Baryon Operators of Higher Twist in QCD and Nucleon Distribution Amplitudes

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Abstract:

We develop a general theoretical framework for the description of higher–twist baryon operators which makes maximal use of the conformal symmetry of the QCD Lagrangian. The conformal operator basis is constructed for all twists. The complete analysis of the one-loop renormalization of twist-4 operators is given. The evolution equation for three-quark operators of the same chirality turns out to be completely integrable. The spectrum of anomalous dimensions coincides in this case with the energy spectrum of the twist-4 subsector of the $SU(2,2)$ Heisenberg spin chain. The results are applied to give a general classification and calculate the scale dependence of subleading twist-4 nucleon distribution amplitudes that are relevant for hard exclusive reactions involving a helicity flip. In particular we find an all-order expression (in conformal spin) for the contributions of geometric twist-3 operators to the (light-cone) twist-4 nucleon distribution amplitudes, which are usually referred to as Wandzura–Wilczek terms.

Contents

1 Introduction

Higher-twist effects in hard processes in QCD generically correspond to corrections to physical observables that are suppressed by powers of the hard scale. They are important in order to achieve high accuracy, and interesting because higher-twist corrections are sensitive to fine details of the hadron structure. A theoretical description of higher twist effects within QCD factorization involves contributions of a large number of local operators which are much more numerous compared to the leading twist so that the choice of a proper operator basis is important. This choice is not unique, as exemplified by the two existing classical approaches to the twist-4 effects in deep-inelastic lepton hadron scattering [1, 2]. The "transverse" basis of Ref. [2] leads to simpler coefficient functions whereas the "longitudinal" basis of Ref. [1] (see also [3]) allows for a parton-model-like interpretation [4].

The renormalization of higher-twist operators corresponds to the scale dependence of the physical observables. For twist three, the corresponding study is essentially completed. The anomalous dimension matrix for baryon operators was first calculated in [5], for chiral-even quark-antiquark-gluon and three-gluon operators in [6], and for chiral-odd in [7]. The structure of the spectrum of twist-three anomalous dimensions is well understood [8, 9, 10, 11, 12, 13, 14, 15] and in some cases explicit WKB-type expansions are available that allow to calculate anomalous dimensions to arbitrary accuracy: The size of the mixing matrix plays the role of the expansion parameter. Beyond twist three much less is known. Up to now, anomalous dimensions have only been calculated for a few operators of lowest dimension (e.g. [16, 17]). In addition, the structure of the most singular parts of the mixing kernels for small values of the Bjorken variable that are relevant for the contribution of two-pomeron cuts in high-energy scattering processes was considered in [18, 19].

The modern approach for the calculation of leading-order anomalous dimensions of higher-twist operators makes maximal use of the conformal symmetry of the renormalization group equations. Historically, the importance of conformal symmetry in the present context was first understood for the leading twist pion distribution amplitude and it was instrumental for the proof of QCD factorization for the pion form factor [20, 21, 22]. A general formalism was developed in [23] for the special class of so-called quasipartonic operators that are built of "plus" components of quark and gluon fields. For each twist, the set of quasipartonic operators is closed under renormalization and the renormalization group (RG) equation can be written in a Hamiltonian form that involves two-particle kernels given in terms of two-particle Casimir operators of the collinear subgroup $SL(2,\mathbb{R})$ of the conformal group. In this formulation symmetries of the RG equations become explicit. Moreover, for a few important cases the corresponding three-particle quantum-mechanical problem turns out to be completely integrable and in fact equivalent to a specific Heisenberg spin chain [8]. An almost complete understanding achieved at present of the renormalization of twist-three operators is due to all these formal developments, see [24, 25] for a review and further references.

The goal of this paper is to generalize some of the above techniques to the situation where not all contributing operators are quasipartonic, as it proves to be the case starting with twist four. Apart of the needs of practical applications to QCD phenomenology, our work is fuelled by the recent study [26, 27] where it was shown that diagonal part of one-loop QCD RG equations (for arbitrary twist) can be written in a Hamiltonian form in terms of quadratic Casimir operators of the full conformal group $SO(4,2)$ instead of its collinear subgroup. Moreover, all kernels can be obtained from the known kernels for the collinear $SL(2,\mathbb{R})$ subgroup [23]. Although much of the formalism appears to be general, in this paper we concentrate on the simplest example of non-quasipartonic twist-four baryon operators that contain two "plus" and one "minus" quark field, schematically

$$q_+q_-q_+\,,$$

and their mixing with (quasipartonic) four-particle operators involving a gluon field, of the type

$$q_+q_+q_+F_{+\perp}.$$

Our main results can be summarized as follows.

First, we construct a complete conformal operator basis for arbitrary twist, with "good" transformation properties. We then specialize to the case of twist-4 baryonic operators, calculate all one-loop evolution kernels including the mixing with four-particle operators involving a gluon field, and check that the kernels are $SL(2)$ invariant, as expected. The operators involving three quark fields with the same chirality do not mix with the operators involving both chiral and antichiral quarks, so that these two cases can be considered separately. The evolution equation for three-quark operators of the same chirality turns out to be completely integrable. The spectrum of anomalous dimensions coincides in this case with the energy spectrum of the twist-4 subsector of the $SU(2,2)$ Heisenberg spin chain, confirming the prediction of [27]. For both cases, we present a detailed study of the spectra of the anomalous dimensions. Finally, these results are applied to give a general classification and calculate the scale dependence of subleading twist-4 nucleon distribution amplitudes that are relevant for hard exclusive reactions involving a helicity flip. In particular we introduce novel four-particle distribution amplitudes involving a gluon field, and derive explicit expressions for the expansion of all distribution amplitudes in contributions of multiplicatively renormalizable operators in first three orders of the conformal expansion. As a byproduct of our analysis, we give an expression for the contributions of geometric twist-3 operators to the (light-cone) twist-4 nucleon distribution amplitudes, which are usually referred to as Wandzura–Wilczek terms.

The presentation is organized as follows. We begin in Sect. 2 with a short exposition of the spinor formalism that is used throughout our work. This formalism is standard in the studies of SUSY theories but is used rarely by the QCD community so we felt that a short summary is necessary. Next, conformal transformation properties of the fields are considered in some detail. A complete basis of one-particle light-ray operators is constructed for chiral quark and self-dual gluon fields in QCD, cf. Eq. (2.59), which is one of our main results. In Sect. 3 we specialize to the particular case of baryonic twist-4 operators which are the main subject of the rest of the paper. Renormalization group equations for the light-ray baryonic operators are derived in Sect. 4. We discuss general properties of these equations, give a summary of the relevant conformal invariant evolution kernels, introduce a convenient scalar product on the space of the solutions and, finally, give explicit expressions of the Hamiltonians for all cases of interest. Solutions of the renormalization group equations for twist-4 operators are considered in Sect. 5. For three-quark operators of the same chirality the problem turns out to be completely integrable. We find the corresponding conserved charge and discuss the relation of this result to the approach of [26, 27]. A simple analytic expression is found for the lowest anomalous dimension in the spectrum of chiral quark twist-4 operators with odd number $N = 2k+1$ of covariant derivatives. For other cases the spectra are studied numerically. The results are presented in the Figures and for the first few N also in table form. It turns out that differences between twist-4 and twist-3 operators mostly affect a few lowest eigenstates (for a given N); the upper part of the spectrum is universal: the anomalous dimensions appear to be almost independent on twist and chirality. Explicit expressions for the nucleon distribution amplitudes taking into account first three orders in conformal spin and Wandzura-Wilczek corrections are given in Sect. 6. The final Sect. 7 is reserved for summary and conclusions.

2 Spinors and Conformal Symmetry

For applications it is important to have an operator basis with good transformation properties with respect to the collinear $SL(2,\mathbb{R})$ subgroup of the conformal group. It is well known that analysis of tensor properties of operators is greatly simplified in the spinor representation. Although this formalism is standard, a number of different prescriptions exist in the literature for raising and lowering indices, normalization etc. In order to make our presentation self-contained we choose to begin with a summary of the definitions and basic relations of the spinor algebra, and also introduce some general notation that is used throughout the paper. Our conventions are similar but not identical to the ones accepted in Ref. [28].

2.1 Spinor formalism

The Lorentz group $SO(3,1)$ is locally isomorphic to the group of complex unimodular 2×2 matrices, $SL(2,\mathbb{C})$. To make this explicit, each covariant four-vector x_μ can be mapped to a hermitian matrix x

$$x = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \equiv x_\mu \sigma^\mu \tag{2.1}$$

where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\vec{\sigma}$ are the usual Pauli matrices. A Lorentz transformation $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ corresponds to a rotation $x' = MxM^\dagger$, where $M \in SL(2,\mathbb{C})$, and the homomorphism $\Lambda \to M$ defines a two-dimensional (spinor) representation of Lorentz group, $u' = Mu$. The correspondence between Λ and M is not unique and in general one might consider four representations defined by the homomorphisms $\Lambda \to M, M^*, M^{-1,T}$ and $M^{-1\dagger}$. The vectors from the corresponding representation spaces – spinors – are usually denoted as $u_\alpha, \bar{u}_{\dot\alpha}, u^\alpha$ and $\bar{u}^{\dot\alpha}$, respectively, i.e. $u'_\alpha = M_\alpha{}^\beta u_\beta$, $\bar{u}'_{\dot\alpha} = M^{*\dot\beta}_{\dot\alpha}\,\bar{u}_{\dot\beta}$ etc. The representations M and $M^{-1,T}$ (also M^* and $M^{-1,\dagger}$) are equivalent since $\sigma_2 M = M^{-1,T}\sigma_2$. The intertwining operator σ_2 is proportional to the Levi-Civita tensor ϵ. We define

$$\epsilon_{12} = \epsilon^{12} = -\epsilon_{\dot1\dot2} = -\epsilon^{\dot1\dot2} = 1 \tag{2.2}$$

and accept the following rule for raising and lowering of spinor indices (cf. [28])

$$u^\alpha = \epsilon^{\alpha\beta}u_\beta\,, \qquad u_\alpha = u^\beta\epsilon_{\beta\alpha}\,, \qquad \bar{u}^{\dot\alpha} = \bar{u}_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}\,, \qquad \bar{u}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{u}^{\dot\beta}\,, \tag{2.3}$$

which is consistent with (2.2). Note that $\epsilon_\alpha{}^\beta = -\epsilon^\beta{}_\alpha = \delta_\alpha^\beta$ and $\epsilon^{\dot\alpha}{}_{\dot\beta} = -\epsilon_{\dot\beta}{}^{\dot\alpha} = \delta^{\dot\alpha}_{\dot\beta}$.

When it is not displayed explicitly it is implied that undotted indices are contracted "up–down", $(uv) \overset{\text{def}}{=} u^\alpha v_\alpha = -u_\alpha v^\alpha$ and dotted ones "down–up", $(\bar{u}\bar{v}) \overset{\text{def}}{=} \bar{u}_{\dot\alpha}\bar{v}^{\dot\alpha} = -\bar{u}^{\dot\alpha}\bar{v}_{\dot\alpha}$

Next, we define $(u_\alpha)^* = \bar{u}_{\dot\alpha}$ and $(u^\alpha)^* = \bar{u}^{\dot\alpha}$ that is, again, consistent with (2.2) and results in $(uv)^* = (\bar{v}\bar{u})$. The Fierz transformation for Weyl spinors reads

$$(u_1u_2)(v_1v_2) = (u_1v_1)(u_2v_2) - (u_1v_2)(u_2v_1) \tag{2.4}$$

which is a consequence of the identity

$$\epsilon_{ab}\epsilon_{cd} = \epsilon_{ac}\epsilon_{bd} - \epsilon_{ad}\epsilon_{bc} \tag{2.5}$$

In addition to $\sigma^\mu_{\alpha\dot\beta} = (\mathbb{1}, \vec{\sigma})$ it is convenient to introduce $(\bar\sigma^\mu)^{\dot\alpha\beta} = (\mathbb{1}, -\vec{\sigma})$ so that $(\bar\sigma^\mu)^{\dot\alpha\beta} = (\sigma^\mu)^{\beta\dot\alpha}$, and define $\bar{x} = x_\mu\bar\sigma^\mu$, cf. (2.1). One easily finds that

$$a_\mu = \frac{1}{2}(a\bar\sigma_\mu)_\alpha{}^\alpha = \frac{1}{2}(\bar{a}\sigma_\mu)^{\dot\alpha}{}_{\dot\alpha}\,, \qquad a_\mu b^\mu = \frac{1}{2}a_{\alpha\dot\alpha}\bar{b}^{\dot\alpha\alpha}\,.$$

For completeness, we give below some useful identities involving σ_μ matrices:

$$\sigma^\mu_{\alpha\dot\alpha}\,(\bar\sigma^\nu)^{\dot\alpha\alpha} = 2g^{\mu\nu}\,, \qquad \sigma^\mu_{\alpha\dot\alpha}\,\bar\sigma^{\dot\beta\beta}_\mu = 2\delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta}\,, \tag{2.6}$$

$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta\,, \qquad (\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta}\,. \tag{2.7}$$

Generators of the Lorentz group read

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{2}[\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu]_\alpha{}^\beta\,, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}[\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta}\,, \tag{2.8}$$

or, in the explicit form

$$\sigma^{0i} = -i\sigma^i \,, \qquad \sigma^{ik} = i\epsilon^{ikj}\sigma^j \,, \qquad \bar\sigma^{0i} = i\sigma^i \,, \qquad \bar\sigma^{ik} = i\epsilon^{ikj}\sigma^j \,, \tag{2.9}$$

They satisfy the self-duality relations

$$\sigma^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\rho\omega}\sigma_{\rho\omega} \,, \qquad \bar\sigma^{\mu\nu} = -\frac{i}{2}\varepsilon^{\mu\nu\rho\omega}\bar\sigma_{\rho\omega} \,. \tag{2.10}$$

where $\epsilon_{0123} = 1$.

A four-dimensional Dirac bispinor is written as

$$q = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix} \,, \qquad \bar q = (\chi^\beta, \bar\psi_{\dot\alpha}) \tag{2.11}$$

and the γ_μ matrices take the form

$$\gamma^\mu = \begin{pmatrix} 0 & [\sigma^\mu]_{\alpha\dot\beta} \\ [\bar\sigma^\mu]^{\dot\alpha\beta} & 0 \end{pmatrix} \,, \qquad \not{a} = \begin{pmatrix} 0 & a_{\alpha\dot\beta} \\ \bar a^{\dot\alpha\beta} & 0 \end{pmatrix} \,. \tag{2.12}$$

For the common $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the charge conjugation matrix $C = i\gamma^2\gamma^0$ one finds

$$\sigma^{\mu\nu} = \begin{pmatrix} [\sigma^{\mu\nu}]_\alpha{}^\beta & 0 \\ 0 & [\bar\sigma^{\mu\nu}]^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \,, \qquad \gamma_5 = \begin{pmatrix} -\delta_\alpha^\beta & 0 \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix} \,, \qquad C = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \,. \tag{2.13}$$

Irreducible representations of the Lorentz group are labeled by two spins $(s, \bar s)$. The representation space is spanned by tensors $T_{\alpha_1\ldots\alpha_{2s},\dot\beta_1\ldots\dot\beta_{2\bar s}}$ which are symmetric in dotted and undotted indices separately. In particularly, the Weyl spinors ψ (chiral) and $\bar\chi$ (antichiral) belong to the representations $(1/2, 0)$ and $(0, 1/2)$, respectively, whereas the Dirac spinor transforms as $(1/2, 0) \oplus (0, 1/2)$.

The gluon strength tensor $F_{\mu\nu}$ transforms as $(1, 0) \oplus (0, 1)$ and can be decomposed as

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}\right) \tag{2.14}$$

where $f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ are chiral and antichiral symmetric tensors, $f^* = \bar f$, which belong to the representations $(1, 0)$ and $(0, 1)$, respectively. One obtains

$$f_{\alpha\beta} = \frac{i}{4}\sigma^{\mu\nu}_{\alpha\beta}F_{\mu\nu} \,, \qquad \bar f_{\dot\alpha\dot\beta} = -\frac{i}{4}\bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}F_{\mu\nu} \,, . \tag{2.15}$$

or in terms of the gauge field $A_{\dot\alpha\alpha}$:

$$f_{\alpha\beta} = \frac{1}{4}\left(D_\alpha{}^{\dot\alpha}\bar A_{\dot\alpha\beta} + D_\beta{}^{\dot\alpha}\bar A_{\dot\alpha\alpha}\right) \,, \qquad \bar f_{\dot\alpha\dot\beta} = \frac{1}{4}\left(\bar D_{\dot\alpha}{}^\alpha A_{\alpha\dot\beta} + \bar D_{\dot\beta}{}^\alpha A_{\alpha\dot\alpha}\right) \,, \tag{2.16}$$

where the covariant derivative is defined as $D_\mu = \partial_\mu - igA_\mu$. The expressions for $F^{\mu\nu}$ and the dual strength tensor $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ are

$$F^{\mu\nu} = \frac{i}{2}\left(\sigma^{\mu\nu}_{\alpha\beta}f^{\alpha\beta} - \bar{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta}\bar{f}^{\dot\alpha\dot\beta}\right), \qquad \widetilde{F}^{\mu\nu} = \frac{1}{2}\left(\sigma^{\mu\nu}_{\alpha\beta}f^{\alpha\beta} + \bar{\sigma}^{\mu\nu}_{\dot\alpha\dot\beta}\bar{f}^{\dot\alpha\dot\beta}\right). \tag{2.17}$$

The Dirac equation for the quark fields reads

$$\bar{D}^{\dot\alpha\alpha}\psi_\alpha(x) = 0\,, \qquad D_{\alpha\dot\alpha}\bar{\chi}^{\dot\alpha}(x) = 0 \tag{2.18}$$

where the covariant derivative is defined as $D_\mu = \partial_\mu - igA_\mu$. The equation of motion (EOM) for the fields $f, \bar{f}$ becomes

$$\bar{D}_{\dot\beta}{}^{\alpha} f^a_{\alpha\beta} = g\left(\bar{\psi}_{\dot\beta}T^a\psi_\beta + \chi_\beta T^a\bar{\chi}_{\dot\beta}\right), \qquad D_\beta{}^{\dot\alpha}\bar{f}^a_{\dot\alpha\dot\beta} = g\left(\bar{\psi}_{\dot\beta}T^a\psi_\beta + \chi_\beta T^a\bar{\chi}_{\dot\beta}\right). \tag{2.19}$$

The class of the operators which are proportional to the equation of motion is closed under renormalization (for a more precise statement see e.g. Ref. [29]). On-shell matrix elements of such operators vanish and one can consider two operators which difference is an EOM operator as being equivalent.

The equation

$$T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_n} = \sigma^{\mu_1}_{\alpha_1\dot\beta_1}\ldots\sigma^{\mu_n}_{\alpha_n\dot\beta_n}T_{\mu_1\ldots\mu_n} \tag{2.20}$$

establishes the relation between generic tensors in the usual vector and spinor representations. The symmetrization over spinor indices is most conveniently achieved contracting the open indices with an auxiliary spinor ξ. We define

$$T_\xi = \xi_1^\alpha\ldots\xi^{\alpha_n}\,T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}}\,\bar{\xi}^{\dot\beta_1}\ldots\bar{\xi}^{\dot\beta_{\bar n}}\,. \tag{2.21}$$

In particular

$$\psi_\xi = (\xi\psi) = \xi^\alpha\psi_\alpha \qquad f_\xi = \xi^\alpha\xi^\beta f_{\alpha\beta}\,,$$
$$\bar{\chi}_\xi = (\bar{\chi}\bar{\xi}) = \bar{\chi}_{\dot\alpha}\bar{\xi}^{\dot\alpha} \qquad \bar{f}_\xi = \bar{f}_{\dot\alpha\dot\beta}\bar{\xi}^{\dot\alpha}\bar{\xi}^{\dot\beta}\,, \tag{2.22}$$

etc.

It is obvious that a symmetric tensor $T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}}$ can unambiguously be restored from the convolution T_ξ by applying multiple derivatives over ξ. We define

$$\partial_\beta\xi^\alpha = \frac{\partial}{\partial\xi^\beta}\xi^\alpha = \epsilon_\beta{}^\alpha = \delta^\alpha_\beta\,, \qquad \bar{\partial}^{\dot\beta}\bar{\xi}_{\dot\alpha} = \frac{\partial}{\partial\bar{\xi}_{\dot\beta}}\bar{\xi}_{\dot\alpha} = \epsilon^{\dot\beta}{}_{\dot\alpha} = \delta^{\dot\beta}_{\dot\alpha}\,, \tag{2.23}$$

so that

$$T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}} = \frac{(-1)^{\bar n}}{n!\,\bar{n}!}\frac{\partial}{\partial\xi^{\alpha_1}}\ldots\frac{\partial}{\partial\xi^{\alpha_n}}\frac{\partial}{\partial\bar{\xi}^{\dot\beta_1}}\ldots\frac{\partial}{\partial\bar{\xi}^{\dot\beta_{\bar n}}}T_\xi\,. \tag{2.24}$$

Note that the rule for raising and lowering of indices for derivatives over spinor variables is different from that for the spinors themselves, cf. Eq. (2.3):

$$\frac{\partial}{\partial\xi^\beta} = \epsilon_{\beta\alpha}\frac{\partial}{\partial\xi_\alpha}\,, \qquad \frac{\partial}{\partial\bar{\xi}_{\dot\beta}} = \epsilon^{\dot\beta\dot\alpha}\frac{\partial}{\partial\bar{\xi}^{\dot\alpha}}\,. \tag{2.25}$$

2.2 Conformal symmetry

It is known that the QCD enjoys conformal symmetry at the classical level. Although this symmetry is broken in the full quantum theory, it leads to strong constraints on the form of (one-loop) operator counterterms and will be quite useful in the subsequent analysis. The action on the generators of the conformal group on the fundamental fields in the spinor representation, $\Phi = (\Phi_\xi, \bar{\Phi}_\xi)$ with $\Phi_\xi = \{\psi_\xi, \chi_\xi, f_\xi\}$ and $\bar{\Phi}_\xi = \{\bar{\psi}_\xi, \bar{\chi}_\xi, \bar{f}_\xi\}$, takes the form [30]

$$
\begin{aligned}
i[\mathbf{P}_{\alpha\dot\alpha}, \Phi(x)] =& \partial_{\alpha\dot\alpha}\Phi(x) \equiv iP_{\alpha\dot\alpha}\Phi(x)\,, && \text{(2.26a)}\\
i[\mathbf{D}, \Phi(x)] =& \frac{1}{2}\left(x_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} + 2t + \xi^\alpha\frac{\partial}{\partial\xi^\alpha} + \bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\xi_{\dot\alpha}}\right)\Phi(x) \equiv iD\,\Phi(x)\,, && \text{(2.26b)}\\
i[\mathbf{M}_{\alpha\beta}, \Phi(x)] =& \frac{1}{4}\left(x_{\alpha\dot\gamma}\partial_\beta{}^{\dot\gamma} + x_{\beta\dot\gamma}\partial_\alpha{}^{\dot\gamma} - 2\xi_\alpha\frac{\partial}{\partial\xi^\beta} - 2\xi_\beta\frac{\partial}{\partial\xi^\alpha}\right)\Phi(x) \equiv iM_{\alpha\beta}\Phi(x)\,, && \text{(2.26c)}\\
i[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \Phi(x)] =& \frac{1}{4}\left(x_{\gamma\dot\alpha}\partial^\gamma{}_{\dot\beta} + x_{\gamma\dot\beta}\partial^\gamma{}_{\dot\alpha} - 2\bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\xi^{\dot\beta}} - 2\bar\xi_{\dot\beta}\frac{\partial}{\partial\bar\xi^{\dot\alpha}}\right)\Phi(x) \equiv i\bar{M}_{\dot\alpha\dot\beta}\Phi(x)\,, && \text{(2.26d)}\\
i[\mathbf{K}_{\alpha\dot\alpha}, \Phi(x)] =& \left(x_{\alpha\dot\gamma}x_{\gamma\dot\alpha}\,\partial^{\gamma\dot\gamma} + 2tx_{\alpha\dot\alpha} + 2\xi_\alpha \bar{x}_{\dot\alpha}{}^\beta\frac{\partial}{\partial\xi^\beta} + 2\bar\xi_{\dot\alpha}x_{\alpha\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}}\right)\Phi(x) \equiv iK_{\alpha\dot\alpha}\Phi(x)\,, && \text{(2.26e)}
\end{aligned}
$$

where $\partial_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ and $t = 1$ is the *geometric* twist [31]: for the field with canonical scaling dimension $\ell^{\rm can}$ and Lorentz spin $(s, \bar{s})$ it is defined as $t = \ell^{\rm can} - s - \bar{s}$. Note that we use boldface letters for the generators acting on quantum fields to distinguish them from the corresponding differential operators acting on the field coordinates. The transformations of the gauge field $A_\xi = A_{\alpha\dot\alpha}\xi^\alpha\bar\xi^{\dot\alpha}$ are given by the same expressions with $t = 0$.

