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A FORMULATION OF QUANTUM FIELD THEORY REALIZING A SEA OF INTERACTING DIRAC PARTICLES

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Abstract. In this survey article we explain a few ideas behind the framework of the fermionic projector and summarize recent results which clarify the connection to quantum field theory. The fermionic projector is introduced, which describes the physical system by a collection of Dirac states, including the states of the Dirac sea. Formulating the interaction by an action principle for the fermionic projector, we obtain a consistent description of interacting quantum fields which reproduces all results of perturbative quantum field theory. Moreover, we find a new mechanism for the generation of boson masses and obtain small corrections to the field equations which violate causality.

1. Introduction and Motivation

In order to give the negative-energy solutions of the Dirac equation a meaningful physical interpretation, Dirac proposed that in the vacuum all states of negative energy should be occupied by particles forming the so-called *Dirac sea* [1, 2]. His idea was that the homogeneous and isotropic Dirac sea configuration of the vacuum should not be accessible to measurements, but deviations from this uniform configuration should be observable. Thus particles are described by occupying additional states having positive energy, whereas "holes" in the Dirac sea can be observed as anti-particles. Moreover, Dirac noticed in [2] that deviations from the uniform sea configuration may also be caused by the interaction with an electromagnetic field. In order to analyze this effect, he first considered a formal sum over all vacuum sea states

$$R(t,\vec{x};t',\vec{x}') = \sum_{l \text{ occupied}} \Psi_l(t,\vec{x})\, \overline{\Psi_l(t',\vec{x}')}\,. \tag{1.1}$$

He found that this sum diverges if the space-time point $(t,\vec{x})$ lies on the light cone centered at $(t',\vec{x}')$ (i.e. if $(t-t')^2 = |\vec{x}-\vec{x}'|^2$). Next, he inserted an electromagnetic potential into the Dirac equation,

$$\big(i\partial\!\!\!/ + eA\!\!\!/(t,\vec{x}) - m\big)\Psi_l(t,\vec{x}) = 0\,.$$

He proceeded by decomposing the resulting sum (1.1) as

$$R = R_a + R_b\,, \tag{1.2}$$

where R_a is again singular on the light cone, whereas R_b is a regular function. The dependence of R_a and R_b on the electromagnetic potential can be interpreted as describing a "polarization of the Dirac sea" caused by the non-uniform motion of the sea particles in the electromagnetic field.

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When setting up an interacting theory, one faces the problem that the total charge density of the sea states is given by the divergent expression

$$\sum_{l \text{ occupied}} e\, \overline{\Psi_l(t,\vec{x})} \gamma^0 \Psi_l(t,\vec{x}) \,.$$

Thus the Dirac sea has an infinite charge density, making it impossible to couple it to a Maxwell field. Similarly, the Dirac sea has an infinite negative energy density, leading to divergences in Einstein's equations. Thus before formulating the field equations, one must subtract the infinite contribution of the Dirac sea to the current and the energy-momentum tensor. This is accomplished in perturbative quantum field theory by the following standard procedure: First, one formally replaces and reinterprets the creation and annihilation operators of the negative-energy states of the free field theory. By Wick ordering one then obtains a positive definite Dirac Hamiltonian on the fermionic Fock space (see for example [9]). After quantizing the electromagnetic field, the interaction can be described perturbatively in terms of Feynman diagrams. This procedure makes it possible to compute the S-matrix of a scattering process and gives rise to the loop corrections, in excellent agreement with experiments.

In the perturbative description, the Dirac sea no longer appears. Therefore, it is a common view that the Dirac sea is merely a historical relic which is no longer needed in modern quantum field theory. However, this view is too simple because removing the Dirac sea by infinite counter terms entails conceptual problems. The basic shortcoming can already be understood from the representation (1.2) of the Dirac sea in an electromagnetic field (for a more detailed discussion see [6, Section 4.6]). Since the singular term R_a involves $\not{A}$, the counter term needed to compensate the infinite charge density of the Dirac sea must depend on the electromagnetic potential. But then it is no longer clear how precisely this counter term is to be chosen. In particular, should the counter term include R_b, or should R_b not be compensated and instead enter the Maxwell equations? Taking this ambiguity seriously, one concludes that the procedure of subtracting infinite charge or energy densities is not a fully convincing concept. Similarly, infinite counter terms are also needed in order to treat the divergences of the Feynman loop diagrams. Dirac himself was uneasy about these infinities, as he expressed later in his life in a lecture on quantum electrodynamics [3, Chapter 2]:

> "I must say that I am very dissatisfied with the situation, because this so-called good theory does involve neglecting infinities which appear in its equations ... in an arbitrary way. This is not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small – not neglecting it just because it is infinitely great and you do not want it!"

