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Large Time Existence for Thin Vibrating Plates

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Abstract

We construct strong solutions for a nonlinear wave equation for a thin vibrating plate described by nonlinear elastodynamics. For sufficiently small thickness we obtain existence of strong solutions for large times under appropriate scaling of the initial values such that the limit system as $h \to 0$ is either the nonlinear von Kármán plate equation or the linear fourth order Germain-Lagrange equation. In the case of the linear Germain-Lagrange equation we even obtain a convergence rate of the three-dimensional solution to the solution of the two-dimensional linear plate equation.

Key words: Wave equation, plate theory, von Kármán, nonlinear elasticity, dimension reduction, singular perturbation
AMS-Classification: Primary: 74B20, Secondary: 35L20, 35L70, 74K20

1 Introduction

In the present contribution we study the nonlinear wave equation for a thin vibrating plate (or beam if $d = 2$). The plate is assumed to be of small but positive thickness $h > 0$ and satisfies the equations of three-dimensional nonlinear elastodynamics.

In order to explain the result and the model under consideration, let us start by recalling some facts and results for the corresponding variational problems, see [7] for further details. We consider the elastic energy

$$\tilde{E}^h(z) = \frac{1}{h} \int_{\Omega_h} \big(W(\nabla z(x)) - f^h \cdot (z(x) - x) \big) \, dx,$$

where $\Omega_h = \Omega' \times (-\frac{h}{2}, \frac{h}{2})$ is the reference configuration of the thin plate, $\Omega' \subset \mathbb{R}^{d-1}$, $d = 2, 3$, is a suitable bounded domain, and $z \colon \Omega_h \to \mathbb{R}^d$ is the deformation of the plate. For simplicity, we will restrict ourselves to the case

$d = 3$ in this introduction. Rescaling Ω_h to $\Omega = \Omega' \times (-\frac{1}{2}, \frac{1}{2})$, we obtain the rescaled energy

$$E^h(y) = \int_\Omega \left(W(\nabla_h y(x)) - f^h \cdot \left(y(x) - \begin{pmatrix} x_1 \\ x_2 \\ hx_3 \end{pmatrix} \right) \right) dx,$$

where $y(x) = z(x', hx_3)$ with $x' = (x_1, x_2)$ and $\nabla_h = (\partial_{x_1}, \partial_{x_2}, \frac{1}{h}\partial_{x_3})$. The limit as $h \to 0$ depends on the asymptotic behaviour of f^h. More precisely, let f^h be of order h^α. If $\alpha = 2$, then the energy E^h is of order h^β with $\beta = 2$. The rescaled energy $\frac{1}{h^2}E^h$ converges as $h \to 0$ to the elastic energy from the geometrically fully nonlinear Kirchhoff theory in the sense of Γ-convergence. To the authors' knowledge there are no results on existence of solutions for the corresponding dynamic wave equation or on regularity of non-minimizing equilibria. Indeed even the precise definition of equilibrium is not completely clear since the isometry constraint $\nabla\overline{y}^T\nabla\overline{y} = \mathrm{Id}$ for the limit map $\overline{y}\colon \Omega' \to \mathbb{R}^3$ makes the problem very rigid; see Hornung [9, 10] for recent progress. If $\alpha > 2$ and $\beta = 2\alpha - 2$, then the limit energy can be described as

$$\frac{\Lambda_\alpha}{2} \int_{\Omega'} Q_2 \left(\varepsilon(U) + \frac{\nabla V \otimes \nabla V}{2} \right) dx' + \frac{1}{24} \int_{\Omega'} Q_2(\nabla^2 V)\, dx',$$

where $\varepsilon(U) = \mathrm{sym}\,(\nabla U)$,

$$U = \lim_{h\to 0} \frac{1}{h^\gamma} \left(\begin{pmatrix} y_1^h \\ y_2^h \end{pmatrix} - \mathrm{Id}' \right), \quad V = \lim_{h\to 0} \frac{1}{h^\delta} y_3^h, \tag{1.1}$$

$$\delta = \alpha - 2, \quad \gamma = \begin{cases} 2(\alpha - 2) & \text{if } 2 < \alpha \leq 3 \\ \alpha - 1 & \text{if } \alpha > 3 \end{cases}, \tag{1.2}$$

where $\mathrm{Id}'(x) = (x_1, x_2)^T$ and $Q_2\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is related to $Q_3(F) := D^2W(\mathrm{Id})(F, F)$ by

$$Q_2(G) = \min_{a\in\mathbb{R}^3} Q_3(G + a \otimes e_3 + e_3 \otimes a).$$

Here

$$\Lambda_\alpha = \begin{cases} +\infty & \text{if } 2 < \alpha < 3, \\ 1 & \text{if } \alpha = 3, \\ 0 & \text{if } \alpha > 3. \end{cases}$$

Thus for $2 < \alpha < 3$ one has the "geometrically linear" constraint $2\varepsilon(U) + \nabla V \otimes \nabla V = 0$, which again has so far prevented the rigorous study of the associated dynamic wave equation or non-minimizing equilibria. For $\alpha = 3$ (and therefore $\beta = 4$) one obtains the von Kármán plate theory and for $\alpha > 3$ (and therefore $\beta > 4$) one obtains a linear Euler-Lagrange equation (linear Germain-Lagrange theory), which for isotropic materials reduces to the biharmonic equation.

Here we study the cases $\alpha = 3, \beta = 4$ and $\alpha > 3, \beta = 2\alpha - 2 > 4$ in the dynamic situation. The equations of elastodynamics arise from the Lagrangian

$$\frac{1}{h}\int_{\Omega_h}\left(\frac{|z_t|^2}{2} - W(\nabla z(x)) + f^h z\right)dx = \int_{\Omega}\left(\frac{|y_t|^2}{2} - W(\nabla_h y(x)) + f^h y\right)dx$$

and solutions formally preserve the total energy

$$\int_{\Omega}\left(\frac{|y_t|^2}{2} + W(\nabla_h y(x)) - f^h y\right)dx \tag{1.3}$$

In view of (1.1)-(1.2) we expect that

$$\begin{aligned} &y_3 \sim h, && \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \mathrm{Id}' \sim h^2 && \text{for } \alpha = 3, \beta = 4 \\ &y_3 \sim h^{\alpha-2}, && \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \mathrm{Id}' \sim h^{\alpha-1} && \text{for } \alpha > 3, \beta = 2\alpha - 2 > 4 \end{aligned}$$

The idea to balance the kinetic and potential energy in (1.3) suggests to rescale time as $\tau = ht$ if $\alpha = 3$. Then the total energy becomes

$$E_{\mathrm{tot}} = h^4 \int_{\Omega}\left(\frac{|\partial_\tau \frac{y}{h}|^2}{2} + \frac{1}{h^4}W(\nabla_h y(x)) - \frac{f_3^h}{h^3}\frac{y_3}{h}\right)dx$$

and with $f_h = h^{-3} f_3^h e_3$ the evolution equations is

$$\frac{1}{h^2}\partial_\tau^2 y - \frac{1}{h^4}\mathrm{div}_h DW(\nabla_h y) = \frac{1}{h} f_h.$$

or equivalently

$$\partial_\tau^2 y - \frac{1}{h^2}\mathrm{div}_h DW(\nabla_h y) = h f_h, \tag{1.4}$$

where $f_h \sim 1$ as $h \to 0$. In the case $\alpha = 3$ we will show existence of strong solutions of (1.4) for well-prepared and small data in a natural scaling with respect to h and time $\tau \in (0, T_0)$ with $T_0 > 0$ sufficiently small. In particular we assume that the rescaled f_h is small, cf. Section 3.1 below. – Note that the small time interval $(0, T_0)$ for τ turns over to a large time interval $(0, T_0 h^{-1})$ in the original times scale for t. In the case $\alpha > 3$, we will use the same time scale. Then we are able to show existence of strong solutions for $\tau \in (0, T)$ for any $T > 0$ provided that $f_h \sim h^{\alpha-3}$ and suitable initial data, cf. Section 3.1 below. We note that this time scale is subcritical for this scaling of f_h. In this case we are even able to construct the leading term of the solution $y = y_h$ as $h \to 0$ provided $W(F) = \mathrm{dist}(F, SO(3))^2$, cf. Section 4.

Together with [1] this shows that after the natural time rescaling and for well prepared data of the correct size solutions of the 3-d nonlinear elastodynamics converge to solutions of the dynamic von Kármán equation or linear von Kármán equation depending on the size of the data. We note that a similar result in the case of stationary solutions was shown by Monneau [18] if the limit system are the von Kármán plate equations. Ge, Kruse and Marsden [8] have taken an alternative and very general approach to study the limit from three-dimensional elasticity to shells and rods by establishing convergence of the underlying Hamiltionian structure. This suggests, but does not prove the convergence of the corresponding dynamical problems (see e.g. recent work by Mielke [17] for the question on the relation of the convergence of the Hamiltonian and the convergence of the resulting dynamical problems). General information and many further references on the dynamics of lower-dimensional nonlinear elastic structures can be found in the book by Antman [3]. For results on existence of weak and strong solutions of the non-stationary von Kármán plate equations we refer to e.g. Chen and Wahl [5], Koch and Lasiecka [12], Lasiecka [15], Koch and Stahel [13]. For a survey on results and open problem of nonlinear elasticity, stationary and non-stationary, we refer to Ball [4].

Let us explain the strategy of our proof and the main difficulties. Basically, the strong solutions are constructed by the usual energy method as presented e.g. in the books by Majda [16] and Dafermos [6]. (For a more abstract and general version see e.g. the classical paper by Hughes et al. [11].) The starting point in the method is the conservation of energy:

$$\frac{d}{dt}\left(\frac{1}{2}\|\partial_t y(t)\|^2_{L^2(\Omega)} + \frac{1}{h^2}\int_\Omega W(\nabla_y y)\,dx + \int_\Omega h f_h \cdot y(t)\,dx\right) = 0$$

which follows from (1.4) by multiplication with $\partial_t y$ under appropriate boundary conditions. (Here and in the following we replace τ by t.) Moreover, differentiating (1.4) with respect to x one gets a control of

$$\frac{d}{dt}\left(\frac{1}{2}\|\partial_t\partial_x^\beta y(t)\|^2_{L^2(\Omega)} + \frac{1}{h^2}\int_\Omega D^2W(\nabla_h y)\partial_x^\beta\nabla_h y : \partial_x^\beta \nabla_h y\,dx\right) = R_\beta, \quad (1.5)$$

where the remainder term R_β can be controlled with the aid of the Gronwall inequality once the left hand side controls $\partial_x^\beta\nabla_h y$ suitably. To this end it is essential to have the coercive estimate

$$\frac{1}{h^2}\int_\Omega D^2W(\nabla_h y)\nabla_h w : \nabla_h w\,dx \geq c_0\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} \quad (1.6)$$

where $\varepsilon_h(w) = \operatorname{sym}(\nabla_h w)$, cf. (3.34) below. By Korn's inequality in the present h-dependent version we have

$$\|\nabla_h w\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^2(\Omega)},$$

cf. Lemma 2.1 below. Therefore we will have one order of h better decay of the symmetric part of $\nabla_h y$ than for the full gradient/the skew-symmetric part. To obtain (1.6) (and similar estimates) it will be essential that

$$\frac{1}{h}\|\varepsilon_h(y) - I\|_{L^\infty} + \|\nabla_h y - I\|_{L^\infty} \le \varepsilon h$$

for some sufficiently small $\varepsilon > 0$ and to treat the symmetric and asymmetric part carefully in a Taylor expansion of $D^2W(\nabla_h y)$ around I, cf. Sections 2 and 3.3 for the details.

Several technical difficulties arise from the fact that we are dealing with natural boundary conditions at the upper and lower boundary $x_d = \pm\frac{1}{2}$. In tangential direction we assume periodic boundary conditions. First of all, in this situation it is easy to differentiate in tangential and temporal direction to obtain (1.5) with $\partial_x^\beta w$ replaced by $\partial_z^\beta w$, where $z = (x', t)$ and $x' = (x_1, \ldots, x_{d-1})$. Therefore we are using anisotropic L^2-Sobolev spaces of sufficiently high order to control $\nabla_h y$ in L^∞. In particular, one of the basic spaces is

$$\tilde{V}(\Omega) = \left\{u \in L^2(\Omega) : \nabla u, \partial_{x_j}\nabla u \in L^2(\Omega), j = 1, \ldots, d-1\right\} \hookrightarrow L^\infty(\Omega)$$

if $d = 2, 3$. Note that $\tilde{V}(\Omega)$ is slightly larger than $H^2(\Omega)$ and that $u \in H^2(\Omega)$ if and only if $u \in \tilde{V}(\Omega)$ and $\partial_{x_d}^2 u \in L^2(\Omega)$. Moreover, since we are dealing with natural boundary conditions, we want to keep the equation in divergence form. Therefore we do not use the identity

$$\mathrm{div}_h DW(\nabla_h y) = D^2W(\nabla_h y) \cdot \nabla_h^2 y$$

to obtain a quasi-linear system. Instead we differentiate (1.4) with respect to time and solve

$$\partial_t w - \frac{1}{h^2}\mathrm{div}_h \left(D^2W(\nabla_h y)\nabla_h w\right) = h\partial_t f,$$

where $w = \partial_t y$. Unfortunately we cannot solve the latter equation directly since we are missing suitable control of $\partial_{x_j} y$, $j = 1, \ldots, d-1$. Therefore we first replace $D^2W(\nabla_h y)$ by suitably smoothed coefficients $A_n(\nabla_h y)$ and construct a solution w_n, y_n, respectively, for the smoothed systems. Once we have one solution y_n at hand, we can differentiate it with respect to x_j, $j = 1, \ldots, d-1$ and get a solutions of

$$\partial_t w_j^n - \frac{1}{h^2}\mathrm{div}_h A_n(\nabla_h y_n)\nabla_h w_j^n = h\partial_{x_j} f,$$

where $w_j^n = \partial_{x_j} y_n$. These solutions w_j^n satisfy the same estimates as $w_n = \partial_t y_n$ (uniformly in $0 < h \le h_0$ and the smoothing parameter $n \in \mathbb{N}$). Then we can pass to the limit $n \to \infty$ to obtain a solution of the original system.

The structure of the article is as follows: In Section 2 we introduce some notation and derive some preliminary results. Our main result is presented

in Section 3.1. Afterwards we introduce our approximate system used for the construction of strong solutions in Section 3.2. The essential results for the linearized system are derived in Section 3.3, which are applied in Section 3.4 to obtain a strong solution first locally in time for fixed $n \in \mathbb{N}$. Then uniform bounds in T, h and $n \in \mathbb{N}$ are shown in Section 3.5 and our main result is shown by first extending the solution for small times in $[0,T]$ for some $T > 0$ given and then passing to the limit $n \to \infty$. Finally, in Section 4 we derive a first order asymptotic expansion as $h \to 0$ in the case that the limit system is linear, i.e., $\beta > 4$, and $W(F) = \operatorname{dist}(F, SO(d))^2$.
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2 Notation and Preliminaries

For any measurable set $M \subseteq \mathbb{R}^N$ the inner product of $L^2(M)$ (w.r.t. to Lebesgue measure) is denoted by $(.,.)_M$. Moreover, $H^k(\Omega)$, $k \in \mathbb{N}_0$, denotes the usual L^2-Sobolev spaces. If X is a Banach space, then the vector-valued variants of $L^2(M)$ and $H^k(M)$ are denoted by $L^2(M;X)$, $H^k(M;X)$, respectively. Furthermore, $C^k([0,T];X)$, $k \in \mathbb{N}_0$, denotes the space of all k-times continuously differentiable functions $f\colon [0,T] \to X$.

For the following $\Omega = (-L,L)^{d-1} \times (-\frac{1}{2}, \frac{1}{2})$, $\Omega' = (-L,L)^{d-1}$, $d = 2,3$, $x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$, let $\nabla_h = (\nabla_{x'}, \frac{1}{h}\partial_{x_d})^T$, $\nabla_{x,t} = (\partial_t, \nabla_x)$ and let

$$\varepsilon_h(w) = \operatorname{sym}(\nabla_h w), \qquad \varepsilon(w) = \varepsilon_1(w),$$

if $w\colon M \subset \mathbb{R}^d \to \mathbb{R}^d$ is a suitable vector field. Here $\operatorname{sym} A = \frac{1}{2}(A + A^T)$ and we denote $\operatorname{skew} A := \frac{1}{2}(A - A^T)$. Moreover, we denote $z = (t, x')$, where $z_0 = t$ and $z_j = x_j$ for $j = 1, \ldots, d-1$.

For $s > 0$, $s \notin \mathbb{N}_0$, we define L^2-Bessel potential spaces

$$H^s(\Omega) = \{f \in L^2(\Omega) : f = F|_\Omega \text{ for some } F \in H^s(\mathbb{R}^d)\}$$

as usual by restriction, equipped with the quotient norm. Since Ω is a Lipschitz domain, there is a continuous extension operator E such that $E\colon H^k(\Omega) \to H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$, cf. Stein [20, Chapter VI, Section 3.2]. Hence $H^s(\Omega)$, $s \geq 0$, is retract of $H^s(\mathbb{R}^d)$ and we obtain the usual interpolation properties, cf. e.g. [21]. In particular, we have

$$(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,2} = H^s(\Omega), \qquad s = (1-\theta)s_0 + \theta s_1, \tag{2.1}$$

for all $\theta \in (0,1)$, $s \geq 0$, where $(.,.)_{\theta,p}$ denotes the real interpolation method.

If $0 < T \leq \infty$ and X is a Banach space, then $BUC([0,T];X)$ is the space of all bounded and uniformly continuous functions $f\colon [0,T) \to X$. Now let

X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ densely. Then

$$W^1_p(0,T;X_0) \cap L^p(0,T;X_1) \hookrightarrow BUC([0,T];(X_0,X_1)_{1-\frac{1}{p},p}) \tag{2.2}$$

for all $1 \leq p < \infty$ continuously, cf. Amann [2, Chapter III, Theorem 4.10.2]. If $X_0 = H$ is a Hilbert space and H is identified with its dual, then $X_1 \hookrightarrow H \hookrightarrow X_1'$ and

$$\frac{1}{2}\frac{d}{dt}\|f\|_H^2 = \langle \frac{d}{dt}f(t), f(t)\rangle_{X_1',X_1} \qquad \text{for almost all } t \in [0,T] \tag{2.3}$$

provided that $f \in L^p(0,T;X_1)$ and $\frac{d}{dt}f \in L^{p'}(0,T;X_1')$, $1 < p < \infty$, cf. Zeidler [23, Proposition 23.23]. In particular, (2.3) implies

$$\sup_{t\in[0,T]} \|f(t)\|_H^2 \leq 2\left(\|\partial_t f\|_{L^2(0,T;X_1')}\|f\|_{L^2(0,T;X_1)} + \|f(0)\|_H^2\right). \tag{2.4}$$

Replacing $f(t)$ by $tf(t)$ and $(T-t)f(T-t)$, one easily derives from the latter estimate

$$\sup_{t\in[0,T]} \|f(t)\|_H \leq C_T \|f\|^{\frac{1}{2}}_{H^1(0,T;X_1')}\|f\|^{\frac{1}{2}}_{L^2(0,T;X_1)} \tag{2.5}$$

for some $C_T > 0$ depending on $T > 0$.

