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Ideles in higher dimension

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IDELES IN HIGHER DIMENSION

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Abstract. We propose a notion of idele class groups of finitely generated fields using the concept of Parshin chains. This new class group allows us to give an idelic interpretation of the higher class field theory of Kato and Saito.

1. Historical Background and Introduction

Let F be a one-dimensional global field, $X = Spec(\mathcal{O}_F)$. The classical henselian finite idele group of F is

$$\mathrm{I}_c(F) = \varinjlim_{S \subset |X|} \left(\prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times \right), \tag{1}$$

where the direct limit is over all finite subsets $S \subset |X|$ and F_v is the henselization of F at v. Here $|X|$ denotes the set of closed points of the scheme X. The corresponding classical class group is defined to be

$$\mathrm{C}_c(F) = \mathrm{I}(F)/F^\times. \tag{2}$$

The properties of these topological group, or more precisely their complete analogs, were first studied by Chevalley and Weil in the 1930s.

The first one to realize that in order to generalize these idelic groups to higher dimensions one has to use higher Milnor K-theory [14] was Parshin [15]. He was able to write down an idele class group for two-dimensional schemes using a concept called Parshin chains today. Parshin chains are higher dimensional analogs of the places of a one-dimensional global field. A few years later a slightly different two-dimensional class group was studied in detail by Kato and Saito who proved the basic properties of two-dimensional class field theory; for an overview see [9]. Several people tried to generalize this idelic approach to class field theory to higher dimensions. In fact Beilinson had generalized adeles to arbitrary dimensions [1], but the theory of ideles remained mysterious.

Kato and Saito were able to define a class group for higher dimensional schemes using Nisnevich cohomology of some Milnor K-sheaf [10], but it was not clear why this group should be idelic, i.e. why it should have a simple presentation in terms of Milnor K-groups of higher local fields. In this note we propose a notion of ideles for higher dimensional schemes and we will explain why the higher Kato-Saito class group is indeed idelic.

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Further applications of our idelic methods will be given elsewhere. In particular we refer to [12] for applications to higher cohomological Hasse principles in the sense of Kato [6].

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2. Some motivation: inverse limits versus restricted products

As we recalled in Section 1 the class group of a one-dimensional global field is usually defined using restricted products. It is fundamental to our higher dimensional generalizations to define the class group in terms of inverse limits instead of restricted products. In order to motivate this approach we discuss the one-dimensional case in this section. For simplicity we restrict to non-archimedean ideles.

Let F be a one-dimensional global field, i.e. a number field or a function field in one variable over a finite field. Let X be $\mathrm{Spec}(\mathcal{O}_F)$. Then in (1) and (2) we defined in the classical way the idele group $\mathrm{I}_c(F)$ and the idele class group $\mathrm{C}_c(F)$. For us a relative version of these groups will be pivotal.

Definition 2.1. For a nonempty open subscheme $U \subset X$ the relative idele group is defined as

$$\mathrm{I}(U \subset X) = \bigoplus_{x \in |U|} \mathbb{Z} \oplus \bigoplus_{v \in X \setminus U} F_v^\times$$

with the direct sum topology. The relative idele class group is defined as

$$\mathrm{C}(U \subset X) = \mathrm{coker}[F^\times \to \mathrm{I}(U \subset X)]$$

with the quotient topology.

To formalize our approach consider the category of pairs $U \subset X$, where U is a nonempty open subscheme of a normal connected one-dimensional proper scheme X over $\mathrm{Spec}(\mathbb{Z})$. A morphism $f : (V \subset Y) \to (U \subset X)$ in this category is a dominant morphisms of schemes $f : Y \to X$ such that $f(V) \subset U$. It is easy to check that I and C form covariant functors from this category to the category of topological abelian groups.

Definition 2.2. For a global one-dimensional field F and $X = \mathrm{Spec}(\mathcal{O}_F)$ we define the idele group as the topological group

$$\mathrm{I}(F) = \varprojlim_U \mathrm{I}(U \subset X)$$

and the idele class group as the topological group

$$\mathrm{C}(F) = \varprojlim_U \mathrm{C}(U \subset X),$$

where the inverse limit is over all nonempty open subschemes $U \subset X$.

The central observation of this section is that the idele and class groups defined in Section 1 coincide with the groups defined here.

Proposition 2.3. *The natural maps*

$$\mathrm{I}_c(F) \to \mathrm{I}(F) \quad \text{and} \quad \mathrm{C}_c(F) \to \mathrm{C}(F) \tag{3}$$

are isomorphisms of topological groups.

Proof. Indeed, for arbitrary finite $S \subset |X|$ and a nonempty open subscheme $U \subset X$ there are continuous homomorphism of topological groups

$$\prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times \longrightarrow \mathrm{I}(U \subset X)$$

and

$$\left(\prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times \right) / F_S^\times \longrightarrow \mathrm{C}(U \subset X).$$

Taking the direct limit on the left side and the inverse limit on the right side we get the continuous maps in (3). It is elementary to check that the first map in (3) is an isomorphism. But as one checks that

$$0 \to F^\times \to \mathrm{I}(F) \to \mathrm{C}(F) \to 0$$

is topologically exact, the second map in (3) is also an isomorphism. □

Our aim in the following sections is to transfer this procedure to the definition of an idele class group of a finitely generated field F. Namely, we will define a relative idele class group of a proper model X of F relative to some open subscheme $U \subset X$ using only direct sums. Then we take the inverse limit over all dense open subschemes $U \subset X$ in order to define the idele class group of F.

