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On Valuation Spectra

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INTRODUCTION

We have seen in the last decade how important it is to switch from the consideration of particular orderings of fields to a study of the set of all orderings of all residue class fields of a commutative ring A, i.e., the real spectrum $\operatorname{Sper} A$.

Now why not do the same with valuations? This leads to the definition of "valuation spectra." In principle the points of the valuation spectrum $\operatorname{Spv} A$ should be pairs $(\mathfrak{p}, v)$ consisting of a prime ideal $\mathfrak{p}$ of A, i.e., a point of $\operatorname{Spec} A$, and a Krull valuation v of the residue class field $qf(A/\mathfrak{p})$. Different valuations of $qf(A/\mathfrak{p})$ which have the same valuation ring are identified.

M. J. de la Puente has written a thesis under the guidance of G. Brumfiel at Stanford about such a valuation spectrum $\operatorname{Spv} A$ (which she calls the "Riemann surface" of A [**Pu**]. Without being aware of the work of Puente (which had not yet appeared), one of us (R.H.) in 1987 started a thorough investigation of valuation spectra [**Hu**, Chap. I]. Puente and Huber both arrive at the same definition of $\operatorname{Spv} A$.

The motivations of Puente (and Brumfiel) and Huber are different. Puente and Brumfiel want to use valuation spectra for compactification of affine algebraic varieties. Here we should also mention a recent paper by N. Schwartz [**S**], where he uses a related "absolute value spectrum" (which he also calls the "valuation spectrum") for the same purpose. The authors of the present article have been driven by some striking analogies between semialgebraic geometry and rigid analytic geometry, a subject started by John Tate (cf. [**BGR**], [**FP**]). This led Huber to a new "abstract" approach to rigid analytic geometry by use of "analytic spectra," which are natural descendents of valuation spectra [**Hu**]. (Only recently have we become aware of the extensive work of V. Berkovich [**Be**],

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who studies rank 1 valuations of Banack algebras and applies his theory to rigid analytic geometry. This is another "abstract" approach to rigid geometry.)

Since Huber's abstract rigid geometry is close in spirit to abstract real algebraic geometry, it is not surprising that these two theories can be "mixed." One result of such a mixture is Huber's recent paper on semirigid functions [**Hu**$_4$], which permits studies of real phenomena of rigid analytic varieties. As has been amply demonstrated by the Spanish school (Andradas, Ruiz, ... cf. also their article in this volume), semianalytic geometry is amenable to methods from abstract real algebraic geometry. We have high hopes that the same will turn out to be true of semirigid geometry.

The spaces on which the semirigid functions are defined are derivates of *real* valuation spectra. The *real valuation spectrum* $\operatorname{Sperv} A$ of a commutative ring is a refinement of the real spectrum $\operatorname{Sper} A$. Its points are the triples $(\mathfrak{p}, P, C)$ with $\mathfrak{p} \in \operatorname{Spec} A$, P an ordering on $qf(A/\mathfrak{p})$, and C a convex subring of $qf(A/\mathfrak{p})$ with respect to P. Notice that $(\mathfrak{p}, P)$ is a point of $\operatorname{Sper} A$ and $(\mathfrak{p}, C)$ is a point of $\operatorname{Spv} A$. In this way $\operatorname{Sperv} A$ may be viewed as a natural subspace of the fibre product of $\operatorname{Sper} A$ and $\operatorname{Spv} A$ over $\operatorname{Spec} A$.

Real valuation spectra are indispensable in real *rigid* geometry. They seem to be also a valuable tool in real *algebraic* geometry, as is indicated by the very frequent occurence of real valuation rings in arguments in this area. All this has motivated us to give several talks about valuation spectra in the Ragsquad seminar and also talks about semirigid functions, both in the Ragsquad seminar and at the AMS conference at San Francisco in January '91. This is also the motivation for the present article.

In this article we intend to give a comprehensive account of basic facts about valuation spectra, as defined in [**Hu**]. We also give some applications to algebraic geometry in order to demonstrate that valuation spectra are already useful there. We have decided not to go on to real valuation spectra and real geometry in this article, because we want to keep the picture as simple as possible. (A [very] brief treatment of real valuation spectra can be found in §1 of [**Hu**$_4$].) Once the reader has obtained a firm grasp of valuation spectra and a feeling about possible applications in algebraic geometry, he or she will have no difficulty understanding real valuation spectra, and will hopefully be able to explore applications in real algebraic geometry. The reader will also find the door open to abstract rigid geometry, which is a very extensive—but useful—enlargement of classical rigid geometry.

Thinking about applications of valuation spectra in algebraic or real algebraic geometry, we should remember that valuations played a central role in Zariski's approach from the late 1930's, building up algebraic geometry by algebraic means. Later this role was reduced by Grothendieck and others in favour of prime ideals. Valuations survived, for example, in various valuative criteria and the resolution of singularities, but lost their dominance in algebraic geometry. Recently, in the Ragsquad seminar and elsewhere, we experienced a revived interest in Zariski's work. This should not be surprising since valuations occur so frequently and in such a natural way in real algebraic geometry.

Valuation spectra may be viewed as a refinement of Zariski spectra. We hope that Chapter 4 of the present article will convince the reader that this refinement, which brings us closer to Zariski's work, can be useful for problems of very different type in algebraic geometry.

1. The valuation spectrum of a ring

1.1. Definition of the valuation spectrum. Let A be a ring. (All rings are tacitly assumed to be commutative with unit element.) We recall the definition of a valuation of A. Let Γ be a totally ordered commutative group written additively. We adjoin an element ∞ to Γ and extend the addition and the ordering of Γ to $\Gamma_\infty :=\Gamma\cup\{\infty\}$ by $\alpha+\infty=\infty+\alpha=\infty$ and $\alpha\leq\infty$ for every $\alpha\in\Gamma_\infty$.

DEFINITION. [**B**, VI.3.1]. A *valuation* of A with values in Γ_∞ is a mapping $v\colon A\to\Gamma_\infty$ such that

i) $v(x+y)\geq\min(v(x),v(y))$ for all $x,y\in A$,
ii) $v(x\cdot y)=v(x)+v(y)$ for all $x,y\in A$,
iii) $v(0)=\infty$ and $v(1)=0$.

Let $v\colon A\to\Gamma_\infty$ be a valuation. The subgroup of Γ generated by $\{v(a)|a\in A,\ v(a)\neq\infty\}$ is called the *value group* of v and is denoted by Γ_v. The valuation is called *trivial* if $\Gamma_v=\{0\}$. The convex subgroup $c\Gamma$ of Γ generated by $\{v(a)|a\in A,\ v(a)\leq 0\}$ is called the *characteristic subgroup* of v. The set $\operatorname{supp}(v):=v^{-1}(\infty)$ is a prime ideal of A and is called the *support* of v. The valuation v factorizes uniquely in $A\xrightarrow{g}qf(A/\operatorname{supp}(v))\xrightarrow{\bar{v}}\Gamma_\infty$ where g is the canonical mapping and $\bar{v}$ is a valuation of the quotient field $qf(A/\operatorname{supp}(v))$ of $A/\operatorname{supp}(v)$. The valuation ring of $\bar{v}$ is denoted by $A(v)$.

Two valuations v and w of A are called equivalent if the following equivalent conditions are satisfied

i) For all $a,b\in A$, $v(a)\geq v(b)$ iff $w(a)\geq w(b)$.
ii) There is an isomorphism $f\colon(\Gamma_v)_\infty\longrightarrow(\Gamma_w)_\infty$ with $w=f\circ v$.
iii) $\operatorname{supp}(v)=\operatorname{supp}(w)$ and $A(v)=A(w)$.

Remark. The model theoretic result that the theory of algebraically closed fields with non-trivial valuation-divisibility relation has elimination of quantifiers implies that the equivalence classes of valuations of A correspond bijectively to the elementary equivalence classes of ring homomorphisms from A to non-trivial valued algebraically closed fields.

DEFINITION. i) $S(A)$ denotes the set of all equivalence classes of valuations of A. (In the following we often do not distinguish between a valuation and its equivalence class.)
ii) $K(A)$ denotes the boolean algebra of subsets of $S(A)$ generated by the subsets of the form $\{v\in S(A)|v(a)\geq v(b)\}\quad(a,b\in A)$.

We equip $S(A)$ with the topology $\mathcal{T}$ generated by the sets of the form $\{v \in S(A) | v(a) \geq v(b) \neq \infty\}$ $(a, b \in A)$, and call the topological space $\operatorname{Spv} A :=(S(A), \mathcal{T})$ the *valuation spectrum* of A. This notation is justified by the following proposition.

PROPOSITION (1.1.1). *Spv A is a spectral space. $K(A)$ is the boolean algebra of constructible subsets of Spv A.*

PROOF. Every valuation v of A defines a binary relation $|_v$ on A by

$$a|_v b: \iff v(a) \leq v(b).$$

Two valuations v and w of A are equivalent if and only if $|_v = |_w$. Therefore we have an injective mapping $\varphi\colon S(A) \to \mathcal{P}(A \times A)$, $v \mapsto |_v$. ($\mathcal{P}(A \times A)$ denotes the power set of $A \times A$.) We equip $\{0, 1\}$ with the discrete topology and $\mathcal{P}(A \times A) = \{0, 1\}^{A \times A}$ with the product topology. Then $\mathcal{P}(A \times A)$ is a compact Hausdorff space. The image $\operatorname{im}(\varphi)$ of φ is closed in $\mathcal{P}(A \times A)$ since $\operatorname{im}(\varphi)$ is the set of all binary relations following conditions

1) $a|b$ or $b|a$.
2) If $a|b$ and $b|c$ then $a|c$.
3) If $a|b$ and $a|c$ then $a|b + c$.
4) If $a|b$ then $ac|bc$.
5) If $ac|bc$ and $0 \nmid c$ then $a|b$.
6) $0 \nmid 1$. $\square$

We equip $S(A)$ with the topology such that φ is a topological embedding. Then $S(A)$ is a compact Hausdorff space and $K(A)$ is the set of all subsets of $S(A)$ which are open and closed. Now Proposition (1.1.1) follows from Hochster's result [**H**, Prop. 7]. For convenience we recall this result in the following lemma.

LEMMA (1.1.2 [**H**]). *Let (X, T) be a quasi-compact topological space and $\mathcal{L}$ be the set of all subsets of X which are open and closed. Let $\mathcal{T}$ be a topology of X such that $\mathcal{T}$ is generated by elements of $\mathcal{L}$ and $(X, \mathcal{T})$ is a T_0-space. Then $(X, \mathcal{T})$ is a spectral space and $\mathcal{L}$ is the set of all constructible subsets of $(X, \mathcal{T})$.*

Let $f\colon A \to B$ be a ring homomorphism. Then f induces a mapping $Spv(f)$: $Spv B \to Spv A$. (We often write $v|A$ instead of $Spv(f)(v)$.) $Spv(f)$ is continuous, even more, $Spv(f)$ is spectral.

Remark (1.1.3). i) If K is a field then $\operatorname{Spv} K$ is the abstract Riemann surface of [**ZS**, VI.17] (with the difference that in [**ZS**] the trivial valuation is excluded).

ii) The set M of all trivial valuations of A is a pro-constructible subset of $\operatorname{Spv} A$.

iii) The support mapping $\operatorname{supp} : \operatorname{Spv} A \to \operatorname{Spec} A$, $v \mapsto \operatorname{supp}(v)$ is spectral. The restriction of supp to the set M of all trivial valuations of A is a homeomorphism from M to $\operatorname{Spec} A$.

1.2. Specializations in the valuation spectrum. Let A be a ring and $v\colon A \longrightarrow \Gamma_\infty$ a valuation of A. To every convex subgroup H of Γ we have the mappings

$$v/H\colon A \longrightarrow (\Gamma/H)_\infty, \quad a \mapsto \begin{cases} v(a) \bmod H & \text{if } v(a) \neq \infty \\ \infty & \text{if } v(a) = \infty\,, \end{cases}$$

$$v|H\colon A \longrightarrow H_\infty, \quad a \mapsto \begin{cases} v(a) & \text{if } v(a) \in H \\ \infty & \text{if } v(a) \notin H\,. \end{cases}$$

One can easily check:

LEMMA (1.2.1). i) *v/H is a valuation of A and v/H is a generalization of v in $\mathrm{Spv}\,A$.*

ii) *$v|H$ is a valuation of A iff $c\Gamma \subseteq H$, and in this case $v|H$ is a specialization of v in $\mathrm{Spv}\,A$.*

The generalizations of v in $\mathrm{Spv}\,A$ of the form $v|H$ are called the *secondary generalizations* of v, and the specializations of v in $\mathrm{Spv}\,A$ of the form $v|H$ are called the *primary specializations* of v. A valuation w of A is called a *generalized primary specialization* of v if w is a primary specialization of v or if $c\Gamma_v = \{0\}$, w is trivial and $\mathrm{supp}\,(v|c\Gamma_v) \subseteq \mathrm{supp}\,(w)$ (in the latter case we have by (1.1.3 iii) a chain of specializations $v \succ v|c\Gamma_v \succ w$).

Remark (1.2.2). Let $A \xrightarrow{g} qf(A/\mathrm{supp}\,(v)) \xrightarrow{\bar v} \Gamma_\infty$ be the canonical factorization of v and H a convex subgroup of Γ. Let $\mathfrak{p}$ be the prime ideal $\{x \in A(v) | \bar v(x) > H\}$ of $A(v)$. Then

i) $\mathrm{supp}\,(v/H) = \mathrm{supp}\,(v)$ and $A(v/H) = A(v)_\mathfrak{p}$.

ii) $g(A) \subseteq A(v)_\mathfrak{p}$ iff $c\Gamma \subseteq H$. Let us assume $c\Gamma \subseteq H$. Then g induces a mapping $A \to K\colon = A(v)_\mathfrak{p}/\mathfrak{p}$, and $Q\colon = A(v)/\mathfrak{p}$ is a valuation ring of the field K. This ring homomorphism of A into the valued field (K, Q) induces the valuation $v|H$ on A.

A subset T of A is called *v-convex* if for all $a, b, c \in A$ holds: $v(a) \geq v(c) \geq v(b)$, $a \in T$, $b \in T \Longrightarrow c \in T$. (If $0 \in T$ this means: $v(c) \geq v(b)$, $b \in T \Longrightarrow c \in T$.)

LEMMA (1.2.3). *The supports of the primary specializations of v are the v-convex prime ideals of A.*

PROOF. Let $\mathfrak{p}$ be a v-convex prime ideal of A. Then $v(A\backslash\mathfrak{p}) < v(\mathfrak{p})$, especially $v(A\setminus\mathfrak{p}) \subseteq \Gamma$. Let G be the subgroup of Γ generated by $v(A\setminus\mathfrak{p})$. Then $v(\mathfrak{p}) > G$. (Indeed, assume to the contrary $g \geq v(c)$ for some $g \in G$ and $c \in \mathfrak{p}$. Since $v(A\setminus\mathfrak{p})$ is additively closed, there exist $a, b \in A\setminus\mathfrak{p}$ with $v(a) - v(b) = g$. Then $v(a) \geq v(bc)$, in contradiction to $v(A\setminus\mathfrak{p}) < v(\mathfrak{p})$.) Let H be the convex hull of G in Γ. Then H is a convex subgroup of Γ with $v(\mathfrak{p}) > H$ and $v(A\setminus\mathfrak{p}) \subseteq H$, hence $c\Gamma \subseteq H$ and $\mathfrak{p} = \mathrm{supp}\,(v|H)$. $\square$

Now we can describe all specializations of v in $\operatorname{Spv} A$.

PROPOSITION (1.2.4). *Every specialization of v is a secondary specialization of a generalized primary specialization of v, and also a primary specialization of a secondary specialization of v.*

PROOF. i) Let $w \in \operatorname{Spv} A$ be a specialization of v. We show that w is a secondary specialization of a generalized primary specialization of v. If $c\Gamma_v = \{0\}$ and $v(a) \leq 0$ for each $a \in A \setminus \operatorname{supp}(w)$ then the trivial valuation u of A with $\operatorname{supp}(u) = \operatorname{supp}(w)$ is a generalized primary specialization of v and w is a secondary specialization of u. It remains to consider the case that $c\Gamma_v \neq \{0\}$ or $v(a) > 0$ for some $a \in A \setminus \operatorname{supp}(w)$. We notice for arbitrary $a, b \in A$.

(1) If $v(a) \geq v(b)$, $w(a) \neq \infty$ and $w(b) = \infty$, then $v(a) = v(b) \neq \infty$. (Indeed, we have $w(b) \geq w(a) \neq \infty$ and hence $v(b) \geq v(a) \neq \infty$.) First we show that $\operatorname{supp}(w)$ is v- convex. Let $x, y \in A$ with $y \in \operatorname{supp}(w)$ and $v(x) \geq v(y)$. We have to show $x \in \operatorname{supp}(w)$. Assume to the contrary $x \notin \operatorname{supp}(w)$. Then

(2) $v(x) \geq v(y)$, $w(x) \neq \infty$, $w(y) = \infty$.
We deduce from (1) and (2)

(3) $v(x) = v(y) \neq \infty$.
By our supposition there exists a $a \in A$ with (I) $v(a) < 0$ or (II) $v(a) > 0$ and $w(a) \neq \infty$. In case (I) we have $v(x) \geq v(ay)$, $w(x) \neq \infty$, $w(ay) = \infty$ (by (2)) and hence $v(x) = v(ay)$ (by (1)), in contradiction to (3). In case (II) we have $v(ax) \geq v(y)$, $w(ax) \neq \infty$, $w(y) = \infty$ (by (2)) and hence $v(ax) = v(y)$ (by (1)), in contradiction to (3). Thus we have proved that $\operatorname{supp}(w)$ is v-convex. By (1.2.3) there exists a primary specialization u of v with $\operatorname{supp}(u) = \operatorname{supp}(w)$. We show that w is a secondary specialization of u. Since $\operatorname{supp}(u) = \operatorname{supp}(w)$, it suffices to show: If $a, b \in A$ with $w(a) \geq w(b) \neq \infty$ then $u(a) \geq u(b)$. Let a, b be elements of A with $w(a) \geq w(b) \neq \infty$. Since w is a specialization of v, we have $v(a) \geq v(b)$ and hence $u(a) \geq u(b)$ (since u is a primary specialization of v).

ii) Let w be a specialization of v. We show that w is a primary specialization of a secondary specialization of v. By i), w is the secondary specialization of a generalized primary specialization u of v. If u is a primary specialization of v, then the assertion follows from the subsequent Lemma (1.2.5 ii). Now assume that u is not a primary specialization of v. By the subsequent Lemma (1.2.6) there exists a primary generalization w' of w with $\operatorname{supp}(w') = \operatorname{supp}(v|c\Gamma_v)$. Then w' is a secondary specialization of $v|c\Gamma_v$. By (1.2.5 ii) there exists some $v' \in \operatorname{Spv} A$ such that v' is a secondary specialization of v and a primary generalization of w'. Then w is a primary specialization of v'. $\square$

LEMMA (1.2.5). *Let w be a primary specialization of v.*

i) *Let v' be a secondary specialization of v.*
Then there exists a unique secondary specialization w' of w such that w' is a primary specialization of v'.

ii) *Let w' be a secondary specialization of w.*
Then there exists a secondary specialization v' of v such that w' is a primary specialization of v'.

iii) *Let v' be a secondary generalization of v.*
Then there exists a unique secondary generalization w' of w such that w' is a generalized primary specialization of v'.

iv) *Let w' be a secondary generalization of w.*
Then there exists a secondary generalization v' of v such that w' is a primary specialization of v'.