In the following $\mathcal{L}^n(V)$, $n \in \mathbb{N}$, denotes the space of all n-linear mappings $A\colon V^n \to \mathbb{R}$ for a vector space V. Moreover, if $A \in \mathcal{L}^n(V)$, $n \geq 2$, and $x_1, \ldots, x_k \in V$, $1 \leq k \leq n$, then $A[x_1,\ldots,x_k] \in \mathcal{L}^{n-k}(V)$ is defined by $A[x_1,\ldots,x_k](x_{k+1},\ldots,x_n) = A(x_1,\ldots,x_n)$ for all $x_{k+1},\ldots,x_n \in V$.

We introduce the scaled inner product

$$A :_h B = \frac{1}{h^2}\operatorname{sym} A : \operatorname{sym} B + \operatorname{skew} A : \operatorname{skew} B, \quad A, B \in \mathbb{R}^{d\times d}, 0 < h \leq 1,$$

and $|A|_h = \sqrt{A :_h A}$ where $A : B = \sum_{i,j=1}^d a_{ij}b_{ij}$. This choice of inner product is motivated by the Korn inequality in thin domains, see Lemma 2.1 below. Of course, $:_1$ coincides with the usual inner product $:$ on $\mathbb{R}^{d\times d}$ and therefore $|A|_1 = |A|$. For $W \in \mathcal{L}^n(\mathbb{R}^{d\times d})$ we define the induced scaled norm by

$$|W|_h = \sup_{|A_j|_h \leq 1, j=1,\ldots,n} |W(A_1,\ldots,A_n)|.$$

Note that, since $|A|_h \geq |A|_1 = |A|$ for all $A \in \mathbb{R}^{d\times d}$, we have $|W|_h \leq |W|_1 =: |W|$ for any $W \in \mathcal{L}^n(\mathbb{R}^{d\times d})$ and $0 < h \leq 1$.

As usual we identify $\mathcal{L}^1(\mathbb{R}^{d\times d}) = (\mathbb{R}^{d\times d})'$ with $\mathbb{R}^{d\times d}$. But one has to be careful whether this representation is taken with respect to the usual scalar product $:$ on $\mathbb{R}^{d\times d}$ or with respect to $:_h$, i.e., $W \in \mathcal{L}^1(\mathbb{R}^{d\times d})$ is identified with $A \in \mathbb{R}^{d\times d}$ such that

$$W(B) = A :_h B \qquad \text{for all } B \in \mathbb{R}^{d\times d}.$$

If nothing else is mentioned, we identify $(\mathbb{R}^{d\times d})'$ and $\mathbb{R}^{d\times d}$ using the standard inner product $:$. In particular, if $W \in C^1(U)$, $U \subset \mathbb{R}^{d\times d}$ and $A \in U$, then $DW(A) \in (\mathbb{R}^{d\times d})' \cong \mathbb{R}^{d\times d}$ coincides with

$$DW(A) : B = \frac{d}{dt} W(A + tB)\Big|_{t=0} \qquad \text{for all } B \in \mathbb{R}^{d\times d}.$$

Furthermore, $W \in \mathcal{L}^2(\mathbb{R}^{d\times d})$ is usually identified with the linear mapping $\tilde{W} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ defined by

$$\tilde{W} A : B = W(A, B) \qquad \text{for all } A, B \in \mathbb{R}^{d\times d}.$$

Finally, we denote by

$$\|W\|_{L^p_h(M;\mathcal{L}^n(\mathbb{R}^{d\times d}))} \equiv \|W\|_{L^p_h(M)} = \left(\int_M |W(x)|^p_h \, dx \right)^{\frac{1}{p}}$$

if $1 \le p < \infty$ and with the obvious modifications if $p = \infty$. Here $M \subseteq \mathbb{R}^d$ is measurable. Moreover, for $f \in L^p(M; \mathbb{R}^{d\times d})$ the scaled norm $\|f\|_{L^p_h(M;\mathbb{R}^{d\times d})} \equiv \|f\|_{L^p_h(M)}$ is defined in the same way.

We now state the relevant Korn inequality in thin domains.

Lemma 2.1 *There is a constant C such that*

$$\|\nabla_h u\|_{L^2(\Omega)} \le C \left\| \frac{1}{h} \varepsilon_h(u) \right\|_{L^2(\Omega)} \tag{2.6}$$

for all $0 < h \le 1$ and $u \in H^1(\Omega)^d$ such that $u|_{x_j=-L} = u|_{x_j=L}$, $j = 1, \ldots, d-1$.

Proof: For clamped boundary conditions the Korn inequality in thin domains was proved by Kohn and Vogelius [14, Prop. 4.1]. They mention that the result also holds without boundary conditions, modulo infinitesimal rigid motions. For the convenience of the reader we provide a proof of Lemma 2.1.

First we prove the case $d = 2$. Let $\Omega_h := (-L, L)^{d-1} \times (-\frac{h}{2}, \frac{h}{2})$ and let $u \in H^1(\Omega_h; \mathbb{R}^2)$ satisfy the boundary conditions $u|_{x_j=-L} = u|_{x_j=L}$, $j = 1, \ldots, d-1$. First of all by a simple scaling in x_d, (2.6) is equivalent to

$$\|\nabla u\|_{L^2(\Omega_h)} \le \frac{C}{h} \|(\nabla u)_{sym}\|_{L^2(\Omega_h)} \tag{2.7}$$

Let N_h be the integer part of $\frac{2L}{h}$ and let $\ell_h := \frac{2L}{N_h}$. We set $J_h := \{-L + k\ell_h : k = 0, \ldots, N_h - 1\}$. By applying Korn inequality on the set $(a, a+\ell_h) \times (-\frac{h}{2}, \frac{h}{2})$ for every $a \in J_h$, we can construct a piecewise constant function $A : (-L, L) \to \mathbb{M}^{2\times 2}$ such that $A(x_d)$ is skew-symmetric and

$$\int_{\Omega_h} |\nabla u - A|^2 \, dx \le C \int_{\Omega_h} |\varepsilon(u)|^2 \, dx. \tag{2.8}$$

Note that, since $\frac{\ell_h}{h}$ is bounded from above and from below, we can use the same Korn inequality constant on each set $(a, a+\ell_h)\times(-\frac{h}{2}, \frac{h}{2})$.

We claim that

$$\int_{\Omega_h} |A(x_1) - A_0|^2\,dx \leq \frac{C}{h^2}\int_{\Omega_h} |\varepsilon(u)|^2\,dx. \tag{2.9}$$

where $A_0 := A(-L)$.

Let us fix $a \in J_h$ and let $b := a + \lambda\ell_h$, with $\lambda \in \{0, 1\}$. By applying Korn inequality on the set $(a, a+2\ell_h)\times(-\frac{h}{2}, \frac{h}{2})$ we have that there exists $\tilde{A} \in \mathbb{M}^{2\times 2}$ such that

$$\int_{(a,a+2\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\nabla u - \tilde{A}|\,dx \leq C\int_{(a,a+2\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2\,dx.$$

From this inequality we deduce

$$\begin{aligned} h\ell_h|A(b) - \tilde{A}|^2 &\leq 2\int_{(b,b+\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\nabla u - A(x_1)|^2\,dx \\ &\quad +2\int_{(b,b+\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\nabla u - \tilde{A}|^2\,dx \\ &\leq C\int_{(a,a+2\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2\,dx. \end{aligned}$$

Combining the previous inequality for $\lambda = 0$ and $\lambda = 1$, we obtain

$$\begin{aligned} h\ell_h|A(a) - A(b)|^2 &\leq 2h\ell_h(|A(a) - \tilde{A}|^2 + |A(b) - \tilde{A}|^2) \\ &\leq C\int_{(a,a+2\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2\,dx. \end{aligned}$$

As A is constant on each interval $(a, a+\ell_h)$, this is equivalent to say that

$$\int_{(a,a+\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |A(x_1+\ell_h) - A(x_1)|^2\,dx \leq C\int_{(a,a+2\ell_h)\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2\,dx. \tag{2.10}$$

Let us set $I_{k,j} := -L + \ell_h(k, k+j)$. By convexity we have the following estimate:

$$\begin{aligned} \int_{\Omega_h} |A(x_1) - A_0|^2\,dx &= h\sum_{k=0}^{N_h-1}\int_{I_{k,1}} |A(x_1) - A_0|^2\,dx_1 \\ &= h\sum_{k=0}^{N_h-1}\int_{I_{k,1}} \Big|\sum_{m=0}^{k-1}\big(A(x_1 - m\ell_h) - A(x_1 - (m+1)\ell_h)\big)\Big|^2\,dx_1 \\ &\leq h\sum_{k=0}^{N_h-1} k\sum_{m=0}^{k-1}\int_{I_{k,1}} \Big|A(x_1 - m\ell_h) - A(x_1 - (m+1)\ell_h)\Big|^2\,dx_1. \end{aligned}$$

By (2.10) we deduce

$$\int_{\Omega_h} |A(x_1) - A_0|^2 \, dx \leq \sum_{k=0}^{N_h-1} k \sum_{m=0}^{k-1} C \int_{I_{k-m-1,2}\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2 \, dx.$$

It is easy to see that for every $k = 0, \dots, N_h - 1$

$$\sum_{m=0}^{k-1} \int_{I_{k-m-1,2}\times(-\frac{h}{2},\frac{h}{2})} |\varepsilon(u)|^2 \, dx \leq 2 \int_{\Omega_h} |\varepsilon(u)|^2 \, dx.$$

Therefore, we conclude that

$$\int_{\Omega_h} |A(x_1) - A_0|^2 \, dx \leq C N_h^2 \int_{\Omega_h} |\varepsilon(u)|^2 \, dx,$$

which proves claim (2.9).

Combining (2.8) and (2.9), we conclude that for every $u \in H^1(\Omega_h; \mathbb{R}^2)$ there exists a constant skew-symmetric $A_0 \in \mathbb{M}^{2\times 2}$ such that

$$\int_{\Omega_h} |\nabla u - A_0|^2 \, dx \leq \frac{C}{h^2} \int_{\Omega_h} |\varepsilon(u)|^2 \, dx.$$

Since

$$\int_{\Omega_h} \Big| \frac{1}{|\Omega_h|} \int_{\Omega_h} (\mathrm{skw} \nabla u) \, dx - A_0 \Big|^2 \, dx \leq \int_{\Omega_h} |(\mathrm{skw} \nabla u) - A_0|^2 \, dx,$$

we also have that

$$\int_{\Omega_h} \Big| \nabla u - \frac{1}{|\Omega_h|} \int_{\Omega_h} (\mathrm{skw} \nabla u) \Big|^2 \, dx \leq \frac{C}{h^2} \int_{\Omega_h} |\varepsilon(u)|^2 \, dx \tag{2.11}$$

for every $u \in H^1(\Omega_h; \mathbb{R}^2)$.

Now, if u is periodic in tangential direction, then

$$\begin{aligned} \int_{\Omega_h} \Big| \frac{1}{|\Omega_h|} \int_{\Omega_h} (\mathrm{skw} \nabla u) \Big|^2 \, dx &= \int_{\Omega_h} \Big| \frac{1}{|\Omega_h|} \int_{\Omega_h} \partial_2 u_1 \Big|^2 \, dx \\ &= \int_{\Omega_h} \Big| \frac{1}{|\Omega_h|} \int_{\Omega_h} (\partial_2 u_1 + \partial_1 u_2) \Big|^2 \, dx \\ &\leq \int_{\Omega_h} |\varepsilon(u)|^2 \, dx, \end{aligned}$$

which, together with (2.11), provides us with the desired inequality.

In order to prove the case $d = 3$, we use that (2.6) for $d = 2$ implies

$$\left\| \begin{pmatrix} \partial_{x_j} \\ \frac{1}{h} \partial_{x_3} \end{pmatrix} \begin{pmatrix} u_j \\ u_3 \end{pmatrix} \right\|_{L^2(\Omega)} \leq \frac{C}{h} \left\| \left(\begin{pmatrix} \partial_{x_j} \\ \frac{1}{h} \partial_{x_3} \end{pmatrix} \begin{pmatrix} u_j \\ u_3 \end{pmatrix} \right)_{sym} \right\|_{L^2(\Omega)} \leq \frac{C}{h} \|(\nabla_h u)_{sym}\|_{L^2(\Omega)}$$

for $j = 1,2$ and any $u \in H^1(\Omega)^3$. Moreover, applying Korn's inequality in $(-L,L)^2$ with periodic boundary conditions, we obtain

$$\|\nabla_{x'} u'\|_{L^2(\Omega)} \le C\|(\nabla_{x'} u')_{sym}\|_{L^2(\Omega)} \le C\|(\nabla_x u)_{sym}\|_{L^2(\Omega)},$$

where $u' = (u_1, u_2)^T$. Altogether this proves (2.6) for $d = 3$. ■

Remark 2.2 *The latter lemma shows that* $\left\|\frac{1}{h}\varepsilon_h(u)\right\|_{L^2(\Omega)}$ *is equivalent to* $\|\nabla_h u\|_{L^2_h(\Omega)}$ *with constants independent of* $0 < h \le 1$.

We denote by $H^1_h(\Omega)$ the space $H^1_{per}(\Omega)^d \cap \{u : \int_\Omega u(x)\,dx = 0\}$ equipped with the norm

$$\|u\|_{H^1_h(\Omega)} = \left\|\frac{1}{h}\varepsilon_h(u)\right\|_{L^2(\Omega)}, \qquad u \in H^1(\Omega)^d$$

and $H^{-1}_h(\Omega)$ its dual space with norm

$$\|f\|_{H^{-1}_h(\Omega)} = \sup\left\{\left|\langle f,\varphi\rangle_{H^{-1}_h,H^1_h}\right| : u \in H^1_h(\Omega) \text{ with } \left\|\frac{1}{h}\varepsilon_h(u)\right\|_{L^2} = 1\right\}.$$

Furthermore, we denote

$$H^m_{per}(\Omega) = \left\{f \in H^m(\Omega) : \partial^\alpha_x f|_{x_j=-L} = \partial^\alpha_x f|_{x_j=L}, j = 1,\ldots,d-1, |\alpha| \le m-1\right\}.$$

Throughout this contribution the following anisotropic variant of $H^m_{per}(\Omega)$ will be important:

$$\begin{aligned} H^{m_1,m_2}(\Omega) \;=\; & \big\{u \in L^2(\Omega) : \nabla^k_{x'}\partial^l_{x_d} u \in L^2(\Omega), k = 0,\ldots,m_1, l = 0,\ldots,m_2\,, \\ & \partial^\alpha_{x'}\partial^l_{x_d} u|_{x_j=-L} = \partial^\alpha_{x'}\partial^l_{x_d} u|_{x_j=L}, j \le d-1, |\alpha| \le m_1 - 1, l \le m_2\big\} \end{aligned}$$

where $m_1 \in \mathbb{N}, m_2 \in \mathbb{N}_0$. The spaces are equipped with the inner product

$$(f,g)_{H^{m_1,m_2}} = \sum_{|\alpha|\le m_1, k=0,\ldots,m_2} (\partial^\alpha_{x'}\partial^k_{x_d} f, \partial^\alpha_{x'}\partial^k_{x_d} g)_{L^2(\Omega)}$$

Please note that periodic boundary conditions are included in the spaces $H^{m_1,m_2}(\Omega)$ in contrast to the space $H^m(\Omega)$, where we denote them by a subscript "*per*" in order to be consistent with the usual definition of $H^m(\Omega)$.

Similarly, an anisotropic variant of L^p will be useful:

$$L^{p,q}(\Omega) \;=\; \left\{u\colon \Omega \to \mathbb{R} : \|u(x_1,.)\|_{L^q(-\frac{1}{2},\frac{1}{2})} \in L^p((-L,L)^{d-1})\right\}$$

where $1 \le p,q \le \infty$ equipped with the norm

$$\|f\|_{L^{p,q}} = \left\|\|u(x_1,.)\|_{L^q(-\frac{1}{2},\frac{1}{2})}\right\|_{L^p((-L,L)^{d-1})}.$$

We note that from the usual Hölder inequality it follows that

$$\|fg\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)},$$

for all $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Lemma 2.3 *Let $d = 2, 3$. Then*

$$H^{1,0}(\Omega) \hookrightarrow L^{p,2}(\Omega), \quad H^{2,0}(\Omega) \hookrightarrow L^{\infty,2}(\Omega), \quad H^1(\Omega) \hookrightarrow L^{4,\infty}(\Omega)$$

continuously for $p = \infty$ if $d = 2$ and any $1 \leq p < \infty$ if $d = 3$. Finally, let

$$V(\Omega) := H^{1,1}(\Omega) \cap H^{2,0}(\Omega).$$

Then $V(\Omega) \hookrightarrow C^0(\overline{\Omega})$ continuously.