3. Parshin chains

Let in the following X be an integral scheme proper over $\mathbb{Z}$ of dimension d. Fix an effective Weil divisor D of X. Set $U = X - D$ and $F = k(X)$. In our exposition we follow [10, Section 1.6] closely.

Definition 3.1.

- A chain on X is a sequence of points $P = (p_0, \dots, p_s)$ of X such that
$$\overline{\{p_0\}} \subset \overline{\{p_1\}} \subset \cdots \subset \overline{\{p_s\}}.$$
- A Parshin chain on X is a chain $P = (p_0, \dots, p_s)$ on X such that $\dim(p_i) = i$ for $0 \le i \le s$.
- A Parshin chain on the pair $(U \subset X)$ is a Parshin chain $P = (p_0, \dots, p_s)$ on X such that $p_i \in D$ for $0 \le i < s$ and such that $p_s \in U$.
- The dimension $\dim(P)$ of a chain $P = (p_0, \dots, p_s)$ is defined to be $\dim(p_s)$.

Using repeated henselizations one defines the finite product of henselian local rings $\mathcal{O}^h_{X,P}$ of a chain $P = (p_0, \dots, p_s)$ on X as follows: If $s = 0$ set $\mathcal{O}^h_{X,P} = \mathcal{O}^h_{X,p_0}$. If $s > 0$ assume that $\mathcal{O}^h_{X,P'}$ has already been defined for chains of the form $P' = (p_0, \dots, p_{s-1})$.

Given a chain $P=(p_0,\dots,p_s)$ let P' be the chain $(p_0,\dots,p_{s-1})$. Denote the ring $\mathcal{O}^h_{X,P'}$ by R and denote by T the finite set of primes ideals of R lying over p_s. Then we set

$$\mathcal{O}^h_{X,P}=\prod_{\mathfrak{p}\in T} R^h_{\mathfrak{p}}.$$

Finally, for a chain $P=(p_0,\dots,p_s)$ on X we let $k(P)$ be the finite product of the residue fields of $\mathcal{O}^h_{X,P}$. Observe that if P is a Parshin chain the ring $k(P)$ is a finite product of higher henselian local fields of dimension $\dim(P)$.

Definition 3.2. A lifted chain on X is a sequence $P=(p_0,\dots,p_s)$ such that $p_0\in X$ and $p_{i+1}\in \operatorname{Spec}(\mathcal{O}^h_{(p_0,\dots,p_i)})$, where the latter henselian spectrum is defined as a successive henselization as above.

4. Idele groups

In this section we define higher dimensional relative idele groups. By X we will always denote an integral scheme proper over $\mathbb{Z}$ of dimension d. Furthermore, D will be an effective Weil divisor on X and $U=X-D$.

We recall the definition of Milnor K-theory.

Definition 4.1. For a commutative unital ring R let $T(R^\times)$ be the tensor algebra over the units of R. Let $I\subset T(R^\times)$ be the two-sided ideal generated by elements $a\otimes(1-a)$ with $a,1-a\in R^\times$. Then the Milnor K-ring of R is the graded ring

$$K^M_*(R)=T(R^\times)/I.$$

The image of $a_1\otimes\cdots\otimes a_n$ in $K^M_n(R)$ is denoted by $\{a_1,\dots,a_n\}$.

Definition 4.2. If R is a discrete valuation ring with quotient field F and maximal ideal $\mathfrak{p}\subset R$ we let $K^M_n(F,m)\subset K^M_n(F)$ be the subgroup generated by $\{1+\mathfrak{p}^m,R^\times,\dots,R^\times\}$ for an integer $m\ge 0$. The topology of $K^M_n(F)$ generated by these subgroups will be called the canonical topology.

Let $\mathcal{P}$ be the set of Parshin chains on the pair $(U\subset X)$. For a Parshin chain $P=(p_0,\dots,p_d)\in\mathcal{P}$ of dimension d let $D(P)$ be multiplicity of the prime divisor $\overline{\{p_{d-1}\}}$ in D.

Definition 4.3. The idele group of the pair $(U\subset X)$ is defined as

$$\mathrm{I}(U\subset X)=\bigoplus_{P\in\mathcal{P}} K^M_{\dim(P)}(k(P)).$$

We endow this group with the topology generated by the open subgroups

$$\bigoplus_{\substack{P\in\mathcal{P}\\ \dim(P)=d}} K^M_d(k(P),D(P))\subset \mathrm{I}(U\subset X)$$

where D runs through all effective Weil divisors with support $X-U$. The idele group of X relative to the fixed effective divisor D with complement U is defined as

$$\mathrm{I}(X,D)=\operatorname{coker}[\bigoplus_{\substack{P\in\mathcal{P}\\ \dim(P)=d}} K^M_d(k(P),D(P))\to \mathrm{I}(U\subset X)].$$

The topology on $\mathrm{I}(U \subset X)$ is obviously a very strong topology but we will see in Section 6 that it induces the correct topology on the idele class group.