In the applications of QCD to high-energy scattering the separation of transverse and longitudinal degrees of freedom proves to be essential. It is conveniently achieved by the introduction of two independent light-like vectors

$$
\begin{aligned}
n_{\alpha\dot\alpha} &= \lambda_\alpha\bar\lambda_{\dot\alpha}\,, \qquad n^2 = 0\,,\\
\tilde{n}_{\alpha\dot\alpha} &= \mu_\alpha\bar\mu_{\dot\alpha}\,, \qquad \tilde{n}^2 = 0\,,
\end{aligned} \tag{2.27}
$$

which we choose to be normalized to

$$
(\mu\lambda) = -(\lambda\mu) = 1\,, \qquad (n\cdot\tilde{n}) = 1/2\,. \tag{2.28}
$$

Without loss of generality one can take

$$
\begin{aligned}
\lambda^\alpha &= (1,0)\,, \qquad \lambda_\alpha = (0,1)\,,\\
\mu^\alpha &= (0,1)\,, \qquad \mu_\alpha = (-1,0)\,.
\end{aligned} \tag{2.29}
$$

Then, for example

$$\partial^{2\dot{2}} = 2(n\cdot\partial)\,, \qquad\qquad \partial^{1\dot{1}} = 2(\tilde{n}\cdot\partial) \tag{2.30}$$

are the derivatives in the two chosen light-like directions whereas the remaining two, $\partial^{1\dot{2}}$ and $\partial^{2\dot{1}}$, are the derivatives in the transverse plane.

Fast moving hadrons can be viewed as a collection of partons that move in the same direction, say $\tilde{n}_\mu$. Whenever this picture applies, quantum fields "living" on the light ray

$$\Phi(x) \to \Phi(zn) \tag{2.31}$$

play a special role. Such light-ray fields can be viewed as generating functions for local operators that arise through the (formal) Taylor expansion

$$\Phi(z) \equiv \Phi(zn) = \sum_k \frac{z^k}{k!}(n\partial)^k\Phi(0) = \sum_k \frac{z^k}{2^k k!}(\partial^{2\dot{2}})^k\Phi(0)\,. \tag{2.32}$$

Note that all local operators on the r.h.s. of (2.32) have the same collinear twist as the field Φ itself since each $\partial^{2\dot{2}}$ derivative adds one unit of dimension and spin projection, simultaneously. We will use a shorthand notation $\Phi(z)$ for $\Phi(nz)$ in what follows.

With the restriction to light-ray operators the four-dimensional conformal transformations are reduced to the collinear subgroup $SL(2,\mathbb{R})$ corresponding to projective (Möbius) transformations of the line $x = zn$:

$$z \to \frac{az+b}{cz+d}\,, \qquad ab - cd = 1\,,$$

where a, b, c, d are real numbers. The generators of the collinear subgroup, $S_\pm, S_0$ can be chosen as

$$S_+ = \frac{i}{2}(\mu\, K\, \bar{\mu})\,, \quad S_- = -\frac{i}{2}(\lambda\, P\, \bar{\lambda})\,, \quad S_0 = \frac{i}{2}\Big(D - \mu^\alpha\lambda^\beta M_{\alpha\beta} - \bar{M}_{\dot{\alpha}\dot{\beta}}\bar{\mu}^{\dot{\alpha}}\bar{\lambda}^{\dot{\beta}}\Big)\,, \tag{2.33}$$

or, using the convention in Eq. (2.29),

$$S_+ = \frac{i}{2}K_{2\dot{2}} = \frac{i}{2}K^{1\dot{1}}\,, \quad S_- = -\frac{i}{2}P^{2\dot{2}} = -\frac{i}{2}P_{1\dot{1}}\,, \quad S_0 = \frac{i}{2}\Big(D - M_{21} - \bar{M}_{\dot{1}\dot{2}}\Big)\,. \tag{2.34}$$

The explicit expressions are

$$S_+ = \frac{1}{2}x_{2\dot{\gamma}}x_{\gamma\dot{2}}\partial^{\gamma\dot{\gamma}} + x_{2\dot{2}}\left(t + \xi^\beta\frac{\partial}{\partial\xi^\beta} + \bar{\xi}^{\dot{\beta}}\frac{\partial}{\partial\bar{\xi}^{\dot{\beta}}}\right) - x_{\beta\dot{2}}\,\xi^\beta\frac{\partial}{\partial\xi^2} - x_{2\dot{\beta}}\,\bar{\xi}^{\dot{\beta}}\frac{\partial}{\partial\bar{\xi}^{\dot{2}}}\,, \tag{2.35a}$$

$$S_- = -\frac{1}{2}\partial^{2\dot{2}}\,, \tag{2.35b}$$

$$S_0 = \frac{1}{2}\left(x_{2\dot{2}}\partial^{2\dot{2}} + \frac{1}{2}\left(x_{2\dot{1}}\partial^{2\dot{1}} + x_{1\dot{2}}\partial^{1\dot{2}}\right) + t + \xi^1\frac{\partial}{\partial\xi^1} + \bar{\xi}^{\dot{1}}\frac{\partial}{\partial\bar{\xi}^{\dot{1}}}\right)\,. \tag{2.35c}$$

They obey the standard commutation relations

$$[S_+, S_-] = 2S_0\,, \qquad\qquad [S_0, S_\pm] = \pm S_\pm\,. \tag{2.36}$$

In addition, there exist two operators that commute with all $SL(2,\mathbb{R})$ generators:

$$E = i\Big(D + M_{21} + \bar{M}_{\dot{1}\dot{2}}\Big) = x_{1\dot{1}}\partial^{1\dot{1}} + \frac{1}{2}\Big(x_{2\dot{1}}\partial^{2\dot{1}} + x_{1\dot{2}}\partial^{1\dot{2}} + 2t\Big) + \xi^2\frac{\partial}{\partial\xi^2} + \bar{\xi}^{\dot{2}}\frac{\partial}{\partial\bar{\xi}^{\dot{2}}}\,, \tag{2.37}$$

$$H = i(\bar{M}_{\dot{1}\dot{2}} - M_{21}) = \frac{1}{2}\left(x_{2\dot{1}}\partial^{2\dot{1}} - x_{1\dot{2}}\partial^{1\dot{2}} + \xi^1\frac{\partial}{\partial\xi^1} - \xi^2\frac{\partial}{\partial\xi^2} - \bar{\xi}^{\dot{1}}\frac{\partial}{\partial\bar{\xi}^{\dot{1}}} + \bar{\xi}^{\dot{2}}\frac{\partial}{\partial\bar{\xi}^{\dot{2}}}\right). \tag{2.38}$$

E is usually called the collinear twist operator: collinear twist E counts the dimension of the field minus spin projection, as opposed to the geometric twist t which is dimension minus spin. In a slight abuse of language we will refer to H as the helicity operator; the name can be justified by observing that for "good" components of the fields (see below) the eigenvalue of H coincides with helicity of the corresponding one-particle state.

A light-ray operator with definite collinear twist E transforms according to the irreducible representation of the $SL(2,\mathbb{R})$ group with the conformal spin

$$j = \ell^{\rm can} - E/2\,. \tag{2.39}$$

In particular the $SL(2)$ generators acquire their canonical form

$$S_+ = z^2\partial_z + 2jz\,, \qquad S_0 = z\partial_z + j\,, \qquad S_- = -\partial_z\,, \tag{2.40}$$

i.e. first order differential operators acting on functions of the light-cone coordinate z. The finite form of the group transformations is

$$[T^j(g^{-1})\Phi](z) = \frac{1}{(cz+d)^{2j}}\Phi\left(\frac{az+b}{cz+d}\right), \qquad g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}. \tag{2.41}$$

For example, a chiral field ψ should be decomposed as

$$\psi(z) = \lambda\,\psi_-(z) - \mu\,\psi_+(z)\,, \tag{2.42}$$

where

$$\psi_+(z) = \lambda^\alpha\psi_\alpha(z) \equiv \psi_1(z)\,, \quad [E\psi_+](z) = \psi_+(z)\,, \quad [H\psi_+](z) = \frac{1}{2}\psi_+(z)\,,$$
$$\psi_-(z) = \mu^\alpha\psi_\alpha(z) \equiv \psi_2(z)\,, \quad [E\psi_-](z) = 2\psi_-(z)\,, \quad [H\psi_-](z) = -\frac{1}{2}\psi_-(z)\,. \tag{2.43}$$

Note that iM_{21} appearing in (2.37), (2.38) counts the difference in the number of "first" and "second" spinor indices, which is nothing but the Lorentz spin projection on the light-ray direction. In particular ψ_+ and ψ_- correspond to spin projections $+1/2$ and $-1/2$, respectively. Using explicit expressions in Eq. (2.26) it is easy to check that the

	ψ_+	ψ_-	$\bar\chi_+$	$\bar\chi_-$	f_{++}	f_{+-}	f_{--}
j	1	1/2	1	1/2	3/2	1	1/2
E	1	2	1	2	1	2	3
H	1/2	$-1/2$	$-1/2$	1/2	1	0	-1

Table 1: The $SL(2,\mathbb{R})$ spin and twist for the fundamental fields

fields ψ_+ and ψ_- indeed transform according to Eq. (2.40) with the conformal spin $j = 1$ and $j = 1/2$, respectively.

Similarly, for the anti-chiral field $\bar\chi$ we define the "plus" and "minus" projections as

$$\bar\chi_+ = \bar\chi_{\dot\alpha}\bar\lambda^{\dot\alpha} , \qquad \bar\chi_- = \bar\chi_{\dot\alpha}\bar\mu^{\dot\alpha} \tag{2.44}$$

and for the self-dual vector field $f_{\alpha\beta}$

$$f_{++}(z) = \lambda^\alpha\lambda^\beta f_{\alpha\beta}(z) , \qquad f_{+-}(z) = \lambda^\alpha\mu^\beta f_{\alpha\beta}(z) , \qquad f_{--}(z) = \mu^\alpha\mu^\beta f_{\alpha\beta}(z) . \tag{2.45}$$

The projections for the conjugate fields are defined as $\bar\psi_\pm = (\psi_\pm)^*$ etc. The $SL(2,\mathbb{R})$ quantum numbers of the fundamental fields — conformal spin, (collinear) twist and helicity— are collected in Table 1.

The plus components of the fields, $\Phi_+(z) = \Phi_\lambda(zn) = \{\psi_+(z), \chi_+(z), f_{++}(z)\}$, and their anti-chiral counterparts — "good" components in conventional terminology — have the lowest twist. The product of the plus fields taken at the different points on the light-ray

$$\Phi_+(z_1)\bar\Phi_+(z_2)\ldots\Phi_+(z_N)$$

serves as a generating function* for the so-called quasipartonic operators [6]. An operator constructed from N "plus" fields has collinear twist E equal to N which is the lowest possible twist for $N-$particle operators. The set of $N-$particle quasipartonic operators is closed under renormalization at the one-loop level. The renormalization group equation can be reinterpreted as a Schrödinger equation where the scale μ plays the role of time. The corresponding Hamiltonian contains pairwise interactions only and can be written in terms of the two-particle Casimir operators of the collinear conformal group [6].

The light-ray $N-$particle operators containing minus components of the fields, $\{\psi_-, \bar\chi_-, f_{+-}, \ldots\}$, have twist larger than N and provide one with examples of operators that are not quasipartonic. Renormalization of non-quasipartonic operators in QCD has never been studied systematically, to the best of our knowledge. On this way, there are two complications.

First, the number of fields ("particles") is not conserved. To one-loop accuracy, the mixing matrix of operators with a given twist E has a block-triangular structure as the operators with less fields can mix with ones containing more fields but not vice versa. Operators with the maximum possible number of fields $N = E$ are quasipartonic.

*For the moment we ignore the color structure and all issues related to gauge invariance.

Second, operators involving minus field components can mix with operators of the same twist containing minus, $\partial^{1\dot{1}}$, or transverse, $\partial^{1\dot{2}}, \partial^{2\dot{1}}$, derivatives. These operators, therefore, also must be included. The problem is that transverse derivatives generally do not have good transformation properties with respect to the $SL(2,\mathbb{R})$ group. In concrete applications it may be possible to get rid of such operators using EOM and exploiting the specific structure of the matrix elements of interest, e.g. if there is no transverse momentum transfer between the initial and the final state. Two well known examples are the twist-four contributions to the deep-inelastic scattering (DIS) [1] and to meson distributions amplitudes [32, 33]. The main problem as far as the operator renormalization is concerned is that after this reduction conformal symmetry becomes obscured. This procedure is also not universal and probably cannot be applied beyond twist four.

In this work we suggest a different, general approach based on the construction of a complete conformal operator basis for all twists. In this basis, the $SL(2,\mathbb{R})$ symmetry of the renormalization group equations is manifest. To begin with, we will explain our construction on the example of a free chiral field ψ, the extension to the other fields is straightforward.

Let us examine the action of the $SL(2,\mathbb{R})$ generators in Eq. (2.40) on the light-ray operator with a transverse or "minus" derivative, $[\partial^{1\dot{2}}\psi_\pm](z)$, $[\partial^{2\dot{1}}\psi_\pm](z)$ and $[\partial^{1\dot{1}}\psi_\pm](z)$. It is easy to see that S_0 and S_- retain their form, and complications only arise in the case of S_+ which is related to special conformal transformations:

$$i[K_{\alpha\dot{\alpha}}\psi_\beta](x) = \left(x_{\alpha\dot{\gamma}}x_{\gamma\dot{\alpha}}\,\partial^{\gamma\dot{\gamma}} + 4x_{\alpha\dot{\alpha}}\right)\psi_\beta(x) - 2x_{\beta\dot{\alpha}}\psi_\alpha(x)\,, \tag{2.46}$$

cf (2.26). In particular

$$\begin{aligned} i[K_{2\dot{2}}\psi_-](x) &= \left(x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}} + 2x_{2\dot{2}}\right)\psi_-(x)\,,\\ i[K_{2\dot{2}}\psi_+](x) &= \left(x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}} + 4x_{2\dot{2}}\right)\psi_+(x) - 2x_{1\dot{2}}\psi_-(x)\,. \end{aligned} \tag{2.47}$$

The action of the "spin-up" generator $S_+ = iK_{2\dot{2}}/2$ on the light-ray operator with a transverse derivative follows readily from Eq. (2.47) observing that, e.g.

$$\Big[\mathbf{K}_{2\dot{2}}, [\partial^{2\dot{1}}\psi_\pm]\Big](z) \equiv \Big(\partial^{2\dot{1}}\Big[K_{2\dot{2}}\psi_\pm\Big](x)\Big)_{x=zn}$$

and taking into account that $\partial^{\alpha\dot{\alpha}}x_{\beta\dot{\beta}} = 2\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}$ and $x_{2\dot{2}} = z$. One obtains

$$\begin{aligned} S_+[\partial^{2\dot{1}}\psi_+](z) &= (z^2\partial_z + 3z)[\partial^{2\dot{1}}\psi_+](z)\,,\\ S_+[\partial^{1\dot{1}}\psi_+](z) &= (z^2\partial_z + 2z)[\partial^{1\dot{1}}\psi_+](z)\,,\\ S_+[\partial^{1\dot{2}}\psi_+](z) &= (z^2\partial_z + 3z)[\partial^{1\dot{2}}\psi_+](z) - 2\psi_-(z) \end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
S_+[\partial^{2\dot{1}}\psi_-](z) &= (z^2\partial_z + 2z)[\partial^{2\dot{1}}\psi_-](z)\,,\\
S_+[\partial^{1\dot{1}}\psi_-](z) &= (z^2\partial_z + z)[\partial^{1\dot{1}}\psi_-](z)\,,\\
S_+[\partial^{1\dot{2}}\psi_-](z) &= (z^2\partial_z + 2z)[\partial^{1\dot{2}}\psi_-](z)\,.
\end{aligned} \tag{2.49}$$

We see that the generator S_+ take the standard from (2.40) for all cases except for $[\partial^{1\dot{2}}\psi_+](z)$. Fortunately, this derivative can be eliminated with the help of EOM (2.18):

$$\partial^{1\dot{2}}\psi_+ = -\partial^{2\dot{2}}\psi_-\,, \qquad\qquad \partial^{2\dot{1}}\psi_- = -\partial^{1\dot{1}}\psi_+\,. \tag{2.50}$$

The first equation in (2.50) allows to replace all occurrences of $\partial^{1\dot{2}}\psi_+(z)$ by $-\partial^{2\dot{2}}\psi_-(z) = -\partial_z\psi_-(z)$. The second one, in principle, can be used in either direction since $\partial^{2\dot{1}}\psi_-$ and $\partial^{1\dot{1}}\psi_+$ both have "good" transformation properties. It turns out, however, that eliminating $\partial^{2\dot{1}}\psi_-$ in favor of $\partial^{1\dot{1}}\psi_+$ is advantageous since it leads to a simpler complete operator basis in a general situation and we adopt this option for what follows. The remaining four independent operators $\partial^{1\dot{1}}\psi_+$, $\partial^{1\dot{1}}\psi_-$, $\partial^{2\dot{1}}\psi_+$ and $\partial^{1\dot{2}}\psi_-$ transform according to the irreducible representations of the collinear conformal group with spin $j = 1$, $j = 1/2$, $j = 3/2$ and $j = 1$, respectively. Note that the "minus" derivative does not change the conformal spin of the light-ray operator, whereas a "good" transverse derivative increases the spin by 1/2.

The above construction can be generalized for an arbitrary number of derivatives. It is easy to verify that the following fields

$$\begin{aligned}
\psi_+^{(j,m)}(z) =& [(\partial^{2\dot{1}})^{2j-2}(\partial^{1\dot{1}})^{2m}\psi_+](z)\,,\\
\psi_-^{(j,m)}(z) =& [(\partial^{1\dot{2}})^{2j-1}(\partial^{1\dot{1}})^{2m}\psi_-](z)
\end{aligned} \tag{2.51}$$

transform according to the spin-j representation of the $SL(2,\mathbb{R})$ group, Eq. (2.40). All other combinations of derivatives can be reduced to this basis with the help of EOM. In particular, all pairs $\partial^{1\dot{2}}\partial^{2\dot{1}}$ can be replaced by $\partial^{2\dot{2}}\partial^{1\dot{1}}$ which is a consequence of (2.50).

Next, we consider which modifications have to be done in the (interacting) gauge theory. In the first place we have to replace ordinary derivatives by the covariant ones. This is achieved by modifying the definition of the light-ray operator (2.32) to include the factor

$$\Phi(z) \to [0,z]\Phi(z)\,, \tag{2.52}$$

where

$$[0,z] = \mathrm{Pexp}\left[-\frac{1}{2}igz\int_0^1 du\, A^{2\dot{2}}(uz)\right] \tag{2.53}$$

is the light-like Wilson line in the appropriate (fundamental or adjoint) representation of the color group. In this way the Taylor expansion goes over covariant derivatives:

$$[0,z]\Phi(z) = \sum_k \frac{z^k}{2^k k!}(D^{2\dot{2}})^k \Phi(0)\,. \tag{2.54}$$

In what follows the $[0,z]$–factors are not shown for brevity, but they are always implied. Note that dropping the gauge links can be viewed as going over to the Fock-Schwinger gauge $x_{\alpha\dot{\alpha}}A^{\alpha\dot{\alpha}}(x)=0$, $A^{\alpha\dot{\alpha}}(0)=0$, or, alternatively, the light-cone gauge $A^{2\dot{2}}=0$.

In addition, we have to replace ordinary derivatives by covariant ones in Eqs. (2.51). All relations which we have used to reduce an arbitrary combination of derivatives to this particular form hold true up to commutator terms $[D^{\alpha\dot{\alpha}}, D^{\beta\dot{\beta}}]$ which can be expressed in terms of gluon field strength. Such terms contain two or more fundamental light-ray fields and do not affect the proof of completeness in the one-particle sector which we are considering at present.

We still have to check, however, that the replacement $\partial \to D$ does not spoil transformation properties of the basis fields (2.51). The special conformal transformation for the gauge field A takes the form

$$i[K_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}}](x) = x_{\alpha\dot{\gamma}}x_{\gamma\dot{\alpha}}\partial^{\gamma\dot{\gamma}}A_{\beta\dot{\beta}}(x) + 2\left(x_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}}(x) - x_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}(x)\right) - 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(x^{\gamma\dot{\gamma}}A_{\gamma\dot{\gamma}}(x))\,, \tag{2.55}$$

and for the components of interest becomes

$$\begin{aligned}
i[K_{2\dot{2}}A^{1\dot{1}}](x) =& x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}}A^{1\dot{1}}(x)\,,\\
i[K_{2\dot{2}}A^{2\dot{1}}](x) =& \left(x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}} + 2x_{2\dot{2}}\right)A^{2\dot{1}}(x) - 2x^{2\dot{1}}A_{2\dot{2}}(x)\,,\\
i[K_{2\dot{2}}A^{1\dot{2}}](x) =& \left(x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}} + 2x_{2\dot{2}}\right)A^{1\dot{2}}(x) - 2x^{1\dot{2}}A_{2\dot{2}}(x)\,.
\end{aligned} \tag{2.56}$$

Using these expressions it is easy to check (by induction in m and j) that, e.g. for $\psi_+^{j,m}(x) = \left(D^{2\dot{1}}\right)^{2j-2}\left(D^{1\dot{1}}\right)^{2m}\psi_+(x)$, the transformation is

$$i\left[K_{2\dot{2}}, \psi_+^{j,m}\right](x) = \left(x_{2\dot{\gamma}}x_{\gamma\dot{2}}\,\partial^{\gamma\dot{\gamma}} + 2jx_{2\dot{2}}\right)\psi_+^{j,m}(x) + x_{1\dot{2}}\mathcal{G}\,, \tag{2.57}$$

where $\mathcal{G}$ is a light-ray operator containing some combination of the chiral field ψ, gauge field A and derivatives. This inhomogeneous term vanishes on the light-cone, $x = zn$, so that one ends up with

$$S_+\,\psi_+^{j,m}(z) = (z^2\partial_z + 2jz)\psi_+^{j,m}(z)\,, \tag{2.58}$$

as required.

The complete basis of one-particle light-ray operators for chiral quark and self-dual gluon fields in QCD contains seven fields:

$$
\begin{aligned}
\psi_{+}^{(j,m)}(z) &= \big(D^{2\dot 1}\big)^{2j-2}\big(D^{1\dot 1}\big)^{2m}\psi_1(z)\,,\\
\psi_{-}^{(j,m)}(z) &= \big(D^{1\dot 2}\big)^{2j-1}\big(D^{1\dot 1}\big)^{2m}\psi_2(z)\,,\\
\bar\chi_{+}^{(j,m)}(z) &= \big(D^{1\dot 2}\big)^{2j-2}\big(D^{1\dot 1}\big)^{2m}\bar\chi_{\dot 1}(z)\,,\\
\bar\chi_{-}^{(j,m)}(z) &= \big(D^{2\dot 1}\big)^{2j-1}\big(D^{1\dot 1}\big)^{2m}\bar\chi_{\dot 2}(z)\,,\\
f_{++}^{(j,m)}(z) &= \big(D^{2\dot 1}\big)^{2j-3}\big(D^{1\dot 1}\big)^{2m}f_{11}(z)\,,\\
f_{--}^{(j,m)}(z) &= \big(D^{1\dot 2}\big)^{2j-1}\big(D^{1\dot 1}\big)^{2m}f_{22}(z)\,,\\
f_{+-}^{(1,m)}(z) &= \big(D^{1\dot 1}\big)^{2m}f_{12}(z)\,.
\end{aligned}
\tag{2.59}
$$

The field carrying the superscript j transforms according to the representation T^j of the $SL(2,\mathbb{R})$ group, see Eq. (2.41). Note that ordering of the covariant derivatives in (2.59) does not affect the transformation properties. The twist E and helicity H take the following values:

$$
\begin{aligned}
E\,\psi_{\pm}^{(j,m)} &= \big(2j+4m\mp 1\big)\psi_{\pm}^{(j,m)}\,, & E\,\bar\chi_{\pm}^{(j,m)} &= \big(2j+4m\mp 1\big)\bar\chi_{\pm}^{(j,m)}\,,\\
E\,f_{\pm\pm}^{(j,m)} &= \big(2j+4m\mp 2\big)f_{\pm\pm}^{(j,m)}\,, & E\,f_{+-}^{(1,m)} &= \big(2+4m\big)f_{+-}^{(1,m)}\,,
\end{aligned}
\tag{2.60}
$$

$$
\begin{aligned}
H\,\psi_{\pm}^{(j,m)} &= \pm\left(2j-1\mp\frac{1}{2}\right)\psi_{\pm}^{(j,m)}\,, & H\,\bar\chi_{\pm}^{(j,m)} &= \mp\left(2j-1\mp\frac{1}{2}\right)\psi_{\pm}^{(j,m)}\,,\\
H\,f_{\pm,\pm}^{(j,m)} &= \pm\left(2j-1\mp 1\right)f_{\pm,\pm}^{(j,m)}\,, & H\,f_{+-}^{(1,m)} &= 0\,,
\end{aligned}
\tag{2.61}
$$

The basis fields in the antichiral sector can be defined as $\bar\psi_{+}^{(s,m)}=(\psi_{+}^{(s,m)})^*$ and similarly for all other cases.