The dissatisfaction about the treatment of the Dirac sea in perturbative quantum field theory was my original motivation for trying to set up a quantum field theory in which the Dirac sea is not removed by counter terms, but is taken into account all the way, thus realizing Dirac's idea of a "sea of interacting particles". The key step for making this idea precise is to describe the interaction by a new type of action principle, which has the desirable property that the divergent terms in (1.1) drop out of the equations, making it unnecessary to subtract any counter terms. This action principle was first introduced in [5]. More recently, in [6] it was analyzed in detail for a system of Dirac seas in the simplest possible configuration referred to as a single sector. Furthermore, the connection to entanglement and second quantization was clarified in [7]. Putting these results together, we obtain a consistent formulation of quantum field theory which is satisfying conceptually and reproduces all results of perturbative quantum field theory. Moreover, our approach gives surprising results which go beyond standard quantum field theory, like a mechanism for the generation of boson masses and small corrections to the field equations which violate causality. The aim of the present paper is to explain a few ideas behind the framework of the fermionic projector and to review the present status of the theory.

2. An Action Principle for the Fermionic Projector in Space-Time

We now introduce our action principle on a formal level (for the analytic justification and more details see [6, Chapter 2]). Similar to (1.1), we describe our fermion system for any points x and y in Minkowski space by the so-called *kernel of the fermionic projector*

$$P(x,y) = -\sum_{l \text{ occupied}} \Psi_l(x)\, \overline{\Psi_l(y)}\,, \tag{2.1}$$

where by the occupied states we mean the sea states except for the anti-particle states plus the particle states. For any x and y, we introduce the *closed chain* A_{xy} by

$$A_{xy} = P(x,y)\, P(y,x)\,. \tag{2.2}$$

It is a 4×4-matrix which can be considered as a linear operator on the Dirac wave functions at x. For such a linear operator A we define the *spectral weight* $|A|$ by

$$|A| = \sum_{i=1}^{4} |\lambda_i|\,,$$

where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of A counted with algebraic multiplicities. We define the *Lagrangian* $\mathcal{L}$ by

$$\mathcal{L}_{xy}[P] = |A_{xy}^2| - \frac{1}{4}\, |A_{xy}|^2\,. \tag{2.3}$$

Integrating over space-time, we introduce the functionals

$$\mathcal{S}[P] \;=\; \iint \mathcal{L}_{xy}[P]\, d^4x\, d^4y \qquad \text{and} \qquad \mathcal{T}[P] \;=\; \iint |A_{xy}|^2\, d^4x\, d^4y\,. \tag{2.4}$$

Our action principle is to

$$\text{minimize } \mathcal{S} \text{ for fixed } \mathcal{T}\,, \tag{2.5}$$

under variations of the wave functions Ψ_l which preserve the normalization with respect to the space-time inner product

$$<\!\Psi|\Phi\!> = \int \overline{\Psi(x)}\Phi(x)\, d^4x\,. \tag{2.6}$$

The action principle (2.5) is the result of many thoughts and extensive calculations carried out over several years. The considerations which eventually led to this action principle are summarized in [5, Chapter 5]. Here we only make a few comments. We first note that the factor 1/4 in (2.3) is merely a convention, as the value of this factor can be arbitrarily changed by adding to $\mathcal{S}$ a multiple of the constraint $\mathcal{T}$. Our convention has the advantage that for the systems under consideration here, the Lagrange multiplier of the constraint vanishes, making it possible to disregard the constraint in the following discussion. Next, we point out that taking the absolute value of an eigenvalue of the closed chain is a non-linear (and not even analytic) operation, so that our Lagrangian is not quadratic. As a consequence, the corresponding Euler-Lagrange equations are *nonlinear*. Our Lagrangian has the property that it vanishes if A is a multiple of the identity matrix. Furthermore, it vanishes if the eigenvalues of A form a complex conjugate pair. These properties are responsible for the fact that the singularities on the light cone discussed in the introduction drop out of the Euler-Lagrange equations. Moreover, it is worth noting that the action involves only the fermionic wave functions, but *no bosonic fields* appear at this stage. The interaction may be interpreted as a direct particle-particle interaction of all the fermions, taking into account the sea states. We finally emphasize that our action involves no coupling constants nor any other free parameters.

