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An action principle for an interacting Fermion system and its analysis in the continuum limit

Felix Finster

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AN ACTION PRINCIPLE FOR AN INTERACTING FERMION SYSTEM AND ITS ANALYSIS IN THE CONTINUUM LIMIT

FELIX FINSTER

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Abstract. We introduce and analyze a system of relativistic fermions in a space-time continuum, which interact via an action principle as previously considered in a discrete space-time. The model is defined by specifying the vacuum as a sum of Dirac seas corresponding to several generations of elementary particles. The only free parameters entering the model are the fermion masses. We find dynamical field equations if and only if the number of generations equals three. In this case, the dynamics is described by a massive axial potential coupled to the Dirac spinors. The coupling constant and the rest mass of the axial field depend on the regularization; for a given regularization method they can be computed as functions of the fermion masses. The bosonic mass term arises as a consequence of a symmetry breaking effect, giving an alternative to the Higgs mechanism. In addition to the standard loop corrections of quantum field theory, we find new types of correction terms to the field equations which violate causality. These non-causal corrections are too small for giving obvious contradictions to physical observations, but they might open the possibility to test the approach in future experiments.

Contents

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1. Introduction

In [13] it was proposed to formulate physics based on a new action principle in space-time. On the fundamental level, this action principle is defined in so-called discrete space-time for a finite collection of projectors in an indefinite inner product space (see also [16]). An effect of spontaneous symmetry breaking [15] leads to the emergence of a discrete causal structure (see [7] for an explanation in simple examples), which for many space-time points and many particles should go over to the usual causal structure of Minkowski space (for the connection between discrete and continuum space-times we also refer to [23, 15, 16] and the survey article [20]). Furthermore, on a more phenomenological level, it is shown in [13, Chapters 4–8] that the action can also be analyzed in Minkowski space in the so-called *continuum limit*, where the interaction is described effectively by classical gauge fields coupled to second-quantized Dirac fields. Finally, in [18] the existence of minimizers of our action principle is proved in various situations, both in discrete and in continuum space-times.

Apart from deriving the general formalism of the continuum limit, in [13, Chapters 4–8] it is shown that for a suitable system involving 24 Dirac seas, the resulting effective gauge group as well as the coupling of the effective gauge fields to the Dirac fields have striking similarities to the standard model. However, the detailed form of the effective interaction so far has not been worked out.

This work is the first of a series of papers devoted to the detailed analysis of our action principle in the continuum limit and to the derivation of the effective field equations. In order to make the presentation as clear and easily accessible as possible, our procedure is to begin with small systems, which are composed of only a few Dirac seas, and then to gradually build up larger and more complicated systems. In the present paper, we consider a system of several Dirac seas (corresponding to several "generations" of elementary particles) in the simplest possible configuration referred to as a *single sector*. The only free parameters entering the model are the masses of the Dirac particles of each generation. However, we do not specify the form of the interaction, which is completely determined by our action principle. Also, we do not put in coupling constants nor the masses of gauge bosons. The analysis of the model in the continuum limit reveals that we get dynamical field equations if and only if the number of generations equals three. In this case, the dynamics can be described by a *massive axial potential* A_a coupled to the Dirac equation. The corresponding Dirac and field equations (stated for notational simplicity for one Dirac particle) become

$$\boxed{(i\partial\!\!\!/ + \gamma^5 A\!\!\!/_\text{a} - m)\Psi = 0\,, \qquad C_0\, j^k_\text{a} - C_2\, A^k_\text{a} = 12\pi^2\, \overline{\Psi}\gamma^5\gamma^k\Psi\,,}$$

where $j^k_\text{a} = \partial^k{}_l A^l_\text{a} - \Box A^k_\text{a}$ is the corresponding axial current. The coupling constant and the rest mass of the axial gauge field are described by the constants C_0 and C_2, which for a given regularization method can be computed as functions of the fermion masses. The mass term of the gauge field arises as a consequence of a symmetry breaking effect, giving an alternative to the Higgs mechanism. The field equations involve surprising corrections which challenge the standard model of elementary particle physics: First, the field equations involve additional convolution terms of the form

$$-f_{[0]} * j^k_\text{a} + 6 f_{[2]} * A^k_\text{a}\,,$$

where $f_{[p]}$ are explicit Lorentz invariant distributions. These convolution terms give rise to small corrections which violate causality. Moreover, we get new types of higher order corrections to the field equations. We also find additional potentials which are non-dynamical in the sense that they vanish away from the sources.

In order to make the paper self-consistent, we introduce our fermion systems and the continuum limit from the basics. However, to avoid an excessive overlap with previous work, we present a somewhat different point of view, where instead of considering a discrete space-time or a space-time continuum of finite volume, we work exclusively in Minkowski space. Furthermore, we always restrict attention to a single sector. For clarity, we omit the more technical aspects of the regularization, relying instead on results from the corresponding chapters of the book [13].

The paper is organized as follows. In Chapter 2 we introduce our action principle in a space-time continuum. In Chapters 3–5 we review and adapt the methods for analyzing this action principle in the continuum as developed in [13]. More precisely, in Chapter 3 we describe the vacuum by a system of regularized Dirac seas. We list all the assumptions on the vacuum state, either motivating them or explaining how they can be justified. In Chapter 4 we construct more general fermion configurations in Minkowski space by modifying and perturbing the vacuum state, also introducing particles and gauge fields. We also outline the mathematical methods for analyzing the unregularized fermionic projector with interaction. In Chapter 5, we explain how interacting systems are to be regularized, and how to treat the regularization in an effective way. This leads us to the formalism of the continuum limit, which allows us to analyze our action principle in the continuum, taking into account the unknown regularization details by a finite number of free parameters. In the following Chapters 6–10 the continuum limit of our action principle is worked out in detail; this is the main part of the paper where we present our new results. Chapter 6 is devoted to the leading singularities of the Euler-Lagrange equations on the light cone, where the vacuum contributions (§6.1) are modified by phases coming from the chiral gauge potentials (§6.2). The next lower orders of singularities are analyzed in Chapter 7. Then the currents of the gauge fields come into play, and we also get a mass term corresponding to the axial gauge field (§7.1). Furthermore, we find a corresponding contribution of the Dirac current (§7.2). A priori, the different current terms are not comparable, because the gauge currents have logarithmic poles on the light cone (§7.3). But provided that the number of generations is at least three, these logarithmic poles can be compensated by a local axial transformation, as is developed in §7.4–§7.6.

Chapter 8 is devoted to the derivation and analysis of the field equations. In §8.1 we show that the Euler-Lagrange equations corresponding to our action principle give rise to relations between the Dirac and gauge currents. If the number of generations equals three, we thus obtain field equations for the axial gauge potential (see Theorem 8.2). These field equations involve non-causal correction terms, which are analyzed and discussed in §8.2 and §8.3. In §8.4 we explain how the standard loop corrections of quantum field theory appear in our framework, and how loop corrections of the non-causal terms could be obtained. In §8.5 we get a connection to the Higgs mechanism and explain why in our approach no Higgs particle appears. We finally compute the coupling constant and the rest mass of the axial field for a few simple regularizations (§8.6).

In Chapter 9 we analyze and discuss further potentials and fields, including scalar and pseudoscalar potentials, bilinear potentials, as well as the gravitational field and a conformal axial field. In Chapter 10 we consider nonlocal potentials, which can be used to satisfy the Euler-Lagrange equations to higher order in an expansion near the origin (see Theorem 10.5).

In order not to interrupt the explanations in the main chapters by longer calculations, the more technical parts are worked out in the appendices. Appendix A supplements the estimates needed for the derivation of the Euler-Lagrange equations in the continuum limit in §5.2. All the calculations in the formalism of the continuum limit as needed in Chapters 6–9 are combined in Appendix B, which also reviews the general method as developed in [13, Appendix G]. All the formulas given in this appendix have been obtained with the help of computer algebra. In Appendix C the local axial transformation is worked out non-perturbatively, thus putting the analysis in §7.6 on a fully convincing basis. In Appendix D we compute and analyze the smooth contributions to the fermionic projector as needed in §8.1; this is done by modifying a resummation technique first introduced in [11]. Finally, in Appendix E we outline how our constructions and results can be extended to the setting where the Dirac seas involve weight factors, as was proposed in [17] and [22].

2. An Action Principle for Fermion Systems in Minkowski Space

In relativistic quantum mechanics, a fermionic particle is described by a Dirac wave function Ψ in Minkowski space $(M, \langle ., . \rangle)$. In order to describe a many-particle system, we consider an operator P on the Dirac wave functions and interpret the vectors in the image of P as the occupied fermionic states of the system (for a discussion of the Pauli exclusion principle and the connection to the fermionic Fock space formalism see [13, Chapter 3 and Appendix A]). We assume that P has an integral representation

$$(P\Psi)(x) = \int_M P(x,y)\; \Psi(y)\; d^4y \tag{2.1}$$

with an integral kernel $P(x,y)$. Moreover, we assume for technical simplicity that $P(x,y)$ is continuous in both arguments x and y; then the integral in (2.1) is clearly well-defined if for the domain of definition of P we choose for example the space $C^\infty_0(M)^4$ of smooth wave functions with compact support. Moreover, we assume that P is symmetric with respect to the Lorentz invariant inner product

$$<\Psi|\Phi> = \int_M \overline{\Psi(x)} \Phi(x)\; d^4x\;, \tag{2.2}$$

where $\overline{\Psi} \equiv \Psi^\dagger \gamma^0$ is the usual adjoint spinor ($\Psi^\dagger$ is the complex conjugate spinor). In other words, we demand that

$$<P\Psi|\Phi> = <\Psi|P\Phi> \qquad \text{for all } \Psi, \Phi \in C^\infty_0(M)^4\;. \tag{2.3}$$

This condition can also be expressed in terms of the kernel by

$$P(x,y)^* \equiv \gamma^0 P(x,y)^\dagger \gamma^0 = P(y,x) \quad \text{for all } x,y \in M\;, \tag{2.4}$$

where the dagger denotes the transposed, complex conjugate matrix. We refer to P as the *fermionic projector*. The vectors in the image of P are referred to as the *wave functions* of our system. We point out that for the moment, these wave functions do not need to be solutions of a Dirac equation.

For any space-time points x and y, we next introduce the *closed chain* A_{xy} by

$$A_{xy} = P(x,y)\, P(y,x)\;. \tag{2.5}$$

It is a 4×4-matrix which can be considered as a linear operator on the wave functions at x. For any such linear operator A we define the *spectral weight* $|A|$ by

$$|A| = \sum_{i=1}^{4} |\lambda_i| \,, \tag{2.6}$$

where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of A counted with algebraic multiplicities. For any $x, y \in M$ we define the *Lagrangian* $\mathcal{L}$ by

$$\mathcal{L}_{xy}[P] = |A_{xy}^2| - \frac{1}{4}\, |A_{xy}|^2 \,. \tag{2.7}$$

Integrating over space-time, we can furthermore introduce the functionals

$$\boxed{\begin{aligned} \mathcal{S}[P] &\overset{\text{formally}}{=} \iint_{M \times M} \mathcal{L}_{xy}[P]\; d^4x\, d^4y \\ \mathcal{T}[P] &\overset{\text{formally}}{=} \iint_{M \times M} |A_{xy}|^2\; d^4x\, d^4y \,. \end{aligned}} \tag{2.8}$$

These expressions are only formal because the integrands need not decay for large x or y, and thus integrals may be infinite (similar as in classical field theory, where the space-time integral over the Lagrangian diverges without imposing suitable decay properties at infinity). The functional $\mathcal{S}$ is referred to as the *action*. Our variational principle is to minimize $\mathcal{S}$ under the constraint that $\mathcal{T}$ is kept fixed. For this minimization we vary the fermionic projector in the following sense. In order to prevent trivial minimizers, the variation should preserve the normalization of the wave functions. This normalization should be performed with respect to the Lorentz invariant inner product (2.2). However, we do not want to assume that this inner product is finite for the wave functions Ψ in the image of P (indeed, for physical wave functions, the inner product $<\Psi|\Psi>$ is in general infinite because the time integral diverges). Our method for avoiding the divergences in (2.2) and (2.8) is to consider variations which outside a compact set are the identity.

Definition 2.1. *An operator U on the Dirac wave functions is called* **unitary in a compact region** *if*

(i) $<U\Psi \,|\, U\Psi> = <\Psi \,|\, \Psi>$ *for all compactly supported* Ψ.

(ii) *The operator* $V := U - \mathbb{1}$ *has the representation*

$$(V\Psi)(x) = \int_M v(x,y)\, \Psi(y)\, d^4y$$

with a smooth integral kernel $v(x,y)$ which has compact support, i.e. there is a compact set $K \subset M$ such that

$$v(x,y) = 0 \qquad \textit{unless } x \in K \textit{ and } y \in K \,.$$

Thus introducing a variation of the wave functions by the transformation $\Psi \to U\Psi$, all the wave functions are changed only in the compact region $K \subset M$, in such a way that all inner products in this region, i.e. all the integrals

$$\int_K \overline{\Psi(x)}\, \Phi(x)\, d^4x \,,$$

remain unchanged. Having introduced a well-defined notion of "varying the fermionic projector while respecting the inner product (2.2)", we can now specify what we mean by a minimizer.

Definition 2.2. *A fermionic projector P of the form* (2.1) *is a* **minimizer** *of the variational principle*

$$\text{minimize } \mathcal{S} \text{ for fixed } \mathcal{T} \tag{2.9}$$

if for any operator U which is unitary in a compact region and satisfies the constraint

$$\int_M d^4x \int_M d^4y \left(|A_{xy}[P]|^2 - |A_{xy}[UPU^{-1}]|^2 \right) = 0 \,, \tag{2.10}$$

the functional $\mathcal{S}$ satisfies the inequality

$$\int_M d^4x \int_M d^4y \left(\mathcal{L}_{xy}[UPU^{-1}] - \mathcal{L}_{xy}[P] \right) \;\geq\; 0 \,. \tag{2.11}$$

We point out that, since U changes the wave functions only inside a compact set K, the integrands in (2.10) and (2.11) clearly vanish if x *and* y are outside K. However, is is not obvious that the integrals over the region $x \in K$ and $y \in M \setminus K$ (and similarly $x \in M \setminus K$ and $y \in M$) exist. By writing (2.10) and (2.11) we implicitly demand that the integrand in (2.10) and the negative part of the integrand in (2.11) should be in $L^1(M \times M, \mathbb{R})$.

Before going on, we briefly discuss this action principle and bring it into the context of previous work. We first remark that, in contrast to [13, 16], we here ignore the condition that P should be idempotent. This is done merely to simplify the presentation, anticipating that the idempotence condition will not be of relevance in this paper. The action principle (2.9) was first introduced in a discrete space-time in [13, §3.5]. Apart from the obvious replacement of sums by integrals, the action here differs from that in [13, §3.5] only by an irrelevant multiple of the constraint $\mathcal{T}$. This has the advantage that the Lagrangian (2.7) coincides with the so-called critical case of the auxiliary Lagrangian as introduced in [16]; this is the case relevant in our setting of one sector. Note that this Lagrangian is symmetric (see [16, equation (13)]) and non-negative,

$$\mathcal{L}_{xy}[P] = \mathcal{L}_{yx}[P] \qquad \text{and} \qquad \mathcal{L}_{xy}[P] \geq 0 \,.$$

Moreover, the action principle (2.9) can be regarded as an infinite volume limit of the variational principle in [18, Chapter 3] (possibly also in the limit where the number of particles tends to infinity). In the special case of homogeneous systems, our variational principle is closely related to the variational principle in infinite volume as considered in [18, Chapter 4]. Working with unitary transformations in a compact region, we can make sense of the action principle even in infinite space-time volume without assuming homogeneity; this procedure can be seen in analogy to considering variations of compact support in the Lagrangian formulation of classical field theory (like a variation $\delta A \in C^\infty_0(M)$ of the electromagnetic potential in classical electrodynamics).