5. Functoriality and reciprocity

Let X and X' be integral schemes proper over $\mathbb{Z}$ of dimension d and let $U \subset X$ and $U' \subset X'$ be nonempty open subschemes. If $f : X' \to X$ is a finite morphism and if it induces a morphism of open subschemes $U' \to U$ we define a continuous homomorphism

$$f_* : \mathrm{I}(U' \subset X') \to \mathrm{I}(U \subset X)$$

by the following procedure: Let $\mathcal{P}$ resp. $\mathcal{P}'$ be the set of Parshin chains on the pair $(U \subset X)$ resp. $(U' \subset X')$. If $P = (p_0, \dots, p_s) \in \mathcal{P}$ and $P' = (p'_0, \dots, p'_{s'}) \in \mathcal{P}'$ satisfy $s \le s'$ and $f(p'_i) = p_i$ for $0 \le i \le s$ then we let

$$f_*^{P' \to P} : K^M_{s'}(k(P')) \longrightarrow K^M_s(k(P))$$

be the composition of the residue symbol

$$\partial : K^M_{s'}(k(P')) \longrightarrow K^M_s(k(P''))$$

with $P'' = (p'_0, \dots, p'_s)$ and the norm map of Milnor K-groups

$$N : K^M_s(k(P'')) \longrightarrow K^M_s(k(P)).$$

For the definition of these maps see for example [3, Chapters 7 and 8]. Otherwise we let $f_*^{P' \to P}$ be the zero map. Then f_* is defined as the sum of the maps $f_*^{P' \to P}$ for all $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$. It follows from [10, Lemma 4.3] that the map f_* is continuous. It is also easy to see that this construction of a pushforward map is functorial.

Lemma 5.1. *If $f : X' \to X$ is an isomorphism and if U is regular the pushforward map*

$$f_* : \mathrm{I}(U' \subset X') \longrightarrow \mathrm{I}(U \subset X)$$

is surjective and a topological quotient.

Proof. In fact for a Parshin chain $P = (p_0, \dots, p_s)$ on the pair $(U \subset X)$ we choose successively for $i > s$ regular points p_i of $\mathrm{Spec}(\mathcal{O}_{X,p_{i-1}})$ of dimension i in X until we get a Parshin chain $P' = (p_0, \dots, p_{s'})$ on the pair $(U' \subset X' = X)$ for some $s' \ge s$. Then the map

$$f_*^{P' \to P} : K^M_{s'}(k(P')) \longrightarrow K^M_s(k(P))$$

is surjective. □

We can now define the idele group $\mathrm{I}(F; X)$ of the finitely generated field F with respect to the integral scheme X proper over $\mathbb{Z}$ with $k(X) = F$.

Definition 5.2. We set

$$\mathrm{I}(F; X) = \varprojlim_{U \subset X} \mathrm{I}(U \subset X)$$

where the inverse limit is over all nonempty open subschemes $U \subset X$. We endow $\mathrm{I}(F; X)$ with the inverse limit topology.

Next we will consider reciprocity maps and we will assume for simplicity that $F = k(X)$ contains a totally imaginary number field if $\mathrm{char}(F) = 0$, because otherwise we had to take archimedean Parshin chains into account, a consideration we leave to the reader. It follows from the usual properties of higher local fields [10, Section 3.7.2] that there exists a continuous functorial prereciprocity homomorphism

$$r : \mathrm{I}(F; X) \longrightarrow \mathrm{Gal}(F)^{ab}$$

which has dense image according to the generalized Cebotarev density theorem [19].

6. Class groups

Let again X be an integral scheme proper over $\mathbb{Z}$ of dimension d and let $U \subset X$ be a nonempty open subscheme. Let furthermore D be an effective divisor with support $X - U$. By $\mathcal{P}$ we denoted the set of Parshin chains on the pair $(U \subset X)$.

Definition 6.1. A Q-chain on $(U \subset X)$ is defined as a chain $P = (p_0, \ldots, p_{s-2}, p_s)$ on X for $1 \le s \le d$ such that $\dim(p_i) = i$ for $i \in \{0, 1, \ldots, s-2, s\}$ and such that

- $p_i \in D$ for $0 \le i \le s-2$,
- $p_s \in U$.

We denote the set of Q-chains on $(U \subset X)$ by $\mathcal{Q}$. Then there exists a natural homomorphism

$$Q : \bigoplus_{P \in \mathcal{Q}} K^M_{\dim(P)}(k(P)) \longrightarrow \mathrm{I}(U)$$

defined as follows. If $P' = (p_0, \ldots, p_{s-2}, p_s)$ is a Q-chain and $P = (p_0, \ldots, p_{s-2}, p_{s-1}, p_s)$ is a Parshin chain on X we let $Q^{P' \to P}$ be defined as

- the map $K^M_s(k(P')) \to K^M_s(k(P))$ induced on Milnor K-groups by the ring homomorphism $k(P') \to k(P)$ if $p_{s-1} \in D$,
- the residue symbol $K^M_s(k(P')) \to K^M_{s-1}(k(P''))$ where $P'' = (p_0, \ldots, p_{s-1})$ if $p_{s-1} \in U$.

We let the map Q be the sum of all these $Q^{P' \to P}$.

Definition 6.2. The idele class group $\mathrm{C}(U \subset X)$ is defined as the cokernel of Q endowed with the quotient topology. The idele class group of X relative to the effective divisor D is defined as the analogous quotient

$$\mathrm{C}(X, D) = \mathrm{coker}[\bigoplus_{P \in \mathcal{Q}} K^M_{\dim(P)}(k(P)) \to \mathrm{I}(X, D)].$$

If $f : X' \to X$ is a finite dominant morphism and if it induces a morphism of open subschemes $U' \to U$ as in the previous section we get an induced continuous homomorphism

$$f_* : \mathrm{C}(U' \subset X') \longrightarrow \mathrm{C}(U \subset X).$$

Definition 6.3. Let the class group $\mathrm{C}(F; X)$ of F with respect to the integral scheme X proper over $\mathbb{Z}$ with $k(X) = F$ be defined as

$$\mathrm{C}(F; X) = \varprojlim_{U} C(U \subset X).$$

Observe that the continuous homomorphism $\mathrm{I}(F;X) \to \mathrm{C}(F;X)$ has dense image, but it is not clear whether it is surjective.

