyset) \xrightarrow[h_n(i)]{} h_n(M,\emptyset) \xrightarrow[h_n(j)]{} h_n(M,A) \xrightarrow[\partial_n(X,A)]{} h_{n-1}(A,\emptyset) \to \ldots$$
is exact. Here i and j denote the inclusion maps $(A,\emptyset) \to (M,\emptyset)$ and $(M,\emptyset) \to (M,A)$.
2) If (M,A) is a pair of spaces and U is an open locally semialgebraic subset of M with $\overline{U} \subset A$ then the inclusion map $j : (M \setminus U, A \setminus U) \to (M,A)$ induces, for every $n \in \mathbb{Z}$, an isomorphism $h_n(j)$.
3) For the one-point space $*$ we have $h_n(*,\emptyset) = 0$ for $n \neq 0$ and $h_0(*,\emptyset) \cong G$.
4) For every family of spaces $(M_\alpha | \alpha \in I)$ and every $n \in \mathbb{Z}$ the natural map from the direct sum of the groups $h_n(M_\alpha,\emptyset)$ to the group $h_n(\amalg(M_\alpha | \alpha \in I),\emptyset)$ is an isomorphism.

Let us call a family $(h_n, \partial_n | n \in \mathbb{Z})$ with the properties indicated in the theorem, together with a fixed isomorphism $\varepsilon : G \xrightarrow{\sim} h_0(*,\emptyset)$, a <u>homology theory over</u> R <u>with coefficients in</u> G. Theorem 7.1 states that such a homology theory exists for every G and R. Then it is pretty evident that, up to isomorphism, this homology theory $((h_n, \partial_n | n \in \mathbb{Z}), \varepsilon)$ is unique. Namely, if (X,A) is a tame pair of complexes over R, then one deduces from the functorial long exact sequences (cf. property 1) of the pairs (X,A) and $(\mathrm{co}\,X, \mathrm{co}\,A)$ together with the fact that $\mathrm{co}\,X$ and $\mathrm{co}\,A$ are strong deformation retracts of X and A, that the inclusion map from $(\mathrm{co}\,X, \mathrm{co}\,A)$ to (X,A) induces an isomorphism
$$h_n(\mathrm{co}\,X, \mathrm{co}\,A) \xrightarrow{\sim} h_n(X,A)$$
for every $n \in \mathbb{Z}$. Then it can be shown as in topology (cf. [ES], [Mi]) that $h_n(\mathrm{co}\,X, \mathrm{co}\,A)$ is isomorphic to the simplicial homology $H_n(K,L;G)$ of the abstract simplicial complex $(K,L) = (K(\mathrm{co}\,X), K(\mathrm{co}\,A))$. In particular $h_n(X,A) = 0$ if $n < 0$ and if $n > \dim X$.

Henceforth we denote the group $h_n(M,A)$ by $H_n(M,A;G)$ and call it the n-th homology group of the pair (M,A) with coefficients in G. We write $H_n(M,G)$ instead of $H_n(M,\emptyset;G)$. Given a map $f:(M,A)\to(N,B)$ we often denote all the homomorphisms $h_n(f)$ by f_*.

From our discussion of uniqueness of homology it is evident that the groups $H_n(M,A;G)$ obey a universal coefficient theorem as common in the simplicial and singular homology theories, cf. [Spa, p. 222].

Theorem 7.2 (First and second main theorem for homology).

a) Let $\tilde{R}$ be a real closed field containing R. Then we have, for every pair of spaces (M,A) and every $n\in\mathbb{Z}$, an isomorphism

$$\kappa : H_n(M,A;G) \xrightarrow{\sim} H_n(M(\tilde{R}),A(\tilde{R});G),$$

functorial with respect to (M,A), such that the diagram

$$\begin{array}{ccc} H_n(M,A;G) & \xrightarrow{\partial_n(M,A)} & H_{n-1}(A,G) \\ \Big\downarrow \kappa & & \Big\downarrow \kappa \\ H_n(M(\tilde{R}),A(\tilde{R});G) & \xrightarrow[\partial_n(M(\tilde{R}),A(\tilde{R}))]{} & H_{n-1}(A(\tilde{R}),G) \end{array}$$

commutes.

b) For every pair of spaces (M,A) over $\mathbb{R}$ and every $n\in\mathbb{Z}$ we have a functorial isomorphism

$$\lambda : H_n(M,A;G) \xrightarrow{\sim} H_n(M,A;G)_{top}$$

such that the diagram

$$\begin{array}{ccc} H_n(M,A;G) & \xrightarrow{\partial_n(M,A)} & H_{n-1}(A,G) \\ \Big\downarrow \lambda & & \Big\downarrow \lambda \\ H_n(M,A;G)_{top} & \xrightarrow[\partial_n(M,A)_{top}]{} & H_{n-1}(A,G)_{top} \end{array}$$

commmutes. Here $H_n(M,A;G)_{top}$ denotes the n-th singular homology group of the pair of topological spaces (M,A) with coefficients in

G and $\partial_{n,top}$ denotes the boundary map in singular homology.

It is now evident that we can transfer a great number of results from singular homology theory to spaces over any real closed field R, cf. the discussion for homotopy groups in §6. For a concrete example, namely Alexander-Poincaré duality, see [DK_3, §5]. A more subtle method for transfering results from singular to semialgebraic homology is also discussed in [DK_3] (cf. §7 - §8 of that paper).

It is clear that, for any space M over R, the group $H_o(M,\mathbb{Z})$ is the free abelian group over the set $\pi_o(M)$ of connected components of M, and $H_o(M,G)$ is the tensor product of $H_o(M,\mathbb{Z})$ with G over $\mathbb{Z}$.

For any $n \in \mathbb{Z}$ we denote by $\tilde{H}_n(M,G)$ the kernel of the homomorphism from $H_n(M,G)$ to $H_n(*,G)$ induced by the map from M to the one point space ("reduced homology groups"). If $n \neq 0$ then $\tilde{H}_n(M,G) = H_n(M,G)$, while we have a split exact sequence

$$0 \to \tilde{H}_o(M,G) \to H_o(M,G) \to G \to 0.$$

If x_o is any point in M then the natural map from $\tilde{H}_o(M,G)$ to $H_o(M,x_o;G)$ is an isomorphism.

Henceforth we denote the homology groups $H_n(M,A;\mathbb{Z})$, $H_n(M,\mathbb{Z})$, $\tilde{H}_n(M,\mathbb{Z})$ simply by $H_n(M,A)$, $H_n(M)$, $\tilde{H}_n(M)$. We want to compare these groups with the homotopy groups (or sets) $\pi_n(M,A,x_o)$, $\pi_n(M,x_o)$.

For any $n \geq 0$ and any real closed base field R the group $H_n(I^n,\partial I^n)$ is isomorphic to $\mathbb{Z}$. We choose a coherent system of generators $z_n \in H_n(I^n,\partial I^n)$, as indicated in [Spa, p. 388]. {z_n maps to the fundamental class of the complete semialgebraic manifold ∂I^n with its standard orientation under the isomorphism ∂ from $H_n(I^n,\partial I^n)$ to $\tilde{H}_{n-1}(\partial I^n)$ provided $n \geq 1$.}

Given a pair (M,A,x_o) of pointed spaces we define a Hurêwicz-map ($n \geq 1$, or $n=0$ and $A = \{x_o\}$)

$$\varphi : \pi_n(M,A,x_o) \to H_n(M,A)$$

by $\varphi[f] := f_*(Z_n)$, for every map $f : (I^n, \partial I^n, J_{n-1}) \to (M,A,x_o)$. This map is well defined since $f_*(Z_n)$ depends only on the homotopy class of f, in fact only on the free homotopy class of $f : (I^n, \partial I^n) \to (M,A)$. Clearly the map φ depends on the triple (M,A,x_o) in a functorial way. Copying arguments in [Spa, VII, §4] we obtain

Theorem 7.3. a) If $n \geq 2$, or if $n=1$ and $A = \{x_o\}$, the Hurêwicz map is a group homomorphism.

b) If $n \geq 1$, the square

$$\begin{tikzcd}
\pi_n(M,A,x_o) \arrow[r, "\partial"] \arrow[d, "\varphi"] & \pi_{n-1}(A,x_o) \arrow[d, "\varphi"] \\
H_n(M,A) \arrow[r, "\partial"] & H_{n-1}(A)
\end{tikzcd}$$

commutes.

Since the image $\varphi[f]$ of an element $[f] \in \pi_n(M,A,x_o)$ only depends on the free homotopy class of f, it is evident that for any path u in A from a point x_o to a point x_1 the triangle

$$\begin{tikzcd}
\pi_n(M,A,x_1) \arrow[rr, "u_\#"] \arrow[dr, "\varphi"'] & & \pi_n(M,A,x_o) \arrow[dl, "\varphi"] \\
& H_n(M,A) &
\end{tikzcd}$$

commutes ($n \geq 1$). Similarly, for any path v in M from a point x_o to a point x_1 the triangle

$$\begin{tikzcd}
\pi_n(M,x_1) \arrow[rr, "v_\#"] \arrow[dr, "\varphi"'] & & \pi_n(M,x_o) \arrow[dl, "\varphi"] \\
& \widetilde{H}_n(M) &
\end{tikzcd}$$

commutes ($n \geq 0$). Here we have identified both $H_n(M,x_o)$ and $H_n(M,x_1)$ with $\tilde{H}_n(M)$ in the canonical way.