Clearly, our setting is very different from the conventional formulation of physics. We have no fermionic Fock space, nor any bosonic fields. Although the expression (2.1) resembles the two-point function, the n-point functions are not defined in our setting. More generally, it seems inappropriate and might even be confusing to use notions from quantum field theory, which have no direct correspondence here. Thus one should be willing to accept that we are in a new mathematical framework where we describe the physical system on the fundamental level by the fermionic projector with kernel (2.1). The connection to quantum field theory is not obvious at this stage, but will be established in what follows.

We finally remark that our approach of working with a nonlinear functional on the fermionic states has some similarity to the "non-linear spinor theory" by Heisenberg et al [4], which got attention in the 1950s but apparently did not turn out to be successful. We point out that our action (2.5) is completely different from the equation $\partial\!\!\!/ \Psi \pm l^2 \gamma^5 \gamma^j \Psi \,(\overline{\Psi} \gamma_j \gamma^5 \Psi) = 0$ considered in [4]. Thus there does not seem to be a connection between these approaches.

3. Intrinsic Formulation in a Discrete Space-Time

Our action principle has the nice feature that it does not involve the differentiable, topological or causal structure of the underlying Minkowski space. This makes it possible to drop these structures, and to formulate our action principle intrinsically in a discrete space-time. To this end, we simply replace Minkowski space by a finite point set M. To every space-time point we associate the *spinor space* as a four-dimensional complex vector space endowed with an inner product of signature $(2, 2)$, again denoted by $\overline{\Psi}\Phi$. A *wave function* Ψ is defined as a function which maps every space-time point $x \in M$ to a vector $\Psi(x)$ in the corresponding spinor space. For a (suitably orthonormalized) finite family of wave functions $\Psi_1, \ldots, \Psi_f$ we then define the kernel of the fermionic projector in analogy to (2.1) by

$$P(x, y) = -\sum_{l=1}^{f} \Psi_l(x) \overline{\Psi_l(y)}\,.$$

Now the action principle can be introduced again by (2.4)–(2.6) if we only replace the space-time integrals by sums over M.

The formulation in discrete space-time is a possible approach for physics on the Planck scale. The basic idea is that the causal and metric structure should be induced on the space-time points by the fermionic projector as a consequence of a spontaneous symmetry breaking effect. In non-technical terms, this *structure formation* can be understood by a self-organization of the wave functions as described by our action principle. More specifically, a discrete notion of causality is introduced as follows:

Definition 3.1. (causal structure) *Two space-time points $x, y \in M$ are called* **timelike** *separated if the spectrum of the product $P(x,y)P(y,x)$ is real. Likewise, the points are* **spacelike** *separated if the spectrum of $P(x,y)P(y,x)$ forms two complex conjugate pairs having the same absolute value.*

We refer the reader interested in the spontaneous structure formation and the connection between discrete and continuum space-times to the survey paper [8] and the references therein. The only point of relevance for what follows is that in the discrete formulation, our action principle is finite and minimizers exist. Thus there is a fundamental setting where the physical equations are intrinsically defined and have regular solutions without any divergences.