3. Assuming a Vacuum Minimizer

Apart from the general existence results in [16, 18] and the simple examples in [7, 18], almost nothing is known about the minimizers of our action principle. Therefore, before we can do physics, we need to assume the existence of a special minimizer which describes a physically meaningful vacuum. In this chapter, we compile our assumptions on this vacuum minimizer, and we outline in which sense and to what extent these assumptions have been justified in [17, 22]. At the end of this chapter, we will explain how to work with these assumptions in practice.

Taking Dirac's original concept seriously, we want to describe the vacuum by "completely filled Dirac seas" corresponding to the masses $m_1, \ldots, m_g$ of g generations of elementary particles (later we will set $g = 3$, but for the moment it is preferable not to specify the number of generations). Thus our first ansatz for the integral kernel of the fermionic projector of the vacuum is the Fourier transform of the projectors $\frac{1}{2m_\beta}(\not{k} + m_\beta)$ on the Dirac states on the lower mass shells,

$$P(x,y) = \sum_{\beta=1}^{g} \int \frac{d^4k}{(2\pi)^4} \, (\not{k} + m_\beta) \, \delta(k^2 - m_\beta^2) \, \Theta(-k^0) \, e^{-ik(x-y)} \, . \tag{3.1}$$

(Here Θ is the Heaviside function, and $k(x-y)$ is a short notation for the Minkowski inner product $\langle k, x-y \rangle$. The slash denotes contraction with the Dirac matrices, thus $\not{k} = k_j \gamma^j$. We always work in natural units $\hbar = c = 1$, and for the signature of the Minkowski inner product we use the convention $(+ - - -)$.) We always index the masses in increasing order,

$$m_1 < m_2 < \ldots < m_g \, . \tag{3.2}$$

The Fourier integral (3.1) is well-defined as a distribution. If the vector $y - x$ is spacelike or timelike, the integral (3.1) exists even pointwise. However, if the vector $y - x$ is null, the distribution $P(x,y)$ is singular (for details see [13, §2.5]). In physical terms, these singularities occur if y lies on the light cone centered at x. Thus we refer to the singularities on the set where $(x-y)^2 = 0$ as the *singularities on the light cone*. As a consequence of these singularities, the pointwise product in (2.5) is ill-defined, and the Lagrangian (2.7) has no mathematical meaning. In order to resolve this problem, one needs to introduce an *ultraviolet regularization*. In position space, this regularization can be viewed as a "smoothing" on a microscopic length scale. It seems natural to identify this microscopic length scale with the *Planck length* ℓ_P. Likewise, in momentum space the regularization corresponds to a cutoff or decay on the scale of the *Planck energy* $E_P = \ell_P^{-1}$. Clearly, the Planck scale is extremely small compared to the length scale ℓ_{macro} of macroscopic physics, and thus it seems reasonable to expand in powers of $\ell_P / \ell_{\text{macro}}$. However, such an expansion would not be mathematically meaningful, because Taylor series can be performed only in continuous variables (but not in a constant, no matter how small). Therefore, it is preferable to denote the regularization length by the variable ε, which may vary in the range $0 < \varepsilon \ll \ell_{\text{macro}}$ (clearly, ε can be always be thought of as being of the order of the Planck length). We are thus led to a one-parameter family of regularizations. We assume that these regularized Dirac sea configurations are all minimizers. We also compile all assumptions on the regularization as introduced in [13, Chapter 4].

Assumption 3.1. (regularized Dirac sea vacuum) There is a family $(P^\varepsilon)_{\varepsilon > 0}$ of fermionic projectors whose kernels $P^\varepsilon(x,y)$ (as defined by (2.1)) have the following properties:

(i) Every $P^\varepsilon(x,y)$ is a *minimizer* in the sense of Definition 2.2.
(ii) Every $P^\varepsilon(x,y)$ is *homogeneous*, i.e. it depends only on the variable $\xi := y - x$.
(iii) Taking its Fourier transform,

$$P^\varepsilon(x,y) = \int \frac{d^4p}{(2\pi)^4} \, \hat{P}^\varepsilon(p) \, e^{-ip(x-y)} \, , \tag{3.3}$$

$\hat{P}^\varepsilon$ is a distribution with a *vector-scalar structure*, i.e.

$$\hat{P}^\varepsilon(k) = (v_j^\varepsilon(k)\, \gamma^j \;+\; \phi^\varepsilon(k)\, \mathbb{1})\, f^\varepsilon(p) \tag{3.4}$$

with a vector field v^ε, a scalar field ϕ^ε and a distribution f^ε, which are all real-valued.

(iv) If the regularization is removed, P^ε goes over to P (as given by (3.1)), i.e.

$$\lim_{\varepsilon \searrow 0} \hat{P}^\varepsilon(k) = \hat{P}(k) := \sum_{\beta=1}^{g} (k\!\!\!/ + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)$$

with convergence in the distributional sense.

The assumptions so far seem natural and are easy to state. The assumption (iii) can even be derived in the Hamiltonian framework by minimizing the physical energy of a regularized Dirac sea configuration, describing the electrostatic interaction in the mean field approximation [28, 29]. To understand the following assumptions, one should notice that the singularities of $P(x, y)$ on the light cone arise because its Fourier transform $\hat{P}(k)$ is supported on the mass shells $k^2 = m_\beta^2$, which are hypersurfaces being asymptotic to the mass cone $k^2 = 0$ (for details see [13, §4.2]). Thus in order to control the behavior of P^ε near the light cone, we need to make suitable assumptions on $P^\varepsilon(\omega, \vec{k})$ for $\omega \approx -|\vec{k}| \sim \varepsilon^{-1}$.

(v) We assume that the distribution $\hat{P}^\varepsilon$ is supported on hypersurfaces described by graphs, i.e. the distribution f^ε in (3.4) should be of the form

$$f^\varepsilon(\omega, \vec{k}) = \sum_{\beta=1}^{g} \delta\Big(\omega + |\vec{k}| + \alpha_\beta(\vec{k})\Big) \;. \tag{3.5}$$

These hypersurfaces should be asymptotic to the mass cone in the sense that

$$\alpha_\beta(\vec{k}) \sim \varepsilon \qquad \text{if } |\vec{k}| \sim \varepsilon^{-1} \;.$$

Except for these singularities, $\hat{P}^\varepsilon(k)$ is so regular that the singularities as $\varepsilon \searrow 0$ of $P^\varepsilon(x, y)$ on the light cone are completely described by the behavior of $\hat{P}^\varepsilon(k)$ on the hypersurfaces (3.5), up to corrections of higher order in ε. We refer to this assumption as the *restriction to surfaces states*.

(vi) On the hypersurfaces (3.5) and for $|\vec{k}| \sim \varepsilon^{-1}$, the vector field v^ε in (3.4) should be parallel to k, up to a small error term. More precisely, decomposing v^ε as

$$v^\varepsilon = s^\varepsilon(k)\, k + \vec{w}^\varepsilon(k)$$

with a scalar function s^ε, the vector field $\vec{w}^\varepsilon$ should be bounded by

$$|\vec{w}^\varepsilon(k)| < \varepsilon_{\text{shear}} \qquad \text{where} \qquad \varepsilon_{\text{shear}} \ll 1 \;. \tag{3.6}$$

Referring to the effect of this assumption in position space, we say that the *vector component is null on the light cone*.

(vii) The functions in (3.4) either vanish, $\phi^\varepsilon(k) = 0 = v^\varepsilon(k)$, or else $\phi^\varepsilon(k) > 0$ and the vector field v^ε is time-like and past-directed. Furthermore,

$$v^\varepsilon(p)^2 = \phi^\varepsilon(p)^2 \;.$$

For a discussion of the assumptions (v) and (vi) we refer to [13, Chapter 4]. The condition (vii) requires a brief explanation. This assumption is clearly satisfied without regularization (3.1) (in which case we choose $v(p) = p/(2\omega)$ and ϕ a positive function which on the mass shells takes the values $m_\alpha/(2\omega)$). A closely related condition was first proposed in [13, Chapter 4] as the assumption of *half-occupied surface states*. This condition was motivated by the wish to realize the Dirac sea configurations with as few occupied states as possible, noting that the condition (vii) implies that the matrix $\hat{P}^\varepsilon(k)$ has rank at most two. Furthermore, the condition (vii) implies that the image of the matrix $\hat{P}^\varepsilon(k)$ is negative definite with respect to the inner product $\overline{\Psi}\Phi$. From the mathematical point of view, this definiteness ensures that $\hat{P}^\varepsilon$ can be regarded as a *negative definite measure* on an indefinite inner product space as introduced in [18, Chapter 4], and this assumption is crucial for the general compactness result [18, Theorem 4.2]. Thus the physical intuition and the mathematical requirements fit together. Moreover, in the case when $\hat{P}^\varepsilon(k)$ does not vanish, we can choose a suitably normalized orthogonal basis $(\Psi_{p,1}, \Psi_{p,2})$ of the image of $\hat{P}^\varepsilon(k)$ such that $(2\pi)^4 \hat{P}^\varepsilon(k) = -\Psi_{k,1}\overline{\Psi_{k,1}} - \Psi_{k,2}\overline{\Psi_{k,2}}$. Substituting this representation into the Fourier integral (3.3) and using (3.5), we obtain

$$P^\varepsilon(x,y) = -\sum_{\beta=1}^{g} \int_{\mathbb{R}^3} d\vec{k} \sum_{a=1,2} \Psi_{\vec{k}\beta a}(x)\, \overline{\Psi_{\vec{k}\beta a}(y)}\,,$$

where $\Psi_{\vec{k}\beta a(x)} = \Psi_{p,a}\, e^{-ipx}$ for $p = (-|\vec{k}| - \alpha_\beta(\vec{k}), \vec{k})$. This representation is helpful because it shows that the regularized fermionic projector of the vacuum is composed of negative-energy wave functions; the index a can be thought of as describing the two spin orientations.

We next outline the approach taken to justify the above assumptions. In [17] a class of regularizations is constructed for which the action remains finite when the regularization is removed (more precisely, this is done by proving that the constructed regularizations satisfy the so-called assumption of a distributional $\mathcal{M}P$-product). These regularizations are spherically symmetric, but they break the Lorentz symmetry. However, after suitably removing the regularization, we obtain a well-defined Lorentz invariant action principle. This Lorentz invariant action principle is analyzed in [22], and it is shown that for certain values of the masses and the so-called weight factors (which for simplicity we do not consider in the main text of this paper; however, see Appendix E), the Dirac sea configuration (3.1) is indeed a minimizer, in a sense made precise using the notion of state stability. Following these results, "good candidates" for satisfying the above assumptions are obtained by regularizing the state stable Dirac sea configurations from [22] according to to the regularization scheme in [17]. The remaining task for giving a rigorous justification of Assumption 3.1 is to use the freedom in choosing the regularization such as to obtain a minimizer in the sense of Definition 2.2. This task seems difficult and has not yet been accomplished. In [18, Theorem 4.2] the existence of minimizers is proved within the class of homogeneous fermionic projectors; but this is considerably weaker than being a minimizer in the sense of Definition 2.2. In technical terms, the main difficulty is to quantify the influence of the spherically symmetric regularization on the action, even taking into account contributions which remain finite when the regularization is removed. Despite this difficult and technically challenging open problem, it is fair to say that the results of [17, 22] show that Dirac sea configurations tend to make our action small, thus explaining why Assumption 3.1 is a reasonable starting point for the continuum analysis.

We finally explain how to work with the above assumptions in practice. Ideally, the fields v^ε, ϕ^ε and the distribution f^ε in (3.4) could be determined by minimizing our action (2.9), thus giving detailed information on P^ε. Such a minimization process is indeed possible (see [18, Theorem 4.2] for a general existence result and [23] for a lattice formulation), but so far has not been analyzed in sufficient depth. Thus for the time being, there is a lot of freedom to choose the functions in (3.4). Our program is not to make a specific choice but to consider instead general functions v^ε, ϕ^ε and f^ε. Our subsequent analysis will clearly depend on the choice of these functions, and our task is to look for conclusions which are robust to regularization details. This so-called *method of variable regularization* (which is worked out in detail in [13, §4.1]) leads to the formalism of the continuum limit which will be explained in Chapter 5 below.

4. Introducing an Interaction

Our next goal is to generalize the regularized fermionic projector P^ε of the previous chapter so as to include an interaction. Postponing the treatment of the regularization to Chapter 5, we shall now extend the definition of the fermionic projector of the vacuum (3.1) to the case with interaction. We outline the methods developed in [10, 11, 12]; see also [13, Chapter 2].

4.1. A Dirac Equation for the Fermionic Projector.

First, it is useful to recover (3.1) as a solution of a Dirac equation: Replacing the ordinary sum in (3.1) by a direct sum, we introduce the so-called *auxiliary fermionic projector* P^{aux} by

$$P^{\text{aux}}(x,y) = \bigoplus_{\beta=1}^{g} \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \tag{4.1}$$

(thus $P^{\text{aux}}(x,y)$ is represented by a $4g \times 4g$-matrix). It is a solution of the free Dirac equation

$$(i\not{\partial} - mY)\, P^{\text{aux}}(x,y) = 0\,, \tag{4.2}$$

where the *mass matrix* Y is composed of the rest masses corresponding to the three generations,

$$mY = \bigoplus_{\beta=1}^{g} m_\beta \tag{4.3}$$

(here $m > 0$ is an arbitrary mass parameter which makes Y dimensionless and will be useful for expansions in the mass parameter). The fermionic projector of the vacuum is obtained from P^{aux} by taking the *partial trace* over the generations defined by

$$P = \sum_{\alpha,\beta=1}^{g} (P^{\text{aux}})^\alpha_\beta\,. \tag{4.4}$$

The obvious idea for introducing an interaction is to replace the free Dirac equation (4.2) by a Dirac equation with interaction,

$$(i\not{\partial} + \mathcal{B} - mY)\, P^{\text{aux}}(x,y) = 0\,, \tag{4.5}$$

where $\mathcal{B}$ is a general perturbation operator, and to introduce the fermionic projector again by taking the partial trace (4.4). In order to ensure that the resulting fermionic projector is again symmetric (2.3), we generalize the inner product (2.2) to the wave functions of the auxiliary Dirac equation by setting

$$<\Psi_{\text{aux}}|\Phi_{\text{aux}}> = \sum_{\beta=1}^{g} \int_M \overline{\Psi^\beta_{\text{aux}}(x)}\, \Phi^\beta_{\text{aux}}(x)\, d^4x\,, \tag{4.6}$$

and demand that the auxiliary fermionic projector is symmetric with respect to this new inner product,

$$<P^{\text{aux}}\Psi_{\text{aux}}|\Phi_{\text{aux}}> = <\Psi_{\text{aux}}|P^{\text{aux}}\Phi_{\text{aux}}> \quad \text{for all } \Psi_{\text{aux}}, \Phi_{\text{aux}} \in C_0^\infty(M)^{4g} . \tag{4.7}$$

In order to obtain a coherent framework, we shall always assume that the auxiliary Dirac operator is also symmetric,

$$<(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi_{\text{aux}}|\Phi_{\text{aux}}> = <\Psi_{\text{aux}}|(i\partial\!\!\!/ + \mathcal{B} - mY)\Phi_{\text{aux}}> . \tag{4.8}$$

This equation gives a condition for the operator $\mathcal{B}$ describing the interaction. Apart from this condition and suitable regularity and decay assumptions, the operator $\mathcal{B}$ can be chosen arbitrarily; in particular, it can be time dependent. In typical applications, $\mathcal{B}$ is a multiplication or differential operator composed of *bosonic potentials* and fields. The choices of $\mathcal{B}$ relevant for this work will be discussed in §4.5 below.