Proof: The first embedding follows from $H^1(\Omega') \hookrightarrow L^p(\Omega')$ and the second from $H^2(\Omega') \hookrightarrow L^\infty(\Omega')$ since $d = 2, 3$ and $\Omega' = (-L, L)^{d-1}$. The third embedding follows from

$$H^1(-\tfrac{1}{2}, \tfrac{1}{2}; L^2(\Omega')) \cap L^2(-\tfrac{1}{2}, \tfrac{1}{2}; H^1(\Omega')) \hookrightarrow BUC([-\tfrac{1}{2}, \tfrac{1}{2}]; H^{\frac{1}{2}}(\Omega'))$$

and $H^{\frac{1}{2}}(\Omega') \hookrightarrow L^4(\Omega')$. Finally, the last embedding follows from

$$\begin{aligned} &L^2(-\tfrac{1}{2}, \tfrac{1}{2}; H^{1+k}((-L, L)^{d-1})) \cap H^1(-\tfrac{1}{2}, \tfrac{1}{2}; H^1((-L, L)^{d-1})) \\ &\hookrightarrow BUC([-\tfrac{1}{2}, \tfrac{1}{2}]; H^{1+\frac{k}{2}}((-L, L)^{d-1})) \hookrightarrow C^0(\overline{\Omega}) \end{aligned}$$

where $k = d - 2$ because of (2.2) and Sobolev embeddings. ■

Remark 2.4 *The spaces $H^{1,0}(\Omega)$ and $V(\Omega)$ are the fundamental spaces, which will be used to solve the evolution equation. We note that*

$$f \in V(\Omega) \quad \Leftrightarrow \quad f, \nabla f \in H^{1,0}(\Omega).$$

Most of the time we will estimate $f \in V(\Omega)$ by the h-dependent norm

$$\|f\|_{V_h} := \|(f, \nabla_h f)\|_{H^{1,0}(\Omega)}.$$

Because of the embedding $V(\Omega) \hookrightarrow L^\infty(\Omega)$, we are able to show that $V(\Omega)$ is an algebra with respect to point-wise multiplication. More precisely, we obtain:

Corollary 2.5 *Let $d = 2, 3$. Then there is some $C = C(\Omega) > 0$ such that*

$$\|(u_1 \cdot u_2, \nabla_h (u_1 \cdot u_2)\|_{H^{1,0}(\Omega)} \leq C \|(u_1, \nabla_h u_1)\|_{H^{1,0}(\Omega)} \|(u_2, \nabla_h u_2)\|_{H^{1,0}(\Omega)} \quad (2.12)$$

for all $u_1, u_2 \in V(\Omega)$ uniformly in $0 < h \leq 1$. Moreover, if $F \in C^2(\overline{U})$ for some open $U \subset \mathbb{R}^N$, $N \in \mathbb{N}$, and $u \in V(\Omega)^N$, then for every $R > 0$ there is some $C(R)$ independent of u such that

$$\|(F(u), \nabla_h F(u))\|_{H^{1,0}(\Omega)} \leq C(R) \qquad \text{if } \|(u, \nabla_h u)\|_{H^{1,0}(\Omega)} \leq R \qquad (2.13)$$

uniformly in $0 < h \leq 1$ and if $u(x) \in \overline{U}$ for all $x \in \overline{\Omega}$.

Proof: First of all (2.12) follows from (2.13) by first considering $\|u_1\|_{V_h}, \|u_2\|_{V_h} \leq 1$ and $F(u_1, u_2) = u_1 \cdot u_2$ together with a scaling argument.

Hence it only remains to prove (2.13). First of all,

$$\begin{aligned} \partial_{x_j} F(u) &= DF(u) \partial_{x_j} u \\ \partial_{x_j} \partial_{x_k} F(u) &= DF(u) \partial_{x_j} \partial_{x_k} u + D^2 F(u)(\partial_{x_j} u, \partial_{x_k} u) \end{aligned}$$

where $DF(u), D^2F(u)$ are uniformly bounded since $u \in C^0(\overline{\Omega})$ and $u(x) \in \overline{U}$ for all $x \in \overline{\Omega}$. Therefore $\nabla_h F(u) \in L^2(\Omega)$ can be easily estimated. Hence it only remains to consider the second order derivatives. To this end we use that

$$\begin{aligned} \left\| D^2 F(u)(\partial_{x_j} u, \frac{1}{h} \partial_{x_d} u) \right\|_{L^2(\Omega)} &\leq C \|\partial_{x_j} u\|_{L^{4,\infty}(\Omega)} \left\| \frac{1}{h} \partial_{x_d} u \right\|_{L^{4,2}(\Omega)} \\ &\leq C \|\partial_{x_j} u\|_{H^1(\Omega)} \left\| \frac{1}{h} \partial_{x_d} u \right\|_{H^{1,0}(\Omega)} \leq C'(R) \|(u, \nabla_h u)\|_{H^{1,0}(\Omega)} \end{aligned}$$

for all $j = 1, \ldots, d-1$ due to Lemma 2.3. Similarly,

$$\left\| D^2 F(u)(\partial_{x_j} u, \partial_{x_k} u) \right\|_{L^2(\Omega)} \leq C'(R) \|(u, \nabla_h u)\|_{H^{1,0}(\Omega)}$$

for all $j, k = 1, \ldots, d-1$. From these estimates the statement of the corollary easily follows. ■

For the following let $W \colon B_r(I) \subset \mathbb{R}^{d \times d} \to \mathbb{R}$ be a smooth function for some $r > 0$ which is frame invariant, i.e., $W(RF) = W(F)$ for every $F \in \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ and $R \in SO(d)$, and such that $DW(I) = 0$ and $D^2W(I) \colon \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ is positive definite on symmetric matrices. Moreover, we set $\widetilde{W}(G) = W(I + G)$. The estimates of derivatives of $D^2 \widetilde{W}(\nabla_h u)$ will be essential for the proof of our main result and will be based on the following lemma:

Lemma 2.6 *There is some constant $C > 0$, $\varepsilon > 0$, and $A \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^3(\mathbb{R}^{d \times d}))$ such that for all $G \in \mathbb{R}^{d \times d}$ with $|G| \leq \varepsilon$ we have*

$$D^3 \widetilde{W}(G) = D^3 \widetilde{W}(0) + A(G),$$

where

$$|D^3\widetilde{W}(0)|_h \leq Ch \qquad \text{for all } 0 < h \leq 1,$$
$$|A(G)| \leq C|G| \quad \text{for all } |G| \leq \varepsilon.$$

Proof: First of all, if $|G| \leq \varepsilon$ for $\varepsilon > 0$ sufficiently small, we can use a polar decomposition $I + G = RU$, where $R \in SO(d)$ and U is symmetric and positive definite such that $U^2 = (I+G)^T(I+G)$. From frame invariance we conclude that $W(I+G) = W(U) = \widehat{W}(U^2) = \widehat{W}(I + 2\operatorname{sym} G + G^T G)$ for some smooth $\widehat{W}\colon V \subset \mathbb{R}^{d\times d} \to \mathbb{R}$, where V is some open neighborhood of I. For this proof we denote $A_s = \operatorname{sym} A$. Straight-forward calculations yield

$$\begin{aligned}
DW(F)(H) &= D\widehat{W}(U^2)(2H_s + H^T G + G^T H)\\
D^2W(F)(H_1,H_2) &= D^2\widehat{W}(U^2)(2H_{1,s} + H_1^T G + G^T H_1, 2H_{2,s} + H_2^T G + G^T H_2)\\
&\quad + D\widehat{W}(U^2)(H_1^T H_2 + H_2^T H_1)
\end{aligned}$$

and

$$\begin{aligned}
&D^3W(F)(H_1,H_2,H_3) =\\
&\quad D^3\widehat{W}(U^2)(2H_{1,s} + H_1^T G + G^T H_1, 2H_{2,s} + H_2^T G + G^T H_2, 2H_{3,s} + H_3^T G + G^T H_3)\\
&\quad + D^2\widehat{W}(U^2)(H_1^T H_2 + H_2^T H_1, 2H_{3,s} + H_3^T G + G^T H_3)\\
&\quad + D^2\widehat{W}(U^2)(H_1^T H_3 + H_3^T H_1, 2H_{2,s} + H_2^T G + G^T H_2)\\
&\quad + D^2\widehat{W}(U^2)(H_2^T H_3 + H_3^T H_2, 2H_{1,s} + H_1^T G + G^T H_1)
\end{aligned}$$

where $F = I + G$. From the latter identities the statements immediately follow. ■

For the following we denote

$$\begin{aligned}
\|A\|_{H_h^{m_1,m_2}} &:= \left(\sum_{|\alpha|\leq m_1, j=0,\ldots,m_2} \|\partial_{x'}^\alpha \partial_{x_d}^j A\|_{L_h^2(\Omega)}^2 \right)^{\frac{1}{2}}\\
\|A\|_{H_h^m} &:= \left(\sum_{|\alpha|\leq m} \|\partial_x^\alpha A\|_{L_h^2(\Omega)}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

where $m, m_1, m_2 \in \mathbb{N}_0$ and $A \in H^{m_1,m_2}(\Omega)^{d\times d}$, $A \in H^m(\Omega)^{d\times d}$, respectively.

Corollary 2.7 *There are some $\varepsilon, C > 0$ such that*

$$\begin{aligned}
&\|D^3\widetilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)}\\
&\quad \leq Ch \left(\|Y_1\|_{H_h^{1,1}} + \|Y_1\|_{H_h^{2,0}} \right) \|Y_2\|_{L_h^2(\Omega)} \|Y_3\|_{L_h^2(\Omega)} \qquad (2.14)
\end{aligned}$$

for all $Y_1 \in V(\Omega)^{d\times d}, Y_2, Y_3 \in L^2(\Omega)^{d\times d}$, $0 < h \leq 1$ *and* $\|Z\|_{L^\infty(\Omega)} \leq \min(\varepsilon, h)$ *and*

$$\begin{aligned} &\|D^3\widetilde{W}(Z)(Y_1, Y_2, Y_3)\|_{L^1(\Omega)} \\ &\quad\leq Ch \|Y_1\|_{H^1_h(\Omega)} \|Y_2\|_{H^{1,0}_h(\Omega)} \|Y_3\|_{L^2_h(\Omega)} \end{aligned} \tag{2.15}$$

for all $Y_1 \in H^1(\Omega)^{d\times d}, Y_2 \in H^{1,0}(\Omega)^{d\times d}, Y_3 \in L^2(\Omega)^{d\times d}$, $0 < h \leq 1$ *and* $Z \in L^\infty(\Omega)^{d\times d}$ *with* $\|Z\|_{L^\infty(\Omega)} \leq \min(\varepsilon, h)$.

Proof: The statement follows directly from Lemma 2.6, Korn's inequality due to Lemma 2.1, and Lemma 2.3. ■

3 Long-Time Existence for Thin Sticks/Plates

3.1 Main Result

We consider

$$\partial_t^2 u_h - \frac{1}{h^2}\mathrm{div}_h D\widetilde{W}(\nabla_h u_h) = f_h h^{1+\theta} \qquad \text{in } \Omega \times I \tag{3.1}$$

where $\widetilde{W}(G) = W(I+G)$, $\Omega = (-L, L)^{d-1} \times (-\frac{1}{2}, \frac{1}{2})$, $\beta = 4 + 2\theta$, which is equivalent to $\theta = \alpha - 3$, and $I = [0, T_*]$ for some T_* together with the initial and boundary conditions

$$D\widetilde{W}(\nabla_h u_h)e_d\Big|_{x_d=\pm\frac{1}{2}} = 0 \tag{3.2}$$

$$u_h|_{x_j=L} = u_h|_{x_j=-L}, \qquad j = 1, \ldots, d-1, \tag{3.3}$$

$$(u_h, \partial_t u_h)|_{t=0} = (u_{0,h}, u_{1,h}). \tag{3.4}$$

Here we assume that $W\colon B_r(I) \to \mathbb{R}$ is a smooth function for some $r > 0$ which is frame invariant, i.e., $W(RF) = W(F)$ for every $F \in \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ and $R \in SO(d)$, and such that $DW(I) = 0$ and $D^2W(I)\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is positive definite on symmetric matrices. – Note that the latter condition implies that $D^2W(I)$ is elliptic in the sense of Legendre-Hadamard:

$$(D^2W(I)a\otimes b) : a\otimes b \geq c_0|a|^2|b|^2 \qquad \text{for all } a, b \in \mathbb{R}^d \tag{3.5}$$

for some $c_0 > 0$.

Theorem 3.1 *Let* $\theta \geq 0$, $0 < T < \infty$, *let* $f_h \in W^2_1(0,T;H^{1,0}) \cap W^1_1(0,T;H^{2,0})$, $0 < h \leq 1$, *and let* $u_{0,h} \in H^{1,3}(\Omega)^d \cap H^{4,0}(\Omega)^d$, $u_{1,h} \in H^{1,2}(\Omega)^d \cap H^{3,0}(\Omega)^d$ *such that*

$$D\widetilde{W}(\nabla_h u_{0,h})e_d|_{x_d=\pm\frac{1}{2}} = D^2\widetilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}e_d|_{x_d=\pm\frac{1}{2}} = 0,$$

and

$$\max_{k=0,1}\left\|\left(\frac{1}{h}\varepsilon_h(u_{1+k,h}),u_{2+k,h}\right)\right\|_{H^{2-k,0}(\Omega)} \leq Mh^{1+\theta} \quad (3.6)$$

$$\left\|\left(\frac{1}{h}\varepsilon_h(f_h|_{t=0}),\partial_t f_h|_{t=0}\right)\right\|_{H^{1,0}} + \max_{|\gamma|\leq 1}\|\partial_z^\gamma f_h\|_{W_1^1(0,T;H^{1,0})} \leq M \quad (3.7)$$

uniformly in $0 < h \leq 1$*, where*

$$u_{2,h} = h^{1+\theta}f_h|_{t=0} + \frac{1}{h^2}\operatorname{div}_h D\widetilde{W}(\nabla_h u_{0,h}), \quad (3.8)$$

$$u_{3,h} = h^{1+\theta}\partial_t f_h|_{t=0} + \frac{1}{h^2}\operatorname{div}_h\left(D^2\widetilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}\right). \quad (3.9)$$

If $\theta > 0$*, then there is some* $h_0 \in (0,1]$ *and* C *depending on* M *such that for every* $0 < h \leq h_0$ *there is a unique solution* $u_h \in C^3([0,T];H^{1,0}) \cap C^0([0,T];H^{2,2}\cap H^{4,0})$ *of (3.1)-(3.4) satisfying*

$$\max_{|\gamma|\leq 1}\left\|\left(\partial_t^2\partial_z^\gamma u_h, \nabla_{x,t}\partial_z^\gamma\frac{1}{h}\varepsilon_h(u_h)\right)\right\|_{C([0,T];H^{1,0})} \leq Ch^{1+\theta} \quad (3.10)$$

uniformly in $0 < h \leq h_0$*. If* $\theta = 0$*, the same statement holds with* $h_0 = 1$ *provided that* $M, T > 0$ *are sufficiently small.*

3.2 Approximate System

In order to construct a solution to (3.1)-(3.4), we first construct solutions for an approximate system.

To this end, let $P_n^h\colon L^2(\Omega)\to L^2(\Omega)$, $n\in\mathbb{N}$, be the projection on the eigenspaces of the eigenvalues not exceeding n of $-\Delta_h$ with domain $\mathcal{D}(-\Delta_h) = \{u \in H^2_{per}(\Omega) : \partial_{x_d}u|_{x_d=\pm\frac{1}{2}} = 0\}$. Then P_n^h are orthogonal projections such that

$$\begin{aligned}\|\nabla_h P_n^h f\|_{L^2(\Omega)} &= \|(-\Delta_h)^{\frac{1}{2}}P_n f\|_{L^2(\Omega)} = \|P_n(-\Delta_h)^{\frac{1}{2}}f\|_{L^2(\Omega)} \\ &\leq \|(-\Delta_h)^{\frac{1}{2}}f\|_{L^2(\Omega)} = \|\nabla_h f\|_{L^2(\Omega)}. \quad (3.11)\end{aligned}$$

The same is also true for $H^{m,0}(\Omega)$ since $-\Delta_h$ and therefore also P_n^h and $(-\Delta_h)^{\frac{1}{2}}$ commute with tangential derivative $\partial_{x'}^\beta$. Furthermore, the previous estimates imply

$$\|\partial_{x_d}P_n^h f\|_{L^2(\Omega)} \leq h\|\nabla_h f\|_{L^2(\Omega)} \leq \|\nabla f\|_{L^2(\Omega)}$$

and therefore

$$\|\nabla P_n^h f\|_{L^2(\Omega)} \leq 2\|\nabla f\|_{L^2(\Omega)}.$$

Moreover, by standard elliptic theory all eigenfunctions are smooth. Therefore $\mathcal{R}(P_n^h) \subseteq C^\infty(\overline{\Omega})$. Finally, there is a constant $C_n > 0$ such that

$$\|P_n^h f\|_{H^2(\Omega)} \leq C_n \|f\|_{L^2(\Omega)} \tag{3.12}$$

uniformly in $0 < h \leq 1$ since

$$\begin{aligned} \|P_n^h f\|_{H^2(\Omega)} &\leq C\left(\|f\|_{L^2(\Omega)} + \|\Delta P_n^h f\|_{L^2(\Omega)}\right) \\ &\leq C\left(\|f\|_{L^2(\Omega)} + \|\Delta_h P_n^h f\|_{L^2(\Omega)}\right) \\ &\leq C\left(\|f\|_{L^2(\Omega)} + n\|P_n^h f\|_{L^2(\Omega)}\right) \leq C_n \|f\|_{L^2(\Omega)} \end{aligned}$$

where we have used that $\|\Delta_h e_j\|_{L^2(\Omega)} = \lambda_j \|e_j\|_{L^2(\Omega)} \leq n\|e_j\|_{L^2(\Omega)}$ for each eigenfunction e_j to some eigenvalue $\lambda_j \leq n$ and

$$c_1 \|\Delta_h u\|_{L^2(\Omega)} \leq \|\Delta_{x'} u\|_{L^2(\Omega)} + \frac{1}{h^2}\left\|\partial_{x_d}^2 u\right\|_{L^2(\Omega)} \leq c_2 \|\Delta_h u\|_{L^2(\Omega)}$$

for any $0 < h \leq 1$. For notational simplicity we will write P_n instead of P_n^h in the following.