PROOF. We prove only ii). By (1.2.2) there is a prime ideal $\mathfrak{p}$ of $A(v)$ such that $(qf(A/\mathrm{supp}\,(w)),\ A(w)) \subseteq (A(v)_{\mathfrak{p}}/\mathfrak{p},\ A(v)/\mathfrak{p})$ is an extension of valued fields. Let B be a valuation ring of $A(v)_{\mathfrak{p}}/\mathfrak{p}$ with $B \subseteq A(v)/\mathfrak{p}$ and $B \cap qf(A/\mathrm{supp}\,(w)) = A(w')$. Let v' be the valuation of A with $\mathrm{supp}\,(v') = \mathrm{supp}\,(v)$ and $A(v') = \lambda^{-1}(B)$ where λ is the canonical mapping $A(v) \longrightarrow A(v)/\mathfrak{p}$. Then v' is a secondary specialization of v and a primary generalization of w'. $\square$

LEMMA (1.2.6). *Let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p} \subseteq supp\,(v)$. Then there exists a primary generalization w of v with $\mathfrak{p} = supp\,(w)$.*

PROOF. Let $(B, \mathfrak{m})$ be a valuation ring of $qf(A/\mathfrak{p})$ which dominates the local ring $(A/\mathfrak{p})_{\mathrm{supp}\,(v)/\mathfrak{p}}$. Let C be a valuation ring of $B/\mathfrak{m}$ with $C \cap qf(A/\mathrm{supp}\,(v)) = A(v)$. Let w be the valuation of A with $\mathrm{supp}\,(w) = \mathfrak{p}$ and $A(w) = \lambda^{-1}(C)$ where $\lambda\colon B \to B/\mathfrak{m}$ is the canonical mapping. Then w is a primary generalization of v. $\square$

For later use we remark:

LEMMA (1.2.7). *Assume that $supp\,(v)$ is a maximal ideal of A. Then a valuation w of A is a primary generalization of v if and only if $w(a) \geq 0$ for all $a \in A$ with $v(a) \geq 0$ and $w(a) > 0$ for all $a \in A$ with $v(a) > 0$.*

PROOF. Put $G = \{w \in \mathrm{Spv}\, A | w(a) \geq 0$ for all $a \in A$ with $v(a) \geq 0$ and $w(a) > 0$ for all $a \in A$ with $v(a) > 0\}$. Then $v \in G$ and G is closed under primary specializations and primary generalizations. Let $w \in G$ be given. Then $w(a) > 0$ for all elements a of the maximal ideal $\mathfrak{m}$ of A. This implies $w(a) > c\Gamma_w$ for all $a \in \mathfrak{m}$. (Indeed, if $a \in \mathfrak{m}$ and $x \in A$, then $w(ax) > 0$.) Hence $w|c\Gamma_w \in \{g \in G | \mathfrak{m} = \mathrm{supp}\,(g)\} = \{v\}$. $\square$

1.3. Some other topologies on $S(A)$. Let A be a ring. Beside the topology $\mathcal{T}$ from (1.1), there are other useful topologies on the set $S(A)$, for example the topologies $\mathcal{T}'$ and $\mathcal{T}''$ with

$\mathcal{T}' :=$ topology generated by the sets $\{v \in S(A) | v(a) > v(b)\}$, $\quad a, b \in A$,

$\mathcal{T}'' :=$ topology generated by $\mathcal{T} \cup \mathcal{T}'$.

We put $\operatorname{Spv}' A := (S(A), \mathcal{T}')$ and $\operatorname{Spv}'' A := (S(A), \mathcal{T}'')$.

Proposition (1.1.1) and (1.1.2) imply

PROPOSITION (1.3.1). *$Spv' A$ and $Spv'' A$ are spectral spaces. $K(A)$ is the set of constructible subsets of both $Spv' A$ and $Spv'' A$.*

Let us study the specializations in $\operatorname{Spv}' A$ and $\operatorname{Spv}'' A$. First we consider $\operatorname{Spv}' A$. Obviously, for a valuation $v: A \to \Gamma_\infty$, the valuations v/H (H a convex subgroup of Γ) and $v|H$ (H a convex subgroup of Γ with $c\Gamma \subseteq H$) are specializations of v in $\operatorname{Spv}' A$. We call v/H a secondary specialization of v and $v|H$ a primary specialization of v. Similarly to (1.2.4) one can prove

PROPOSITION (1.3.2). *Every specialization of a point v in $Spv' A$ is a secondary specialization of a primary specialization of v, and also a primary specialization of a secondary specialization of v.*

Propositions (1.2.4) and (1.3.2) imply

PROPOSITION (1.3.3). *Let v and w be points of $Spv'' A$. Then w is a specialization of v in $Spv'' A$ if and only if there exists a convex subgroup H of Γ_v with $c\Gamma_v \subseteq H$ and $w = v|H$.*

Remark (1.3.4). i) If K is a field, then $\operatorname{Spv}' K$ is the inverse spectral space to $\operatorname{Spv} K$ in the sense of [**H**, Prop. 8].

ii) The support mapping supp : $\operatorname{Spv}' A \to \operatorname{Spec} A$ is spectral.

iii) Let M be the set of all trivial valuations of A. Then M is closed in $\operatorname{Spv}' A$ and $\operatorname{supp} | M: M \to \operatorname{spec} A$ is a homeomorphism if we equip $\operatorname{spec} A$ with the constructible topology of the spectral space $\operatorname{Spec} A$.

Let us motivate the topologies $\mathcal{T}, \mathcal{T}', \mathcal{T}''$. Let k be an algebraically closed field complete with respect to a rank 1 valuation $\alpha: k \to \Gamma_\infty$. In rigid analytic geometry one associates to every (affine) variety $X = \operatorname{Spec} E$ over k an analytic space whose underlying "topological space" is the set $X(k)$ of k-rational points of X equipped with a Grothendieck topology G [**BGR**], [**FP**]. The admissible open sets of G are sets of the form

$$(*) \quad \{x \in X(k) | \alpha(f_i(x)) \geq \alpha(g_i(x)) \neq \infty \quad \text{for } i = 1, \ldots, n\} \quad \text{with } f_i, g_i \in E.$$

(Notice that weak inequalities $\geq$ are used in order to define the admissible open sets.) The description $(*)$ of admissible open sets suggests to work with the topology $\mathcal{T}$. As is shown in [**Hu**], there is a strong relation between $(X(k), G)$ and $\operatorname{Spv} E$.

Concerning the topology $\mathcal{T}'$ there is, for example, the following application: In [**Be**], Berkovich constructs to $\operatorname{Spec} E$ an analytic space but instead of $(X(k), G)$ he uses the topological subspace $\{v \in \operatorname{Spv}' E | v$ has rank 1 and $v|k = \alpha\}$ of $\operatorname{Spv}' E$.

We are interested in $\mathcal{T}''$ since there are applications of Spv$''$ in algebraic geometry and analytic geometry (cf. (4.2) and [**Hu$_2$**]). The spectrum Spv$''$ has many properties in common with the real spectrum, for example:

a) If K is a field, then any constructible subset of $\operatorname{Spv}'' K$ is open.
b) The specializations of a point in $\operatorname{Spv}'' A$ form a chain and are uniquely determined by their supports.
c) Let k be an algebraically closed field, α a nontrivial valuation of k and E a finitely generated k-algebra. By $\operatorname{Spv}''(\alpha, E)$ we denote the pro-constructible subspace $\{v \in \operatorname{Spv}'' E \mid \alpha = v|k\}$ of $\operatorname{Spv}'' E$. Then a constructible subset L of $\operatorname{Spv}''(\alpha, E)$ is open if and only if $L \cap (\operatorname{Spec} E)(k)$ is open in the strong topology of $(\operatorname{Spec} E)(k)$ induced by α (cf. (3.2)).
d) If A is universally catenary, then we have a curve selection lemma for $\operatorname{Spv}'' A$ (cf. (2.3)).
e) If the topological space $\operatorname{Spec} A$ is noetherian, then the closure of a constructible subset of $\operatorname{Spv}'' A$ is constructible (cf. (2.2)).

But Spv$''$ has a big disadvantage in comparison with Spv and Spv$'$. Namely, $\operatorname{Spv}'' A$ is disconnected if $\dim A \geq 1$, whereas $\operatorname{Spv} A$ is connected iff $\operatorname{Spv}' A$ is connected iff $\operatorname{Spec} A$ is connected. Even in the geometric situation we have: Let α be a henselian valuation of a field k and A a finitely generated k-algebra. Then $\operatorname{Spv}''(\alpha, A)$ has infinitely many connected components if $\dim A \geq 1$, but $\operatorname{Spv}(\alpha, A)$ is connected iff $\operatorname{Spv}'(\alpha, A)$ is connected iff $\operatorname{Spec} A$ is connected. (Remark: Let Z be the set of closed points of $\operatorname{Spv}'' A$ (resp. $\operatorname{Spv}''(\alpha, A)$). For every $z \in Z$, let $G(z)$ be the set of generalizations of z in $\operatorname{Spv}'' A$ (resp. $\operatorname{Spv} \diamond''(\alpha, A)$). Then $(G(z) | z \in Z)$ is the family of connected components of $\operatorname{Spv}'' A$ (resp. $\operatorname{Spv}''(\alpha, A)$).)

Remark (1.3.5). Schwartz uses in [**S**] a modification of $\mathcal{T}'$, namely the topology T of $S(A)$ generated by the sets $\{v \in S(A) | \infty \neq v(a) > v(b)\}$, $\{v \in S(A) | \infty \neq v(a)\}$ $(a, b \in A)$. We have (M denotes the set of trivial valuations of A):

i) T is weaker than $\mathcal{T}'$, and $T | S(A) \setminus M = \mathcal{T}' | S(A) \setminus M$.
ii) M is closed in $(S(A), T)$ and $\operatorname{supp} : (M, T|M) \to \operatorname{Spec} A$ is a homeomorphism.
iii) $(S(A), T)$ is a spectral space and $K(A)$ is the set of constructible subsets of $(S(A), T)$.
iv) Let v and w be valuations of A. Then w is a specialization of v in $(S(A), T)$ if and only if w is a specialization of v in $\operatorname{Spv}' A$ or w is a trivial valuation with $\operatorname{supp}(v) \subseteq \operatorname{supp}(w)$.

PROOF. ii) is trivial, and iii) follows from (1.1.1) and (1.1.2).

i) The mapping $\operatorname{Spv}'A \to (S(A), T), v \mapsto v$ is spectral. By [**S**, Prop. 26], $\operatorname{Spv}'A$ and $(S(A), T)$ have the same specializations on $S(A) \setminus M$. Hence $T'|S(A) \setminus M = T|S(A) \setminus M$.

iv) If w is a specialization of v in $\operatorname{Spv}'A$, then w is a specialization of v in $(S(A), T)$ by i), and if w is trivial with $\operatorname{supp}(v) \subseteq \operatorname{supp}(w)$, then w is specialization of v in $(S(A), T)$ by definition of T. Conversely, assume that w is a specialization of v in $(S(A), T)$. If v is trivial, then w is trivial and $\operatorname{supp}(v) \subseteq \operatorname{supp}(w)$ by ii). Assume that v is not trivial. If w is not trivial, then w is a specialization of v in $\operatorname{Spv}'A$ by i), and if w is trivial, then $\operatorname{supp}(v) \subseteq \operatorname{supp}(w)$ since $\operatorname{supp} : (S(A), T) \to \operatorname{Spec} A$ is continuous. □

2. Some general results on the valuation spectrum

2.1. Morphisms. By (1.1.1) we know the constructible subsets of the valuation spectrum. Then the following proposition is an immediate consequence of the fact that the theory of algebraically closed fields with non-trivial valuation-divisibility relation has elimination of quantifiers [**P**, 4.17].

PROPOSITION (2.1.1). *Let $f\colon A \to B$ be a ring homomorphism of finite presentation and let L be a constructible subset of $\operatorname{Spv} B$. Then $\operatorname{Spv}(f)(L)$ is a constructible subset of $\operatorname{Spv} A$.*

Let $f\colon A \to B$ be a ring homomorphism. We want to study the relation between the specializations (resp. generalizations) of a point v in $\operatorname{Spv} B$ and the specializations (resp. generalizations) of $\operatorname{Spv}(f)(v)$ in $\operatorname{Spv} A$. By (1.2.4) it suffices to consider secondary specializations (resp. secondary generalizations) and primary specializations (resp. primary generalizations). Concerning the secondary specializations (resp. secondary generalizations), we have the following trivial remark.

Remark (2.1.2). Let v be a point of $\operatorname{Spv} B$ and $w:=\operatorname{Spv}(f)(v)$. Let $S(v)$ (resp. $G(v)$) be the set of all secondary specializations (resp. secondary generalizations) of v in $\operatorname{Spv} B$, analogously $S(w)$ (resp. $G(w)$). Then $\operatorname{Spv}(f)\colon \operatorname{Spv} B \to \operatorname{Spv} A$ induces surjective mappings $S(v) \to S(w)$ and $G(v) \to G(w)$.

Let v be a point of $\operatorname{Spv} B$. We call $\operatorname{Spv}(f)$ *primarily generalizing* at v if for every primary generalization y of $\operatorname{Spv}(f)(v)$ in $\operatorname{Spv} A$ there is a primary generalization x of v in $\operatorname{Spv} B$ with $y = \operatorname{Spv}(f)(x)$. We call $\operatorname{Spv}(f)$ *universally primarily generalizing* at v if, for every base extension $g\colon C \to C \otimes_A B$ of f and every point w of $\operatorname{Spv} C \otimes_A B$ lying over v, the mapping $\operatorname{Spv}(g)$ is primarily generalizing at w. Analogously we define (*universally*) *primarily specializing.* With this definition we have

PROPOSITION (2.1.3). *Let v be a valuation of B. Then the following conditions are equivalent.*

i) *$\operatorname{Spv}(f)$ is universally primarily generalizing at v.*

ii) *Spec(f) is universally generalizing at supp(v).*

PROOF. ii) follows from i) by (1.2.6). Let us assume ii). Let t be a primary generalization of $s\colon = \operatorname{Spv}(f)(v)$ in $\operatorname{Spv} A$. We have to show that there exists a primary generalization w of v in $\operatorname{Spv} B$ with $t = \operatorname{Spv}(f)(w)$. By (1.2.2) there exist valuation rings A', C and a ring homomorphism $h\colon A \to A'$ such that $C \subseteq A'$, $qf(C) = qf(A')$ and $\operatorname{Spv}(h)(t') = t$ and $\operatorname{Spv}(h)(s') = s$ where t' and s' are the points of $\operatorname{Spv} A'$ given by the valuation rings C and $C/\mathfrak{m}_{A'}$. Let $f'\colon A' \to A' \otimes_A B =: B'$ be the ring homomorphism induced by f. Let v' be a valuation of B' with $v'|B = v$ and $v'|A' = s'$. It suffices to show that there is a primary generalization w' of v' in $\operatorname{Spv} B'$ with $w'|A' = t'$. Let $\mathfrak{p}$ be a prime ideal of B' with $f'^{-1}(\mathfrak{p}) = \{0\}$ and $\mathfrak{p} \subseteq \operatorname{supp}(v')$. By (1.2.6) there exists a primary generalization w' of v' with $\mathfrak{p} = \operatorname{supp}(w')$. Then $w'|A' = t'$ since s' has only one primary generalization in $\operatorname{Spv} A'$ with support $\{0\}$. $\square$

COROLLARY (2.1.4). *If f is flat and finitely presented then the mappings $Spv(f)\colon Spv B \to Spv A$, $Spv'(f)\colon Spv' B \to Spv' A$ and $Spv''(f)\colon Spv'' B \to Spv'' A$ are open.*

PROPOSITION (2.1.5). *Let v be a valuation of B. If Spec(f) is universally specializing at supp(v), then Spv(f) is universally primarily specializing at v.*

PROOF. Let t be a primary specialization of $s\colon = \operatorname{Spv}(f)(v)$ in $\operatorname{Spv} A$. We have to show that there exists a primary specialization w of v in $\operatorname{Spv} B$ with $t = \operatorname{Spv}(f)(w)$. By (1.2.2) there exists a valuation ring D of $K\colon = qf(A/\operatorname{supp}(s))$ such that D contains $A(s)$ and the image of the mapping $A \to K$ and t is induced by the mapping of A into the valued field $(D/\mathfrak{m}_D,\ A(s)/\mathfrak{m}_D)$. Let E be a valuation ring of $F\colon = qf(B/\operatorname{supp}(v))$ with $B(v) \subseteq E$ and $E \cap K = D$. Since $\operatorname{Spec}(f)$ is universally specializing at $\operatorname{supp}(v)$, E contains the image of the mapping $B \to F$. Hence we have a mapping of B to the valued field $(E/\mathfrak{m}_E,\ B(v)/\mathfrak{m}_E)$, which induces a valuation w of B. Then w is a primary generalization of v with $t = \operatorname{Spv}(f)(w)$. $\square$

COROLLARY (2.1.6). i) *If f is integral, then Spv(f) is universally primarily specializing at every point.*

ii) *If f is integral and injective, A integral and normal, and B integral, then Spv(f) is universally primarily generalizing at every point.*

PROOF. i) follows from (2.1.5). The assumptions of ii) imply that $\operatorname{Spec}(f)$ is universally generalizing at every point [**EGA**, IV, 14.4.2]. Hence ii) follows from (2.1.3). $\square$

COROLLARY (2.1.7). *We consider the mappings $Spv(f)\colon Spv B \to Spv A$, $Spv'(f)\colon Spv' B \to Spv' A$ and $Spv''(f)\colon Spv'' B \to Spv'' A$.*

i) *If f is integral, then there are no specializations in the fibres of $Spv(f)$, $Spv'(f)$, $Spv''(f)$.*

ii) *If f is integral, then the mappings $Spv(f)$, $Spv'(f)$, $Spv''(f)$ are closed.*

iii) *If f is integral, injective and finitely presented, A integral and normal, and B integral, then the mappings $Spv(f), Spv'(f), Spv''(f)$ are open.*

PROOF. i) is obvious, since there are no specializations in the fibres of f; ii) and iii) follow from (2.1.6). □

2.2. *Closure of constructible subsets.*

PROPOSITION (2.2.1). *Let A be a ring such that the topological space $\operatorname{Spec} A$ is noetherian and let L be a constructible subset of $Spv''A$. Then the closure $\bar{L}$ of L in $Spv''A$ is constructible.*

The analogous statements for $\operatorname{Spv} A$ and $\operatorname{Spv}'A$ are not true. Examples:

i) The closure of $\{v \in \operatorname{Spv}\mathbb{C}[T] | v(2) > 0,\ v(T) \geq 0\}$ in $\operatorname{Spv}\mathbb{C}[T]$ is not constructible (by the results of (3.2)).

ii) The closure of $\{v \in \operatorname{Spv}'\mathbb{Z} | v(2) > 0\}$ in $\operatorname{Spv}'\mathbb{Z}$ is not constructible.