The next lemma is an immediate consequence of Lemma 5.1.

Lemma 6.4. *If $U' \subset U$ and U is regular the map*

$$\mathrm{C}(U' \subset X) \longrightarrow \mathrm{C}(U \subset X)$$

is surjective and a topological quotient.

As we mentioned in Section 4 the induced topology on $\mathrm{C}(F;X)$ is the correct topology. A first result which justifies this is the following. Its proof is similar to the proof of [10, Thm. 2.5].

Theorem 6.5. *Let $X' \hookrightarrow X$ be a closed integral subscheme such that $U' = U \cap X' \neq \emptyset$ and assume U is regular. Then the canonical map*

$$\mathrm{C}(U' \subset X') \longrightarrow \mathrm{C}(U \subset X)$$

is continuous.

Proof. If $\dim(X) - \dim(X') > 1$ choose a point $x \in U - U'$ such that the generic point of X' lies on the regular locus of $\overline{\{x\}}$. By descending induction on $\dim(X) - \dim(X')$ we can replace X by $\overline{\{x\}}$. In order to guarantee that $U \cap \overline{\{x\}}$ is regular we have to shrink U, which is feasible because of Lemma 6.4. So assume without restriction $\dim(X) - \dim(X') = 1$. Then it suffices to show:

Lemma 6.6. *Let A be a two-dimensional excellent normal henselian local rings with maximal ideal $\mathfrak{m}$, quotient field F and let $\mathfrak{q}, \mathfrak{p}_1, \dots, \mathfrak{p}_r$ be distinct prime ideals of codimension 1. Set $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Given integers $m_1, \dots m_r \geq 0$ there exists $m \geq 0$ such that*

$$\phi : K^M_{n-1}(k(\mathfrak{q}), m) \longrightarrow \mathrm{C}(A, \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r})$$

is the zero map, where

$$\mathrm{C}(A, \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r}) = \operatorname{coker}[K^M_n(F) \longrightarrow \bigoplus_{\mathfrak{p}_i \in S} K^M_n(F_{\mathfrak{p}_i})/K^M_n(F_{\mathfrak{p}_i}, m_i) \oplus \bigoplus_{\mathfrak{p} \notin S} K^M_{n-1}(k(\mathfrak{p}))].$$

Also we can choose m large enough so that ϕ remains the zero map if we replace A by a finite étale extension.

Proof. Choose $\pi \in A$ with $v_{\mathfrak{q}}(\pi) = 1$ and $v_{\mathfrak{p}_i}(\pi) = 0$ for $1 \leq i \leq r$. Let $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_t$ be the prime ideals of codimension 1 containing π and different from $\mathfrak{q}$. Let $\pi_{\mathfrak{q}}$ be a prime element of the discretely valued field $k(\mathfrak{q})$. Choose $\pi_i \in A - \mathfrak{q}$ with $v_{\mathfrak{p}_i}(\pi_i) = 1$ for $1 \leq i \leq t$ and let $m_i = 1$ for $r < i \leq t$. Now choose $m \geq 0$ such that

$$\pi_{\mathfrak{q}}^m \mathcal{O}_{k(\mathfrak{q})} \subset \pi_1^{m_1} \cdots \pi_t^{m_t} A/\mathfrak{q}. \tag{4}$$

Given a symbol $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{n-1}\} \in K^M_{n-1}(k(\mathfrak{q}), m)$ with $\bar{a}_1 \in 1 + \pi_{\mathfrak{q}}^m \mathcal{O}_{k(\mathfrak{q})}$ lift $\bar{a}_1$ according to (4) and lift the other $\bar{a}_i$ to A arbitrarily. Then one sees that the image of

$$\{\pi, a_1, \dots, a_{n-1}\} \in K^M_n(F) \quad \text{in} \quad \bigoplus_{\mathfrak{p}_i \in S} K^M_n(F_{\mathfrak{p}_i})/K^M_n(F_{\mathfrak{p}_i}, m_i) \oplus \bigoplus_{\mathfrak{p} \notin S} K^M_{n-1}(k(\mathfrak{p}))$$

is equal to $\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-1}\} \in K^M_{n-1}(k(\mathfrak{q}))$. □

□

The next corollary follows immediately from Theorem 6.5.

Corollary 6.7. *For a Parshin chain P on the pair $(U \subset X)$ with regular U the map*

$$K^M_{\dim(P)}(k(P)) \longrightarrow \mathrm{C}(U \subset X) \tag{5}$$

is continuous if we endow the Milnor K-group of $k(P)$ with the canonical topology of Definition 4.2.

It seems likely that the stronger fact is true that the map (5) is continuous with respect to Kato's 'topology' on Milnor K-groups, see [5].

Corollary 6.8. *The image of the natural map*

$$\bigoplus_{x \in |U|} \mathbb{Z} \longrightarrow \mathrm{C}(U \subset X)$$

is dense if U is regular.

Here $|U|$ denotes the set of all closed points of U.