4.2. **The Interacting Dirac Sea.** Clearly, the Dirac equation (4.5) has many different solutions, and thus in order to determine P^{aux}, we need to specify of which one-particle states P^{aux} should be composed. In the vacuum (4.1), this can be done by taking all the negative-energy solutions, i.e. all states on the lower mass shells $\{k^2 = m_\beta^2, k^0 < 0\}$. Unfortunately, the concept of negative energy does not carry over to the situation of a time-dependent interaction (4.5), because in this case the energy of the Dirac wave functions is not conserved; this is the so-called *external field problem* (see [13, §2.1]). The clue for resolving this problem is the observation that the negative-energy states in (4.1) can be characterized alternatively using the causality of the Dirac Green's functions in a specific way. This causal approach generalizes to the situation (4.5) and allows to extend the concept of the Dirac sea to the time-dependent setting. It gives rise to a unique definition of a fermionic projector P^{sea} in terms of a power series in $\mathcal{B}$. More precisely, the so-called *causal perturbation expansion* expresses P^{sea} as sums of operator products

$$P^{\text{sea}} = \sum_{k=0}^\infty \sum_{\alpha=0}^{\alpha_{\max}(k)} c_\alpha \, C_{1,\alpha} \, \mathcal{B} \, C_{2,\alpha} \, \mathcal{B} \, \cdots \, \mathcal{B} \, C_{k+1,\alpha} \, , \tag{4.9}$$

where the factors $C_{l,\alpha}$ are the Green's functions or fundamental solutions of the free Dirac equation (4.2), and the c_α are combinatorial factors (for details see [10] and [13, §2.2, §2.3]; for a more recent account on idempotence and unitarity questions see [21]). In the language of Feynman diagrams, each summand in (4.9) is a tree diagram. These tree diagrams are all finite, provided that $\mathcal{B}$ satisfies suitable regularity and decay assumptions at infinity (see [13, Lemma 2.2.2.]).

4.3. **Introducing Particles and Anti-Particles.** The fermionic projector P^{sea} is interpreted as a generalization of completely filled Dirac seas to the interacting situation (4.5). In order to bring particles and anti-particles into the system, we add the projectors on states $\Psi_1, \ldots, \Psi_{n_f}$ which are *not* contained in the image of the operator P^{sea} (the particle states) and subtract the projectors on states $\Phi_1, \ldots, \Phi_{n_a}$ which are in the image of P^{sea} (the anti-particle states),

$$P^{\text{aux}}(x,y) = P^{\text{sea}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_f} \Psi_k(x) \overline{\Psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_a} \Phi_l(x) \overline{\Phi_l(y)} \, . \tag{4.10}$$

Finally, the fermionic projector is again obtained by taking the partial trace over the generations (4.4).

The wave functions in (4.10) are normalized such that they are orthonormal with respect to the usual integral over the probability density,

$$\int_{\mathbb{R}^3} (\overline{\Psi_k} \gamma^0 \Psi_l)(t, \vec{x})\, d\vec{x} = \delta_{kl} = \int_{\mathbb{R}^3} (\overline{\Phi_k} \gamma^0 \Phi_l)(t, \vec{x})\, d\vec{x}\,. \tag{4.11}$$

The factors $\pm\frac{1}{2\pi}$ in (4.10) are needed for the proper normalization of the fermionic states with respect to the inner product (2.2). The fact that this last inner product is indefinite has the effect that the normalization factors in (4.10) have the opposite sign from what one would have expected in Hilbert spaces. These normalization conditions are derived in [13, §2.6] by considering the system in finite 3-volume and taking the infinite volume limit. For the dependence of the normalization constants on the global geometry of space-time see [24].

4.4. **The Light-Cone Expansion and Resummation.** We now outline the methods for analyzing the fermionic projector in position space (for details see [11, 12]). The following notion is very useful for describing the structure of the singularities on the light cone.

Definition 4.1. *A distribution $A(x,y)$ on $M \times M$ is of the order $\mathcal{O}((y-x)^{2p})$, $p \in \mathbb{Z}$, if the product*

$$(y-x)^{-2p}\, A(x,y)$$

is a regular distribution (i.e. a locally integrable function). It has the **light-cone expansion**

$$A(x,y) = \sum_{j=g_0}^{\infty} A^{[j]}(x,y)$$

with $g_0 \in \mathbb{Z}$ if the distributions $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p \geq g$,

$$A(x,y) - \sum_{j=g_0}^{p} A^{[j]}(x,y) \qquad \textit{is of the order } \mathcal{O}((y-x)^{2p+2})\,.$$

Thus the light-cone expansion is an expansion in the orders of the singularity on the light cone. As the main difference to a Taylor expansion, the expansion parameter $(y-x)^2$ vanishes for any fixed x on an unbounded set, namely the whole light cone centered at x. In this sense, the light-cone expansion is a *nonlocal* expansion.

For a convenient formulation of the light-cone expansion of the fermionic projector, it is helpful to work with a *generating function*, i.e. a power series in a real parameter $a > 0$ whose coefficients are functions in $(y-x)^2$ which are of increasing order on the light cone. The first ansatz for such a generating function is the Fourier transform $T_a(x,y)$ of the lower mass shell with $k^2 = a$,

$$T_a(x,y) = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - a)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{4.12}$$

Carrying out the Fourier integral and expanding the resulting Bessel functions, one obtains

$$\begin{aligned} T_a(x,y) = &-\frac{1}{8\pi^3} \left(\frac{\mathrm{PP}}{\xi^2} + i\pi\delta(\xi^2)\,\varepsilon(\xi^0) \right) \\ &+ \frac{a}{32\pi^3} \sum_{j=0}^\infty \left(\log|a\xi^2| + c_j + i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0) \right) \frac{(-1)^j}{j!\,(j+1)!} \frac{(a\xi^2)^j}{4^j}\,, \end{aligned} \tag{4.13}$$

where we again used the abbreviation $\xi = y - x$, and ϵ denotes the sign function (i.e. $\epsilon(x) = 1$ if $x \geq 0$ and $\epsilon(x) = -1$ otherwise). The real coefficients c_j are given explicitly in [13, §2.5]. Unfortunately, due to the factor $\log|a\xi^2|$, the expression (4.13) is not a power series in a. In order to bypass this problem, we simply remove the logarithms in a by subtracting suitable counter terms,

$$T^{\mathrm{reg}}_a(x,y) := T_a(x,y) - \frac{a}{32\pi^3}\,\log|a| \sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!} \frac{(a\xi^2)^j}{4^j}\,. \tag{4.14}$$

The resulting distribution T^{reg}_a is a power series in a, and it is indeed the right choice for our generating function. We denote its coefficients by

$$T^{(n)} = \left(\frac{d}{da} \right)^n T^{\mathrm{reg}}_a \Big|_{a=0} \qquad (n = 0, 1, 2, \ldots) \tag{4.15}$$

and also introduce $T^{(-1)}$ via the distributional equation

$$\frac{\partial}{\partial x^k} T^{(0)}(x,y) = \frac{1}{2}\,(y-x)_k\, T^{(-1)}(x,y)\,. \tag{4.16}$$

Combining Fourier techniques with methods of hyperbolic partial differential equations, one can perform the light-cone expansion of each summand of the perturbation series (4.9). After suitably rearranging all the resulting contributions, one can partially carry out the infinite sums. This so-called *resummation* gives rise to an expansion of the interacting fermionic projector of the form

$$\begin{aligned} P^{\mathrm{sea}}(x,y) = &\sum_{n=-1}^\infty \sum_k m^{p_k} (\text{phase-inserted nested line integrals}) \times T^{(n)}(x,y) \\ &+ P^{\mathrm{le}}(x,y) + P^{\mathrm{he}}(x,y)\,. \end{aligned} \tag{4.17}$$

Here the n-summands describe the different orders of the singularities on the light cone, whereas the k-sum describes all contributions to a given order on the light cone. The phase-inserted nested line integrals involve $\mathcal{B}$ and its partial derivatives, possibly sandwiched between time-ordered exponentials of chiral potentials. Since these nested line integrals are smooth functions in x and y, the series in (4.17) is a light-cone expansion in the sense of Definition 4.1, provided that the k-sum is finite for every n. This is indeed the case if $\mathcal{B}$ is composed of scalar, pseudoscalar and chiral potentials [12], whereas for a more general perturbation operator $\mathcal{B}$ this condition still needs to be verified. This expansion is *causal* in the sense that it depends on $\mathcal{B}$ and its partial derivatives only along the line segment $\overline{xy}$. The contributions P^{le} and P^{he}, on the other hand, are not causal but depend instead on the global behavior of $\mathcal{B}$ in space-time. They can be written as a series of functions which are all smooth in x and y. Their different internal structure gives rise to the names *non-causal low energy contribution* and *non-causal high energy contribution*, respectively.

For an introduction to the light-cone expansion and the needed mathematical methods we refer to [11] and [12] or to the exposition in [13, §2.5]. The formulas of the light-cone expansion needed in this work are compiled in Appendix B.

4.5. **Clarifying Remarks.** The above constructions require a few explanations. We first point out that, although we are working with one-particle wave functions, the ansatz for the fermionic projector (4.10) describes a many-particle quantum state. In order to get a connection to the Fock space formalism, one can take the wedge product of the wave functions Ψ_k and Φ_l to obtain a vector in the fermionic Fock space (for details see [13, Appendix A]). We conclude that (4.10) describes *second-quantized fermions*. For the description of entangled states see [19].

One should keep in mind that at this stage, the form of the potential $\mathcal{B}$ has not been specified; it can be an arbitrary operator. Indeed, we regard the operator $\mathcal{B}$ only as a device for modifying or perturbing the fermionic projector. We do not want to preassume which of these perturbations are physically relevant; instead, we want to select the relevant perturbations purely on the basis of whether they are admissible for minimizers of our action principle (2.9). In other words, our action principle should decide how the physical interaction looks like, even quantitatively in the sense that our action principle should determine the corresponding field equations. Following this concept, we should choose $\mathcal{B}$ as general as possible, even allowing for potentials which are usually not considered in physics. We now give a brief overview over the potentials which will be of relevance in the present work. The simplest example is to choose an electromagnetic potential[1],

$$\mathcal{B} = \not{A} \,. \tag{4.18}$$

More generally, one can choose *chiral potentials*, which may be non-diagonal in the generations,

$$\mathcal{B} = \chi_L \,\not{A}_R + \chi_R \,\not{A}_L \,, \tag{4.19}$$

where $A_{L/R} = (A^i_{L/R})^\alpha_\beta$ with generation indices $\alpha, \beta = 1, \ldots, g$ and a vector index $i = 0, \ldots, 3$ (here $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5)$ are the chiral projectors, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the usual pseudoscalar matrix). To describe a *gravitational field*, one needs to choose for $\mathcal{B}$ a differential operator of first order; more precisely,

$$\mathcal{B} = \mathcal{D} - i\not{\partial} \,, \tag{4.20}$$

where $\mathcal{D}$ is the Dirac operator in the presence of a gravitational field.

The above choices of $\mathcal{B}$ are of course motivated by physical fields observed in nature. However, we point out that we do not assume any field equations. Thus the electromagnetic potential in (4.18) does not need to satisfy Maxwell's equations, in (4.19) we do not assume the Yang-Mills equations, and in (4.20) the Einstein equations are not imposed. This is because, as already pointed out above, our goal is to derive the classical field equations from our action principle (2.9).

Apart from the above choices of $\mathcal{B}$ motivated from physics, one can also choose other physically less obvious operators, like for example *scalar* or *pseudoscalar potentials*,

$$\mathcal{B} = \Phi + i\gamma^5 \Xi \tag{4.21}$$

[1] For convenience we shall always omit the coupling constant e in the Dirac equation. Our convention is obtained from the usual choice $\mathcal{B} = e\not{A}$ by the transformation $A \to e^{-1}A$. The coupling constant clearly reappears in the Maxwell equations, which we write in natural units and with the Heaviside-Lorentz convention as $\partial_{jk}A^k - \Box A_k = e^2\overline{\Psi}\gamma_k\Psi$. As usual, the fine structure constant is given by $\alpha = e^2/(4\pi)$.

with $\Phi = \Phi^\alpha_\beta$, $\Xi = \Xi^\alpha_\beta$ and $\alpha, \beta = 1, \ldots, g$. Furthermore, one can consider a *scalar differential operator*,

$$\mathcal{B} = i\Phi^j \partial_j \,,$$

or a higher order differential operator. More specifically, we will find a *pseudoscalar differential potential* useful,

$$\mathcal{B} = \gamma^5 \left(v^j \partial_j + \partial_j v^j \right) \,.$$

It is worth noting that one does not need to restrict attention to differential operators. Indeed, $\mathcal{B}$ can also be an integral operator, in which case we talk of *nonlocal potentials*. Clearly, one can also take linear combinations of all the above operators $\mathcal{B}$.

Next, it is worth noting that for the moment, we consider $\mathcal{B}$ as a-priori given, and thus at this stage, our system consists of Dirac particles moving in an *external field*. However, our action principle (2.9) will give relations between the potentials contained in $\mathcal{B}$ and the Dirac wave functions in (4.10), and thus these potentials will be influenced by the Dirac wave functions. This leads to a mutual coupling of the potentials to the the Dirac wave functions, giving rise to a fully interacting system. We also point out that the potentials and fields contained in $\mathcal{B}$ should be regarded as *classical*. Indeed, in this paper we will always work with classical bosonic fields. However, as is worked out in [19], the framework of the fermionic projector also allows for the description of second-quantized bosonic fields.

4.6. **Relation to Other Approaches.** Having outlined our approach, we can now give a short review of related works. In order to get a connection to our description of the Dirac sea in §4.2, we begin with the construction of quantum fields in an external field. Historically, this problem was first analyzed in the *Fock space formalism*. Klaus and Scharf [33, 34] considered the Fock representation of the electron-positron field in the presence of a static external field. They noticed that the Hamiltonian needs to be regularized by subtracting suitable counter terms which depend on the external field. They also noticed that the electron-positron field operators in the external field form a Fock representation on the standard Fock space of free fields only if the external field satisfies a certain regularity condition. This regularity condition is quite restrictive and excludes many cases of physical interest (in particular a magnetic field [36] and a Coulomb potential [32]). These results show that different external fields in general give rise to nonequivalent Fock representations of the electron-positron field operators. More recently, in [27, 26] the vacuum state was constructed for a system of Dirac particles with electrostatic interaction in the Bogoliubov-Dirac-Fock approximation. The conclusion of this analysis is that for mathematical consistency, one must take into account all the states forming the Dirac sea. Furthermore, the interaction mixes the states in such a way that it becomes impossible to distinguish between the particle states and the states of the Dirac sea.

In the time-dependent setting, Fierz and Scharf [9] proposed that the Fock representation should be adapted to the external field as measured by a local observer. Then the Fock representation becomes time and observer dependent. This implies that the distinction between particles and anti-particles no longer has an invariant meaning, but it depends on the choice of an observer. In this formulation, the usual particle interpretation of quantum states only makes sense for the in- and outgoing scattering states, but it has no invariant meaning for intermediate times. This picture has been confirmed in a rigorous self-consistent manner in [30]. For a related approach which allows for the construction of quantum fields in the presence of an external magnetic field see [6]. In all the above approaches, the Dirac sea leads to divergences, which must be treated by an ultraviolet regularization and suitable counter terms.

As an alternative to working with Fock spaces, one can use the so-called *point splitting renormalization method*, which is particularly useful for renormalizing the expectation value of the energy-momentum tensor [3]. The idea is to replace a function of one variable $T(x)$ by a two-point distribution $T(x, y)$, and to take the limit $y \to x$ after subtracting suitable singular distributions which take the role of counter terms. Analyzing the singular structure of the counter terms leads to the so-called *Hadamard condition* (see for example [25]). Reformulating the Hadamard condition for the two-point function as a local spectral condition for the wave front set [39] turns out to be very useful for the axiomatic formulation of free quantum fields in curved space-time. As in the Fock space formalism, in the point splitting approach the particle interpretation depends on the observer. This is reflected mathematically by the fact that the Hadamard condition specifies the two-point distribution only up to smooth contributions, thus leaving the smooth particle wave functions undetermined. For a good introduction to free quantum fields in curved space-time we refer to the recent book [1].