To motivate the approximation, we note that

$$\begin{aligned} D\widetilde{W}(\nabla_h u) &= D^2\widetilde{W}(0)\nabla_h u + \int_0^1 D^3\widetilde{W}(\tau\nabla_h u)[\nabla_h u, \nabla_h u](1-\tau)\, d\tau, \\ &= D^2\widetilde{W}(0)\nabla_h u + F'(\nabla_h u), \end{aligned} \tag{3.13}$$

by Taylor's expansion since $D\widetilde{W}(0) = DW(I) = 0$. Therefore we define the approximations

$$\begin{aligned} F_n(\nabla_h u) &:= D^2\widetilde{W}(0)\nabla_h u + \int_0^1 P_n D^3\widetilde{W}(P_n\tau\nabla_h u)[P_n\nabla_h u, P_n\nabla_h u](1-\tau)\, d\tau \\ &\equiv D^2\widetilde{W}(0)\nabla_h u + F_n'(P_n\nabla_h u) \end{aligned} \tag{3.14}$$

where $n \in \mathbb{N}$. Replacing $D\widetilde{W}(\nabla_h u)$ by $F_n(\nabla_h u)$ in (3.1), we obtain the following approximate system.

$$\partial_t^2 u_h^n - \frac{1}{h^2}\mathrm{div}_h F_n(\nabla_h u_h^n) = f_h h^{1+\theta} \qquad \text{in } \Omega \times (0,T) \tag{3.15}$$

$$F_n(\nabla_h u_h^n) e_d|_{x_d = \pm\frac{1}{2}} = 0 \tag{3.16}$$

$$u_h^n|_{x_j = L} = u_h^n|_{x_j = -L}, \qquad j = 1, \ldots, d-1, \tag{3.17}$$

$$(u_h^n, \partial_t u_h^n)|_{t=0} = (u_{0,h}^n, u_{1,h}^n). \tag{3.18}$$

Here $u_{0,h}^n, u_{1,h}^n$ will be chosen as solutions of

$$u_{2,h} = h^{1+\theta} f_h|_{t=0} + \frac{1}{h^2}\mathrm{div}_h F_n(\nabla_h u_{0,h}^n) \tag{3.19}$$

$$u_{3,h} = h^{1+\theta}\partial_t f_h|_{t=0} + \frac{1}{h^2}\mathrm{div}_h\left(DF_n(\nabla_h u_{0,h}^n)\nabla_h u_{1,h}^n\right) \tag{3.20}$$

together with $\int_\Omega u_{j,h}^n \, dx = \int_\Omega u_{j,h} \, dx$, $j = 0, 1$, and the boundary conditions

$$F_n(\nabla_h u_{0,h}^n) e_d|_{x_d=\pm\frac{1}{2}} = DF_n(\nabla_h u_{0,h}^n) \nabla_h u_{1,h}^n e_d|_{x_d=\pm\frac{1}{2}} = 0, \tag{3.21}$$

where $u_{2,h}, u_{3,h}$ are as in Theorem 3.1. We will show that (3.19)-(3.21) has a unique solution $u_{0,h}^n, u_{1,h}^n$ satisfying

$$\max_{k=0,1} \left\| \left(\frac{1}{h} \varepsilon_h(u_{k,h}^n), \nabla \frac{1}{h} \varepsilon_h(u_{k,h}^n), \nabla_h^2 u_{k,h}^n \right) \right\|_{H^{2-k,0}(\Omega)} \leq C_0 M h^{1+\theta} \tag{3.22}$$

for all $0 < h \leq h_0$ provided that $h_0 \in (0, 1]$ is sufficiently small in the case $\theta > 0$ and provided that $M > 0$ is sufficiently small (and $h_0 = 1$) if $\theta = 0$. Here C_0 is some universal constant.

Remark 3.2 *If $k = 0$, the statement follows from Proposition 3.8 below. If $k = 1$ and $u_{0,h}$ already constructed, the statement follows from Lemma 3.5 below and the Lemma of Lax-Milgram.*

The main step now consists in solving (3.15)-(3.18) under the same assumptions as in Theorem 3.1 and showing uniform bounds in $n \in \mathbb{N}$ and $0 < h \leq h_0$. More precisely, we show

Theorem 3.3 *Let $\theta \geq 0$, $0 < T < \infty$, and let $u_{2,h}, u_{3,h}, f_h$ be as in Theorem 3.1 . If $\theta > 0$, then there are $h_0 \in (0, 1]$ and C depending only on M, T such that for every $0 < h \leq h_0$ and $n \in \mathbb{N}$ there are unique solutions $u_{0,n}^h$ and $u_{1,n}^h$ of (3.19)-(3.21) and a unique solution u_h^n of (3.15)-(3.18) satisfying*

$$\max_{|\gamma| \leq 1} \left\| \left(\partial_t^2 \partial_z^\gamma u_h^n, \nabla_{x,t} \partial_z^\gamma \frac{1}{h} \varepsilon_h(u_h^n) \right) \right\|_{C([0,T]; H^{1,0})} \leq \; C h^{1+\theta} \tag{3.23}$$

uniformly in $0 < h \leq h_0$, $n \in \mathbb{N}$. If $\theta = 0$, the same is true provided $M, T > 0$ are sufficiently small.

Once Theorem 3.3 is proved, the main Theorem 3.1 is easily proved by passing to the limit $n \to \infty$ for a suitable subsequence using the uniform bounds due to (3.23).

For simplicity we will write u_n instead of u_h^n in the following.

In order to construct a solution to (3.15)-(3.18), we differentiate (3.15)-(3.18) once with respect to time. This yields the system

$$\partial_t^2 w_n - \frac{1}{h^2} \mathrm{div}_h \left(A_n(\nabla_h u_n) \nabla_h w_n \right) = \partial_t f_h h^{1+\theta} \qquad \text{in } \Omega \times (0, T), \tag{3.24}$$

$$A_n(\nabla_h u_n) \nabla_h w_n e_d|_{x_d=\pm\frac{1}{2}} = 0 \tag{3.25}$$

$$w_n|_{x_j=L} = w_n|_{x_j=-L}, \qquad j = 1, \ldots, d-1, \tag{3.26}$$

$$(w_n, \partial_t w_n)|_{t=0} = (w_{0,n}, w_{1,n}). \tag{3.27}$$

for $w_n = \partial_t u_n \equiv \partial_t u_h^n$, where

$$\begin{aligned} &A_n(\nabla_h u_n)\nabla_h w \\ &= D^2\widetilde{W}(0)\nabla_h w + \int_0^1 P_n D^3\widetilde{W}(P_n\tau\nabla_h u_n)[P_n\nabla_h u_n, P_n\nabla_h w]\,d\tau, \quad (3.28) \\ &= D^2\widetilde{W}(0)\nabla_h w + P_n\left(D^2\widetilde{W}(P_n\nabla_h u_n) - D^2\widetilde{W}(0)\right)P_n\nabla_h w \end{aligned}$$

since $F_n(\nabla_h u_n) - D^2\widetilde{W}(0)\nabla_h u_n = P_n D\widetilde{W}(P_n\nabla_h u_n) - P_n D^2\widetilde{W}(0)P_n\nabla_h u_n$ due to (3.13)-(3.14). – We note that $A_n(\nabla_h u_n)$ defines a symmetric operator on $L^2(\Omega)^{d\times d}$ since $P_n^* = P_n$ and that

$$DA_n(\nabla_h u_n)[\nabla_h v, \nabla_h w] = P_n D^3\widetilde{W}(P_n\nabla_h u_n)[P_n\nabla_h v, P_n\nabla_h w].$$

Moreover, we have $w_{0,n} = u_{1,n} \equiv u_{1,h}^n$ and

$$\begin{aligned} w_{1,n} &= u_{2,h}^n \equiv u_{2,h} = f_h h^{1+\theta}|_{t=0} + \frac{1}{h^2}\mathrm{div}_h\left(DW(\nabla_h u_{0,h})\right), \\ w_{2,n} &= u_{3,h}^n \equiv u_{3,h} = \partial_t f_h h^{1+\theta}|_{t=0} + \frac{1}{h^2}\mathrm{div}_h\left(D^2W(\nabla_h u_{0,h})\nabla_h u_{1,h}\right), \end{aligned}$$

provided (3.19)-(3.20) hold. Hence $(w_{1,n}, w_{2,n}) \equiv (w_1, w_2)$ are independent of $n \in \mathbb{N}_0$ and $u_{0,n}, u_{1,n}$. First we will solve (3.24)-(3.27) for w_n and small times (depending on $n \in \mathbb{N}$) provided that $0 < h \leq h_0$ is sufficiently small $h_0 \in (0,1]$ if $\theta > 0$ and that $M > 0$ is sufficiently small if $\theta = 0$ (independent of $n \in \mathbb{N}$). Here u_n is determined by w_n via

$$u_n(x,t) = u_{0,h}^n(x) + \int_0^t w_n(x,\tau)\,d\tau. \quad (3.29)$$

Afterwards we will derive uniform bounds in $0 < h \leq h_0$, $n \in \mathbb{N}$, and $t \in (0,T)$.

Finally, we note that, if w_n solves (3.24)-(3.27) and u_n is determined by (3.29), then u_n solves (3.15)-(3.18) since (3.24) implies

$$\partial_t^2 u_n - \frac{1}{h^2}\mathrm{div}_h F_n(\nabla_h u_n) = f_h h^{1+\theta} + c,$$

where $c = c(x)$ is independent of t, and the initial condition $\partial_t w_n|_{t=0} = \partial_t^2 u_n|_{t=0} = w_{1,n}$ implies $c \equiv 0$ by the choice of $w_{1,n}$.

3.3 Estimates for the Linearized Operator

Recall that $z = (t, x')$ with the convention that $z_0 = t$ and $z_j = x_j$ for $j = 1, \dots, d-1$. Moreover, recall that $\nabla_z = \nabla_{t,x'} = (\partial_t, \nabla_{x'})$. Furthermore,

$P_n = P_n^h$, $n \in \mathbb{N}$, $0 < h \leq 1$, denotes the smoothing operator defined above and we set $P_\infty = I$.

Let u_h for some $0 < h \leq 1$ be given such that

$$\max_{|\gamma|\leq 1} \left\| \left(\frac{1}{h}\varepsilon_h(\partial_z^\gamma u_h), \nabla_{x,t}\frac{1}{h}\varepsilon_h(\partial_z^\gamma u_h) \right) \right\|_{C([0,T];H^{1,0})} \leq Rh \tag{3.30}$$

where $R \in (0, R_0]$ for some $0 < R_0 \leq 1$ to be determined later. We note that (3.30) implies that $\frac{1}{h}\varepsilon_h(\partial_z^\gamma u_h) \in C([0,T]; V(\Omega))$ and $\partial_t^2 \frac{1}{h}\varepsilon_h(u_h) \in C([0,T]; H^{1,0})$. For the following we denote

$$\|f\|_{V_h} = \|(f, \nabla_h f)\|_{H^{1,0}},$$

where $f \in V(\Omega)$. Of course $\|f\|_V \leq \|f\|_{V_h}$ for all $0 < h \leq 1$.

Because of $\|\nabla_h P_n f\|_{L^2} \leq \|\nabla_h f\|_{L^2}$, Korn's inequality (2.6), and since P_n commutes with derivatives with respect to $z = (t, x')$, (3.30) implies

$$\begin{aligned} \max_{|\gamma|\leq 1} &\left\| \left(\partial_z^\gamma P_n \nabla_h u_h, \partial_z^\gamma P_n \frac{1}{h}\varepsilon_h(u_h) \right) \right\|_{L^\infty(0,T;V)} \\ &+ \left\| \left(\partial_t^2 P_n \nabla_h u_h, \partial_t^2 P_n \frac{1}{h}\varepsilon_h(u_h) \right) \right\|_{L^\infty(0,T;H^{1,0})} \leq C_1 Rh \end{aligned} \tag{3.31}$$

for some $C_1 \geq 1$ depending only on the constant in the Korn inequality.

Remark 3.4 *The analysis in the following will be mainly based on (3.31). Therefore we will assume throughout this section that (3.31) holds for some given $\nabla_h u_h$, $n \in \mathbb{N}_0 \cup \{\infty\}$, and $0 < h \leq 1$. – Of course, if $\nabla_h u_h$ satisfies the stronger estimate (3.30), we will have (3.31) for any $n \in \mathbb{N} \cup \{\infty\}$.*

Because of $V(\Omega) \hookrightarrow L^\infty(\Omega)$, cf. Lemma 2.3, (3.31) implies in particular

$$\left\| \left(P_n \nabla_h u_h, P_n \frac{1}{h}\varepsilon_h(u_h) \right) \right\|_{L^\infty(0,T;L^\infty \cap V)} \leq MRh, \tag{3.32}$$

where M depends only on Ω. Recall that $\widetilde{W}(A) = W(I + A)$ for all $A \in \mathbb{R}^{d\times d}$. In order to evaluate $D\widetilde{W}(P_n \nabla_h u_h)$, we will assume that $R_0 > 0$ is so small that $\widetilde{W} \in C^\infty(\overline{B_{MR_0}(0)})$ and $MR_0 \leq \varepsilon$, where $\varepsilon > 0$ is as in Corollary 2.7.

Using (3.32) and (2.14), we obtain

$$\begin{aligned} &\left| \frac{1}{h^2} \int_0^1 \left(D^3\widetilde{W}(\tau P_n \nabla_h u_h(t))[P_n \nabla_h u_h(t), P_n \nabla_h v], P_n \nabla_h w \right)_{L^2(\Omega)} d\tau \right| \\ &\leq C_0' \frac{1}{h} \left\| \left(P_n \nabla_h u_h, P_n \frac{1}{h}\varepsilon_h(u_h) \right) \right\|_{V(\Omega)} \left\| \frac{1}{h}\varepsilon_h(v) \right\|_{L^2(\Omega)} \left\| \frac{1}{h}\varepsilon_h(w) \right\|_{L^2(\Omega)} \\ &\leq C_0 R \left\| \frac{1}{h}\varepsilon_h(v) \right\|_{L^2(\Omega)} \left\| \frac{1}{h}\varepsilon_h(w) \right\|_{L^2(\Omega)} \end{aligned} \tag{3.33}$$

uniformly in $v, w \in H^1_{per}(\Omega)^d$, $0 \le t \le T$, $0 < h \le 1$.

In particular, we derive

$$\begin{aligned}
&\frac{1}{h^2}(A_n(\nabla_h u_h(t))\nabla_h v, \nabla_h v)_{L^2(\Omega)} = \frac{1}{h^2}(D^2\widetilde{W}(0)\nabla_h v, \nabla_h v)_{L^2(\Omega)} \\
&\qquad + \frac{1}{h^2}\int_0^1 \left(D^3\widetilde{W}(\tau P_n \nabla_h u_h(t))[P_n\nabla_h u_h(t), P_n \nabla_h v], P_n \nabla_h v\right)_{L^2(\Omega)} d\tau \\
&\ge \ (c_0 - C_0 R_0)\left\| \frac{1}{h}\varepsilon_h(v)\right\|^2_{L^2(\Omega)},
\end{aligned}$$

uniformly in $v \in H^1_{per}(\Omega)^d$, $t \in [0,T]$, $0 < T < \infty$ $0 < R \le R_0$, $0 < h \le 1$, where $c_0 > 0$ depends only on $D^2\widetilde{W}(0)$ and Ω. Hence, if $R_0 \in (0,1]$ is sufficiently small, we have

$$\frac{1}{h^2}(A_n(\nabla_h u_h(t))\nabla_h v, \nabla_h v)_{L^2(\Omega)} \ge \frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(v)\right\|^2_{L^2(\Omega)} \tag{3.34}$$

for all $v \in H^1_{per}(\Omega)^d$, $t \in [0,T]$, $0 < h \le 1$, $0 < R \le R_0$, and u_h satisfying (3.31), where c_0 is as above and depends only on $D^2\widetilde{W}(0)$ and Ω. By the same kind of expansion for $D^2\widetilde{W}$ and estimates one shows

$$\left|\frac{1}{h^2}(\partial_{z_j} A_n(\nabla_h u_h(t))\nabla_h v, \nabla_h w)_{L^2(\Omega)}\right| \le C' R \left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2}\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^2} \tag{3.35}$$

for all $v, w \in H^1_{per}(\Omega)^d$, $j = 0, \dots, d-1$ uniformly in $0 < h \le 1$, $t \in [0,T]$, $0 < R \le R_0$, $0 < T < \infty$. Therefore

$$\begin{aligned}
&\frac{1}{h^2}(A_n(\nabla_h u_h(t))\nabla_h v, \nabla_h v)_{H^{1,0}(\Omega)} \\
&\ge \ \frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(v)\right\|^2_{L^2(\Omega)} + \sum_{j=1}^{d-1}\frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(\partial_{x_j} v)\right\|^2_{L^2(\Omega)} \\
&\qquad - \sum_{j=1}^{d-1}\left|\frac{1}{h^2}(\partial_{x_j} A_n(\nabla_h u_h(t))\nabla_h v, \nabla_h \partial_{x_j} v)_{L^2(\Omega)}\right| \\
&\ge \ \left(\frac{c_0}{2} - C R_0\right)\left\|\frac{1}{h}\varepsilon_h(v)\right\|^2_{H^{1,0}(\Omega)} \ge \frac{c_0}{3}\left\|\frac{1}{h}\varepsilon_h(v)\right\|^2_{H^{1,0}(\Omega)}
\end{aligned} \tag{3.36}$$

for all $v \in H^{1,1}(\Omega)^d$ if $0 < R \le R_0$ for some $R_0 \in (0,1]$ sufficiently small.