The proof that the fields in (2.59) form a complete basis in the one-particle sector essentially follows the above discussion of a chiral field. To this end one can consider the derivatives as commuting ones and assume that the fundamental fields satisfy "free" EOM. One has to demonstrate that all possible combinations of derivatives acting on the self-dual strength tensor can be reduced to the combinations appearing in (2.59). This can be achieved by inspection. The first step, as above, is to get rid of all pairs $D^{1\dot 2}D^{2\dot 1}$ replacing them by $D^{2\dot 2}D^{1\dot 1}\to\partial_z D^{1\dot 1}$, and then check that all remaining combinations can be rewritten in the desired form, e.g.

$$
\begin{aligned}
&\big(D^{1\dot 2}\big)^{2k}\big(D^{1\dot 1}\big)^{2m}f_{12}(z)\to-\partial_z\big(D^{1\dot 2}\big)^{2k-1}\big(D^{1\dot 1}\big)^{2m}f_{22}(z)+O(f^2)\\
&\big(D^{2\dot 1}\big)^{2k}\big(D^{1\dot 1}\big)^{2m}f_{12}(z)\to-\big(D^{2\dot 1}\big)^{2k-1}\big(D^{1\dot 1}\big)^{2m+1}f_{22}(z)+O(f^2)\,,
\end{aligned}
\tag{2.62}
$$

etc.

Finally, taking a color-singlet product of the basis fields defined in Eq. (2.59), $\Phi^{j,m} = \{\psi_\pm^{j,m}, \ldots, f_{+-}^{(j=1,m)}\}$, and their antichiral counterparts, $\bar{\Phi}^{j,m}$, at different light-ray positions $z_1, \ldots, z_N$, one obtains a complete basis of gauge-invariant N-particle operators

$$\mathcal{O}(z_1, \ldots, z_N) = \Phi^{j_1,m_1}(z_1) \ldots \Phi^{j_N,m_N}(z_N) \tag{2.63}$$

that transform according to the representation $T^{j_1} \otimes \ldots \otimes T^{j_N}$ of the collinear conformal group $SL(2,\mathbb{R})$ and serve as generating functions for towers of the local operators of twist $E = E_1 + \ldots + E_N$. If $E > N$ then these operators get mixed under renormalization with the operators of the same twist E and the number of fields ranging from N to E. Hence the mixing matrix has a block-triangular form. The anomalous dimensions are determined by the diagonal blocks only, the off-diagonal blocks are, however, important for the construction of multiplicatively renormalizable operators. The premium and main rationale for using the conformal basis (2.59) is that the $SL(2,\mathbb{R})$ symmetry imposes severe constraints on the form of the kernels and also allows one to apply many of the technical tools that were developed earlier for quasipartonic operators. The explicit construction of this basis presents one of the main results of this paper.

Since the maximum light-cone spin projection coincides, obviously, with the Lorentz spin, quasipartonic operators have definite geometric twist $T = E = N$. On the other hand, non-quasipartonic operators contain both $T = E$ contributions and those with a lower twist, $T < E$. Operators with different values of T do not mix. Thus, introducing operators with different geometrical twist would bring the mixing matrix in the block-diagonal form at the cost, however, that the $SL(2,\mathbb{R})$ symmetry of the kernels is lost. A better strategy is to separate the (highest) geometric twist of interest by imposing the appropriate symmetry conditions on the solutions of the renormalization group equation for the operators in (2.63) and maintain the $SL(2,\mathbb{R})$ covariance.

For illustration, let us consider a simple example: renormalization of twist-3 operators that one encounters in the study of chiral odd pion distribution amplitudes [32]. The complete set includes in this case three $E = 3$ light-ray operators

$$\begin{aligned}
\mathcal{O}_1(z_1, z_2) &= \chi_+(z_1)\psi_-(z_2)\,, \\
\mathcal{O}_2(z_1, z_2) &= \chi_-(z_1)\psi_+(z_2)\,, \\
\mathcal{O}_3(z_1, z_2, z_3) &= \chi_+(z_1)\bar{f}_{++}(z_2)\psi_+(z_3)
\end{aligned} \tag{2.64}$$

that transform according to the representations $T^{j=1} \otimes T^{j=1/2}$, $T^{j=1/2} \otimes T^{j=1}$ and $T^{j=1} \otimes T^{j=3/2} \otimes T^{j=1}$ of the collinear conformal group $SL(2,\mathbb{R})$, respectively. The renormalization group equation can be written, schematically, as

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right\} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{pmatrix} = -\frac{\alpha_s}{2\pi} \begin{pmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} & \mathcal{H}_{1,3} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} & \mathcal{H}_{2,3} \\ 0 & 0 & \mathcal{H}_{3,3} \end{pmatrix} \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{pmatrix}\,, \tag{2.65}$$

where the kernels $\mathcal{H}_{2,3}$ have simple $SL(2,\mathbb{R})$ transformation properties. The operator $\mathcal{O}_3$ is quasipartonic with $T = E = 3$, the other two are non-quasipartonic and do not have definite geometric twist.

For example, consider a generic local operator obtained by the Taylor expansion of the light-ray operator $\mathcal{O}_1$

$$\lambda^\alpha\lambda^{\alpha_1}\dots\lambda^{\alpha_k}\,\mu^\beta\,\bar\lambda^{\dot\alpha_1}\dots\bar\lambda^{\dot\alpha_k}\chi_\alpha D_{\alpha_1\dot\alpha_1}\dots D_{\alpha_k\dot\alpha_k}\psi_\beta$$

It is symmetric in all dotted indices, but does not have definite symmetry in the undotted: β can be either symmetrized or antisymmetrized with (either) one of the $\alpha,\alpha_1,\dots\alpha_k$. These two possibilities correspond to picking up contributions of different geometric twist $T=2$ and $T=3$, respectively. Going over to local operators is in fact not necessary as the separation of contributions of different symmetry can be achieved in the nonlocal form:

$$\mathcal{O}_{1,2}^{T=2} = \lambda^\alpha\frac{\partial}{\partial\mu^\alpha}\mathcal{O}_{1,2} = \chi_+(z_1)\psi_+(z_2)\,,$$
$$\mathcal{O}_{1,2}^{T=3} = \epsilon^{\alpha\beta}\frac{\partial}{\partial\mu^\beta}\frac{\partial}{\partial\lambda^\alpha}\mathcal{O}_{1,2}\,. \tag{2.66}$$

It is convenient to introduce

$$\mathcal{O}_\pm(z_1,z_2) = \mathcal{O}_1(z_1,z_2)\pm\mathcal{O}_2(z_1,z_2)\,. \tag{2.67}$$

The operator $\mathcal{O}_-$ can be written as

$$\mathcal{O}_-(z_1,z_2) = (\lambda\mu)\,\chi^\alpha(z_1)\psi_\alpha(z_2) \tag{2.68}$$

and is pure twist $T=3$, whereas for $\mathcal{O}_+$ inverting the relations in (2.66) one obtains after some algebra

$$\mathcal{O}_+(z_1,z_2) = \mu^\alpha\frac{\partial}{\partial\lambda^\alpha}\int_0^1 d\tau\,\mathcal{O}_+^{T=2}(\tau z_1,\tau z_2) - (\mu\lambda)\int_0^1 d\tau\,\tau\,\mathcal{O}_+^{T=3}(\tau z_1,\tau z_2)\,. \tag{2.69}$$

Using $\mathcal{O}_-$ and $\mathcal{O}_+^{T=3}$ as basis fields instead of $\mathcal{O}_1$ and $\mathcal{O}_2$ one can avoid the contamination by twist-two operators altogether. The problem is that the renormalization group equation will in this form involve linear combinations of the kernels (2.65), of the type $\mathcal{H}_{11}\pm\mathcal{H}_{12}$ etc., with different $SL(2,\mathbb{R})$ transformation properties.

In the present case the problem can be simplified drastically using the operator identities [32] that allow to rewrite both two-particle operators $\mathcal{Q}_+^{T=3}$ and $\mathrm{Q}_3^{T=3}$ in terms of $\mathcal{O}_3$ (up to a local term $\chi^\alpha(0)\psi_\alpha(0)$ [32]) so that they do not need to be considered separately. In this way the matrix renormalization group equation (2.65) is reduced to the single term $\mathcal{H}_{33}$. Unfortunately, a similar reduction to the quasipartonic sector does not hold in the general situation.

3 Complete Operator Basis for twist-4

After this general discussion we proceed to the systematic study of the renormalization of QCD baryon operators of twist-4. From the theory side, this is the simplest example where non-quasipartonic operators enter nontrivially and cannot be excluded by using EOM; a generalization of the approach of Ref. [23] to such situations is our primary goal. The main application of twist-4 baryon operators to QCD phenomenology has been to the studies of hard exclusive reactions involving helicity flip, for example the Pauli electromagnetic form factor of the nucleon $F_2(Q^2)$ using pQCD factorization [34] or light-cone sum rules [35]. The necessary nonperturbatuve input is given in this case by the three nucleon (proton) distribution amplitudes Φ_4, Ψ_4 and Ξ_4 defined in Ref. [36] as matrix elements of twist-4 three-quark operators. They can be represented in spinor notation as follows:

$$\begin{aligned}
\langle 0|\epsilon^{ijk}\psi_+^{u,i}(z_1)\bar\chi_+^{u,j}(z_2)\psi_-^{d,k}(z_3)|P\rangle &= -\frac{1}{4}(\mu\lambda)\, m_N N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\Phi_4(x)\,,\\
\langle 0|\epsilon^{ijk}\,\bar\chi_+^{u,i}(z_1)\,\psi_-^{u,j}(z_2)\,\psi_+^{d,k}(z_3)\,|P\rangle &= -\frac{1}{4}(\mu\lambda)\, m_N N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\Psi_4(x)\,,\\
\langle 0|\epsilon^{ijk}\psi_-^{u,i}(z_1)\psi_+^{u,j}(z_2)\psi_+^{d,k}(z_3)|P\rangle &= -\frac{1}{4}(\mu\lambda)\, m_N N_+^\downarrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\Xi_4(x)\,. \qquad (3.1)
\end{aligned}$$

For comparison, the leading twist-3 distribution amplitude can be defined as

$$\langle 0|\epsilon^{ijk}\psi_+^{u,i}(z_1)\bar\chi_+^{u,j}(z_2)\psi_+^{d,k}(z_3)|P\rangle = \frac{1}{2}(pn)\, N_+^\downarrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\Phi_3(x)\,, \qquad (3.2)$$

and only involves "plus" quark fields. In addition, we introduce three independent twist-4 four-particle distribution amplitudes involving a gluon field:

$$\begin{aligned}
\langle 0|ig\epsilon^{ijk}\psi_+^{u,i}(z_1)\bar\chi_+^{u,j}(z_2)[\bar f_{++}(z_4)\psi_+^{d}(z_3)]^k|P\rangle &= \frac{1}{4} m_N (pn)^2 N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Phi_4^g(x)\,,\\
\langle 0|ig\epsilon^{ijk}\,\bar\chi_+^{u,i}(z_1)\,[\bar f_{++}(z_4)\psi_+^{u}(z_2)]^j\,\psi_+^{d,k}(z_3)\,|P\rangle &= \frac{1}{4} m_N (pn)^2 N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Psi_4^g(x)\,,\\
\langle 0|ig\epsilon^{ijk}[\bar f_{++}(z_4)\psi_+^{u}(z_1)]^i\,\psi_+^{u,j}(z_2)\,\psi_+^{d,k}(z_3)|P\rangle &= \frac{1}{4} m_N (pn)^2 N_+^\downarrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Xi_4^g(x)\,.
\end{aligned} \qquad (3.3)$$

Here the first and the second superscripts of $\psi^{a,i}$ ($\bar\chi^{a,i}$) are the flavor, $a = u, d$, and color, $i = 1, 2, 3$ indices of the quark field, respectively; m_N and p_μ are the nucleon mass and momentum, and $N^{\uparrow(\downarrow)} = (1/2)(1 \pm \gamma_5)\, N(p)$ is the antichiral (chiral) part of the Dirac spinor. In order to keep the auxiliary spinors λ, μ dimensionless we choose

$$n = m_N^{-1}\lambda\otimes\bar\lambda, \qquad \tilde n = m_N\,\mu\otimes\bar\mu\,.$$

The distribution amplitudes depend on the set of parton momentum fractions $x = \{x_1, \ldots, x_n\}$ and the integration measure is defined as

$$\int \mathcal{D}x = \int_0^1 dx_1 \ldots dx_n\, \delta(1 - \sum x_k)\,. \qquad (3.4)$$

We hope that using the same notation $\int \mathcal{D}x$ for the three-particle and the four-particle integration measure will not create confusion. We keep the factors $(\mu\lambda) = 1$ on r.h.s. of Eq. (3.1) to maintain the balance between the spinors μ and λ on the both sides.

The scale dependence of the distribution amplitudes is driven by the renormalization of the nonlocal light-ray operators on the l.h.s. of Eq. (3.1). Our first task is to construct the complete operator basis. Note that Ξ_4 involves chiral quarks only whereas Φ_4 and Ψ_4 involve both chiral and antichiral fields. Since chirality is conserved in QCD perturbation theory, there is no mixing and the two cases (pure chirality and mixed chirality operators) can be considered separately.

3.1 Chiral operators

The distribution amplitude Ξ_4 is related to the matrix element of the operators with collinear twist $E = 4$ and helicity $H = 1/2$

$$\begin{aligned} Q_1(z_1, z_2, z_3) =&\epsilon^{ijk}\psi_-^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ Q_2(z_1, z_2, z_3) =&\epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_-^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ Q_3(z_1, z_2, z_3) =&\epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,\psi_-^{c,k}(z_3)\,. \end{aligned} \tag{3.5}$$

For the discussion of the renormalization it is convenient to consider the three quark fields of different flavor in which case the three light-ray operators in (3.5) are independent. We do not assign flavor indices to Q_i assuming that the flavors are always ordered as in the above expressions.

The three-quark operators in (3.5) mix with quasipartonic operators containing an additional gluon field $F_{+,\mu\bar\lambda} = -(\mu\lambda)\bar f_{++}$

$$\begin{aligned} G_1(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}(\mu\lambda)\,[\bar f_{++}(z_4)\psi_+^{a}(z_1)]^i\,\psi_+^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ G_2(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}(\mu\lambda)\,\psi_+^{a,i}(z_1)\,[\bar f_{++}(z_4)\psi_+^{b}(z_2)]^j\,\psi_+^{c,k}(z_3)\,,\\ G_3(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}(\mu\lambda)\,\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,[\bar f_{++}(z_4)\psi_+^{c}(z_3)]^k\,, \end{aligned} \tag{3.6}$$

which are, however, not all independent because of the identity

$$G_1(z_1, z_2, z_3, z_4) + G_2(z_1, z_2, z_3, z_4) + G_3(z_1, z_2, z_3, z_4) = 0\,. \tag{3.7}$$

Recall that ψ_-, ψ_+ and $\bar f_{++}$ have conformal spins $j = 1/2$, $j = 1$ and $j = 3/2$, respectively. Hence Q_1 transforms according to the representation $T^{(1/2)} \otimes T^{(1)} \otimes T^{(1)}$, of the $SL(2,\mathbb{R})$ group (and similar for Q_2, Q_3), whereas G_i belong to the $T^{(1)}\otimes T^{(1)}\otimes T^{(1)}\otimes T^{(3/2)}$ representation..

For the matrix elements of the operators Q_i and G_i between the vacuum and proton state there are more relations that follow from identity of the two u-quarks in the proton and the requirement that the nucleon has isospin 1/2. Let

$$
\begin{aligned}
q_i(z_1, z_2, z_3) &= \langle 0 | Q_i(z_1, z_2, z_3) | P \rangle \,, \\
g_i(z_1, z_2, z_3, z_4) &= \langle 0 | G_i(z_1, z_2, z_3, z_4) | P \rangle \,,
\end{aligned} \tag{3.8}
$$

where we put $a = b = u$ and $c = d$. One finds

$$
\begin{aligned}
q_2(z_1, z_2, z_3) &= q_1(z_2, z_1, z_3) \,, \\
q_3(z_2, z_3, z_1) &= -q_1(z_1, z_2, z_3) - q_1(z_1, z_3, z_2)
\end{aligned} \tag{3.9}
$$

and similarly

$$
\begin{aligned}
g_2(z_1, z_2, z_3, z_4) &= g_1(z_2, z_1, z_3, z_4) \,, \\
g_3(z_2, z_3, z_1, z_4) &= -g_1(z_1, z_2, z_3, z_4) - g_1(z_1, z_3, z_2, z_4) \,.
\end{aligned} \tag{3.10}
$$

Taking into account the identity (3.7) it follows that the remaining independent function g_1 satisfies the symmetry relation

$$
g_1(z_1, z_2, z_3, z_4) - g_1(z_3, z_2, z_1, z_4) = g_1(z_3, z_1, z_2, z_4) - g_1(z_2, z_1, z_3, z_4) \,. \tag{3.11}
$$

Going over to the momentum fraction representation, we define a new twist-4 four-particle proton distribution amplitude Ξ_4^g in Eq. (3.3).

3.2 Mixed chirality operators

The distribution amplitudes Φ_4 and Ψ_4 are given by the matrix elements of the operators with collinear twist $E = 4$ and helicity $H = -1/2$. There exist three independent three-quark operators with these quantum numbers:

$$
\begin{aligned}
\mathcal{Q}_1(z_1, z_2, z_3) =& \epsilon^{ijk} \, \psi_-^{a,i}(z_1) \, \psi_+^{b,j}(z_2) \, \bar{\chi}_+^{c,k}(z_3) \,, \\
\mathcal{Q}_2(z_1, z_2, z_3) =& \epsilon^{ijk} \, \psi_+^{a,i}(z_1) \, \psi_-^{b,j}(z_2) \, \bar{\chi}_+^{c,k}(z_3) \,, \\
\mathcal{Q}_3(z_1, z_2, z_3) =& \frac{1}{2} \epsilon^{ijk} \, \psi_+^{a,i}(z_1) \, \psi_+^{b,j}(z_2) \, [\bar{\chi}_+^{3/2}]^{c,k}(z_3) \,,
\end{aligned} \tag{3.12}
$$

where $\bar{\chi}_+^{3/2} \equiv \bar{\chi}_+^{(3/2,0)} = -(\mu D \bar{\lambda}) \bar{\chi}_+ \equiv -D_{\mu\dot{\lambda}} \bar{\chi}_+$, cf. Eq. (2.59). In addition there are three operators containing an extra gluon field

$$
\begin{aligned}
\mathcal{G}_1(z_1, z_2, z_3, z_4) =& ig \epsilon^{ijk} \, (\mu\lambda) \, [\bar{f}_{++}(z_4) \psi_+^a(z_1)]^i \, \psi_+^{b,j}(z_2) \, \bar{\chi}_+^{c,k}(z_3) \,, \\
\mathcal{G}_2(z_1, z_2, z_3, z_4) =& ig \epsilon^{ijk} \, (\mu\lambda) \, \psi_+^{a,i}(z_1) \, [\bar{f}_{++}(z_4) \psi_+^b(z_2)]^j \, \bar{\chi}_+^{c,k}(z_3) \,, \\
\mathcal{G}_3(z_1, z_2, z_3, z_4) =& ig \epsilon^{ijk} \, (\mu\lambda) \, \psi_+^{a,i}(z_1) \, \psi_+^{b,j}(z_2) \, [\bar{f}_{++}(z_4) \bar{\chi}_+^c(z_3)]^k \,,
\end{aligned} \tag{3.13}
$$

which, again, are subjected to the constraint

$$\mathcal{G}_1(z_1,z_2,z_3,z_4)+\mathcal{G}_2(z_1,z_2,z_3,z_4)+\mathcal{G}_3(z_1,z_2,z_3,z_4)=0\,. \tag{3.14}$$

Thus there are two independent distribution amplitudes in this case, Φ_4^g and Ψ_4^g, which can be defined as in Eq. (3.3).

The operators $\mathcal{Q}_1(z_2,z_3,z_1)$ and $\mathcal{Q}_2(z_1,z_3,z_2)$ with the choice of flavors $b=d$ and $a=c=u$ enter directly the definition of Ψ_4 and Φ_4 in (3.1). For practical applications it can be convenient to introduce an additional distribution amplitude related to the operator $\mathcal{Q}_3$

$$\frac{1}{2}\langle 0|\epsilon^{ijk}\psi_+^{u,i}(z_1)[\bar{\chi}_+^{3/2}]^{u,j}(z_2)\psi_+^{d,k}(z_3)|P\rangle =\frac{i}{4}(\mu\lambda)(pn)m_N N_+^{\uparrow}\int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, D_4(x)\,, \tag{3.15}$$

The amplitude D_4 is not independent and can be reduced to the combination of other amplitudes using EOM. For the matrix elements in coordinate space

$$\varphi_k(z_1,z_2,z_3)=\langle 0|\mathcal{Q}_k(z_1,z_2,z_3)|P\rangle\,,\quad \varphi_k^g(z_1,z_2,z_3,z_4)=\langle 0|\mathcal{G}_k(z_1,z_2,z_3,z_4)|P\rangle\,, \tag{3.16}$$

this relation reads

$$\varphi_3(z_1,z_2,z_3)=\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3)+\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) -\frac{1}{2}\int_0^1 d\tau\,\Big(z_{13}\,\varphi_1^g(z_1,z_2,z_3,z_{13}^\tau)+z_{23}\,\varphi_2^g(z_1,z_2,z_3,z_{23}^\tau)\Big), \tag{3.17}$$

where we use the notation

$$z_{ik}=z_i-z_k\,,\qquad \bar\tau=1-\tau\,,\qquad z_{ik}^\tau=z_i\bar\tau+z_k\tau. \tag{3.18}$$

Going to the momentum fractions one derives

$$\begin{aligned} D_4(x_1,x_2,x_3)=&x_3\Phi_4(x_1,x_2,x_3)+x_1\Psi_4(x_2,x_1,x_3)\\ &+\frac{1}{2}\bigg(\int_0^{x_2}\frac{dx}{x}\Phi_4^g(x_1,x_2-x,x_3,x)-\int_0^{x_3}\frac{dx}{x}\Phi_4^g(x_1,x_2,x_3-x,x)\\ &+\int_0^{x_1}\frac{dx}{x}\Psi_4^g(x_2,x_1-x,x_3,x)-\int_0^{x_3}\frac{dx}{x}\Psi_4^g(x_2,x_1,x_3-x,x)\bigg). \end{aligned} \tag{3.19}$$

4 Renormalization Group Equations

4.1 General properties

The matrix of anomalous dimensions for a set of local operators O_i is defined as

$$\gamma_{ik}=Z_{ij}^{-1}\mu\frac{\partial}{\partial\mu}Z_{jk}\,,\qquad\qquad [O]_i=Z_{ik}^{-1}O_k^B, \tag{4.1}$$

where $[O]_i$, O_i^B are renormalized and bare operators, respectively. The renormalized operators satisfy the matrix RG equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{ik}(g)\right) [O]_k = 0\,, \tag{4.2}$$

which, equivalently, can be cast in the form of an integro-differential equation for the generating functions (light-ray operators):

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma(g)\right) [O](z_1, \ldots, z_N) = 0\,. \tag{4.3}$$

Here γ is an integral operator, which we write as

$$\gamma = \frac{\alpha_s}{2\pi}\,\mathbb{H}\,, \tag{4.4}$$

where $\alpha_s = g^2/4\pi$. In what follows we will refer to the evolution kernel $\mathbb{H}$ as the Hamiltonian. It is determined by one-loop counterterms to the nonlocal operator.

For the both cases of pure chirality and mixed chirality operators we will be dealing with three light-ray three-quark operators, (3.5) or (3.12), and three operators with an additional gluon field, (3.6) or (3.13), respectively. Thus $\mathbb{H}$ is in both cases a 6×6 matrix which can be written as

$$\mathbb{H} = \begin{pmatrix} \mathbb{H}_q & \mathbb{H}_{qg} \\ 0 & \mathbb{H}_g \end{pmatrix}. \tag{4.5}$$

Each block, $\mathbb{H}_q, \mathbb{H}_g, \mathbb{H}_{qg}$, is a 3×3 matrix where the entries are integral operators. The diagonal blocks, $\mathbb{H}_q$ and $\mathbb{H}_g$, are given by the sum of two-particle operators, whereas the off-diagonal block $\mathbb{H}_{qg}$ describes $3 \to 2$ transitions. The explicit expressions will be given below.