4. Bosonic Currents Arising from a Sea of Interacting Dirac Particles

In preparation for analyzing our action principle, we need a systematic method for describing the kernel of the fermionic projector in position space. In the vacuum, the formal sum in (2.1) is made precise as the Fourier integral of a distribution supported on the lower mass shell,

$$P^{\text{sea}}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \tag{4.1}$$

(where Θ is the Heaviside function). In order to introduce particles and anti-particles, one occupies (suitably normalized) positive-energy states or removes states of the sea,

$$P(x,y) = P^{\text{sea}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_f} \Psi_k(x) \overline{\Psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_a} \Phi_l(x) \overline{\Phi_l(y)}\,. \tag{4.2}$$

Next we want to modify the physical system so as to describe a general interaction. To this end, it is useful to regard $P(x,y)$ as the integral kernel of an operator P on the wave functions, i.e.

$$(P\Psi)(x) := \int P(x,y)\, \Psi(y)\, d^4y\,.$$

Since we want to preserve the normalization of the fermionic states with respect to the inner product (2.6), the interacting fermionic projector $\tilde{P}$ can be obtained from the vacuum fermionic projector P by the transformation

$$\tilde{P} = U P U^{-1}$$

with an operator U which is unitary with respect to the inner product (2.6). The calculation

$$0 = U(i\not{\partial} - m) P U^{-1} = U(i\not{\partial} - m) U^{-1} \tilde{P}$$

shows that $\tilde{P}$ is a solution of the Dirac equation

$$(i\not{\partial} + \mathcal{B} - m)\tilde{P} = 0 \qquad \text{where} \qquad \mathcal{B} := iU\not{\partial}U^{-1} - i\not{\partial}\,.$$

This consideration shows that we can describe a general interaction by a potential $\mathcal{B}$ in the Dirac equation, provided that $\mathcal{B}$ is an operator of a sufficiently general form. It can be a multiplication or differential operator, but it could even be a nonlocal operator. The usual bosonic potentials correspond to special choices of $\mathcal{B}$. This point of view is helpful because then the bosonic potentials no longer need to be considered as fundamental physical objects. They merely become a technical device for describing specific variations of the Dirac sea.

In order to clarify the structure of $\tilde{P}$ near the light cone, one performs the so-called *causal perturbation expansion* and the *light-cone expansion*. For convenience omitting the tilde, one gets in analogy to (1.2) a decomposition of the form

$$P^{\text{sea}}(x,y) = P^{\text{sing}}(x,y) + P^{\text{reg}}(x,y)\,, \tag{4.3}$$

where $P^{\text{sing}}(x,y)$ is a distribution which is singular on the light cone and can be expressed explicitly by a series of terms involving line integrals of $\mathcal{B}$ and its partial derivatives along the line segment $\overline{xy}$. The contribution P^{reg}, on the other hand, is a smooth function which is noncausal in the sense that it depends on the global behavior of $\mathcal{B}$ in space-time. It can be decomposed further into so-called low-energy and high-energy contributions which have a different internal structure.

For simplicity, we here omit all further details and only explain one point which is important for the physical understanding. As mentioned above, the line integrals appearing in P^{sing} also involve partial derivatives of $\mathcal{B}$. In the case when $\mathcal{B} = \not{A}$ is an electromagnetic potential (or similarly a general gauge field), one finds that P^{sing} involves the electromagnetic field tensor and the electromagnetic current. More specifically, the contribution to P^{sing} involving the electromagnetic current takes the form

$$-\frac{e}{16\pi^3} \int_0^1 (\alpha-\alpha^2)\gamma_k \left(\partial^k{}_l A^l - \Box A^k\right)\Big|_{\alpha y+(1-\alpha x)} \lim_{\varepsilon \searrow 0} \log\Big((y-x)^2 + i\varepsilon\,(y^0-x^0)\Big)\,. \tag{4.4}$$

The appearance of this contribution to the fermionic projector can be understood similar to the "polarization of the Dirac sea" mentioned in the introduction as being a result of the non-uniform motion of the sea particles in the electromagnetic field. This contribution influences the closed chain (2.2) and thus has an effect on our action principle (2.5). In this way, the electromagnetic current also enters the corresponding Euler-Lagrange equations. In general terms, one can say that in our formulation, the bosonic currents arise in the physical equations only as a consequence of the collective dynamics of the particles of the Dirac sea.