As mentioned at the beginning of §4.5, in our approach the connection to the Fock space formalism is obtained by choosing a basis of the image of the fermionic projector and taking the wedge product of the basis vectors (for details see [13, Appendix A] or [19]). If in this construction the states of the Dirac sea are taken into account, we get precisely the framework in [9]. The connection to the Hadamard condition is even closer. Indeed, considering the light-cone expansion locally for y near x, the summands in (4.17) coincide precisely with the singular distributions in the Hadamard construction. Since the non-causal contributions P^{he} and P^{le} are smooth functions, we conclude that the integral kernel of the fermionic projector satisfies the Hadamard condition, provided that the perturbation expansions for P^{he} and P^{le} converge (a subtle technical problem which we do not want to enter here). Thus in a given external field, $P^{\text{sea}}(x, y)$ can be interpreted as the two-point function, and using the methods of [39, 1] one could construct the corresponding free quantum field theory.

A major difference of our approach is that our framework allows for the description of an *interacting theory*, where the coupling of the fermions to bosonic fields and the back-reaction of the bosonic fields to the fermions is taken into account. In this setting, the interaction is described by our action principle (2.8). The mathematical framework is no longer equivalent to standard quantum field theory. In particular, $P(x, y)$ *cannot be interpreted as the two-point function* of a corresponding quantum field theory, simply because the notions of quantum field theory can no longer be used. But we still get a connection to the Feynman diagrams of quantum field theory (as will be explained in §8.4 below).

Another major difference of our approach is that the distribution P^{sea} as defined by the causal perturbation expansion (4.9) distinguishes a unique state which can be interpreted as the fermionic vacuum state where all Dirac seas are completely filled. Thus working relative to this distinguished state, there is a unique *observer independent particle interpretation*, even at intermediate times (see [10, Section 5] for a discussion of this point). At first sight, this distinguished particle interpretation might seem of purely academic interest, because P^{sea} is defined globally in space-time and is thus not accessible to a local observer. However, our action principle (2.8) does have access to quantities defined globally in space-time, and in this way the distinguished particle interpretation enters the physical equations. More precisely, $P^{\rm sea}$ drops out of the Euler-Lagrange equations corresponding to our action principle, up to terms which are well-defined and explicitly computable, even determining the smooth contributions. In this way, the arbitrariness of working modulo smooth contributions (in the Hadamard condition) or modulo regular counter terms (in the Fock space formalism) is removed. The corresponding smooth contributions to the physical equations will be analyzed in §8.1 and Appendix D. They are nonlocal and violate causality, as will be explained in §8.2.

A frequently asked question is how our approach relates to Connes' *noncommutative geometry* [5]. In particular, can our approach be thought of as a Lorentzian version of noncommutative geometry? Clearly, both approaches have in common that the Dirac operator plays a central role. Moreover, the light-cone expansion is the Lorentzian analog of local expansions of the resolvent near the diagonal. A major difference is that instead of considering the whole spectrum of the Dirac operator, we only consider the eigenspaces corresponding to the masses m_α of the Dirac particles of our system. Furthermore, we only take "half the eigenspaces" by constructing Dirac seas and build in additional particle and anti-particle states (4.10). Another major difference concerns the mathematical structure of our action principle (2.8). Namely, this action cannot be thought of as a spectral action, because it is impossible to express it in terms of spectral properties of the Dirac operator. This is obvious from the fact that in (2.7) and (2.8) we perform a nonlinear (and even non-analytic) transformation of the kernel $P(x,y)$ before integrating over x and y. As a consequence, there is no connection to a regularized trace or Hilbert-Schmidt norm of P. The specific form of our action principle makes it possible to regard the structures of Minkowski space as emerging from a self-organization of the wave functions in discrete space time (see [20]), an idea which has no correspondence in noncommutative geometry. On the other hand, noncommutative geometry has deep connections to Riemannian geometry, index theory and number theory. We conclude that despite superficial similarities, the aims, ideas and methods of our approach are quite different from those in noncommutative geometry.

5. The Continuum Limit

In Chapter 3 we described the vacuum by a family of regularized fermionic projectors P^ε. Our next goal is to use the information on the regularized vacuum to also regularize the fermionic projector with interaction. We cannot expect that this information will suffice to determine the interacting fermionic projector in all details, because it is unknown how the interaction affects the fermionic projector on the microscopic scale. But as shown in [13, Chapter 4 and Appendix D], there is a canonical method to regularize the formulas of the light-cone expansion (4.17). This method also gives a meaning to composite expressions as needed for the analysis of the action principle introduced in Chapter 2. In particular, it allows us to analyze the corresponding Euler-Lagrange equations in the continuum, taking into account the unknown regularization details by a finite number of free parameters. We now outline this method, relying for all technical issues on the detailed analysis in [13]. The method in §5.2 is a major improvement and simplification of the techniques in [13, Appendix F].

5.1. **Weak Evaluation on the Light Cone.** Our method is based on the physically reasonable *assumption of macroscopic potentials and wave functions* which states that both the bosonic potentials in (4.5) and the fermionic wave functions in (4.10) vary only on the macroscopic scale and are thus almost constant on the Planck scale. Then the idea is to try to regularize the perturbation expansion (4.9) in such a way that the interaction modifies the fermionic projector also only on the macroscopic scale. As exemplified in [13, Appendix D] in the perturbation expansion to first order, this idea can be realized by demanding that the perturbation expansion should be gauge invariant and satisfies a causality condition. Performing the light-cone expansion for the thus regularized perturbation expansion and using the form of the regularized vacuum minimizers as specified in Assumption 3.1, one obtains a simple regularization scheme for the continuum fermionic projector (4.17), which we now describe.

The non-causal contributions P^{le} and P^{he}, which are already smooth in x and y, are not regularized. Likewise, the smooth nested line-integrals are not regularized. Thus we only regularize the singularities of the factors $T^{(n)}$ on the light cone, and this is done by the replacement rule

$$m^p\, T^{(n)} \to m^p\, T^{(n)}_{[p]}\,, \tag{5.1}$$

where the factors $T^{(n)}_{[p]}(\xi)$ are smooth functions defined by Fourier integrals involving the functions v^ε, ϕ^ε and f^ε in the ansatz (3.4). If the partial trace is taken, we clarify the handling of the generation index by accents. More precisely, we extend the replacement rule (5.1) to

$$\sum_{\alpha,\beta,\gamma_1,\ldots,\gamma_{p-1}=1}^{g} m^p\, \underbrace{Y^\alpha_{\gamma_1} \cdots Y^{\gamma_1}_{\gamma_2} \cdots Y^{\gamma_{p-1}}_\beta}_{p \text{ factors } Y}\, T^{(n)} \to m^p\, \acute{Y} Y \cdots \grave{Y}\, T^{(n)}_{[p]}\,, \tag{5.2}$$

and in particular[2]

$$\sum_{\alpha,\beta=1}^{g} m^0\, \delta^\alpha_\beta\, T^{(n)} \to m^0\, g\, T^{(n)}_{[0]} \qquad \text{and} \qquad \sum_{\alpha,\beta=1}^{g} m\, Y^\alpha_\beta\, T^{(n)} \to m\, \hat{Y}\, T^{(n)}_{[1]}\,.$$

Fortunately, the rather complicated detailed form of the factors $T^{(n)}_{[p]}$ will not be needed here, because these functions can always be treated symbolically using the following simple calculation rules. In computations one may treat the $T^{(n)}_{[p]}$ like complex functions. However, one must be careful when tensor indices of factors $\not{\xi}$ are contracted with each other. Naively, this gives a factor ξ^2 which vanishes on the light cone and thus changes the singular behavior on the light cone. In order to describe this effect correctly, we first write every summand of the light cone expansion (4.17) such that it involves at most one factor $\not{\xi}$ (this can always be arranged using the anti-commutation relations of the Dirac matrices). We now associate every factor $\not{\xi}$ to the corresponding factor $T^{(n)}_{[p]}$. In simple calculations, this can be indicated by putting brackets around

[2] In contrast to the convention in [13], here we always write out the factors g counting the number of generations (in [13], the factor g was absorbed into the factors $T^{(n)}_{[0]}$ and $\overline{T^{(n)}_{[0]}}$). The shorter notation in [13] has the disadvantage that reinserting the factors of g in the end is a potential source of confusion and may lead to computational errors. In the convention here, the factors $T^{(n)}_\circ$ without regularization always coincide with the distributions (4.15) and (4.16).

the two factors, whereas in the general situation we add an index to the factor $\xi\!\!\!/$, giving rise to the replacement rule

$$m^p \, \xi\!\!\!/ \, T^{(n)} \to m^p \, \xi\!\!\!/^{(n)}_{[p]} \, T^{(n)}_{[p]} \, .$$

The factors $\xi\!\!\!/$ which are contracted to other factors $\xi\!\!\!/$ are called *inner factors*. The contractions of inner factors can be handled with the so-called *contraction rules*

$$(\xi\!\!\!/^{(n)}_{[p]})^j \, (\xi\!\!\!/^{(n')}_{[p']})_j = \frac{1}{2} \left(z^{(n)}_{[p]} + z^{(n')}_{[p']} \right) \tag{5.3}$$

$$(\xi\!\!\!/^{(n)}_{[p]})^j \, \overline{(\xi\!\!\!/^{(n')}_{[p']})_j} = \frac{1}{2} \left(z^{(n)}_{[p]} + \overline{z^{(n')}_{[p']}} \right) \tag{5.4}$$

$$z^{(n)}_{[p]} \, T^{(n)}_{[p]} = -4 \left(n \, T^{(n+1)}_{[p]} + T^{(n+2)}_{\{p\}} \right) , \tag{5.5}$$

which are to be complemented by the complex conjugates of these equations. Here the factors $z^{(n)}_{[p]}$ can be regarded simply as a book-keeping device to ensure the correct application of the rule (5.5). The factors $T^{(n)}_{\{p\}}$ have the same scaling behavior as the $T^{(n)}_{[p]}$, but their detailed form is somewhat different; we simply treat them as a new class of symbols[3]. In cases where the lower index does not need to be specified we write $T^{(n)}_\circ$. After applying the contraction rules, all inner factors ξ have disappeared. The remaining so-called *outer factors* ξ need no special attention and are treated like smooth functions.

Next, to any factor $T^{(n)}_\circ$ we associate the *degree* $\deg T^{(n)}_\circ$ by

$$\deg T^{(n)}_\circ = 1 - n \, .$$

The degree is additive in products, whereas the degree of a quotient is defined as the difference of the degrees of numerator and denominator. The degree of an expression can be thought of as describing the order of its singularity on the light cone, in the sense that a larger degree corresponds to a stronger singularity (for example, the contraction rule (5.5) increments n and thus decrements the degree, in agreement with the naive observation that the function $z = \xi^2$ vanishes on the light cone). Using formal Taylor expansions, we can expand in the degree. In all our applications, this will give rise to terms of the form

$$\eta(x,y) \, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad \text{with } \eta(x,y) \text{ smooth} \, . \tag{5.6}$$

Here the quotient of the two monomials is referred to as a *simple fraction*.

A simple fraction can be given a quantitative meaning by considering one-dimensional integrals along curves which cross the light cone transversely away from the origin $\xi = 0$. This procedure is called *weak evaluation on the light cone*. For our purpose, it suffices to integrate over the time coordinate $t = \xi^0$ for fixed $\vec{\xi} \neq 0$. Using the symmetry under reflections $\xi \to -\xi$, it suffices to consider the upper light cone $t \approx |\vec{\xi}|$.

[3]We remark that the functions $T^{(n)}_{\{p\}}$ will be of no relevance in this paper, because they contribute to the EL equations only to degree three and lower; see §9.4.

The resulting integrals will diverge if the regularization is removed. The leading contribution for small ε can be written as

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt\; \eta(t,\vec{\xi})\; \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\; \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\; \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \;\approx\; \eta(|\vec{\xi}|,\vec{\xi})\; \frac{c_{\text{reg}}}{(i|\vec{\xi}|)^L}\; \frac{\log^r(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}}\,, \tag{5.7}$$

where L is the degree and c_{reg}, the so-called *regularization parameter*, is a real-valued function of the spatial direction $\vec{\xi}/|\vec{\xi}|$ which also depends on the simple fraction and on the regularization details (the error of the approximation will be specified below). The integer r describes a possible logarithmic divergence; we postpone its discussion until when we need it (see §7.3). Apart from this logarithmic divergence, the scalings in both ξ and ε are described by the degree.

When analyzing a sum of expressions of the form (5.6), one must know if the corresponding regularization parameters are related to each other. In this respect, the *integration-by-parts rules* are important, which are described symbolically as follows. On the factors $T^{(n)}_\circ$ we introduce a derivation ∇ by

$$\nabla T^{(n)}_\circ = T^{(n-1)}_\circ\,.$$

Extending this derivation with the Leibniz and quotient rules to simple fractions, the integration-by-parts rules states that

$$\nabla \left(\frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\; \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\; \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \right) = 0\,. \tag{5.8}$$

These rules give relations between simple fractions (the name is motivated by the fact that when evaluating (5.8) weakly on the light cone (5.7), the rules state that the integral over a derivative vanishes). Simple fractions which are not related to each by the integration-by-parts rules are called *basic fractions*. As shown in [13, Appendix E], there are no further relations between the basic fractions. Thus the corresponding *basic regularization parameters* are linearly independent.

We next specify the error of the above expansions. By not regularizing the bosonic potentials and fermionic wave functions, we clearly miss the

$$\text{higher orders in } \varepsilon/\ell_{\text{macro}}\,. \tag{5.9}$$

Furthermore, in (5.7) we must stay away from the origin, meaning that we neglect the

$$\text{higher orders in } \varepsilon/|\vec{\xi}|\,. \tag{5.10}$$

The higher oder corrections in $\varepsilon/|\vec{\xi}|$ depend on the fine structure of the regularization and thus seem unknown for principal reasons. Neglecting the terms in (5.9) and (5.10) also justifies the formal Taylor expansion in the degree. Finally, we disregard the higher order corrections in the parameter $\varepsilon_{\text{shear}}$ in (3.6).

The above symbolic computation rules give a convenient procedure to evaluate composite expressions in the fermionic projector, referred to as the *analysis in the continuum limit*: After applying the contraction rules and expanding in the degree, we obtain equations involving a finite number of terms of the form (5.6). By applying the integration-by-parts rules, we can arrange that all simple fractions are basic fractions. We evaluate weakly on the light cone (5.7) and collect the terms according to their scaling in ξ. Taking for every given scaling in ξ only the leading pole in ε, we obtain equations which involve linear combinations of smooth functions and basic regularization parameters. We consider the basic regularization parameters as empirical parameters describing the unknown microscopic structure of space-time. We thus end up with equations involving smooth functions and a finite number of free parameters. We point out that these free parameters cannot be chosen arbitrarily because they might be constrained by inequalities (see the discussion after [13, Theorem E.1]). Also, the values of the basic regularization parameters should ultimately be justified by an analysis of vacuum minimizers of our variational principle (as discussed at the end of Chapter 3).