Proof. Fix $0 \le l \le \dim(X)$. Let $\mathcal{P}_{\le l}$ be the set of Parshin chain P on $(U \subset X)$ of dimension at most l and let $\mathcal{Q}_{\le l}$ be the set of Q-chains on $(U \subset X)$ of dimension at most l. We show by descending induction on $0 \le l \le \dim(X)$ that the image I_l of the map

$$\bigoplus_{P \in \mathcal{P}_{\le l}} K^M_{\dim(P)}(k(P)) \longrightarrow \mathrm{C}(U \subset X)$$

is dense. It suffices to show that I_{l-1} is dense in I_l. By Corollary 6.7 this follows from the fact that the image of the map

$$\bigoplus_{P \in \mathcal{Q}_{\le l} - \mathcal{Q}_{\le l-1}} K^M_l(k(P)) \longrightarrow \bigoplus_{P \in \mathcal{P}_{\le l} - \mathcal{P}_{\le l-1}} K^M_l(k(P))$$

has dense image where each summand on the right hand side has the canonical topology, see Definition 4.2. The latter density follows from the next lemma, whose simple proof is left to the reader.

Lemma 6.9. *Given a finite family of inequivalent discrete valuations $v_1, \ldots, v_s$ on F the map*

$$K^M_n(F) \longrightarrow \bigoplus_{v_i} K^M_n(F_{v_i})$$

has dense image, where F_{v_i} is the henselization of F at v_i.

□

7. Wiesend's class group

For an integer $l > 0$ denote by $\mathcal{P}_{\le l}$ the subset of the set of Parshin chains $\mathcal{P}$ on the pair $(U \subset X)$ consisting of all chains P with $\dim(P) \le l$ and let $\mathcal{Q}_{\le l}$ be the set of all Q-chains P on $(U \subset X)$ with $\dim(P) \le l$.

Replacing $\mathcal{P}$ by $\mathcal{P}_{\le 1}$ and $\mathcal{Q}$ by $\mathcal{Q}_{\le 1}$ in the definition of the idele class group we get Wiesend's class group.

Definition 7.1. Wiesend's idele group of U is defined as the topological group

$$\mathrm{I}_W(U) = \bigoplus_{P \in \mathcal{P}_{\le 1}} K^M_{\dim(P)}(k(P)).$$

Here for $\dim(P) = 1$ we put the canonical topology on $K^M_1(k(P)) = k(P)^\times$. Wiesend's idele class group is defined as

$$\mathrm{C}_W(U) = \mathrm{coker}[\bigoplus_{P \in \mathcal{Q}_{\le 1}} K^M_{\dim(P)}(k(P)) \to \mathrm{I}_W(U)]$$

with the quotient topology.

Note that Wiesend's idele group and idele class group of U do not depend on the compactification $U \subset X$. For more details on these topological groups see [21], [13] and [11]. The reader should be warned that the map $\mathrm{I}_W(U) \to \mathrm{I}(U \subset X)$ is not continuous for $\dim(X) \ge 2$, but Theorem 6.5 shows that the induced map on idele class groups is continuous.

Proposition 7.2. *If U is regular the natural homomorphism $\mathrm{C}_W(U) \to \mathrm{C}(U \subset X)$ is continuous.*

One of the most difficult open problems in Wiesend's class field theory is to construct a theory of moduli. For this it would be important first of all to give an explicit description of the kernel of the map $\mathrm{C}_W(U) \to \mathrm{C}(X, D)$ to the idele class group with modulus D, at least if X is regular and D is a simple normal crossing divisor.

8. Comparison Theorem (Nisnevich topology)

For us a comparison theorem is a statement which compares a class group defined using sheaf cohomology to an idele class group as defined in Section 6. In this section we discuss comparison theorems for the Nisnevich topology and in the next section we discuss a comparison theorem for the étale topology. All sheaf cohomology groups in this section are with respect to the Nisnevich topology.

Let K be a number field resp. a finite field. By $\mathcal{O}_K$ we denote the ring of integers of K resp. the finite field K itself. Let $X/\mathcal{O}_K$ be an integral proper scheme. We start with a local comparison theorem. For this we have to develop a local version of the idele class group, which is in complete analogy with the global theory. Given a lifted chain P, see Definition 3.2, on X let X_P be the scheme $\mathrm{Spec}(\mathcal{O}^h_{X,P}) - \mathfrak{m}_P$ where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}^h_{X,P}$. Fix an open subset U_P of X_P which is the complement of an effective Weil divisor D_P.

The idele group of the pair $(U_P \subset X_P)$ is defined as

$$\mathrm{I}(U_P \subset X_P) = \bigoplus_{P' \in \mathcal{P}} K^M_{\dim(P')}(k(P')).$$

Here $\mathcal{P}$ is the set of Parshin chains on the pair $(U_P \subset X_P)$, which is defined verbatim the same way as in the global case. The dimension $\dim(P')$ is the dimension of of the composite chain (P, P') on X. The idele class groups $\mathrm{C}(U_P \subset X_P)$ and $\mathrm{C}(X_P, D_P)$ are defined as the obvious quotients of $\mathrm{I}(U_P \subset X_P)$, compare Definition 6.2.