In order to prove (2.2.1) we need the following lemma.

LEMMA (2.2.2). *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let L be the set of all points of $Spv''A$ which have a specialization with support $\mathfrak{m}$ (i.e. $supp(v|c\Gamma_v) = \mathfrak{m}$). Let π be the mapping $L \to Spv''\kappa$, $v \mapsto v|c\Gamma_v$ (here we identify the subspace $\{v \in Spv''A | supp(v) = \mathfrak{m}\}$ of $Spv''A$ with $Spv''\kappa$). Then*

i) $L = \{v \in Spv''A | v(a) > 0$ *for all* $a \in \mathfrak{m}\}$.

ii) *π is spectral.*

iii) *If K is a constructible subset of L, then $\pi(K)$ is a constructible subset of $Spv''\kappa$.*

PROOF. i) If $\operatorname{supp}(v|c\Gamma_v) = \mathfrak{m}$, then $v(a) > c\Gamma_v$ for all $a \in \mathfrak{m}$, especially $v(a) > 0$. Conversely, let v be a valuation of A with $v(a) > 0$ for all $a \in \mathfrak{m}$. Then, for all $x \in A \setminus \operatorname{supp}(v)$ and $a \in \mathfrak{m}$, $v(xa) > 0$. Hence $v(a) > c\Gamma_v$ for all $a \in \mathfrak{m}$, which means $\operatorname{supp}(v|c\Gamma_v) = \mathfrak{m}$.

ii) Let f be an element of κ and a an element of A with $f = a \bmod \mathfrak{m}$. Then $\pi^{-1}(\{v \in \operatorname{Spv}''\kappa | v(f) \geq 0\}) = \{v \in L | v(a) \geq 0\}$. Hence π is spectral.

iii) Let K be a constructible subset of L. By ii), $\pi(K)$ is pro-constructible in $\operatorname{Spv}''\kappa$. Let v be an element of K. We have to show that there is a constructible subset W of $\operatorname{Spv}''\kappa$ with $\pi(v) \in W \subseteq \pi(K)$. Choose $a_i, b_i, c_i, d_i \in A$ $(i = 1, \ldots, n)$ such that $v \in \{x \in L | x(a_i) \geq x(b_i),\ x(c_i) > x(d_i)$ for $i = 1, \ldots, n\} \subseteq K$. Then $d_i \notin \operatorname{supp}(v)$ for $i = 1, \ldots, n$. We may assume $b_i \notin \operatorname{supp}(v)$ for $i = 1, \ldots, m$ and $b_i \in \operatorname{supp}(v)$ for $i = m+1, \ldots, n$. Then $a_i \in \operatorname{supp}(v)$ for $i = m+1, \ldots, n$. By (1.2.2) there exists a valuation ring B of $qf(A/\operatorname{supp}(v))$ which dominates $A/\operatorname{supp}(v)$ and contains $A(v)$. Let k be the residue field of B and $f\colon A \to qf(A/\operatorname{supp}(v))$, $g\colon B \to k$, $h\colon \kappa \to k$ the canonical mappings. We have $\lambda_i\colon = \frac{f(a_i)}{f(b_i)} \in A(v) \subseteq B$ for $i = 1, \ldots, m$

and $\mu_i := \frac{f(c_i)}{f(d_i)} \in A(v) \subseteq B$ for $i = 1,\dots,n$. Put $S := \{x \in \operatorname{Spv} k | x(g(\lambda_i)) \geq 0$ for $i = 1,\dots,m$ and $x(g(\mu_i)) > 0$ for $i = 1,\dots,n\}$. By the subsequent lemma, $\operatorname{Spv}''(h)(S)$ is constructible in $\operatorname{Spv}''\kappa$. We have $\pi(v) \in \operatorname{Spv}''(h)(S) \subseteq \pi(K)$. □

LEMMA (2.2.3). *Let $E \hookrightarrow F$ be an extension of fields and L a constructible subset of $\operatorname{Spv} F$. Then the image of L under the mapping $\operatorname{Spv} F \to \operatorname{Spv} E$ is constructible in $\operatorname{Spv} E$.*

PROOF. We choose a field G and a constructible subset M of $\operatorname{Spv} G$ such that $E \hookrightarrow G \hookrightarrow F$, G is finitely generated over E and L is the preimage of M under the mapping $f\colon \operatorname{Spv} F \to \operatorname{Spv} G$. Since f is surjective, we have to show that $g(M)$ is constructible in $\operatorname{Spv} E$ where g is the mapping $\operatorname{Spv} G \to \operatorname{Spv} E$. Let A be a finitely generated E-subalgebra of G with $G = qf(A)$ and $h\colon \operatorname{Spv} A \to \operatorname{Spv} E$ be the canonical mapping. Let N be a constructible subset of $\operatorname{Spv} A$ such that N is closed under primary generalizations in $\operatorname{Spv} A$ and $M = N \cap \operatorname{Spv} G$. Then $g(M) = h(N)$. (Indeed, let $v \in N$ be given. By (1.2.6), there exists a primary generalization w of v in $\operatorname{Spv} A$ with $\{0\} = \operatorname{supp}(w)$. Then $w \in M$ and $g(w) = h(v)$.) Now (2.1.1) shows that $g(M)$ is constructible in $\operatorname{Spv} E$. □

Now we prove (2.2.1). We use ideas from **[Ru]**. $\bar{L}$ is pro-constructible in $\operatorname{Spv}''A$. Let v be an element of $\bar{L}$. We have to show that there exists a constructible subset M of $\operatorname{Spv}''A$ with $v \in M \subseteq \bar{L}$. Put $F = qf(A/\operatorname{supp}(v))$. Applying (2.2.2 iii) to the local ring $A_{\operatorname{supp}(v)}$, we obtain a constructible subset N of $\operatorname{Spv}''F$ with $v \in N \subseteq \bar{L}$. Let M be an open constructible subset of $\operatorname{Spv}''A/\operatorname{supp}(v)$ with $M \cap \operatorname{Spv}''F = N$. By (1.2.6), M is contained in the closure of N. Hence $v \in M \subseteq \bar{L}$. Since $\operatorname{Spec} A$ is noetherian, M is a constructible subset of $\operatorname{Spv}''A$.

2.3. Curve selection lemma. We have the following abstract version of the curve selection lemma. Concrete versions will be deduced from it in (3.2.6) and $[\mathbf{Hu_2}]$.

PROPOSITION (2.3.1). *Let A be a noetherian ring and v a point of $\operatorname{Spv}''A$. We assume that A is universally catenary or that A is local and henselian with maximal ideal $\operatorname{supp}(v)$. Put $T = \{w \in \operatorname{Spv}''A | w$ specializes to $v\}$ and $T_0 = \{w \in T | ht(\operatorname{supp}(v)/\operatorname{supp}(w)) \leq 1\}$. Then T is the closure of T_0 in the constructible topology of $\operatorname{Spv}''A$.*

PROOF. We may assume that A is local with maximal ideal $\mathfrak{m} = \operatorname{supp}(v)$. Let L be a non-empty constructible subset of T. We have to show $L \cap T_0 \neq \emptyset$. We may assume $L = \{x \in T | x(a_i) \geq x(b_i)$ and $x(c_i) > x(d_i)$ for $i = 1,\dots,n\}$ with $a_i, b_i, c_i, d_i \in A$. Let w be an element of L. Assume $ht(\mathfrak{m}/\operatorname{supp}(w)) \geq 2$. Then we will show that there exists a $u \in \operatorname{Spv}''A$ with

a) $\operatorname{supp}(w) \subsetneqq \operatorname{supp}(u)$

b) $u \in L$.

Then we are done, since $\dim A$ is finite. $\square$

Without loss of generality we can assume that A is an integral domain and $\operatorname{supp}(w) = \{0\}$. Furthermore we may assume $b_i \neq 0$ for $i = 1, \dots, m$ and $b_i = 0$ for $i = m+1, \dots, n$ (which implies $a_i = 0$ for $i = m+1, \dots, n$). We have $d_i \neq 0$ for $i = 1, \dots, n$.

Let B be the subring $A\left[\frac{a_i}{b_i},\ i = 1, \dots, m; \frac{c_i}{d_i}, i = 1, \dots, n\right]$ of $qf(A)$, and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the morphism of schemes induced by the inclusion $A \subseteq B$. Then we have

(1) There exists a valuation $\bar{v}$ of B with the following properties
- i) $\bar{v}\left(\frac{a_i}{b_i}\right) \geq 0$ for $i = 1, \dots, m$
- ii) $\bar{v}\left(\frac{c_i}{d_i}\right) > 0$ for $i = 1, \dots, n$
- iii) $v = \bar{v}|A$
- iv) $\operatorname{supp}(\bar{v})$ is a closed point of the fibre $f^{-1}(\mathfrak{m})$.

PROOF. The valuation ring $A(w)$ of $qf(A) = qf(B)$ defines a valuation $\bar{w}$ of B with $w = \bar{w}|A$ and $\bar{w}(x) \geq 0$ for every $x \in \{\frac{a_i}{b_i} | i = 1, \dots, m\} \cup \{\frac{c_i}{d_i} | i = 1, \dots, n\}$. Hence the characteristic subgroup $c\Gamma_{\bar{w}}$ is the convex hull of $c\Gamma_w$ in $\Gamma_{\bar{w}} \leq$.

Since v is a specialization of w in $\operatorname{Spv}''A$, there exists a smallest convex subgroup H of Γ_w with $c\Gamma_w \subseteq H$ and $v = w|H$. Let $\bar{H}$ be the convex hull of H in $\Gamma_{\bar{w}}$. Then $c\Gamma_{\bar{w}} \subseteq \bar{H}$. Hence we have a specialization $s\colon = \bar{w}|\bar{H}$ of $\bar{w}$ in $\operatorname{Spv}''B$ with $v = s|A$. $\square$

Now we distinguish two cases.

First case: v is trivial. Let $\bar{v}$ be a trivial valuation of B such that $\operatorname{supp}(s) \subseteq \operatorname{supp}(\bar{v})$ and $\operatorname{supp}(\bar{v})$ is closed in $f^{-1}(\mathfrak{m})$. Then, clearly i), iii), iv) are satisfied. Since $s\left(\frac{c_i}{d_i}\right) > 0$ for $i = 1, \dots, n$ and s is trivial, we have $\frac{c_i}{d_i} \in \operatorname{supp}(s) \subseteq \operatorname{supp}(\bar{v})$ for $i = 1, \dots, n$. Hence ii) is fulfilled.

Second case: v is non-trivial. Then the existence of a valuation $\bar{v}$ of B satisfying i)–iv) follows from the fact that s fulfills i), ii), iii) and the result that the theory of algebraically closed fields with non-trivial valuation-divisibility relation has elimination of quantifiers ([**P**, 4.17], cf. (3.2.3)).

Put $h = \left(\prod_{i=1}^{m} b_i\right) \cdot \left(\prod_{i=1}^{n} d_i\right) \in A$. Then we have

(2) There exists a $t \in \operatorname{Spv}''B$ with
- i) t is a generalization of $\bar{v}$,
- ii) $h \notin \operatorname{supp}(t)$,
- iii) $\{0\} \neq \operatorname{supp}(t)$.

PROOF. First we observe that $ht(\operatorname{supp}(\bar{v})) \geq 2$. Indeed, if A is universally catenary, then the dimension formula $ht(\mathfrak{m}) + \operatorname{trdeg}(B|A) = ht(\operatorname{supp}(\bar{v})) + \operatorname{trdeg}(B/\operatorname{supp}(\bar{v})|A/\mathfrak{m})$ [**EGA**, IV.5.6.1] gives $ht(\operatorname{supp}(\bar{v})) = ht(\mathfrak{m}) \geq 2$, since $qf(A) = qf(B)$ and $B/\operatorname{supp}(\bar{v})$ is algebraic over $A/\mathfrak{m}$ (the latter by (1 iv)). Now assume that A is henselian. If $\operatorname{supp}(\bar{v})$ has a proper generalization $\mathfrak{q}$ in $f^{-1}(\mathfrak{m})$, then $\{0\} \subsetneqq \mathfrak{q} \subsetneqq \operatorname{supp}(v)$ and hence $ht(\operatorname{supp}(\bar{v})) \geq 2$. If $\operatorname{supp}(\bar{v})$ has no proper generalization in $f^{-1}(\mathfrak{m})$, then $\operatorname{supp}(\bar{v})$ is isolated in $f^{-1}(\mathfrak{m})$ (by (1 iv)) and hence $B_{\operatorname{supp}(\bar{v}}$ is finite over A, which implies $ht(\operatorname{supp}(\bar{v})) = ht(\mathfrak{m}) \geq 2$.

Now, as $ht(\operatorname{supp}(\bar{v})) \geq 2$, the equivalence of a) and f) in [**EGA**, IV.10.5.1] shows that the localization $(B_{\operatorname{supp}(\bar{v})})_h$ is not a field. This means that there exists a prime ideal $\mathfrak{p}$ of B with $\{0\} \neq \mathfrak{p}$, $h \notin \mathfrak{p}$ and $\mathfrak{p} \subseteq \operatorname{supp}(\bar{v})$. By (1.2.6), there exists a generalization of $\bar{v}$ in $\operatorname{Spv}''B$ with support $\mathfrak{p}$. This shows (2). □

We claim that the conditions a) and b) are satisfied with $u: = t|A$. Since $A_h = B_h$, (2 ii) and (2 iii) imply $\operatorname{supp}(w) = \{0\} \subsetneqq \operatorname{supp}(u)$. Since d_i is a divisor of h, we have $d_i \notin \operatorname{supp}(t)$ by (2 ii). Then (1 ii) and (2 i) give $u(c_i) > u(d_i)$ for $i = 1, \ldots, n$. (1 i) and (2 i) imply $u(a_i) \geq u(b_i)$ for $i = 1, \ldots, m$. Note that $u(a_i) \geq u(b_i)$ for $i = m+1, \ldots, n$, since $a_i = b_i = 0$ for $i = m+1, \ldots, n$. According to (1 iii) and (2 i), u is a generalization of v in $\operatorname{Spv}''A$. Hence $u \in L$.

2.4. Connected components. In this paragraph we study the connected components of pro-constructible subsets of valuation spectra $\operatorname{Spv} A$.

We begin with a general remark on connected components of spectral spaces.

LEMMA (2.4.1). *Let $(X_i|i \in I)$ be a cofiltered system of spectral spaces such that all transition maps $X_i \to X_j$ are spectral. Let X be the projective limit of $(X_i|i \in I)$ in the category of topological spaces. Then*

i) *X is spectral. Each clopen (= closed and open) subset of X is the preimage of a clopen subset of some X_i. In particular, if each X_i is connected, then X is connected.*

ii) *Let Z be a connected component of X. For each $i \in I$, let Z_i be the connected component of X_i containing the image of Z. Then $Z = \varprojlim_{i \in I} Z_i \subseteq X$.*

iii) *For each $i \in I$, let Z_i be a connected component of X_i such that $\varphi(Z_i) \subseteq Z_j$ for each transition map $\varphi: X_i \to X_j$. Then $\varprojlim_{i \in I} Z_i \subseteq X$ is a connected component of X.*

iv) *Let Y be a spectral space and Z a connected component of Y. Then Z is the intersection of the clopen subsets of Y containing Z.*

PROOF. i) follows from (1.1.2) and [$\mathbf{B}_1$, I.9.6].

ii) We have $Z \subseteq \varprojlim Z_i \subseteq X$. By i), $\varprojlim Z_i$ is connected. Hence $Z = \varprojlim Z_i$.

iii) By i), $T: = \varprojlim Z_i$ is connected. Let Z be the connected component of X containing T. Then the image of Z in X_i is contained in Z_i. Hence $T = Z$.

iv) Let A be a ring with $Y \cong \operatorname{Spec} A$. We have $\operatorname{Spec} A \cong \varprojlim_j \operatorname{Spec} A_j$, where each A_j is a finitely generated $\mathbb{Z}$-algebra. Now the assertion follows from ii). □

PROPOSITION (2.4.2). *Let K be a field and D, E subsets of K. We consider the pro-constructible subset $L = \{v \in \mathrm{Spv}\, K | v(d) \geq 0$ for all $d \in D$ and $v(e) > 0$ for all $e \in E\}$ of $\mathrm{Spv}\, K$. Let A be the integral closure in K of the subring generated by $D \cup E$. Then there is a canonical bijection from the set of clopen subsets of $\mathrm{Spec}\, A/E \cdot A$ to the set of clopen subsets of L. In particular, the connected components of $\mathrm{Spec}\, A/E \cdot A$ correspond to the connected components of L.*

In order to prove (2.4.2), we first recall Zariski's representation of the valuation spectrum of a field as a projective limit of schemes [**ZS**, VI.17]:

LEMMA (2.4.3). *Let A be a ring, K a field and $s\colon A \to K$ a ring homomorphism. Let I be the following category. The objects are the triples (X, f, g) with X an integral scheme, $f\colon X \to \mathrm{Spec}\, A$ a projective morphism and $g\colon \mathrm{Spec}\, K \to X$ a dominant morphism such that $\mathrm{Spec}(s) = f \circ g$. The morphisms $(X, f, g) \to (X', f', g')$ are the morphisms of schemes $h\colon X \to X'$ with $g' = h \circ g$ (and hence $f = f' \circ h$). Let c be the functor from I to the category of topological spaces which assigns to $(X, f, g) \in I$ the topological space $|X|$ underlying X. Put $Y = \{v \in \mathrm{Spv}\, K | s(A) \subseteq K(v)\}$. For every $i = (X, f, g) \in I$, we have a continuous mapping $\varphi_i\colon Y \to |X|$. Namely, if v is an element of Y and if $t\colon \mathrm{Spec}\, K(v) \to \mathrm{Spec}\, A$ and $\bar{g}\colon \mathrm{Spec}\, K(v) \to X$ are the extensions of $\mathrm{Spec}(s)$ and g with $t = f \circ \bar{g}$, then $\varphi_i(v)$ is defined to be the image of the closed point of $\mathrm{Spec}\, K(v)$ under $\bar{g}$.*

With these arrangements we have: $(Y, (\varphi_i | i \in I))$ is the projective limit of c.