5. The Continuum Limit, the Field Equations

We now outline the method for analyzing our action principle for the fermionic projector (4.3). Since P^{sing} is a distribution which is singular on the light cone, the pointwise product $P(x,y)\,P(y,x)$ is ill-defined. Thus in order to make mathematical sense of the Euler-Lagrange equations corresponding to our action principle, we need to introduce an ultraviolet regularization. Such a regularization is not a conceptual problem because the setting in discrete space-time in Section 3 can be regarded as a special regularization. Thus in our approach, a specific, albeit unknown regularization should have a fundamental significance. Fortunately, the details of this regularization are not needed for our analysis. Namely, for a general class of regularizations of the vacuum Dirac sea (for details see [6, Chapter 3] or [5, Chapter 4]), the Euler-Lagrange equations have a well-defined asymptotic behavior when the regularization is removed. In this limit, the Euler-Lagrange equations give rise to differential equations involving the particle and anti-particle wave functions as well as the bosonic potentials and currents, whereas the Dirac sea disappears. This construction is subsumed under the notion *continuum limit*.

In the recent paper [6], the continuum limit was analyzed in detail for systems which in the vacuum are described in generalization of (4.1) by a sum of Dirac seas,

$$P^{\mathrm{sea}}(x,y) \;=\; \sum_{\beta=1}^{g} \int \frac{d^4k}{(2\pi)^4}\, (\slashed{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,.$$

Such a configuration is referred to as a *single sector*. The parameter g can be interpreted as the number of generations of elementary particles. It turns out that in the case $g = 1$ of one Dirac sea, the continuum limit gives equations which are only satisfied in the vacuum, in simple terms because the logarithm in current terms like (4.4) causes problems. In order to get non-trivial differential equations, one must assume that there are exactly *three generations* of elementary particles. In this case, the logarithms in the current terms of the three Dirac seas can compensate each other, as is made precise by a uniquely determined so-called local axial transformation. Analyzing the possible operators $\mathcal{B}$ in the corresponding Dirac equation in an exhaustive way (including differential and nonlocal operators), one finds that the dynamics is described completely by an *axial potential* A_{a} coupled to the Dirac spinors. We thus obtain the coupled system

$$(i\slashed{\partial} + \gamma^5 \slashed{A}_{\mathrm{a}} - m)\Psi = 0\,, \qquad C_0\, j^k_{\mathrm{a}} - C_2\, A^k_{\mathrm{a}} = 12\pi^2\, \overline{\Psi}\gamma^5\gamma^k\Psi\,, \tag{5.1}$$

where $j^k_{\mathrm{a}} = \partial^k{}_l A^l_{\mathrm{a}} - \Box A^k_{\mathrm{a}}$ is the corresponding axial current. Here the constants C_0 and C_2 are empirical parameters which take into account the unknown microscopic structure of space-time. For a given regularization method, these constants can even be computed as functions of the fermion masses.

6. A New Mechanism for the Generation of Boson Masses

The term $C_2\, A^k_{\mathrm{a}}$ in (5.1) gives the axial field a rest mass $M = \sqrt{C_2/C_0}$. This bosonic mass term is surprising, because in standard gauge theories a boson can be given a mass only by the Higgs mechanism of spontaneous symmetry breaking. We now explain how the appearance of the mass term in (5.1) can be understood on a non-technical level (for more details see [6, §6.2 and §8.5]).

In order to see the connection to gauge theories, it is helpful to consider the behavior of the Dirac operator and the fermionic projector under gauge transformations. We begin with the familiar gauge transformations of electrodynamics, for simplicity in the case $m = 0$ of massless fermions. Thus assume that we have a pure gauge potential $A = \partial\Lambda$ with a real function $\Lambda(x)$. This potential can be inserted into the Dirac operator by the transformation

$$i\slashed{\partial} \;\to\; e^{i\Lambda(x)} i\slashed{\partial}\, e^{-i\Lambda(x)} = i\slashed{\partial} + (\slashed{\partial}\Lambda)\,,$$

showing that the electromagnetic potential simply describes the phase transformation $\Psi(x) \to e^{i\Lambda(x)}\Psi(x)$ of the wave functions. Since the multiplication operator $U = e^{i\Lambda}$ is unitary with respect to the inner product (2.6), it preserves the normalization of the fermionic states. Thus in view of (2.1), the kernel of the fermionic projector transforms according to

$$P(x,y) \;\to\; e^{i\Lambda(x)} P(x,y)\, e^{-i\Lambda(y)}\,.$$

When forming the closed chain (2.2), the phase factors drop out. This shows that our action principle is *gauge invariant* under the local $U(1)$-transformations of electrodynamics.