If $n \geq 2$ then we denote by $\pi_n^+(M,A,x_o)$ the quotient of $\pi_n(M,A,x_o)$ by the normal subgroup generated by all elements $u_\#(x)x^{-1}$ with x running through $\pi_n(M,A,x_o)$ and $[u]$ running through $\pi_1(A,x_o)$. The first triangle above tells us, in the case $x_1 = x_o$, that φ induces a group homomorphism

$$\varphi^+ : \pi_n^+(M,A,x_o) \to H_n(M,A) \qquad (n \geq 2).$$

Similarly, if $n \geq 1$, then we denote by $\pi_n^*(M,x_o)$ the quotient of $\pi_n(M,x_o)$ by the normal subgroup generated by all elements $v_\#(x)x^{-1}$ with $x \in \pi_n(M,x_o)$, $[v] \in \pi_1(M,x_o)$. Notice that $\pi_1^*(M,x_o)$ is just the factor commutator group of $\pi_1(M,x_o)$. The second triangle above shows for $x_o = x_1$ that the Hurêwicz map φ induces a group homomorphism

$$\varphi^* : \pi_n^*(M,x_o) \to H_n(M) \qquad (n \geq 1).$$

Theorem 7.4 (Absolute and relative Hurêwicz theorem).

a) Let $n \geq 1$. Assume that $\pi_r(M,x_o) = 0$ for $0 \leq r \leq n-1$. Then the map φ^* from $\pi_n^*(M,x_o)$ to $H_n(M)$ is an isomorphism.

b) Let $n \geq 2$. Assume that M and A are connected and that $\pi_r(M,A,x_o) = 0$ for $1 \leq r \leq n-1$. Then the map φ^+ from $\pi_n^+(M,A,x_o)$ to $H_n(M,A)$ is an isomorphism.

N.B. Part a) tells us in the case $n = 1$ that, for every connected space M, the factor commutator group of $\pi_1(M,x_o)$ is canonically isomorphic to $H_1(M)$. If $n \geq 2$ then, under the assumptions there, $\pi_n^*(M,x_o) = \pi_n(M,x_o)$.

We obtain Theorem 7.4 by starting from the well-known Hurêwicz theorems in topology ([Spa, p. 397f], [W, p. 178]) and using our main theorems in homotopy (Th. 6.3, Th. 6.4) and homology (Th. 7.2) together with the following obvious facts:

1) Let $\widetilde{R}$ be a real closed field containing R and (M,A,x_o) a pair of pointed spaces over R. Then the square

$$\begin{tikzcd}
\pi_n(M,A,x_o) \arrow[r, "\varphi"] \arrow[d, "\kappa"', "\cong"] & H_n(M,A) \arrow[d, "\kappa"', "\cong"] \\
\pi_n(M(\widetilde{R}),A(\widetilde{R}),x_o) \arrow[r, "\varphi"] & H_n(M(\widetilde{R}),A(\widetilde{R}))
\end{tikzcd}$$

commutes.

2) Assume that $R=\mathbb{R}$. Identifying the semialgebraic homotopy and homology groups with the corresponding homotopy and homology groups of topological spaces, the semialgebraic Hurêwicz homomorphism

$$\varphi : \pi_n(M,A,x_o) \to H_n(M,A)$$

is the same map as the topological Hurêwicz homomorphism.

Similarly, one can transfer other results about the Hurêwicz homomorphisms to spaces over any real closed field. In particular, one can transfer the following theorem due to Hopf and Fox (cf. [W, p. 248]): If $n \geq 2$ and $\pi_r(M,x_o) = 0$ for $0 \leq r \leq n-1$, then the Hurêwicz map from $\pi_{n+1}(M,x_o)$ to $H_{n+1}(M)$ is surjective.

Semialgebraic homology groups are quite useful in our homotopy theory. This is already evident by the following complement to Theorem 6.6. It can be obtained by transfer from a well-known theorem of J.H.C. Whitehead ([Spa, p. 399], [W, p. 181]).

<u>Theorem 7.5.</u> Let $f: M \to N$ be a map between connected spaces over R. Let x_o be a point of M and $y_o := f(x_o)$. Given a natural number n we consider the induced homomorphisms

$$\pi_q(f) : \pi_q(M,x_o) \to \pi_q(N,y_o), \quad H_q(f) : H_q(M) \to H_q(N),$$

for $1 \leq q \leq n$.

a) If $\pi_q(f)$ is bijective for $1 \leq q < n$ and surjective for $q = n$, then $H_q(f)$ is bijective for $1 \leq q < n$ and surjective for $q = n$.

b) Assume that $\pi_1(M, x_o) = 0$ and $\pi_1(N, y_o) = 0$. Then, conversely, if $H_q(f)$ is bijective for $1 \leq q < n$ and surjective for $q = n$, the homomorphism $\pi_q(f)$ is bijective for $1 \leq q < n$ and surjective for $q = n$.

If the space M is semialgebraic then mapping cylinders are at our disposal, and Theorem 7.5 is an easy consequence of the relative Hurewicz theorem. But the theorem holds in general.

§8 - Homotopy groups of ends

This section refers to the theory of ends of a space M developed in II, §9 and §11. Given an absolute end λ of M we want to define the q-th homotopy group of λ as the projective limit of the homotopy groups $\pi_q((M\setminus K)_\lambda, x)$ with K running through the set $\mathcal{T}_c(M)$ of partially complete (locally semialgebraic) subsets of M and suitable base points $x \in (M\setminus K)_\lambda$. {Recall that $(M\setminus K)_\lambda$ is the connected component of $M\setminus K$ determined by λ, cf. II, §9, Def. 4}. We cannot choose a common base point in all the sets $(M\setminus K)_\lambda$ since they have an empty intersection. The most natural idea seems to be to choose the base points x on some proper incomplete path α with $\varepsilon(\alpha) = \lambda$ (cf. II, §9, Def. 6). This leads to the following definition.

Let $\alpha : [0,1[\to M$ be a proper incomplete path and let $\lambda = \varepsilon(\alpha)$ be the end determined by α. We consider the set $\Lambda(\alpha)$ consisting of all pairs (K,t) with $K \in \mathcal{T}_c(M)$, $t \in [0,1[$, $\alpha([t,1[) \subset M\setminus K$. We define a partial ordering on $\Lambda(\alpha)$ by

$$(K,t) \leq (L,u) \Leftrightarrow K \subset L,\ t \leq u.$$

If $(K,t) \leq (L,u)$ then α yields, for every $q \geq 1$, a natural homomorphism

$$\rho^{(L,u)}_{(K,t)} : \pi_q(M\setminus L, \alpha(u)) \to \pi_q(M\setminus K, \alpha(t)),$$

namely the map induced by the inclusion $(M\setminus L, \alpha(u)) \to (M\setminus K, \alpha(u))$ and the path $s \mapsto ((1-s)t + su)$ from $\alpha(t)$ to $\alpha(u)$. With these transition maps $\rho^{(L,u)}_{(K,u)}$ we have, for every $q \geq 1$, an inverse system $(\pi_q(M\setminus K, \alpha(t)) \mid (K,t) \in \Lambda(\alpha))$ of groups.

<u>Definition 1.</u> <u>The</u> q-th <u>homology group</u> $\pi_q(\lambda,\alpha)$ <u>of the end</u> λ <u>based at</u> α is the projective limit of this system $(q \geq 1)$. In short,

$$\pi_q(\lambda,\alpha) = \varprojlim_{(K,t)\in\Lambda(\alpha)} \pi_q(M\setminus K, \alpha(t)).$$

These groups behave functorially with respect to partially proper maps.

Indeed, if $f: M \to N$ is a partially proper map, then $\beta := f \circ \alpha$ is a proper incomplete path in N which determines the end $f_*(\lambda)$, cf. II, §9. Every index $(L,s) \in \Lambda(\beta)$ yields an index $(f^{-1}(L),s) \in \Lambda(\alpha)$, and f induces, for every $q \geq 1$, a homomorphism

$$f_* : \pi_q(M \setminus f^{-1}(L), \alpha(s)) \to \pi_q(N \setminus L, \beta(s)).$$

These homomorphisms are compatible with the transition maps. Thus we obtain a unique homomorphism

$$f_* : \pi_q(\lambda, \alpha) \to \pi_q(f_*(\lambda), \beta)$$

such that, for every $(L,s) \in \Lambda(\beta)$, the square

$$\begin{tikzcd}
\pi_q(\lambda,\alpha) \arrow[r, "f_*"] \arrow[d] & \pi_q(f_*(\lambda),\beta) \arrow[d] \\
\pi_q(M\setminus f^{-1}(L),\alpha(s)) \arrow[r, "f_*"'] & \pi_q(N\setminus L,\beta(s))
\end{tikzcd}$$

with the canonical projections as vertical arrows, commutes.