Now we come to the proof of (2.4.2). Let V denote the subspace $\{\mathfrak{p} \in \mathrm{Spec}\, A | E \subseteq \mathfrak{p}\}$ of $\mathrm{Spec}\, A$. We have $v(a) \geq 0$ for every $v \in L$, $a \in A$. Let φ be the mapping $L \to V$, $v \mapsto \mathrm{supp}\,(w | c\Gamma_w)$ with $w \colon = v|A \in \mathrm{Spv}\, A$. The following two properties, i) and ii) of φ, show that $U \mapsto \varphi^{-1}(U)$ gives a bijection from the set of clopen subsets of V to the set of clopen subsets of L.

i) φ is spectral, specializing and surjective.

ii) Each fibre of φ is connected.

To i): For every $f \in A$, $\varphi^{-1}(D(f)) = \{v \in L | v(f) \leq 0\}$. Hence φ is spectral. Let v be an element of L and $\mathfrak{q}$ a specialization of $\varphi(v)$. Then the trivial valuation of A with support $\mathfrak{q}$ is a generalized primary specialization of $v|A$. Hence by (1.2.4) and (2.1.2), there exist a specialization w of v in L with $\varphi(v) = \mathfrak{q}$. The surjectivity of φ follows from (1.2.6).

To ii): We have to show that, for every local subring B of K which is integrally closed in K, the subset $\{v \in \mathrm{Spv}\, K | K(v)$ dominates $B\} \subseteq \mathrm{Spv}\, K$ is connected. By (2.4.1 i) we may assume that B is noetherian. Then the assertion follows from (2.4.1 i), (2.4.3) and [**EGA**, III.4.3.5].

PROPOSITION (2.4.4). *Let A be a ring, A_0 a subring of A and I an ideal of A_0 such that A_0 is henselian along I [**EGA**, IV.18.5.5]. We consider the subspace $L = \{v \in \mathrm{Spv}\, A | v(a) \geq 0$ for all $a \in A_0$ and $v(a) > 0$ for all $a \in I\} \subseteq \mathrm{Spv}\, A$. Let*

$\lambda\colon L \to \operatorname{Spec} A$ be the support mapping. Then $U \mapsto \lambda^{-1}(U)$ gives a bijection from the set of clopen subsets of $\operatorname{Spec} A$ to the set of clopen subsets of L. In particular, the connected components of $\operatorname{Spec} A$ correspond to the connected components of L.

PROOF. The assertion follows from the following two properties of λ.

i) λ is spectral, generalizing and surjective.

ii) Each fibre of λ is connected.

To i): Since L is closed under primary generalizations, λ is generalizing by (1.2.6). In order to show the surjectivity of λ, let $\mathfrak{p} \in \operatorname{Spec} A$ be given. Since A_0 is henselian along I, and thus I lies in the Jacobson radical of A_0, $\mathfrak{p} \cap A_0$ specializes to a prime ideal $\mathfrak{q} \in \operatorname{Spec} A_0$ with $I \subseteq \mathfrak{q}$. By (1.2.6) there exists a $w \in \operatorname{Spec} A_0$ such that $\mathfrak{p} \cap A_0 = \operatorname{supp}(w)$ and the trivial valuation of A_0 with support $\mathfrak{q}$ is a primary specialization of w. Let v be a valuation of A with $\mathfrak{p} = \operatorname{supp}(v)$ and $w = v|A_0$. Then $v \in L$.

To ii): Let $\mathfrak{p}$ be a prime ideal of A. Let B be the integral closure of A_0 in $qf(A/\mathfrak{p})$. Then $\operatorname{Spec} B/I \cdot B$ is connected, since A_0 is henselian along I and $\operatorname{Spec} B$ is connected. We conclude from (2.4.2) that $\lambda^{-1}(\mathfrak{p})$ is connected. $\square$

Remark (2.4.5). Let A be a ring, A_0 a subring of A and I an ideal of A_0. But now we do not assume that A_0 is henselian along I. So we cannot apply (2.4.4) directly. But it is obvious what we have to do. Let $(\bar{A}_0, \bar{I})$ be a henselization of (A_0, I) [**R**, XI.2]. We consider the tensor product

$$\begin{array}{ccc} \bar{A}\colon = \bar{A}_0 \otimes_{A_0} A & \xleftarrow{f} & A \\ \bar{i} \uparrow & & \uparrow i \\ \bar{A}_0 & \xleftarrow[f_0]{} & A_0 . \end{array}$$

Then $\bar{A}_0$ is henselian along $\bar{I}$, and $\bar{i}$ is injective. Put $L = \{v \in \operatorname{Spv} A | v(a) \geq 0$ for all $a \in A_0$ and $v(a) > 0$ for all $a \in I\}$ and $\bar{L} = \{v \in \operatorname{Spv} \bar{A} | v(a) \geq 0$ for all $a \in \bar{A}_0$ and $v(a) > 0$ for all $a \in \bar{I}\}$. Then we have: The mapping $\operatorname{Spv}(f)$ induces a homeomorphism $g\colon \bar{L} \to L$. (Application: If A_0 is noetherian and A of finite type over A_0, then L has finitely many connected components.)

PROOF. We show that g is bijective and generalizing. Since $\bar{I} = I \cdot \bar{A}_0$, $\bar{L}$ is closed under generalizations in $\operatorname{Spv}(f)^{-1}(L)$. Then (2.1.3) and (2.1.2) imply that g $g(v_1) = g(v_2)$. Let K_i be an algebraic closure of $qf(\bar{A}/\operatorname{supp}(v_i))$ and A_i a valuation ring of K_i extending $\bar{A}(v_i)$ $(i = 1, 2)$. Let $h_i\colon \bar{A} \to K_i$ be the canonical ring homomorphism $(i = 1, 2)$. Since $g(v_1) = g(v_2)$, there exists an isomorphism $h\colon K_1 \to K_2$ with $h(A_1) = A_2$ and $h \circ h_1 \circ f = h_2 \circ f$. We consider the ring homomorphisms $g_i\colon = h_i \circ \bar{i}\colon \bar{A}_0 \to A_i$ $(i = 1, 2)$. Then $(h \circ g_1) \circ f_0 = g_2 \circ f_0$. Since A_2 is henselian and $f_0\colon (A_0, I) \to (\bar{A}_0, \bar{I})$ a henselization of (A_0, I), we conclude $h \circ g_1 = g_2$. Now $h \circ h_1 \circ f = h_2 \circ f$ and $h \circ h_1 \circ \bar{i} = h_2 \circ \bar{i}$ imply

$h \circ h_1 = h_2$, and therefore $v_1 = v_2$. A similar argument (representing a $v \in L$ by a henselian valuation ring ...) shows that g is surjective. $\square$

If X is an irreducible normal complex analytic space, L a connected open subset of X, and M a closed complex analytic subspace of X with $\dim M < \dim X$, then $L \setminus M$ is connected. In the next proposition we prove an analogous result for the valuation spectrum.

PROPOSITION (2.4.6). *Let A be a normal integral domain, L a connected pro-constructible subset of $\operatorname{Spv} A$ which is closed under primary generalizations, and M a subset of $\operatorname{Spv} A$ such that there is a $a \in A \setminus \{0\}$ with $a \in \operatorname{supp}(v)$ for all $v \in M$. Then $L \setminus M$ is connected, too.*

PROOF. Put $T = \{v \in L | a \in \operatorname{supp}(v)\}$. By (1.2.6), $L \setminus T$ is dense in L. In particular, $L \setminus T$ is dense in $L \setminus M$. Hence it suffices to show that $L \setminus T$ is connected. Assume to the contrary that $L \setminus T$ is not connected. Let $L \setminus T = U_1 \cup U_2$ be a partition of $L \setminus T$ into non-empty closed subsets. Since L is connected and $L \setminus T$ dense in L, there exists a $t \in T$ having generalizations in U_1 and U_2. Then by (1.2.4), t has primary generalizations in U_1 and U_2. Let $f\colon A \to B$ be the strict henselization of A at $\operatorname{supp}(t)$ [**R**, VIII.2]. We consider the mapping $g = \operatorname{Spv}(f)\colon \operatorname{Spv} B \to \operatorname{Spv} A$. Let s be a valuation of B with $t = g(s)$. Put $C = \{b \in B | s(b) \geq 0\}$ and $I = \{b \in B | s(b) > 0\}$. Then C is a subring of B and I is an ideal of C. More precisely, C is a local ring with maximal ideal I. Since B and $B(s)$ are henselian and $B(s)$ is integrally closed in the residue field of B, C is henselian. Put $G = \{v \in \operatorname{Spv} B | v(c) \geq 0$ for all $c \in C$ and $v(i) > 0$ for all $i \in I\}$. B is integral, since A is normal. Hence $\{\mathfrak{p} \in \operatorname{Spec} B | f(a) \notin \mathfrak{p}\}$ is connected. Now we know by (2.4.4) that $H\colon = \{v \in G | f(a) \notin \operatorname{supp}(v)\}$ is connected. According to (1.2.7) and (2.1.3), $g(G)$ is the set of primary specializations of t in $\operatorname{Spv} A$. Hence $H \subseteq g^{-1}(U_1) \cup g^{-1}(U_2)$, $g^{-1}(U_1) \cap g^{-1}(U_2) = \emptyset$, $g^{-1}(U_1) \cap H \neq \emptyset$, $g^{-1}(U_2) \cap H \neq \emptyset$, in contradiction to the connectedness of H. $\square$

In the rest of this paragraph and in §3.4 we will investigate the following question: Let $f\colon A \to B$ be a ring homomorphism of finite presentation and let L be a pro-constructible subset of $\operatorname{Spv} A$ such that every constructible subset of L has finitely many connected components. Under what conditions has every constructible subset of $\operatorname{Spv}(f)^{-1}(L) \subseteq \operatorname{Spv} B$ finitely many connected components, too? For example, we will show that every constructible subset of $\operatorname{Spv}(f)^{-1}(L)$ has finitely many connected components if A is a Nagata ring [**M**, Ch. 12] and L is closed under primary generalizations or if every valuation $v \in L$ is non-trivial. But in general, not every constructible subset of $\operatorname{Spv}(f)^{-1}(L)$ has finitely many connected components, as the following example shows: Let A be a noetherian ring, $B = A[T]$ the polynomial ring in one variable over A, $f\colon A \to B$ the canonical ring homomorphism and L the set of all trivial valuations of A. Then L is pro-constructible in $\operatorname{Spv} A$ and every constructible subset of L has finitely many connected components (1.1.3). But in $M\colon = \{v \in \operatorname{Spv}(f)^{-1}(L) | v(T) > 0$ and

$v(T) \neq \infty\}$ there are no proper specializations, and hence M is totally disconnected. (M is homeomorphic to L equipped with the constructible topology.)

LEMMA (2.4.7). *Let A be a ring such that the topological space $\operatorname{Spec} A$ is noetherian. Let L be a pro-constructible subset of $\operatorname{Spv} A$ which is closed under primary generalizations. We consider the following two conditions:*

i) *For every residue field K of A, every constructible subset of $L \cap \operatorname{Spv} K$ has finitely many connected components.*

ii) *Every constructible subset of L has finitely many connected components.*

Then i) *implies* ii). *If, for every prime ideal $\mathfrak{p}$ of A, the set $\{x \in \operatorname{Spec} A/\mathfrak{p} | (A/\mathfrak{p})_x$ is normal$\}$ contains a nonempty open subset of $\operatorname{Spec} A/\mathfrak{p}$, then* ii) *implies* i).

PROOF. Assume i). Let M be a constructible subset of L. In order to prove that M has finitely many connected components we show that, for every $x \in M$, there is a connected constructible subset of M containing x. Let $x \in M$ be given. Put $K = qf(A/\operatorname{supp}(x))$. By assumption there exists a connected constructible subset T of $M \cap \operatorname{Spv} K$ containing x. Let Z be a constructible subset of $\operatorname{Spv}(A/\operatorname{supp}(x))$ such that $Z \cap L \cap \operatorname{Spv} K = T$ and Z is closed under primary generalizations in $\operatorname{Spv}(A/\operatorname{supp}(x))$. Let λ: $\operatorname{Spv}(A/\operatorname{supp}(x)) \to \operatorname{Spec}(A/\operatorname{supp}(x))$ be the support mapping. Since $Z \cap L \cap \operatorname{Spv} K \subseteq M$, there exists a non-empty open subset U of $\operatorname{Spec}(A/\operatorname{supp}(x))$ with $S:= Z \cap L \cap \lambda^{-1}(U) \subseteq M$. Since S is closed under primary generalizations in $\operatorname{Spv}(A/\operatorname{supp}(x))$ and $T = S \cap \operatorname{Spv} K$ is connected, we conclude by (1.2.6) that S is connected. Since $\operatorname{Spec} A$ is noetherian, S is constructible in L. We have $x \in S$ by construction of S.

Now assume ii). Let $\mathfrak{p}$ be a prime ideal of A such that the set $\{x \in \operatorname{Spec} A/\mathfrak{p}| (A/\mathfrak{p})_x$ is normal$\}$ contains a non-empty open affine subset U of $\operatorname{Spec} A/\mathfrak{p}$. Put $K = qf(A/\operatorname{supp}(x))$. Let M be a constructible subset of $L \cap \operatorname{Spv} K$. Let λ: $\operatorname{Spv} A/\mathfrak{p} \to \operatorname{Spec} A/\mathfrak{p}$ be the support mapping. Choose a constructible subset Z of $\lambda^{-1}(U)$ such that $Z \cap L \cap \operatorname{Spv} K = M$ and Z is closed under primary generalizations in $\lambda^{-1}(U)$. Since $\operatorname{Spec} A$ is noetherian, $Z \cap L$ is constructible in L. So by assumption, $Z \cap L$ has finitely many connected components $L_1, \ldots, L_n$. Each L_1 is pro-constructible in $\lambda^{-1}(U)$ and closed under primary generalizations in $\lambda^{-1}(U)$. Hence by (2.4.6), $L_i \cap \lambda^{-1}(V)$ is connected for every open subset V of U. Put

$$M_i = \bigcap_{\substack{V \subseteq U \text{ open,} \\ V \neq \emptyset}} L_i \cap \lambda^{-1}(V).$$

Then M_i is connected (by (2.4.1 i)) and $M = \bigcup_{i=1}^{n} M_i$. Hence M has finitely many connected components. $\square$

In §3.4 we will prove

LEMMA (2.4.8). *Let $E \hookrightarrow F$ be a finitely generated extension of fields. We consider the mapping $g\colon Spv F \to Spv E$. Let L be a pro-constructible subset of $Spv E$.*

i) *If L has finitely many connected components, then $g^{-1}(L)$ has finitely many connected components, too. More precisely: Assume that L is connected. Then if F is purely transcendental over E, then $g^{-1}(L)$ is connected, and if F is finite over E, then the number of connected components of $g^{-1}(L)$ is at most $[F\colon E]_s$ (the separable degree of F over E).*

ii) *If every constructible subset of L has finitely many connected components, then every constructible subset of $g^{-1}(L)$ has finitely many connected components.*

Lemmata (2.4.7) and (2.4.8) imply

COROLLARY (2.4.9). *Let A be a Nagata ring and $f\colon A \to B$ a ring homomorphism of finite type. Let L be a pro- constructible subset of $Spv A$ such that L is closed under primary generalizations and every constructible subset of L has finitely many connected components. Then every constructible subset of $Spv(f)^{-1}(L) \subseteq Spv B$ has finitely many connected components.*

EXAMPLE. Let $A = S^{-1}B$ be the localization of a finitely generated $\mathbb{Z}$-algebra B by a multiplicative system $S \subseteq B$. Then every constructible subset of $\operatorname{Spv} A$ has finitely many connected components.

In §3.4 we will also prove

PROPOSITION (2.4.10). *Let $f\colon A \to B$ be a ring homomorphism of finite presentation and let L be a pro-constructible subset of $Spv A$ such that every constructible subset of L has finitely many connected components and every valuation $v \in L$ is non-trivial. Then every constructible subset of $Spv(f)^{-1}(L)$ has finitely many connected components, too.*

Remark (2.4.11). If K is a field, then $\operatorname{Spv}' K$ is the dual spectral space to $\operatorname{Spv} K$. Hence (2.4.2) and (2.4.8) remain true if we write Spv' instead of Spv. Then the proofs of (2.4.4), (2.4.5), (2.4.6) and (2.4.7) show that these results are also true for Spv'. One can show that (2.4.10) is true for Spv', even without the assumption that every $v \in L$ is non-trivial.

3. Valuation spectrum of rings over fields

3.1. Affine schemes over fields. Let k be a field, α a valuation of k and A a k-algebra. We put $S(\alpha, A) = \{v \in S(A) \mid v|k = \alpha\}$. Then $S(\alpha, A) = \{v \in S(A) \mid v(a) \geq 0$ for all $a \in k$ with $\alpha(a) \geq 0$ and $v(a) > 0$ for all $a \in k$ with $\alpha(a) > 0\}$. We equip $S(\alpha, A)$ with the subspace topologies of $\operatorname{Spv} A$, $\operatorname{Spv}' A$, $\operatorname{Spv}'' A$, and denote the resulting topological spaces by $\operatorname{Spv}(\alpha, A)$, $\operatorname{Spv}'(\alpha, A)$, $\operatorname{Spv}''(\alpha, A)$,

$$\begin{aligned}
\operatorname{Spv}(\alpha, A) &= (S(\alpha, A), \mathcal{T}|S(\alpha, A)) \\
\operatorname{Spv}'(\alpha, A) &= (S(\alpha, A), \mathcal{T}'|S(\alpha, A)) \\
\operatorname{Spv}''(\alpha, A) &= (S(\alpha, A), \mathcal{T}''|S(\alpha, A)).
\end{aligned}$$

Then $\operatorname{Spv}(\alpha, A)$, $\operatorname{Spv}'(\alpha, A)$, $\operatorname{Spv}''(\alpha, A)$ are convex pro-constructible subspaces of $\operatorname{Spv} A$, $\operatorname{Spv}' A$, $\operatorname{Spv}'' A$, are closed under primary specializations and primary generalizations, and their constructible topologies coincide.