For the local version of the Nisnevich cohomological class group of Kato and Saito we need a relative Milnor K-sheaf. First of all let $\mathcal{K}^M_*$ be the Nisnevich sheaf associated to the Milnor presheaf defined in Section 4. For an effective Cartier divisor with corresponding ideal sheaf $\mathcal{I}_P$ on X_P whose associated Weil divisor is D_P set

$$\mathcal{K}^M_*(\mathcal{O}_{X_P}, \mathcal{I}_P) = \ker[\mathcal{K}^M_*(\mathcal{O}_{X_P}) \to \mathcal{K}^M_*(\mathcal{O}_{X_P}/\mathcal{I}_P)]$$

thought of as a Nisnevich sheaf on X_P.

Let d be the dimension of X_P, d_P be the dimension of P and d_X be the dimension of X.

Definition 8.1. Given an effective Cartier divisor with ideal sheaf $\mathcal{I}_P$ set

$$\mathrm{C}_{\mathrm{Nis}}(X_P, \mathcal{I}_P) = H^d(X_P, \mathcal{K}^M_{d_X}(\mathcal{O}_{X_P}, \mathcal{I}_P)).$$

This local Kato-Saito class group was studied by Sato in [18]. Note that it gives a useful definition for higher class field theory only if P is a Parshin chain, i.e. if the residue field $k(P)$ is a higher local field. Otherwise there does not exist a reciprocity map.

For an arbitrary lifted chain P and for $x \in (X_P)_0$ there exist natural homomorphisms

$$\mathrm{C}_{\mathrm{Nis}}(X_{(P,x)}, \mathcal{I}_{(P,x)}) \to \mathrm{C}_{\mathrm{Nis}}(X_P, \mathcal{I}_P) \quad \text{and} \quad \mathrm{C}(X_{(P,x)}, D_{(P,x)}) \to \mathrm{C}(X_P, D_P). \tag{6}$$

Now we can state our local comparison theorem:

Theorem 8.2. *For lifted chains P on X such that $U_P = X_P - D_P$ is (formally) smooth over $\mathcal{O}_K$, where $\mathcal{I}_P$ is the ideal sheaf of an effective Cartier divisor with associated Weil divisor D_P, there exist natural isomorphisms*

$$\mathrm{C}_{\mathrm{Nis}}(X_P, \mathcal{I}_P) \cong \mathrm{C}(X_P, D_P)$$

compatible with the change of chain morphisms (6).

Proof. Let $\mathcal{F}$ be the Nisnevich sheaf $\mathcal{K}^M_{d+d_P+1}(\mathcal{O}_{X_P}, \mathcal{I}_P)$. We construct the isomorphism by induction on $d = \dim(X_P)$. If $d = 0$ its definition is clear. If $d > 0$ consider the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X_P^{(p)}} H^{p+q}_x(X_P, \mathcal{F}) \Longrightarrow H^{p+q}(X_P, \mathcal{F}).$$

It is well known, see [10, Section 1.2], that it degenerates to an isomorphism between $\mathrm{C}_{\mathrm{Nis}}(X_P, \mathcal{I}_P)$ and the cokernel of the map

$$\bigoplus_{x \in (X_P)_1} H^{d-1}_x(X_P, \mathcal{F}) \longrightarrow \bigoplus_{x \in (X_P)_0} H^d_x(X_P, \mathcal{F}). \tag{7}$$

For simplicity of notation we restrict to $d > 1$ for the rest of the proof. The case $d = 1$ will be obvious from our argument. Clearly for $x \in (X_P)_1$

$$H_x^{d-1}(X_P, \mathcal{F}) = H^{d-2}(X_{(P,x)}, \mathcal{F}) \cong \mathrm{C}(X_{(P,x)}, D_{(P,x)})$$

and for $x \in (X_P)_0$

$$H_x^{d}(X_P, \mathcal{F}) = H^{d-1}(X_{(P,x)}, \mathcal{F}) \cong \mathrm{C}(X_{(P,x)}, D_{(P,x)})$$

by the induction assumption. Here $D_{(P,x)}$ is the induced Weil divisor on $X_{(P,x)}$ and we denote the pullback of $\mathcal{F}$ to $X_{(P,x)}$ by the same symbol. Furthermore, if $x \in U_P$ we have

$$\mathrm{C}_{\mathrm{Nis}}(X_{(P,x)}, \mathcal{F}) = K^M_{\dim(P,x)}(k(P,x)).$$

by Gersten's conjecture which was proven in this case in [7]. So the induction assumption says that $\mathrm{C}(X_P, D_P)$ is isomorphic to the cokernel of the map

$$\bigoplus_{x \in (U_P)_1} H_x^{d-1}(X_P, \mathcal{F}) \longrightarrow \bigoplus_{x \in (X_P)_0} H_x^{d}(X_P, \mathcal{F}).$$

In order to prove the theorem we have to show that the image of the latter map is the same as the image of the map in (7). Or in terms of ideles we have to show that the map

$$\bigoplus_{x \in (D_P)_1} \mathrm{C}(X_{(P,x)}, D_{(P,x)}) \longrightarrow \mathrm{C}(X_P, D_P) \tag{8}$$

vanishes. It follows abstractly from the fact that the map (7) is well-defined that the map (8) is well-defined, i.e. maps to the direct sum, and we will see the concrete reason for this in terms of ideles below.