How does the group $\pi_q(\lambda,\alpha)$ depend on the choice of the incomplete path α "in" the end λ? Assume that α and β are two proper incomplete paths in M and that $H : [0,1[\times I \to M$ is a <u>proper homotopy from</u> α <u>to</u> β, i.e. H is a proper map with $H(-,0) = \alpha$ and $H(-,1) = \beta$. For any $K \in \mathcal{T}_c(M)$ the preimage $H^{-1}(K)$ is a complete semialgebraic subset of $[0,1[\times I$. Thus $H^{-1}(K)$ is contained in $[0,c]\times I$ for some $c \in]0,1[$. For every $t \in]c,1[$ the path $s \mapsto H(t,s)$ runs in $M \setminus K$ from the point $\alpha(t)$ to the point $\beta(t)$. We see that α and β determine the same end λ of M. Moreover, for every $t \in]c,1[$ and $q \geq 1$, we have a homomorphism

$$H(t,-)_{\#} : \pi_q(M \setminus K, \beta(t)) \to \pi_q(M \setminus K, \alpha(t)) .$$

If $u > t$ is a second parameter in $]c,1[$ then the square

$$\begin{tikzcd}
\pi_q(M\setminus K,\beta(u)) \arrow[r, "H(u,-)_{\#}"] \arrow[d, "\rho^{(K,u)}_{(K,t)}"'] & \pi_q(M\setminus K,\alpha(u)) \arrow[d, "\rho^{(K,u)}_{(K,t)}"] \\
\pi_q(M\setminus K,\beta(t)) \arrow[r, "H(t,-)_{\#}"'] & \pi_q(M\setminus K,\alpha(t))
\end{tikzcd}$$

commutes, since the vertical arrows are just the maps $\gamma_{\#}$, $\delta_{\#}$ induced by the paths

$$\gamma(s) = H((1-s)t + s\,u, 1), \quad \delta(s) = H((1-s)t + s\,u, 0)$$

and since H maps the whole rectangle $[t,u]\times I$ into $M \setminus K$. It is now evident that the homomorphisms $H(t,-)_{\#}$ for the various $K \in \mathcal{T}_c(M)$ and admitted $t \in]0,1[$ all fit together and yield a well defined homomorphism

$$H_{\#} : \pi_q(\lambda,\beta) \to \pi_q(\lambda,\alpha).$$

If $\Phi : [0,1[\times I\times I \to M$ is a proper homotopy relative $[0,1[\times\{0,1\}$ from H to a second proper homotopy $H' : [0,1[\times I \to M$ from α to β then $H_{\#} = H'_{\#}$ as is easily verified.

If $G : [0,1[\times I \to M$ is a proper homotopy from β to a third proper incomplete path γ in M, then the composite homotopy $H*G$ is again proper and the homomorphisms $H_{\#}\circ G_{\#}$ and $(H*G)_{\#}$ from $\pi_q(\lambda,\gamma)$ to $\pi_q(\lambda,\alpha)$ are equal. Now we see by the usual argument that the inverse homotopy $H^{-1} : (t,s) \mapsto H(t,1-s)$ from β to α, which is again proper, yields a homomorphism $(H^{-1})_{\#}$ from $\pi_q(\lambda,\alpha)$ to $\pi_q(\lambda,\beta)$ which is inverse to $H_{\#}$. In particular $H_{\#}$ is an isomorphism.

Definition 2. The fundamental groupoid $\Pi(\lambda)$ of the end λ of M is the category whose objects are the proper incomplete paths $\alpha : [0,1[\to M$ with $\varepsilon(\alpha) = \lambda$ and whose morphisms from an object α to an object β are the proper homotopy classes $\langle H\rangle$ relative $[0,1[\times\{0,1\}$ of proper homotopies H from α to β. Notice that every morphism in the category $\Pi(\lambda)$ is an isomorphism, the inverse of $\langle H\rangle$ being the class $\langle H^{-1}\rangle$. Thus $\Pi(\lambda)$ is indeed a groupoid.

We summarize our observations on the maps $H_{\#}$ as follows.

Proposition 8.1. For every end λ of M and every $q \in \mathbb{N}$ the assignment $\alpha \mapsto \pi_q(\lambda,\alpha)$, $\langle H\rangle \mapsto H_{\#}$ is a functor from the fundamental groupoid $\Pi(\lambda)$

to the category of groups.

The following theorem states in particular that the fundamental groupoid $\Pi(\lambda)$ is connected, i.e. for any two objects α,β in $\Pi(\lambda)$ there exists a morphism from α to β. This implies that, for given $q \geq 1$, the groups $\pi_q(\lambda,\alpha)$, with α running through the proper incomplete paths "in" λ, are all isomorphic.

<u>Theorem 8.2.</u> Let $\alpha,\beta : [0,1[\rightrightarrows M$ be proper incomplete paths in M determining the same end λ. Let $M \subset P$ be a pure completion (cf. II, §9, Def. 2) of M, and let $\bar{\alpha},\bar{\beta} : [0,1] \rightrightarrows P$ denote the unique extensions of α,β to paths in P. Then there exists a homotopy

$$\bar{H} : (I,[0,1[,\{1\}) \times I \to (P,M,P \setminus M)$$

with $\bar{H}(-,0) = \bar{\alpha}, \bar{H}(-,1) = \bar{\beta}$. The restriction of $\bar{H}$ to $[0,1[\times I$ is a proper homotopy from α to β.

<u>Proof.</u> Once we have constructed the homotopy $\bar{H}$ the last sentence of the theorem will be obvious since $\bar{H}^{-1}(M) = [0,1[\times I$. Let $A := P(\lambda)$ be the connected component of $P \setminus M$ determined by λ (II, 9.24). Then $x := \bar{\alpha}(1)$ and $y := \bar{\beta}(1)$ are points in A. We choose a path $\gamma : [0,1] \to A$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then we choose a simultaneous triangulation of P and the sets M, $\gamma([0,1])$, $\bar{\alpha}([0,1])$, $\bar{\beta}([0,1])$, and we regard P as a closed locally finite complex with these sets as subcomplexes.

The points x and y are vertices of P. There exists a finite sequence of vertices $(x_i \mid 0 \leq i \leq n)$ in $\gamma([0,1])$ such that $x_o = x$, $x_n = y$ and $[x_{i-1},x_i]$ is a closed 1-simplex of P contained in $\gamma([0,1])$ for $1 \leq i \leq n$. For every $i \in \{1,\dots,n\}$ we choose an open simplex σ_i in M which has $]x_{i-1},x_i[$ as a face. This is possible since M is dense in P.

Henceforth we denote the set of paths

$$\delta : ([0,1],[0,1[,\{1\}) \to (P,M,P\setminus M)$$

by $\Gamma(P,M)$. For any two such paths δ_1, δ_2 we write $\delta_1 \simeq \delta_2$ if there exists a homotopy from δ_1 to δ_2 regarded as maps between triples of spaces. In this notation our claim is $\overline{\alpha} \simeq \overline{\beta}$.

If z is a point of M and w is a point of $P\setminus M$ such that the open line segment $]z,w[$ is contained in some open simplex of M, then the linear path from z to w is an element of $\Gamma(P,M)$. We denote this path by zw. We make the following simple but crucial observation: If z_1,z_2 are points in the open star $St_M(x_i)$ for some $i \in \{0,\dots,n\}$ (cf. II, §7), then $z_1x_i \simeq z_2x_i$. Indeed, since the dense pair of spaces (P,M) is assumed to be pure the complex M is certainly maximal (cf. II, §5, Def. 4). Thus any two open simplices in $St_M(x_i)$ are connectable in $St_M(x_i)$ by a chain of open simplices in the sense of II, §7, Def. 4 (cf. II, Prop. 7.3). It is then easy to write down a piecewise linear homotopy from z_1x_i to z_2x_i.

In particular, $\hat{\sigma}_ix_i \simeq \hat{\sigma}_{i+1}x_i$ for every $i \in \{1,\dots,n-1\}$ since the barycenters $\hat{\sigma}_i$ and $\hat{\sigma}_{i+1}$ of σ_i and σ_{i+1} are both lying in $St_M(x_i)$. Obviously we also have $\hat{\sigma}_ix_{i-1} \simeq \hat{\sigma}_ix_i$ for $i \in \{1,\dots,n\}$.

There exists a unique vertex p in the one dimensional subcomplex $\alpha([0,1[)$ of P such that $]p,x[$ is an edge of P. Let a denote the midpoint of $]p,x[$. This is a point in $St_M(x)$. It is easily verified that $\overline{\alpha} \simeq ax$. Similarly $\overline{\beta} \simeq by$ for a suitable point $b \in St_M(y)$. By the observation above we have $ax \simeq \hat{\sigma}_1x_o$ and $by \simeq \hat{\sigma}_nx_n$. Thus altogether we have $\overline{\alpha} \simeq \overline{\beta}$. q.e.d.

We return to the discussion of the functoriality of the groups $\pi_q(\lambda,\alpha)$. Let $f,g : M \rightrightarrows N$ be two partially proper maps and let $F : M\times I \to N$ be a <u>partially proper</u> homotopy from f to g. For every proper incomplete path

α in M the map

$$H := F\circ(\alpha\times id) : [0,1[\times I \to N$$

is a <u>proper</u> homotopy from $f\circ\alpha$ to $g\circ\alpha$. Thus, if α determines the end λ of M, then $f\circ\alpha$ and $g\circ\alpha$ determine the same end $\mu = f_*(\lambda) = g_*(\lambda)$ of N. Moreover, we have the following proposition.

<u>Proposition 8.3.</u> The triangle

$$\begin{tikzcd} & \pi_q(\mu, f\circ\alpha) \\ \pi_q(\lambda,\alpha) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_q(\mu, g\circ\alpha) \arrow[uu, "H_\#"'] \end{tikzcd}$$

commutes.

The proof may safely be left to the reader. This proposition implies the invariance of the groups $\pi_q(\lambda,\alpha)$ under partially proper homotopy equivalences.

We want to compare the homotopy group $\pi_q(\lambda,\alpha)$ with the homotopy group $\pi_q(P(\lambda),\overline{\varphi\circ\alpha}(1))$ for a given pure completion $\varphi: M \to P$. {Again $P(\lambda)$ denotes the end of M in P determined by λ, and $\overline{\varphi\circ\alpha}$ denotes the extension of $\varphi\circ\alpha$ to a path in P.} We are forced to assume that $P(\lambda)$ has complexity ≤ 1, i.e. is locally complete. Notice that this condition only depends on λ but not on the chosen pure completion φ, cf. II.9.25.