In view of the later considerations in §8.1, we point out that the above calculation rules are valid only *modulo smooth contributions* to the fermionic projector. This can be understood from the fact that these rules only deal with the terms of the series in (4.17), but they do not take into account the smooth non-causal high and low energy contributions. But the above calculation rules affect these smooth contributions as well. To give a simple example, we consider the distribution $T^{(0)}$, which according to (4.13)–(4.15) is given by

$$T^{(0)} = -\frac{1}{8\pi^3} \left(\frac{\text{PP}}{\xi^2} \,+\, i\pi\delta(\xi^2)\, \varepsilon(\xi^0) \right).$$

Multiplying by $z = \xi^2$ in the distributional sense gives a constant

$$zT^{(0)} = -\frac{1}{8\pi^3}\,. \tag{5.11}$$

On the other hand, the contraction rule (5.5) yields

$$z^{(0)}_{[0]} T^{(0)}_{[p]} = -4 T^{(2)}_{\{p\}}\,. \tag{5.12}$$

The last relation gives much finer information than the distributional equation (5.11), which is essential when we want to evaluate composite expressions weakly on the light cone (5.7). However, the constant term in (5.11) does not appear in (5.12). The way to think about this shortcoming is that this constant term is smooth and can thus be taken into account by modifying the corresponding low energy contribution $P^{\text{le}}(x,y)$ in (4.17). Indeed, this situation is not as complicated as it might seem at first sight. Namely, the smooth contributions to the fermionic projector need special attention anyway and must be computed using the resummation technique explained in Appendix D. When performing this resummation, we can in one step also compute all the smooth contributions which were not taken into account by the formalism of the continuum limit. Thus altogether we have a convenient method where we first concentrate on the singularities on the light cone, whereas the neglected smooth contributions will be supplemented later when performing the resummation.

We note that the above procedure needs to be modified for the description of *gravity*, because in this case we need relations between terms involving different powers of the Planck length ε. We postpone this generalization to a future paper.

5.2. **The Euler-Lagrange Equations in the Continuum Limit.** We now return to the action principle of Chapter 2. Our goal is to bring the conditions for a minimizer (2.10) and (2.11) into a form suitable for the analysis in the continuum limit. We begin by considering a smooth family $P(\tau)$ of fermionic projectors and compute the corresponding first variation of the action. We differentiate (2.11) with respect to τ, treating the constraint (2.10) with a Lagrange multiplier. For convenience, we introduce the functional

$$\mathcal{S}_\mu[P] \stackrel{\text{formally}}{=} \iint_{M\times M} \mathcal{L}_\mu[A_{xy}]\, d^4x\, d^4y \qquad \text{with} \qquad \mathcal{L}_\mu[A] = |A^2| - \mu |A|^2 \,. \tag{5.13}$$

Choosing $\mu = \frac{1}{4}$ gives precisely the action (2.8), whereas by allowing a general $\mu \in \mathbb{R}$ we take into account the Lagrange multiplier. We thus obtain the condition

$$0 = \delta \mathcal{S}_\mu[P] = \iint_{M\times M} \mathrm{Re}\,\mathrm{Tr}\left\{ \nabla \mathcal{L}_\mu[A_{xy}]\, \delta P(x,y) \right\} d^4x\, d^4y \,, \tag{5.14}$$

where $\delta P := P'(0)$. Here we consider $P(y,x)$ via

$$P(y,x) = P(x,y)^* \equiv \gamma^0 P(x,y)^\dagger \gamma^0$$

as a function of $P(x,y)$, and ∇ denotes the gradient where the real and imaginary parts of $P(x,y)$ are treated as independent variables, i.e.

$$(\nabla f)^\alpha_\beta := \frac{\partial f}{\partial \mathrm{Re} P(x,y)^\beta_\alpha} - i \frac{\partial f}{\partial \mathrm{Im} P(x,y)^\beta_\alpha} \,, \tag{5.15}$$

and $\alpha, \beta = 1, \ldots, 4$ are spinor indices. Introducing the integral operator R with kernel

$$R(y,x) := \nabla \mathcal{L}_\mu[A_{xy}] \,, \tag{5.16}$$

we can write (5.14) as a trace of an operator product,

$$\delta \mathcal{S}_\mu[P] = \mathrm{Re}\,\mathrm{tr}\,(R\, \delta P) \,.$$

In order to get rid of the real part, it is convenient to replace R by its symmetric part. More precisely, introducing the symmetric operator Q with kernel

$$Q(x,y) = \frac{1}{4} \left(R(x,y) + R(x,y)^* \right) , \tag{5.17}$$

we can write the variation as

$$\delta \mathcal{S}_\mu[P] = 2\ \mathrm{tr}\,(Q\, \delta P) \,. \tag{5.18}$$

As explained before Definition 2.1, we want to vary the fermionic projector by unitary transformations in a compact region. Thus the family of fermionic projectors $P(\tau)$ should be of the form

$$P(\tau) = U^{-1}(\tau) P U(\tau) \tag{5.19}$$

with a smooth family $U(\tau)$ of unitary transformations in a fixed compact region K (see Definition 2.1) with $U(0) = \mathbb{1}$. Then the operator $B = -iU'(0)$ has the integral representation

$$(B\Psi)(x) = \int_M B(x,y)\, \Psi(y)\, d^4y$$

with a smooth compactly supported integral kernel $B \in C^\infty_0(K \times K, \mathbb{C}^{4\times 4})$. Differentiating (5.19) yields that $\delta P = i[P, B]$, and substituting this identity into (5.18) and cyclically commuting the operators inside the trace, we can rewrite the condition (5.14) as

$$0 = \mathrm{tr}\,([P,Q]B) \,.$$

Since B is arbitrary, we obtain the Euler-Lagrange (EL) equations

$$\boxed{[P,Q] = 0 \,,} \tag{5.20}$$

stating that two operators in space-time should commute. For more details on the derivation of the EL equations we refer to [13, §3.5].

When analyzing the commutator (5.20) in the continuum limit, the kernel $Q(x,y)$ can be evaluated weakly using the formula (5.7). The subtle point is that, according to (5.10), this weak evaluation formula only applies if x and y stay apart. But writing the commutator in (5.20) with integral kernels,

$$[P,Q](x,y) = \int_M \Big(P(x,z)\, Q(z,y) - Q(x,z)\, P(z,y) \Big)\, d^4z\,, \tag{5.21}$$

we also integrate over the regions $z \approx y$ and $z \approx x$ where the kernels $Q(z,y)$ and $Q(x,z)$ are ill-defined. There are several methods to resolve this difficulty, which all give the same end result. The cleanest method is the method of *testing on null lines*. We now explain the ideas and results of this last method, referring for the rigorous derivation to Appendix A (for other methods of testing see [13, Appendix F]). The idea is to take the expectation value of the commutator in (5.21) for two wave functions Ψ_1 and Ψ_2, one being in the kernel and one in the image of the operator P. Thus

$$P\Psi_1 = 0 \qquad \text{and} \qquad \Psi_2 = P\Phi \tag{5.22}$$

for a suitable wave function Φ. Then, using the symmetry of P with respect to the indefinite inner product (2.2), we find

$$<\!\Psi_1\,|\,[P,Q]\,\Phi\!> \,=\, <\!P\Psi_1\,|\,Q\Phi\!> \;-\; <\!\Psi_1\,|\,QP\Phi\!> \,=\, -\; <\!\Psi_1\,|\,Q\Psi_2\!> \;. \tag{5.23}$$

Now the commutator has disappeared, and the EL equations (5.20) give rise to the condition

$$0 = \,<\!\Psi_1 | Q\, \Psi_2\!> \,= \iint_{M\times M} Q(x,y)\; \Psi_1(x)\; \Psi_2(y)\; d^4x\, d^4y\,. \tag{5.24}$$

The hope is that by choosing suitable wave functions Ψ_1 and Ψ_2 of the form (5.22) having disjoint supports, we can evaluate the expectation value (5.24) weakly on the light cone (5.7), thus making sense of the EL equations in the continuum limit.

The basic question is to what extent the constraints (5.22) restrict the freedom in choosing the wave functions Ψ_1 and Ψ_2. For clarity, we here explain the situation in the simplified situation where P is composed of one free Dirac sea of mass m,

$$P(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \tag{5.25}$$

(but Q can be a general operator for which the methods of Chapter 5 apply). The generalization to several generations and a P with general interaction is worked out in Appendix A. In order to extract information from (5.24) and (5.7), it is desirable that the wave functions Ψ_1 and Ψ_2 are as much as possible localized in space-time. For the wave function Ψ_1, this requirement is easy to fulfill by removing a strip of width $\Delta\omega$ around the lower mass shell in momentum space. For example, we can construct a wave function supported near the origin by choosing for a given parameter $\delta > 0$ a smooth function η supported in the ball of radius δ in Euclidean $\mathbb{R}^4$ and setting

$$\Psi_1(x) = \int \frac{d^4k}{(2\pi)^4}\, \hat{\eta}(k)\; \chi_{\mathbb{R}\setminus[-\Delta\omega,\Delta\omega]}\Big(k^0 + \sqrt{|\vec{k}|^2+m^2}\Big)\; e^{-ikx}\,, \tag{5.26}$$

where $\hat{\eta}$ is the Fourier transform of η, and χ_I is the characteristic function defined by $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise. In the limit $\Delta\omega \searrow 0$, the characteristic function in (5.26) becomes the identity, so that Ψ_1 goes over to η. Moreover, for any $\Delta\omega > 0$, the function Ψ_1 is indeed in the kernel of the operator P, because it vanishes on the lower mass shell. Thus by choosing $\Delta\omega$ sufficiently small, we can arrange that Ψ_1 is arbitrarily close to η and satisfies the condition in (5.22) (indeed,

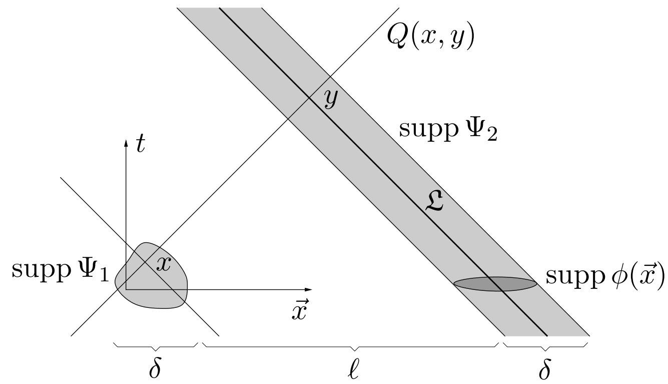

FIGURE 1. Intersection of the null line $\mathfrak{L}$ with the singular set of $Q(x, y)$.

in finite space-time volume one cannot choose $\Delta\omega$ arbitrarily small, leading to small corrections which will be specified in Appendix A; see Remark A.4).

The construction of Ψ_2 is a bit more difficult because Ψ_2 must lie in the image of P, and thus it must be a negative-energy solution of the Dirac equation $(i\partial\!\!\!/ - m)\Psi_2 = 0$. Due to current conservation, it is obviously not possible to choose Ψ_2 to be localized in space-time; the best we can do is to localize in space by considering a wave packet. According to the Heisenberg Uncertainty Principle, localization in a small spatial region requires large momenta, and thus we are led to considering an *ultrarelativistic wave packet* of negative energy moving along a null line $\mathfrak{L}$, which does not intersect the ball $B_\delta(0) \subset \mathbb{R}^4$ where Ψ_1 is localized. By a suitable rotation and/or a Lorentz boost of our reference frame $(t, \vec{x})$, we can arrange that

$$\mathfrak{L} = \{(\tau, -\tau + \ell, 0, 0) \text{ with } \tau \in \mathbb{R}\}$$

with $\ell > 0$. For Ψ_2 we take the ansatz

$$\Psi_2 = (i\partial\!\!\!/ + m) \left(e^{-i\Omega(t+x)}\, \phi(t + x - \ell, y, z) \right) + (\text{small corrections})\,, \tag{5.27}$$

where the smooth function ϕ is supported in $B_\delta(\vec{0}) \subset \mathbb{R}^3$, and the frequency $\Omega < 0$ as well as the length scales δ and ℓ are chosen in the range

$$\varepsilon \ll |\Omega|^{-1} \ll \delta \ll \ell, \ell_{\text{macro}}, m^{-1}\,. \tag{5.28}$$

The small corrections in (5.27) are due to the non-zero rest mass, the dispersion and the condition that Ψ_2 must have no contribution of positive-energy (for details see Appendix A).

Except for the small corrections to be specified in Appendix A, the support of the wave function Ψ_1 in (5.26) lies in $B_\delta(0)$, and thus it is disjoint from the support $B_\delta(\mathfrak{L})$ of the wave function Ψ_2 in (5.27). Hence the integrals in (5.24) only involve the region $x \neq y$ where $Q(x, y)$ is well-defined in the continuum limit. Furthermore, the null line $\mathfrak{L}$ intersects the null cone around x in precisely one point y for which $|\xi^0| = |\vec{\xi}| \sim \ell$ (see Figure 1). Since this intersection is transverse, we can evaluate the expectation value (5.24) with the help of (5.7). In view of the freedom in choosing the parameter ℓ and the direction of $\mathfrak{L}$, we conclude that (5.7) itself must vanish,

$$\boxed{Q(x, y) = 0 \quad \text{if evaluated weakly on the light cone}\,.} \tag{5.29}$$

The above consideration is made rigorous in Appendix A. More precisely, in Proposition A.2, the above arguments are extended to the setting involving several generations and a general interaction, and the scaling of the correction terms in (5.27) is specified to every order in perturbation theory. This proposition applies to our action principle (2.9) and all interactions to be considered here, thus justifying (5.29) in all cases of interest in this paper. Moreover, in Remark A.4 we consider the corrections to (5.26) which arise if the lifetime of the universe is finite. Using that this lifetime can be estimated by the time from the big bang as known from experiments, we show that the correction to (5.26) can indeed be neglected for our universe.

To summarize, we saw that within the formalism of the continuum limit, the commutator in (5.20) vanishes only if $Q(x,y)$ itself is zero. This result is the strongest condition we could hope for, because in view of (5.18) it implies that arbitrary first variations of the action vanish, even if we disregard the constraint that P must be a projector. We refer to (5.29) as the *Euler-Lagrange equations in the continuum limit*.

We finally remark that by replacing the null lines by null geodesics, the above method could immediately be generalized to situations involving a gravitational field. However, the estimates of Appendix A would become more demanding.

6. The Euler-Lagrange Equations to Degree Five

We proceed with the analysis of the EL equations in the continuum limit (5.29) using the methods outlined in Chapters 4 and 5. For clarity, we begin in the vacuum and then introduce more and more interaction terms. Furthermore, we consider the contributions to the EL equations to decreasing degree on the light cone. In this chapter, we consider the most singular contributions of degree five. The contributions of degree four will be analyzed in Chapter 7, whereas the contributions to even lower degree are discussed in Chapter 9.

We point out that many results of this chapter are already contained in [13, Chapters 5 and 6], albeit in a more general setting, while mainly restricting attention to one generation. In order to lay consistent foundations for the new calculations of Chapters 7–9, we here present all calculations in a self-contained way.

6.1. The Vacuum.

To perform the light-cone expansion of the fermionic projector of the vacuum, we first pull the Dirac matrices out of the Fourier integral (4.1) and use (4.12) to obtain

$$P^{\text{aux}}(x,y) = \bigoplus_{\beta=1}^{g} (i\partial\!\!\!/_x + m_\beta)\, T_a(x,y)\Big|_{a=m_\beta^2}\,.$$

After removing the logarithmic mass terms by the replacement $T_a \to T_a^{\text{reg}}$, the light-cone expansion reduces to a Taylor expansion in the mass parameter a. Restricting attention to the leading degree on the light cone, it suffices to consider the first term of this expansion. Using (4.16) and taking the partial trace (4.4), we obtain for the regularized fermionic projector (for the factors of g see Footnote 2 on page 19)

$$P(x,y) = \frac{ig}{2}\, \xi\!\!\!/\, T^{(-1)}_{[0]} + (\deg < 2)\,, \tag{6.1}$$

where for notational convenience we omitted the indices ${}^{-1}_{[0]}$ of the factor ξ, and where the bracket $(\deg < 2)$ stands for terms of degree at most one.