LEMMA (3.1.1). *Let (K, β) be a valued field extending (k, α). We consider the canonical mappings $Spv(\beta, A \otimes_k K) \to Spv(\alpha, A)$, $Spv'(\beta, A \otimes_k K) \to Spv'(\alpha, A)$, $Spv''(\beta, A \otimes_k K) \to Spv''(\alpha, A)$.*

i) *They are surjective and spectral.*

ii) *If K is algebraic over k, then they are open and closed and map constructible subsets to constructible subsets.*

iii) *If (K, β) is a henselization of (k, α), then they are homeomorphisms.*

PROOF. i) is obvious.

ii) Let L be a constructible subset of $\operatorname{Spv}(\beta, A \otimes_k K)$. We choose a finite extension F of k with $k \hookrightarrow F \hookrightarrow K$ and a constructible subset M of $\operatorname{Spv}(\beta|F, A \otimes_k F)$ with $L = p^{-1}(M)$ where $p\colon \operatorname{Spv}(\beta, A \otimes_k K) \to \operatorname{Spv}(\beta|F, A \otimes_k F)$ is the canonical mapping. Since p is surjective, we have $r(L) = q(M)$ with $r\colon \operatorname{Spv}(\beta, A \otimes_k K) \to \operatorname{Spv}(\alpha, A)$ and $q\colon \operatorname{Spv} A \otimes_k F \to \operatorname{Spv} A$. $\operatorname{Spv}(\beta|F, A \otimes_k F)$ is open, closed and constructible in $q^{-1}(\operatorname{Spv}(\alpha, A))$. Hence by (2.1.1), $r(L)$ is constructible in $\operatorname{Spv}(\alpha, A)$. If L is open (resp. closed), then we choose M open (resp. closed) in $\operatorname{Spv}(\beta|F, A \otimes_k F)$. By (2.1.4) (resp. (2.1.7 ii)), we obtain that $r(L)$ is open (resp. closed) in $\operatorname{Spv}(\alpha, A)$.

iii) follows from (2.4.5) and (2.4.11). □

PROPOSITION (3.1.2). *Let (K, β) be a henselization of (k, α). Then*

i) *There is a canonical bijection between the set of clopen subsets of $Spv(\alpha, A)$ and the set of clopen subsets of $Spec\, A \otimes_k K$. Especially, the connected components of $Spv(\alpha, A)$ correspond to the connected components of $Spec\, A \otimes_k K$.*

ii) *There is a canonical bijection between the set of clopen subsets of $Spv'(\alpha, A)$ and the set of clopen subsets of $Spec\, A \otimes_k K$. Especially, the connected components of $Spv'(\alpha, A)$ correspond to the connected components of $Spec\, A \otimes_k K$.*

PROOF. By (3.1.1 iii), $\operatorname{Spv}(\beta, A \otimes_k K) \xrightarrow{\sim} \operatorname{Spv}(\alpha, A)$ and $\operatorname{Spv}'(\beta, A \otimes_k K) \xrightarrow{\sim} \operatorname{Spv}(\alpha, A)$. Hence we may assume $(K, \beta) = (k, \alpha)$, i.e. α is henselian. Then the assertion follows from (2.4.4) and (2.4.11). □

PROPOSITION (3.1.3). *If A is finitely generated over k, then every constructible subset of $Spv(\alpha, A)$ or $Spv'(\alpha, A)$ has finitely many connected components.*

PROOF. (2.4.9) and (2.4.11). □

PROPOSITION (3.1.4). *If the topological space* $Spec\,A$ *is noetherian, then the closure of a constructible subset of* $Spv''(\alpha, A)$ *is constructible.*

PROOF. Let L be a constructible subset of $\mathrm{Spv}''(\alpha, A)$. We choose a constructible subset M of $\mathrm{Spv}''A$ with $L = M \cap \mathrm{Spv}''(\alpha, A)$. Let $\bar{L}$ (resp. $\bar{M}$) be the closure of L (resp. M) in $\mathrm{Spv}''(\alpha, A)$ (resp. $\mathrm{Spv}''A$). Then $\bar{L} = \bar{M} \cap \mathrm{Spv}''(\alpha, A)$, since $\mathrm{Spv}''(\alpha, A)$ is closed under generalizations in $\mathrm{Spv}''A$. Now (2.2.1) implies that $\bar{L}$ is constructible in $\mathrm{Spv}''(\alpha, A)$. □

The (combinatorial) dimension $\dim X$ of a spectral space X is the supremum of lengths of chains of specializations in X.

PROPOSITION (3.1.5). i) $\dim Spv''A = \dim Spv''(\alpha, A) = \dim Spec\,A$.

ii) *Assume that* α *is non-trivial. Then* $\dim Spv(\alpha, A) = \dim Spv'(\alpha, A) = \sup\,\{trdeg(K|k)\ |K \text{ residue field of } A\}$. *In particular, if* A *is finitely generated over* k*, then* $\dim Spv(\alpha, A) = \dim Spv'(\alpha, A) = \dim Spec\,A$.

PROOF. i) follows from (1.2.6).

ii) Put $s = \sup\{\mathrm{trdeg}\,(K|k)|K$ residue field of $A\}$. Obviously, $\dim \mathrm{Spv}\,(\alpha, A) \geq s$ and $\dim \mathrm{Spv}'(\alpha, A) \geq s$. We show $\dim \mathrm{Spv}\,(\alpha, A) \leq s$ and $\dim \mathrm{Spv}'(\alpha, A) \leq s$. Let g be the length of a chain of specializations in $\mathrm{Spv}\,(\alpha, A)$. Then by (1.2.4) and (1.2.5 ii), there exist a chain of specializations $v_0 \succ v_1 \succ \cdots \succ v_m = w_0 \succ w_1 \succ \cdots \succ w_n$ in $\mathrm{Spv}\,(\alpha, A)$ such that $v_0 \succ v_1 \succ \cdots \succ v_m$ are secondary specializations, $w_0 \succ w_1 \succ \cdots \succ w_n$ are primary specializations and $g = m + n$. Let $\sum$ be the value group of v_m, Γ the value group of α and K the residue field of A at $\mathrm{supp}\,(v_m)$. There exist convex subgroups $G_0, \dots, G_m$ and $H_0, \dots, H_n$ of $\sum$ with $G_0 \supsetneqq G_1 \supsetneqq \cdots \supsetneqq G_m = \{0\}$, $G_i \cap \Gamma = \{0\}$ for $i = 0, \dots, m$ and $\sum = H_0 \supsetneqq H_1 \supsetneqq \cdots \supsetneqq H_n \supseteq \Gamma$. Then $g = m + n \leq \dim_{\mathbb{Q}}(\sum/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \leq \mathrm{trdeg}\,(K|k) \leq s$. Hence $\dim \mathrm{Spv}\,(\alpha, A) \leq s$. Similarly, one can show $\dim \mathrm{Spv}'(\alpha, A) \leq s$. □

Remark (3.1.6). Assume that k is algebraically closed, α non-trivial and A finitely generated over k. Let $f\colon A \to B$ be an etale ring homomorphism. We consider the mappings $g\colon \mathrm{Spv}\,(\alpha, B) \to \mathrm{Spv}\,(\alpha, A)$, $g'\colon \mathrm{Spv}'(\alpha, B) \to \mathrm{Spv}'(\alpha, A)$ and $g''\colon \mathrm{Spv}''(\alpha, B) \to \mathrm{Spv}''(\alpha, A)$ induced by f. Let x be a point of $S(\alpha, B)$. If $\mathrm{supp}\,(x)$ is a maximal ideal of B, then g, g', g'' are local homeomorphisms at x. But if $\mathrm{supp}\,(x)$ is not a maximal ideal of B, then in general g, g', g'' are not local homeomorphisms at x. Example: Let n be a natural number with $n \geq 2$ and $\mathrm{char}\,(k) \nmid n$. We consider the mapping $g\colon \mathrm{Spv}\,(\alpha, k[T]) \to \mathrm{Spv}\,(\alpha, k[T])$ induced by the k-algebra homomorphism $f\colon k[T] \to k[T]$, $T \mapsto T^n$. Let v be the Gauss valuation of $k[T]$ extending α, i.e. $v(a_0 + a_1 T + \cdots + a_m T^m) = \min\{\alpha(a_i)|i = 0, \dots, m\}$. Then there exists no constructible subset L of $\mathrm{Spv}\,(\alpha, k[T])$ such that $v \in L$ and $g|L$ is injective. Indeed, let L be a constructible subset of $\mathrm{Spv}\,(\alpha, k[T])$ with $v \in L$. The extension of fields $k(T) \hookrightarrow k(T)$ induced by f is galois, and

$v \circ \mu = v$ for every $\mu \in$ Gal $(k(T)|k(T))$. Hence $g^{-1}(g(v)) = \{v\} \subseteq L$. This implies that there is a constructible subset M of $\operatorname{Spv}(\alpha, k[T])$ with $g(v) \in M$ and $g^{-1}(M) \subseteq L$. By the subsequent Proposition (3.2.3), there exists a $w \in M$ such that $\operatorname{supp}(w)$ is a maximal ideal of $k[T]$ and $\operatorname{supp}(w) \neq T \cdot k[T]$. Then $g^{-1}(w) \subseteq L$ and $\# g^{-1}(w) = n$.

3.2. Semialgebraic sets. Let k be a field, $\alpha\colon k \twoheadrightarrow \Gamma_\infty$ a non-trivial valuation of k and A a k-algebra.

Let $\tilde{\Gamma}$ be the divisible hull of Γ. We put

$$\begin{aligned} \operatorname{Max}(\alpha, A) = \{v\colon A \to \tilde{\Gamma}_\infty | v &\text{ is a valuation of A,}\\ A/\operatorname{supp}(v) &\text{ is algebraic over } k,\\ v &\text{ extends } \alpha\colon k \to \Gamma_\infty \end{aligned}$$

and equip $\operatorname{Max}(\alpha, A)$ with the weakest topology such that, for every $a \in A$, the mapping $\operatorname{Max}(\alpha, A) \to \tilde{\Gamma}_\infty$, $v \mapsto v(a)$ is continuous where $\tilde{\Gamma}_\infty$ carries the order-induced topology.

LEMMA (3.2.1). i) *Assume that k is algebraically closed and A is generated over k by $a_1, \dots, a_n \in A$. Equip k with the valuation topology of α and k^n with the product topology. Then $Max(\alpha, A) \to k^n$, $v \mapsto (a_1 \, mod\, supp\,(v), \dots, a_n \, mod\, supp\,(v))$ is a topological embedding.*

ii) *The canonical mappings $Max(\alpha, A) \to Spv A$, $Max(\alpha, A) \to Spv' A$, $Max(\alpha, A) \to Spv'' A$ are topological embeddings.*

PROOF. i) is obvious.

ii) We show that the canonical mapping $\varphi\colon \operatorname{Max}(\alpha, A) \to \operatorname{Spv} A$ is a topological embedding. Obviously φ is injective. Let $a \in A$ and $\gamma \in \tilde{\Gamma}_\infty$ be given. Put $U = \{v \in \operatorname{Max}(\alpha, A) | v(a) < \gamma\}$ and $V = \{v \in \operatorname{Max}(\alpha, A) | v(a) > \gamma\}$. We have to show that there exist open subsets U' and V' of $\operatorname{Spv} A$ with $U = U' \cap \operatorname{Max}(\alpha, A)$ and $V = V' \cap \operatorname{Max}(\alpha, A)$. We may assume $\gamma \neq \infty$. For every $\delta \in \tilde{\Gamma}$ we choose $n(\delta) \in \mathbb{N}$ and $k(\delta) \in k^*$ with $n(\delta) \cdot \delta = \alpha(k(\delta))$ and put $U(\delta) = \{v \in \operatorname{Spv} A | v(a^{n(\delta)}) \leq v(k(\delta))\}$ and $V(\delta) = \{v \in \operatorname{Spv} A | v(a^{n(\delta)}) \geq v(k(\delta))\}$. Then $U(\delta)$, $V(\delta)$ are open in $\operatorname{Spv} A$ and we have

$$U = \operatorname{Max}(\alpha, A) \cap \bigcup_{\substack{\delta \in \tilde{\Gamma} \\ \delta < \gamma}} U(\delta),$$

$$V = \operatorname{Max}(\alpha, A) \cap \bigcup_{\substack{\delta \in \tilde{\Gamma} \\ \delta > \gamma}} V(\delta).$$

Let $a, b \in A$ be given and put $U = \{v \in \operatorname{Spv} A | v(a) \geq v(b) \neq \infty\}$. We have to show that $\varphi^{-1}(U)$ is open in $\operatorname{Max}(\alpha, A)$. Let $x \in \varphi^{-1}(U)$ be given. We consider the element $c\colon = \frac{a}{b} \in A(x) \subseteq A/\operatorname{supp}(x)$. Since the residue field of $A(x)$ is algebraic over the residue field of $k(\alpha)$, there exists a monic polynomial

$p(T) = T^n + e_1 T^{n-1} + \cdots + e_n \in k(\alpha)[T]$ such that $p(c)$ is contained in the maximal ideal of $A(x)$. Put $d := a^n + e_1 ba^{n-1} + e_2 b^2a^{n-2} + \cdots + e_nb^n \in A$. Then $x(d) > x(b^n)$, and hence we can choose an element $\gamma \in \tilde{\Gamma}$ with $x(d) > \gamma > x(b^n)$. Put $V := \{v \in \mathrm{Max}(\alpha, A) | v(d) > \gamma > v(b^n)\}$. Then V is open in $\mathrm{Max}(\alpha, A)$ and $x \in V \subseteq \varphi^{-1}(U)$. Thus we have proved that $\mathrm{Max}(\alpha, A) \to \mathrm{Spv}\, A$ is a topological embedding. Obviously, $\mathrm{Max}(\alpha, A) \to \mathrm{Spv}'A$ is a topological embedding. Hence $\mathrm{Max}(\alpha, A) \to \mathrm{Spv}''A$ is a topological embedding, too. □

Remark (3.2.2). Since $\mathrm{Max}(\alpha, A) \to \mathrm{Spv}\, A$ is continuous, the topology of $\mathrm{Max}(\alpha, A)$ is the weakest topology on $\mathrm{Max}(\alpha, A)$ such that, for all $a \in A$, the mapping $\mathrm{Max}(\alpha, A) \to \tilde{\Gamma}_\infty$, $v \mapsto v(a)$ is continuous where we now equip $\tilde{\Gamma}_\infty$ with the topology generated by the sets $\{\gamma\}$, $\{x \in \tilde{\Gamma}_\infty | x > \gamma\}$ with $\gamma \in \tilde{\Gamma}$.

The theory of algebraically closed fields with non-trivial valuation-divisibility relation has elimination of quantifiers [**P**, 4.17]. This implies

PROPOSITION (3.2.3). *Assume that A is finitely generated over k. Then $Spv''(\alpha, A)$ is the closure of $Max(\alpha, A)$ in the constructible topology of $Spv''A$.*

COROLLARY (3.2.4). *If A is finitely generated over k, then the closure of $Max(\alpha, A)$ in the constructible topology of $Spv''A$ is closed under generalization and specialization in $Spv''A$.*

DEFINITION. If A is finitely generated over k, we call a subset S of $\mathrm{Max}(\alpha, A)$ *semialgebraic* if S is a finite boolean combination of sets of the type $\{v \in \mathrm{Max}(\alpha, A) | v(a) > v(b)\}$ $(a, b \in A)$.

By (3.2.3) there is a canonical bijection $S \mapsto \tilde{S}$ from the set of semialgebraic subsets of $\mathrm{Max}(\alpha, A)$ onto the set of constructible subsets of $\mathrm{Spv}''(\alpha, A)$, namely $\tilde{S}$ is the unique constructible subset of $\mathrm{Spv}''A$ with $S = \tilde{S} \cap \mathrm{Max}(\alpha, A)$.

PROPOSITION (3.2.5). *Assume that A is finitely generated over k. Let L be a constructible subset of $Spv''(\alpha, A)$ and let $\bar{L}$ be the closure of L in $Spv''(\alpha, A)$. Then*

i) *$\bar{L}$ is constructible in $Spv''(\alpha, A)$, and $\bar{L} \cap Max(\alpha, A)$ is the closure of $L \cap Max(\alpha, A)$ in $Max(\alpha, A)$.*
ii) *L is closed (resp. open) in $Spv''(\alpha, A)$ if and only if $L \cap Max(\alpha, A)$ is closed (resp. open) in $Max(\alpha, A)$.*

PROOF. i) By (3.1.4), $\bar{L}$ is constructible in $\mathrm{Spv}''(\alpha, A)$. By (3.2.1 ii) and (3.2.3), $\bar{L} \cap \mathrm{Max}(\alpha, A)$ is the closure of $L \cap \mathrm{Max}(\alpha, A)$ in $\mathrm{Max}(\alpha, A)$.
ii) follows from i) and (3.2.3). □

Proposition (3.2.5) means that the operation $\sim$ commutes with the closure operations in $\mathrm{Max}(\alpha, A)$ and $\mathrm{Spv}''(\alpha, A)$.

PROPOSITION (3.2.6). *We assume that A is finitely generated over k. Let S be a semialgebraic subset of $Max(\alpha, A)$ and let $x \in Max(\alpha, A)$ be a point of the closure of S in $Max(\alpha, A)$. Then there exist a finitely generated, 1-dimensional,* *regular k-algebra B, a k-algebra homomorphism $f\colon A \to B$, an open subset U of $Max(\alpha, B)$ and a point $x_0 \in U$ such that $g(x_0) = x$ and $g(U \setminus \{x_0\}) \subseteq S$ where $g\colon Max(\alpha, B) \to Max(\alpha, A)$ is the mapping induced by f.*

PROOF. We distinguish the cases $x \in S$ and $x \notin S$. First assume $x \in S$. Let K be the residue field at $\operatorname{supp}(x)$ and β the valuation of K induced by x. Let f be the k-algebra homomorphism $A \to K[T]$, $a \mapsto a \bmod \operatorname{supp}(x) \in K \subseteq K[T]$. Put $U = \operatorname{Max}(\beta, K[T]) \subseteq \operatorname{Max}(\alpha, K[T])$. Then U is open in $\operatorname{Max}(\alpha, K[T])$ and $g(U) = \{x\}$. Now assume $x \notin S$. Let L be a constructible subset of $\operatorname{Spv}''A$ with $L \cap \operatorname{Max}(\alpha, A) = S$. Then x lies in the closure of L in $\operatorname{Spv}''A$. By (2.3.1) there exists a generalization v of x in $\operatorname{Spv}''A$ with $v \in L$ and $ht(\operatorname{supp}(x)/\operatorname{supp}(v)) = 1$. Let B be the normalization of $A/\operatorname{supp}(v)$ and $f\colon A \to B$ the canonical ring homomorphism. Then B is a finitely generated, 1–dimensional, regular k-algebra. Put $h\colon = \operatorname{Spv}''(f)$. There exist a $x_0 \in \operatorname{Max}(\alpha, B)$ and a generalization v_0 of x_0 in $\operatorname{Spv}''B$ with $x = h(x_0)$ and $v = h(v_0)$. Then $\{x_0\}$ is constructible in $\operatorname{Spv}''(\alpha, B)$ and v_0 is the unique proper generalization of x_0 in $\operatorname{Spv}''(\alpha, B)$. Since $v_0 \in h^{-1}(L)$, there exists an open constructible subset V of $\operatorname{Spv}''(\alpha, B)$ with $x_0 \in V$ and $V \setminus \{x_0\} \subseteq h^{-1}(L)$. Put $U = V \cap \operatorname{Max}(\alpha, B)$. Then U is a neighbourhood of x_0 in $\operatorname{Max}(\alpha, B)$ with $h(U \setminus \{x_0\}) \subseteq S$. □

3.3. An example. Let k be an algebraically closed field and $\alpha\colon k \twoheadrightarrow \Gamma_\infty$ a non-trivial valuation of k. In this section, we study $\operatorname{Spv}(\alpha, k[T])$ where $k[T]$ is the polynomial ring in one variable over k. (Analogous results hold for $\operatorname{Spv}'(\alpha, k[T])$.)