We regard φ as an inclusion. The set $P(\lambda)$ is open in its closure $\overline{P(\lambda)}$ in P. Thus the set $\partial P(\lambda) = \overline{P(\lambda)} \setminus P(\lambda)$ is a member of $\mathcal{T}_c(M)$. If some $K \in \mathcal{T}_c(M)$ contains $\partial P(\lambda)$ then $P(\lambda)$ is closed in $P\setminus K$. By II, Theorem 11.7, there exists a set $L \in \mathcal{T}_c(M)$ with $L \supset K$ such that the closure $\tilde{U} = \overline{U} \setminus \partial P(\lambda)$ in $P \setminus \partial P(\lambda)$ of the connected component U of $P\setminus L$

containing $P(\lambda)$ is a "mapping cylinder neighbourhood" of $P(\lambda)$. This means, there exists a proper map $h : \tilde{\partial}U \to P(\lambda)$, $\tilde{\partial}U := \partial U \setminus \partial P(\lambda)$, such that the triple $(\tilde{U}, \tilde{\partial}U, P(\lambda))$ is isomorphic to $(Z(h), (\tilde{\partial}U)\times 0, P(\lambda))$ with $Z(h)$ the mapping cylinder of h. Let $t \in [0,1[$ be a parameter with $\alpha([t,1[) \subset M \setminus L$. Since $(U, P(\lambda))$ is isomorphic to $(Z(h) \setminus (\tilde{\partial}U)\times 0, P(\lambda))$ the natural map $(q \geq 1)$

$$\pi_q(P(\lambda), \bar{\alpha}(1)) \to \pi_q(U, \bar{\alpha}(1)) = \pi_q(P \setminus L, \bar{\alpha}(1))$$

is an isomorphism. Composing the inverse of this map with

$$\pi_q(M \setminus L, \alpha(t)) \to \pi_q(P \setminus L, \alpha(t)) \xrightarrow{\sim} \pi_q(P \setminus L, \bar{\alpha}(1)),$$

where the last arrow is $(\bar{\alpha}|[t,1])_{\#}^{-1}$, we obtain, for every $q \geq 1$, a natural homomorphism

$$\sigma_{(L,t)} : \pi_q(M \setminus L, \alpha(t)) \to \pi_q(P(\lambda), \bar{\alpha}(1)).$$

Let $\Lambda'(\alpha)$ denote the set of all pairs $(L,t) \in \Lambda(\alpha)$ with L having the above properties. $\Lambda'(\alpha)$ is cofinal in $\Lambda(\alpha)$. Thus, for every $q \geq 1$, the group $\pi_q(\lambda, \alpha)$ is the projective limit of the groups $\pi_q(M \setminus L, \alpha(t))$ with $(L,t) \in \Lambda'(\alpha)$. Now it is easily seen that the transition maps between these groups are compatible with the homomorphisms $\sigma_{(L,t)}$. Thus the $\sigma_{(L,t)}$ yield a natural homomorphism

$$\sigma : \pi_q(\lambda, \alpha) \to \pi_q(P(\lambda), \bar{\alpha}(1)).$$

We abandon the interpretation of φ as an inclusion map and denote this homomorphism by

$$\sigma_\varphi : \pi_q(\lambda, \alpha) \to \pi_q(P(\lambda), \overline{\varphi \circ \alpha}(1)).$$

If $\psi : M \to Q$ is a second pure completion of M and $f : Q \xrightarrow[M]{} P$ is a morphism from ψ to φ then f restricts to a proper surjective map $g : Q(\lambda) \twoheadrightarrow P(\lambda)$. It is easily checked that, for every $q \geq 1$, the triangle

$$(8.4)\qquad \begin{tikzcd} & \pi_q(Q(\lambda),\overline{\psi\circ\alpha}(1)) \arrow[dd, "g_*"] \\ \pi_q(\lambda,\alpha) \arrow[ur, "\sigma_\psi"] \arrow[dr, "\sigma_\varphi"'] & \\ & \pi_q(P(\lambda),\overline{\varphi\circ\alpha}(1)) \end{tikzcd}$$

commutes.

We recall that every completion of M is dominated by a completion $M \hookrightarrow P$ such that the end $P(\lambda)$ in P is collared (II, Th. 11.6). Assume that our given completion $\varphi : M \hookrightarrow P$ is of this special type. Then there exists, as is easily seen, a cofinal subset $\Lambda''(\alpha)$ of $\Lambda'(\alpha)$ such that, for every $(L,t) \in \Lambda''(\alpha)$, the connected component U of $P(\lambda)$ in $P \setminus \varphi(L)$ is an "open collar" of $P(\lambda)$. This implies that $\sigma_{(L,t)}$ is an isomorphism. We conclude

<u>Proposition 8.5.</u> If the end $P(\lambda)$ of M in P is collared, then

$$\sigma_\varphi : \pi_q(\lambda,\alpha) \to \pi_q(P(\lambda),\overline{\varphi\circ\alpha}(1))$$

is an isomorphism for every $q \geq 1$.

Now we are well prepared to give an application of our theory of homotopy groups of ends.

<u>Theorem 8.6</u> (Homotopy invariance of collared ends). Let $\varphi : M \hookrightarrow P$ and $\psi : M \hookrightarrow Q$ be completions of M such that both ends $P(\lambda)$ and $Q(\lambda)$ determined by a given absolute end λ of M in P and Q are collared. Then the spaces $P(\lambda)$ and $Q(\lambda)$ are homotopy equivalent.

<u>Proof.</u> There exists a completion $\chi : M \hookrightarrow S$ dominating φ and ψ such that $S(\lambda)$ is collared. Thus it suffices to consider the case that there exists a morphism $f : Q \xrightarrow[M]{} P$ from ψ to φ. We choose a proper incomplete

path α in M with $\varepsilon(\alpha) = \lambda$. In the commutative diagram (8.4) both maps σ_ψ and σ_φ are isomorphisms by Proposition 8.5. Thus g_* is an isomorphism for every $q \geq 1$. By Whitehead's theorem 6.6 the map $g : Q(\lambda) \to P(\lambda)$ is a homotopy equivalence. q.e.d.

In the course of this study it turned out that it sometimes is natural to consider partially proper homotopy classes of partially proper maps. It would be desirable to have analogues of our two main theorems on homotopy classes for partially proper homotopy classes. Up to now we do not know how to prove such theorems.

Appendix A (to Chapter I): Abstract locally semialgebraic spaces.

This appendix is meant only for those readers who are familiar with the notion of the real spectrum Sper A of a commutative ring A, cf. [CR] and the literature cited there, and for an introduction to real spectra also [BCR, Chap. 7] and parts of the articles [L], [Br], [K]. We regard here Sper A as a topological space, neglecting its usual structure sheaf of "abstract Nash functions". In fact we will introduce below a new - much bigger - structure sheaf on Sper A , the sheaf of "abstract semialgebraic functions".

The appendix is not necessary for an understanding of the other parts of this book in any technical sense. Our goal is to explain how the category of locally semialgebraic spaces over a real closed field R fits into the framework of "abstract" semialgebraic geometry, a now rapidly developing area. In particular we will see that our locally semialgebraic spaces are objects equivalent to suitable locally ringed spaces over genuine - albeit not Hausdorff - topological spaces. This view point leads to a better understanding of various examples of locally semialgebraic spaces given in Chapter I. We will see that, although some of these spaces look pathological at first glance, they nevertheless are equivalent to subspaces of very honest real spectra and thus may be regarded as "occuring in nature".

Let V be an affine algebraic variety over a real closed field R. As usual we denote by R[V] the ring of algebraic functions (= "polynomial functions") on V and by V(R) the set of R-rational points of V. The set V(R) is a subset of the real spectrum Sper R[V] , and the subspace topology of V(R) in Sper R[V] coincides with the usual strong topology of V(R). There is a canonical 1-1-correspondence between the semialgebraic subsets of V(R) and the constructible subsets of Sper R[V] .

Namely, if $M \subset V(R)$ is semialgebraic, and if we describe M by equalities and inequalities between functions in $R[V]$, then these equalities and inequalities also define a constructible subset $\tilde{M}$ of $\operatorname{Sper} R[V]$. The set $\tilde{M}$ is the unique constructible subset of $\operatorname{Sper} R[V]$ with $\tilde{M} \cap V(R) = M$, cf. [CR, §5]. If M is a one point set $\{x\}$, then $\tilde{M} = M$.

Now we consider a fixed semialgebraic subset M of $V(R)$. A semialgebraic subset X of M is open (resp. closed) in M if and only if $\tilde{X}$ is open (resp. closed) in $\tilde{M}$ [CR, §5]. Moreover the open constructible subsets of $\tilde{M}$ are precisely those open subsets which are quasicompact. Thus the family $\mathring{\gamma}(M)$ of open semialgebraic subsets and then also the family $\gamma(M)$ of all semialgebraic subsets of M is completely determined by the topology of $\tilde{M}$.

The points of $\tilde{M}$ can be interpreted as the ultrafilters of the Boolean lattice $\gamma(M)$ in a canonical way, cf. [Br, p. 260], [CaC, §1]. Then the constructible subset $\tilde{X}$ of $\tilde{M}$ corresponding to a semialgebraic subset X of M is just the set of all ultrafilters F with $X \in F$. In particular ($X = \{x\}$) a point x of M is identified with the principal ultrafilter F_x generated by x.

The open quasicompact subsets of $\tilde{M}$ are a basis of the topology of $\tilde{M}$ [CR, §5] and, since the space $\tilde{M}$ itself is quasicompact, every covering of $\tilde{M}$ by open subsets has a finite subcovering. Thus it is evident that there is a 1-1-correspondence $F \leftrightarrow \tilde{F}$ between the sheaves F on the semialgebraic space M and the sheaves $\tilde{F}$ on the topological space $\tilde{M}$, given by the formula $F(U) = \tilde{F}(\tilde{U})$ with U running through $\mathring{\gamma}(M)$. In particular the sheaf $\mathcal{O}_M$ of semialgebraic functions on M corresponds to a sheaf of rings $\tilde{\mathcal{O}}_M$ on $\tilde{M}$. The pair $(\tilde{M}, \tilde{\mathcal{O}}_M)$ is a locally ringed space in the usual topological sense. It is an example of an <u>abstract semialgebraic space</u>, to be defined now.