One of the three light-ray operators involving a gluon field, e.g. $G_3(z_1, z_2, z_3, z_4)$ (3.6) and $\mathcal{G}_3(z_1, z_2, z_3, z_4)$ (3.13), can be excluded from consideration with the help of the operator identity (3.7). Let

$$\mathbb{O}^{\rm chiral}(\vec{z}) = \begin{pmatrix} Q_1(z_1, \ldots, z_3) \\ Q_2(z_1, \ldots, z_3) \\ Q_3(z_1, \ldots, z_3) \\ G_1(z_1, \ldots, z_4) \\ G_2(z_1, \ldots, z_4) \end{pmatrix} \tag{4.6}$$

be the vector of the remaining five independent chiral operators, and similar for mixed chirality. This vector satisfies the RG equation with the modified Hamiltonian

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi}\,\widetilde{\mathbb{H}}\right) \mathbb{O}(\vec{z}) = 0\,. \tag{4.7}$$

where $\widetilde{\mathbb{H}}$ is a 5×5 matrix such that

$$[\widetilde{\mathbb{H}}_q]_{ik} = [\mathbb{H}_q]_{ik}\,, \qquad\qquad i,k=1,2,3$$
$$[\widetilde{\mathbb{H}}_g]_{ik} = [\mathbb{H}_g]_{ik} - [\mathbb{H}_g]_{i3}\,, \qquad\qquad i,k=1,2$$
$$[\widetilde{\mathbb{H}}_{qg}]_{ik} = [\mathbb{H}_{qg}]_{ik} - [\mathbb{H}_{qg}]_{i3}\,, \qquad\qquad i=1,2,3,\quad k=1,2 \tag{4.8}$$

The nonlocal operator $\mathbb{O}(\vec{z})$ can be expanded over a complete basis of local operators

$$\mathbb{O}(\vec{z}) = \sum_{N,q} \Psi_{N,q}(\vec{z})\,\mathbb{O}_{N,q}\,, \tag{4.9}$$

where operator $\mathbb{O}_{N,q}$ has canonical dimension $N+9/2$ and q enumerates independent local operators of the same dimension. If a local operator $\mathbb{O}_{N,q}$ satisfies the RG equation $(\mu\partial_\mu + \beta(g)\partial_g + \gamma_{N,q})\,\mathbb{O}_{N,q} = 0$ then the corresponding "coefficient" function $\Psi_{N,q}(\vec{z})$ is the eigenvector of the Hamiltonian $\widetilde{\mathbb{H}}$

$$[\widetilde{\mathbb{H}}\Psi_{N,q}](\vec{z}) = E_{N,q}\Psi_{N,q}(\vec{z})\,, \tag{4.10}$$

and $\gamma_{N,q} = (\alpha_s/2\pi)E_{N,q}$ so that

$$\mathbb{O}_{N,q}(\mu_2) = \left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}\right)^{E_{N,q}/\beta_0} \mathbb{O}_{N,q}(\mu_1)\,, \tag{4.11}$$

where $\beta_0 = 11/3N_c - 2/3n_f$.

The coefficient functions of local operators $\Psi_{N,q}(\vec{z})$ are homogeneous polynomials of three variables

$$\Psi^{(i)}_{N,q}(\vec{z}) = \sum_{\substack{k_1,\ldots,k_3\\ k_1+k_2+k_3=N}} \psi^{(i)N,q}_{k_1\,k_2\,k_3}\, z_1^{k_1}\, z_2^{k_2}\, z_3^{k_3}\,, \tag{4.12}$$

for the quark components, $i=1,2,3$, and four variables

$$\Psi^{(i)}_{N,q}(\vec{z}) = \sum_{\substack{k_1,\ldots,k_4\\ k_1+\ldots+k_4=N-2}} \psi^{(i)N,q}_{k_1\,k_2\,k_3\,k_4}\, z_1^{k_1}\, z_2^{k_2}\, z_3^{k_3}\, z_4^{k_4} \tag{4.13}$$

for the quark-gluon components, $i=4,5$.

4.2 Conformally invariant evolution kernels

The structure of the Hamiltonian is severely constrained by conformal symmetry. It follows from the group theory that a nontrivial operator mapping the representation $T^{j_1}\otimes T^{j_2}$ to $T^{i_1}\otimes T^{i_2}$ only exists if the difference $i_1+i_2-j_1-j_2$ is an integer. If

$i_1+i_2=j_1+j_2$ a conformally invariant operator K can be written in the form (see Ref. [37] for details)

$$[K^{i_1 i_2}_{j_1 j_2}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_0^1 d\beta\,\bar\alpha^{i_1+j_1-2}\,\alpha^{i_2-j_2}\,\bar\beta^{i_2+j_2-2}\,\beta^{i_1-j_1}\,\kappa\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right)\varphi(z_{12}^\alpha,z_{21}^\beta)\,, \tag{4.14}$$

where the notation follows Eq. (3.18); the function $\kappa(x)$ is arbitrary.

The two-particle one-loop kernels fall in two groups. The kernels in the first group involve a single integration, $\kappa(x)\sim\delta(x)$, and their form is completely fixed (up to prefactor) by the conformal spins of the fields. For the case that the conformal spins are conserved, $i_i=j_1$ and $i_2=j_2$, one important example is[†]

$$[\mathcal{H}^v_{12}\varphi](z_1,z_2)=\int_0^1\frac{d\alpha}{\alpha}\Big\{\bar\alpha^{2j_1-1}[\varphi(z_1,z_2)-\varphi(z_{12}^\alpha,z_2)]+\bar\alpha^{2j_2-1}[\varphi(z_1,z_2)-\varphi(z_1,z_{21}^\alpha)]\Big\}\,. \tag{4.15}$$

This structure is specific for gauge theories and arises from the diagrams involving the gluon field in the light-like Wilson lines (in covariant gauges).

Another kernel of this type is the "exchange" kernel, $\mathcal{H}^e$. It maps $T^{j_1}\otimes T^{j_2}\to T^{j_2}\otimes T^{j_1}$ and is fixed by conformal symmetry up to a prefactor [37]. Assuming that $j_1>j_2$:

$$[\mathcal{H}^e_{12}\varphi](z_1,z_2)=\int_0^1 d\alpha\,\bar\alpha^{2j_2-1}\,\alpha^{2(j_1-j_2)-1}\,\varphi(z_{12}^\alpha,z_2)\,. \tag{4.16}$$

The two-particle kernels (4.15), (4.16) depend implicitly on the conformal spins of the fields which they act on; this dependence will always be implied.

The kernels (4.15) and (4.16) are both known from the studies of leading twist operators. In addition, we will need

$$[\mathcal{H}^d_{12}\varphi](z_1,z_2)=\varphi(z_{12}^\alpha,z_{21}^\beta)=\int_0^1 d\alpha\,\bar\alpha^{2j_1-1}\,\alpha^{2j_2-1}\,\varphi(z_{12}^\alpha,z_{12}^\alpha)\,, \tag{4.17}$$

which corresponds to the "diagonal" mapping without the spin exchange: $T^{j_1}\otimes T^{j_2}\to T^{j_1}\otimes T^{j_2}$ with $\kappa(x)=\delta(1-x)$.

The kernels in the second group retain both integrations as in (4.14) and usually involve a theta–function $\kappa(x)\sim\theta(1-x)$ or $\kappa(x)\sim\theta(x-1)$ which restricts the region of integration to $\alpha+\beta\le 1$ and $1\le\alpha+\beta$, respectively. We define two more kernels $\mathcal{H}^\pm$ by

$$[\mathcal{H}^+_{12}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\bar\alpha^{2j_1-2}\,\bar\beta^{2j_2-2}\,\varphi(z_{12}^\alpha,z_{21}^\beta)\,, \tag{4.18}$$

$$[\mathcal{H}^-_{12}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_{\bar\alpha}^1 d\beta\,\bar\alpha^{2j_1-2}\,\bar\beta^{2j_2-2}\,\varphi(z_{12}^\alpha,z_{21}^\beta)\,. \tag{4.19}$$

[†]Here and below the subscripts $\mathcal{H}_{ik}$, $i,k=1,2,3,4$, indicate that the kernel acts on the coordinates of the i-th and k-th partons (particles).

The off-diagonal block $\mathbb{H}_{qg}$ maps the $SL(2,\mathbb{R})$ representations with a different number of fields $T^1 \otimes T^1 \otimes T^{3/2} \to T^{j_1} \otimes T^{j_2}$, where either $j_1 = 1/2$, $j_2 = 1$ or $j_1 = 1/2$, $j_2 = 1$ so that in the both cases $j_1 + j_2 = 3/2$. The general form of the corresponding kernel consistent with conformal symmetry is

$$[Rf](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \bar{\beta}^{2j_2-1}\beta^{2j_1-1} r\left(\frac{\alpha\gamma}{\bar{\alpha}\bar{\gamma}}, \frac{\gamma\bar{\beta}}{\beta\bar{\gamma}}\right) f(z_{12}^\alpha, z_{21}^\gamma, z_{21}^\beta)\,. \tag{4.20}$$

where $r(x, y)$ is an arbitrary function. In one-loop diagrams the following kernels appear:

$$[\mathcal{V}^{(1)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \, \frac{\bar{\alpha}\bar{\beta}}{\alpha} \, f(z_{12}^\alpha, z_2, z_{21}^\beta)\,, \tag{4.21}$$

$$[\mathcal{V}^{(2)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, \beta \, f(z_{12}^\alpha, z_2, z_{21}^\beta)\,, \tag{4.22}$$

$$[\mathcal{V}^{(3)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\beta \int_\beta^1 d\gamma \, \frac{\beta\bar{\gamma}}{\gamma} \left(\frac{\bar{\gamma}}{\gamma} - 2\frac{\bar{\beta}}{\beta}\right) f(z_1, z_{21}^\gamma, z_{21}^\beta)\,, \tag{4.23}$$

$$[\mathcal{V}^{(4)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\beta \, \bar{\beta} \left\{ f(z_1, z_2, z_{21}^\beta) + \frac{\bar{\beta}}{\beta} \int_0^\beta d\gamma f(z_1, z_{21}^\gamma, z_{21}^\beta) \right\}, \tag{4.24}$$

$$[\mathcal{V}^{(a)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \int_0^{\bar{\alpha}} d\gamma \, \bar{\beta} \, f(z_{12}^\alpha, z_{21}^\gamma, z_{21}^\beta)\,, \tag{4.25}$$

$$[\mathcal{V}^{(b)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \int_0^{\bar{\alpha}} d\gamma \, \bar{\beta} \, f(z_{12}^\alpha, z_{21}^\gamma, z_{21}^\beta)\,, \tag{4.26}$$

$$[\mathcal{V}^{(c)}_{12(3)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \int_0^\beta d\gamma \, \bar{\beta} \, f(z_{12}^\alpha, z_{21}^\gamma, z_{21}^\beta)\,, \tag{4.27}$$

which correspond to the choices

$$r^{(1)}(x, y) = \delta(x)\,\theta(x/y - 1), \qquad r^{(2)}(x, y) = \delta(y)\,\theta(1 - x/y), \tag{4.28}$$

$$r^{(3)}(x, y) = \delta(x)\,\theta(y - 1)\left(\frac{1}{y} - 2\right), \qquad r^{(4)}(x, y) = \theta(1 - y)\delta(x/y)\,(1 + \delta(y)), \tag{4.29}$$

$$r^{(a)}(x, y) = \theta(1 - x)\theta(x/y - 1), \qquad r^{(b)}(x, y) = \theta(1 - x)\theta(1 - x/y), \tag{4.30}$$

$$r^{(c)}(x, y) = \theta(1 - y)\theta(1 - x/y)\,. \tag{4.31}$$

4.3 The scalar product

We will be looking for solutions of the Schrödinger equation (4.10) that are polynomials in light-cone variables z_k. The scalar product on this space can be constructed as follows. First of all, we allow z_k to take complex values. It is convenient at this point to go over from the $SL(2, R)$ group to $SU(1, 1)$ which has the same algebra. In particular, the $SU(1,1)$ generators have the same form. The $SU(1,1)$ invariant scalar product is defined as [38]

$$\langle f_1, f_2\rangle_j = \int_{|z|<1} D_j z\, \overline{f_1(z)}\, f_2(z)\,, \qquad D_j z = \frac{2j-1}{\pi}(1-|z|^2)^{2j-2} d^2 z\,, \tag{4.32}$$

where j is the conformal spin, $j \geq 1/2$, and functions $f_k(z)$ are polynomials in z; $\overline{f(z)} = (f(z))^*$ stands for complex conjugation. For the special case $j = 1/2$ the $SU(1,1)$ scalar product takes the form

$$\langle f_1, f_2\rangle_{j=1/2} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \overline{f_1(e^{i\varphi})}\, f_2(e^{i\varphi})\,. \tag{4.33}$$

One easily finds $||z^n||^2 \equiv \langle z^n, z^n\rangle = \dfrac{\Gamma(2j)n!}{\Gamma(2j+n)}$. For polynomials of several variables $f(z_1,\ldots,z_n)$ which belong to the tensor product $T^{j_1}\otimes\ldots\otimes T^{j_n}$, the scalar product is given by

$$\langle f_1, f_2\rangle = \left(\prod_{k=1}^{n}\int_{|z_k|<1} D_{j_k}\, z_k\right)\overline{f_1(z_1,\ldots,z_n)}\, f_2(z_1,\ldots,z_n)\,. \tag{4.34}$$

It can be shown that the diagonal blocks (three-particle and four-particle) of the evolution Hamiltonians are self-adjoint with respect to the following scalar product:

$$\langle\Psi_1|\Psi_2\rangle = \sum_i^5 \langle\Psi_1^{(i)}|\Omega_{ik}\Psi_2^{(k)}\rangle\,, \tag{4.35}$$

where the matrix Ω is

$$\Omega = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0 & 0\\ 0 & 0 & a_3 & 0 & 0\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 1 & 2\end{pmatrix}\,. \tag{4.36}$$

The coefficients $a_i = a$ for the chiral operators and $a_1 = a_2 = 2a_3 = a$ for the mixed ones, where $a > 0$ is an arbitrary constant. We put $a = 1$ in what follows. It is tacitly implied that $\langle\Psi_1^{(i)}|\Omega_{ik}\Psi_2^{(k)}\rangle$, in (4.36) is given by the $SU(1,1)$ scalar product (4.34) with the spins j_k equal to the conformal spins of $\Psi_1^{(i)}$.

We would like to emphasize that the choice of the scalar product depends on the problem under consideration, see, e.g. [39] for a discussion.

4.4 Chiral operators

We are now in a position to write the explicit expression for the Hamiltonian that governs the RG equation for the chiral operators (3.5), (3.6). The quark block $\mathbb{H}_q^{\psi\psi\psi}$ can be written in terms of the two kernels defined in (4.15) and (4.16):

$$\mathbb{H}_q^{\psi\psi\psi} = \left(1 + \frac{1}{N_c}\right) \mathcal{H}_q^{\psi\psi\psi} \tag{4.37}$$

with

$$\mathcal{H}_q^{\psi\psi\psi} = \begin{pmatrix} \mathrm{H} & \mathcal{H}_{12}^e & \mathcal{H}_{13}^e \\ \mathcal{H}_{21}^e & \mathrm{H} & \mathcal{H}_{23}^e \\ \mathcal{H}_{31}^e & \mathcal{H}_{32}^e & \mathrm{H} \end{pmatrix}, \tag{4.38}$$

where

$$\mathrm{H} = \mathcal{H}_{12}^v + \mathcal{H}_{23}^v + \mathcal{H}_{31}^v - \frac{1}{2}\,. \tag{4.39}$$

As discussed in Sect. 3 the Hamiltonian $\mathbb{H}_q$ describes renormalization of the three quark operators of both geometrical twist $T = 3$ and $T = 4$. On the other hand, the Hamiltonian for the twist-3 operator $\psi_+\psi_+\psi_+$ is known from Ref. [8], where it is called $H_{3/2}$. Thus one obtains a nontrivial consistency condition[‡]

$$(\mathbb{H}_q^{\psi\psi\psi})_{k1} + (\mathbb{H}_q^{\psi\psi\psi})_{k2} + (\mathbb{H}_q^{\psi\psi\psi})_{k3} = H_{3/2}\,, \qquad k = 1,2,3\,, \tag{4.40}$$

which, alternatively, can be used to restore $\mathbb{H}_q$ from the known result [8] for $H_{3/2}$.

The "gluon" block $\mathbb{H}_g$ describes the renormalization of four-particle quasi-partonic operators and can be restored from the results existing in the literature [6, 23, 40]. Though the baryon operators in question make sense for $N_c = 3$ only, it is convenient to separate the terms with the different color factor:

$$\mathbb{H}_g^{\psi\psi\psi\bar{f}} = N_c\,\mathbb{H}_g^{(1)} + \mathbb{H}_g^{(0)} + \frac{1}{N_c}\,\mathbb{H}_g^{(-1)} + \frac{21}{2}\,, \tag{4.41}$$

where the last term (a constant) stands for the self-energy type contributions. For the diagonal elements we find

$$\begin{aligned} [\mathbb{H}_g^{(1)}]_{kk} =& \mathcal{H}_{k4}^v - 2\,\mathcal{H}_{k4}^+\,, \\ [\mathbb{H}_g^{(0)}]_{kk} =& \mathcal{H}_{k+1,k-1}^v + \mathcal{H}_{k+1,4}^v + \mathcal{H}_{k-1,4}^v - 2\left(\mathcal{H}_{k+1,4}^+ + \mathcal{H}_{k-1,4}^+ + \mathcal{H}_{k+1,4}^- + \mathcal{H}_{k-1,4}^-\right)\,, \\ [\mathbb{H}_g^{(-1)}]_{kk} =& \mathcal{H}_{12}^v + \mathcal{H}_{23}^v + \mathcal{H}_{31}^v - 2\,\mathcal{H}_{k4}^-\,. \end{aligned} \tag{4.42}$$

In these expressions $k = 1,2,3$ and $k, k \pm 1$ appearing in the subscripts of the kernels on the r.h.s. refer to arguments of the quark fields. If $k + 1 = 4$ or $k - 1 = 0$ the corresponding subscript should be put to 1 or 3, respectively.

[‡] $H_{3/2}$ on the r.h.s. of (4.40) is given by the sum of three "v"-type kernels for the conformal spins pairs $\{j_i, j_k\} = \{1,1\}$ whereas H^0 on the l.h.s. contains one kernel with $\{j_i, j_k\} = \{1,1\}$ and two kernels with $\{j_i, j_k\} = \{1,1/2\}$. The "e"-type kernels appearing in the non-diagonal entries $(\mathbb{H}_q)_{12} + (\mathbb{H}_q)_{13}$ correct for this difference.

The off-diagonal matrix elements $i, k = 1, 2, 3$, $i \neq k$, read

$$
\begin{aligned}
[\mathbb{H}_g^{(0)}]_{ik} =& \mathcal{H}_{ik}^v - \mathcal{H}_{k4}^v + 2\,\mathcal{H}_{k4}^+ - \frac{1}{2}\,,\\
[\mathbb{H}_g^{(-1)}]_{ik} =& -2\,\mathcal{H}_{k4}^-\,.
\end{aligned}
\tag{4.43}
$$

The identity in Eq. (3.7) implies the relations

$$
\begin{aligned}
&[\mathbb{H}_g]_{11} - [\mathbb{H}_g]_{13} + [\mathbb{H}_g]_{21} - [\mathbb{H}_g]_{23} = [\mathbb{H}_g]_{33} - [\mathbb{H}_g]_{31}\,,\\
&[\mathbb{H}_g]_{22} - [\mathbb{H}_g]_{23} + [\mathbb{H}_g]_{12} - [\mathbb{H}_g]_{13} = [\mathbb{H}_g]_{33} - [\mathbb{H}_g]_{32}\,.
\end{aligned}
\tag{4.44}
$$

Finally, for the off-diagonal quark-gluon block we find

$$
\mathbb{H}_{qg}^{\text{chiral}} = -\frac{1}{2}\mathcal{H}_{qg}^{\text{chiral}}\,,
\tag{4.45}
$$

with

$$
\begin{aligned}
[\mathcal{H}_{qg}^{\text{chiral}}]_{kk} =& \frac{1}{N_c}\left(\mathcal{V}^{(1)}_{k,k+1,(4)} + \mathcal{V}^{(1)}_{k,k-1,(4)}\right) - \mathcal{V}^{(2)}_{k,k+1,(4)} - \mathcal{V}^{(2)}_{k,k-1,(4)}\,,\\
[\mathcal{H}_{qg}^{\text{chiral}}]_{ik} =& \mathcal{V}^{(1)}_{ik(4)} + \mathcal{V}^{(2)}_{ik(4)}\,,
\end{aligned}
\tag{4.46}
$$

where, as above, the subscripts $k, k \pm 1$ take the values $1, 2, 3$.

4.5 Mixed chirality operators

For the mixed chirality operators (3.12), (3.13) the quark block is

$$
\mathbb{H}_q^{\psi\psi\chi} = \left(1 + \frac{1}{N_c}\right)\mathcal{H}_q^{\psi\psi\chi}
\tag{4.47}
$$

where

$$
\mathcal{H}_q^{\psi\psi\chi} = \begin{pmatrix}
\mathrm{H} + \mathcal{H}_{13}^d - \mathcal{H}_{23}^+ & \mathcal{H}_{12}^e & z_{13}\mathcal{H}_{13}^+\\
\mathcal{H}_{21}^e & \mathrm{H} + \mathcal{H}_{23}^d - \mathcal{H}_{13}^+ & z_{23}\mathcal{H}_{23}^+\\
z_{13}^{-1}(\mathbb{I} - 2\mathcal{H}_{13}^d) & z_{23}^{-1}(\mathbb{I} - 2\mathcal{H}_{23}^d) & \mathrm{H} - 2(\mathcal{H}_{13}^+ + \mathcal{H}_{23}^+) + 3
\end{pmatrix}.
\tag{4.48}
$$

Similar to Eq. (4.40) one can derive the consistency relations

$$
\begin{aligned}
&[\mathbb{H}_q^{\psi\psi\chi}]_{11} + [\mathbb{H}_q^{\psi\psi\chi}]_{12} - [\mathbb{H}_q^{\psi\psi\chi}]_{13}\,\partial_{z_3} = H_{1/2}\,,\\
&[\mathbb{H}_q^{\psi\psi\chi}]_{33}\,\partial_{z_3} - [\mathbb{H}_q^{\psi\psi\chi}]_{31} - [\mathbb{H}_q^{\psi\psi\chi}]_{32} = \partial_{z_3}\,H_{1/2}\,,
\end{aligned}
\tag{4.49}
$$

where the Hamiltonian $H_{1/2}$ describes the renormalization of the leading twist-three nucleon distribution amplitude [9].