To obtain higher regularity, we will use:

Lemma 3.5 *There are constants $C_0 > 0, R_0 \in (0,1]$ independent of $n \in \mathbb{N} \cup \{\infty\}$ (and $R \in (0, R_0]$) such that, if $w \in H^{1,2}(\Omega)^d \cap H^{3,0}(\Omega)^d$ solves*

$$-\frac{1}{h^2}\operatorname{div}_h(A_n(\nabla_h u_h(t))\nabla_h w) \ = \ f \qquad in\ \mathcal{D}'(\Omega)$$

for some $f \in H^{1,0}(\Omega)$ *and* $0 < h \leq 1$ *and* $P_n \nabla_h u$ *satisfies (3.32) for* $0 < R \leq R_0$*, then we have*

$$\left\| \left(\nabla \frac{1}{h} \varepsilon_h(w), \nabla_h^2 w \right) \right\|_{H^{1,0}(\Omega)} \leq C_0 \left(\|h^2 f\|_{H^{1,0}(\Omega)} + \left\| \frac{1}{h} \varepsilon_h(w) \right\|_{H^{2,0}(\Omega)} \right). \quad (3.37)$$

If additionally

$$e_d \cdot A_n(\nabla_h u_h(t)) \nabla_h w|_{x_d = \pm\frac{1}{2}} = 0, \quad (3.38)$$

then

$$\max_{j=0,1} \left\| \left(\nabla_h^{1+j} w, \nabla^j \frac{1}{h} \varepsilon_h(w) \right) \right\|_{H^{1,0}(\Omega)} \leq C_0 \|f\|_{H^{1,0}(\Omega)}. \quad (3.39)$$

Proof: Let $0 < R_0 \leq 1$ be at least as small as above. First of all, since $A_n(0) = D^2\widetilde{W}(0)$, we obtain

$$\begin{aligned} \operatorname{div}_h(A_n(0)\nabla_h w) &= \frac{1}{h} \partial_{x_d} (D^2\widetilde{W}(0) \nabla_h w)_d + \operatorname{div}_{x'} (D^2\widetilde{W}(0) \nabla_h w)' \\ &= \frac{1}{h^2} (D^2\widetilde{W}(0) \partial_{x_d}^2 w \otimes e_d)_d + \frac{1}{h} (D^2\widetilde{W}(0) \partial_{x_d} (\nabla_{x'}, 0) w)_d + \operatorname{div}_{x'} (D^2\widetilde{W}(0) \nabla_h w)' \end{aligned}$$

where $A' = (a_{ij})_{i=1,\ldots d, j=1,\ldots d-1}$ for $A \in \mathbb{R}^{d\times d}$. We note that the second and third term consists of terms of $\nabla_{x'} \nabla_h w$. Moreover,

$$(D^2\widetilde{W}(0) \partial_{x_d}^2 w \otimes e_d)_d = M \partial_{x_d}^2 w$$

for some symmetric positive definite matrix M, which follows from the Legendre-Hadamard condition (3.5). Hence

$$\frac{1}{h^2} \partial_{x_d}^2 w = M^{-1} \left(\operatorname{div}_h (Q \nabla_h w) - \frac{1}{h} (Q \partial_{x_d} (\nabla_{x'}, 0) w)_d - \operatorname{div}_{x'} (Q \nabla_h w)' \right)$$

for $Q = D^2\widetilde{W}(0)$ and therefore

$$\begin{aligned} &\left\| \frac{1}{h^2} \partial_{x_d}^2 w \right\|_{H^{1,0}(\Omega)} \\ &\leq C_0 \left(\left\| \operatorname{div}_h \left(D^2\widetilde{W}(0) \nabla_h w \right) \right\|_{H^{1,0}(\Omega)} + \|\nabla_{x'} \nabla_h w\|_{H^{1,0}(\Omega)} \right). \end{aligned}$$

Thus Korn's inequality and $\|\partial_{x_d} \frac{1}{h} \varepsilon_h(w)\|_{L^2(\Omega)} \leq \|\nabla_h^2 w\|_{L^2(\Omega)}$ yields

$$\begin{aligned} &\left\| \left(\nabla \frac{1}{h} \varepsilon_h(w), \nabla_h^2 w \right) \right\|_{H^{1,0}(\Omega)} \\ &\leq C_0 \left(\left\| \operatorname{div}_h \left(D^2\widetilde{W}(0) \nabla_h w \right) \right\|_{H^{1,0}(\Omega)} + \left\| \nabla_{x'} \frac{1}{h} \varepsilon_h(w) \right\|_{H^{1,0}(\Omega)} \right). \quad (3.40) \end{aligned}$$

Next we use that

$$
\begin{aligned}
&\operatorname{div}_h\left(A_n(\nabla_h u_h)\nabla_h w\right)\\
&\quad= \operatorname{div}_h\left(D^2\widetilde{W}(0)\nabla_h w\right) + \int_0^1 \operatorname{div}_h\left(P_n D^3\widetilde{W}(\tau P_n^h\nabla_h u_h)[P_n^h\nabla_h u_h, P_n^h\nabla_h w]\right)d\tau\\
&\quad\equiv \operatorname{div}_h\left(D^2\widetilde{W}(0)\nabla_h w\right) + \operatorname{div}_h\left(G(P_n^h\nabla_h u_h)[P_n^h\nabla_h u_h, P_n^h\nabla_h w]\right),
\end{aligned}
$$

where $G \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^3(\mathbb{R}^{d\times d}))$ for some suitable $\varepsilon > 0$. Hence Corollary 2.5 implies

$$
\begin{aligned}
&\|G(P_n^h\nabla_h u_h)[P_n^h\nabla_h u_h, P_n^h\nabla_h w]\|_{V_h}\\
&\quad\le C\|G(P_n^h\nabla_h u_h)\|_{V_h}\|P_n^h\nabla_h u_h\|_{V_h}\|P_n^h\nabla_h w\|_{V_h}\\
&\quad\le CR_0\left\|\left(\nabla_h w, \nabla_h^2 w\right)\right\|_{H^{1,0}}.
\end{aligned}
$$

where $\|f\|_{V_h} = \|(f,\nabla_h f)\|_{H^{1,0}}$ and we have used (3.11) as well as (3.32). Hence

$$
\begin{aligned}
&\left\|\operatorname{div}_h\left(D^2\widetilde{W}(0)\nabla_h w\right)\right\|_{H^{1,0}(\Omega)}\\
&\quad\le \|\operatorname{div}_h\left(A_n(\nabla u_h)\nabla_h w\right)\|_{H^{1,0}(\Omega)}\\
&\qquad + \left\|\nabla_h(G(P_n^h\nabla_h u_h)[P_n^h\nabla_h u_h, P_n^h\nabla_h w]\right\|_{H^{1,0}(\Omega)}\\
&\quad\le \left\|h^2 f\right\|_{H^{1,0}(\Omega)} + CR_0\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{H^{1,0}(\Omega)}. \qquad (3.41)
\end{aligned}
$$

Combining the latter estimate with (3.40) for sufficiently small $R_0 \in (0,1]$, we obtain (3.37).

Now, if additionally (3.38), then

$$
\frac{1}{h^2}(A_n(\nabla_h u_h)\nabla_h w, \nabla_h\varphi)_{H^{1,0}} = (f,\varphi)_{H^{1,0}}
$$

for all $\varphi \in V(\Omega)^d$. Hence, choosing $\varphi = \partial_{x'}^{2\gamma} w_0$ with $w_0 = w - \frac{1}{|\Omega|}\int_\Omega w\,dx$ and $|\gamma| \le 1$ and using integration by parts, we obtain by (3.36), (3.37), and (3.42) below

$$
\begin{aligned}
&\sup_{|\gamma|\le 1}\left\|\partial_{x'}^\gamma \frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}(\Omega)}^2\\
&\quad\le C_0\|f\|_{H^{1,0}(\Omega)}\max_{|\gamma|\le 1}\left\|\partial_{x'}^{2\gamma} w_0\right\|_{H^{1,0}(\Omega)} + CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^1(\Omega)}\max_{|\gamma|\le 1}\|\partial_{x'}^\gamma w_0\|_{H^{1,0}(\Omega)}\\
&\quad\le C_0\left(\|f\|_{H^{1,0}(\Omega)} + R\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{2,0}(\Omega)}\right)\max_{|\gamma|\le 1}\left\|\partial_{x'}^\gamma \frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}(\Omega)}.
\end{aligned}
$$

Thus, choosing R_0 sufficiently small, we obtain

$$
\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{2,0}(\Omega)} \le C_0\|f\|_{H^{1,0}(\Omega)}
$$

with $C_0 > 0$ depending only on Ω. This finishes the proof. ■

Lemma 3.6 *Let $P_n \nabla u_h(t)$ satisfy (3.31) for some $n \in \mathbb{N} \cup \{\infty\}$, $0 < h \leq 1$, $t \in [0,T]$, and $0 < R \leq R_0$, where $R_0 \in (0,1]$ is so small that all previous conditions are satisfied. Then*

$$\left|\frac{1}{h^2}\left((\partial_z^\beta A_n(\nabla_h u_h(t)))\nabla_h w, \nabla_h v\right)_\Omega\right| \leq CR \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{|\beta|-1}(\Omega)} \left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2(\Omega)} \quad (3.42)$$

if $1 \leq |\beta| \leq 2$ and

$$\left|\frac{1}{h^2}\left((\partial_z^\beta A_n(\nabla_h u_h(t)))\nabla_h w, \nabla_h v\right)_\Omega\right| \leq CR \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{V(\Omega)} \left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2(\Omega)} \quad (3.43)$$

if $|\beta| = 3$ and $\beta \neq 3e_0$, i.e., $\partial_z^\beta \neq \partial_t^3$. Moreover,

$$\left|\frac{1}{h^2}\left((\partial_t \partial_{x_j} A_n(\nabla_h u_h(t)))\nabla_h w, \nabla_h v\right)_\Omega\right| \leq CR \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}(\Omega)} \left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2(\Omega)} \quad (3.44)$$

for all $j = 1, \ldots, d-1$. The constants C are independent of $\nabla_h u_h(t), w, v, h, n, R$.

Proof: If $|\beta| = 1$, then (3.42) is just (3.35). Next let $|\beta| = 2$. Then for $j,k = 0,\ldots,d-1$

$$\begin{aligned}\partial_{z_j}\partial_{z_k} A_n(\nabla_h u_h) &= P_n D^3 \widetilde{W}(P_n \nabla_h u_h)(P_n \partial_{z_j}\partial_{z_k} \nabla_h u_h) \\ &\quad + P_n D^4 \widetilde{W}(P_n \nabla_h u_h)(P_n \partial_{z_j} \nabla_h u_h, P_n \partial_{z_k} \nabla_h u_h),\end{aligned}$$

where

$$\left\|P_n \partial_{z_j}\partial_{z_k} \frac{1}{h}\varepsilon_h(u_h)\right\|_{H^{1,0}(\Omega)} \leq C_0 \left\|P_n \nabla_z \frac{1}{h}\varepsilon_h(u_h)\right\|_{V(\Omega)} \leq C_0 R h$$

due to (3.31). Together with (2.15) the latter estimate implies (3.42) in the case $|\beta| = 2$. Moreover, (3.44) is proved in the same way using that $\partial_t \partial_{x_j} \frac{1}{h}\varepsilon_h(u_h) \in H^1(\Omega)$ is uniformly bounded and again (2.15).

Finally, if $|\beta| = 3$ with $\partial_z^\beta \neq \partial_t^3$, we use that

$$\begin{aligned}\partial_{z_j}\partial_{z_k}\partial_{z_l} A_n(\nabla_h u_h) &= P_n D^3 \widetilde{W}(P_n \nabla_h u_h)[P_n \partial_{z_j}\partial_{z_k}\partial_{z_l} \nabla_h u_h] \\ &\quad + P_n D^4 \widetilde{W}(P_n \nabla_h u_h)[P_n \partial_{z_j}\partial_{z_l} \nabla_h u_h, P_n \partial_{z_k} \nabla_h u_h] \\ &\quad + P_n D^4 \widetilde{W}(P_n \nabla_h u_h)[P_n \partial_{z_j} \nabla_h u_h, P_n \partial_{z_k}\partial_{z_l} \nabla_h u_h] \\ &\quad + P_n D^4 \widetilde{W}(P_n \nabla_h u_h)[P_n \partial_{z_l} \nabla_h u_h, P_n \partial_{z_j}\partial_{z_k} \nabla_h u_h] \\ &\quad + P_n D^5 W(P_n \nabla_h u_h)[P_n \partial_{z_j} \nabla_h u_h, P_n \partial_{z_k} \nabla_h u_h, P_n \partial_{z_l} \nabla_h u_h]\end{aligned}$$

Since $P_n\nabla_z\nabla_h u_h \in L^\infty(\Omega)$ and $P_n\nabla_z^2\nabla_h u_h \in H^{1,0}(\Omega) \hookrightarrow L^{4,2}(\Omega)$ are of order CRh due to (3.31), the estimates of all parts in

$$\frac{1}{h^2}\left((\partial_z^\beta A_n(\nabla_h u_h))\nabla_h w, \nabla_h v\right)_\Omega$$

which come from terms involving $D^4\widetilde{W}$ or $D^5\widetilde{W}$ can be done in a straight forward manner by

$$CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^{4,\infty}}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2} \leq C'R\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^1}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2}$$

uniformly in $0 < h \leq 1$ and $n \in \mathbb{N}_0 \cup \{\infty\}$. It only remains to estimate the part involving the $D^3\widetilde{W}$-term: To this end we use that (3.31) and (2.14) imply

$$\begin{aligned}
&\left|\frac{1}{h^2}\left((D^3\widetilde{W}(P_n\nabla_h u_h))[P_n\partial_z^\beta\nabla_h u_h, P_n\nabla_h w], P_n\nabla_h v\right)_\Omega\right| \\
&\leq \frac{C_0}{h}\left\|\partial_z^\beta\frac{1}{h}\varepsilon_h(u_h)\right\|_{L^2(\Omega)}\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla_h w\right)\right\|_{V(\Omega)}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2(\Omega)} \\
&\leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{V(\Omega)}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{L^2(\Omega)}
\end{aligned}$$

Altogether we obtain (3.43). ∎

Corollary 3.7 *Let $P_n\nabla_h u_h(t)$, $n \in \mathbb{N} \cup \{\infty\}$, $0 < h \leq 1$, $0 < R \leq R_0$ be as in Lemma 3.6. Then we have*

$$\begin{aligned}
\left|\frac{1}{h^2}\left((\partial_t A_n(\nabla_h u_h(t)))\nabla_h w, \nabla_h v\right)_{H^{1,0}}\right| &\leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{H^{1,0}} \\
\left|\frac{1}{h^2}\left((\partial_t^2 A_n(\nabla_h u_h))\nabla_h w, \nabla_h v\right)_{H^{1,0}}\right| &\leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{V(\Omega)}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{H^{1,0}}
\end{aligned}$$

uniformly in $0 < h \leq 1$, $0 < R \leq R_0$, $n \in \mathbb{N}_0 \cup \{\infty\}$, $t \in [0, T]$, and $0 < T < \infty$.

Proof: Because of (3.35), we have

$$\left|\frac{1}{h^2}\left((\partial_t A_n(\nabla_h u_h))\nabla_h w_j, \nabla_h v_j\right)_{L^2(\Omega)}\right| \leq CR\left\|\frac{1}{h}\varepsilon_h(w_j)\right\|_{L^2}\left\|\frac{1}{h}\varepsilon_h(v_j)\right\|_{L^2}$$

for $j = 0, \ldots, d-1$ and $(w_0, v_0) = (w, v)$, $(w_j, v_j) = (\partial_{x_j} w, \partial_{x_j} v)$ if $j = 1, \ldots, d-1$. Moreover,

$$\begin{aligned}
&\left|\frac{1}{h^2}\left((\partial_t\partial_{x_j} A_n(\nabla_h u_h))\nabla_h w, \nabla_h v_j\right)_{L^2(\Omega)}\right| \\
&\leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}}\left\|\frac{1}{h}\varepsilon_h(v_j)\right\|_{L^2} \leq C'R\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{1,0}}\left\|\frac{1}{h}\varepsilon_h(v)\right\|_{H^{1,0}}
\end{aligned}$$

for $j = 1, \ldots, d-1$ due to (3.44). Altogether this implies the first estimate.

Similarly, (3.42) yields

$$\begin{aligned}
&\left| \frac{1}{h^2} \left((\partial_t^2 A_n(\nabla_h u_h)) \nabla_h w_j, \nabla_h v_j \right)_{L^2(\Omega)} \right| \\
&\quad \leq \; CR \left\| \frac{1}{h} \varepsilon_h(w_j) \right\|_{H^1} \left\| \frac{1}{h} \varepsilon_h(v_j) \right\|_{L^2} \leq C'R \left\| \frac{1}{h} \varepsilon_h(w) \right\|_{V(\Omega)} \left\| \frac{1}{h} \varepsilon_h(v) \right\|_{H^{1,0}}
\end{aligned}$$

where (w_j, v_j) are as above. Finally, (3.43) implies

$$\begin{aligned}
&\left| \frac{1}{h^2} \left((\partial_t^2 \partial_{x_j} A_n(\nabla_h u_h)) \nabla_h w, \nabla_h v_j \right)_{L^2(\Omega)} \right| \\
&\quad \leq \; CR \left\| \frac{1}{h} \varepsilon_h(w) \right\|_{V(\Omega)} \left\| \frac{1}{h} \varepsilon_h(v_j) \right\|_{L^2} \leq C'R \left\| \frac{1}{h} \varepsilon_h(w) \right\|_{V(\Omega)} \left\| \frac{1}{h} \varepsilon_h(v) \right\|_{H^{1,0}}
\end{aligned}$$

This shows the second estimates. ■

Next we will show solvability of (3.19) and the estimate (3.22) for $k = 0$.

Proposition 3.8 *Let $0 < h \leq 1$, P_n^h, $n \in \mathbb{N}$, be the smoothing operators from above, let $P_\infty^h = I$, and let F_n be defined as in (3.14). Then there are constants $C_0 > 0, M_0 \in (0,1]$ independent of $n \in \mathbb{N} \cup \{\infty\}$ such that for any $f \in H^{2,0}(\Omega)^d$ with $\|f\|_{H^{2,0}} \leq M_0 h$ and $\int_\Omega f \, dx = 0$ there is a solution $w \in H^{2,2}(\Omega)^d \cap H^{4,0}(\Omega)^d$, which is unique up to a constant, such that*

$$\frac{1}{h^2} \left(F_n(\nabla_h w), \nabla_h \varphi \right)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \tag{3.45}$$

for all $\varphi \in H^1_{per}(\Omega)^d$ and

$$\left\| \left(\frac{1}{h} \varepsilon_h(w), \nabla \frac{1}{h} \varepsilon_h(w), \nabla_h^2 w \right) \right\|_{H^{2,0}(\Omega)} \leq C_0 \|f\|_{H^{2,0}(\Omega)}. \tag{3.46}$$

for some $C_0 > 0$ independent of h, f, n.