Let A be an arbitrary commutative ring (with 1) and K be a constructible subset of Sper A . Recently there has been introduced a sheaf $\mathcal{O}_K$ of "abstract semialgebraic functions" on K, cf. [B_1, §3], [Sch], [D_1, §1]. The sections of this sheaf may be considered as "functions" which are continuous and whose graphs are closed and constructible, cf. [D_1, §1]. In the case $A = R[V]$, $K = \widetilde{M}$ from above, the sheaf $\mathcal{O}_K$ coincides with $\widetilde{\mathcal{O}}_M$. So these functions are the abstract analogues of the semialgebraic functions on semialgebraic spaces over real closed fields [D_1, 1.9].

We call a locally ringed space $(K,\mathcal{O}_K)$ as above a <u>semialgebraic subspace</u> of Sper A . If one desires to establish an "abstract" semialgebraic geometry, as people now do, then it is quite natural to study locally ringed spaces which locally look like semialgebraic subspaces of real spectra.

<u>Definitions 1</u> [D_2, I, §1]. a) An <u>abstract affine semialgebraic space</u> is a locally ringed space which is isomorphic to a semialgebraic subspace of a real spectrum.

b) An <u>abstract locally semialgebraic space</u> is a locally ringed space $(X,\mathcal{O}_X)$ which has an open covering $(X_\alpha | \alpha \in I)$ such that, for every $\alpha \in I$, the space $(X_\alpha, \mathcal{O}_X | X_\alpha)$ is an abstract affine semialgebraic space and the intersection $X_\alpha \cap X_\beta$ is quasicompact for any two indices $\alpha, \beta \in I$. {We are only interested in "quasiseparated" spaces.} If, in addition, the topological space X is quasicompact then $(X,\mathcal{O}_X)$ is called an abstract <u>semialgebraic</u> space. Since the X_α above are quasicompact, this just means that the index set I can be chosen finite.

c) A <u>locally semialgebraic map</u> between abstract locally semialgebraic spaces $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ is a morphism $(f,\mathcal{J})$ in the category of locally ringed spaces, i.e. $f: X \to Y$ is a continuous map and $\mathcal{J}: f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a homomorphism of sheaves of rings such that, for every $x \in X$, the homomorphism $\mathcal{J}_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local. If the spaces $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$

are semialgebraic then, of course, we call such a morphism $(f,\mathfrak{I})$ more briefly a <u>semialgebraic map</u>.

The locally semialgebraic spaces and maps over a real closed field, as defined in I, §1 will henceforth often be called <u>geometric</u> locally semialgebraic spaces and maps in order to distinguish them from the <u>abstract</u> spaces and maps defined now. The reader will perceive the close analogy between the definitions here and in I, §1.

Abstract locally semialgebraic spaces have first been defined by N. Schwartz [Sch]. He called these spaces "real closed spaces".

We give two examples of abstract locally semialgebraic maps.

<u>Examples A1.</u> i) Let $\varphi : A \to B$ be a ring homomorphism. φ induces a continuous map $f = \operatorname{Sper}\varphi : \operatorname{Sper} B \to \operatorname{Sper} A$, cf. [CR]. Let $\mathcal{O}_{\operatorname{Sper} A}$ and $\mathcal{O}_{\operatorname{Sper} B}$ denote the sheaves of abstract semialgebraic functions on $\operatorname{Sper} A$ and $\operatorname{Sper} B$. "Composition with f" yields a homomorphism $\mathcal{O}_{\operatorname{Sper} A} \to f_*\mathcal{O}_{\operatorname{Sper} B}$ of sheaves of rings, cf. [D_1, 1.8], hence a homomorphism $\mathfrak{I}: f^*\mathcal{O}_{\operatorname{Sper} A} \to \mathcal{O}_{\operatorname{Sper} B}$. This homomorphism $\mathfrak{I}$ is local on the stalks. Thus $(f,\mathfrak{I})$ is a semialgebraic map between the semialgebraic spaces $(\operatorname{Sper} A, \mathcal{O}_{\operatorname{Sper} A})$ and $(\operatorname{Sper} B, \mathcal{O}_{\operatorname{Sper} B})$. We denote this morphism by $\operatorname{Sper}\varphi$.

ii) Let $(f,\mathfrak{I}) : (M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ be a semialgebraic map (in the sense of I, §1) between semialgebraic subspaces $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$ of affine varieties V and W over some real closed field R. We consider the semialgebraic subspaces $(\tilde{M},\tilde{\mathcal{O}}_M)$, $(\tilde{N},\tilde{\mathcal{O}}_N)$ of $\operatorname{Sper} R[V]$ and $\operatorname{Sper} R[W]$ corresponding to $(M,\mathcal{O}_M)$ and $(N,\mathcal{O}_N)$. f yields a map $\tilde{f}$ from $\tilde{M}$ to $\tilde{N}$ as follows: If F is a point of $\tilde{M}$, i.e. an ultrafilter in $\mathscr{T}(M)$, then $\tilde{f}(F)$ is the filter in $\mathscr{T}(N)$ generated by the sets $f(X)$ with X running through F. It turns out that $\tilde{f}(F)$ is again an ultrafilter [Br, p. 261]. It is easily checked that this map $\tilde{f}: \tilde{M} \to \tilde{N}$ is continuous. For any $X \in \mathcal{T}(N)$ we have $\tilde{f}^{-1}(\tilde{X}) = f^{-1}(X)^{\sim}$. For any sheaf $\mathcal{F}$ on the semialgebraic space N we have $f^*(\mathcal{F})^{\sim} = \tilde{f}^*(\tilde{\mathcal{F}})$. The homomorphism $\mathcal{J}: f^*(\mathcal{O}_N) \to \mathcal{O}_M$ corresponds to a homomorphism $\tilde{\mathcal{J}}: \tilde{f}^*(\tilde{\mathcal{O}}_M) \to \tilde{\mathcal{O}}_M$ of sheaves of rings over $\tilde{M}$ which turns out to be local on the stalks. Thus our semialgebraic map $(f,\mathcal{J}): (M,\mathcal{O}_M) \to (N,\mathcal{O}_N)$ yields an abstract semialgebraic map $(\tilde{f},\tilde{\mathcal{J}}): (\tilde{M},\tilde{\mathcal{O}}_M) \to (\tilde{N},\tilde{\mathcal{O}}_N)$.

It is easily seen that the morphisms $(\tilde{f},\tilde{\mathcal{J}})$ in this last example A.1.ii are precisely all abstract semialgebraic maps from $(\tilde{M},\mathcal{O}_{\tilde{M}})$ to $(\tilde{N},\mathcal{O}_{\tilde{N}})$. This implies that we have an isomorphism $(M,\mathcal{O}_M) \mapsto (\tilde{M},\mathcal{O}_{\tilde{M}})$, $(f,\mathcal{J}) \mapsto (\tilde{f},\tilde{\mathcal{J}})$ from the category of affine semialgebraic spaces over R to a full subcategory of the category of abstract affine semialgebraic spaces. It is an easy matter to prove that this equivalence extends in a unique way to an isomorphism $(M,\mathcal{O}_M) \mapsto (\tilde{M},\mathcal{O}_{\tilde{M}})$, $(f,\mathcal{J}) \mapsto (\tilde{f},\tilde{\mathcal{J}})$ from the category of all locally semialgebraic spaces over R to a full subcategory of the category of abstract locally semialgebraic spaces. We call $(\tilde{M},\mathcal{O}_{\tilde{M}})$ the <u>abstraction</u> of the locally semialgebraic space $(M,\mathcal{O}_M)$ and $(\tilde{f},\tilde{\mathcal{J}})$ the abstraction of the locally semialgebraic map $(f,\mathcal{J})$.

Many of the facts about $(\tilde{M},\mathcal{O}_{\tilde{M}})$ for M affine semialgebraic, mostly mentioned above, generalize readily (in a suitable way) to the case that M is locally semialgebraic. The points of $\tilde{M}$ can be interpreted as those ultrafilters of the Boolean lattice $\mathcal{T}(M)$ of locally semialgebraic subsets of M which contain semialgebraic sets. The points of M are identified with the principal ultrafilters, and thus M is a subset of $\tilde{M}$ in a natural way. $\tilde{\mathcal{O}}_M$ is a sheaf of R-algebras since $\mathcal{O}_M$ is a sheaf of R-algebras. The points of M are precisely those $x \in \tilde{M}$ where the residue class field $k(x)$ of $\tilde{\mathcal{O}}_{M,x}$ coincides with R. For the other points $x \in \tilde{M}$ the fields $k(x)$ are real closed fields containing R. The points of M are closed in $\tilde{M}$, but usually $\tilde{M}$ contains also other closed points.

The subspace topology of M in $\tilde{M}$ is just the strong topology (cf. I, §3) on M. For every open semialgebraic subset U of M there exists a unique quasicompact open subset $\tilde{U}$ of $\tilde{M}$ with $\tilde{U} \cap M = U$. More generally, for every open locally semialgebraic subset U of M there exists a unique retrocompact open subset $\tilde{U}$ of $\tilde{M}$ with $\tilde{U} \cap M = U$. (A subset Z of a topological space X is called retrocompact in X if $Z \cap V$ is quasicompact for every quasicompact open subset V of X [EGA I*.0.2.3]). In this way we obtain a bijection $U \mapsto \tilde{U}$ from the set $\dot{\mathcal{T}}(M)$ to the set of all retrocompact open subsets of $\tilde{M}$, which maps $\dot{\gamma}(M)$ onto the set of all quasicompact open subsets of $\tilde{M}$. The bijection $U \mapsto \tilde{U}$ preserves finite intersections and unions (in fact "locally finite" intersections and unions). The points of $\tilde{U}$ are precisely the ultrafilters $F \in \tilde{M}$ with $U \in F$.