The gluon block $\mathbb{H}_g^{\psi\psi\chi\bar{f}}$ can be expanded in powers of N_c in the same way as in (4.41), with

$$[\mathbb{H}_g^{\psi\psi\chi\bar{f},(1)}]_{kk} = \mathcal{H}_{k4}^v - 2\,(1-\delta_{k3})\mathcal{H}_{k4}^+\,, \tag{4.50}$$

$$\begin{aligned}
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{11} = \mathcal{H}_{23}^v + \mathcal{H}_{24}^v + \mathcal{H}_{34}^v - \mathcal{H}_{23}^+ - 2\mathcal{H}_{24}^+ - 2\mathcal{H}_{24}^- + P_{34}\mathcal{H}_{43}^e\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{22} = \mathcal{H}_{13}^v + \mathcal{H}_{14}^v + \mathcal{H}_{34}^v - \mathcal{H}_{13}^+ - 2\mathcal{H}_{14}^+ - 2\mathcal{H}_{14}^- + P_{34}\mathcal{H}_{43}^e\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{33} = \mathcal{H}_{12}^v + \mathcal{H}_{14}^v + \mathcal{H}_{24}^v - 2(\mathcal{H}_{14}^+ + \mathcal{H}_{24}^+ + \mathcal{H}_{14}^- + \mathcal{H}_{24}^-)\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{12} = \mathcal{H}_{12}^v - \mathcal{H}_{24}^v + 2\mathcal{H}_{24}^+ - \frac{1}{2}\,, \qquad [\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{21} = \mathcal{H}_{21}^v - \mathcal{H}_{14}^v + 2\mathcal{H}_{14}^+ - \frac{1}{2}\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{j3} = \mathcal{H}_{j3}^v - \mathcal{H}_{34}^v - \mathcal{H}_{j3}^+ - \frac{1}{2}\,, \qquad [\mathbb{H}_g^{\psi\psi\chi\bar{f},(0)}]_{3j} = \mathcal{H}_{3j}^v - \mathcal{H}_{j4}^v - \mathcal{H}_{j3}^+ + 2\mathcal{H}_{j4}^+ - \frac{1}{2}\,,
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(-1)}]_{kk} = \mathcal{H}_{12}^v + \mathcal{H}_{13}^v + \mathcal{H}_{23}^v - \mathcal{H}_{13}^+ - \mathcal{H}_{23}^+ - 2(1-\delta_{k,3})\mathcal{H}_{k4}^- + \delta_{k3}P_{34}\mathcal{H}_{43}^e\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(-1)}]_{12} = -2\mathcal{H}_{24}^-\,, \qquad [\mathbb{H}_g^{\psi\psi\chi\bar{f},(-1)}]_{21} = -2\mathcal{H}_{14}^-\,,\\
&[\mathbb{H}_g^{\psi\psi\chi\bar{f},(-1)}]_{j3} = P_{34}\mathcal{H}_{43}^e\,, \qquad [\mathbb{H}_g^{\psi\psi\chi\bar{f},(-1)}]_{3j} = -2\mathcal{H}_{14}^-\,,
\end{aligned} \tag{4.52}$$

where P_{34} is the permutation operator $P_{34}f(z_1,z_2,z_3,z_4) = f(z_1,z_2,z_4,z_3)$, $k=1,2,3$ and $j=1,2$. The entries $[\mathbb{H}_g^{\psi\psi\chi\bar{f}}]_{ik}$ satisfy the same constraint (4.44). Finally, for the off-diagonal quark-gluon block we find

$$\mathbb{H}_{qg}^{\text{mixed}} = -\frac{1}{2}\mathcal{H}_{qg}^{\text{mixed}}\,, \tag{4.53}$$

where

$$\begin{aligned}
&\left[\mathcal{H}_{qg}^{\text{mixed}}\right]_{jk} = \left[\mathcal{H}_{qg}^{\text{chiral}}\right]_{jk} + [\Delta\mathcal{H}_{qg}]_{jk}\,, \qquad j,k=1,2\\
&[\mathcal{H}_{qg}^{\text{mixed}}]_{3k} = \frac{2}{z_{k3}}\left(\mathcal{V}_{k3(4)}^{(b)} - \frac{1}{3}\mathcal{V}_{k3(4)}^{(a)} - \frac{1}{2}\mathcal{V}_{k3(4)}^{(3)} + \frac{1}{2}\mathcal{V}_{k3(4)}^{(4)}\right)\,, \qquad k=1,2\\
&[\mathcal{H}_{qg}^{\text{mixed}}]_{33} = -2\sum_{j=1}^{2}\frac{1}{z_{j3}}\left(\mathcal{V}_{j3(4)}^{(a)} - \frac{1}{3}\mathcal{V}_{j3(4)}^{(b)} + \frac{4}{3}\mathcal{V}_{j3(4)}^{(c)} + \frac{1}{6}\mathcal{V}_{j3(4)}^{(3)} + \frac{1}{2}\mathcal{V}_{j3(4)}^{(4)}\right)\,,
\end{aligned} \tag{4.54}$$

$\mathcal{H}_{qg}^{\text{chiral}}$ is given in Eq. (4.46) and

$$\begin{aligned}
&[\Delta\mathcal{H}_{qg}]_{12} = [\Delta\mathcal{H}_{qg}]_{21} = 0\,,\\
&[\Delta\mathcal{H}_{qg}]_{jj} = \frac{1}{3}\mathcal{V}_{j3(4)}^{(a)} - \mathcal{V}_{j3(4)}^{(b)}\,,\\
&[\Delta\mathcal{H}_{qg}]_{j3} = \mathcal{V}_{j3(4)}^{(a)} - \frac{1}{3}\mathcal{V}_{j3(4)}^{(b)} + \frac{4}{3}\mathcal{V}_{j3(4)}^{(c)}\,,
\end{aligned} \tag{4.55}$$

for $j=1,2$.

5 Renormalization Group Equations II: Solutions

The Hamiltonian (4.7) has block-triangular structure. In order to find its eigenvalues it is, therefore, sufficient to consider the diagonal quark and quark-gluon blocks separately.

5.1 Chiral quark operators

5.1.1 Permutation symmetry

The three-quark Hamiltonian $\mathcal{H}_q^{\psi\psi\psi}$ is given by Eq. (4.38) with the kernels $\mathcal{H}^v$, $\mathcal{H}^e$ defined in (4.15), (4.16), respectively. It is easy to check that $\mathcal{H}_q^{\psi\psi\psi}$ commutes with the generator of cyclic permutations

$$[\mathcal{P}, \mathcal{H}_q^{\psi\psi\psi}] = 0\,, \qquad \mathcal{P} = P_a \otimes P_z\,, \qquad \mathcal{P}^3 = 1\,, \tag{5.1}$$

where P_a permutes the quark quantum numbers and P_z the quark coordinates, respectively:

$$\begin{aligned} P_a\, \Psi^{(i)}_{N,q}(z_1,z_2,z_3) &= \Psi^{(i+1)}_{N,q}(z_1,z_2,z_3)\,, \\ P_z\, \Psi^{(i)}_{N,q}(z_1,z_2,z_3) &= \Psi^{(i)}_{N,q}(z_3,z_1,z_2)\,. \end{aligned} \tag{5.2}$$

The eigenfunctions of $\mathcal{H}_q^{\psi\psi\psi}$ can always be chosen to have definite parity with respect to the cyclic permutations:

$$\mathcal{P}\, \Psi^{\varepsilon}_{N,q} = \varepsilon\, \Psi^{\varepsilon}_{N,q}\,, \qquad \varepsilon \in \{1, e^{i2\pi/3}, e^{-i2\pi/3}\}\,. \tag{5.3}$$

The (vector) eigenfunction $\Psi^{(\epsilon)}_{N,q}$ can be written in terms of the single function $\psi^{(\epsilon)}_{N,q}$ as

$$\Psi^{\varepsilon}_{N,q}(z_1,z_2,z_3) = \begin{pmatrix} \varepsilon^0\; \psi^{\varepsilon}_{N,q}(z_1,z_2,z_3) \\ \varepsilon^1\; \psi^{\varepsilon}_{N,q}(z_2,z_3,z_1) \\ \varepsilon^2\; \psi^{\varepsilon}_{N,q}(z_3,z_1,z_2) \end{pmatrix}. \tag{5.4}$$

The eigenfunctions of different parity are orthogonal with respect to the quark part of the scalar product (4.35), $\langle \Psi^{\varepsilon} | \Psi^{\varepsilon'} \rangle \sim \delta_{\varepsilon\varepsilon'}$, whereas for the eigenfunctions of the same parity one gets

$$\langle \Psi^{\varepsilon}_{N,q} | \Psi^{\varepsilon}_{N,q'} \rangle = 3\, \langle \psi^{\varepsilon}_{N,q} | \psi^{\varepsilon}_{N,q'} \rangle\,. \tag{5.5}$$

The scalar product on the r.h.s. is given by Eq. (4.34) for the spins $j_1 = 1/2$, $j_2 = j_3 = 1$.

In addition to the symmetry under cyclic permutations, the Hamiltonian $\mathcal{H}_q^{\psi\psi\psi}$ commutes with the permutation operator $\mathcal{P}_{12} = P_a^{(12)} \otimes P_z^{(12)}$ defined as

$$P_a^{(12)} \Psi^{(1)}(\vec{z}) = \Psi^{(2)}(\vec{z})\,, \quad P_a^{(12)} \Psi^{(2)}(\vec{z}) = \Psi^{(1)}(\vec{z})\,, \quad P_z^{(12)} \Psi^{(i)}(z_1,z_2,z_3) = \Psi^{(i)}(z_2,z_1,z_3)\,. \tag{5.6}$$

Since $\mathcal{P}_{12}\,\mathcal{P} = \mathcal{P}^{-1}\,\mathcal{P}_{12}$ one concludes that $\mathcal{P}_{12}\,\Psi^{\varepsilon}_{N,q} \sim \Psi^{\varepsilon^{-1}}_{N,q}$. Thus the eigenvalues of the Hamiltonian $\mathcal{H}^{\psi\psi\psi}_q$ in the sectors with $\varepsilon = e^{\pm i2\pi/3}$ coincide.

Furthermore, if $\Psi^{\varepsilon}_{N,q}$ is the eigenfunction of the Hamiltonian $\mathcal{H}^{\psi\psi\psi}_q$ with the eigenvalue $\mathcal{E}^{\varepsilon}_{N,q}$, then $\psi^{\varepsilon}_{N,q}$ is the eigenfunction with the same eigenvalue of the effective Hamiltonian $\mathcal{H}(\varepsilon)$,

$$\mathcal{H}(\varepsilon)\,\psi^{\varepsilon}_{N,q}(z_1,z_2,z_3) = \mathcal{E}^{\varepsilon}_{N,q}\,\psi^{\varepsilon}_{N,q}(z_1,z_2,z_3)\,, \tag{5.7}$$

where

$$\mathcal{H}(\varepsilon) = \mathcal{H}_{3/2} - \mathcal{H}^{e}_{12}(1-\varepsilon P_z^{-1}) - \mathcal{H}^{e}_{13}(1-\varepsilon^{-1}P_z)\,. \tag{5.8}$$

Here $\mathcal{H}_{3/2}$ is the evolution Hamiltonian for the twist-3 chiral quark operator [8],

$$\mathcal{H}_{3/2} = \mathcal{H}^{v}_{12} + \mathcal{H}^{v}_{23} + \mathcal{H}^{v}_{31} + \frac{3}{2}\,, \tag{5.9}$$

where the three kernels $\mathcal{H}^{v}_{ik}$ are all given by (4.15) for the conformal spins $j_i = j_k = 1$.

As was explained earlier the Hamiltonian $\mathcal{H}^{\psi\psi\psi}_q$, alias $\mathcal{H}(\varepsilon)$, describes the evolution of chiral operators of both geometrical twist-4 and twist-3. The twist-3 eigenfunctions $\Psi^{\rm tw-3}_{N,q}(z_1,z_2,z_3)$ which belong to the sector with $\mathcal{P}$-parity ε, have the same parity with respect to cyclic permutations of the coordinates P_z alone, $P_z\,\Psi^{\rm tw-3}_{N,q} = \varepsilon\,\Psi^{\rm tw-3}_{N,q}$. From the explicit expression in (5.8) it follows, obviously, that on such functions $\mathcal{H}(\varepsilon)$ reduces to $\mathcal{H}_{3/2}$, as expected.

It follows that the eigenvalues (and eigenfunctions) of $\mathcal{H}^{\psi\psi\psi}_q$ can be separated in three symmetry classes corresponding to the renormalization group equation for:

- twist-3 operators;
- twist-4 operators with $\mathcal{P}$-parity $\varepsilon = 1$;
- twist-4 operators with $\mathcal{P}$-parity $\varepsilon = e^{\pm i2\pi/3}$.

In the application to the nucleon distribution amplitudes we are interested in the eigenfunctions that satisfy the relations in (3.9) (otherwise, the matrix elements of the corresponding local operators between the vacuum and nucleon state vanish). One can easily verify that the following combination of the eigenfunctions with $\varepsilon = e^{+2i\pi/3}$ and $\varepsilon = e^{-2i\pi/3}$ has the required symmetry property:

$$\Psi_{N,q}(\vec{z}) \sim (1+\mathcal{P}_{12})\;\Psi^{\varepsilon}_{N,q}(\vec{z})\,. \tag{5.10}$$

The twist-4 eigenfunctions with $\varepsilon = 1$ do not satisfy the relation (3.9). They are relevant e.g. for the distribution amplitudes of baryons with isospin $I = 3/2$.

5.1.2 Complete Integrability

The Hamiltonian (4.38) possesses a hidden integral of motion. Let

$$S_{ik} = \partial_k (z_k - z_i) \equiv (\partial/\partial z_k)(z_k - z_i)\,. \tag{5.11}$$

It is easy to see that S_{ik} acts as the intertwining operator between the representations $T^{j_k=1/2} \otimes T^{j_i=1}$ and $T^{j_k=1} \otimes T^{j_i=1/2}$:

$$S_{ik} T^{j_k=1/2} \otimes T^{j_i=1} = T^{j_k=1} \otimes T^{j_i=1/2} S_{ik}\,. \tag{5.12}$$

We define two-particle (3×3 matrix) operators $[Q^{\pm}_{ik}]^{i'k'}$, where $i < k$ and $i', k' = 1, 2, 3$, by

$$[Q^{\pm}_{ik}]^{ik} = S_{ik}\,, \qquad\qquad [Q^{\pm}_{ik}]^{ki} = S_{ki} \tag{5.13}$$

and all other off-diagonal matrix elements being zero. For the diagonal matrix elements we put $[Q^{\pm}_{ik}]^{ii} = [Q^{\pm}_{ik}]^{kk} = \frac{1}{2}$ and

$$[Q^{+}_{ik}]^{jj} = \frac{1}{2} + S_{ik} \qquad\qquad [Q^{-}_{ik}]^{jj} = \frac{1}{2} + S_{ki}, \tag{5.14}$$

for j different from i and k. For example, explicit expressions for $Q^{\pm}_{12}$ are

$$Q^{+}_{12} = \frac{1}{2}\mathbb{I} + \begin{pmatrix} 0 & S_{12} & 0 \\ S_{21} & 0 & 0 \\ 0 & 0 & S_{12} \end{pmatrix}, \qquad Q^{-}_{12} = \frac{1}{2}\mathbb{I} + \begin{pmatrix} 0 & S_{12} & 0 \\ S_{21} & 0 & 0 \\ 0 & 0 & S_{21} \end{pmatrix}. \tag{5.15}$$

The two-particle Casimir operators $\widehat{J}^2_{ik}$ can be written in terms of $Q^{\pm}_{ik}$ as[§]

$$\widehat{J}^2_{ik} = \frac{1}{2}\{Q^{+}_{ik}, Q^{-}_{ik}\} - \frac{1}{4} \tag{5.16}$$

The Hamiltonian $\mathcal{H}^{\psi\psi\psi}_q$ can be represented in the form

$$\mathcal{H}^{\psi\psi\psi}_q = \mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{31} + \frac{3}{2}\,, \tag{5.17}$$

where

$$\mathcal{H}_{12} = 2\Big[\psi(\widehat{J}^2_{12}) - \psi(2)\Big] = \begin{pmatrix} \mathcal{H}^v_{12} - 1 & \mathcal{H}^e_{12} & 0 \\ \mathcal{H}^e_{12} & \mathcal{H}^v_{12} - 1 & 0 \\ 0 & 0 & \mathcal{H}^v_{12} \end{pmatrix} \tag{5.18}$$

and similarly for $\mathcal{H}_{23}, \mathcal{H}_{31}$.

[§]It is easy to see that the anticommutator $\widehat{J}^2_{ik}$ of the $Q^{\pm}_{ik}$ operators is $SL(2,\mathbb{R})$ invariant whereas the $Q^{\pm}_{ik}$ themselves are not.

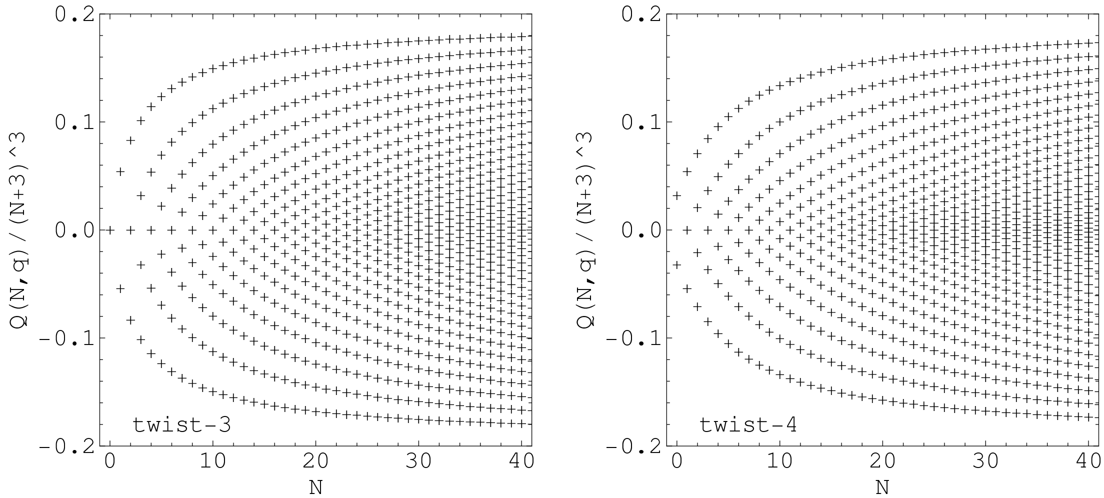

Figure 1: The spectrum of the conserved charge, $Q_3/(N+3)^3$, for twist-3 and twist-4 chiral quark operators.

Finally, let[¶]

$$\widehat{Q}_3 = \frac{i}{2}[\widehat{J}^2_{12}, \widehat{J}^2_{23}]\,. \tag{5.19}$$

The operator $\widehat{Q}_3$ commutes with the Hamiltonian $\mathcal{H}_q^{\psi\psi\psi}$:

$$[\widehat{Q}_3, \mathcal{H}_q^{\psi\psi\psi}] = 0 \tag{5.20}$$

and defines, therefore, a nontrivial integral of motion. To prove (5.20) it is sufficient to show that

$$[(\mathcal{H}_{ik}), \widehat{Q}_3] = i(\widehat{J}^2_{kj} - \widehat{J}^2_{ji})\,, \tag{5.21}$$

where $j \neq i, k$, and the pair-wise Hamiltonians $(\mathcal{H}_{ik})$ are given in Eq. (5.18). This can be done by calculating both sides of the relation (5.21) in the conformal basis, see Refs. [9, 15] for the details.

It is straightforward to check that the operator $\widehat{Q}_3$ commutes with the operator of cyclic permutations $\mathcal{P}$ and anticommutes with $\mathcal{P}_{12}$,

$$[\widehat{Q}_3, \mathcal{P}] = \{\widehat{Q}_3, \mathcal{P}_{12}\} = 0\,. \tag{5.22}$$

Eq. (5.22) together with (5.20) imply that all eigenstates of the Hamiltonian are double degenerate, except for the states which are annihilated by $\widehat{Q}_3$ i.e. $\widehat{Q}_3\Psi = 0$, cf. [9]. The spectrum of $\widehat{Q}_3$ is shown and compared with the corresponding spectrum of the twist-3 conserved charge [9] in Fig. 1.

[¶] Note that $[\widehat{J}^2_{12}, \widehat{J}^2_{23}] = [\widehat{J}^2_{13}, \widehat{J}^2_{12}] = [\widehat{J}^2_{23}, \widehat{J}^2_{13}]$.

It was shown in Ref. [27] that the spectrum of one-loop anomalous dimensions of (anti)chiral composite operators in QCD, i.e. operators constructed from the (anti)chiral fields and their derivatives, coincides with the spectrum of a certain (integrable) $SU(2,2)$-invariant spin chain. The Hamiltonian (5.17) can be viewed as the restriction of the general $SU(2,2)$ spin chain Hamiltonian on the subspace with twist $E=4$ and helicity $H=1/2$.

5.1.3 The spectrum of anomalous dimensions

A short-distance expansion of the nonlocal operator $\mathbb{O}(\vec{z})$, see Eq. (4.9), runs over a complete set of local operators $\mathbb{O}_{N,q}$ including operators with total derivatives. It is clear that in order to find the anomalous dimensions the operators with total derivatives can be omitted. The operators without total derivatives can be singled out by their properties under conformal transformations: they transform according to (2.41) and are usually referred to as conformal operators. The coefficient functions $\Psi_{N,q}(\vec{z})$ corresponding to the conformal operators satisfy the following constraints

$$(\partial_1+\partial_2+\partial_3)\Psi_{N,q}(\vec{z})=0\,, \tag{5.23}$$

$$\partial_1\Psi^{(1)}_{N,q}(\vec{z})+\partial_2\Psi^{(2)}_{N,q}(\vec{z})+\partial_3\Psi^{(3)}_{N,q}(\vec{z})=0\,, \tag{5.24}$$

that follows from the requirement that such operators correspond to highest weights of the corresponding representation. To get the first equation, we apply the operator $P^{2\dot{2}}$ which generates shifts along the "plus" light-cone direction to Eq. (4.9). One gets

$$\sum_{N,q}\Big[(\partial_1+\partial_2+\partial_3)\Psi_{N,q}(\vec{z})\Big]\,\mathbb{O}_{N,q}=\sum_{N,q}\Psi_{N,q}(\vec{z})\,\partial_+\mathbb{O}_{N,q}\,. \tag{5.25}$$

The r.h.s. of this identity contains only operators with total derivatives, hence the coefficients of the conformal operators on the l.h.s. must vanish.

To derive Eq. (5.24) we apply the transverse derivative $P^{1\dot{2}}$ to the nonlocal operator of leading twist–3: $\mathbb{O}^{tw-3}(\vec{z})=\epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\psi_+^{c,k}(z_3)$. Taking into account that $i[\mathbf{P}^{1\dot{2}}\psi_+](z)=-2\partial_z\psi_-(z)$, one obtains

$$i[\mathbf{P}^{1\dot{2}},\mathbb{O}^{tw-3}(\vec{z})]=-2\sum_{k=1}^{3}\frac{\partial}{\partial z_k}Q_k(\vec{z})=-2\sum_{N,q}\left(\sum_{k=1}^{3}\frac{\partial}{\partial z_k}\Psi^{(k)}_{N,q}(\vec{z})\right)\mathbb{Q}_{N,q}\,, \tag{5.26}$$

where $Q_k(\vec{z})$ are defined in Eq. (3.5). Again, since the l.h.s. only contains operators with the total derivatives, the coefficients of conformal operators on the r.h.s. have to vanish.

The spectrum of the Hamiltonian $\mathcal{H}_q^{\psi\psi\psi}$ can be studied using powerful Quantum Inverse Scattering Methods (QISM) [41] which is however beyond the scope of present paper. For this work we adopted a "brute-force" method, calculating $\mathcal{H}_q^{\psi\psi\psi}$ in the basis of functions

$$e_{N,k}(z_1,z_2,z_3)=\frac{(z_1-z_2)^k\,(z_1-z_3)^{N-k}}{k!(N-k)!)}\,, \tag{5.27}$$

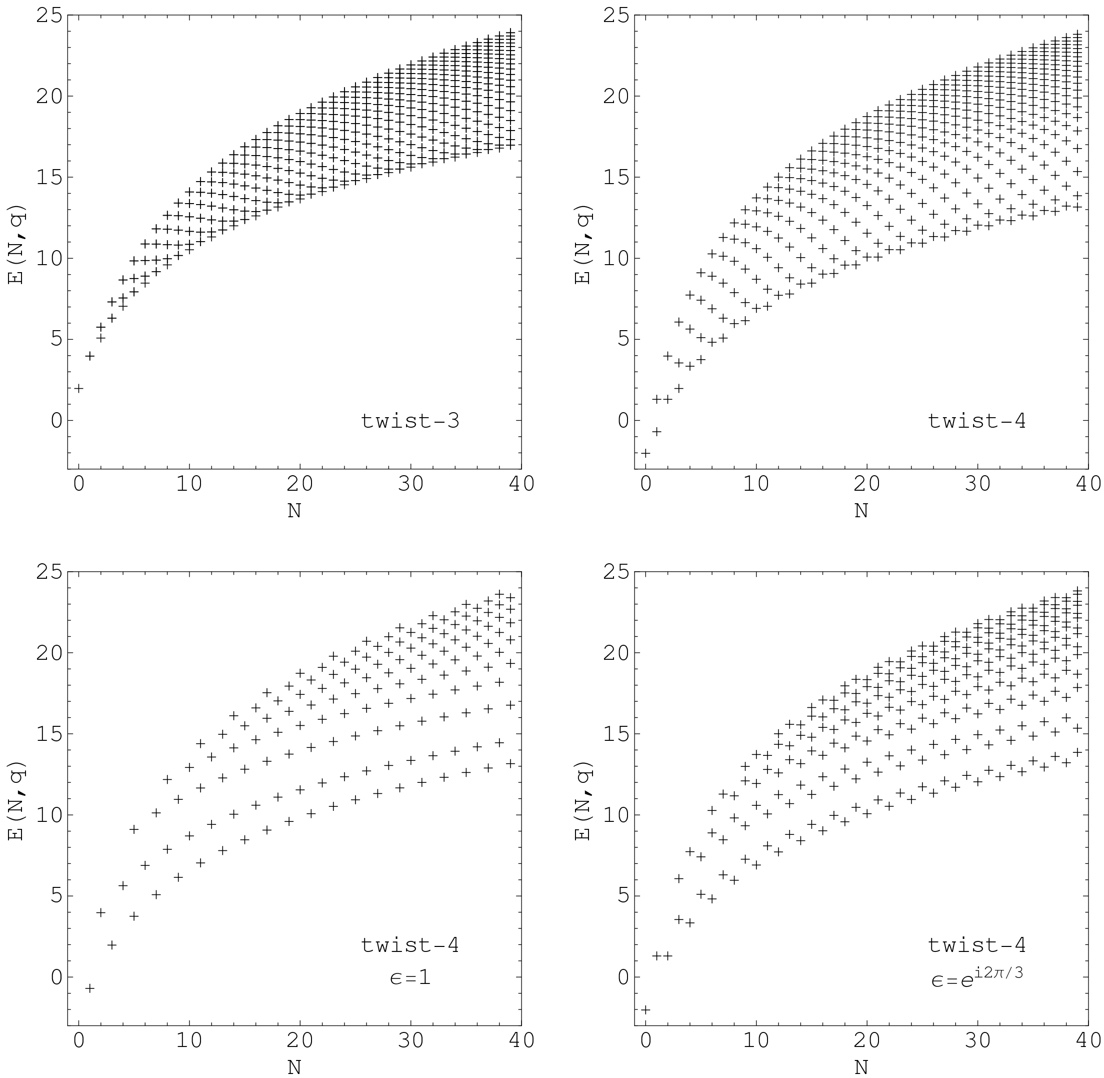

Figure 2: The spectrum of the Hamiltonian $\mathbb{H}_q^{\psi\psi\psi} = (1 + 1/N_c)\mathcal{H}_q^{\psi\psi\psi}$.

$$\mathcal{H}_q^{\psi\psi\psi} e_{N,k} = \sum_{k'=0}^{N} (\mathcal{H}_q^{\psi\psi\psi})_{k'k}\, e_{N,k'}\,, \tag{5.28}$$

and diagonalizing the resulting $(N+1)\times(N+1)$ matrix $(\mathcal{H}_q^{\psi\psi\psi})_{k'k}$ numerically. The translation invariance of $e_{N,k}$ guarantees that the constraint in Eq. (5.23) is satisfied identically.