We consider $\operatorname{Max}(\alpha, k[T])$ as a subspace of $\operatorname{Spv}(\alpha, k[T])$ (by (3.2.1 ii)) and identify $\operatorname{Max}(\alpha, k[T])$ with k (by (3.2.1 i)).

By (3.2.3), the points of $\operatorname{Spv}(\alpha, k[T])$ correspond to the ultrafilters of semialgebraic subsets of k. Our first aim is to define a subset C of the set of semialgebraic subsets of k and to show that the points of $\operatorname{Spv}(\alpha, k[T])$ correspond to the filters of C.

DEFINITION. i) Let S be a subset of k. S is called an *o-disk* if there exist $a \in k$ and $\gamma \in \Gamma$ with $S = \mathbb{B}^+(a, \gamma)\colon = \{x \in k | \alpha(x - a) \geq \gamma\}$. S is called a *c-disk* if $|S| = 1$, or if there exist $a \in k$ and $\gamma \in \Gamma$ with $S = \mathbb{B}^-(a, \gamma)\colon = \{x \in k | \alpha(x - a) > \gamma\}$. S is called a *disk* if S is an o-disk or a c-disk.

ii) Let S be a subset of $\operatorname{Spv}(\alpha, k[T])$. S is called an *o-disk* if there exist $a \in k$, $b \in k^*$ with $S = \{v \in \operatorname{Spv}(\alpha, k[T]) | v(T - a) \geq v(b)\}$. S is called a *c-disk* if there exist $a, b \in k$ with $S = \operatorname{Spv}(\alpha, k[T]) \setminus \{v \in \operatorname{Spv}(\alpha, k[T]) | v(b) \geq v(T - a) \neq \infty\}$. S is called a *disk* if S is an o-disk or a c-disk.

By $S \mapsto \tilde{S}$ we have a bijection from the set of o-disks (resp. c-disks) of k to the set of o-disks (resp. c-disks) of $\operatorname{Spv}(\alpha, k[T])$). If A, B are two disks of k (resp. $\operatorname{Spv}(\alpha, k[T])$), then $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$.

DEFINITION. Let C be the set of disks of k. A *filter* of C is a subset F of C such that

a) If $A \in F$ and $B \in C$ with $A \subseteq B$, then $B \in F$.
b) If $A, B \in F$, then $A \cap B \in F$.

Remark (3.3.1). i) To $a \in k$, $b \in k^*$ there exists a unique valuation $v = v(a, \alpha(b))$ of $k(T)$ extending α such that $v\left(\frac{T-a}{b}\right) \geq 0$ and the image t of $\frac{T-a}{b}$ in the residue field of v is transcendental over the residue field $\bar{k}$ of α [**B**, VI.10.1, Prop. 2]. We have $v\colon k(T) \to \Gamma_\infty$,

$$v\Big(\sum_{i=0}^{n} a_i(T-a)^i\Big) = \min\{\alpha(a_i) + i \cdot \alpha(b) | i = 0, \dots, n\}.$$

The residue field of v is $\bar{k}(t)$. Conversely, if v is an extension of α to $k(T)$ such that the residue field of v is a proper extension of $\bar{k}$, then $v = v(a, \gamma)$ for some $a \in k$, $\gamma \in \Gamma$.

ii) Let M be a minor subset of Γ (i.e. if $x \in M$, $y \in \Gamma$ with $y \leq x$ then $y \in M$) and a an element of k. Then there exists, up to equivalence, a unique valuation $v = v(a, M)$ of $k(T)$ extending α such that $v(T-a) \notin \Gamma$ and $M = \{\gamma \in \Gamma | \gamma < v(T-a)\}$ (cf. [**B**, VI.10.1, Prop. 1]). v can be constructed as follows. We consider Γ as a subgroup of $\Gamma \oplus \mathbb{Z}$ by $\gamma = (\gamma, 0)$, and extend the ordering of Γ to the group ordering of $\Gamma \oplus \mathbb{Z}$ such that $M = \{\gamma \in \Gamma | \gamma < (0,1)\}$. Then $v\colon k(T) \to (\Gamma \oplus \mathbb{Z})_\infty$,

$$v\Big(\sum_{i=0}^{n} a_i(T-a)^i\Big) = \min\{\alpha(a_i) + (0,i) | i = 0, \dots, n\}.$$

The residue field of v is equal to that of α. Conversely, if v is an extension of α to $k(T)$ such that $\Gamma_v \supsetneqq \Gamma$, then $v = v(a, M)$ for some $a \in k$ and minor subset M of Γ.

PROPOSITION (3.3.2). *Let F be a filter of C. Then there exists an unique point $\varphi(F) \in Spv(\alpha, k[T])$ with $F = \{S \in C | \varphi(F) \in \tilde{S}\}$. Distinguishing four cases, we can give a precise description of $\varphi(F)$.*

I) *If there exists a $a \in k$ with $F = \{S \in C | a \in S\}$, then $\varphi(F)$ is the point of $Spv(\alpha, k[T])$ with support $(T-a) \cdot k[T]$, i.e. $\varphi(F) = a$.*
II) *If there exist $a \in k$, $\gamma \in \Gamma$ with $F = \{S \in C | \mathbb{B}^+(a,\gamma) \subseteq S\}$, then $\varphi(F) = v(a, \gamma)$.*
III) *If $\bigcap_{S \in F} S = \emptyset$, then $\varphi(F)$ is an immediate extension of α to $k(T)$ and can be constructed as follows. Let $\frac{p(T)}{q(T)} \in k(T)^*$ be given. Choose $S \in F$, which is disjoint to the zero set of $p(T) \cdot q(T)$. Then there exists a $\gamma \in \Gamma$ with $\alpha\left(\frac{p(x)}{q(x)}\right) = \gamma$ for every $x \in S$, and we have $\varphi(F)\left(\frac{p(T)}{q(T)}\right) = \gamma$.*

IV) *Assume that F is not of type* I *or* II *or* III. *Choose $\alpha \in \bigcap_{S\in F} S$ and put* $M = \{\gamma \in \Gamma | \mathbb{B}^-(a,\gamma) \in F\}$. *Then $\varphi(F) = v(a, M)$.*

The mapping $F \mapsto \varphi(F)$ is a bijection from the set of filters of C to the set $Spv(\alpha, k[T])$.

PROOF. 1) For every $a \in k$, there exists obviously an unique $x \in \mathrm{Spv}\,(\alpha, k[T])$ with $\{S \in C | a \in S\} = \{S \in C | x \in \tilde{S}\}$. Namely, x is the point of $\mathrm{Spv}\,(\alpha, A)$ with support $(T-a)\cdot k[T]$. Hence, in the following steps 2) and 3) we deal only with filters F such that $|S| > 1$ for every $S \in F$ and with points x of $\mathrm{Spv}\,(\alpha, k[T])$ such that $\{0\} = \mathrm{supp}\,(x)$.

2) Let F be a filter of C. Then there exists at most one $v \in \mathrm{Spv}\,(\alpha, k[T])$ with $F = \{S \in C | v \in \tilde{S}\}$. Indeed, let such a v be given. For every $a \in k$ put $M_a = \{\gamma \in \Gamma | v(T-a) \geq \gamma\}$ and $N_a = \{\gamma \in \Gamma | v(T-a) > \gamma\}$. Then M_a and N_a are uniquely determined since $M_a = \{\gamma \in \Gamma | v \in \mathbb{B}^+(a,\gamma)^\sim\}$ and $N_a = \{\gamma \in \Gamma | v \in \mathbb{B}^-(a,\gamma)^\sim\}$. M_a and N_a are minor subsets of Γ with $N_a \subseteq M_a$. If $M_a = N_a$ for some $a \in k$, then $v(T-a) \notin \Gamma$, and hence v is uniquely determined by (3.3.1 ii). So assume $N_a \subsetneqq M_a$ for all $a \in k$. Then $v(T-a) \in \Gamma$ and $v(T-a) = \max M_a$ for every $a \in k$. Hence v is uniquely determined on the set $\{T - a | a \in k\} \subseteq k[T]$, and therefore v is uniquely determined.

3) Let F be a filter of type II, III or IV, and let $\varphi(F)$ be the valuation as defined in II, III or IV. One can easily check that $F = \{S \in C | \varphi(F) \in \tilde{S}\}$.

4) By 1), 2), 3) we have at every filter F of C an unique point $\varphi(F) \in \mathrm{Spv}\,(\alpha, k[T])$ with $F = \{S \in C | \varphi(F) \in \tilde{S}\}$. Hence we have a mapping φ from the set of filters of C to the set $\mathrm{Spv}\,(\alpha, k[T])$. Obviously, φ is injective. To show the surjectivity of φ, let $x \in \mathrm{Spv}\,(\alpha, k[T])$ be given. Then $F = \{S \in C | x \in \tilde{S}\}$ is a filter of C with $x = \varphi(F)$.

5) Let v be a valuation as in III. It remains to show that v is an immediate extension of α. We have $\Gamma_v = \Gamma$ by construction of v. We deduce from (3.3.1 i) the injectivity of φ and II that the residue field of v is equal to that of α. $\square$

COROLLARY (3.3.3). *The boolean algebra of constructible subsets of $Spv\,(\alpha, k[T])$ is generated by the disks of $Spv\,(\alpha, k[T])$.*

PROOF. Let B be the boolean algebra generated by the disks of $\mathrm{Spv}\,(\alpha, k[T])$. Let L be a constructible subset of $\mathrm{Spv}\,(\alpha, k[T])$ and let x be a point in the complement of L. By (3.3.2) there exists, for every $y \in L$, an element M of B with $y \in M$ and $x \notin M$. Hence there exists a $K \in B$ with $L \subseteq K$ and $x \notin K$. That means

$$L = \bigcap_{\substack{K\in B\\ L\subseteq K}} K$$

which implies $L \in B$.

A subset S of $\mathrm{Spv}(\alpha, k[T])$ is called a *generalized disk* if we can write

$$(*) \qquad S = B \setminus \bigcup_{i=1}^{n} B_i$$

where $n \in \mathbb{N}_0$ and $B, B_1, \ldots, B_n$ are disks or $\mathrm{Spv}(\alpha, k[T])$. S is called a *generalized o-disk* if there exists a representation $(*)$ where B is an o-disk or $\mathrm{Spv}(\alpha, k[T])$ and $B_1, \ldots, B_n$ are c-disks, and S is called a *generalized c-disk* if there exists a representation $(*)$ where B is a c-disk or $\mathrm{Spv}(\alpha, k[T])$ and $B_1, \ldots, B_n$ are o-disks. (Note that we can assume $B_i \subseteq B$ for $i = 1, \ldots, n$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.) $\square$

The following corollary is a reformulation of (3.3.3).

COROLLARY (3.3.4). *Every constructible subset of $\mathrm{Spv}(\alpha, k[T])$ is a finite and disjoint union of generalized disks.*

Next we consider the specializations in $\mathrm{Spv}(\alpha, k[T])$. We call $\mathbb{B}^+(a, \lambda)$ the associated o-disk to the c-disk $\mathbb{B}^-(a, \lambda)$. A c-disk U of k is called associated to an o-disk V of k if V is associated to U. Every o-disk of k is the disjoint union of its associated c-disks.

PROPOSITION (3.3.5). i) *A point x of $\mathrm{Spv}(\alpha, k[T])$ has a proper generalization in $\mathrm{Spv}(\alpha, k[T])$ if and only if there exists a c-disk B of k with $x = \varphi(\{S \in C | B \subseteq S\})$ or an o-disk B of k with $x = \varphi(\{S \in C | B \subsetneqq S\})$. In the following points* ii), iii), iv), *we describe these generalizations.*

ii) *Let a be an element of k. Then $a = \varphi(\{S \in C | a \in S\})$ has a unique proper generalization in $\mathrm{Spv}(\alpha, k[T])$, namely the point $\varphi(\{S \in C | \{a\} \subsetneqq S\})$, and this is a primary generalization.*

iii) *Let B be a c-disk of k with $|B| > 1$ and let D be the associated o-disk. Then $\varphi(\{S \in C | B \subseteq S\})$ has a unique proper generalization in $\mathrm{Spv}(\alpha, k[T])$, namely the point $\varphi(\{S \in C | D \subseteq S\})$, and this is a secondary generalization.*

iv) *Let B be an o-disk of k. Then $\varphi(\{S \in C | B \subsetneqq S\})$ has a unique proper generalization in $\mathrm{Spv}(\alpha, k[T])$, namely the point $\varphi(\{S \in C | B \subseteq S\})$, and this is a secondary generalization.*

PROOF. Let x be a point of $\mathrm{Spv}(\alpha, k[T])$. We consider the generalizations of x.

1) Assume that $\mathrm{supp}(x)$ is the maximal ideal $(T - a) \cdot k[T]$ of $k[T]$. Then x has no proper secondary generalization in $\mathrm{Spv}(\alpha, k[T])$, and hence according to (1.2.4) every generalization of x in $\mathrm{Spv}(\alpha, k[T])$ is primary. Since the local ring $k[T]_{\mathrm{supp}(x)}$ is a valuation ring of dimension 1, x has a unique proper primary generalization v (cf. (1.2.2 ii)). One easily checks that v corresponds to the filter $\{S \in C | \{a\} \subsetneqq S\}$.

2) Assume supp $(x) = \{0\}$. Then every generalization v of x in Spv $(\alpha, k[T])$ is a secondary generalization, i.e. $v = x/H$ where H is a convex subgroup of Γ_x with $H \cap \Gamma = \{0\}$. If x is of type II or III in (3.3.2), then $\Gamma_x = \Gamma$ and hence $v = x$. So we assume that x is of type IV in (3.3.2), $x = \varphi(F) = v(a, M)$. There exists a non-trivial convex subgroup H of Γ_x with $H \cap \Gamma = \{0\}$ if and only there exists a $h \in \Gamma_x$ such that $0 < h < \gamma$ for every non-negative $\gamma \in \Gamma$, and that holds true if and only if M has a greatest element or $\Gamma \setminus M$ has a smallest element. In both cases there is only one non-trivial convex subgroup H of Γ_x with $H \cap \Gamma = \{0\}$. M has a greatest element if and only if there exists a c-disk B of k such that $|B| > 1$ and $F = \{S \in C | B \subseteq S\}$. $\Gamma \setminus M$ has a smallest element if and only if there exists an o-disk B of k with $F = \{S \in C | B \subsetneqq S\}$. Let γ be the greatest element of M (resp. smallest element of $\Gamma \setminus M$). Then $v = x/H = v(a, M)/H = v(a, \gamma)$. Hence by (3.3.2 II), v is the point given in iii) and iv). □

The following proposition is a reformulation of (3.3.5).

PROPOSITION (3.3.6). i) *A point x of Spv$(\alpha, k[T])$ has a proper specialization in Spv$(\alpha, k[T])$ if and only if there exists a $a \in k$ with $x = \varphi(\{S \in C | \{a\} \subsetneqq S\})$ or an o-disk B of k with $x = \varphi(\{S \in C | B \subseteq S\})$. These specializations are described in the following points* ii) *and* iii).

ii) *Let a be an element of k. Then $\varphi(\{S \in C | \{a\} \subsetneqq S\})$ has an unique specialization in Spv$(\alpha, k[T])$, namely the point $a \in$ Spv$(\alpha, k[T])$, and this specialization is primary.*

iii) *Let B be an o-disk of k. We consider the point $x = \varphi(\{S \in C | B \subseteq S\})$. Let $B_i, i \in I$ be the c-disks associated to B. Then $\varphi(\{S \in C | B_i \subseteq S\})$, $i \in I$ and $\varphi(\{S \in C | B \subsetneqq S\})$ are the specializations of x in Spv$(\alpha, k[T])$. All these specializations are secondary.*

The following two corollaries are consequences of (3.3.5) and (3.3.6).

COROLLARY (3.3.7). i) *Let B be a c-disk of Spv$(\alpha, k[T])$. Then there exist unique points $x \in B$ and $y \in$ Spv$(\alpha, k[T]) \setminus B$ such that y is a generalization of x.*

ii) *Let B be an o-disk of Spv$(\alpha, k[T])$. Then there exist unique points $x \in B$ and $y \in$ Spv$(\alpha, k[T]) \setminus B$ such that y is a specialization of x.*

COROLLARY (3.3.8). i) *A generalized disk is closed in Spv$(\alpha, k[T])$ if and only if it is a generalized c- disk.*

ii) *A generalized disk is open in Spv$(\alpha, k[T])$ if and only if it is a generalized o-disk.*

PROPOSITION (3.3.9).

i) *Every open constructible subset of Spv$(\alpha, k[T])$ is a finite and disjoint union of generalized o-disks.*

ii) *Every closed constructible subset of Spv*$(\alpha, k[T])$ *is a finite and disjoint union of generalized c-disks.*

PROOF. Let L be an open constructible subset of $\operatorname{Spv}(\alpha, k[T])$. By (3.3.4), $L = L_1 \cup \cdots \cup L_n$ where $L_1, \ldots, L_n$ are generalized disks. Since L is closed under generalizations in $\operatorname{Spv}(\alpha, k[T])$, we conclude from (3.3.4) and (3.3.5) that, for every $i = 1, \ldots, n$, there exists a generalized o-disk M_i with $L_i \subseteq M_i \subseteq L$. Assertion ii) can be proved analogously. □

One can conclude from (3.3.4) and (3.3.5):

PROPOSITION (3.3.10). *A constructible subset of Spv*$(\alpha, k[T])$ *is connected if and only if it is a generalized disk.*

Then (3.3.4) and (3.3.10) imply

COROLLARY (3.3.11). *Every constructible subset of Spv*$(\alpha, k[T])$ *has finitely many connected components.*

3.4. Supplement to (2.4). In this paragraph we want to give a proof of (2.4.8) and (2.4.10).

LEMMA (3.4.1). *Let* $f\colon X \to Y$ *be a spectral mapping between spectral spaces. Let* Z *be a clopen subset of a fibre* $F = f^{-1}(y)$ *of* f*, and let* U *be a constructible subset of* X *containing* Z*. Then there exist a constructible subset* V *of* U *and a constructible subset* W *of* Y *such that* $Z = F \cap V$ *and* V *is a clopen subset of* $f^{-1}(W)$*.*