As for any abstract locally semialgebraic space the quasicompact open subsets of $\tilde{M}$ are a basis of the topology of $\tilde{M}$. Thus it is again clear that we have an isomorphism $F \mapsto \tilde{F}$ from the category of sheaves F on the generalized topological space M (cf. I, §1, Def. 1) to the category of sheaves on the topological space $\tilde{M}$. Of course, $\tilde{\mathcal{O}}_M = \mathcal{O}_{\tilde{M}}$.

A sheaf $\mathcal{G}$ on $\tilde{M}$ is - more or less - determined by its family of stalks $(\mathcal{G}_x | x \in \tilde{M})$. But there are sheaves of abelian groups $\mathcal{F}$ on M which are not zero while all stalks $\mathcal{F}_x$, $x \in M$, are zero, cf. [D_2, Example I.1.7]. Thus if you want to argue stalk by stalk, as it is often convenient in sheaf theory, then usually you have to work on $\tilde{M}$ instead of M (cf. [D_1], [D_2] for some striking examples).

If $(M_\alpha | \alpha \in I)$ is the family of connected components of M (I, §3) then, as is easily seen, $(\tilde{M}_\alpha | \alpha \in I)$ is the family of connected components of the topological space $\tilde{M}$.

Let $\tilde{M}^{max}$ denote the topological subspace of $\tilde{M}$ consisting of the closed points of $\tilde{M}$. There is a close connection between some of the properties of $(M,\mathcal{O}_M)$ studied in I, §4-§7 and certain standard properties of the topological space $\tilde{M}^{max}$ (which may hold or not hold). We give examples of this connection. The statements are all proven in [D_2, Chap. I, §4, §5]. As is usual in topology the properties "compact", "locally compact", "paracompact" are understood to include "Hausdorff".

Examples A.2. i) If $(M,\mathcal{O}_M)$ is regular then $\tilde{M}^{max}$ is Hausdorff. The converse is true if $(M,\mathcal{O}_M)$ is assumed to be taut [D_2, I.4.1].

ii) $(M,\mathcal{O}_M)$ is regular and taut if and only if $\tilde{M}^{max}$ is locally compact [D_2, I.4.5].

iii) $(M,\mathcal{O}_M)$ is affine semialgebraic if and only if $\tilde{M}^{max}$ is compact. This is an easy consequence of (ii).

iv) $(M,\mathcal{O}_M)$ is regular and paracompact if and only if $\tilde{M}^{max}$ is paracompact (in the topological sense).

v) $(M,\mathcal{O}_M)$ is connected if and only if $\tilde{M}^{max}$ is connected. In general, if $(M_\alpha|\alpha\in I)$ is the family of connected components of M, then $\tilde{M}_\alpha \cap \tilde{M}^{max} = \tilde{M}_\alpha^{max}$ since $\tilde{M}_\alpha$ is closed in $\tilde{M}$. Thus $(\tilde{M}_\alpha^{max}|\alpha\in I)$ is the family of connected components of $\tilde{M}^{max}$.

These examples show that $\tilde{M}^{max}$ is a well behaved Hausdorff space under mild conditions on $(M,\mathcal{O}_M)$. (N.B. These are only conditions on the families of subsets $\mathring{\gamma}(M)$, $\mathring{\mathcal{T}}(M)$ of the set M). The examples also give new evidence that not "separated" but "regular" is the reasonable separation property for geometric locally semialgebraic spaces.

How close is the connection between $\tilde{M}$ and its subspace $\tilde{M}^{max}$? This subspace contains the set M. For every non empty open semialgebraic subset U of M, the open subset $\tilde{U}$ of M contains points of $\tilde{M}^{max}$, namely the points of U. Thus $\tilde{M}^{max}$ is dense in $\tilde{M}$. By the same argument $\tilde{M}^{max}$ is even "very dense" in $\tilde{M}$ in the following sense: The abstraction $\tilde{X}$ of any non empty semialgebraic subset X of M meets the set $\widetilde{M}^{max}$.

Assume that $(M,\mathcal{O}_M)$ is regular. Then every semialgebraic subset U of M is affine. This implies that the specializations of a point $x\in\widetilde{M}$ form a chain, since - as is well known [CR, p. 32] - the specializations of a point in a real spectrum form a chain. (A point y <u>specializes</u> x if $y\in\overline{\{x\}}$ [EGA I*,0.2.1]). In particular, there is a unique closed point $r(x)\in\widetilde{M}^{max}$ which specializes x. It turns out that the map $r:\widetilde{M}\to\widetilde{M}^{max}$ is continuous if and only if $(M,\mathcal{O}_M)$ is also taut [D_2, I.4.3]. Thus for $(M,\mathcal{O}_M)$ taut and regular the space $\widetilde{M}^{max}$ is a retract of $\widetilde{M}$.

Since $\widetilde{M}^{max}$ is often such an honest topological space and is very dense in $\widetilde{M}$, why not replace the abstraction $(\widetilde{M},\mathcal{O}_{\widetilde{M}})$ of $(M,\mathcal{O}_M)$ by the locally ringed space $(\widetilde{M}^{max},i^*\mathcal{O}_{\widetilde{M}})$, with $i:\widetilde{M}^{max}\hookrightarrow\widetilde{M}$ the inclusion? Doing this we may run into sheaf theoretic difficulties since the intersection of an open quasicompact subset of $\widetilde{M}$ with $\widetilde{M}^{max}$ is usually not quasicompact. Thus we have to consider in $\widetilde{M}^{max}$ "more" open coverings than in $\widetilde{M}$. But there is an even more serious obstruction: If $(\widetilde{f},\widetilde{\mathcal{J}})$ is the abstraction of a locally semialgebraic map $(f,\mathcal{J}):(M,\mathcal{O}_M)\to(N,\mathcal{O}_N)$, then usually $\widetilde{f}$ will <u>not</u> map $\widetilde{M}^{max}$ into $\widetilde{N}^{max}$. Thus the topological space $\widetilde{M}^{max}$ does not depend in a naive functorial way on $(M,\mathcal{O}_M)$. This makes $\widetilde{M}^{max}$ less useful than $\widetilde{M}$.

Henceforth we usually omit the structure sheaf $\mathcal{O}_X$ in the notation $(X,\mathcal{O}_X)$ of an abstract locally semialgebraic space and the second component $\mathcal{J}$ in the notation $(f,\mathcal{J})$ of a locally semialgebraic map, as we did with geometric spaces and maps in most parts of this book. In particular, for any commutative ring A, the notation $\operatorname{Sper} A$ now means the real spectrum of A equipped with the sheaf of abstract semialgebraic functions (instead of the sheaf of abstract Nash functions, although Nash functions are very important as soon as we quit the purely semialgebraic setting). We also say more briefly "space" and "map" instead of "locally semialgebraic space" and "locally semialgebraic map".

Schwartz has proved in [Sch], among many other things that, for any two abstract maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the fibre product $X_1 \times_Y X_2$ exists in the category of abstract spaces. We give a description of the fibre product in three special cases.

Examples A.3. i) Let $\varphi : C \to A$ and $\psi : C \to B$ be ring homomorphisms. Then the fibre product of the spaces $\operatorname{Sper} A$ and $\operatorname{Sper} B$ over $\operatorname{Sper} C$ with respect to the semialgebraic maps $\operatorname{Sper}\varphi$ and $\operatorname{Sper}\psi$ (cf. Example A.1.i) is the space $\operatorname{Sper} A \otimes_C B$, the fibre product projections from $\operatorname{Sper} A \otimes_C B$ to $\operatorname{Sper} A$ and $\operatorname{Sper} B$ being the maps induced by the canonical ring homomorphism $A \to A \otimes_C B$ and $B \to A \otimes_C B$.

ii) Let

$$\begin{array}{ccc} M & \xrightarrow{p_1} & M_2 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow[f_1]{} & N \end{array}$$

be a cartesian square in the category of geometric spaces over a real closed field R (cf. I, Prop. 3.5). Then

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{p}_1} & \tilde{M}_2 \\ {\scriptstyle \tilde{p}_2}\downarrow & & \downarrow{\scriptstyle \tilde{f}_2} \\ \tilde{M}_1 & \xrightarrow[\tilde{f}_1]{} & \tilde{N} \end{array}$$

is a cartesian square in the category of abstract locally semialgebraic spaces. In short,

$$(M_1 \times_N M_2)^{\sim} = \tilde{M}_1 \times_{\tilde{N}} \tilde{M}_2 .$$

iii) Let M be a geometric space over R and let S be a real closed field containing R. Then the fibre product of $\tilde{M}$ and $\operatorname{Sper} S$ with respect to the obvious maps $\tilde{M} \to \operatorname{Sper} R$ and $\operatorname{Sper} S \to \operatorname{Sper} R$ is the abstraction $\widetilde{M(S)}$ of the space $M(S)$ over S obtained from M by base field extension (I, 2.10). In short,

$$\tilde{M} \times_{\operatorname{Sper} R} \operatorname{Sper} S = \widetilde{M(S)}.$$

The image of an ultrafilter $F \in \widetilde{M(S)}$ under the projection from $\widetilde{M(S)}$ to $\tilde{M}$ is the ultrafilter $G \in \tilde{M}$ consisting of all $X \in \mathcal{T}(M)$ with $X(S) \in F$.