The results are presented in Fig. 2. The spectra of twist-3 and twist-4 operators are shown in the upper left and upper right panels, respectively. The two lower panels show the twist-4 spectra in the sectors with $\varepsilon = 1$ and $\varepsilon = e^{i2\pi/3}$ separately. In addition, the numerical values of the eigenvalues for $N \le 6$ are collected in Table 2.

N	$E_{N,0}$	$E_{N,1}$	$E_{N,2}$	$E_{N,3}$
0	-2^*	-	-	-
1	$-\frac{2}{3}$	$\frac{4}{3}^*$	-	-
2	$\frac{4}{3}^*$	4	-	-
3	2	$\frac{29-\sqrt{57}}{6}^*$	$\frac{29+\sqrt{57}}{6}^*$	-
4	$\frac{167-3\sqrt{481}}{30}^*$	$\frac{17}{3}$	$\frac{167+3\sqrt{481}}{30}^*$	-
5	$\frac{34}{9}$	$\frac{77}{15}^*$	$\frac{67}{9}^*$	$\frac{137}{15}$
6	3.633418^*	311/60	6.687457^*	7.724361^*

Table 2: Anomalous dimensions of twist-4 chiral-quark operators in units of $\alpha_s/(2\pi)$; N is the total number of covariant derivatives. The entries marked with an asterisk correspond to the operators with $\mathcal{P}$-parity $\varepsilon = e^{\pm i2\pi/3}$ and the remaining ones to $\varepsilon = 1$. All anomalous dimensions except for the lowest ones, $E_{N,0}$, for odd $N = 2k+1$, are double degenerate.

The general features of the twist-3 and twist-4 spectra are similar. For both cases all eigenvalues are double degenerate (see above), except for those corresponding to zero eigenvalue of the conserved charge $\widehat{Q}_3\Psi_{N,q} = Q_3\Psi_{N,q}$, $Q_3 = 0$. These special eigenvalues turn out to be the lowest ones in the spectrum and can be found explicitly. There are two series of such states, one in the twist-3 sector and one in the twist-4 sector.

The twist-3 eigenstates with $Q_3 = 0$ were studied in Ref. [9]. They exist for even N, are invariant under cyclic permutations and have the energy

$$E^{tw-3}_{Q_3=0}(N) = \left(1+\frac{1}{N_c}\right)\left\{4\big[\psi(N+3)-\psi(2)\big]-\frac{1}{2}\right\}, \qquad N-\text{even}\,, \tag{5.29}$$

where $\psi(x)$ is the Euler ψ–function. The twist-4 eigenstates with $Q_3 = 0$ exist for odd N. They are also invariant and cyclic permutations ($\varepsilon = 1$) and have the energy

$$E^{tw-4}_{Q_3=0}(N) = \left(1+\frac{1}{N_c}\right)\left\{4\left[\psi\left(\frac{N+3}{2}\right)-\psi(2)\right]-\frac{1}{2}\right\}, \qquad N-\text{odd}. \tag{5.30}$$

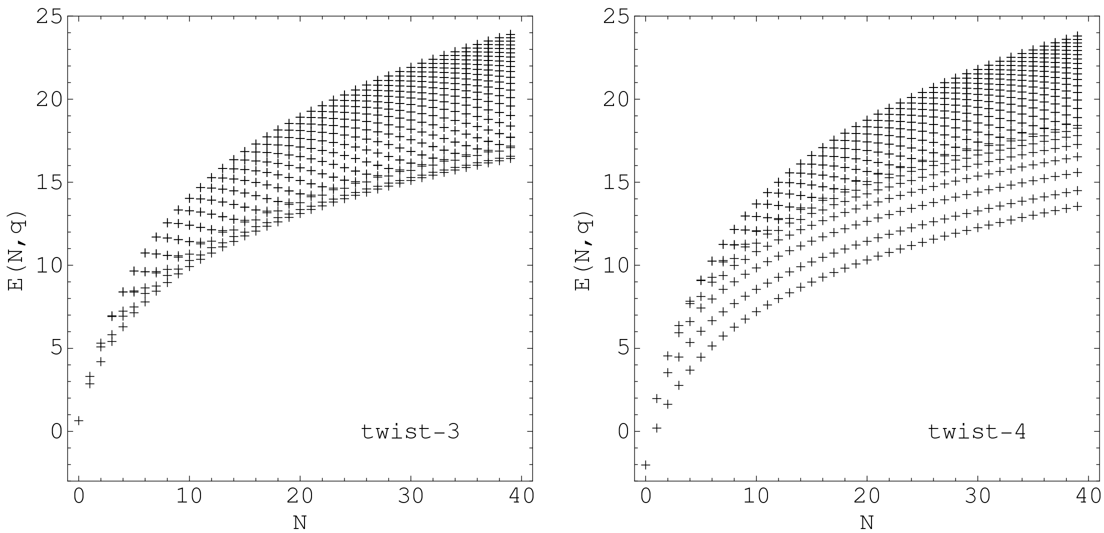

Figure 3: The spectrum of the Hamiltonian $\mathbb{H}_q^{\psi\psi\bar{\chi}} = (1 + 1/N_c)\, \mathcal{H}_q^{\psi\psi\bar{\chi}}$.

The appearance of $(N+3)/2$ as argument of the Euler ψ–function is characteristic for systems involving half-integer conformal spins, cf. [12, 42]

As is seen from Fig. 2 the upper parts of the twist-3 and twist-4 spectra are very similar to each other and are in fact interlacing. In particular the line of the largest eigenvalues is the same for both twists and is given by [9]

$$E_{\max}(N) = \left(1 + \frac{1}{N_c}\right)\left\{6\ln N - 3\ln 3 + 6\gamma_E + \frac{3}{2} + \mathcal{O}(1/N)\right\}. \tag{5.31}$$

The distance between the neighboring eigenvalues is in both cases $\mathcal{O}(1/N)$ in the upper part and $\mathcal{O}(1/\ln^2 N)$ in the lower part of the spectra, respectively.

5.2 Mixed chirality quark operators

The analysis of operators of mixed chirality goes along the same lines. The eigenfunctions of conformal twist-4 operators have to satisfy the constraint in Eq. (5.23) and in addition

$$\frac{\partial}{\partial z_1}\Psi^{(1)}_{N,q}(\vec{z}) + \frac{\partial}{\partial z_2}\Psi^{(2)}_{N,q}(\vec{z}) = \Psi^{(3)}_{N,q}(\vec{z})\,, \tag{5.32}$$

instead of (5.24). The spectra of the operators of geometric twist-3 and twist-4 are shown in Fig. 3. They are non-degenerate in both cases, in difference to the chiral operators. The numerical values of the twist-4 eigenvalues for $N \le 4$ are collected in Table 3.

The eigenvalues in the upper part of the spectra for twist-3 and twist-4 are very close to each other and to the corresponding eigenvalues for chiral operators, cf. Fig. 2. In particular, to the $\mathcal{O}(1/N)$ accuracy the line of the largest eigenvalues does not depend on chirality. It is the same for both twists and is given by Eq. (5.31).

N	$E_{N,0}$	$E_{N,1}$	$E_{N,2}$	$E_{N,3}$	$E_{N,4}$
0	-2	-	-	-	-
1	$2/9$	2	-	-	-
2	$\frac{2(14-\sqrt{43})}{9}$	$32/9$	$\frac{2(14+\sqrt{43})}{9}$	-	-
3	$\frac{197-\sqrt{5089}}{45}$	$\frac{49-\sqrt{73}}{9}$	$\frac{49+\sqrt{73}}{9}$	$\frac{197+\sqrt{5089}}{45}$	-
4	3.706620	$\frac{589-\sqrt{11161}}{90}$	6.634936	$\frac{589+\sqrt{11161}}{90}$	7.858442

Table 3: Anomalous dimensions of twist-4 quark operators of mixed chirality in units of $\alpha_s/(2\pi)$; N is the total number of covariant derivatives.

5.3 Quark-gluon operators

Anomalous dimensions of four-particle twist-4 quark-gluon operators correspond to the eigenvalues of the Hamiltonians $\mathbb{H}_g^{\psi\psi\psi\bar{f}}$ and $\mathbb{H}_g^{\psi\psi\chi\bar{f}}$ for the pure and mixed quark chirality cases, respectively.

The Hamiltonian $\mathbb{H}^{\psi\psi\psi\bar{f}}$ is given explicitly in Eqs. (4.41)–(4.43). It has the same symmetries as $\mathbb{H}^{\psi\psi\psi}$ and its eigenfunctions can be classified by parity with respect to the cyclic permutation $\mathcal{P}$ of the three quarks (5.1), $\varepsilon = 1, e^{\pm i2\pi/3}$ Due to the identity (3.7) we are interested in eigenfunctions which satisfy the restriction

$$\sum_{i=1}^{3} \Psi_{N,q}^{(i)}(z_1, z_2, z_3, z_4) = 0\,. \tag{5.33}$$

The eigenfunctions belonging to the sectors with $\varepsilon = e^{\pm i2\pi/3}$ have the same eigenvalues, $E_{N,q}(\varepsilon) = E_{N,q}(\varepsilon^{-1})$, and are related to each other as $\Psi^{\varepsilon^{-1}}_{N,q}(\vec{z}) = \mathcal{P}_{12}\Psi^{\varepsilon}_{N,q}(\vec{z})$. As in the case of quark operators, in the applications to nucleon distribution amplitudes we are interested in the eigenfunctions of particular symmetry, cf. (3.10). One can easily verify that the combination $\Psi_{N,q}(\vec{z}) = (1 + \mathcal{P}_{12})\,\Psi^{\varepsilon}_{N,q}(\vec{z})$ with $\varepsilon = e^{i2\pi/3}$ has all the necessary properties. In turn, the eigenvalues with $\varepsilon = 1$ are not degenerate and the corresponding operators (eigenfunctions) are relevant e.g. for the Δ-baryon.

In order to calculate the spectrum we have used the basis of functions

$$e_{N,k,m}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_4)^k\,(z_2 - z_4)^m (z_3 - z_4)^{N-2-k-m}}{k!\,m!\,(N-2-k-m)!} \tag{5.34}$$

and diagonalized the resulting $N(N-1)\times N(N-1)$ matrix numerically. The translation invariance of $e_{N,k.m}$ corresponds to the restriction to conformal operators, cf. Eq. (5.23).

The combined spectrum of three-quark and three-quark-gluon chiral operators is shown in Fig. 4. (all parities) and in Fig. 5 ($\varepsilon = e^{i2\pi/3}$ only). Note that the quark-gluon spectrum is much more dense as for given N there are $\mathcal{O}(N)$ quark and $\mathcal{O}(N^2)$ quark-gluon eigenstates. Also, it is seen that for large N the spectra overlap significantly. It is

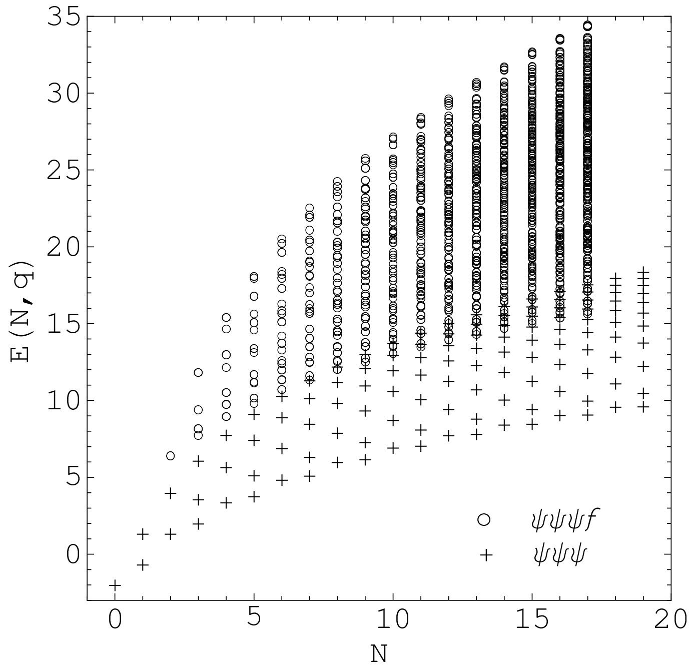

Figure 4: The combined spectrum of the Hamiltonians $\mathbb{H}_q^{\psi\psi\psi}$ (crosses) and $\mathbb{H}_g^{\psi\psi\psi\bar{f}}$ (open circles), all parities.

easy to show that for $N \to \infty$ the eigenvalues lie within the bands

$$\frac{16}{3}\ln N < E_{N,q}^{\psi\psi\psi} < 8\ln N\,, \qquad \frac{16}{3}\ln N < E_{N,q}^{\psi\psi\psi\bar{f}} < 14\ln N\,. \tag{5.35}$$

Note that the lowest four-particle quark-gluon eigenvalue has the same logarithmic asymptotic as the lowest three-quark one, so that they are separated at most by a constant. This suggests that for most of the quark eigenstates there is strong mixing with the quark-gluon ones, see the next Section.

The combined spectrum of three-quark and three-quark-gluon operators with mixed chirality is shown in Fig. 6. It is similar to the chiral case, except that all eigenvalues are nondegenerate.

5.4 Multiplicatively renormalizable operators

For the construction of multiplicatively renormalizable operators one has to take into account the off-diagonal quark-gluon blocks $\mathbb{H}_{qg}^{\rm chiral}$ and $\mathbb{H}_{qg}^{\rm mixed}$ for the chiral and mixed cases, respectively. Note that the complete (reduced) evolution Hamiltonian $\widetilde{\mathbb{H}}$ is not Hermitian, therefore its eigenfunctions are not mutually orthogonal.

In order to find the multiplicatively renormalizable operators we adopt the following procedure. The nonlocal operator $\mathbb{O}(\vec{z})$ can be expanded over a complete basis of multiplicatively renormalizable local operators (4.9) with the coefficient functions $\Psi_{N,q}(\vec{z})$

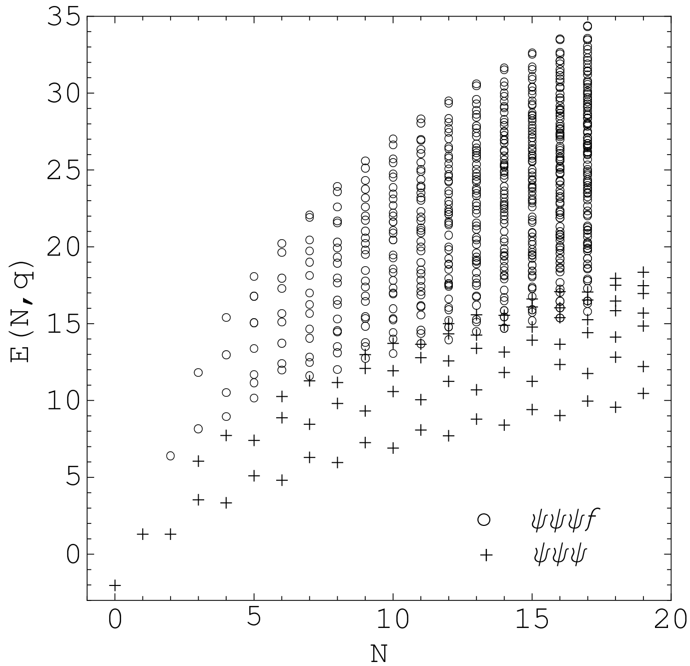

Figure 5: The same as in Fig. 4, but for $\varepsilon = e^{i2\pi/3}$ only. All eigenvalues are double–degenerate.

that are eigenfunctions of the Hamiltonian $\widetilde{\mathbb{H}}$

$$[\widetilde{\mathbb{H}}\Psi_{N,q}](\vec{z}) = E_{N,q}\Psi_{N,q}(\vec{z})\,, \tag{5.36}$$

cf. (4.10). Let $\Psi^{\dagger}_{N,q}$ be the eigenfunctions of the adjoint operator $\widetilde{\mathbb{H}}^{\dagger}$

$$[\widetilde{\mathbb{H}}^{\dagger}\Psi^{\dagger}_{N,q}](\vec{z}) = E_{N,q}\Psi^{\dagger}_{N,q}(\vec{z})\,. \tag{5.37}$$

The functions $\Psi_{N,q}(\vec{z})$ and $\Psi^{\dagger}_{N,q}(\vec{z})$ form a bi-orthogonal system:

$$\langle\Psi^{\dagger}_{N',q'}|\Psi_{N,q}\rangle = \delta_{NN'}\delta_{qq'}\,. \tag{5.38}$$

Eigenstates corresponding to different eigenvalues are orthogonal with respect to the scalar product (4.35). The multiplicatively renormalizable local operators $\mathbb{O}_{N,q}$ can be obtained as the scalar product of the nonlocal operator with the eigenfunction of the adjoint operator

$$\mathbb{O}_{N,q} = \langle\Psi^{\dagger}_{N,q}|\mathbb{O}\rangle. \tag{5.39}$$

With the increasing N the expressions rapidly become very cumbersome so that in this work we present explicit results for $N = 0, 1, 2$ only.

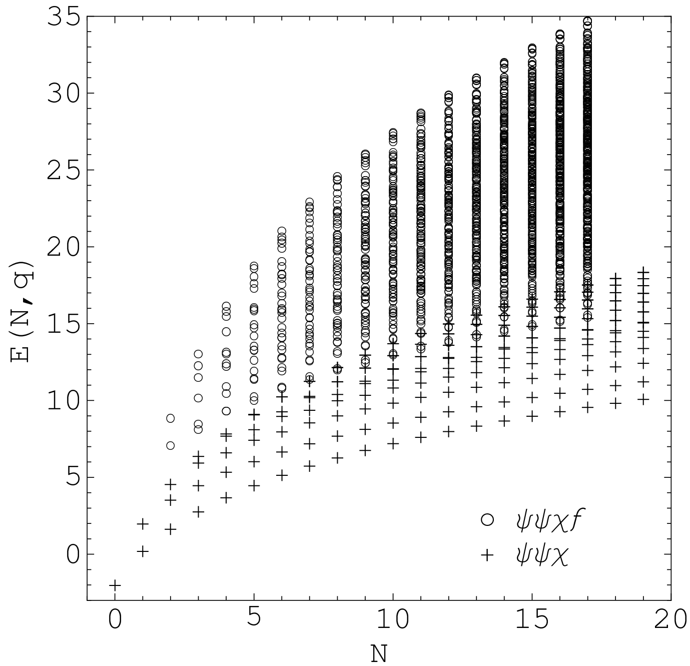

Figure 6: The combined spectrum of the Hamiltonians $\mathbb{H}_q^{\psi\psi\chi}$ (crosses) and $\mathbb{H}_g^{\psi\psi\chi\bar{f}}$ (open circles).

To begin with, consider chiral operators. Let

$$\begin{aligned}
Q_1^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_-^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,\\
Q_2^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_-^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,\\
Q_3^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_-^c]^k\,,
\end{aligned} \tag{5.40}$$

and

$$\begin{aligned}
G_1^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(n\cdot D)^{k_4}\bar{f}_{++}(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,\\
G_2^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}(\mu\lambda)\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_4}\bar{f}_{++}(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,
\end{aligned} \tag{5.41}$$

As discussed above, the eigenfunctions alias multiplicatively renormalizable operators can always be chosen in this case to have definite parity with respect to the cyclic permutations (5.3). We remind that the spectrum for $\varepsilon = e^{\pm i2\pi/3}$ is double degenerate. Any linear combination of the corresponding eigenfunctions satisfies the evolution equation. We present the results corresponding to the operators of definite symmetry under permutation of the first and the second quark

$$\Psi^{\pm}_{N,q} = (1 \pm \mathcal{P}_{12})\Psi^{\varepsilon}_{N,q} \tag{5.42}$$

which are more convenient for applications than those with definite $\varepsilon = e^{\pm i2\pi/3}$. One obtains

$$\begin{aligned}
\mathbb{O}^{chiral,+}_{0,0} =& Q^{(000)}_1 + Q^{(000)}_2 - 2Q^{(000)}_3 \,, \\
\mathbb{O}^{chiral,+}_{1,1} =& Q^{(100)}_1 - \frac{3}{2}Q^{(010)}_1 + Q^{(001)}_1 - \frac{3}{2}Q^{(100)}_2 + Q^{(010)}_2 + Q^{(001)}_2 + \frac{1}{2}Q^{(100)}_3 \\
&+ \frac{1}{2}Q^{(010)}_3 - 2Q^{(001)}_3 \,, \\
\mathbb{O}^{chiral,+}_{2,0} =& Q^{(200)}_1 + 2Q^{(020)}_1 + Q^{(002)}_1 - 4Q^{(110)}_1 + 2Q^{(101)}_1 - 4Q^{(011)}_1 + 2Q^{(200)}_2 \\
&+ Q^{(020)}_2 + Q^{(002)}_2 - 4Q^{(110)}_2 - 4Q^{(101)}_2 + 2Q^{(011)}_2 - 3Q^{(200)}_3 - 3Q^{(020)}_3 \\
&- 2Q^{(002)}_3 + 8Q^{(110)}_3 + 2Q^{(101)}_3 + 2Q^{(011)}_3 - \frac{7}{12}G^{(0000)}_1 - \frac{7}{12}G^{(0000)}_2 \,, \\
\mathbb{O}^{g,chiral,+}_{2,0} =& \frac{3}{2}\left(G^{(0000)}_1 + G^{(0000)}_2\right) \,, \qquad E^{g,chiral}_{2,0} = 19/3
\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}
\mathbb{O}^{chiral,-}_{0,0} =& Q^{(000)}_1 - Q^{(000)}_2 \,, \\
\mathbb{O}^{chiral,-}_{1,1} =& 3Q^{(100)}_1 + \frac{1}{2}Q^{(010)}_1 - 2\,Q^{(001)}_1 - \frac{1}{2}Q^{(100)}_2 - 3\,Q^{(010)}_2 + 2\,Q^{(001)}_2 \\
&- \frac{5}{2}Q^{(100)}_3 + \frac{5}{2}Q^{(010)}_3 \,, \\
\mathbb{O}^{chiral,-}_{2,0} =& Q^{(200)}_1 + \frac{4}{3}Q^{(020)}_1 + \frac{5}{3}Q^{(002)}_1 - 2\,Q^{(101)}_1 - 4\,Q^{(011)}_1 - \frac{4}{3}Q^{(200)}_2 - Q^{(020)}_2 - \frac{5}{3}Q^{(002)}_2 \\
&+ 4Q^{(101)}_2 + 2Q^{(011)}_2 + \frac{1}{3}Q^{(200)}_3 - \frac{1}{3}Q^{(020)}_3 - 2Q^{(101)}_3 + 2Q^{(011)}_3 \\
&- \frac{7}{36}G^{(0000)}_1 + \frac{7}{36}G^{(0000)}_2 \,, \\
\mathbb{O}^{g,chiral,-}_{2,0} =& \frac{1}{2}\left(G^{(0000)}_1 - G^{(0000)}_2\right) \,, \qquad E^{g,chiral}_{2,0} = 19/3 \,.
\end{aligned} \tag{5.44}$$

The multiplicatively renormalizable operators of the lowest dimension in the $\varepsilon = 1$ sector are

$$
\begin{aligned}
\mathbb{O}_{1,0}^{chiral,1} =& Q_1^{(010)} - Q_1^{(001)} - Q_2^{(100)} + Q_2^{(001)} + Q_3^{(100)} - Q_3^{(010)}\,,\\
\mathbb{O}_{2,1}^{chiral,1a} =& Q_1^{(020)} - Q_1^{(002)} - 6Q_1^{(110)} + 6Q_1^{(101)} - Q_2^{(200)} + Q_2^{(002)} + 6Q_2^{(110)} - 6Q_2^{(011)}\\
&+ Q_3^{(200)} - Q_3^{(020)} - 6Q_3^{(101)} + 6Q_3^{(011)}\,,\\
\mathbb{O}_{2,1}^{chiral,1b} =& Q_1^{(200)} - \frac{1}{2}Q_1^{(020)} - \frac{1}{2}Q_1^{(002)} - Q_1^{(110)} - Q_1^{(101)} + 2\,Q_1^{(011)}\\
&- \frac{1}{2}Q_2^{(200)} + Q_2^{(020)} - \frac{1}{2}Q_2^{(002)} - Q_2^{(110)} + 2\,Q_2^{(101)} - Q_2^{(011)}\\
&- \frac{1}{2}Q_3^{(200)} - \frac{1}{2}Q_3^{(020)} + Q_3^{(002)} + 2\,Q_3^{(110)} - Q_3^{(101)} - Q_3^{(011)}\,.
\end{aligned}
\tag{5.45}
$$

The operators $\mathbb{O}_{2,1}^{chiral,1a}$ and $\mathbb{O}_{2,1}^{chiral,1b}$ have the same anomalous dimension $E_{2,1}^{\rm chiral} = 4$, cf. Table 2, so that any linear combination of them is multiplicatively renormalizable as well. Note also that for $N = 2$ there is no quark-gluon operator, the reason being that for $\varepsilon = 1$ the cyclic permutation symmetry picks up the combination which is forbidden because of the condition in Eq. (3.7).