PROOF. One can prove (3.4.1) by use of (2.4.1 i). But we give here another proof. Let P (resp. Q) be the specializations (resp. generalizations) of points of $F \setminus Z$ in X. Then $P \cup Q$ is a pro-constructible subset of X with $Z \cap (P \cup Q) = \emptyset$. Let R be a constructible subset of X with $Z \subseteq R \subseteq U \setminus (P \cup Q)$. Let S (resp. T) be the set of specializations (resp. generalizations) of points of R. Then $H\colon = (S \cup T) \setminus R$ is a pro-constructible subset of X with $H \cap F = \emptyset$. Hence $f(H)$ is a pro-constructible subset of Y with $y \notin f(H)$. Let W be a constructible subset of Y with $y \in W$ and $f(H) \cap W = \emptyset$. Put $V = R \cap f^{-1}(W)$. Then $V \subseteq U \cap f^{-1}(W)$, $Z = F \cap V$ and V is closed under specializations and generalizations in $f^{-1}(W)$, hence V is clopen in $f^{-1}(W)$. □

COROLLARY (3.4.2). *Let* $f\colon A \to B$ *be a flat, quasi-finite and finitely presented ring homomorphism. Let* L *be a pro-constructible subset of* $\operatorname{Spv} A$ *such that every constructible subset of* L *has finitely many connected components. Then every constructible subset of* $\operatorname{Spv}(f)^{-1}(L) \subseteq \operatorname{Spv} B$ *has finitely many connected components.*

PROOF. By [**EGA**, IV.18.12.13] there exist ring homomorphisms $g\colon C \to B$ and $h\colon A \to C$ such that $f = g \circ h$, h is finite and $\operatorname{Spec}(g)\colon \operatorname{Spec} B \to \operatorname{Spec} C$ is an open embedding of schemes. We consider $\operatorname{Spv} B$ as an open subspace of $\operatorname{Spv} C$ via $\operatorname{Spv}(g)$. Let U be a constructible subset of $\operatorname{Spv}(f)^{-1}(L)$ and x an element of U. We have to show that there exists a constructible subset V of U which is connected and contains x. Put $p = \operatorname{Spv}(h)\colon \operatorname{Spv} C \to \operatorname{Spv} A$. By (3.4.1) there exist a constructible subset W of L and a constructible subset V of U such that $\{x\} = p^{-1}(p(x)) \cap V$ and V is a clopen subset of $p^{-1}(W)$. Since W has finitely many connected components, we may assume that W is connected. We claim that then V is connected, too. Assume by way of contradiction that there exists a decomposition $V = V_1 \cup V_2$ into nonempty clopen subsets. Let $q\colon V \to W$ be the restriction of p. By (2.1.7 ii), q is closed and by (2.1.4) q is open. Hence $q(V_1)$ and $q(V_2)$ are clopen subsets of W. So $q(V_1) = q(V_2)$, in contradiction to $\#q^{-1}(q(x)) = 1$ and $V_1 \cap V_2 = \emptyset$. □

LEMMA (3.4.3). *Let k be an algebraically closed field and α the trivial valuation of k. We consider $\mathit{Spv}(\alpha, k[T])$, where $k[T]$ is the polynomial ring in one variable over k. Let v_0 be the trivial valuation of $k(T)$, v_1 the valuation of $k(T)$ with valuation ring $k[T^{-1}]_{T^{-1}\cdot k[T^{-1}]}$, and, for every $a \in k$, $t(a)$ the valuation of $k(T)$ with valuation ring $k[T]_{(T-a)\cdot k[T]}$ and $s(a)$ the trivial valuation of $k[T]$ with support $(T-a)\cdot k[T]$. A subset L of $\mathit{Spv}(\alpha, k[T])$ is called a generalized disk if $|L| \le 1$ and $L \neq \{v_0\}$ or if $L = \{s(a), t(a)\}$ for some $a \in k$ or if $L = \mathit{Spv}(\alpha, k[T]) \setminus M$ where M is a finite subset of $\mathit{Spv}(\alpha, k[T]) \setminus \{v_0\}$. Then*

i) *$\mathit{Spv}(\alpha, k[T]) = \{v_0, v_1\} \cup \{s(a), t(a) | a \in k\}$.*

ii) *Every subset L of $\mathit{Spv}(\alpha, k[T]) \setminus \{v_0\}$ with $|L| \le 1$ is constructible, namely $\{v_1\} = \{v \in \mathit{Spv}(\alpha, k[T]) | v(T) < 0\}$, $\{s(a)\} = \{v \in \mathit{Spv}(\alpha, k[T]) | v(T-a) = \infty\}$ and $\{t(a)\} = \{v \in \mathit{Spv}(\alpha, k[T]) | v(T-a) > 0$ and $v(T-a) \neq \infty\}$.*

iii) *A subset L of $\mathit{Spv}(\alpha, k[T])$ is constructible if and only if there exists a finite subset M of $\mathit{Spv}(\alpha, k[T]) \setminus \{v_0\}$ with $L = M$ or $L = \mathit{Spv}(\alpha, k[T]) \setminus M$.*

iv) *The proper specializations in $\mathit{Spv}(\alpha, k[T])$ are $v_0 \succ t(a) \succ s(a)$ $(a \in k)$ and $v_0 \succ v_1$.*

v) *A constructible subset of $\mathit{Spv}(\alpha, k[T])$ is connected if and only if it is a generalized disk.*

PROOF. iii) Let L be a constructible subset of $\operatorname{Spv}(\alpha, k[T])$. If $v_0 \notin L$, then L is finite by ii), and if $v_0 \in L$, then $\operatorname{Spv}(\alpha, k[T]) \setminus L$ is finite by ii).

v) Every constructible subset of $\operatorname{Spv}(\alpha, k[T])$ containing v_0 is connected since $v_0 \succ x$ for every $x \in \operatorname{Spv}(\alpha, k[T])$. □

In the following we use the following notation: Let k be a field and α a valuation of k. We call a subset L of $\operatorname{Spv}(\alpha, k[T])$ a generalized disk if $\lambda^{-1}(L)$

is a generalized disk of $\operatorname{Spv}(\bar{\alpha}, \bar{k}[T])$ where $\bar{k}$ is an algebraic closure of k, $\bar{\alpha}$ an extension of α to $\bar{k}$ and $\lambda\colon \operatorname{Spv}(\bar{\alpha}, \bar{k}[T]) \to \operatorname{Spv}(\alpha, k[T])$ the canonical mapping. Since λ is surjective, (3.3.10) and (3.4.3 v) imply that generalized disks are connected.

PROOF OF (2.4.8). i) Let L be a connected pro-constructible subset of $\operatorname{Spv} E$. First assume that $F = E(T)$ is a transcendental extension of E. By (3.1.2), $\operatorname{Spv}(\alpha, E[T])$ is connected for every $\alpha \in L$. Then by (2.4.6) and (2.4.1 i), $\operatorname{Spv}(\alpha, E(T)) = g^{-1}(\alpha)$ is connected. Since g is open (by (2.2.3) and (2.1.2)), we conclude that $g^{-1}(L)$ is connected. Now consider the case that F is finite over E. Assume that the number of connected components of $g^{-1}(L)$ is greater than $[F\colon E]_s$. Then there exists a decomposition $g^{-1}(L) = M_1 \cup \cdots \cup M_n$ of $g^{-1}(L)$ into non-empty clopen subsets $M_1, \ldots, M_n$ with $n > [F\colon E]_s$. Since g is open and closed, we have $L = g(M_1) = \cdots = g(M_n)$. Hence $\# g^{-1}(x) \geq n$ for every $x \in L$, contradiction.

ii) Let L be a pro-constructible subset of $\operatorname{Spv} E$ such that every constructible subset of L has finitely many connected components. We will show that every constructible subset of $g^{-1}(L)$ has finitely many connected components. By (3.4.2) we may assume that F is purely transcendental over E, and then by induction we may assume that $F = E(T)$ has transcendence degree 1 over E. We consider the canonical mapping $f\colon \operatorname{Spv} E[T] \to \operatorname{Spv} E$. By (2.4.7), it suffices to show that every constructible subset of $f^{-1}(L)$ has finitely many connected components. Let M be a constructible subset of $f^{-1}(L)$ and x an element of M. We will show that there exists a connected constructible subset of M containing x. Let $\bar{E}$ be the algebraic closure of E. We consider the commutative diagram (with canonical morphisms)

$$\begin{array}{ccc}
\operatorname{Spv} \bar{E}[T] & \xrightarrow{h_T} & \operatorname{Spv} E[T] \\
{\scriptstyle \bar{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\operatorname{Spv} \bar{E} & \xrightarrow[h]{} & \operatorname{Spv} E\,.
\end{array}$$

Let $\bar{x}$ be a point of $\operatorname{Spv} \bar{E}[T]$ lying over x. Put $\bar{y} = \bar{f}(\bar{x}) \in \operatorname{Spv} \bar{E}$. We distinguish five cases.

First case: $\bar{y}$ is non-trivial. By (3.3.4) there exists a generalized disk B of $\operatorname{Spv}(\bar{y}, \bar{E}[T]) = \bar{f}^{-1}(\bar{y})$ with $\bar{x} \in B \subseteq h_T^{-1}(M)$. Let $\bar{e}$ be an element of $\bar{E}$ such that the point of $\operatorname{Spv}(\bar{y}, \bar{E}[T])$ with support $(T - \bar{e}) \cdot \bar{E}[T]$ is contained in B. Choose a description of $B \subseteq \operatorname{Spv}(\bar{y}, \bar{E}[T])$ by polynomials $p_1, \ldots, p_n \in \bar{E}[T]$ of the form $p_i = T - a$ or $p_i = a$ with $a \in \bar{E}$. Let K be the subfield of $\bar{E}$ generated by $E, \bar{e}$ and the coefficients of $p_1, \ldots, p_n$. We consider the commutative diagram

$$(*)\qquad \begin{array}{ccccc} \operatorname{Spv} \bar{E}[T] & \xrightarrow{j_T} & \operatorname{Spv} K[T] & \xrightarrow{i_T} & \operatorname{Spv} E[T] \\ \bar{f}\downarrow & & \downarrow p & & \downarrow f \\ \operatorname{Spv} \bar{E} & \xrightarrow[j]{} & \operatorname{Spv} K & \xrightarrow[i]{} & \operatorname{Spv} E\,. \end{array}$$

Put $y = j(\bar{y})$. There exists a constructible subset U of $\operatorname{Spv} K[T]$ such that, for every $z \in \operatorname{Spv} K$, $p^{-1}(z) \cap U \subseteq p^{-1}(z) = \operatorname{Spv}(z, K[T])$ is a generalized disk of $\operatorname{Spv}(z, K[T])$ and $j_T(B) = p^{-1}(y) \cap U$. Then $p^{-1}(y) \cap U \subseteq i_T^{-1}(M)$. Hence there exists a constructible subset V of $i^{-1}(L)$ with $y \in V$ and $p^{-1}(V) \cap U \subseteq i_T^{-1}(M)$. Let $s\colon \operatorname{Spv} K \to \operatorname{Spv} K[T]$ be the mapping induced by the K-algebra homomorphism $K[T] \to K$, $T \mapsto \bar{e}$. Then s is a section of p with $s(y) \in U$. Making V smaller, we can assume $s(V) \subseteq U$. By (3.4.2), V has finitely many connected components. Hence we may assume that V is connected. Then since $p^{-1}(z) \cap U$ is connected for every $z \in V$ and $s(V) \subseteq p^{-1}(V) \cap U$ is connected, we obtain that $H\colon = p^{-1}(V) \cap U$ is connected. Now $i_T(H)$ is a connected constructible subset of M which contains x.

Second case: $\bar{y}$ is trivial and $\bar{x}$ is the trivial valuation of $\bar{E}(T)$. Then x is the trivial valuation of $E(T)$. Hence $x \succ z$ for every $z \in \operatorname{Spv} E[T]$, which implies that M is connected.

Third case: $\bar{y}$ is trivial and $\bar{x}(T) < 0$. Let y be the trivial valuation of E. Then x is the unique point v of $\operatorname{Spv}(y, E[T]) = f^{-1}(y)$ with $v(T) < 0$. Put $U = \{v \in \operatorname{Spv} E[T] | v(T) < 0\}$. Since $f^{-1}(y) \cap U \subseteq M$, there exists a constructible subset V of L with $y \in V$ and $f^{-1}(V) \cap U \subseteq M$. For every $z \in V$, $f^{-1}(z) \cap U$ is connected and contains a point which is a specialization of x. Hence $f^{-1}(V) \cap U$ is connected.

Fourth case: $\bar{y}$ is trivial and there exists a $\bar{e} \in \bar{E}$ with $\bar{x}(T - \bar{e}) > 0$ and $\bar{x}(T - \bar{e}) \neq \infty$. We consider the diagram $(*)$ with $K = E(\bar{e})$. Put $y = j(\bar{y})$ and $U = \{v \in \operatorname{Spv} K[T] | v(T - \bar{e}) > 0 \text{ and } v(T - \bar{e}) \neq \infty\}$. Then $p^{-1}(y) \cap U = \{j_T(\bar{x})\} \subseteq i_T^{-1}(M)$. Hence there exists a constructible subset V of $i^{-1}(L)$ with $y \in V$ and $p^{-1}(V) \cap U \subseteq i_T^{-1}(M)$. For every $z \in V$, $f^{-1}(z) \cap U$ is connected and contains a point which is a specialization of $j_T(x)$. Hence $H\colon = p^{-1}(V) \cap U$ is connected. Then $i_T(H)$ is a connected constructible subset of M with $x \in i_T(H)$.

Fifth case: $\bar{y}$ is trivial and there exists a $\bar{e} \in \bar{E}$ with $\bar{x}(T - \bar{e}) = \infty$. We consider the diagram $(*)$ with $K = E(\bar{e})$. Put $y = j(\bar{y})$ and $U = \{v \in \operatorname{Spv} K[T] | v(T - \bar{e}) = \infty\}$. Then $p^{-1}(y) \cap U = \{j_T(\bar{x})\} \subseteq i_T^{-1}(M)$. Hence there exists a constructible subset V of $i^{-1}(L)$ with $y \in V$ and $p^{-1}(V) \cap U \subseteq i_T^{-1}(M)$. Since every point of $H\colon = p^{-1}(V) \cap U$ is a specialization of $j_T(\bar{x}) \in H$, H is connected. Hence $i_T(H)$ is a connected constructible subset of M which contains x. $\square$

PROOF OF (2.4.10). We may assume that B is a polynomial ring over A and then by induction we may assume that $B = A[T]$ is the polynomial ring in one variable over A and $f\colon A \to B$ is the canonical ring homomorphism. Let M be a constructible subset of $\operatorname{Spv}(f)^{-1}(L)$ and x an element of M. We show that there exists a connected constructible subset of M which contains x. Let $(C_i | i \in I)$ be a filtered inductive system of flat, quasi-finite and finitely presented A-algebras such that $C := \varinjlim_{i \in I} C_i$ is a local ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}$ lies over $\operatorname{supp}(x) \cap A$ and $C/\mathfrak{m}$ is algebraically closed. We consider the commutative diagram

$$\begin{tikzcd} \operatorname{Spv} C[T] \arrow[r, "h_T"] \arrow[d, "f"'] & \operatorname{Spv} A[T] \arrow[d] \\ \operatorname{Spv} C \arrow[r, "h"'] & \operatorname{Spv} A. \end{tikzcd}$$

Let y be a point of $\operatorname{Spv} C[T]$ with $x = h_T(y)$ and $\operatorname{supp}(y) \cap C = \mathfrak{m}$. Put $z = f(y)$. By (3.3.4), there exists a generalized disk D of $f^{-1}(z)$ with $y \in D \subseteq h_T^{-1}(M)$. Choose a representation of D by polynomials $p_1, \ldots, p_n \in C/\mathfrak{m}[T]$ such that every p_i is of the form $p_i = T - \bar{a}_i$ or $p_i = \bar{a}_i$ with $\bar{a}_i \in C/\mathfrak{m}$, and choose a $\bar{b} \in C/\mathfrak{m}$ such that the point of $f^{-1}(z)$ with support $(T - \bar{b}) \cdot C/\mathfrak{m}[T]$ is contained in D. Let $b, a_1, \ldots, a_n \in C$ be representatives of $\bar{b}, \bar{a}_1, \ldots, \bar{a}_n$, and choose a $k \in I$ with $b, a_1, \ldots, a_n \in C_k$. Now, using the commutative diagram

$$\begin{tikzcd} \operatorname{Spv} C[T] \arrow[r, "j_T"] \arrow[d, "f"'] & \operatorname{Spv} C_k[T] \arrow[r, "i_T"] \arrow[d, "p"] & \operatorname{Spv} A[T] \arrow[d] \\ \operatorname{Spv} C \arrow[r, "j"'] & \operatorname{Spv} C_k \arrow[r, "i"'] & \operatorname{Spv} A, \end{tikzcd}$$

we can continue with the arguments of the first case in the proof of (2.4.8 ii). □

4. Applications of the valuation spectrum to algebraic geometry

4.1. Etale cohomology. In this short section we pursue two aims. First we indicate that a conjecture of Artin stated in [**SGA**, XII.6.13] is equivalent to the vanishing of the cohomology of constant torsion sheaves on certain pro-constructible subsets of the valuation spectrum of algebraically closed fields, and secondly we sketch that this equivalence allows us to prove both these statements. We are interested in the results of this paragraph also because they are very useful in order to study etale cohomology of rigid analytic varieties (cf. [**Hu**$_3$]). Precise proofs for all that is described in this section are contained in [**Hu**$_1$].

Let X be a topological space, Y a closed subspace of X and F an abelian sheaf on X such that $H^i(U, F) = 0$ for all $i \in \mathbb{N}$ and all open subsets U of X. We assume X is paracompact or X is a normal spectral space (in the sense of [**CC**]). Then $H^i(Y, F|Y) = 0$ for all $i \in \mathbb{N}$. In [**SGA**, XII.6.13], Artin conjectured that also the analogous statement for etale cohomology of affine schemes is true, more precisely:

(*) Let X be an affine scheme, Y a closed subscheme of X and F an abelian sheaf on X_{et} which is flasque (i.e. $H^i(U,F)_{et} = 0$ for every $i \in \mathbb{N}$ and every $U \in X_{et}$) and torsion. Then $H^i(Y, F|Y)_{et} = 0$ for every $i \in \mathbb{N}$.

Let us make two statements about the cohomology of the valuation spectrum of algebraically closed fields.

(**) Let K be an algebraically closed field and D, E subsets of K. Put $X = \{v \in \operatorname{Spv} K | v(d) \geq 0 \text{ for all } d \in D \text{ and } v(e) > 0 \text{ for all } e \in E\}$ and equip X with the subspace topology of $\operatorname{Spv} K$. Then $H^i(X, F) = 0$ for every $i \in \mathbb{N}$ and every abelian torsion group F.

(**)′ Let K be an algebraically closed field and D, E subsets of K. Put $X = \{v \in \operatorname{Spv}' K | v(d) \geq 0 \text{ for all } d \in D \text{ and } v(e) > 0 \text{ for all } e \in E\}$ and equip X with the subspace topology of $\operatorname{Spv}' K$. Then $H^i(X, F) = 0$ for every $i \in \mathbb{N}$ and every abelian torsion group F.