We next consider an axial potential $A_{\rm a}$ as appearing in (5.1). A pure gauge potential $A_{\rm a} = \partial \Lambda$ can be generated by the transformation

$$i\partial\!\!\!/ \;\to\; e^{i\gamma^5 \Lambda(x)} i\partial\!\!\!/ \, e^{i\gamma^5 \Lambda(x)} = i\partial\!\!\!/ + \gamma^5 (\partial\!\!\!/ \Lambda)\,,$$

suggesting that the kernel of the fermionic projector should be transformed according to

$$P(x,y) \;\to\; e^{-i\gamma^5 \Lambda(x)} P(x,y)\, e^{-i\gamma^5 \Lambda(x)}\,.$$

The main difference compared to the electromagnetic case is that now the transformation operator $U = e^{-i\gamma^5 \Lambda(x)}$ is *not* unitary with respect to the inner product (2.6). This leads to the technical complication that we need to be concerned about the normalization of the fermionic states. More importantly, the phases no longer drop out of the closed chain, because

$$\begin{aligned} A_{xy} &\to \Big(e^{-i\gamma^5 \Lambda(x)} P(x,y)\, e^{-i\gamma^5 \Lambda(x)} \Big) \Big(e^{-i\gamma^5 \Lambda(y)} P(y,x)\, e^{-i\gamma^5 \Lambda(x)} \Big) \\ &= e^{-i\gamma^5 \Lambda(x)} P(x,y)\, e^{-2i\gamma^5 \Lambda(y)} P(y,x)\, e^{-i\gamma^5 \Lambda(x)}\,. \end{aligned}$$

This shows that in general, our action is not invariant under axial gauge transformations. As a consequence, the appearance of the axial potential in the field equations does not contradict gauge invariance.

A more detailed analysis shows that the above axial transformation indeed changes only the phases of the eigenvalues λ_i of the closed chain, and these phases drop out when taking their absolute values as appearing in the closed chain. But repeating the above argument in the case $m > 0$ of massive fermions, one finds additional contributions proportional to $m^2 A_a$ which even affect the absolute values $|\lambda_i|$. These contributions are responsible for the bosonic mass term in the field equations.

In simple terms, the bosonic mass arises because the corresponding potential does not describe a local symmetry of our system. More specifically, an axial gauge transformation changes the relative phase of the left- and right-handed components of the fermionic projector. This relative phase does change the physical system and is thus allowed to enter the physical equations. In order to get a closer connection to the Higgs mechanism, one can say that the axial gauge symmetry is spontaneously broken by the states of the Dirac sea, because they distinguish the relative phase of the left- and right-handed components of the fermionic projector.

7. The Feynman Diagrams and the Standard Loop Corrections

In the continuum limit, we obtain the nonlinear system of hyperbolic partial differential equations (5.1). The bosonic potential is classical, whereas the fermions form a Hartree-Fock state (obtained by taking the wedge product of the one-particle wave functions in (4.2)). As worked out in [6, §8.4], treating the nonlinear hyperbolic system perturbatively gives rise to all Feynman diagrams which do not involve fermion loops. Taking into account that by exciting sea states we can describe pair creation and annihilation processes, we also obtain all diagrams involving fermion loops. We thus obtain full agreement with perturbative quantum field theory.

Clearly, the above perturbation expansion gives back the divergences of quantum field theory, making it necessary to renormalize (see for example [9]). Also, it is again not clear whether the renormalized perturbation series converges. Thus the basic technical problems of quantum field theory are not solved by our approach. But at least, the divergences no longer cause conceptual problems, as can be understood in two different ways: One way to argue is that the intrinsic formulation in discrete space-time gives a fundamental explanation why one should work with an ultraviolet regularization. If such a regularization is present, the loop diagrams are all finite, and the renormalization changes the masses and coupling constants only by a finite and non-zero factor. Alternatively, one may argue that the divergences in perturbative quantum field theory simply show that it is not appropriate to expand the interacting theory in a perturbation expansion. This point of view is sustained by the fact that rewriting (5.1) as a nonlinear symmetric hyperbolic system (see for example [10, Chapter 16]) yields a well-posed Cauchy problem, giving a strong indication that at least the divergences of the bosonic loop diagrams do not necessarily correspond to an actual blow-up of the solutions.