Next we consider the operators involving quark fields of mixed chirality. Let

$$
\begin{aligned}
\mathcal{Q}_1^{(k_1,k_2,k_3)} =& \epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_-^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,,\\
\mathcal{Q}_2^{(k_1,k_2,k_3)} =& \epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_-^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,,\\
\mathcal{Q}_3^{(k_1,k_2,k_3)} =& \frac{1}{2}\epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^{3/2,c}]^k\,,
\end{aligned}
\tag{5.46}
$$

where $\bar\chi_+^{3/2} \equiv \bar\chi_+^{(3/2,0)} = -(\mu D\bar\lambda)\bar\chi_+ \equiv -D_{\mu\dot\lambda}\bar\chi_+$, cf. Eq. (2.59), and

$$
\begin{aligned}
\mathcal{G}_1^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(n\cdot D)^{k_4}\bar f_{++}(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,,\\
\mathcal{G}_2^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_4}\bar f_{++}(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,.
\end{aligned}
\tag{5.47}
$$

In the mixed case there is no additional symmetry and all eigenvalues are non-degenerate.

The multiplicatively renormalizable operators of the lowest dimension read

$$\begin{aligned}
\mathbb{O}^{mixed}_{0,0} =& \mathcal{Q}_1^{(000)} - \mathcal{Q}_2^{(000)}\,,\\
\mathbb{O}^{mixed}_{1,0} =& \mathcal{Q}_1^{(100)} + \mathcal{Q}_1^{(010)} - \frac{3}{2}\mathcal{Q}_1^{(001)} - \mathcal{Q}_2^{(100)} - \mathcal{Q}_2^{(010)} + \frac{3}{2}\mathcal{Q}_2^{(001)}\,,\\
\mathbb{O}^{mixed}_{1,1} =& \mathcal{Q}_1^{(100)} - \mathcal{Q}_1^{(010)} + \frac{1}{2}\mathcal{Q}_1^{(001)} - \mathcal{Q}_2^{(100)} + \mathcal{Q}_2^{(010)} + \frac{1}{2}\mathcal{Q}_2^{(001)} + \mathcal{Q}_3^{(000)}\,,\\
\mathbb{O}^{mixed}_{2,1} =& \mathcal{Q}_1^{(200)} - \mathcal{Q}_1^{(020)} - \frac{2}{3}\mathcal{Q}_1^{(002)} - 2\mathcal{Q}_1^{(101)} + 3\mathcal{Q}_1^{(011)} - \mathcal{Q}_2^{(200)} + \mathcal{Q}_2^{(020)} - \frac{2}{3}\mathcal{Q}_2^{(002)}\\
&- 2\mathcal{Q}_2^{(011)} + 3\mathcal{Q}_2^{(101)} + \mathcal{Q}_3^{(100)} + \mathcal{Q}_3^{(010)} - \frac{4}{3}\mathcal{Q}_3^{(001)} + \frac{91}{282}\mathcal{G}_1^{(0000)} + \frac{91}{282}\mathcal{G}_2^{(0000)}\,,\\
\begin{pmatrix}\mathbb{O}^{mixed}_{2,0}\\ \mathbb{O}^{mixed}_{2,2}\end{pmatrix} =& \mathcal{Q}_1^{(200)} + \mathcal{Q}_1^{(020)} + \frac{4}{27}\left(4 \pm \sqrt{43}\right)\mathcal{Q}_1^{(002)} + \frac{4}{9}\left(-5 \pm \sqrt{43}\right)\mathcal{Q}_1^{(110)}\\
&+ \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)\mathcal{Q}_1^{(101)} - \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)\mathcal{Q}_1^{(011)} - \mathcal{Q}_2^{(200)} - \mathcal{Q}_2^{(020)}\\
&- \frac{4}{27}\left(4 \pm \sqrt{43}\right)\mathcal{Q}_2^{(002)} - \frac{4}{9}\left(-5 \pm \sqrt{43}\right)\mathcal{Q}_2^{(110)} - \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)\mathcal{Q}_2^{(011)}\\
&+ \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)\mathcal{Q}_2^{(101)} + \frac{1}{9}\left(19 \mp 2\sqrt{43}\right)\left[\mathcal{Q}_3^{(100)} - \mathcal{Q}_3^{(010)}\right]\\
&+ \frac{1}{234}\left(33 \mp 16\sqrt{43}\right)\left(\mathcal{G}_1^{(0000)} - \mathcal{G}_2^{(0000)}\right).\\
\mathbb{O}^{g,mixed}_{2,0} =& \mathcal{G}_1^{(0000)} - \mathcal{G}_2^{(0000)}\,, \qquad E^{g,mixed}_{2,0} = 7\,,\\
\mathbb{O}^{g,mixed}_{2,1} =& 3\left(\mathcal{G}_1^{(0000)} + \mathcal{G}_2^{(0000)}\right)\,, \qquad E^{g,mixed}_{2,1} = 79/9\,.
\end{aligned} \tag{5.48}$$

The corresponding eigenvalues (anomalous dimensions) are listed in Table 3.

Closing this section we want to emphasize that all of the above results are written for generic quark flavors and can be applied to the study of baryon states with arbitrary quantum numbers.

6 Nucleon Distribution Amplitudes

6.1 The leading twist-3 distribution amplitude

For completeness and for further use in Sec. 6.3.2 we write down the expansion for the leading twist-3 distribution amplitude $\Phi_3(x,\mu)$ (3.2)

$$\Phi_3(x,\mu) = x_1 x_2 x_3 \sum_{N,q} c_{Nq}\,\phi_{Nq}(\mu)\,P_{Nq}(x)\,, \tag{6.1}$$

where μ stands for the scale dependence. The polynomials $P_{N,q}(x)$ are related to the eigenfunctions $\Psi_{N,q}$ of the Hamiltonian $H_{1/2}$ (4.49) and are given by

$$P_{N,q}(x_1,x_2,x_3) = \langle e^{\sum x_k z_k} | \Psi_{Nq}(\vec{z})\rangle\,, \tag{6.2}$$

where $\langle \ldots \rangle$ is the $SU(1,1)$-invariant scalar product (4.34) corresponding to the spins $j_1 = j_2 = j_3 = 1$. The normalization constants $c_{N,q}$ are defined as

$$c_{Nq}^{-1} = ||\Psi_{Nq}(\vec{z})||^2/\Gamma(2N+6) = \int \mathcal{D}x\, x_1x_2x_3|P_{Nq}|^2\,. \tag{6.3}$$

Finally, the scale-dependent coefficients $\phi_{N,q}(\mu)$ are defined as reduced matrix elements of the multiplicatively renormalizable twist-3 operators

$$\begin{aligned}\mathbb{O}^{tw-3}(\vec{z}) =&\epsilon^{ijk}\psi_+^{u,i}(z_1)\bar{\chi}_+^{u,j}(z_2)\psi_+^{d,k}(z_3)\,,\\ \mathbb{O}^{tw-3}_{Nq}(\mu) =&P_{Nq}(\partial_z)\,\mathbb{O}^{tw-3}(z,\mu)|_{\vec{z}=0},\end{aligned} \tag{6.4}$$

$$\langle 0|\mathbb{O}^{tw-3}_{Nq}(\mu)|N\rangle = +\frac{1}{2}(pn)N_+^{\downarrow}\,(-ipn)^N\,\phi_{Nq}(\mu)\,. \tag{6.5}$$

Taking into account Eq. (6.3) one can project the coefficients ϕ_{Nq} as follows

$$\phi_{Nq}(\mu) = \int \mathcal{D}x\, P_{N,q}(x)\,\Phi_3(x,\mu). \tag{6.6}$$

The first few terms in the expansion (6.1) read

$$\begin{aligned}\Phi_3(x_1,x_2,x_3) =&120x_1x_2x_3\Big[\phi_0^{(2/3)} + 42\,\phi_{1,0}^{(26/9)}P_{1,0}(x) + 14\,\phi_{1,1}^{(10/3)}P_{1,1}(x) + \frac{63}{10}\,\phi_{2,0}^{(38/9)}P_{2,0}(x)\\ &+\frac{63}{2}\,\phi_{2,1}^{(46/9)}P_{2,1}(x) + \frac{9}{5}\,\phi_{2,2}^{(16/3)}P_{2,2}(x) + \ldots\Big],\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}P_{1,0}(x) =&\frac{1}{2}(x_1-x_3)\,,\\ P_{1,1}(x) =&\frac{1}{2}(x_1+x_3-2x_2)\,,\\ P_{2,0}(x) =&3x_1^2-3x_1x_2+2x_2^2-6x_1x_3-3x_2x_3+3x_3^2\,,\\ P_{2,1}(x) =&(x_1-x_3)(x_1+x_3-3x_2)\,,\\ P_{2,2}(x) =&x_1^2+x_3^2-12x_1x_3+9x_1x_2+9x_2x_3-6x_2^2\,.\end{aligned} \tag{6.8}$$

The superscript in $\phi_{N,q}^{(E_{Nq})}$ shows the corresponding anomalous dimension:

$$\phi_{N,q}^{(E_{Nq})}(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{E_{Nq}/\beta_0}\phi_{N,q}^{(E_{Nq})}(\mu_0)\,. \tag{6.9}$$

These expressions agree with the ones existing in the literature, e.g. [43, 9, 44, 45]. In notations of Ref. [36]

$$\phi_3^0 = \phi_0^{(2/3)}\,,\qquad \phi_3^- = 12\phi_{1,0}^{(26/9)} - \frac{7}{2}\phi_{1,1}^{(10/3)}\,,\qquad \phi_3^+ = 12\phi_{1,0}^{(26/9)} + \frac{21}{2}\phi_{1,1}^{(10/3)}\,. \tag{6.10}$$

6.2 Twist-4 distribution amplitudes: General formalism

The construction of twist-4 distribution amplitudes is somewhat more cumbersome because $\mathbb{O}(\vec{z})$ (4.6) involves five independent light-ray operators (for both chiral and mixed cases), and also because triangular mixing between quark and quark gluon operators makes the Hamiltonian non-hermitian. Solving Eq. (5.36) one obtains the expansion of $\mathbb{O}(\vec{z})$ in the form

$$\mathbb{O}(\vec{z}) = \sum_{N,q} C_{Nq} \Psi_{Nq}(\vec{z})\, \mathbb{O}_{Nq}\,, \tag{6.11}$$

where[‖]

$$\mathbb{O}_{Nq} = \langle \Psi^\dagger_{Nq} | \mathbb{O} \rangle\,, \qquad\qquad C^{-1}_{Nq} = \langle \Psi^\dagger_{Nq} | \Psi_{Nq} \rangle\,. \tag{6.12}$$

We remind that $\Psi^\dagger_{Nq}$ are the eigenfunctions of the adjoint Hamiltonian (with respect to the scalar product (4.35)), cf. Eq. (5.37). It is convenient to split the sum in (6.11) in two: one contains the "three-quark operators", and the other one – three-quark-gluon operators:

$$\mathbb{O}(\vec{z}) = \sum_{N,q} A_{nq} \Psi_{Nq}(\vec{z})\, \mathbb{O}_{Nq} + \sum_{N \geq 2,q} B_{nq} \Psi^g_{Nq}(\vec{z})\, \mathbb{O}^g_{Nq}\,. \tag{6.13}$$

Due to a block-triangular form of the Hamiltonian $\widetilde{\mathbb{H}}$, Eq. (4.5), only the first three ("quark") components of the coefficient functions $\Psi^a_{Nq}(\vec{z}), a = 1, 2, \ldots, 5$ corresponding to the "quark" operators are nonzero, $\Psi^{a=4,5}_{N,q} = 0$. On the other hand, all five components of the coefficient functions Ψ^g_{Nq} are nonzero, in general.

For the eigenfunctions of the adjoint operator $\widetilde{\mathbb{H}}^\dagger$, (5.37), the situation is the opposite: the "quark" eigenfunctions have all components nonzero, whereas "quark" components of the quark-gluon eigenfunctions vanish, $\Psi^{g,\dagger,a=1,2,3}_{Nq} = 0$. Since the diagonal blocks of the Hamiltonian (4.5) are self-adjoint operators one can choose the eigenfunctions Ψ and $\Psi^\dagger$ as follows

$$\Psi_{Nq}(\vec{z}) = \begin{pmatrix} \overrightarrow{\Psi}_{Nq}(\vec{z}) \\ \overrightarrow{0} \end{pmatrix}, \qquad \Psi^\dagger_{Nq}(\vec{z}) = \begin{pmatrix} \overrightarrow{\Psi}_{Nq}(\vec{z}) \\ \overrightarrow{\widetilde{\Psi}}{}^\dagger_{Nq}(\vec{z}) \end{pmatrix},$$

$$\Psi^g_{Nq}(\vec{z}) = \begin{pmatrix} \overrightarrow{\widetilde{\Psi}}{}^g_{Nq}(\vec{z}) \\ \overrightarrow{\Psi}{}^g_{Nq}(\vec{z}) \end{pmatrix}, \qquad \Psi^{g,\dagger}_{Nq}(\vec{z}) = \begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{\Psi}{}^g_{Nq}(\vec{z}) \end{pmatrix}. \tag{6.14}$$

Here $\overrightarrow{\Psi}_{Nq}(\vec{z})$, $\overrightarrow{\widetilde{\Psi}}{}^g_{Nq}(\vec{z})$ are three–dimensional "vectors", whereas $\overrightarrow{\Psi}{}^g_{Nq}(\vec{z})$, $\overrightarrow{\widetilde{\Psi}}_{Nq}(\vec{z})$ are two–dimensional. It follows from (6.13) and (6.12) that the only remaining "daggered"

[‖]The coefficients C_{Nq} are added (cf. Eq. (4.9)) to allow for arbitrary normalization for the functions $\Psi_{Nq}, \Psi^\dagger_{Nq}$.

function $\overrightarrow{\widetilde{\Psi}}^{\dagger}_{Nq}(\vec{z})$ can be expressed as follows:

$$\overrightarrow{\widetilde{\Psi}}^{\dagger}_{Nq}(\vec{z}) = -\sum_{q'} B_{Nq'} \langle \overrightarrow{\Psi}_{Nq} | \overrightarrow{\widetilde{\Psi}}^{g}_{Nq'} \rangle \, \overrightarrow{\Psi}^{g}_{Nq}(\vec{z}) \,, \tag{6.15}$$

i.e. explicit construction of the adjoint Hamiltonian is in fact not necessary.

Our task is to write the expansion of the nucleon distribution amplitudes in contributions of multiplicatively renormalizable operators. To this end it is convenient to introduce auxiliary amplitudes

$$\langle 0|\mathbb{O}^a|P\rangle = -\frac{1}{4}(\mu\lambda)(-ipn)^{n_a} N_+^{\downarrow(\uparrow)} \int \mathcal{D}x \, e^{-i(pn)\sum_k x_k z_k} \mathcal{O}^a(x) \,, \quad a = 1, 2, \ldots, 5 \,. \tag{6.16}$$

The r.h.s. of Eq. (6.16) involves the nucleon spinor $N^{\downarrow}(N^{\uparrow})$ for the chiral (mixed) operator, respectively. The factors $(-ipn)^{n_a}$ are introduced for later convenience: For the chiral operators we choose $n = (0, 0, 0, 2, 2)$, and for the operators of mixed chirality $n = (0, 0, 1, 2, 2)$. It will always be clear from the context which operator, chiral or mixed, is considered, so that we do not show it explicitly.

The "standard" nucleon distribution amplitudes introduced in Sec. 3 are related to the amplitudes $\mathcal{O}^{(a)}$ as follows:

$$\Xi_4(x_1, x_2, x_3) = \mathcal{O}^{(1)}(x_1, x_2, x_3) \,, \qquad \Xi_4^g(x_1, x_2, x_3, x_4) = \mathcal{O}^{(4)}(x_1, x_2, x_3, x_4) \tag{6.17}$$

for the chiral case and

$$\begin{aligned}
\Psi_4(x_1, x_2, x_3) =& \mathcal{O}^{(1)}(x_2, x_3, x_1) \,, & \Psi_4^g(x_1, x_2, x_3, x_4) =& \mathcal{O}^{(4)}(x_2, x_3, x_1, x_4), \\
\Phi_4(x_1, x_2, x_3) =& \mathcal{O}^{(2)}(x_1, x_3, x_2) \,, & \Phi_4^g(x_1, x_2, x_3, x_4) =& \mathcal{O}^{(5)}(x_1, x_3, x_2, x_4), \\
D_4(x_1, x_2, x_3) =& \mathcal{O}^{(3)}(x_1, x_3, x_2) & &
\end{aligned} \tag{6.18}$$

for mixed chirality.

We define reduced matrix elements of the multiplicatively renormalizable local operators as

$$\langle 0|\mathbb{O}^a_{Nq}|P\rangle = -\frac{1}{4}(\mu\lambda)(-ipn)^N N_+^{\downarrow(\uparrow)} \, \phi_{Nq} \,. \tag{6.19}$$

It follows from Eqs. (6.16) and (6.12) that

$$\phi_{Nq} = \int \mathcal{D}x \sum_{ab} \overline{P^{a,\dagger}_{Nq}(x)} \, \Omega_{ab} \, \mathcal{O}^b(x) \,. \tag{6.20}$$

Here

$$P^{a,\dagger}_{Nq}(x) = \langle e^{\sum_k x_k z_k} | \Psi^{\dagger,a}_{Nq} \rangle_a \,, \tag{6.21}$$

where $\langle \ldots \rangle_a$ is the $SU(1,1)$ invariant scalar product (4.34) corresponding to the conformal spins of the function $\Psi^{\dagger,a}_{Nq}$. We also define the functions

$$P^a_{Nq}(x) = \langle e^{\sum_k x_k z_k} | \Psi^a_{Nq} \rangle_a . \tag{6.22}$$

The scalar product for the functions Ψ_{Nq}, $\Psi^\dagger_{Nq}$ that corresponds to conformal operators (i.e. $\Psi_{Nq}(z)$ and $\Psi^\dagger_{Nq}(z)$ are shift-invariant polynomials) can be written as

$$\langle \Psi^\dagger_{Nq} | \Psi_{Nq'} \rangle = c_N \int \mathcal{D}x \sum_{ab} \mu^a(x) \, \overline{P^{\dagger,a}_{Nq}(x)} \, \Omega_{ab} \, P^b_{Nq'}(x) \equiv c_N (P^\dagger_{Nq} | P_{Nq'}) \, , \tag{6.23}$$

where $c_N = \Gamma(2N+5)$ and the integration measure is defined as

$$\begin{aligned} &\mu_1(x) = x_2 x_3 \, , \qquad \mu_2(x) = x_1 x_3 \, , \qquad \mu_4(x) = \mu_5(x) = \frac{1}{2} x_1 x_2 x_3 x_4^2 \, , \\ &\mu_3(x) = \begin{cases} x_1 x_2 & \text{the chiral case} \\ \frac{1}{2} x_1 x_2 x_3^2 & \text{the mixed case} \, . \end{cases} \end{aligned} \tag{6.24}$$

Since the functions Ψ_{Nq} and $\Psi^\dagger_{Nq}$ form a bi-orthogonal system one obtains

$$\mathcal{O}^a(x,\mu) = \mu_a(x) \sum_{Nq} c_{Nq} \, \phi_{Nq}(\mu) \, P^a_{Nq}(x) \, , \tag{6.25}$$

where

$$c^{-1}_{Nq} = (P^\dagger_{Nq} | P_{Nq}) = C^{-1}_{Nq} / c_N \, . \tag{6.26}$$

The scale dependence of the reduced matrix elements is given by

$$\phi_{Nq}(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{E_{Nq}/\beta_0} \phi_{Nq}(\mu_0) \, . \tag{6.27}$$

We now specialize to the particular cases of interest.

6.3 Chiral amplitudes Ξ_4, Ξ_4^g

The expansion for the chiral quark, Ξ_4, and gluon, Ξ_4^g nucleon distribution amplitudes reads

$$\begin{aligned} \Xi_4(x,\mu) =& x_2 x_3 \left[\sum_{N,q} a_{Nq} \, \xi_{Nq}(\mu) \, \Pi_{Nq}(x) + \sum_{N \geq 2, q} b_{Nq} \, \xi^g_{Nq}(\mu) \, \widetilde{\Pi}^g_{Nq}(x) \right] , \\ \Xi^g_4(x,\mu) =& \frac{1}{2} x_1 x_2 x_3 x_4^2 \sum_{N \geq 2, q} b_{Nq} \, \xi^g_{Nq}(\mu) \, \Pi^g_{N,q}(x) \, , \end{aligned} \tag{6.28}$$

where $\Pi_{Nq}(x) = P^{a=1}_{Nq}(x)$, $\widetilde{\Pi}^g_{Nq}(x) = \widetilde{P}^{g,a=1}_{Nq}(x)$, $\Pi^g_{N,q}(x) = P^{g,a=1}_{Nq}$ and the expansion coefficients ξ_{Nk}, ξ^g_{Nk} are defined as reduced matrix elements of the multiplicatively renormalizable operators

$$\begin{aligned}\langle 0|\mathbb{O}^{chiral+}_{N,q}(\mu)|N\rangle =& -\frac{1}{4}(\mu\lambda)m_N N^{\downarrow}_{+}\,(-ipn)^N\,\xi_{Nq}(\mu)\,,\\ \langle 0|\mathbb{O}^{g,chiral+}_{N,q}(\mu)|N\rangle =& -\frac{1}{4}(\mu\lambda)m_N N^{\downarrow}_{+}\,(-ipn)^N\,\xi^g_{Nq}(\mu)\,.\end{aligned} \tag{6.29}$$

Explicit expressions for the operators $\mathbb{O}^{chiral+}_{N,q}$, $\mathbb{O}^{g,chiral+}_{N,q}$ for $N \le 2$ are given in Eq. (5.43); the corresponding anomalous dimensions can be found in Table 2.

Taking into account the symmetry properties (3.9), (3.10) one can write the normalization constants as

$$\begin{aligned}a^{-1}_{Nq} =& 2\int \mathcal{D}x\,\mu_1(x)\,\overline{\Pi_{N,q}(x)}\Big(2+P_{23}\Big)\Pi_{N,q}(x)\,,\\ b^{-1}_{Nq} =& 2\int \mathcal{D}x\,\mu_4(x)\,\overline{\Pi^g_{Nq}(x)}\Big(2+P_{23}\Big)\Pi^g_{Nq}(x)\,,\end{aligned} \tag{6.30}$$

where P_{23} is the permutation operator: $P_{23}\Phi(x_1,x_2,x_3) = \Phi(x_1,x_3,x_2)$. Projecting out the contribution of a particular operator one finds

$$\begin{aligned}\frac{1}{2}\xi^g_{Nq} =& \int \mathcal{D}x\,\overline{\Pi^g_{Nq}(x)}\Big(2+P_{23}\Big)\,\Xi^g_4(x)\,,\\ \frac{1}{2}\xi_{Nq} =& \int \mathcal{D}x\,\overline{\Pi_{Nq}(x)}\Big(2+P_{23}\Big)\,\Xi_4(x) - \sum_{q'} m^N_{qq'}\,\xi^g_{q'}\,,\end{aligned} \tag{6.31}$$

where

$$m^N_{qq'} = 2\int \mathcal{D}x\,\mu_1(x)\,\Pi_{Nq}(x)\Big(2+P_{23}\Big)\overline{\widetilde{\Pi}^g_{Nq'}(x)}\,. \tag{6.32}$$

Finally, taking into account the contributions with $N \le 2$ we obtain the distribution amplitudes

$$\begin{aligned}\Xi_4(x,\mu) =& 4x_2x_3\Big[\xi_{0,0}(\mu)\,\Pi_0(x) + 9\,\xi_{1,1}(\mu)\,\Pi_1(x) + 12\,\xi_{2,0}(\mu)\,\Pi_2(x) + \frac{28}{3}\xi^g_{2,0}(\mu)\,\widetilde{\Pi}^g_2(x)\Big],\\ \Xi^g_4(x,\mu) =& \frac{1}{3}8!\,x_1x_2x_3x_4^2\,\xi^g_2\,(\mu)\Pi^g_2(x)\,,\end{aligned} \tag{6.33}$$

where

$$\begin{aligned}\Pi_0(x) =& 1\,,\\ \Pi_1(x) =& x_1 + x_3 - \frac{3}{2}x_2\,,\\ \Pi_2(x) =& x_1^2 - 4x_1x_2 + 2x_2^2 + 2x_1x_3 - 4x_2x_3 + x_3^2\,,\\ \widetilde{\Pi}^g_2(x) =& \frac{43}{2}x_1^2 + 4x_1x_2 - 2x_2^2 - 47x_1x_3 + 4x_2x_3 + \frac{13}{2}x_3^2\,.\end{aligned} \tag{6.34}$$

and $\Pi_2^g(x) = 1/2$. In notation of Ref. [36]

$$\lambda_2 = \xi_{0,0}\,,$$
$$\lambda_1 f_2^d = \frac{4}{15}\xi_{0,0} + \frac{2}{5}\xi_{1,0}\,. \tag{6.35}$$

6.3.1 Mixed chirality amplitudes $\Phi_4, \Psi_4, \Phi_4^g, \Psi_4^g$

The distribution amplitudes Φ_4, Ψ_4, D_4 have collinear twist 4, but receive contributions both from geometric twist-3 and twist-4 operators. Moreover, among twist-4 operators there are descendants of twist-3 operators which matrix elements do not involve new nonperturbative parameters, cf. [46]. In the discussion of the operator renormalization in Sect. 5 such operators were excluded by imposing appropriate symmetry conditions on the solutions of the renormalization group equations, Eqs. (5.23), (5.32), and they are not listed in Sect. 5.4. They do contribute to the distribution amplitudes, however. The contributions to collinear twist-4 amplitudes that can be expressed in terms of the leading twist distribution are usually referred to as Wandzura-Wilczek contributions, with the remaining parts being "genuine" twist-4:

$$\Phi_4(x) = \Phi_4^{WW}(x) \;+\; \Phi_4^{tw-4}(x)\,,$$
$$\Psi_4(x) = \Psi_4^{WW}(x) \;+\; \Psi_4^{tw-4}(x)\,,$$
$$D_4(x) = D_4^{WW}(x) \;+\; D_4^{tw-4}(x)\,. \tag{6.36}$$

In what follows we consider these two types of contributions separately.