Using this formula for the fermionic projector, the closed chain (2.5) becomes

$$A_{xy} = \frac{g^2}{4}\,(\not\xi T^{(-1)}_{[0]})(\overline{\not\xi T^{(-1)}_{[0]}}) + \not\xi(\deg \leq 3) + (\deg < 3)\,, \tag{6.2}$$

where $\overline{\not\xi} := \overline{\xi_j}\gamma^j$. Its trace can be computed with the help of the contraction rules (5.4),

$$\mathrm{Tr}(A_{xy}) = g^2\,(\xi_j\overline{\xi^j})\,T^{(-1)}_{[0]}\,\overline{T^{(-1)}_{[0]}} = \frac{g^2}{2}\,(z+\overline{z})\,T^{(-1)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + (\deg < 3)\,.$$

Next we compute the square of the trace-free part of the closed chain,

$$\begin{aligned}\left(A_{xy} - \frac{1}{4}\,\mathrm{Tr}(A_{xy})\,\mathbb{1}\right)^2 &= \frac{g^4}{16}\left(\not\xi\overline{\not\xi} - \frac{z+\overline{z}}{2}\right)^2\left(T^{(-1)}_{[0]}\overline{T^{(-1)}_{[0]}}\right)^2 \\ &= \frac{g^4}{16}\left(\not\xi\overline{\not\xi}\not\xi\overline{\not\xi} - (z+\overline{z})\,\not\xi\overline{\not\xi} + \frac{1}{4}\,(z+\overline{z})^2\right)\left(T^{(-1)}_{[0]}\overline{T^{(-1)}_{[0]}}\right)^2 \\ &= \frac{g^4}{64}\,(z-\overline{z})^2\,\left(T^{(-1)}_{[0]}\overline{T^{(-1)}_{[0]}}\right)^2.\end{aligned}$$

Combining these formulas, we see that to leading degree, the closed chain is a solution of the polynomial equation

$$\left(A_{xy} - \frac{g^2}{8}\,(z+\overline{z})\,T^{(-1)}_{[0]}\overline{T^{(-1)}_{[0]}}\right)^2 = \left(\frac{g^2}{8}\,(z-\overline{z})\,T^{(-1)}_{[0]}\overline{T^{(-1)}_{[0]}}\right)^2. \tag{6.3}$$

We point out that the calculations so far are only formal, but they have a well-defined meaning in the formalism of the continuum, because to our end formulas we will be able to apply the weak evaluation formula (5.7). Having this in mind, we can interpret the roots of the polynomial in (6.3)

$$\lambda_+ = \frac{g^2}{4}\,(z\,T^{(-1)}_{[0]})\,\overline{T^{(-1)}_{[0]}} \qquad \text{and} \qquad \lambda_- = \frac{g^2}{4}\,T^{(-1)}_{[0]}\,\overline{(z\,T^{(-1)}_{[0]})}$$

as the eigenvalues of the closed chain. Using the contraction rule (5.5), these eigenvalues simplify to

$$\boxed{\lambda_+ = g^2\,T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + (\deg < 3)\,, \qquad \lambda_- = g^2\,T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}} + (\deg < 3)\,.} \tag{6.4}$$

The corresponding spectral projectors, denoted by $F_\pm$, are given by

$$F_+ = \frac{A_{xy} - \lambda_-}{\lambda_+ - \lambda_-}\,, \qquad F_+ = \frac{A_{xy} - \lambda_+}{\lambda_- - \lambda_+}\,;$$

a short calculation yields

$$\boxed{F_\pm = \frac{1}{2}\Big(\mathbb{1} \pm \frac{[\not\xi, \overline{\not\xi}]}{z-\overline{z}}\Big) + \not\xi(\deg \leq 0) + (\deg < 0)\,.} \tag{6.5}$$

Since in the formalism of the continuum limit, the factors z and $\overline{z}$ are treated as two different functions, we do not need to worry about the possibility that the denominator in (6.5) might vanish. Similarly, we can treat ξ and $\overline{\xi}$ simply as two different vectors. Then the matrices F_+ and F_- have rank two, so that the eigenvalues λ_+ and λ_- are both two-fold degenerate. A straightforward calculation yields

$$A_{xy} = \lambda_+ F_+ + \lambda_- F_- + \not\xi(\deg \leq 3) + (\deg < 3)\,, \tag{6.6}$$

showing that our spectral decomposition is indeed complete. An important general conclusion from (6.4) and (6.5) is that in the vacuum, the eigenvalues of the closed chain form a *complex conjugate pair*, and are both *two-fold degenerate*.

We now give the corresponding operator Q which appears in the EL equations of the continuum limit (5.29).

Proposition 6.1. *For the fermionic projector of the vacuum* (6.1)*, the operator Q as defined by* (5.17) *and* (5.16) *takes the form*

$$Q(x,y) = i\not{\xi}\, g^3\,(1-4\mu)\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} + (\deg < 5)\,. \tag{6.7}$$

In order not to distract from the main points, we first discuss the consequences of this result and derive it afterwards. According to the EL equations in the continuum limit (5.29), the expression (6.7) must vanish. This determines the value of the Lagrange multiplier $\mu = \frac{1}{4}$. Thus the action (5.13) reduces to the action in (2.8), and we conclude that

$$P \text{ is a critical point of } \mathcal{S}\,, \tag{6.8}$$

disregarding the constraint $\mathcal{T} = \text{const}$. This result can be understood immediately from the form of the Lagrangian (2.7) and the fact that the eigenvalues of A_{xy} form a complex conjugate pair. Namely, writing the spectral weights in (2.7) via (2.6) as sums over the eigenvalues $\lambda^{xy}_{\pm}$ (both of multiplicity two), we obtain

$$\mathcal{L}_{xy}[P] = \left(|\lambda_+| - |\lambda_-|\right)^2\,.$$

The expression $|\lambda_+| - |\lambda_-|$ clearly vanishes for a complex conjugate pair, and the fact that it appears quadratically is the reason why even first variations of $\mathcal{L}_{xy}[P]$ vanish, explaining (6.8).

The last argument applies whenever the eigenvalues of A_{xy} form a complex conjugate pair, making it possible to show that Q vanishes in a more general sense. First, a straightforward calculation shows that the eigenvalues of the closed chain A_{xy} form a complex conjugate pair to every degree on the light cone (for details see [13, §5.3]), and thus Q vanishes identically in the formalism of the continuum limit. Moreover, going beyond the formalism of the continuum limit, in [17] it is shown that there are regularizations of the vacuum for which the operator Q vanishes up to contributions which stay finite in the limit $\varepsilon \searrow 0$. Furthermore, in [17] it is shown that restricting attention to such regularizations does not give any constraints for the regularization parameters c_{reg} in (5.7). Since we are here interested in the singularities of $Q(x,y)$ in the limit $\varepsilon \searrow 0$ as described by the weak evaluation formula (5.7), we can in what follows assume that in the vacuum, the operator Q vanishes identically.

The remainder of this section is devoted to deriving the result of Proposition 6.1. For the derivation it is preferable to bypass (5.16) by determining Q directly from (5.18). For later use, we assume a more general spectral decomposition of A_{xy} with eigenvectors $\lambda^{xy}_1, \ldots, \lambda^{xy}_4$ and corresponding one-dimensional spectral projectors $F^{xy}_1, \ldots, F^{xy}_4$. This setting can be obtained from (6.6) by choosing pseudo-orthonormal bases in the degenerate eigenspaces and letting F^{xy}_k be the projectors onto the span of these basis vectors. It is convenient to choose these bases according to the following general lemma.

Lemma 6.2. *Assume that for a one-parameter family of fermionic projectors $P(\tau)$ and fixed $x, y \in M$, the matrices A_{xy} and A_{yx} are diagonalizable for all τ in a neighborhood* *of $\tau = 0$, and that the eigenvalues of the matrix $A_{xy}|_{\tau=0}$ are all non-real. Then the unperturbed closed chain A_{xy} has a spectral representation*

$$A_{xy}\big|_{\tau=0} = \sum_{k=1}^{4} \lambda_k^{xy} F_k^{xy} \tag{6.9}$$

with the following properties. The last two eigenvalues and spectral projectors are related to the first two by

$$\lambda_3^{xy} = \overline{\lambda_1^{xy}}\,, \quad F_3^{xy} = (F_1^{xy})^* \qquad \text{and} \qquad \lambda_4^{xy} = \overline{\lambda_2^{xy}}\,, \quad F_4^{xy} = (F_2^{xy})^*\,. \tag{6.10}$$

The first order perturbation $\delta A_{xy} = \partial_\tau A_{xy}|_{\tau=0}$ of the closed chain is diagonal in the bases of the non-trivial degenerate subspaces, i.e.

$$F_k^{xy} (\delta A_{xy}) F_l^{xy} = 0 \qquad \text{if } k \neq l \text{ and } \lambda_k^{xy} = \lambda_l^{xy}. \tag{6.11}$$

The closed chain A_{yx} has a corresponding spectral representation satisfying (6.9)–(6.11) *with all indices 'xy' are replaced by 'yx'. The spectral representations of A_{xy} and A_{yx} are related to each other by*

$$\lambda_k^{xy} = \lambda_k^{yx} \qquad \text{and} \qquad F_k^{xy}\, P(x,y) = P(x,y)\, F_k^{yx}\,. \tag{6.12}$$

Proof. By continuity, the eigenvalues of the matrix A_{xy} are non-real in a neighborhood of $\tau = 0$. Using (2.4), one sees that the matrix A_{xy} is symmetric in the sense that $A_{xy} = A_{xy}^* = \gamma^0 A_{xy}^\dagger \gamma^0$. Hence, using the idempotence of the matrix γ^0 together with the multiplicity of the determinant, we find that

$$\det(A_{xy} - \lambda) = \det(\gamma^0 (A_{xy}^\dagger - \lambda)\gamma^0) = \det(A_{xy}^\dagger - \lambda) = \overline{\det(A_{xy} - \overline{\lambda})}\,. \tag{6.13}$$

Hence if λ is an eigenvalue of the matrix A_{xy}, so is $\overline{\lambda}$. Thus the eigenvalues must form complex conjugate pairs.

We first complete the proof in the case that there are no degeneracies. For any eigenvalue λ of A_{xy} we choose a polynomial $p_\lambda(z)$ with $p_\lambda(\lambda) = 1$ and $p_\lambda(\mu) = 0$ for all other spectral points μ. Then the spectral projector on the eigenspace corresponding to λ, denoted by F_λ^{xy}, is given by

$$F_\lambda^{xy} = p_\lambda(A_{xy})\,. \tag{6.14}$$

Taking the adjoint and possibly after reordering the indices k, we obtain the relations (6.9) and (6.10). The general matrix relation $\det(BC - \lambda) = \det(CB - \lambda)$ (see for example [16, Section 3]) shows that the closed chains A_{xy} and A_{yx} have the same spectrum. Multiplying (6.14) by $P(x,y)$ and iteratively applying the relation

$$A_{xy}\, P(x,y) = P(x,y)\, P(y,x)\, P(x,y) = P(x,y)\, A_{yx}\,,$$

we find that $F_\lambda^{xy}\, P(x,y) = P(x,y)\, F_\lambda^{yx}$. Thus we can label the eigenvalues of the matrix A_{yx} such that (6.12) holds.

In the case with degeneracies, the assumption that A_{xy} is diagonalizable in a neighborhood of $\tau = 0$ allows us to diagonalize δA_{xy} on the degenerate subspaces. This yields (6.11), whereas (6.10) can be arranged by a suitable ordering of the spectral projectors F_k^{xy}. In the degenerate subspaces of A_{yx} we can choose the bases such that (6.9) and (6.10) hold (with 'xy' replaced by 'yx') and that (6.12) is satisfied. It remains to prove that (6.11) also holds for A_{yx}: From (6.11) we know that for any pair l, k with $\lambda^{xy}_l = \lambda^{xy}_k$,

$$\begin{aligned} 0 = F^{xy}_k (\delta A_{xy}) F^{xy}_l &= F^{xy}_k \Big(\delta P(x,y)\, P(y,x) + P(x,y)\, \delta P(y,x) \Big) F^{xy}_l \\ &= F^{xy}_k (\delta P(x,y)) F^{yx}_l\, P(y,x) + P(x,y) F^{yx}_k (\delta P(y,x)) F^{xy}_l \;, \end{aligned}$$

where in the last line we applied the second equation in (6.12). Multiplying by $P(y,x)$ from the left and by $P(x,y)$ on the right, we find

$$0 = P(y,x) F^{xy}_k (\delta P(x,y)) F^{yx}_l\, \lambda^{yx}_l + \lambda^{yx}_k\, F^{yx}_k (\delta P(y,x)) F^{xy}_l\, P(x,y) \,.$$

After dividing by $\lambda^{yx}_l = \lambda^{yx}_k$ (note that the eigenvalues are non-zero because they are assumed to form complex conjugate pairs), we can again use the second equation in (6.12) to obtain

$$\begin{aligned} 0 &= P(y,x) F^{xy}_k (\delta P(x,y)) F^{yx}_l + F^{yx}_k (\delta P(y,x)) F^{xy}_l\, P(x,y) \\ &= F^{yx}_k \Big(P(y,x)\, \delta P(x,y) + \delta P(y,x)\, P(x,y) \Big) F^{yx}_l = F^{yx}_k (\delta A_{yx}) F^{yx}_l \;, \end{aligned}$$

concluding the proof. □

For later use, we next compute the operator Q in the general setting of the previous lemma. Noting that the function $\mathcal{L}_\mu$ in (5.13) depends only on the absolute values of the eigenvalues, we can write

$$\mathcal{L}_\mu[A_{xy}] = \mathcal{L}_\mu(|\lambda^{xy}|_1, \ldots, |\lambda^{xy}|_4) \,.$$

The partial derivatives of the function $\mathcal{L}_\mu(|\lambda^{xy}|_1, \ldots, |\lambda^{xy}|_4)$ will be denoted by D_k.