Then we have

LEMMA (4.1.1). *The following conditions are equivalent:*

i) (*) *holds,*

ii) (**) *holds,*

iii) (**)′ *holds.*

To prove the equivalence of i) and ii), use Zariski's representation of Spv K as a projective limit of schemes (2.4.3) and the proper base change theorem for etale cohomology [**SGA**, XII.5.1]. The equivalence of ii) and iii) follows from a general result of Schwartz [$\mathbf{S_1}$], which says that for every normal spectral space X, every abelian group G and every $n \in \mathbb{N}$, there is a canonical isomorphism $H^n(X, G) \cong H^n(X^*, G)$, where X^* is the inverse spectral space to X (in the sense of [**H**, Prop. 8]. In our situation, $X = \{v \operatorname{Spv}' K \mid v(d) \geq 0$ for all $d \in D$ and $v(e) > 0$ for all $e \in E\}$ is a normal spectral space and $\{v \in \operatorname{Spv}\ K \mid v(d) \geq 0$ for all $d \in D$ and $v(e) > 0$ for all $e \in E\}$ is the inverse spectral space to X.

Then (4.1.1) and the following lemma show that (*), (**) and (**)′ are true.

LEMMA (4.1.2). (**)′ *holds.*

In order to prove (4.1.2) we first show (**)′ for $i = 1$ by using the equivalence of i) and iii) in (4.1.1). Then we proceed by induction on i. For that we use that the cohomological dimension of a normal spectral space X is bounded from above by the combinatorial dimension of X ([**CC**]) and that, if $f \colon X \to Y$ is a spectral and specializing map between spectral spaces and the specializations of every point of X form a chain, then $(R^n f_* F)_y \cong H^n(f^{-1}(y), F|f^{-1}(y))$ for every $n \in \mathbb{N}_0$, $y \in Y$ and abelian sheaf F on X.

4.2. Open morphisms. Let $f \colon X \to Y$ be a morphism of schemes where f is called universally open at a point $x \in X$ if, for every base change $f_{(Y')} \colon X' = X \times_Y Y' \to Y'$ of f and every point $x' \in X\diamond'$ lying over x, the mapping $f_{(Y')}$ is open at x'. If f is of finite presentation at $x \in X$, then f is universally open at x

if and only if f is universally generalizing at x. In the following two propositions we study points at which f is universally generalizing.

PROPOSITION (4.2.1). *Let k be a field, A and B finitely generated k-algebras and $f\colon A \to B$ a k-algebra homomorphism. Then the set of points $x \in \operatorname{Spec} B$ at which $\operatorname{Spec}(f)\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is universally generalizing is constructible in $\operatorname{Spec} B$.*

Let $f\colon X \to Y$ be a morphism of schemes, and let L be the set of points of X at which f is universally open. In [**EGA**, IV.14.3.9], *Grothendieck asked whether L is constructible in X.*

Lemma (4.2.1) says that L is constructible in X if Y is locally of finite type over a field and f is locally of finite type. Parusinski proved a similar result in complex analytic geometry. Namely in [**Pa**] he showed that if $f\colon X \to Y$ is a morphism of complex analytic spaces, then the set of points of X at which f is (universally) open is constructible in X (in the complex analytic sense). It is obvious that one can apply Parusinski's ideas to the algebraic situation, and then one obtains, more general than (4.2.1), that L is constructible in X if Y is locally noetherian and f locally of finite type. The main tool in Parusinski's proof is the flattening technique ([**Hi**], [**RG**]), whereas in our proof of (4.2.1) we use simpler methods, namely the valuation spectrum and some model theory. Also for the proof of the following proposition we use valuations.

PROPOSITION (4.2.2). *Let A, B be noetherian adic rings and $A \to B$ a continuous ring homomorphism. Let $\hat{A}$ and $\hat{B}$ be the completions of A and B. Let $\mathfrak{p}$ be an open prime ideal of B and $\hat{\mathfrak{p}}$ the corresponding open prime ideal of $\hat{B}$. Then the following conditions are equivalent.*

i) *$\operatorname{Spec} B \to \operatorname{Spec} A$ is universally generalizing at $\mathfrak{p}$.*
ii) *$\operatorname{Spec} \hat{B} \to \operatorname{Spec} \hat{A}$ is universally generalizing at $\hat{\mathfrak{p}}$.*

PROOF OF (4.2.1). We need three lemmata.

LEMMA (4.2.3). *Let $f\colon A \to B$ be a ring homomorphism.*

i) *Let x be a point of $\operatorname{Spec} B$. If for all $n \in \mathbb{N}_0$ the morphism $\operatorname{Spec} B[T_1, \ldots, T_n] \to \operatorname{Spec} A[T_1, \ldots, T_n]$ induced by f is generalizing at every point $x' \in \operatorname{Spec} B[T_1, \ldots, T_n]$ lying over x, then $\operatorname{Spec}(f)$ is universally generalizing at x.*
ii) *Let x be a point of $\operatorname{Spv}'' B$. If for all $n \in \mathbb{N}_0$ the mapping $\operatorname{Spv}'' B[T_1, \ldots, T_n] \to \operatorname{Spv}'' A[T_1, \ldots, T_n]$ induced by f is generalizing at every point $x' \in \operatorname{Spv}'' B[T_1, \ldots, T_n]$ lying over x, then $\operatorname{Spv}''(f)$ is universally generalizing at x.*

PROOF. We show ii), whereas i) can be proved analogously. Let $A \to C$ be a ring homomorphism and let y be a point of $\operatorname{Spv}'' C \otimes_A B$ lying over x. We show that $g\colon \operatorname{Spv}'' C \otimes_A B \to \operatorname{Spv}'' C$ is generalizing at y. We represent C

as the inductive limit of finitely generated A-algebras, $C = \varinjlim_{i \in I} C_i$. Let $y_i \in \operatorname{Spv}'' C_i \otimes_A B$ be the image of y under the mapping $\operatorname{Spv}'' C \otimes_A B \to \operatorname{Spv}'' C_i \otimes_A B$. We denote the mapping $\operatorname{Spv}'' C_i \otimes_A B \to \operatorname{Spv}'' C_i$ by g_i. The assumption implies that, for every $i \in I$, the set $G(y_i)$ of generalizations of y_i in $\operatorname{Spv}'' C_i \otimes_A B$ is mapped onto the set $G(g_i(y_i))$ of generalizations of $g(y_i)$ in $\operatorname{Spv}'' C_i$. Since the set $G(y)$ of generalizations of y in $\operatorname{Spv}'' C \otimes_A B$ is the projective limit of $(G(y_i) | i \in I)$ and the set $G(g(y))$ of generalizations of $g(y)$ in $\operatorname{Spv}'' C$ is the projective limit of $(G(g_i(y_i)) | i \in I)$, the mapping $G(y) \to G(g(y))$ is surjective [$\mathbf{B}_1$, I.9.6]. □

LEMMA (4.2.4). *Let k be a field with a non-trivial valuation α. Let A, B be finitely generated k-algebras and $f \colon A \to B$ a k-algebra homomorphism. Let $g \colon \operatorname{Spv}'' B \to \operatorname{Spv}'' A$ and $\bar{g} \colon \operatorname{Max}(\alpha, B) \to \operatorname{Max}(\alpha, A)$ be the mappings induced by f. Let L be a constructible subset of $\operatorname{Spv}''(\alpha, B)$ such that $\bar{g}$ is open at every point of $L \cap \operatorname{Max}(\alpha, B)$. Then g is generalizing at every point of L.*

PROOF. Let $x \in L$ be given. Assume that g is not generalizing at x. Then there is a generalization y of $g(x)$ in $\operatorname{Spv}''(\alpha, A)$ such that no point of $g^{-1}(y)$ specializes to x. Hence there exists an open constructible subset U of $\operatorname{Spv}''(\alpha, B)$ with $x \in U$ and $g^{-1}(y) \cap U = \emptyset$. By assumption $\bar{g}(L \cap U \cap \operatorname{Max}(\alpha, B))$ is contained in the interior of $\bar{g}(U \cap \operatorname{Max}(\alpha, B))$ in $\operatorname{Max}(\alpha, A)$. By (3.2.3) we have $\bar{g}(L \cap U \cap \operatorname{Max}(\alpha, B)) = g(L \cap U) \cap \operatorname{Max}(\alpha, A)$ and $\bar{g}(U \cap \operatorname{Max}(\alpha, B)) = g(U) \cap \operatorname{Max}(\alpha, A)$. Therefore, $g(L \cap U)\operatorname{Max}(\alpha, A)$ is contained in the interior of $g(U) \cap \operatorname{Max}(\alpha, A)$ in $\operatorname{Max}(\alpha, A)$. By (2.1.1), $g(L \cap U)$ and $g(U)$ are constructible subsets of $\operatorname{Spv}''(\alpha, A)$. Now (3.2.5) implies that $g(L \cap U)$ lies in the interior of $g(U)$ in $\operatorname{Spv}''(\alpha, A)$. Since $x \in U \cap L$ and y specializes to $g(x)$, we have $y \in g(U)$, in contradiction to $g^{-1}(y) \cap U = \emptyset$. □

LEMMA (4.2.5). *Let*

$$(*) \qquad \begin{array}{ccc} U & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ V & \xrightarrow[g]{} & Y \end{array}$$

be a commutative diagram of schemes. Let u be a point of U.

i) *If f is universally generalizing at u and g is universally generalizing at $f(u)$, then h is universally generalizing at $i(u)$.*
ii) *Assume that $(*)$ is cartesian and g is universally generalizing at $f(u)$. Then f is universally generalizing at u if and only if h is universally generalizing at $i(u)$.*

PROOF. i) is obvious and ii) follows from i). □

Now we prove (4.2.1). First we study the case that k is algebraically closed and has a non-trivial valuation α. We consider the mappings $g\colon \operatorname{Spv}'' B \to \operatorname{Spv}'' A$ and $\bar{g}\colon \operatorname{Max}(\alpha, B) \to \operatorname{Max}(\alpha, A)$ induced by f. Since the statement "$\bar{g}$ is open at x" can be expressed in the formal language of valued fields by a formula Φ, and since the theory of algebraically closed fields with non-trivial valuation has elimination of quantifiers [**P**, 4.17], there exists a constructible subset L of $\operatorname{Spv}''(\alpha, B)$ such that $L \cap \operatorname{Max}(\alpha, B)$ is the set of points of $\operatorname{Max}(\alpha, B)$ at which $\bar{g}$ is open. We will show that L is the set of points of $\operatorname{Spv}''(\alpha, B)$ at which g is universally generalizing. For every $n \in \mathbb{N}$ we consider the mappings $g_n\colon \operatorname{Spv}'' B[T_1, \dots, T_n] \to \operatorname{Spv}'' A[T_1, \dots, T_n]$ and $p_n\colon \operatorname{Spv}''(\alpha, B[T_1, \dots, T_n]) \to \operatorname{Spv}''(\alpha, B)$. The mapping $\bar{g}_n\colon \operatorname{Max}(\alpha, B[T_1, \dots, T_n]) \to \operatorname{Max}(\alpha, A[T_1, \dots, T_n])$ is open at a point $x \in \operatorname{Max}(\alpha, B[T_1, \dots, T_n])$ if $\bar{g}$ is open at $p_n(x)$. Hence $p_n^{-1}(L)$ is a constructible subset of $\operatorname{Spv}''(\alpha, B[T_1, \dots, T_n])$ such that $\bar{g}_n$ is open at every point of $p_n^{-1}(L) \cap \operatorname{Max}(\alpha, B[T_1, \dots, T_n])$. Then according to (4.2.4), g_n is generalizing at every point of $p_n^{-1}(L)$. Now (4.2.3 ii) implies that g is universally generalizing at every point of L. Conversely, let x be a point of $\operatorname{Spv}''(\alpha, B)$ at which g is universally generalizing. Let $c\colon B \to s$ be a ring homomorphism of B to an algebraically closed field s and β a valuation of s with $\beta | B = x$. We consider the mappings $g'\colon \operatorname{Spv}'' B \otimes_k s \to \operatorname{Spv}'' A \otimes_k s$ and $\bar{g}'\colon \operatorname{Max}(\beta, B \otimes_k s) \to \operatorname{Max}(\beta, A \otimes_k s)$. Let x' be the point $\operatorname{Spv}''(d)(\beta) \in \operatorname{Spv}'' B \otimes_k s$ where d is the ring homomorphism $B \otimes_k s \to s$ with $d(b \otimes e) = c(b) \cdot e$. Then x' is an element of $\operatorname{Max}(\beta, B \otimes_k s)$ and is mapped to x under the projection $p\colon \operatorname{Spv}''(\beta, B \otimes_k s) \to \operatorname{Spv}''(\alpha, B)$. Since g is universally generalizing at x, g' is generalizing at x'. Hence $\bar{g}'$ is open at x'. Since the set of points of $\operatorname{Max}(\beta, B \otimes_k s)$ at which $\bar{g}'$ is open can be described by Φ, we have $x' \in p^{-1}(L)$, i.e. $x \in L$. Thus we have proved that L is the set of points of $\operatorname{Spv}''(\alpha, B)$ at which g is universally generalizing.

Let M be the set of points of $\operatorname{Spec} B$ at which $\operatorname{Spec}(f)$ is universally generalizing. Let $t\colon \operatorname{Spv}''(\alpha, B) \to \operatorname{Spec} B$ be the support mapping. By (2.1.3), $L = t^{-1}(M)$. Since t is surjective and spectral, we deduce that M is constructible in $\operatorname{Spec} B$.

Now we prove (4.2.1) in general. Let K be an extension field of k such that K is algebraically closed and carries a non-trivial valuation. We consider the cartesian square

$$\begin{array}{ccc} \operatorname{Spec} B \otimes_k K & \xrightarrow{\;q\;} & \operatorname{Spec} B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \operatorname{Spec} A \otimes_k K & \xrightarrow[\;p\;]{} & \operatorname{Spec} A\,, \end{array}$$

where g and h are induced by $f\colon A \to B$ and p, q are the canonical morphisms. Let S (bzw. T) be set of points of $\operatorname{Spec} B$ (resp. $\operatorname{Spec} B \otimes_k K$) at which g (resp. h) is universally generalizing. By (4.2.5 ii), we obtain $T = q^{-1}(S)$. We know already that T is constructible in $\operatorname{Spec} B \otimes_k K$. Since q is surjective and spectral, we deduce that S is constructible in $\operatorname{Spec} B$. $\square$

PROOF OF (4.2.2). We have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Spec} \hat{B} & \xrightarrow{i} & \operatorname{Spec} B \\
\downarrow & & \downarrow \\
\operatorname{Spec} \hat{A} & \xrightarrow[g]{} & \operatorname{Spec} A .
\end{array}$$

Since g is flat, (4.2.5 i) shows that i) follows from ii). Now we prove that i) implies ii). For that we use continuous valuations: A valuation v of a topological ring E is called continuous if for every $\gamma \in \Gamma_v$ there exists a neighbourhood U of $0 \in E$ with $v(u) > \gamma$ for every $u \in U$. Let I be an ideal of definition of A. Let C be the ring B equipped with the $I \cdot B$-adic topology. Then $\operatorname{Spec} \hat{B} \to \operatorname{Spec} \hat{A}$ factorizes in $\operatorname{Spec} \hat{B} \xrightarrow{j} \operatorname{Spec} \hat{C} \xrightarrow{k} \operatorname{Spec} \hat{A}$. Since j is flat, it suffices to prove that k is universally generalizing at $\hat{\mathfrak{p}} \cap \hat{C}$. This means we may assume that $I \cdot B$ is an ideal of definition of B. In the following we equip every A-algebra E with the $I \cdot E$-adic topology. Let $\mathfrak{q}$ be a prime ideal of $\hat{B}[X] := \hat{B}[X_1, \ldots, X_n$ with $\mathfrak{q} \cap B = \hat{\mathfrak{p}}$. By (4.2.3 i), it suffices to show that $\operatorname{Spec} \hat{B}[X] \to \operatorname{Spec} \hat{A}[X]$ is generalizing at $\mathfrak{q}$. Let $\mathfrak{r} \in \operatorname{Spec} \hat{A}[X]$ be a proper generalization of $\mathfrak{q} \cap \hat{A}[X]$. We have to show that there exists a prime ideal $\mathfrak{s}$ of $\hat{B}[X]$ with $\mathfrak{s} \subseteq \mathfrak{q}$ and $\mathfrak{s} \cap \hat{A}[X] = \mathfrak{r}$. By [**EGA***, 0.6.5.8], there exists a rank 1 valuation $\hat{v}$ of $\hat{A}[X]$ such that $\operatorname{supp}(\hat{v}) = \mathfrak{r}$ and $\hat{A}[X](\hat{v})$ dominates the localization of $\hat{A}[X]/\mathfrak{r}$ at the prime ideal $(\mathfrak{q} \cap \hat{A}[X])/\mathfrak{r}$. Then $\hat{v}$ is continuous and the trivial valuation of $A[X]$ with support $\mathfrak{q} \cap A[X]$ is a primary specialization of $v := \hat{v}|A[X]$ in $\operatorname{Spv} A[X]$. By (2.1.3) there exists a valuation w of $B[X]$ such that the trivial valuation of $B[X]$ with support $\mathfrak{q} \cap B[X]$ is a primary specialization of w in $\operatorname{Spv} B[X]$ and $w|A[X] = v$. Let H be the convex hull of Γ_v in Γ_w. Since $c\Gamma_w = \{0\}$, we have the valuation $u := w|H$ of $B[X]$. Then u is continuous, $u|A[X] = v$ and $u|c\Gamma_u$ is the trivial valuation of $B[X]$ with support $\mathfrak{q} \cap B[X]$. Since the ring homomorphism $\varphi \colon B[X] \to \hat{B}[X]$ is continuous and $\operatorname{im}(\varphi)$ is dense in $\hat{B}[X]$, u extends to a continuous valuation $\hat{u}$ of $\hat{B}[X]$. Then $\hat{u}|c\Gamma_{\hat{u}}$ lies over $u|c\Gamma_u$. Since $\mathfrak{q}$ is the unique prime ideal of $\hat{B}[X]$ lying over $\mathfrak{q} \cap B[X]$, we have $\operatorname{supp}(\hat{u}) \subseteq \operatorname{supp}(\hat{u}|c\Gamma_{\hat{u}}) = \mathfrak{q}$. Since $\hat{v}$ and $\hat{u}|\hat{A}[X]$ are continuous valuations of $\hat{A}[X]$ and $\hat{v}|A[X] = (\hat{u}|\hat{A}[X])|A[X]$, we have $\hat{v} = \hat{u}|\hat{A}[X]$, especially $\mathfrak{r} = \operatorname{supp}(\hat{v}) = \operatorname{supp}(\hat{u}|\hat{A}[X]) = \operatorname{supp}(\hat{u}) \cap \hat{A}[X]$. Hence $\operatorname{supp}(\hat{u})$ is a prime ideal as desired. $\square$

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