Fix x in $(D_P)_1$. It follows from a local variant of Corollary 6.8 that $\mathrm{C}(X_{(P,x)}, D_{(P,x)})$ is generated by the images of the groups $K^M_{d_p+3}(k(P,x,y)) = \mathrm{C}(X_{(P,x,y)}, D_{(P,x,y)})$ where y runs through the points of dimension 0 in $U_{(P,x)}$. Consider an element $\alpha \in \mathrm{C}(X_{(P,x)}, D_{(P,x)})$ coming from $\mathrm{C}(X_{(P,x,y)}, D_{(P,x,y)})$ and denote by y' the image of y in X_P. Use a local variant of Theorem 6.5 and Lemma 6.9 to find an element β in $K^M_{d_p+3}(k(P,y')) = \mathrm{C}(X_{(P,y')}, D_{(P,y')})$ such that the image of β under the composite map

$$\mathrm{C}(X_{(P,y')}, D_{(P,y')}) \longrightarrow \mathrm{C}(X_{(P,x,y')}, D_{(P,x,y')}) \longrightarrow \mathrm{C}(X_{(P,x)}, D_{(P,x)})$$

is α and for all $x' \neq x$ in $(D_P)_1$ the image of β in $\mathrm{C}(X_{(P,x')}, D_{(P,x')})$ vanishes. Let γ be the image of β under the residue map

$$\mathrm{C}(X_{(P,y')}, D_{(P,y')}) \longrightarrow \bigoplus_{x' \in (U_P)_1} \mathrm{C}(X_{(P,x')}, D_{(P,x')}).$$

By reciprocity the image of α in $\mathrm{C}(X_P, d_P)$ is the same as the image of γ. But it is clear that the latter vanishes.

So we have shown that the natural surjective map $\mathrm{C}(X_P, D_P) \to \mathrm{C}_{\mathrm{Nis}}(X_P, \mathcal{I}_P)$ stemming from the coniveau spectral sequence and the induction assumption is injective. We let this map be our natural isomorphism needed to conclude the induction step. $\square$

The global class group of Kato and Saito is defined as follows. Again X is an integral scheme proper over $\mathcal{O}_K$ of dimension d.

Definition 8.3. Given an effective Cartier divisor with corresponding ideal sheaf $\mathcal{I}$ on X whose associated Weil divisor is D set

$$\mathrm{C}_{\mathrm{Nis}}(X,\mathcal{I}) = H^d(X, \mathcal{K}_d^M(\mathcal{O}_X, \mathcal{I})).$$

Our global comparison theorem reads:

Theorem 8.4. *If $U = X - D$ is smooth over $\mathcal{O}_K$ there exists a natural isomorphism*

$$\mathrm{C}_{\mathrm{Nis}}(X,\mathcal{I}) \cong \mathrm{C}(X,D).$$

Proof. Let $\mathcal{F}$ be the Nisnevich sheaf $\mathcal{K}_d^M(\mathcal{O}_X,\mathcal{I})$. As in the proof of Theorem 8.2 the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H_x^{p+q}(X,\mathcal{F}) \Longrightarrow H^{p+q}(X,\mathcal{F})$$

degenerates to an isomorphism between $\mathrm{C}_{\mathrm{Nis}}(X,\mathcal{I})$ and the cokernel of the map

$$\bigoplus_{x\in X_1} H_x^{d-1}(X,\mathcal{F}) \longrightarrow \bigoplus_{x\in X_0} H_x^{d}(X,\mathcal{F}). \tag{9}$$

By Theorem 8.2 we get an isomorphism between $\mathrm{C}(X,D)$ and the cokernel of the map

$$\bigoplus_{x\in U_1} H_x^{d-1}(X,\mathcal{F}) \longrightarrow \bigoplus_{x\in X_0} H_x^{d}(X,\mathcal{F}). \tag{10}$$

So we have to show that the maps (9) and (10) have the same image. For this it suffices to show that the induced map

$$\bigoplus_{x\in D_1} \mathrm{C}(X_{(x)}, D_{(x)}) \longrightarrow \mathrm{C}(X,D)$$

vanishes, which can be shown similarly to the final part of the proof of Theorem 8.2. $\square$

9. Comparison Theorem (étale topology)

In this section we discuss the local comparison theorem with respect to the étale topology. As the proof is very similar the the proof of the Nisnevich comparison theorem we leave it to the reader. Let the notation be as in the previous section, but let all sheaf cohomology groups be taken with respect to the étale topology.

Consider a lifted chain P of dimension $d_P = \dim(P)$ on integral proper scheme $X/\mathbb{Z}$ of dimension $d_X = \dim(X)$. Let d be the dimension of X_P We define the étale class group of the scheme X_P as follows:

Definition 9.1. For a nonempty open subscheme $j : U_P \subset X_P$ and an integer $m > 0$ which is invertible on U_P set

$$\mathrm{C}_{\text{ét}}(U_P \subset X_P; \mathbb{Z}/m) = H^e(X_P, j_!\, \mathbb{Z}/m(d_X))$$

with $e = 2d + d_P + 1$.

The étale class group was discussed in the work of Saito on the class field theory of henselian local rings [17].

For an arbitrary lifted chain P on X and for $x \in (X_P)_0$ there exists a natural homomorphism

$$\mathrm{C}_{\text{ét}}(X_{(P,x)}, \mathcal{I}_{(P,x)}) \to \mathrm{C}_{\text{ét}}(X_P, \mathcal{I}_P). \tag{11}$$

Using the norm residue isomorphism due to Voevodsky and Rost [20] and an argument analogous to the proof of Theorem 8.2 we prove:

Theorem 9.2. *If m is invertible on $\mathcal{O}^h_{X,P}$ and $\mathcal{O}^h_{X,P}$ is (formally) smooth over $\mathcal{O}_K$ then there exist natural isomorphisms*

$$\mathrm{C}_{\text{ét}}(U_P \subset X_P; \mathbb{Z}/m) \cong \mathrm{C}(U_P \subset X_P)/m \tag{12}$$

compatible with the change of chain morphisms (6) *and* (11).