Lemma 6.3. *Under the assumptions of Lemma 6.2, the operator Q in* (5.18) *is given by*

$$Q(x,y) = \sum_{k=1}^4 D_k \mathcal{L}_\mu\big(|\lambda^{xy}_1|, \ldots, |\lambda^{xy}_4|\big)\, \frac{\overline{\lambda^{xy}_k}}{|\lambda^{xy}_k|}\, F^{xy}_k\, P(x,y) \,. \tag{6.15}$$

Proof. The relation (6.11) allows us to compute the variation of the eigenvalues by a standard first order perturbation calculation without degeneracies,

$$\delta\lambda^{xy}_k = \mathrm{Tr}(F^{xy}_k\, \delta A_{xy}) \,. \tag{6.16}$$

Using that that $\delta|\lambda| = \mathrm{Re}(\overline{\lambda}\, \delta\lambda / |\lambda|)$, we can compute the first variation of this function with the help of (6.16),

$$\delta\mathcal{L}_\mu[A_{xy}] = \mathrm{Re} \sum_{k=1}^4 D_k \mathcal{L}_\mu\big(|\lambda^{xy}_1|, \ldots, |\lambda^{xy}_4|\big)\, \frac{\overline{\lambda^{xy}_k}}{|\lambda^{xy}_k|}\; \mathrm{Tr}(F^{xy}_k\, \delta A_{xy}) \,. \tag{6.17}$$

In the last trace we substitute the identity

$$\delta A_{xy} = \delta P(x,y)\, P(y,x) + P(x,y)\, \delta P(y,x)$$

and cyclically commute the arguments to obtain

$$\begin{aligned} \mathrm{Tr}(F^{xy}_k\, \delta A_{xy}) &= \mathrm{Tr}\left(F^{xy}_k\, P(x,y)\, \delta P(y,x) + P(y,x) F^{xy}_k\, \delta P(x,y) \right) \\ &= \mathrm{Tr}\left(F^{xy}_k\, P(x,y)\, \delta P(y,x) + F^{yx}_k\, P(y,x)\, \delta P(x,y) \right) , \end{aligned}$$

where in the last step we applied (6.12). Substituting this formula into (6.17) and integrating over x and y, we can exchange the names of x and y such that only $\delta P(y,x)$ appears. We thus obtain

$$\delta \mathcal{S}_\mu[P] = 2\,\mathrm{Re} \iint_M d^4x\, d^4y\, \mathrm{Tr}\left(Q(x,y)\, \delta P(y,x)\right) \tag{6.18}$$

with the integral kernel $Q(x,y)$ given by (6.15). Using Lemma 6.2, one sees that the operator corresponding to this integral kernel is symmetric (i.e. $Q(x,y)^* = Q(y,x)$). As a consequence, the integral in (6.18) is real, so that it is unnecessary to take the real part. Comparing with (5.18), we conclude that the operator with kernel (6.15) indeed coincides with the operator Q in (5.18). We note that due to the sum in (6.15), it is irrelevant how the bases were chosen on the degenerate subspaces of A_{xy}. □

Proof of Proposition 6.1. Let us specialize the general formula (6.15) to our spectral representation with eigenvalues (6.4) and spectral projectors (6.5). First, from (5.13) we readily obtain that

$$D_k \mathcal{L}_\mu\big(|\lambda_1^{xy}|, \ldots, |\lambda_4^{xy}|\big) = 2|\lambda_k| - 2\mu \sum_{l=1}^4 |\lambda_l| = 2(1-4\mu)\,|\lambda_-|\,.$$

The product $F_k^{xy}\, P(x,y)$ can be computed with the help of (6.1) and (6.5) as well as the relations

$$[\xi\!\!\!/, \overline{\xi\!\!\!/}]\, \xi\!\!\!/ = 2\langle \overline{\xi}, \xi\rangle\, \xi\!\!\!/ - 2\xi^2\, \overline{\xi\!\!\!/} = -(z-\overline{z})\, \xi\!\!\!/\,,$$

where in the last step we treated the factors $\xi\!\!\!/$ and $\overline{\xi\!\!\!/}$ as outer factors and applied the contraction rules (5.3) and (5.4). We thus obtain

$$F_+^{xy} P(x,y) = (\deg < 2)\,, \qquad F_-^{xy} P(x,y) = \frac{ig}{2}\, \xi\!\!\!/\, T^{(-1)}_{[0]} + (\deg < 2)\,. \tag{6.19}$$

Substituting these formulas into (6.15) and using (6.4), the result follows. □

6.2. **Chiral Gauge Potentials.** We now begin the study of interacting systems by introducing *chiral potentials*. Thus we choose the operator $\mathcal{B}$ in the auxiliary Dirac equation with interaction (4.5) according to (4.19) with two real vector fields A_L and A_R. Sometimes it is convenient to write $\mathcal{B}$ in the form

$$\mathcal{B} = A\!\!\!/_\mathrm{v} + \gamma^5 A\!\!\!/_\mathrm{a} \tag{6.20}$$

with a *vector potential* A_v and an *axial potential* A_a defined by

$$A_\mathrm{v} = (A_L + A_R)/2 \qquad \text{and} \qquad A_\mathrm{a} = (A_L - A_R)/2\,. \tag{6.21}$$

To the considered highest degree on the light cone, the chiral gauge potentials merely describe phase transformations of the left- and right-handed components of the fermionic projector (for details see [12] or [13, §2.5]). More precisely, the fermionic projector is obtained from (6.1) by inserting the phase factors

$$P(x,y) = \frac{ig}{2}\, \left(\chi_L\, e^{-i\Lambda_L^{xy}} + \chi_R\, e^{-i\Lambda_R^{xy}}\right) \xi\!\!\!/\, T^{(-1)}_{[0]} + (\deg < 2)\,, \tag{6.22}$$

where the functions $\Lambda^{xy}_{L/R}$ are integrals of the chiral potentials along the line segment $\overline{xy}$,

$$\Lambda^{xy}_{L/R} = \int_x^y A^j_{L/R}\, \xi_j := \int_0^1 A^j_{L/R}|_{\tau y + (1-\tau)x}\, \xi_j\, d\tau\,. \tag{6.23}$$

Consequently, the closed chain is obtained from (6.2) by inserting phase factors,

$$A_{xy} = \frac{1}{4} (\chi_L \nu_L + \chi_R \nu_R) (\not\xi T^{(-1)}_{[0]}) (\overline{\not\xi T^{(-1)}_{[0]}}) + \not\xi (\deg \le 3) + (\deg < 3) , \tag{6.24}$$

where

$$\nu_L = \overline{\nu_R} = e^{-i(\Lambda^{xy}_L - \Lambda^{xy}_R)} = \exp\Big(-2i \int_x^y A^j_{\mathrm{a}} \, \xi_j \Big) . \tag{6.25}$$

From (6.24) one sees that the matrix A_{xy} is invariant on the left- and right-handed subspaces (i.e. on the image of the operators χ_L and χ_R). On each of these invariant subspaces, it coincides up to a phase with the closed chain of the vacuum (6.2). Using these facts, the eigenvalues $(\lambda^c_s)_{c \in \{L,R\}, s \in \{+,-\}}$ and corresponding spectral projectors F^c_s are immediately computed by

$$\boxed{\lambda^{L/R}_\pm = \nu_{L/R} \, \lambda_s \qquad \text{and} \qquad F^{L/R}_\pm = \chi_{L/R} \, F_\pm} \tag{6.26}$$

with λ_s and F_s as in (6.4) and (6.5). We conclude that the eigenvalues of the closed chain are again complex, but in general they now form two complex conjugate pairs. Since the eigenvalues λ^L_c and λ^R_c differ only by a phase, we see that all eigenvalues have the same absolute value,

$$|\lambda^L_+| = |\lambda^R_+| = |\lambda^L_-| = |\lambda^R_-| . \tag{6.27}$$

Writing the Lagrangian (2.7) as

$$\mathcal{L}_{xy}[P] = \sum_{c,s} |\lambda^c_s|^2 - \frac{1}{4} \Big(\sum_{c,s} |\lambda^c_s| \Big)^2 = \frac{1}{8} \sum_{c,c' \in \{L,R\}} \; \sum_{s,s' \in \{\pm\}} \Big(|\lambda^c_s| - |\lambda^{c'}_{s'}| \Big)^2 \tag{6.28}$$

(where we sum over $c \in \{L,R\}$ and $s \in \{\pm\}$), we find that $\mathcal{L}$ vanishes identically. Since the Lagrangian is quadratic in $|\lambda^c_s| - |\lambda^{c'}_{s'}|$, also first variations of $\mathcal{L}$ vanish, suggesting that the operator $Q(x,y)$ should again vanish identically. This is indeed the case, as is verified immediately by applying Lemmas 6.2 and 6.3. We conclude that for chiral potentials, the EL equations in the continuum limit (5.29) are satisfied to degree five on the light cone.

We end this section by explaining how the line integrals in (6.23) and the phase factors in (6.22) and (6.24) can be understood from an underlying local gauge symmetry (for more details in the general context of non-abelian gauge fields see [13, §6.1]). The local phase transformation $\Psi(x) \to e^{i\Lambda(x)} \Psi(x)$ with a real function Λ describes a unitary transformation of the wave functions (with respect to the inner product (2.2)). Transforming all objects unitarily, we obtain the transformation laws

$$i\not\partial + \mathcal{B} - mY \to e^{i\Lambda(x)} \, (i\not\partial + \mathcal{B} - mY) \, e^{-i\Lambda(x)} = i\not\partial + \mathcal{B} - mY + (\not\partial\Lambda) \tag{6.29}$$

$$P(x,y) \to e^{i\Lambda(x)} \, P(x,y) e^{-i\Lambda(y)} \tag{6.30}$$

$$A_{xy} \to e^{i\Lambda(x)} \, P(x,y) e^{-i\Lambda(y)} \; e^{i\Lambda(y)} \, P(y,x) e^{-i\Lambda(x)} = A_{xy} . \tag{6.31}$$

The transformation of the Dirac operator corresponds to a transformation of the vector and axial potentials by

$$A_{\mathrm{v}} \to A_{\mathrm{v}} + \partial\Lambda \qquad \text{and} \qquad A_{\mathrm{a}} \to A_{\mathrm{a}} . \tag{6.32}$$

These are the familiar gauge transformations of electrodynamics. Using the formula

$$\Lambda(y) - \Lambda(x) = \int_0^1 \frac{d}{d\tau} \Lambda|_{\tau y + (1-\tau) x} \, d\tau = \int_x^y (\partial_j \Lambda) \, \xi^j \, d\tau ,$$

the phases in (6.30) can be described similar to (6.23) in terms of line integrals. This explains why the phase factors in (6.22) describe the correct behavior under gauge transformations. According to (6.31), the closed chain A_{xy} is gauge invariant. This is consistent with the fact that in (6.24) and (6.25) only the axial potential enters, which according to (6.32) is also gauge invariant.

In order to transform the axial potential, one can consider the local transformation $\Psi(x) \to e^{-i\gamma^5 \Lambda(x)} \Psi(x)$. In contrast to the above gauge transformation, this transformation is *not unitary* (with respect to the inner product (2.2)), and the requirement that the Dirac operator and the fermionic projector must be symmetric operators leads us to the transformations

$$\begin{aligned} i\partial\!\!\!/ + \mathcal{B} - mY &\to e^{i\gamma^5\Lambda(x)} \,(i\partial\!\!\!/ + \mathcal{B} - mY)\, e^{i\gamma^5\Lambda(x)} \\ &= i\partial\!\!\!/ + e^{i\gamma^5\Lambda(x)}(\mathcal{B} - mY)e^{i\gamma^5\Lambda(x)} + \gamma^5(\partial\!\!\!/\Lambda) \\ P(x,y) &\to e^{-i\gamma^5\Lambda(x)}\, P(x,y) e^{-i\gamma^5\Lambda(x)} \,. \end{aligned} \tag{6.33}$$

Thus the vector and axial potentials transform as desired by

$$A_{\mathrm{v}} \to A_{\mathrm{v}} \qquad \text{and} \qquad A_{\mathrm{a}} \to A_{\mathrm{a}} + \partial\Lambda$$

(and also the term mY is modified, but this is of no relevance for the argument here). The point is that when we now consider the transformation of the closed chain,

$$A_{xy} \to e^{i\gamma^5\Lambda(x)} P(x,y)\, e^{i\gamma^5\Lambda(y)}\; e^{i\gamma^5\Lambda(y)} P(y,x)\, e^{i\gamma^5\Lambda(x)} \,, \tag{6.34}$$

the local transformations do *not* drop out. This explains why in (6.24) phases involving the axial potentials do appear.

For clarity, we point out that the field tensors and the currents of the chiral gauge potentials also affect the fermionic projector, in a way which cannot be understood from the simple gauge transformation laws considered above. The corresponding contributions to the operator Q will be of degree four, and we shall consider them in the next chapter.

7. The Euler-Lagrange Equations to Degree Four

We come to the analysis of the EL equations to the next lower degree four on the light cone. In preparation, we bring the EL equations into a convenient form.

Lemma 7.1. *To degree four, the EL equations in the continuum limit* (5.29) *are equivalent to the equation*

$$\mathcal{R} := \frac{\Delta(|\lambda^L_-| - |\lambda^R_-|)}{|\lambda_-|}\; g^3\, T^{(0)}_{[0]} T^{(-1)}_{[0]}\; \overline{T^{(-1)}_{[0]}} = 0 + (\deg < 4) \,, \tag{7.1}$$

where Δ *denotes the perturbation of the eigenvalues* (6.26) *to degree two.*

Proof. According to (6.26), the eigenvalues to degree three are all non-real. Since this property is stable under perturbations of lower degree, we can again apply Lemmas 6.2 and 6.3. Noting that before (6.8), we fixed the Lagrange multiplier to $\mu = \frac{1}{4}$, we consider the Lagrangian (2.7), which we now write in analogy to (6.28) as

$$\mathcal{L}_{xy}[P] = \frac{1}{8} \sum_{k,l=1}^{4} (|\lambda^{xy}_k| - |\lambda^{xy}_l|)^2 \,.$$

Then the relation (6.15) can be written as

$$Q(x,y) = \frac{1}{2} \sum_{k,l=1}^{4} \Big\{ |\lambda^{xy}_k| - |\lambda^{xy}_l| \Big\} \, \frac{\overline{\lambda^{xy}_k}}{|\lambda^{xy}_k|} \, F^{xy}_k \, P(x,y) \,.$$

According to (6.27), the curly brackets vanish for the unperturbed eigenvalues. This has the convenient consequence that to degree four, it suffices to take into account the perturbation of the curly brackets, whereas everywhere else we may work with the unperturbed spectral decomposition (6.26),

$$Q(x,y) = \frac{1}{2} \sum_{k,l=1}^{4} \Delta\Big(|\lambda^{xy}_k| - |\lambda^{xy}_l| \Big) \, \frac{\overline{\lambda^{xy}_k}}{|\lambda^{xy}_k|} \, F^{xy}_k \, P(x,y) + (\deg < 4) \,.$$

Using (6.19), we see that we only get a contribution if λ_k equals λ^L_- or λ^R_-. Furthermore, we can apply (6.10), numbering the eigenvalues such that $\overline{\lambda^\pm_L} = \lambda^\mp_R$. We thus obtain

$$Q(x,y) = \sum_{c \in \{L,R\}} \Delta\Big(|\lambda^c_-| - |\lambda^c_+| \Big) \, \frac{\overline{\lambda^c_-}}{|\lambda^c_-|} \, \chi_c \, \frac{i \not{\xi}}{2} \, g \, T^{(-1)}_{[0]} + (\deg < 4) \,. \tag{7.2}$$

The EL equations (5.29) imply that the left- and right-handed components of this expression must vanish separately. Thus, again applying (6.10), we obtain the sufficient and necessary condition

$$\Delta\Big(|\lambda^L_-| - |\lambda^R_-| \Big) \, \frac{\overline{\lambda_-}}{|\lambda_-|} \, g \, T^{(-1)}_{[0]} + (\deg < 4) = 0 \,.$$

The explicit formulas (6.26) and (6.4) yield the result. □

It is important to observe that the EL equations only involve the difference of the absolute values of the left- and right-handed eigenvalues. This can immediately be understood as follows. To the leading degree three, the eigenvalues of A_{xy} form two complex conjugate pairs (see (6.26)). Since this property is preserved under perturbations, we can again write the Lagrangian in the form (6.28). Hence the Lagrangian vanishes identically unless the absolute values of the eigenvalues are different for the two pairs. This explains the term $\Delta(|\lambda^L_-| - |\lambda^R_-|)$ in (7.1).

As explained on page 28, the expression $\Delta(|\lambda^L_-| - |\lambda^R_-|)$ vanishes in the vacuum. Furthermore, the phase factors in (6.26) drop out of this expression. But new types of contributions to the interacting fermionic projector come into play, as we now explain.

7.1. **The Axial Current Terms and the Mass Terms.** An interaction by chiral potentials (4.19) as introduced in §6.2 affects the fermionic projector in a rather complicated way. For clarity, we treat the different terms in succession, beginning with the contributions near the origin $\xi = 0$ (the contributions away from the origin will be considered in Chapter 10). For the Taylor expansion around $\xi = 0$ we note that when evaluated weakly on the light cone (5.7), a simple fraction of degree L has a pole $|\vec{\xi}|^{-L}$. Thus it is reasonable to say that a term of the form (5.6) is of the order k at the origin if the smooth function η vanishes at the origin to the order $k + L$.

Definition 7.2. *An expression of the form* (5.6) *is said to be of* **order** $o(|\vec{\xi}|^k)$ **at the origin** *if the function* η *is in the class* $o((|\xi^0| + |\vec{\xi}|)^{k+L})$.

In the next lemma we specify the contributions to the EL equations to degree four on the light cone, to leading order at the origin.