As these examples show it is sometimes more satisfying from a formal point of view to work in the category of abstract locally semialgebraic spaces instead of the bunch of categories of geometric spaces over the various real closed fields. We have also indicated above that sheaf theory can be better done over abstract spaces instead of geometric spaces. At this point the reader may suspect that the present authors have written the wrong book! Why not abandon geometric spaces altogether and work only in the abstract category?

The abstraction $\tilde{M}$ of a geometric space M contains – if $\dim M > 0$ – many more points than M. While this fact makes $\tilde{M}$ more amenable than M for formal considerations and for sheaf theory, it makes $\tilde{M}$ usually more difficult than M as soon as concrete geometric problems have to be solved.

For example, a homotopy between two geometric maps $f,g : M \rightrightarrows N$ over R is a locally semialgebraic map $F : M \times I \to N$. It lives on the cartesian product of the set M and the unit interval I in R and fulfills the requirements $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for $x \in M$ (cf. III, §1). The abstraction $\tilde{F}$ of F lives on the fibre product $\tilde{M} \times_{\operatorname{Sper} R} \tilde{I}$ (cf. Example A.3.ii). Although $\operatorname{Sper} R$ has only one point this fibre product contains many more points than the cartesian product of the topological spaces $\tilde{M}$ and $\tilde{I}$.

Even $(\tilde{M} \times_{\mathrm{Sper}\, R} \tilde{I})^{max}$ is bigger than the cartesian product of $\tilde{M}^{max}$ and $\tilde{I}^{max}$. Thus the geometric homotopy relation corresponds to a relation between abstract maps which is more complicated than topological homotopy. It seems that this relation bears much more meaning for the abstract maps than topological homotopy, since the results in Chapter III, in particular the two main theorems in its various versions (III.3.1, 4.2, 5.1, 6.3, 6.4), give overwhelming evidence that we have defined homotopy between geometric maps in the right way.

We feel that most proofs <u>in this book</u> cannot be simplified by passing from the geometric spaces to their abstractions. It is usually hopeless to use the standard theorems of general and algebraic topology in order to prove our results, and this not only for the reason that the abstract spaces are almost never Hausdorff.

Quite on the contrary, we believe that our geometric theory will serve as a sound and necessary base to explore abstract spaces. Certainly abstract spaces will become extremely important in semialgebraic geometry, as schemes have become important in algebraic geometry. But, to the best of our knowledge, the main method available up to now to obtain deeper results about abstract spaces is to reduce the problems to the geometric case.

Notice, for example, that the real spectrum of a finitely generated commutative ring is the abstraction of a geometric space over the field R_o of real algebraic numbers. If A is an arbitrary commutative ring then a problem in Sper A which involves only finitely many elements of A can sometimes be reduced to Sper A_o for A_o a suitable finitely generated subring of A and thus reduced to a geometric problem. A good exercise in this direction is to prove Tietze's extension theorem for semialgebraic functions on an abstract affine semialgebraic space starting from the known extension theorem [DK_5, §4] (cf. I.4.15) in the geometric case.

We close this appendix with an "explanation" of Example I.2.9 and its subexamples 2.6, 2.8 in terms of abstract spaces.

Example A.4. Let M be a geometric space and let D be an arbitrary open subset of (the underlying topological space of) $\tilde{M}$. Then D, equipped with the structure sheaf $\tilde{\mathcal{O}}_M|D$, is an abstract locally semialgebraic space. We want to describe D in geometric terms. We choose a covering $(D_\alpha|\alpha\in I)$ of D by quasicompact open sets. Let $X_\alpha := D_\alpha \cap M$. Then $X_\alpha \in \mathring{\mathcal{T}}(M)$ and $D_\alpha = \tilde{X}_\alpha$. We now are in the situation of Example I.2.9. Let X be the inductive limit, considered there, of the open semialgebraic subspaces $X_J := \cup(X_\alpha|\alpha\in J)$ with J running through the finite subsets of I. We have

$$\tilde{X}_J = D_J := \cup(D_\alpha|\alpha\in J).$$

Obviously $(D,\tilde{\mathcal{O}}_M|D)$ is the inductive limit of the spaces $(D_J,\tilde{\mathcal{O}}_M|D_J)$. It is now pretty evident that the abstraction $\tilde{X}$ of the geometric space X is just the open subspace D of $\tilde{M}$. Of course it does not matter with which covering $(D_\alpha|\alpha\in I)$ of D by quasicompact open subsets we have started. If U is any quasicompact open subset of D then $U \subset D_J$ for some finite subset J of I, hence $U \cap M \subset X_J$. Thus adding $U \cap M$ to the family $(X_\alpha|\alpha\in I)$ leads to the same geometric space X.

Conversely, if $(X_\alpha|\alpha\in I)$ is any family of open semialgebraic subsets of M and X is the inductive limit of the open subspaces X_J of M, as considered in Example I.2.9, then the abstraction $\tilde{X}$ of X is the open subspace $D := \cup(\tilde{X}_\alpha|\alpha\in I)$ of $\tilde{M}$. Thus the abstractions of the spaces constructed in Example I.2.9 are precisely all open subspaces of $\tilde{M}$.

Subexample A.5. i) Assume that M is locally complete. Let $\gamma_c(M)$ denote the set of all complete semialgebraic subsets of M. We choose as family $(X_\alpha|\alpha\in I)$ the family of all open semialgebraic subsets U of M with $\bar{U}\in\gamma_c(M)$. This leads to the space $X = M_{loc}$ considered in I, §7, and, for M affine, in Example I.2.6. The points of $\tilde{X}$ are precisely all ultrafilters F in $\mathcal{T}(M)$ with $F\cap\gamma_c(M) \neq \emptyset$. Intuitively, the "points at infinity" of $\tilde{M}$ are those ultrafilters $F\in\tilde{M}$ which contain every set $M\setminus K$ with $K\in\gamma_c(M)$ (cf. our theory of ends in Chapter II and notice that an ultrafilter $F\in\tilde{M}$ which contains a partially complete subset of M also contains a complete subset of M, since F contains semialgebraic subsets). The set of points at infinity is closed in $\tilde{M}$, and $\tilde{M}_{loc}$ is obtained from $\tilde{M}$ by omitting this set.

ii) For clarity we denote geometric and abstract spaces in the old way as pairs involving the structure sheaf. As before, let $(M,\mathcal{O}_M)$ be a geometric space. Let X be any open subset of M in the strong topology. Let $(D_\alpha|\alpha\in I)$ be the family of all quasicompact open subsets of $\tilde{M}$ with $D_\alpha\cap M\subset X$ and let $D := \cup(D_\alpha|\alpha\in I)$. This set D is the largest open subset of $\tilde{M}$ with $D\cap M = X$. Let $X_\alpha := D_\alpha\cap M$, hence $D_\alpha = \tilde{X}_\alpha$. The open subspace $(D,\mathcal{O}_{\tilde{M}}|D)$ of $(\tilde{M},\mathcal{O}_{\tilde{M}})$ is the abstraction of a geometric space $(X,\mathcal{O}_X)$ with underlying set X which is the inductive limit of the spaces $(X_\alpha,\mathcal{O}_M|X_\alpha)$. This is just the space "induced on X" by the space $(M,\mathcal{O}_M)$, as considered in Example I.2.7, since the X_α are precisely all semialgebraic open subsets of M which are contained in X.

In order to understand the message of all these examples the reader should choose for M an old friend, say $M = R^n$. Then $\tilde{M} = \operatorname{Sper} R[T_1,\ldots,T_n]$. The open subsets D of $\operatorname{Sper} R[T_1,\ldots,T_n]$, considered as subspaces are abstractions of geometric spaces X which are regular by I.4.2 but nevertheless may be quite complicated. In particular paracompactness can fail for such a space X. For example, $X = M_{loc}$ is not paracompact if R does not contain a sequence of positive elements converging to zero, even for $n = 1$ (I.4.20). The example I.3.15 shows that $\operatorname{Sper} R[T_1, T_2]$ contains an open subset D such that X is not taut and a fortiori not paracompact, even for $R = \mathbb{R}$.

Appendix B (to Chapter I): Conservation of some properties of spaces and maps under extension of the base field.

In the following S is a real closed field containing our base field R. Contrary to the usage in Chapter III and a large part of Chapter II a "space" over R now means an arbitrary locally semialgebraic separated space over R. Later we shall return to the convention that every space is assumed to be regular and paracompact.

Theorem B.1. Let M be a space over R.

i) $M(S)$ is semialgebraic if and only if M is semialgebraic.

ii) $M(S)$ is affine semialgebraic if and only if M is affine semialgebraic.

iii) $M(S)$ is paracompact if and only if M is paracompact.

iv) $M(S)$ is regular and paracompact if and only if M is regular and paracompact.

Proof. i) Let $(M_\alpha | \alpha \in I)$ be an admissible covering of M by affine open semialgebraic subsets. Then $(M_\alpha(S) | \alpha \in I)$ is an admissible covering of $M(S)$ by open affine semialgebraics. If M is semialgebraic then there exists a finite subset J of I with $M = \cup(M_\alpha | \alpha \in J)$. Then $M(S) = \cup(M_\alpha(S) | \alpha \in J)$, hence also $M(S)$ is semialgebraic. Conversely, if $M(S)$ is semialgebraic, then there exists a finite subset J of I with $M(S) = \cup(M_\alpha(S) | \alpha \in J)$. Intersecting with M we see that $M = \cup(M_\alpha | \alpha \in J)$, and we conclude that M is semialgebraic.

ii) We may already assume that M is semialgebraic. If M is affine then, of course, $M(S)$ is affine. Assume now that $M(S)$ is affine. We want to prove that M is affine. We choose a covering $(M_i | 1 \le i \le r)$ of M by finitely many open affine semialgebraics and induct on r. The case $r = 1$ is trivial. Once we have settled the case $r = 2$, then we are done: We know by induction that $U := M_1 \cup \ldots \cup M_{r-1}$ is affine and then that $M = U \cup M_r$ is affine. Thus we assume henceforth that $r = 2$.