Proof:

Because of (3.14), (3.45) is equivalent to solve

$$\begin{aligned}
\langle L_h w, \varphi \rangle_{H_h^{-1}, H_h^1} &\equiv \frac{1}{h^2} \left(D^2 \widetilde{W}(0) \nabla_h w, \nabla_h \varphi \right)_{L^2(\Omega)} \\
&= \; (f, \varphi)_{L^2(\Omega)} - \frac{1}{h^2} \left(F_n'(\nabla_h w), \nabla_h \varphi \right)_{L^2(\Omega)} \equiv (G_h(w), \nabla \varphi)_{L^2(\Omega)}.
\end{aligned}$$

We will prove the proposition with the aid of the contraction mapping principle. To this end we note that for every $f \in H^{1,0}(\Omega)^d$ and $F \in H^{1,1}(\Omega)^{d \times d} \cap H^{2,0}(\Omega)^{d \times d}$ there is a unique $w \in H^{2,1}(\Omega)^d \cap H^{3,0}(\Omega)^d$ such that

$$\langle L_h w, \varphi \rangle_{H_h^{-1}, H_h^1} = (f, \varphi)_{L^2(\Omega)} + (F, \nabla_h \varphi)_{L^2(\Omega)} \tag{3.47}$$

for all $\varphi \in H^1_h(\Omega)$ because of the Lemma of Lax-Milgram, Korn's inequality, and since L_h commutes with tangential derivatives. The solution satisfies

$$\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{2,0}(\Omega)} \leq C_0\left(\|f\|_{H^{1,0}(\Omega)} + \|F\|_{H^{2,0}_h(\Omega)}\right) \tag{3.48}$$

for some universal $C_0 > 0$. Moreover, (3.47) implies

$$-\frac{1}{h^2}\mathrm{div}_h(D^2\widetilde{W}(0)\nabla_h w) = f - \mathrm{div}_h F \qquad \text{in } \mathcal{D}'(\Omega).$$

Therefore $w \in H^{2,1}(\Omega)^d$ by standard elliptic regularity. Hence Lemma 3.5 together with the previous estimate imply

$$\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{H^{1,0}(\Omega)} \leq C_0\left(\left\|\left(f, h^2\nabla_h F\right)\right\|_{H^{1,0}(\Omega)} + \|F\|_{H^{2,0}_h(\Omega)}\right)$$

for some universal $C_0 > 0$. Using (3.33) and Corollary 2.7, one derives that

$$\|G_h(w_1) - G_h(w_2)\|_{H^{2,0}_h(\Omega)} \leq CM_0\left\|\frac{1}{h}\varepsilon_h(w_1 - w_2)\right\|_{V(\Omega)}$$

for some $C > 0$ provided that

$$\max_{j=1,2}\left\|\left(\frac{1}{h}\varepsilon_h(w_j), \nabla\frac{1}{h}\varepsilon_h(w_j), \nabla_h^2 w_j\right)\right\|_{H^{1,0}(\Omega)} \leq 2C_0M_0 h, \tag{3.49}$$

where $C_0 > 0$ is as (3.48) and $M_0 \in (0,1]$. Here we note that

$$\begin{aligned}
\partial_{x_k} G_h(w_j) &= -\frac{1}{h^2}A'_n(\nabla_h w_j)\nabla_h\partial_{x_k} w_j, \\
\partial_{x_k}\partial_{x_l} G_h(w_j) &= -\frac{1}{h^2}A'_n(\nabla_h w_j)\nabla_h\partial_{x_k}\partial_{x_l} w_j \\
&\quad -\frac{1}{h^2}P_n D^3 W(P_n\nabla_h w_j)[P_n\nabla_h\partial_{x_k} w_j, P_n\nabla_h\partial_{x_l} w_j]
\end{aligned}$$

for all $k, l = 1, \ldots, d-1$, $j = 1, 2$, where $A'_n(\nabla_h w_j) = A_n(\nabla_h w_j) - D^2\widetilde{W}(0)$. To estimate the A'_n-terms one uses (2.14) or (3.33) and to estimate the D^3W-term one uses (2.15).

Furthermore, using Corollary 2.5, one shows in the same way as in the proof of Lemma 3.5, that

$$h^2\|\nabla_h(G_h(w_1) - G_h(w_2))\|_{H^{1,0}(\Omega)} \leq CM_0\left\|\left(\nabla_h^2(w_1 - w_2), \nabla_h(w_1 - w_2)\right)\right\|_{H^{1,0}}$$

for some $C > 0$ provided that (3.49) holds. Hence, if $M_0 \in (0,1]$ is sufficiently small, we obtain that $L_h^{-1}G_h \colon X_h \to X_h$ restricted to $\overline{B_{2C_0M_0h}(0)}$ is a contraction, where X_h is normed by

$$\|w\|_{X_h} := \left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{H^{1,0}(\Omega)}.$$

Therefore we obtain a unique solution w solving (3.45) and satisfying (3.46) with $H^{2,0}(\Omega)$ replaced by $H^{1,0}(\Omega)$. In order to obtain (3.46), one can simply use that $w_j := \partial_{x_j} w$, $j = 1, \dots, d-1$, solves

$$\frac{1}{h^2}\left(A_n(\nabla_h w)\nabla_h w_j, \nabla_h \varphi\right)_{L^2(\Omega)} = (\partial_{x_j} f, \varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in H^1_{per}(\Omega)$$

and apply Lemma 3.5. ■

The following lemma contains the essential estimate for the linearized system to (3.1)-(3.4):

Lemma 3.9 *Let $0 < T < \infty$, $0 < h \le 1$, $0 < R \le R_0$, $n \in \mathbb{N} \cup \{\infty\}$ be given, and let R_0 be as in Lemma 3.5. Assume that u_h satisfies (3.31) and that $f \in W^1_1(0,T;H^{1,0})^d$, $w_0 \in H^{1,2}(\Omega)^d \cap H^{3,0}(\Omega)^d, w_1 \in V(\Omega)^d$. Then there is a unique $w \in C^0([0,T];H^{2,1}(\Omega) \cap H^{3,0}(\Omega))^d \cap C^2([0,T];H^{1,0}(\Omega))^d$ that solves*

$$\begin{aligned} \partial_t^2 w - \frac{1}{h^2}\operatorname{div}_h(A_n(\nabla_h u_h)\nabla_h w) &= f && (3.50)\\ A_n(\nabla_h u_h)\nabla_h w\, e_d|_{x_d=\pm\frac{1}{2}} &= 0 && (3.51)\\ w|_{x_j=L} &= w|_{x_j=-L}, j = 1,\dots,d-1 && (3.52)\\ (w, \partial_t w)|_{t=0} &= (w_0, w_1). && (3.53)\end{aligned}$$

Moreover, there are some constants $C_L, C' \ge 1$ depending only on Ω and W such that

$$\begin{aligned} &\left\| \left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w)\right)\right\|_{C([0,T];H^{1,0})} && (3.54)\\ &\le C_L e^{C'RT}\left(\|f\|_{W^1_1(0,T;H^{1,0})} + \left\|\left(\frac{1}{h}\varepsilon_h(w_1), w_2, f|_{t=0}\right)\right\|_{H^{1,0}}\right)\end{aligned}$$

where

$$w_2 = \frac{1}{h^2}\operatorname{div}_h(A_n(\nabla_h u_h|_{t=0})\nabla_h w_0) + f|_{t=0}. \qquad (3.55)$$

Proof: Existence of a solution $w \in C^1([0,T];H^{1,0}(\Omega)) \cap C^0([0,T];V(\Omega))$ can be obtained by the standard energy method, cf. e.g. [19, Theorem 10.8] with $H = H^{1,0}(\Omega)^d, V = V(\Omega)^d$ and

$$a(t;v,w) = \frac{1}{h^2}(A_n(\nabla_h u_h)\nabla_h v, \nabla_h w)_{H^{1,0}(\Omega)}, \qquad v, w \in V(\Omega)^d,$$

which is bounded and coercive on $V(\Omega)^d$ because of (3.36). If $n \neq \infty$, then $w \in C^2([0,T];H^{1,0}(\Omega)) \cap C^1([0,T];V(\Omega))$ can be obtained by the same technique as in [22, Section 30.1], where we note that

$$\frac{1}{h^2}\operatorname{div}_h((\partial_t A_n(\nabla_h u_h))\nabla_h \cdot)\colon V(\Omega) \to H^{1,0}(\Omega)$$

is a bounded linear operator with operator norm bounded uniformly in $t \in [0,T]$ because of the smoothing operator P_n in the definition of A_n and $\partial_t \nabla_h u_h \in C([0,T]; L^2(\Omega))$. Moreover, $w \in C^0([0,T]; H^{1,2}(\Omega) \cap H^{3,0}(\Omega)))$ follows from (3.50) with $\partial_t^2 w, f \in C^0([0,T]; H^{1,0}(\Omega))$, (3.28) with $n < \infty$ and standard elliptic theory. Finally, if $n = \infty$, then existence of a solution $w \in C^2([0,T]; H^{1,0}(\Omega)) \cap C^0([0,T]; H^{1,2}(\Omega) \cap H^{3,0}(\Omega))$ can be obtained from the case $n \in \mathbb{N}$ by using the uniform bounds due to (3.54) proved below and passing to the limit $n \to \infty$.

Hence the main task is to establish (3.54). First of all, we note that (3.50)-(3.52) imply

$$a(t) := \int_\Omega w(t)\,dx = \int_\Omega w_0\,dx + t\int_\Omega w_1\,dx + \int_0^t (t-\tau)\int_\Omega f(\tau,x)\,dx\,d\tau.$$

Hence, replacing $w(t)$ by $w(t) - a(t)$ and subtracting from (w_0, w_1, f) their mean values with respect to Ω, we can reduce to the case

$$\int_\Omega w_0\,dx = \int_\Omega w_1\,dx = \int_\Omega f(t)\,dx = \int_\Omega w(t)\,dx = 0$$

for all $0 \le t \le T$.

Now we differentiate (3.50) with respect to t and multiply with $\partial_t^2 w$ in $H := H^{1,0}(\Omega)$. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_t^2 w\right\|_H^2 + \frac{1}{h^2}\left(A_n(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\right)_H\right)\\
&\quad\le \left|\left(\partial_t f, \partial_t^2 w\right)_H\right| + \frac{3}{2}\left|\frac{1}{h^2}\left((\partial_t A_n(\nabla_h u_h))\nabla_h\partial_t w, \nabla_h\partial_t w\right)_H\right|\\
&\qquad + \left|\frac{1}{h^2}\left((\partial_t^2 A_n(\nabla_h u_h))\nabla_h w, \nabla_h\partial_t w\right)_H\right| - \frac{1}{h^2}\frac{d}{dt}\left((\partial_t A_n(\nabla_h u_h))\nabla_h w, \nabla_h\partial_t w\right)_H
\end{aligned}$$

in the sense of distributions, where we have used

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2h^2}\left(A_n(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\right)_H + \frac{1}{h^2}\left((\partial_t A_n(\nabla_h u_h))\nabla_h w, \nabla_h\partial_t w\right)_H\right)\\
&\quad= -\frac{1}{2}\frac{d}{dt}\|\partial_t^2 w\|_H^2 + \left(\partial_t f, \partial_t^2 w\right)_H + \frac{3}{2h^2}\left((\partial_t A_n(\nabla_h u_h))\nabla_h\partial_t w, \nabla_h\partial_t w\right)_H\\
&\qquad + \frac{1}{h^2}\left((\partial_t^2 A_n(\nabla_h u_h))\nabla_h w, \nabla_h\partial_t w\right)_H
\end{aligned} \tag{3.56}$$

and (3.51)-(3.52). Due to Corollary 3.7 we have

$$\begin{aligned}
\frac{1}{h^2}\left|\left((\partial_t A_n(\nabla_h u_h))\nabla_h\partial_t w, \nabla_h\partial_t w\right)_H\right| &\le CR\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|_H^2,\\
\frac{1}{h^2}\left|\left((\partial_t^2 A_n(\nabla_h u_h))\nabla_h w, \nabla_h\partial_t w\right)_H\right| &\le CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{V(\Omega)}\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|_H
\end{aligned}$$

for every $t \in [0,T]$. Moreover, because of Corollary 3.7 again,

$$\begin{aligned}
&\sup_{0\leq\tau\leq t}\left|\frac{1}{h^2}\left((\partial_t A_n(\nabla_h u_h(\tau)))\nabla_h w(\tau), \nabla_h\partial_t w(\tau)\right)_H\right| \\
&\quad\leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^\infty(0,t;H)}\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|_{L^\infty(0,t;H)}
\end{aligned}$$

Therefore the previous estimates, (3.36), and Young's inequality imply

$$\begin{aligned}
&\sup_{0\leq\tau\leq t}\left\|\left(\partial_t^2 w(\tau), \frac{1}{h}\varepsilon_h(\partial_t w(\tau))\right)\right\|_H^2 \\
&\quad\leq CR\left\|\left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w)\right)\right\|_{L^2(0,t;H)}^2 + C_0\|\partial_t f\|_{L^1(0,T;H)}^2 \\
&\qquad+C_0\left\|\left(\frac{1}{h}\varepsilon_h(w_1), w_2\right)\right\|_H^2 + CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^\infty(0,t;H)}^2.
\end{aligned}$$

Now

$$\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^\infty(0,t;H)}^2 \leq C_0\left(\max_{j=0,1}\left\|\frac{1}{h}\varepsilon_h(\partial_t^j w)\right\|_{L^2(0,t;H)}^2 + \left\|\frac{1}{h}\varepsilon_h(w_0)\right\|_H^2\right),$$

due to

$$\|f\|_{L^\infty(0,t;H)} \leq C_0\left(\|f\|_{W_2^1(0,t;H)} + \|f|_{t=0}\|_H\right) \tag{3.57}$$

with some $C_0 > 0$ independent of $t > 0$, cf. (2.4), and

$$\begin{aligned}
&\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla\frac{1}{h}\varepsilon_h(w)\right)\right\|_{L^\infty(0,t;H)} \\
&\quad\leq C_0\left(\|f\|_{L^\infty(0,t;H)} + \|\partial_t^2 w\|_{L^\infty(0,t;H)}\right) \\
&\quad\leq C_0\left(\|f\|_{W_1^1(0,t;H)} + \|f|_{t=0}\|_H + \|\partial_t^2 w\|_{L^\infty(0,t;H)}\right)
\end{aligned}$$

due to (3.39) and (3.57) uniformly in $0 \leq t \leq T$. Hence we conclude

$$\begin{aligned}
&\sup_{0\leq\tau\leq t}\left\|\left(\partial_t^2 w(\tau), \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w(\tau))\right)\right\|_H^2 \\
&\quad\leq CR\left\|\left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w)\right)\right\|_{L^2(0,t;H)}^2 \\
&\qquad+C_0\|f\|_{W_1^1(0,T;H)}^2 + C_0\left\|\left(\frac{1}{h}\varepsilon_h(w_1), w_2, f|_{t=0}\right)\right\|_H^2,
\end{aligned}$$

where we have used $R \leq 1$ and (3.55). Therefore the Lemma of Gronwall yields

$$\begin{aligned}
&\left\| \left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w) \right) \right\|^2_{L^\infty(0,T;H)} \\
&\leq C_L e^{C'RT} \left(\left\| \left(\frac{1}{h}\varepsilon_h(w_1), w_2, f|_{t=0} \right) \right\|^2_H + \|f\|^2_{W^1_1(0,T;H)} \right).
\end{aligned}$$

This shows (3.54). ■

Finally, we consider (3.24)-(3.27) with f replaced by $-\mathrm{div}_h f$ in its weak form, namely:

$$\begin{aligned}
-(\partial_t w, \partial_t \varphi)_{Q_T} &+ \frac{1}{h^2}(A_n(\nabla_h u_h)\nabla_h w, \nabla_h \varphi)_{Q_T} \\
&= (f, \nabla_h \varphi)_{Q_T} + \langle w_1, \varphi|_{t=0}\rangle_{H_h^{-1}, H_h^1} && (3.58)\\
w|_{x_j=L} &= w|_{x_j=-L} \quad \text{for all } j = 1, \ldots, d-1 && (3.59)\\
w|_{t=0} &= w_0. && (3.60)
\end{aligned}$$

for all $\varphi \in C^1([0,T]; H^1_{per}(\Omega)^d)$ with $\varphi|_{t=T} = 0$.

Lemma 3.10 *Assume that u_h satisfies (3.31) with $R \in (0, R_0]$ and some given $0 < h \leq 1$, $n \in \mathbb{N} \cup \{\infty\}$ and let $R_0 \in (0,1]$ be so small that* (3.35) *and* (3.34) *hold. Let $w \in H^1(Q_T)^d$ be a solution of (3.58)-(3.60) for some $f \in L^1(0,T; L^2(\Omega)^{d\times d})$, $w_0 \in L^2(\Omega)^d$, and $w_1 \in H^1_{per}(\Omega)^d$ and let $u(t) = \int_0^t w(\tau)\, d\tau$. Then there are some $C_0, C > 0$ independent of w and $0 < T < \infty$ such that*

$$\begin{aligned}
&\left\| \left(w, \frac{1}{h}\varepsilon_h(u) \right) \right\|_{L^\infty(0,T;L^2)} \\
&\leq C_0 e^{CRT} \left(\|f\|_{L^1(0,T;L^2_h)} + \|w_0\|_{L^2} + \|w_1\|_{H_h^{-1}(\Omega)} \right). \qquad (3.61)
\end{aligned}$$

Proof: Let $0 \leq T' \leq T$ and define $\tilde{u}_{T'}(t) = -\int_t^{T'} w(\tau)\, d\tau$. We choose $\varphi = \tilde{u}_{T'}\chi_{[0,T']}$ in (3.58) (after a standard approximation). Then

$$\begin{aligned}
&\frac{1}{2}\|w(T')\|^2_{L^2(\Omega)} + \frac{1}{2h^2}(A_n(\nabla_h u_h)\nabla_h \tilde{u}_{T'}(0), \nabla_h \tilde{u}_{T'}(0))_\Omega \\
&= -\frac{1}{2h^2}((\partial_t A_n(\nabla_h u_h))\nabla_h \tilde{u}_{T'}, \nabla_h \tilde{u}_{T'})_{Q_{T'}} - (f, \nabla_h \tilde{u}_{T'})_{Q_{T'}} \\
&\quad -\langle w_1, \tilde{u}_{T'}(0)\rangle_{H_h^{-1}, H_h^1} + \frac{1}{2}\|w_0\|^2_{L^2(\Omega)}
\end{aligned}$$

Hence (3.34), (3.35), and $\tilde{u}_{T'}(0) = -u(T')$ imply

$$\|w(T')\|^2_{L^2(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(u(T'))\right\|^2_{L^2} \leq CR\left\|\frac{1}{h}\varepsilon_h(\tilde{u}_{T'})\right\|^2_{L^2(Q_{T'})}$$
$$+C\|w_0\|^2_{L^2(\Omega)} + C\left(\|f\|_{L^1(0,T;L^2_h)} + \|w_1\|_{H_h^{-1}}\right)\left\|\frac{1}{h}\varepsilon_h(u)\right\|_{L^\infty(0,T';L^2)}$$

for all $0 \leq T' \leq T$. Since $\tilde{u}_{T'}(t) = -u(T') + u(t)$, we obtain

$$\left\|\frac{1}{h}\varepsilon_h(\tilde{u}_{T'})\right\|^2_{L^2(Q_{T'})} \leq \left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^2(Q_{T'})} + T'\left\|\frac{1}{h}\varepsilon_h(u(T'))\right\|^2_{L^2(\Omega)}.$$

Hence there is some $\kappa > 0$ independent of $R \in (0, R_0]$, $h \in (0,1]$, such that

$$\|w\|^2_{L^\infty(0,T';L^2)} + \left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^\infty(0,T';L^2)}$$
$$\leq CR\left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^2(Q_{T'})} + C_0\left(\|w_0\|^2_{L^2(\Omega)} + \|w_1\|^2_{H_h^{-1}(\Omega)} + \|f\|^2_{L^1(0,T;L^2_h)}\right)$$

provided that $RT' \leq \kappa$. By the lemma of Gronwall we obtain (3.61) for all $0 < T < \infty$ such that $RT \leq \kappa$. Now, if $0 < T < \infty$ with $RT > \kappa$, we apply the latter estimate successively for some $0 = T_0 < T_1 < \ldots < T_N = T$ such that $R(T_{j+1} - T_j) \leq \kappa$, $j = 0, \ldots, N-1$, and $N \leq 2R\kappa^{-1}T$. Hence we obtain

$$\left\|\left(w, \frac{1}{h}\varepsilon_h(u)\right)\right\|_{L^\infty(0,T;L^2)}$$
$$\leq (C_0)^N e^{CRT}\left(\|f\|_{L^1(0,T;L^2_h)} + \|w_0\|_{L^2} + \|w_1\|_{H_h^{-1}(\Omega)}\right),$$

where

$$(C_0)^N \leq \exp\left(2\kappa^{-1}\ln C_0 RT\right) \leq \exp(C_0' RT)$$

since $N \leq 2R\kappa^{-1}T$. This implies (3.61) for some modified C_0, C independent of $R \in (0, R_0]$, $h \in (0,1]$, $0 < T < \infty$. ∎

3.4 Local in Time Existence

For the following we assume that $\theta \geq 0$, $0 < T \leq 1$, and $(u_{2,h}, u_{3,h}), f_h$ are as in Theorem 3.3 and set $w_1 = u_{2,h}, w_2 = u_{3,h}$. Moreover, we assume that $R_0 \in (0,1]$ is so small that all the statements in Section 3.3 are applicable. – Note that $T \leq 1$ is not a restriction for the proof of Theorem 3.3 and Theorem 3.1. By a simple scaling with T^{-1} in time t and h we can always reduce to this case changing $M > 0$ by a certain factor depending on T if necessary. (Of course this finally influence the smallness assumption of $h_0 > 0$ in the case $\theta > 0$ and the starting smallness assumption on M if $\theta = 0$.)