6.3.2 Wandzura-Wilczek contributions Φ_4^{WW}, Ψ_4^{WW}

Let

$$O^{tw-3}(\vec{z},\lambda) \equiv O^{tw-3}(z_1,z_2,z_3;\lambda) = \mathbb{O}^{tw-3}(z_1,z_3,z_2,\lambda) \tag{6.37}$$

be the leading-twist light-ray operator that enters the definition of the leading twist-3 distribution amplitude: In comparison to (6.4) we have replaced $z_2 \leftrightarrow z_3$ and added an argument λ to remind that the "plus" projection is done with respect to this particular spinor.

The short-distance expansion of $O^{tw-3}(\vec{z},\lambda)$ can be written as

$$O^{tw-3}(\vec{z},\lambda) = \sum_{N,q} C_{Nq}\left(\Phi_{Nq}(\vec{z})\,\mathbb{O}_{Nq}^{tw-3}(\lambda) + \frac{1}{4(N+3)}S^+\Phi_{Nq}(\vec{z})\,i[\mathbf{P}_{\lambda\bar{\lambda}},\mathbb{O}_{Nq}^{tw-3}(\lambda)]\right) + \dots\,. \tag{6.38}$$

Here the first term in parenthesis corresponds to the contribution of the conformal twist-3 operators $\mathbb{O}_{Nq}^{tw-3}(\lambda)$ (6.4) which are annihilated by the step-down operator of the $SL(2)$ algebra, so that $i[\mathbf{K}_{\mu\bar{\mu}}, \mathbb{O}^{tw-3}_{Nq}(\lambda)] = 0$. The corresponding coefficient functions $\Phi_{Nq}(z_1, z_2, z_3)$ are shift-invariant homogeneous polynomials of degree N and are related by a simple change of arguments $z_2 \leftrightarrow z_3$ to $\Psi_{Nq}(\vec{z})$ appearing in (6.2). The normalization coefficients are chosen as $C_{Nq} = ||\Phi_{Nq}||^{-2} = c_{Nq}/\Gamma(2N+6)$, cf. (6.3).

The second term in (6.38) corresponds to contribution of the operators that include one total derivative (in "plus" direction). Note that the corresponding coefficient functions can be obtained by application of the three-particle "step-up" operator $S^+ = S^+_{j_1=1} + S^+_{j_2=1} + S^+_{j_3=1} = \sum_k (z_k^2 \partial_{z_k} + 2z_k)$ (2.40) so they are not independent and do not need to be calculated separately. The terms with two and more total derivatives have the similar structure. They are indicated by ellipses.

Omitting contributions of genuine geometric twist-4 operators the expansion of the light-ray collinear twist-4 vector-operator $\mathbb{O}^{mixed}(\vec{z})$ reads to the same accuracy

$$
\begin{aligned}
[\mathbb{O}^a(\vec{z}, \mu, \lambda)]_{tw-3} =& \sum_{N,q} C_{Nq} \Psi^a_{Nq}(\vec{z})\, \mathbb{O}_{Nq}(\mu, \lambda) + \sum_{N\geq 1,q} C_{Nq}\, \widehat{\Psi}^a_{N+1,q}(\vec{z}) \widehat{\mathbb{O}}_{N+1,q}(\mu, \lambda) \\
&+ \sum_{N,q} C_{Nq} \frac{1}{2(2N+5)} S^+ \Psi^a_{Nq}(\vec{z})\, i[\mathbf{P}_{\lambda\bar{\lambda}}, \mathbb{O}_{Nq}(\mu, \lambda)] + \ldots\,, \qquad (6.39)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{O}_{Nq}(\mu, \lambda) =& \frac{1}{N+2} (\mu \partial_\lambda)\, \mathbb{O}^{tw-3}_{Nq}(\lambda)\,, \\
\widehat{\mathbb{O}}_{N+1,q}(\mu, \lambda) =& \frac{1}{4(N+3)^2} \left(i[\mathbf{P}_{\mu\bar{\lambda}}\,, \mathbb{O}^{tw-3}_{Nq}(\lambda))] - \frac{N+2}{2N+5} i[\mathbf{P}_{\lambda\bar{\lambda}}\,, \mathbb{O}^{tw-3}_{Nq}(\mu, \lambda)] \right) \qquad (6.40)
\end{aligned}
$$

are the light-cone conformal operators $i[\mathbf{K}_{\mu\bar{\mu}}, \mathbb{O}_{Nq}(\mu, \lambda)] = i[\mathbf{K}_{\mu\bar{\mu}}, \widehat{\mathbb{O}}_{N+1,q}(\lambda, \mu)] = 0$. (It is tacitly assumed that the generator S^+ in the second line in Eq. (6.39) involves the conformal spins of the functions it acts on.) The operator $\mathbb{O}_{Nq}(\mu, \lambda)$ has geometric twist-3 whereas $\widehat{\mathbb{O}}_{N+1,q}(\mu, \lambda)$ contains both twist-3 and twist-4 contributions.

The task is to find the corresponding (five-component) coefficient functions Ψ_{Nq}, $\widehat{\Psi}_{Nq}$. This can be done by observing that $\mathbb{O}^a(z, \mu, \lambda)$ only depends linearly on μ and essentially reduces to $O^{tw-3}(z, \lambda)$ after a formal substitution $\mu \to \lambda$. Taking into account the identities

$$
\lambda \partial_\mu \mathbb{O}^{(a=1,2)}(\vec{z}, \mu, \lambda) = O^{tw-3}(\vec{z}, \lambda)\,, \qquad \lambda \partial_\mu \mathbb{O}^{(a=3)}(\vec{z}, \mu\lambda) = -\partial_{z_2} O^{tw-3}(\vec{z}, \lambda) \qquad (6.41)
$$

and comparing the two representations in (6.38) and (6.39) one easily gets for Ψ_{Nq}

$$
\Psi^{a=1,2}_{Nq}(\vec{z}) = \Phi_{Nq}(z)\,, \qquad \Psi^{a=3}_{Nq}(\vec{z}) = -\,\partial_3 \Phi_{Nq}(\vec{z})\,, \qquad \Psi^{a=4}_{Nq}(\vec{z}) = \Psi^{a=5}_{Nq}(\vec{z}) = 0\,. \qquad (6.42)
$$

For the functions $\widehat{\Psi}^a_{Nq}$ one obtains, after some algebra

$$
\begin{aligned}
\widehat{\Psi}^{a=1(2)}_{N+1,q}(\vec{z}) =& -\left[S^+ - 2(N+3) z_{1(2)} \right] \Phi_{Nq}(\vec{z})\,, \\
\widehat{\Psi}^{a=3}_{N+1,q}(\vec{z}) =& -\left[2(2N+5) - \left[S^+ - 2(N+3) z_3 \right] \partial_3 \right] \Phi_{Nq}(\vec{z})\,, \qquad (6.43)
\end{aligned}
$$

where $S^+ = S^+_{j_1=1} + S^+_{j_2=1} + S^+_{j_3=1}$. One can easily check that the functions $\widehat{\Psi}_{Nq}$ defined in Eq. (6.43) are shift-invariant polynomials, as they should be. They are normalized as

$$||\Psi_{Nq}||^2 = (N+2)||\Phi_{Nq}||^2\,, \qquad ||\widehat{\Psi}_{N+1,q}||^2 = 2(2N+5)(N+3)^2||\Phi_{Nq}||^2\,. \tag{6.44}$$

Going over to the "P-representation" as defined in Eq. (6.22) we obtain

$$P^{(1)}_{Nq}(x) =\partial_1 x_1\, P^{tw-3}_{Nq}(x)\,, \quad P^{(2)}_{Nq}(x) =\partial_2 x_2\, P^{tw-3}_{Nq}(x)\,, \quad P^{(3)}_{Nq}(x) = -\,2\partial_3\, P^{tw-3}_{Nq}(x) \tag{6.45}$$

and

$$\begin{aligned}
\widehat{P}^{(1)}_{N+1,q}(x) =&\Big(2N+5-X\partial_{x_1}\Big)\, x_1\, P^{tw-3}_{Nq}(x)\,,\\
\widehat{P}^{(2)}_{N+1,q}(x) =&\Big(2N+5-X\partial_{x_2}\Big)\, x_2\, P^{tw-3}_{Nq}(x)\,,\\
\widehat{P}^{(3)}_{N+1,q}(x) =&-2\Big(2N+5-X\partial_3\Big)\, P^{tw-3}_{Nq}(x)\,,
\end{aligned} \tag{6.46}$$

where $X = x_1 + x_2 + x_3$.

Taking appropriate matrix elements we end up with the Wandzura-Wilczek contributions to the auxiliary distributions $\mathcal{O}^{(a)}$ (6.16)

$$[\mathcal{O}^{(a)}(x)]^{WW} = \mu_a(x)\sum_{Nq}\frac{c_{Nq}\,\phi_{Nq}}{(2N+5)}\left(\frac{1}{N+2}P^{(a)}_{Nq}(x) - \frac{1}{N+3}\widehat{P}^{(a)}_{N+1,q}(x)\right), \tag{6.47}$$

where c_{Nq}, ϕ_{Nq} are the twist-3 expansion coefficients, (6.1). One can easily verify the following relation (cf. Eq. (3.19)):

$$[\mathcal{O}^{(3)}(x)]^{WW} = x_1\,[\mathcal{O}^{(1)}(x)]^{WW} + x_2\,[\mathcal{O}^{(2)}(x)]^{WW}\,. \tag{6.48}$$

Finally, taking into account Eqs. (6.18) we obtain

$$\Phi^{WW}_4(x) = -\sum_{N,q}\frac{c_{Nq}\,\phi_{Nq}}{(N+2)(N+3)}\left(N+2-\frac{\partial}{\partial x_3}\right)x_1x_2x_3\; P^{tw-3}_{Nq}(x_1,x_2,x_3)\,, \tag{6.49}$$

$$\Psi^{WW}_4(x) = -\sum_{N,q}\frac{c_{Nq}\,\phi_{Nq}}{(N+2)(N+3)}\left(N+2-\frac{\partial}{\partial x_2}\right)x_1x_2x_3\; P^{tw-3}_{Nq}(x_2,x_1,x_3)\,, \tag{6.50}$$

where the polynomials $P^{tw-3}_{Nq}(x)$ are defined in Eq. (6.1), (6.8). These expressions present one of the main results of this paper.

6.3.3 Genuine twist-4 contributions $\Psi^{tw-4}_4, \Phi^{tw-4}_4, \Psi^g_4, \Phi^g_4$

The expansion of the twist-4 auxiliary amplitudes $\mathcal{O}^a(x)$ (6.16) reads

$$\begin{aligned}
\mathcal{O}^a(x) =&\mu_a(x)\left[\sum_{Nq} A_{Nq}\,\eta_{Nq}\,\mathcal{P}^a_{Nq}(x) + \sum_{N\geq 2,q} B_{Nq}\,\eta^g_{Nq}\,\widetilde{\mathcal{P}}^a_{Nq}(x)\right], \quad a=1,2,3\,,\\
\mathcal{O}^{a+3}(x) =&\frac{1}{2}x_1x_2x_3x_4^2\sum_{N\geq 2,q} B_{Nq}\,\eta^g_{Nq}\,\mathcal{P}^{g,a}_{Nq}(x)\,, \qquad a=1,2\,.
\end{aligned} \tag{6.51}$$

The coefficients $A_{N,q}$ and $B_{N,q}$ are given by

$$A_{Nq}^{-1} = \int \mathcal{D}x\, x_1x_2x_3 \left(\frac{1}{x_1}\,|\mathcal{P}^1_{Nq}(x)|^2 + \frac{1}{x_2}\,|\mathcal{P}^2_{Nq}(x)|^2 + \frac{x_3}{4}|\mathcal{P}^3_{Nq}(x)|^2\right),$$
$$B_{Nq}^{-1} = \int \mathcal{D}x\, \mu_4(x) \sum_{ab} \mathcal{P}^{g,a}_{Nq}(x)\, \omega_{ab}\, \mathcal{P}^{g,b}_{Nq}(x)\,, \tag{6.52}$$

where in the last line $\omega = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Note a useful identity

$$\mathcal{P}^3_{Nq}(x) = \frac{2}{x_3^2} \int_0^{x_3} d\tau\, \tau\, \left(\partial_1 \mathcal{P}^1_{Nq}(x_1, x_2, \tau) + \partial_2 \mathcal{P}^2_{Nq}(x_1, x_2, \tau)\right) \tag{6.53}$$

which is a consequence of Eq. (5.32).

The reduced matrix elements $\eta_{Ng}(\mu)$ and $\eta^g_{Ng}(\mu)$ are defined similar to (6.29), replacing the spin-down spinor $N^{\downarrow}$ by the spin-up one, $N^{\uparrow}$, cf. (6.16), (6.19). The expressions for the operators $\mathbb{O}^{mixed}_{N,q}$, $\mathbb{O}^{g,mixed}_{N,q}$ for $N \leq 2$ and the corresponding anomalous dimensions are given in Eq. (5.48) and Table 3, respectively. They can be isolated by the projection

$$\eta_{Nq} = \sum_{a=1}^{3} \int \mathcal{D}x\, \mathcal{P}^a_{Nq}(x)\, \mathcal{O}^a(x) - \sum_{q'} M^N_{qq'}\, \eta^g_{Nq'}\,,$$
$$\eta^g_{Nq} = \int \mathcal{D}x \sum_{ab} \mathcal{P}^{g,a}_{Nq}(x)\, \omega_{ab}\, \mathcal{G}^b(x)\,, \tag{6.54}$$

where

$$M^N_{qq'} = B_{Nq'} \int \mathcal{D}x\, x_1x_2x_3 \left(\frac{1}{x_1}\mathcal{P}^1_{Nq}(x)\widetilde{\mathcal{P}}^{g,1}_{Nq'}(x) + \frac{1}{x_2}\mathcal{P}^2_{Nq}(x)\widetilde{\mathcal{P}}^{g,2}_{Nq'}(x) + \frac{x_3}{4}\mathcal{P}^3_{Nq}(x)\widetilde{\mathcal{P}}^{g,3}_{Nq'}(x)\right).$$

Taking into account the contributions of local operators with $N \leq 2$ we obtain

$$\begin{aligned}
\Psi_4^{tw-4}(x,\mu) = &\ \ 12x_1x_3\Big[\eta_{0,0}(\mu) + 4\,\eta_{1,0}(\mu)\,\mathcal{P}_{1,0}(x_2,x_3,x_1) + \frac{20}{3}\eta_{1,1}(\mu)\,\mathcal{P}_{1,1}(x_2,x_3,x_1) \\
&+ \frac{5}{2}\left(\frac{11}{2} + \frac{5}{\sqrt{43}}\right)\eta_{2,0}(\mu)\,\mathcal{P}_{2,0}(x_2,x_3,x_1) + \frac{45}{2}\eta_{2,1}(\mu)\,\mathcal{P}_{2,1}(x_2,x_3,x_1) \\
&+ \frac{5}{2}\left(\frac{11}{2} - \frac{5}{\sqrt{43}}\right)\eta_{2,2}(\mu)\,\mathcal{P}_{2,2}(x_2,x_3,x_1) + \frac{140}{117}\eta^g_{2,0}(\mu)\,\widetilde{\mathcal{P}}^g_{2,0}(x_2,x_3,x_1) \\
&+ \frac{70}{47}\eta^g_{2,1}(\mu)\,\widetilde{\mathcal{P}}^g_{2,1}(x_2,x_3,x_1)\Big], \\
\Phi_4^{tw-4}(x,\mu) = &- 12x_1x_2\Big[\eta_{0,0}(\mu) + 4\,\eta_{1,0}(\mu)\,\mathcal{P}_{1,0}(x_3,x_1,x_2) - \frac{20}{3}\eta_{1,1}(\mu)\,\mathcal{P}_{1,1}(x_3,x_1,x_2) \\
&+ \frac{5}{2}\left(\frac{11}{2} + \frac{5}{\sqrt{43}}\right)\eta_{2,0}(\mu)\,\mathcal{P}_{2,0}(x_3,x_1,x_2) - \frac{45}{2}\eta_{2,1}(\mu)\,\mathcal{P}_{2,1}(x_3,x_1,x_2) \\
&+ \frac{5}{2}\left(\frac{11}{2} - \frac{5}{\sqrt{43}}\right)\eta_{2,2}(\mu)\,\mathcal{P}_{2,2}(x_3,x_1,x_2) + \frac{140}{117}\eta^g_{2,0}(\mu)\,\widetilde{\mathcal{P}}^g_{2,0}(x_3,x_1,x_2) \\
&- \frac{70}{47}\eta^g_{2,1}(\mu)\,\widetilde{\mathcal{P}}^g_{2,1}(x_3,x_1,x_2)\Big], \\
\Psi_4^g(x,\mu) = &\ \ \frac{1}{4}8!x_1x_2x_3x_4^2\left[\eta^g_{2,0}(\mu) + \frac{1}{3}\eta^g_{2,1}(\mu)\right], \\
\Phi_4^g(x,\mu) = &- \frac{1}{4}8!x_1x_2x_3x_4^2\left[\eta^g_{2,0}(\mu) - \frac{1}{3}\eta^g_{2,1}(\mu)\right],
\end{aligned} \tag{6.55}$$

where

$$\begin{aligned}
\mathcal{P}_{1,0}(x) =& x_1 + x_2 - \frac{3}{2}x_3\,, \\
\mathcal{P}_{1,1}(x) =& x_1 - x_2 + \frac{1}{2}x_3\,, \\
\mathcal{P}_{2,1}(x) =& x_1^2 - x_2^2 - 2x_1x_3 + 3x_2x_3 - \frac{2}{3}x_3^2\,, \\
\begin{pmatrix}\mathcal{P}_{2,0}(x)\\ \mathcal{P}_{2,2}(x)\end{pmatrix} =& x_1^2 + \frac{4}{9}\left(-5 \pm \sqrt{43}\right)x_1x_2 + x_2^2 + \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)x_1x_3 \\
&- \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)x_2x_3 + \frac{4}{27}\left(4 \pm \sqrt{43}\right)x_3^2\,, \\
\widetilde{\mathcal{P}}^g_{2,0}(x) =& 64x_1^2 - 55x_1x_2 + \frac{11}{2}x_2^2 - 73x_1x_3 + 11x_2x_3 + \frac{17}{2}x_3^2\,, \\
\widetilde{\mathcal{P}}^g_{2,0}(x) =& 16x_1^2 - \frac{1}{3}x_2^2 - 32x_1x_3 + x_2x_3 + 5x_3^2\,.
\end{aligned} \tag{6.56}$$

The expression for $D_4(x,\mu)$ (3.15) can be obtained using the integral representation in Eq. (3.19).

The reduced matrix elements η_{Nq} for $N = 0, 1$ are related to the parameters introduced in Ref. [36] as

$$\begin{aligned} \lambda_1 &= -\,\eta_{00}\,, \\ \lambda_1\, f_1^d &= -\,\frac{1}{6}\phi_{00} - \frac{3}{10}\eta_{00} - \frac{1}{5}\eta_{10} + \frac{1}{3}\eta_{11}\,, \\ \lambda_1\, f_1^u &= -\,\frac{1}{6}\phi_{00} - \frac{1}{10}\eta_{00} - \frac{3}{5}\eta_{10} + \frac{1}{3}\eta_{11}\,. \end{aligned} \tag{6.57}$$

The term in ϕ_{00} is the Wandzura-Wilczek contribution that can be traced to the twist-4 operator containg a transverse derivative of a local twist-3 three-quark operator (and properly symmetrized).

7 Conclusions

The motivation for our study has been to work out efficient techniques for a calculation of anomalous dimensions of generic higher twist operators in QCD. Apart from the applications to QCD phenomenology, this project was fuelled by the recent progress in the understanding of the spectrum of the dilatation operator in the maximally supersymmetric $N = 4$ Yang-Mills theory [26] and, in particular, the work [27] where it was argued that the diagonal part of one-loop QCD RG equations (for arbitrary twist) can be written in a Hamiltonian form in terms of quadratic Casimir operators of the full conformal group $SO(4, 2)$. In simple words, the symmetry under the conformal transformations in the directions orthogonal to the light-cone plane implies existence of relations between the renormalization group equations of different (geometric) twist. This, in turn, suggests that the techniques developed for the description of quasipartonic operators in QCD [23] can be generalized to include non-quasipartonic operators as well. Our goal was to develop a consistent computational framework based on these ideas.

The first step was to construct the appropriate operator basis with "good" transformation properties. We found that the complete basis of one-particle conformal operators for chiral quark and self-dual gluon fields in QCD contains seven light-ray fields, and similar in the anti-chiral sector. An interesting feature of this basis is that it includes some, but not all, transverse derivatives: If the transverse plane is parameterized in terms of a single complex variable as it is usually done in the studies of high-energy scattering, then the basis fields only include holomorphic derivatives acting on holomorphic components of the fields, and vice versa.

Although much of the formalism appears to be general, in this paper we concentrate on the simplest example of non-quasipartonic twist-four baryon operators that contain two "plus" and one "minus" quark field, schematically $q_+q_-q_+$, and their mixing with (quasipartonic) four-particle operators involving a gluon field, of the type $q_+q_+q_+F_{+\perp}$.

For this setup we calculate all one-loop evolution kernels and check that they are $SL(2)$ invariant, as expected. The evolution equation for three-quark operators of the same chirality turns out to be completely integrable. The spectrum of anomalous dimensions coincides in this case with the energy spectrum of the twist-4 subsector of the $SU(2,2)$ Heisenberg spin chain, confirming the prediction of [27]. We find the explicit expression of the corresponding conserved charge and calculate its spectrum. A simple analytic expression is found for the lowest anomalous dimension for chiral quark twist-4 operators with odd number $N = 2k+1$ of covariant derivatives. For other cases the spectra are studied numerically, see Figs. 2–6 and Tables 2,3. It turns out that differences between twist-4 and twist-3 operators and also between twist-4 operators of different chirality mostly affect a few lowest eigenstates (for a given N); the upper part of the spectrum of anomalous dimensions is universal. The spectrum of twist-4 quark operators overlaps strongly with that of quark-gluon operators, apart from a few lowest states.

Finally, these results are applied to give a general classification and calculate the scale dependence of subleading twist-4 nucleon distribution amplitudes that are relevant for hard exclusive reactions involving a helicity flip. In particular we introduce new four-particle distribution amplitudes involving a gluon field, and derive explicit expressions for the expansion of all distribution amplitudes in contributions of multiplicatively renormalizable operators taking into account first three orders in the conformal spin expansion. As a byproduct of our analysis, we give an all-order expression (in conformal spin) for the contributions of geometric twist-3 operators to the (light-cone) twist-4 nucleon distribution amplitudes, which are usually referred to as Wandzura–Wilczek contributions. The applications of these results to phenomenology of hard exclusive reactions will be considered elsewhere.

The techniques suggested in this paper can have a rather broad field of applications, in particular to the calculation of twist-4 corrections to the structure functions of deep-inelastic lepton-hadron scattering. We plan to consider this problem in a separate publication.

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