We choose isomorphisms $\varphi_i : N_i \xrightarrow{\sim} M_i$ from semialgebraic subsets N_i of standard spaces R^{n_i} to M_i $(i = 1,2)$. Let $N_{12} := \varphi_1^{-1}(M_1 \cap M_2)$ and $N_{21} := \varphi_2^{-1}(M_1 \cap M_2)$. These are open semialgebraic subsets of N_1 and N_2 respectively. Let f denote the isomorphism $x \mapsto \varphi_2^{-1} \circ \varphi_1(x)$ from N_{12} onto N_{21}. Since $M(S)$ is affine there exist isomorphisms $\tilde{g}_i$ from $N_i(S)$ onto semialgebraic subsets $\tilde{X}_i$ of S^n $(i = 1,2)$ for some $n \in \mathbb{N}$, such that

$$(\tilde{g}_2 | N_{21}(S)) \circ f_S = \tilde{g}_1 | N_{12}(S)$$

(namely, this means an isomorphism from $M(S)$ onto $\tilde{X}_1 \cup \tilde{X}_2$ which maps $M_i(S)$ onto $\tilde{X}_i$). Using Tarski's principle we know that there exist isomorphisms g_i from N_i onto suitable semialgebraic subsets X_i of R^n with $(g_2|N_{21}) \circ f = g_1|N_{12}$. We then have an isomorphism from M onto $X_1 \cup X_2$ and we see that M is affine. For the argument using Tarski's principle the reader may consult §3 of Chapter III. As in that section one constructs a "parameter space" L whose points represent the polynomial data of a description of pairs (g_1,g_2) of isomorphisms $g_i : N_i \xrightarrow{\sim} X_i \subset R^n$ with $(g_2|N_{21}) \circ f = g_1|N_{12}$ of a fixed "type". If the type is chosen right then $L(S)$ contains a point which describes $(\tilde{g}_1,\tilde{g}_2)$. Thus $L(S)$ is not empty. By Tarski's principle L is not empty, and we are done.

iii) and iv). If M is paracompact (regular and paracompact) then we know from I, §4 that $M(S)$ is paracompact (resp. regular and paracompact) (cf. I.4.4, 4.9). Assume now that $M(S)$ is paracompact and, without loss of generality, that M is connected. Then $M(S)$ is connected. By I.4.17, $M(S)$ is Lindelöf. We choose an admissible covering $(U_n | n \in \mathbb{N})$ of $M(S)$ by open semialgebraic sets. For every $n \in \mathbb{N}$ we choose an open semialgebraic set V_n in M with $U_n \subset V_n(S)$. This is possible by the very definition of the space $M(S)$ (I. Ex. 2.10). If W is any open semialgebraic subset of M then $W(S)$ is covered by finitely many sets U_n, hence also by finitely many sets $V_n(S)$. This implies that W is covered by finitely many sets

V_n, and we conclude that $(V_n|n\in\mathbb{N})$ is an admissible covering of M (cf. I, §1). Thus M is Lindelöf.

We claim that for every $U\in\mathring{\gamma}(M)$ the closure $\overline{U}$ is again semialgebraic. By I, Prop. 4.18, this will imply that M is paracompact. By I, Prop. 3.22.c we know that $\overline{U}(S)=\overline{U(S)}$. This is a semialgebraic set, since M(S) is paracompact (I, Prop. 4.6). We conclude, by (i), that $\overline{U}$ is semialgebraic. Thus M is indeed paracompact.

Assume that, in addition, M(S) is regular. Let U be an open semialgebraic subset of M. Then U(S) is open semialgebraic in M(S). Since M(S) is regular this implies that U(S) is affine hence, by (ii), that U is affine. We conclude that M is regular (cf. I, Prop. 4.7). This finishes the proof of the theorem.

As before we are only interested in regular paracompact spaces. Assertion (iv) in Theorem B.1 gives us the justification to admit only such spaces. Thus, from nom on, a "space" again means a regular paracompact locally semialgebraic space. A "map" between spaces means a locally semialgebraic map.

<u>Proposition B.2.</u> Let M be a space over R.
i) M is partially complete if and only if M(S) is partially complete.
ii) M is complete if and only if M(S) is complete.

<u>Proof.</u> We choose an isomorphism $\varphi: |K|_R \xrightarrow{\sim} M$, with K a strictly locally finite abstract simplicial complex (II, Th. 4.4). Then φ_S is an isomorphism from $|K|_S$ onto M(S). Now M is partially complete (complete) if and only if K is closed (closed and finite) and this also means that M(S) is partially complete (complete). q.e.d.

We digress for a short time into the contents of Chapter II, §9. We choose a completion $j: M\hookrightarrow P$ of M. By the just proved proposition the space $P(S)$ is partially complete, and since $j(M)$ is dense in P also $j(M)(S) = j_S(M(S))$ is dense in $P(S)$ (cf. I, Prop. 3.22). Thus $j_S : M(S) \hookrightarrow P(S)$ is a completion of $M(S)$. We regard M as a subspace of P via j, hence also $M(S)$ as a subspace of $P(S)$. Let $((P_k, M_k) \mid k \geq 0)$ denote the derived sequence of (P,M) (II, §9, Def. 7). It is now evident that $((P_k(S), M_k(S)) \mid k \geq 0)$ is the derived sequence of $(P(S), M(S))$. This implies the following result.

Corollary B.3. M and $M(S)$ have the same complexity. If $(S_k(M) \mid k \geq 0)$ is the LC-stratification of M (cf. II, §9, Def. 8) then $(S_k(M)(S) \mid k \geq 0)$ is the LC-stratification of $M(S)$. In particular, $M(S)$ is locally complete (i.e. has complexity ≤ 1) if and only if M is locally complete.

Remark. Locally complete spaces are regular, and a space M is locally complete if and only if every open (affine) semialgebraic subset U of M is locally complete. This shows that the last statement in Corollary B.3 remains true without the assumption that M is regular and paracompact. (This remark will be needed in $[D_2]$.)

Now we are ready to discuss some properties of maps.

Theorem B.4. Let $f : M \to N$ be a map between spaces over R.

i) f_S is semialgebraic if and only if f is semialgebraic.

ii) f_S is partially proper if and only if f is partially proper.

iii) f_S is proper if and only if f is proper.

iv) f_S is partially finite if and only if f is partially finite.

v) f_S is finite if and only if f is finite.

Proof. i) We choose an admissible covering $(N_\alpha \mid \alpha \in I)$ of N by open semialgebraics. Then $(N_\alpha(S) \mid \alpha \in I)$ is an admissible covering of $N(S)$ by open semialgebraics. The map f_S (resp. f) is semialgebraic iff, for every $\alpha \in I$, the space $f_S^{-1}(N_\alpha(S)) = f^{-1}(N_\alpha)(S)$ (resp. $f^{-1}(N_\alpha)$) is semialgebraic. The assertion now follows from Theorem B.1.i. (Here we did not need the assumption that M and N are regular and paracompact.)

ii) We choose a completion

$$\begin{tikzcd} M \arrow[r, hook, "\varphi"] \arrow[d, "f"'] & P \arrow[d, "g"] \\ N \arrow[r, hook, "\psi"'] & Q \end{tikzcd}$$

of f (cf. II, Prop. 5.1). Then φ_S and ψ_S are completions of M(S) and N(S) as shown above. Thus

$$\begin{tikzcd} M(S) \arrow[r, hook, "\varphi_S"] \arrow[d, "f_S"'] & P(S) \arrow[d, "g_S"] \\ N(S) \arrow[r, hook, "\psi_S"'] & Q(S) \end{tikzcd}$$

is a completion of f_S. Without loss of generality we regard φ and ψ, hence also φ_S and ψ_S, as inclusion maps. f is partially proper if and only if $g^{-1}(N) = M$ and f_S is partially proper if and only if $g_S^{-1}(N(S)) = M(S)$. These conditions are equivalent, since $g_S^{-1}(N(S)) = g^{-1}(N)(S)$.

Assertion (iii) is now evident since a map is proper if and only if it is partially proper and semialgebraic.

Next we prove (v). A map is finite if and only if it is proper and has finite fibres. Taking into account (iii) we may already assume that f and (hence) f_S is proper. Of course, if f_S has finite fibres then a fortiori f has finite fibres. Assume now that f is finite. By Hardt's theorem (II, 6.3) there exists a locally finite partition $(N_\alpha | \alpha \in I)$ of N into semialgebraic sets such that, for every $\alpha \in I$, the restriction $f^{-1}(N_\alpha) \to N_\alpha$ of f is a trivial map with a finite fibre F_α. Then $(N_\alpha(S) | \alpha \in I)$ is a partition of N(S) into semialgebraic sets, and f_S is over each $N_\alpha(S)$ a trivial map with the fibre $F_\alpha(S) = F_\alpha$. Thus f_S has finite fibres.

It remains to prove (iv). We choose a locally finite covering $(A_\alpha | \alpha \in I)$ of M by closed semialgebraics. Then $(A_\alpha(S) | \alpha \in I)$ is a locally finite covering of $M(S)$ by closed semialgebraics. The map f_S is partially finite if and only if, for every index $\alpha \in I$, the restriction $f_S|A_\alpha(S) = (f|A_\alpha)(S)$ is a finite map. This means, as proved before, that, for every α, the map $f|A_\alpha$ is finite, hence that f is partially finite.

q.e.d.

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Symbols

$\mathbb{N}$ denotes the set $\{1,2,3,\dots\}$ of natural numbers.

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