Moreover, let $C_L \geq 1$ be the constant in Lemma 3.9 and let $C_0 \geq 1$ be as in (3.22). Then (3.6)-(3.7), (3.22) imply

$$\begin{aligned}\left\|h^{1+\theta}\partial_t f_h\right\|_{W_1^1([0,T];H^{1,0})} + \max_{k=0,1}\left\|\left(\frac{1}{h}\varepsilon_h(u^n_{k,h}), \nabla\frac{1}{h}\varepsilon_h(u^n_{k,h}), \nabla_h^2 u^n_{k,h}\right)\right\|_{H^{2-k,0}(\Omega)} \\ + \max_{|\gamma|\leq 1}\left\|\left(\frac{1}{h}\varepsilon_h(w_1), w_2, h^{1+\theta}\partial_z^\gamma f_h|_{t=0}\right)\right\|_{H^{1,0}(\Omega)} \leq \tilde{M}h^{1+\theta} \qquad (3.62)\end{aligned}$$

where $\tilde{M} = (2 + C_0)M$. If $\theta > 0$, we can find some $h_0 \in (0, 1]$ (depending on M) such that $R := 6C_L\tilde{M}h_0^\theta \leq R_0$. If $\theta = 0$, we assume that $M > 0$ is so small that $R := 6C_L\tilde{M} \leq R_0$. In this case we set $h_0 = 1$.

Under the latter assumptions we will prove:

Theorem 3.11 *For every $n \in \mathbb{N}$ and $0 < h \leq h_0$ there is some $T_0 > 0$ and a unique $u_h^n \in C^3([0,T']; H^{1,0}) \cap C([0,T]; H^{2,2} \cap H^{4,0})$ solving (3.15)-(3.18) on $(0, T')$ with $T' = \min(T, T_0)$, satisfying*

$$\max_{|\gamma|\leq 1}\left\|\left(\partial_t^2\partial_z^\gamma u_h^n, \tfrac{1}{h}\varepsilon_h(\partial_z^\gamma u_h^n), \nabla_{x,t}\partial_z^\gamma\tfrac{1}{h}\varepsilon_h(u_h^n)\right)\right\|_{C([0,T'];H^{1,0})} \leq 3C_L\tilde{M}h^{1+\theta} \qquad (3.63)$$

uniformly in $0 < h \leq h_0$. Here $T_0 > 0$ depends only on M, n.

We solve (3.15)-(3.18) by solving (3.24)-(3.27). To this end, let $w_n^0 = 0$ and let w_n^{k+1}, $k \in \mathbb{N}_0$, be defined recursively by

$$\partial_t^2 w_n^{k+1} - \frac{1}{h^2}\operatorname{div}_h\left(A_n(\nabla_h u_n^k)\nabla_h w_n^{k+1}\right) = \partial_t f_h h^{1+\theta} \qquad \text{in } \Omega\times(0,T'), \qquad (3.64)$$

$$A_n(\nabla_h u_n^k)\nabla_h w_n^{k+1} e_d\big|_{x_d=\pm\frac{1}{2}} = 0, \qquad (3.65)$$

$$w_n^{k+1}\big|_{x_j=-L} = w_n^{k+1}\big|_{x_j=L},\ j = 1,\ldots,d-1, \qquad (3.66)$$

$$(w_n^{k+1}, \partial_t w_n^{k+1})\big|_{t=0} = (w_{0,n}, w_1), \qquad (3.67)$$

where $w_{0,n} = u_{1,h}^n$,

$$u_n^k(x,t) = u_{0,h}^n(x) + \int_0^t w_n^k(x,\tau)\,d\tau \qquad \text{for all } k \in \mathbb{N}_0, \qquad (3.68)$$

and $u_{0,h}^n, u_{1,h}^n$ solve (3.19),(3.20), respectively, cf. Remark 3.2. The existence of a unique solution w_h^{k+1} follows from Lemma 3.9.

As usual for short time existence of hyperbolic equations, we first show boundedness of $(u_n^k)_{k\in\mathbb{N}}$ in some suitable "high norms" and then convergence of $(u_n^k)_{k\in\mathbb{N}}$ in some "low norms" provided that $T_0 = T_0(n) > 0$ is sufficiently small and $n \in \mathbb{N}$ is fixed.

In order to get "Boundedness in High Norms", we show that $(w_n^k)_{k\in\mathbb{N}}$ satisfies for sufficiently small $T' = T'(n) \in (0,T]$

$$\left\|\left(\partial_t^2 w_n^k, \frac{1}{h}\varepsilon_h(w_n^k), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w_n^k)\right)\right\|_{C([0,T'];H^{1,0})} \leq 2C_L\tilde{M}h^{1+\theta} \tag{3.69}$$

uniformly in $0 < h \leq h_0$, where C_L is the constant in Lemma 3.9. To this end we use:

Lemma 3.12 *There is some $0 < T'(n) \leq \min(1,T)$ depending only on M, n and $\min(1,T)$ such that, if u_n^k, w_n^k satisfy (3.68) and*

$$\left\|\left(\partial_t^2 w_n^k, \frac{1}{h}\varepsilon_h(w_n^k), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w_n^k)\right)\right\|_{C([0,T'];H^{1,0})} \leq 4C_L\tilde{M}h^{1+\theta} \tag{3.70}$$

then the solution w_n^{k+1} of (3.64)-(3.67) satisfies

$$\left\|\left(\partial_t^2 w_n^{k+1}, \frac{1}{h}\varepsilon_h(w_n^{k+1}), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w_n^{k+1})\right)\right\|_{C([0,T'];H^{1,0})} \leq 2C_L\tilde{M}h^{1+\theta}$$

and u_n^k satisfies

$$\max_{|\gamma|\leq 1}\left\|\left(\partial_z^\gamma P_n\nabla_h u_n^k, \partial_z^\gamma P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{L^\infty(0,T';V)} \tag{3.71}$$
$$+\left\|\left(\partial_t^2 P_n\nabla_h u_n^k, \partial_t^2 P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{L^\infty(0,T';H^{1,0})} \leq 6C_1C_L\tilde{M}h^{1+\theta} \leq C_1R_0h$$

uniformly in $0 < h \leq h_0$,

Proof: First of all, if (3.70) holds for some $k \in \mathbb{N}$ and some $T' > 0$, then $P_n\nabla_h u_n^k$ satisfies

$$\left\|\left(\partial_t P_n\nabla_h u_n^k, \partial_t P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{C([0,T'];V)}$$
$$+ \quad \left\|\left(\partial_t^2 P_n\nabla_h u_n^k, \partial_t^2 P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{C([0,T'];H^{1,0})} \leq 4C_1C_L\tilde{M}h^{1+\theta}$$

uniformly in $0 < h \leq h_0$ where $C_1 \geq 1$ can be chosen as the same constant as in (3.31). Moreover, there is some $C_n > 0$ depending only on $n \in \mathbb{N}$ such that $\|P_n^h f\|_{H^2(\Omega)} \leq C_n\|f\|_{L^2(\Omega)}$ uniformly in $0 < h \leq 1$, cf. (3.12). Now, using

$$\nabla_h u_n^k = \nabla_h u_{0,h}^n + \int_0^t \nabla_h w_n^k(\tau)\,d\tau, \tag{3.72}$$

(3.22), (3.70), and the previous estimate, we conclude

$$\max_{|\gamma|\le 1}\left\|\left(\partial_z^\gamma P_n\nabla_h u_n^k, \partial_z^\gamma P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{C([0,T'];V)} \tag{3.73}$$

$$+\left\|\left(\partial_t^2 P_n\nabla_h u_n^k, \partial_t^2 P_n\frac{1}{h}\varepsilon_h(u_n^k)\right)\right\|_{C([0,T'];H^{1,0})} \le (4C_1C_L + 1 + C_n'T')\tilde{M}h^{1+\theta}$$

uniformly in $0 < h \le h_0$, $0 < T' \le T$, $j = 1, \ldots, d-1$ for some C_n' depending on $n \in \mathbb{N}$. Hence, if $T' = T'(n) > 0$ is so small that $1 + C_n'T' \le 2C_1C_L$, we obtain (3.71) uniformly in $0 < h \le h_0$, where $6C_L\tilde{M}h_0^\theta \le R_0$ by the choice of h_0, M.

Because of (3.71), $P_n\nabla_h u_n^k$ satisfies (3.31) for T replaced by $T' = T'(n)$ and given $n \in \mathbb{N}$. Hence we can apply Lemma 3.9 to conclude

$$\begin{aligned}
&\left\|\left(\partial_t^2 w_n^{k+1}, \frac{1}{h}\varepsilon_h(w_n^{k+1}), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w_n^{k+1})\right)\right\|_{C([0,T'],H^{1,0})}\\
&\le\; C_L e^{6C'Mh_0^\theta T'}\Big(h^{1+\theta}\|f_h\|_{W_1^2(0,T';H^{1,0})}\\
&\qquad+\left\|\left(\frac{1}{h}\varepsilon_h(w_1), w_2, h^{1+\theta}\partial_t f_h|_{t=0}\right)\right\|_{H^{1,0}}\Big)\\
&\le\; C_L e^{C'R_0T'}\tilde{M}h^{1+\theta}
\end{aligned} \tag{3.74}$$

for all $T' > 0$ such that $1+C_n'T' \le 2C_1C_L$. Hence, if $T'(n) > 0$ is chosen sufficiently small, we obtain $C_Le^{C'R_0T'} \le 2C_L$. This shows the estimate for w_n^{k+1}. ■

Proof of Theorem 3.11: Let w_k^n, u_k^n be defined as above. Because of the latter lemma (3.69) holds for all $k \in \mathbb{N}$ provided that $T' = T'(n) > 0$ is chosen as in the lemma.

In order to show "contraction in low norms", let $z_n^{k+1} := w_n^{k+1} - w_n^k$ and $Z_n^{k+1} := u_n^{k+1} - u_n^k$. Because of (3.64)-(3.65), z_n^{k+1} solves

$$\begin{aligned}
-(\partial_t z_n^{k+1}, \partial_t\varphi)_{Q_{T'}} + \frac{1}{h^2}(A_n(\nabla_h u_n^k)\nabla_h z_n^{k+1}, \nabla_h\varphi)_{Q_{T'}}&\\
= -\frac{1}{h^2}(A_n(\nabla_h u_n^k) - A_n(\nabla_h u_n^{k-1}))\nabla_h w_n^k, \nabla_h\varphi)_{Q_{T'}}&
\end{aligned}$$

for all $\varphi \in C^1([0,T']; L^2(\Omega))^d \cap C^0([0,T']; H^1(\Omega))^d$ with $\varphi|_{t=T} = 0$ and $(z_h^{k+1}, \partial_t z_h^{k+1})\big|_{t=0} = (0,0)$. Here

$$\begin{aligned}
&\left|\frac{1}{h^2}((A_n(\nabla_h u_n^k) - A_n(\nabla_h u_n^{k-1}))\nabla_h w_n^k, \nabla_h\varphi)_{Q_{T'}}\right|\\
&\le\; \frac{CT'}{h}\left\|\left(P_n\nabla_h w_n^k, P_n\frac{1}{h}\varepsilon_h(w_n^k)\right)\right\|_{L^\infty(0,T';V)}\left\|\frac{1}{h}\varepsilon_h(Z_n^k)\right\|_{L^\infty(0,T';L^2)}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^\infty(0,T';L^2)}\\
&\le\; C'T'\tilde{M}h_0^\theta\left\|\frac{1}{h}\varepsilon_h(Z_n^k)\right\|_{L^\infty(0,T';L^2)}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^\infty(0,T';L^2)}
\end{aligned}$$

because of Corollary 2.7, (3.70), and

$$\frac{1}{h^2}((A_n(\nabla_h u_n^k) - A_n(\nabla_h u_n^{k-1}))\nabla_h w_n^k, \nabla_h \varphi)_\Omega$$
$$= \int_0^1 \left(D^3\widetilde{W}((1-\tau)\nabla_h u_n^{k-1} + \tau\nabla_h u_n^k)[P_n \nabla_h Z_n^k, P_n \nabla_h w_n^k], P_n \nabla_h \varphi \right)_\Omega d\tau.$$

Hence

$$\left\| \left(\partial_t Z_n^{k+1}, \frac{1}{h}\varepsilon_h(Z_n^{k+1}) \right) \right\|_{C([0,T'];L^2)} \leq Ce^{C'\tilde{M}h_0^\theta T'} T' \left\| \frac{1}{h}\varepsilon_h(Z_n^k) \right\|_{C([0,T'];L^2)}$$

due to Lemma 3.10. Therefore there is some $T_0' > 0$ depending only on $\tilde{M}$ such that

$$\left\| \left(\partial_t Z_n^{k+1}, \frac{1}{h}\varepsilon_h(Z_n^{k+1}) \right) \right\|_{L^\infty(0,T';L^2)} \leq \frac{1}{2} \left\| \left(\partial_t Z_n^k, \frac{1}{h}\varepsilon_h(Z_n^k) \right) \right\|_{L^\infty(0,T';L^2)}$$

provided that $0 < T'(n) \leq T_0'$. This shows that $(u_n^k)_{k\in\mathbb{N}}$ converges to some $u_n \in C([0,T'];H^1(\Omega))$ as $k \to \infty$ and $\partial_t u_n^k \to \partial_t u_n$ in $C([0,T'];L^2(\Omega))$ for some sufficiently small $T'(n)$. Because of (3.69) and (3.71), u_n and $w_n := \partial_t u_n$ satisfy the same estimates as u_k^n, w_k^n, respectively. By interpolation $\nabla_h u_n^k \to_{k\to\infty} \nabla_h u_n$ in $L^\infty(Q_{T'})$ strongly. Therefore u_n and w_n solve (3.24)-(3.27). Consequently u_n solves (3.15)-(3.18).

Finally, we have to prove (3.63). We know that u_n constructed above on $(0,T'(n))$ satisfies

$$\left\| \left(\partial_t^3 u_n, \partial_t \frac{1}{h}\varepsilon_h(u_n), \partial_t \nabla_{x,t} \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq 2C_L \tilde{M} h^{1+\theta}$$

due to (3.69). Moreover, using (3.72) and (3.62), we conclude

$$\max_{j=0,1} \left\| \left(\partial_t^{2+j} u_n, \partial_t^j \frac{1}{h}\varepsilon_h(u_n), \partial_t^j \nabla_{x,t} \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq 3C_L \tilde{M} h^{1+\theta}. \quad (3.75)$$

Hence it remains to show that we can replace one time derivative in (3.75) by ∂_{x_j}, $j = 1, \ldots, d-1$. To this end let $w_n^j = \partial_{x_j} u_n$, $j = 1, \ldots, d-1$. Then w_n^j solves

$$\partial_t^2 w_n^j - \frac{1}{h^2}\mathrm{div}_h A_n(\nabla_h u_n)\nabla_h w_n^j = \partial_{x_j} f_h h^{1+\theta} \quad \text{in } \Omega \times (0,T') \quad (3.76)$$
$$A_n(\nabla_h u_n)\nabla_h w_n^j e_d\big|_{x_d=\pm\frac{1}{2}} = 0, \quad (3.77)$$
$$w_n^j\big|_{x_j=L} = w_n^j\big|_{x_j=-L}, \quad j = 1, \ldots, d-1, \quad (3.78)$$
$$(w_n^j, \partial_t w_n^j)\big|_{t=0} = (w_{0,h}^j, w_{1,h}^j) \quad (3.79)$$

with $w_{k,h}^j = \partial_{x_j} u_{k,h}^n$, $k = 0,1$. Hence Lemma 3.12 with $w_n^k = w_n, u_n^k = u_n$, $w_n^{k+1} = w_n^j$, and $\partial_t f$ replaced by $\partial_{x_j} f$ implies the estimates of the remaining terms in (3.63). ■

3.5 Uniform bounds and Proof of Theorem 3.1

We know that u_n constructed above on $(0, T'(n))$ satisfies

$$\max_{|\gamma|\leq 1} \left\| \left(\partial_t^2 \partial_z^\gamma u_n, \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n), \partial_z^\gamma \nabla_{x,t} \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq 3C_L \tilde{M} h^{1+\theta} \quad (3.80)$$

where $\tilde{M} = C_0 M + 2M$ due to Theorem 3.11. In the case $\theta > 0$, it remains to show that we can replace $T'(n)$ by an arbitrary $0 < T \leq 1$ if $0 < h \leq h_0$ for some sufficiently small $0 < h_0 \leq 1$ independent of $n \in \mathbb{N}$. If $\theta = 0$, we will show that $T'(n)$ can be replaced by $0 < T \leq 1$ if M is sufficiently small (independent of $n \in \mathbb{N}_0$).