Proof. Consider the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X_P^{(p)}} H_x^{p+q}(X_P, \mathcal{F}) \Longrightarrow H^{p+q}(X_P, \mathcal{F})$$

where $\mathcal{F} = j_! \, \mathbb{Z}/m(d_X)$. It degenerates to an isomorphism between

$$E_2^{d,d_X} = \operatorname{coker}[\bigoplus_{x \in X_P^{(d-1)}} H_x^{d+d_X-1}(X_P, \mathcal{F}) \to \bigoplus_{x \in X_P^{(d)}} H_x^{d+d_X}(X_P, \mathcal{F})] \tag{13}$$

and $\mathrm{C}_{\text{ét}}(U_P \subset X_P; \mathbb{Z}/m)$, since the groups E_2^{d-1,d_X+1} and E_2^{d-2,d_X+1} vanish. In fact the latter groups do not depend on the open subscheme $U_P \subset X_P$, cf. [18, Sec. 4], and for $U_P = X_P$ they vanish because of the Gersten conjecture proved by Gillet and Levine [4].

Now one makes an induction on $\dim(X_P)$ in order to construct the isomorphism (12). In case $\dim_P(X) = 0$ it is just the norm residue isomorphism [20]. As in the proof of Theorem 8.2 for $\dim(X_P) > 0$ it suffices to show that the cokernel (13) is the same as the cokernel of the map

$$\bigoplus_{x \in U_P^{(d-1)}} H_x^{d+d_X-1}(X_P, \mathcal{F}) \longrightarrow \bigoplus_{x \in X_P^{(d)}} H_x^{d+d_X}(X_P, \mathcal{F}).$$

The argument is very similar to the argument given in the proof of Theorem 8.2, so we leave the details to the reader. □

Remark 9.3. If the chain P consists of a closed point of X Theorem 9.2 and local duality imply the existence of a local reciprocity isomorphism

$$\mathrm{C}(U_P \subset X_P)/m \xrightarrow{\sim} \pi_1^{ab}(U_P)/m.$$

A similar isomorphism was constructed by Sato [18]. It is not difficult to develop an analogous approach to the global reciprocity isomorphism.

10. Class field theory

In this section we reformulate the main results of Kato and Saito in terms of our new class group. Fix a finitely generated field F, which we assume for simplicity to contain a totally imaginary number field if $\mathrm{char}(F) = 0$. Let X be an integral scheme proper over $\mathbb{Z}$ with $k(X) = F$. First of all Kato's reciprocity law [5, Proposition 7] shows that the homomorphism $r : \mathrm{I}(F;X) \to \mathrm{Gal}(F)^{ab}$ factors through a continuous homomorphism $\rho : \mathrm{C}(F;X) \to \mathrm{Gal}(F)^{ab}$, which is called the reciprocity map.

Theorem 10.1 (Isomorphism)**.** *For a finite Galois extension $\phi : \mathrm{Spec}(K) \to \mathrm{Spec}(F)$ which extends to a finite morphism of integral schemes $Y \to X$ with $k(Y) = K$ the map ρ induces an isomorphism*

$$\mathrm{C}(F;X)/\phi_*\mathrm{C}(K;X) \xrightarrow{\sim} \mathrm{Gal}(K/F)^{ab}.$$

For a proof of the Isomorphism Theorem we refer to [10, Thm. 4.6]. The next theorem is a reformulation of the Existence Theorem of Kato-Saito, see [10, Theorem 9.1].

Theorem 10.2 (Existence)**.** *The reciprocity map ρ induces an isomorphism of profinite groups*

$$\mathrm{C}(F;X)^\wedge \xrightarrow{\sim} \mathrm{Gal}(F)^{ab}.$$

Here for a topological group G we denote by $G^\wedge$ the profinite completion of G, i.e. the completion with respect to all cofinite open subgroups.

The Finiteness Theorem of Kato and Saito is probably the deepest result in higher global class field. The central ideas in its proof go back to Bloch [2] and were revisited by Wiesend [22, Section 8]. As we do not give the proof we refer the reader to [16, Theorem 6.1 and 6.2].

Theorem 10.3 (Finiteness)**.** *If $\mathrm{char}(F) = 0$ every open subgroup of $\mathrm{C}(F;X)$ has finite index. If $\mathrm{char}(F) = p > 0$ every open subgroup of $\mathrm{C}(F;X)$ with nontrivial image in $\mathrm{C}(\mathbb{F}_p) = \mathbb{Z}$ has finite index.*

If $\mathrm{char}(F) = 0$ we have an even stronger result due to Wiesend [21] which comprises the previous three theorems. For a complete proof see [13] or [11].

Theorem 10.4. *If X is an integral scheme proper and flat over $\mathrm{Spec}(\mathbb{Z})$ and $U \subset X$ is a regular open subscheme the sequence*

$$0 \longrightarrow \mathrm{C}_W(U)^0 \longrightarrow \mathrm{C}_W(U) \longrightarrow \pi_1^{ab}(U) \longrightarrow 0$$

is topological exact.

Here for a topological group G we denote by G^0 the connected component of the identity element.

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