8. Noncausal Corrections

So far, we disregarded the noncausal contribution P^{reg} in (4.3). Taking this contribution into account, we get small corrections to the field equations which depend on the global behavior of the bosonic potentials in space-time, thus violating causality. To first order in the bosonic potential, we get a correction to the field equation which violates causality in the sense that the future can influence the past, but no interaction in space-like directions is possible. To higher order in the bosonic potential, even space-time points with spacelike separation can influence each other. At first sight, a violation of causality seems worrisome because it contradicts experience and seems to imply logical inconsistencies. However, these non-causal correction terms are only apparent on the Compton scale, and furthermore they are too small for giving obvious contradictions to physical observations. But they might open the possibility to test our approach in future experiments. For a detailed discussion of the causality violation we refer to [6, §8.2 and §8.3].

In order to explain in words how the violation of causality comes about, we point out that in discrete space-time causality does not arise on the fundamental level. But for a minimizer of our action principle, Definition 3.1 gives us a notion a "discrete causal structure." This notion is compatible with our action principle in the sense that space-time points x and y with spacelike separation do not influence each other via the Euler-Lagrange equations. This can be seen as follows: According to our definition, for such space-time points the eigenvalues of the closed chain all have the same absolute value. Using the specific form of the Lagrangian (2.3), this implies that the Lagangian and its first variation vanish. This in turn implies that A_{xy} drops out of the Euler-Lagrange equations. We conclude that our action principle is "causal" in the sense that no spacelike influences are possible. But at this stage, no time direction is distinguished, and therefore there is no reason why the future should not influence the past.

The system of hyperbolic equations (5.1) obtained in the continuum limit is causal in the sense that given initial data has a unique time evolution to the future. Moreover, we have finite propagation speed meaning that no information can travel faster than the speed of light. Thus in the continuum limit we recover the usual notion of causality. However, the contribution P^{reg} in (4.3) is not causal in this strict sense, making it possible that the future influences the past. Moreover, to higher order in the bosonic potential the normalization conditions for the fermions give rise to nonlocal constraints. As a consequence, the bosonic potential may influence $P(x, y)$ even for spacelike distances.

9. Entanglement and Second Quantization

Recall that in the continuum limit we obtain a system of classical bosonic fields coupled to a fermionic Hartree-Fock state. Although this setting gives rise to the Feynman diagrams, it seems too restrictive for describing all quantum effects observed in nature. However, as shown in [7], the framework of the fermionic projector also allows for the description of general second quantized fermionic and bosonic fields. In particular, it is possible to describe entanglement.

The derivation of these results is based on the assumption that space-time should have a non-trivial microstructure. In view of our concept of discrete space-time, this assumption seems natural. Homogenizing the microstructure, one obtains an effective description of the system by a vector in the fermionic or bosonic Fock space. This concept, referred to as the *microscopic mixing of decoherent subsystems*, is worked out in detail in [7], where the methods and results are also discussed with regard to decoherence phenomena and the measurement problem.

10. Conclusions and Outlook

Combining our results, we obtain a formulation of quantum field theory which is consistent with perturbative quantum field theory but has surprising additional features. First, we find a new mechanism for the generation of masses of gauge bosons and obtain new types of corrections to the field equations which violate causality. Moreover, our model involves fewer free parameters, and the structure of the interaction is completely determined by our action principle. Before one can think of experimental tests, one clearly needs to work out a more realistic model which involves all elementary particles and includes all interactions observed in nature. As shown in [5, Chapters 6–8], a model involving 24 Dirac seas is promising because the resulting gauge fields have striking similarity to the standard model. Furthermore, the underlying diffeomorphism invariance gives agreement with the equivalence principle of general relativity. Thus working out the continuum limit of this model in detail will lead to a formulation of quantum field theory which is satisfying conceptually and makes quantitative predictions to be tested in future experiments.

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