To this end we apply Lemma 3.9 for $w_n^j = \partial_{x_j} u_n$ and $w_n = \partial_t u_n$ using the equations for w_n^j, w_n and that (3.30) is valid for $u = u_n$ and $R = 6C_L \widetilde{M} h^\theta \leq R_0$, which implies (3.31) uniformly in $n \in \mathbb{N}_0 \cup \{\infty\}$. Thus we obtain

$$\max_{|\gamma|=1} \left\| \left(\partial_t^2 \partial_z^\gamma u_n, \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n), \nabla_{x,t} \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq C_L e^{C' \tilde{M} h^\theta} \tilde{M} h^{1+\theta}$$

uniformly in $n \in \mathbb{N}$ and $0 < h \leq h_0$. Due to (3.22) and (3.72), we conclude

$$\max_{|\gamma|\leq 1} \left\| \left(\partial_t^2 \partial_z^\gamma u_n, \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n), \nabla_{x,t} \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq 2C_L e^{C' \tilde{M} h^\theta} \tilde{M} h^{1+\theta}$$

If $\theta > 0$, we can now choose $0 < h_0 \leq 1$ so small that

$$\max_{|\gamma|\leq 1} \left\| \left(\partial_t^2 \partial_z^\gamma u_n, \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n), \nabla_{x,t} \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,T'];H^{1,0})} \leq 3C_L \tilde{M} h^{1+\theta}$$

uniformly in $n \in \mathbb{N}$ and $0 < h \leq h_0$. If $\theta = 0$, then we choose $\tilde{M} = (2 + C_0)M$ sufficiently small to obtain the same estimates. Hence we get a uniform bound as long as the solutions exists. Applying the results of Section 3.4 with the only difference that the bound M of the norm of the initial values is replaced by $3C_L \tilde{M}$ and a shift in time by $T'(n)$, we can continue the solution u_h on $[0, \widetilde{T}]$, $\widetilde{T} = \min(T'(n) + T'', T)$, where $T'' > 0$ depends only on n and $3C_L \tilde{M}$. This extended solution can be constructed such that (3.70) is valid. But the arguments of this section show that then even

$$\max_{|\gamma|\leq 1} \left\| \left(\partial_t^2 \partial_z^\gamma u_n, \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n), \nabla_{x,t} \partial_z^\gamma \frac{1}{h}\varepsilon_h(u_n) \right) \right\|_{C([0,\widetilde{T}];H^{1,0})} \leq 3C_L \tilde{M} h^{1+\theta}.$$

Therefore we can repeat this argument finitely many times and can continue the solution for all $t \in [0, T]$.

Using the uniform bounds, it is easy to pass to the limit $n \to \infty$ for a suitable subsequence. Uniqueness of the solution can be shown by similar estimates as in the proof of Theorem 3.11. This finishes the proof of Theorem 3.1.

4 First Order Asymptotics

Throughout this section we assume that

$$f_h(x,t) = \begin{pmatrix} 0 \\ g(x',t) \end{pmatrix},$$

for some given $g \in W^2_1(0,T;H^3_{per}((-L,L)^{d-1})) \cap W^1_1(0,T;H^5_{per}((-L,L)^{d-1}))$. As seen in the proof of Lemma 3.5, we can assume without loss of generality that $\int_{(-L,L)^{d-1}} g(x')dx' = 0$. Moreover, we assume that $0 < \theta \leq 1$.

For simplicity let $W(F) = \operatorname{dist}(F, SO(d))^2$, which implies $D^2W(0)F = \operatorname{sym} F$. Let $L_2 v = \operatorname{div}'\varepsilon'(v)$ and let v be the solution of the $(d-1)$-dimensional wave equation

$$\begin{aligned} \partial_t^2 v + \frac{1}{12}\Delta^2_{x'} v &= g && \text{in } (-L,L)^{d-1} \times (0,T), \\ v|_{x_j=-L} &= v|_{x_j=L} && \text{for } j = 1,\ldots,d-1, \\ (v, \partial_t v)|_{t=0} &= (v_0, v_1) && \text{in } (-L,L)^{d-1}, \end{aligned}$$

where $v_0 \in H^7_{per}((-L,L)^{d-1}), v_1 \in H^5_{per}((-L,L)^{d-1})$. By standard methods the latter system possesses a unique solution

$$\begin{aligned} v \in\, & C^2([0,T];H^5_{per}((-L,L)^{d-1})) \cap C^1([0,T];H^7_{per}((-L,L)^{d-1})) \\ & \cap C^0([0,T];H^9_{per}((-L,L)^{d-1})). \end{aligned}$$

Using v, we define an approximate solution $\tilde{u}^h$ of (3.1)-(3.4) by

$$\begin{aligned} \tilde{u}^h(x,t) \;=\; & h^\theta \begin{pmatrix} 0 \\ hv \end{pmatrix} + h^{2+\theta} \begin{pmatrix} -x_d \nabla_{x'} v \\ 0 \end{pmatrix} + h^{4+\theta} \begin{pmatrix} (\frac{1}{3}x_d^3 - \frac{1}{4}x_d)\nabla_{x'}\Delta_{x'} v \\ 0 \end{pmatrix} \\ & + h^{5+\theta} \begin{pmatrix} 0 \\ (\frac{1}{48}x_d^2 - \frac{1}{24}x_d^4 - \frac{1}{24\cdot 16})\Delta^2_{x'} v \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \nabla_h \tilde{u}^h \;=\; & h^{1+\theta} \begin{pmatrix} 0 & -\nabla_{x'} v \\ \nabla_{x'} v^T & 0 \end{pmatrix} + h^{2+\theta} \begin{pmatrix} -x_d \nabla^2_{x'} v & h(x_d^2 - \frac{1}{4})\nabla_{x'}\Delta_{x'} v \\ 0 & 0 \end{pmatrix} \\ & + h^{4+\theta} \begin{pmatrix} (\frac{1}{3}x_d^3 - \frac{1}{4}x_d)\nabla^2_{x'}\Delta_{x'} v & 0 \\ 0 & (\frac{1}{24}x_d - \frac{1}{6}x_d^3)\Delta^2_{x'} v \end{pmatrix} \\ & + h^{5+\theta} \begin{pmatrix} 0 & 0 \\ (\frac{1}{48}x_d^2 - \frac{1}{24}x_d^4 - \frac{1}{24\cdot 16})\nabla^T_{x'}\Delta^2_{x'} v & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon_h(\tilde{u}^h) = h^{2+\theta} & \begin{pmatrix} -x_d \nabla^2_{x'} v & \frac{h}{2}(x_d^2 - \frac{1}{4})\nabla_{x'}\Delta_{x'} v \\ \frac{h}{2}(x_d^2 - \frac{1}{4})\nabla^T_{x'}\Delta_{x'} v & h^2(\frac{1}{24}x_d - \frac{1}{6}x_d^3)\Delta^2_{x'} v \end{pmatrix} \\ + h^{4+\theta} & \begin{pmatrix} (\frac{1}{3}x_d^3 - \frac{1}{4}x_d)\nabla^2_{x'}\Delta_{x'} v & \frac{h}{2}(\frac{1}{48}x_d^2 - \frac{1}{24}x_d^4 - \frac{1}{24\cdot 16})\nabla_{x'}\Delta^2_{x'} v \\ \frac{h}{2}(\frac{1}{48}x_d^2 - \frac{1}{24}x_d^4 - \frac{1}{24\cdot 16})\nabla^T_{x'}\Delta^2_{x'} v & 0 \end{pmatrix} \end{aligned}$$

and therefore

$$\frac{1}{h}\varepsilon_h(\tilde{u}^h)e_d|_{x_d=\pm\frac{1}{2}} = 0. \tag{4.1}$$

Moreover,

$$\begin{aligned}\frac{1}{h^2}\mathrm{div}_h\varepsilon_h(\tilde{u}^h) &= h^\theta\begin{pmatrix}-x_d\nabla_{x'}\Delta_{x'}v + x_d\nabla_{x'}\Delta_{x'}v\\ \frac{h}{2}(x_d^2-\frac{1}{4})\Delta_{x'}^2v + h(\frac{1}{24}-\frac{1}{2}x_d^2)\Delta_{x'}^2v\end{pmatrix}\\ &\quad +h^{2+\theta}\begin{pmatrix}(\frac{1}{3}x_d^3-\frac{1}{4}x_d)\nabla_{x'}\Delta_{x'}^2v+\frac{1}{2}(\frac{1}{24}x_d-\frac{1}{6}x_d^3)\nabla_{x'}\Delta_{x'}^2v\\ \frac{h}{2}(\frac{1}{48}x_d^2-\frac{1}{24}x_d^4-\frac{1}{24\cdot 16})\Delta_{x'}^3v\end{pmatrix}\\ &\equiv h^{1+\theta}\begin{pmatrix}0\\ -\frac{1}{12}\Delta_{x'}^2v\end{pmatrix}+r_h,\end{aligned}$$

where

$$\|r_h\|_{C([0,T];L^2(\Omega))} \leq Ch^{2+\theta}. \tag{4.2}$$

Thus $\tilde{u}_h$ is a solution of

$$\begin{aligned}\partial_t^2\tilde{u}_h - \frac{1}{h^2}\mathrm{div}_h\left(D^2\widetilde{W}(0)\nabla_h\tilde{u}_h\right) &= f_hh^{1+\theta}-r_h && \text{in } \Omega\times(0,T) \qquad (4.3)\\ \left(D^2\widetilde{W}(0)\nabla_h\tilde{u}_h\right)e_d\Big|_{x_d=\pm\frac{1}{2}} &= 0 \\ \tilde{u}_h|_{x_j=L} &= \tilde{u}_h|_{x_j=-L}, && j=1,\ldots,d-1,\\ (\tilde{u}_h,\partial_t\tilde{u}_h)|_{t=0} &= (\tilde{u}_{0,h},\tilde{u}_{1,h}),\end{aligned}$$

where

$$\begin{aligned}\tilde{u}_{j,h}(x) &= h^{1+\theta}\begin{pmatrix}0\\ hv_j\end{pmatrix}+h^{2+\theta}\begin{pmatrix}-x_d\nabla_{x'}v_j\\ 0\end{pmatrix}+h^{4+\theta}\begin{pmatrix}(\frac{1}{3}x_d^3-\frac{1}{4}x_d)\nabla_{x'}\Delta_{x'}v_j\\ 0\end{pmatrix}\\ &\quad +h^{5+\theta}\begin{pmatrix}0\\ (\frac{1}{48}x_d^2-\frac{1}{24}x_d^4-\frac{1}{24\cdot16})\Delta_{x'}^2v_j\end{pmatrix}, \quad j=0,1.\end{aligned}$$

We will compare this approximate solution with an exact solution of the d-dimensional system (3.1)-(3.4) for an appropriate choice of initial values.

Theorem 4.1 *Let $0<\theta\leq 1$, let v_0, v_1, f, $\tilde{u}_{0,h},\tilde{u}_{1,h}$ and $\tilde{u}_h$ be defined as above. Then for some sufficiently small $h_0\in(0,1]$ there are initial values $(u_{0,h},u_{1,h})$ satisfying (3.6) and such that*

$$\max_{j=0,1}\left\|\frac{1}{h}\varepsilon_h(u_{j,h})-\frac{1}{h}\varepsilon_h(\tilde{u}_{j,h})\right\|_{L^2(\Omega)} \leq Ch^{1+2\theta}.$$

Moreover, if u_h is the solution of (3.1)-(3.4) due to Theorem 3.1, then

$$\left\|\left(\partial_t(u_h-\tilde{u}_h),\frac{1}{h}\varepsilon_h(u_h-\tilde{u}_h)\right)\right\|_{L^\infty(0,T;L^2)} \leq Ch^{1+2\theta}$$

for all $0<h\leq h_0$ and some $C>0$ independent of h.

Proof: We construct the initial values $u_{0,h}, u_{1,h}$ as solution of

$$\begin{aligned}
\frac{1}{h^2}(DW(\nabla_h u_{0,h}), \nabla_h \varphi)_\Omega &= (u_{2,h}, \varphi)_\Omega - (h^{1+\theta} f|_{t=0}, \varphi)_\Omega, \\
\frac{1}{h^2}(D^2W(\nabla_h u_{0,h})\nabla_h u_{1,h}, \nabla_h \varphi)_\Omega &= (u_{3,h}, \varphi)_\Omega - (h^{1+\theta}\partial_t f|_{t=0}, \varphi)_\Omega
\end{aligned}$$

for all $\varphi \in H^1_{per}(\Omega)^d$, where

$$u_{2+j,h} = h^{1+\theta}\begin{pmatrix} 0 \\ -\frac{h}{12}\Delta^2_{x'}\partial_t^j v|_{t=0}\end{pmatrix} + h^{2+\theta}\begin{pmatrix} -\frac{x_d}{12}\nabla_{x'}\Delta^2_{x'}\partial_t^j v|_{t=0} \\ 0\end{pmatrix}, \quad j = 0, 1.$$

We note that $\int_\Omega u_{2+j,h}\, dx = 0$ and

$$\frac{1}{h}\varepsilon_h(u_{2+j,h}) = h^{1+\theta}\begin{pmatrix} -x_d\nabla^2_{x'}\Delta^2_{x'}v_j & 0 \\ 0 & 0\end{pmatrix}$$

by a similar calculation as above. In particular, this implies

$$\left\|\left(\frac{1}{h}\varepsilon_h(u_{2+j,h}), \frac{1}{h}\varepsilon_h(\partial_t^j f|_{t=0})\right)\right\|_{H^{2-j,0}} \leq Ch^{1+\theta}, \qquad j = 0, 1,$$

where we note that f is independent of x_d. Because of Proposition 3.8, the Lemma of Lax-Milgram, and Lemma 3.5, $(u_{0,h}, u_{1,h})$ exist for all $0 < h \leq h_0$ if $h_0 \in (0, 1]$ is sufficiently small and satisfy

$$\left\|\left(\frac{1}{h}\varepsilon_h(u_{1,h}), \frac{1}{h}\varepsilon_h(u_{0,h}), \nabla\frac{1}{h}\varepsilon_h(u_{0,h}), \nabla_h^2 u_{0,h}\right)\right\|_{H^{2,0}(\Omega)} \leq Ch^{1+\theta}.$$

Altogether, $(u_{0,h}, u_{1,h}, u_{2,h}, u_{3,h})$ satisfy (3.6) and (3.8)-(3.9). Moreover, we note that

$$\max_{j=0,1}\left\|\frac{1}{h}\varepsilon_h(u_{j,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{j,h})\right\|_{L^2(\Omega)} \leq Ch^{1+2\theta}$$

since

$$\begin{aligned}
\frac{1}{h^2}(\varepsilon_h(u_{0,h} - \tilde{u}_{0,h}), \varepsilon_h(\varphi))_\Omega &= \frac{1}{h^2}(F'(\nabla_h u_{0,h}), \nabla_h\varphi)_\Omega + (r_{0,h}, \varphi)_\Omega, \\
\frac{1}{h^2}(\varepsilon_h(u_{1,h} - \tilde{u}_{1,h}), \varepsilon_h(\varphi))_\Omega &= \frac{1}{h^2}((DW^2(\nabla_h u_{0,h}) - DW^2(0))\nabla_h u_{1,h}, \nabla_h\varphi)_\Omega \\
&\quad + (r_{1,h}, \varphi)_\Omega
\end{aligned}$$

due to (4.3), where F' is as in (3.13) and $\|(r_{0,h}, r_{1,h})\|_{L^2(\Omega)} \leq Ch^{2+\theta}$. Here we have used that

$$\left|\frac{1}{h^2}\left(F'(\nabla_h u_{0,h}), \nabla_h\varphi\right)_\Omega\right| \leq Ch^{1+2\theta}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}$$

$$\left|\frac{1}{h^2}\left((DW^2(\nabla_h u_{0,h}) - DW^2(0))\nabla_h u_{1,h}, \nabla_h\varphi\right)_\Omega\right| \leq Ch^{1+2\theta}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}$$

for all $\varphi \in H^1_{per}(\Omega)^d$ because of (3.33) and the estimates for $u_{0,h}, u_{1,h}$.

Now let u_h be the solution of (3.1)-(3.4) due to Theorem 3.1 and consider $w_h = \partial_t u_h - \partial_t \tilde{u}_h$. Then w_h solves

$$\begin{aligned}
-(\partial_t w_h, \partial_t \varphi)_{Q_T} &+ \frac{1}{h^2}(DW(\nabla_h u_h)\nabla_h w_h, \nabla_h \varphi)_{Q_T} - (w_{1,h}, \varphi|_{t=0})_\Omega \\
&= -\frac{1}{h^2}\left((DW(\nabla_h u_h) - D^2W(0))\nabla_h \partial_t \tilde{u}_h, \nabla_h \varphi\right)_{Q_T} - (r_h, \varphi)_{Q_T} \\
w|_{x_j=L} &= w|_{x_j=-L}, \quad j = 1, \dots, d-1, \\
w|_{t=0} &= w_{0,h}.
\end{aligned}$$

for all $\varphi \in C^1([0,T]; H^1_{per}(\Omega)^d)$ with $\varphi|_{t=T} = 0$, where $w_{j,h} = \partial_t^{1+j} u_h|_{t=0} - \partial_t^{1+j} \tilde{u}_h|_{t=0}$, $j = 0, 1$ and r_h satisfies (4.2). Moreover,

$$\begin{aligned}
&\left|\frac{1}{h^2}\left((DW(\nabla_h u_h) - D^2W(0))\nabla_h \partial_t \tilde{u}_h, \nabla_h \varphi\right)_\Omega\right| \\
&\leq \frac{C}{h}\left\|\frac{1}{h}\varepsilon_h(u_h)\right\|_{L^\infty(0,T;L^2)}\left\|\frac{1}{h}\varepsilon_h(\partial_t \tilde{u}_h)\right\|_{L^\infty(0,T;V)}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)} \\
&\leq Ch^{1+2\theta}\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}
\end{aligned}$$

due to (3.33). Hence Lemma 3.10 implies

$$\left\|\left(\partial_t(u_h - \tilde{u}_h), \frac{1}{h}\varepsilon_h(u_h - \tilde{u}_h)\right)\right\|_{L^\infty(0,T;L^2)} \leq Ch^{1+2\theta},$$

which proves the theorem. ■

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