Lemma 7.3. *For an interaction described by vector and axial potentials* (4.19)*, the expression* $\mathcal{R}$ *as defined by* (7.1) *takes the form*

$$\mathcal{R} = -i\xi_k \left(j^k_{\mathrm{a}}\, N_1 - m^2 A^k_{\mathrm{a}}\, N_2 \right) + (\deg < 4) + o(|\vec{\xi}|^{-3}) \,, \tag{7.3}$$

where j_{a} *is the axial current*

$$j^k_{\mathrm{a}} = \partial^k_j A^j_{\mathrm{a}} - \Box A^k_{\mathrm{a}} \,, \tag{7.4}$$

and N_1, N_2 *are the simple fractions*

$$N_1 = \frac{g^3}{6\, \overline{T^{(0)}_{[0]}}} \Big[\Big(T^{(0)}_{[0]} T^{(0)}_{[0]} - 2\, T^{(1)}_{[0]} T^{(-1)}_{[0]} \Big) \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big] \tag{7.5}$$

$$N_2 = -\frac{2g\, \hat{Y}^2}{\overline{T^{(0)}_{[0]}}} \Big[T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} \Big(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} + T^{(0)}_{[0]} \overline{T^{(0)}_{[1]}} \Big) - c.c. \Big] \tag{7.6}$$

$$\qquad - \frac{2g^2\, \acute{Y} \grave{Y}}{\overline{T^{(0)}_{[0]}}} \Big[T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} \Big(T^{(0)}_{[2]} \overline{T^{(0)}_{[0]}} + T^{(1)}_{[2]} \overline{T^{(-1)}_{[0]}} \Big) - c.c. \Big] . \tag{7.7}$$

Here "c.c." denotes the complex conjugate of the preceding simple fraction; the accents were defined in (5.2)*.*

In order not to distract from the main ideas, we postpone the proof of this lemma to Appendix B and proceed right away to the physical discussion. From the mathematical point of view, the appearance of the axial current j_{a} is not surprising, because the light-cone expansion of the fermionic projector involves derivatives of the potentials. In physical terms, this shows that the axial potential affects the fermionic projector not only via the phases in (6.22), but also via the Yang-Mills current. The term $-i\xi_k\, j^k_a N_1$ is referred to as the *current term.* The other term $-i\xi_k\, m^2 A^k_a N_2$ could not appear in ordinary Yang-Mills theories because it would not be gauge invariant. However, as pointed out after (6.33), the axial $U(1)$-transformations do *not* correspond to a local gauge symmetry, because they are not unitary. Instead, they describe relative phase transformations of the left- and right-handed components of the fermionic projector, thereby changing the physics of the system. Only the phase transformations (6.31) correspond to a local gauge symmetry, and in view of (6.32), the term $-i\xi_k\, m^2 A^k_a N_2$ is indeed consistent with this local $U(1)$-symmetry.

Since the direction ξ can be chosen arbitrarily on the light cone, the condition (7.1) implies that the bracket in (7.3) must vanish,

$$j^k_{\mathrm{a}}\, N_1 - m^2 A^k_{\mathrm{a}}\, N_2 = 0 \,. \tag{7.8}$$

If N_1 and N_2 could be treated as constants, this equation would go over to a field equation for the axial potential A_{a} with rest mass $m^2 N_2 / N_1$. For this reason, we refer to the term $-i\xi_k\, m^2 A^k_a N_2$ in (7.3) as the *mass term.* It is remarkable that in our framework, the bosonic mass term appears naturally, without the need for the Higgs mechanism of spontaneous symmetry breaking (for a detailed discussion of this point see §8.5). We also point out that the simple fraction N_2 involves the mass matrix Y, and thus the mass term in (7.8) depends on the masses of the fermions of the system.

In order to make the above argument precise, we need to analyze the simple fractions N_1 and N_2 weakly on the light cone. Before this will be carried out in §7.3, we specify how the Dirac current enters the EL equations.

7.2. **The Dirac Current Terms.** As explained in §4.3, the particles and anti-particles of the system enter the auxiliary fermionic projector via (4.10), where we orthonormalize the wave functions according to (4.11). Introducing the left- and right-handed component of the Dirac current by

$$J^i_{L/R} = \sum_{k=1}^{n_f} \overline{\Psi_k} \chi_{R/L} \gamma^i \Psi_k - \sum_{l=1}^{n_a} \overline{\Phi_l} \chi_{R/L} \gamma^i \Phi_l \,,$$

a decomposition similar to (6.21) leads us to define the *axial Dirac current* by

$$J^i_{\rm a} = \sum_{k=1}^{n_f} \overline{\Psi_k} \gamma^5 \gamma^i \Psi_k - \sum_{l=1}^{n_a} \overline{\Phi_l} \gamma^5 \gamma^i \Phi_l \,. \tag{7.9}$$

The next lemma gives the corresponding contribution to the EL equations, to leading order at the origin.

Lemma 7.4. *Introducing the axial Dirac current by the particle and anti-particle wave functions in* (4.10) *leads to a contribution to* $\mathcal{R}$ *of the form*

$$\mathcal{R} \asymp i\xi_k \, J^k_{\rm a} \, N_3 + (\deg < 4) + o(|\vec{\xi}|^{-3}) \,,$$

where

$$N_3 = \frac{g^2}{8\pi} \, \frac{1}{\overline{T^{(0)}_{[0]}}} \Big[T^{(-1)}_{[0]} \, \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big]. \tag{7.10}$$

Here the symbol "$\asymp$" means that we merely give the contribution to $\mathcal{R}$ by the Dirac current, but do not repeat the earlier contributions given in Lemma 7.3. The proof of this lemma is again postponed to Appendix B.

7.3. **The Logarithmic Poles on the Light Cone.** Combining the results of Lemmas 7.1, 7.3 and 7.4, the Euler-Lagrange equations give rise to the equation

$$\xi_k \left(j^k_{\rm a} \, N_1 - m^2 A^k_{\rm a} \, N_2 - J^k_a \, N_3 \right) = 0 \,,$$

which involves the axial potential $A_{\rm a}$ (see (6.20)), the corresponding Yang-Mills current (7.4) and the axial Dirac current (7.9). At first sight, this equation resembles a bosonic field equation, which describes the coupling of the Dirac spinors to the bosonic field and involves a bosonic mass term. However, the situation is not quite so simple, because the factors N_1, N_2 and N_3 (see (7.5)–(7.7) and (7.10)) have a mathematical meaning only when evaluating weakly on the light cone (5.7). Let us analyze the weak evaluation in more detail. The simple fraction N_3 is composed of the functions $T^{(0)}_{[0]}$, $T^{(-1)}_{[0]}$ and their complex conjugates, which according to (4.13)–(4.16) all have poles of the order ξ^{-2} or ξ^{-4}. In particular, no logarithmic poles appear, and thus we may apply (5.7) with $r = 0$ to obtain

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt \; \eta \, \xi_k \, J^k_a \, N_3 \;=\; \frac{c^{\rm reg}_3}{\varepsilon^3 |\vec{\xi}|^4} \, \eta(x) \, \xi_k \, J^k_a(x) + (\deg < 4) + o(|\vec{\xi}|^{-3})$$

with a regularization parameter c_3^{reg}, where we omitted error terms of the form (5.9) and (5.10). The simple fractions N_1 and N_2, on the other hand, involve in addition the functions $T_\circ^{(1)}$ and $\overline{T_\circ^{(1)}}$, which according to (4.13)–(4.15) involve a factor $\log|\xi^2|$ and thus have a *logarithmic pole on the light cone.* As a consequence, in (5.7) we also obtain contributions with $r=1$,

$$\begin{aligned}&\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt\; \eta\, \xi_k \left(j^k_{\text{a}}\, N_1 - m^2 A^k\, N_2\right) \;=\; (\deg < 4) + o(|\vec{\xi}|^{-3})\\ &+ \frac{1}{\varepsilon^3 |\vec{\xi}|^4}\, \eta(x)\, \xi_k \left[j^k_{\text{a}} \left(c_1^{\text{reg}} + d_1^{\text{reg}}\, \log(\varepsilon|\vec{\xi}|)\right) - m^2 A^k_{\text{a}} \left(c_2^{\text{reg}} + d_2^{\text{reg}}\, \log(\varepsilon|\vec{\xi}|)\right)\right],\end{aligned}$$

involving four regularization parameters $c_{1/2}^{\text{reg}}$ and $d_{1/2}^{\text{reg}}$. Combining the above weak evaluation formulas, the freedom in choosing the radius $|\vec{\xi}|$ and the spatial direction $\vec{\xi}/|\vec{\xi}|$ implies that that the logarithmic and non-logarithmic terms must vanish separately,

$$j^k_{\text{a}}\, d_1^{\text{reg}} - m^2 A^k_{\text{a}}\, d_2^{\text{reg}} = 0 \tag{7.11}$$

$$j^k_{\text{a}}\, c_1^{\text{reg}} - m^2 A^k_{\text{a}}\, c_2^{\text{reg}} = J^k_a\, c_3^{\text{reg}}\,, \tag{7.12}$$

where $c_{1/2}^{\text{reg}}$ and $d_{1/2}^{\text{reg}}$ are constants depending on the particular regularization.

Unfortunately, the system of equations (7.11) into (7.12) is overdetermined and is thus too restrictive for physical applications. Namely, at least for generic regularizations, the constants c_1^{reg}, c_3^{reg} and d_1^{reg} are non-zero. Thus solving (7.11) for j_{a} and substituting into (7.12), one obtains an algebraic equation involving J_{a} and A_{a}. This means that either J_{a} must vanish identically, or else the gauge potential A_{a} is fixed to a constant times J_{a} and thus cannot be dynamical. Both cases are not interesting from a physical point of view. The basic reason for this shortcoming is that the bosonic current and mass terms have logarithmic poles on the light cone, whereas the Dirac current terms involve no such logarithms. Our method for overcoming this problem is to insert additional potentials into the Dirac equation, with the aim of compensating the logarithmic poles of the bosonic current and mass terms. Before entering these constructions in §7.4, we now briefly discuss alternative methods for treating the logarithmic poles.

An obvious idea for reducing the system (7.11) and (7.12) to a single equation is to restrict attention to special regularizations where the constants c_i^{reg} and/or d_i^{reg} take special values. In particular, it seems tempting to demand that $d_1^{\text{reg}} = d_2^{\text{reg}} = 0$, so that (7.11) is trivially satisfied, leaving us with the field equations (7.12). This method does not work, as the following consideration shows. Differentiating (4.14) and using (4.13), one sees that

$$32\pi^3\, T_\circ^{(1)} = \log|\xi^2| + c_0 + i\pi\Theta(\xi^2)\,\epsilon(\xi^0)\,. \tag{7.13}$$

Evaluating near the upper light cone $\xi^0 \approx |\vec{\xi}|$, we can apply the relation $\log|\xi^2| = \log\left|\xi^0 + |\vec{\xi}|\right| + \log\left|\xi^0 - |\vec{\xi}|\right|$ to obtain

$$32\pi^3\, T_\circ^{(1)} = \log|2\vec{\xi}| + \log\left|\xi^0 - |\vec{\xi}|\right| + i\pi\Theta(\xi^0 - |\vec{\xi}|) + c_0 + \mathcal{O}(\xi^0 - |\vec{\xi}|)\,. \tag{7.14}$$

When evaluating the corresponding simple fraction weakly (5.7), the first term in (7.14) gives rise to the $\log|\vec{\xi}|$-dependence, the second term gives the $\log\varepsilon$-dependence, whereas all the other terms do not involve logarithms or are of higher order in ε. Obviously, the same is true for the complex conjugate $\overline{T^{(1)}_\circ}$. Since in (5.7) the vector $\vec{\xi}$ is fixed, the vanishing of the $\log|\vec{\xi}|$-dependent contribution to the integral (5.7) implies that the simple fraction still vanishes if the factors $T^{(1)}_\circ$ and $\overline{T^{(1)}_\circ}$ are replaced by constants. Inspecting the $T^{(1)}$-dependence of (7.5) and (7.7) and comparing with (7.10), we find that

$$d_1^{\text{reg}} = 0 \iff d_2^{\text{reg}} = 0 \qquad \text{and} \qquad d_1^{\text{reg}} = 0 \implies c_3^{\text{reg}} = 0\,.$$

Thus if the constants d_1^{reg} and d_2^{reg} in (7.11) vanish, then (7.11) becomes trivial as desired. But then the constant c_3^{reg} in (7.12) is also zero, so that the Dirac current drops out of the field equation. Again, we do not end up with physically reasonable equations.

Sticking to the idea of considering regularizations where the regularization constants have special values, the remaining method is to assume that *all* regularization constants in (7.11) and (7.12) vanish. Then the EL equations would be trivially satisfied to degree four on the light cone, and one would have to proceed to the analysis to degree three on the light cone. This method does not seem to be promising for the following reasons. First, it is not clear whether there exist regularizations for which all the regularization constants in (7.11) and (7.12) vanish. In any case, it seems difficult to satisfy all these conditions, and the resulting regularizations would have to be of a very special form. This would not be fully convincing, because one might prefer not to restrict the class of admissible regularizations at this point. Secondly, there is no reason to believe that the situation to degree three would be better, at least not without imposing additional relations between regularization constants, giving rise to even more constraints for the admissible regularizations.

We conclude that assuming special values for the regularization constants in (7.11) and (7.12) does not seem to be a promising strategy. Thus in what follows we shall *not* impose any constraints on the regularization constants, which also has the advantage that our constructions will apply to *any* regularization. Then the only possible strategy is to try to compensate the logarithmic poles by additional potentials in the Dirac operator.

7.4. **A Pseudoscalar Differential Potential.** Our goal is to compensate the logarithmic poles of the bosonic current and mass terms by inserting additional potentials into the auxiliary Dirac equation (4.5). In order to get contributions of comparable structure, these potentials should involve a vector field v, which should be either equal to the axial potential A_a or the corresponding axial current j_a (see Lemma 7.3). Since contracting the vector index of v with the Dirac matrices would again give rise to chiral potentials (4.19), we now prefer to contract v with partial derivatives. Moreover, since we want to compensate contributions which are odd under parity transformations (i.e. which flip the left- and right-handed components), the resulting operator must involve the pseudoscalar matrix γ^5. The requirement that the Dirac operator should be symmetric with respect to the inner product (2.2) leads us to the ansatz involving an anti-commutator

$$\mathcal{B} = \gamma^5 \left\{ v^j, \partial_j \right\} = 2\gamma^5 v^j \partial_j + \gamma^5 (\partial_j v^j)\,. \tag{7.15}$$

We refer to this ansatz as a *pseudoscalar differential potential*. Our ansatz seems unusual because such differential potentials do not occur in the standard model nor in general relativity. We postpone the physical discussion to the last paragraph of §7.6.

The corresponding leading contribution to the fermionic projector is of the form (for details see equation (B.32) in Appendix B)

$$\begin{aligned} P(x,y) &\asymp \frac{g}{2}\,\gamma^5 \xi_i \left(v^i(y)+v^i(x)\right) T^{(-1)} \\ &\quad + \frac{g}{2}\,\gamma^5 \xi_i \int_x^y \left[\not{\xi}, (\not{\partial} v^i)\right] T^{(-1)} \;+\; (\deg < 2)\,. \end{aligned} \tag{7.16}$$

This contribution has a pole of order ξ^{-4} on the light cone and is therefore much more singular than the desired logarithmic pole. A straightforward calculation shows that (7.16) does contribute to the expression $\mathcal{R}$ in Lemma 7.1, and thus we conclude that (7.16) is not suitable for compensating the logarithmic pole.

The key for making use of the pseudoscalar differential potential (7.15) is to observe that the required logarithmic poles do appear to higher order in a mass expansion. More precisely, to leading order at the origin, the cubic contribution to the fermionic projector is

$$P(x,y) \asymp \frac{m^3}{4}\gamma^5 \left[v^{(3)}_j(x) + \mathcal{O}\big(|\xi^0| + |\vec{\xi}|\big)\right] \left(\not{\xi}\xi^j\, T^{(0)} - 2\gamma^j\, T^{(1)}\right) + (\deg < -1)\,, \tag{7.17}$$

where $v^{(3)}$ is a Hermitian matrix composed of v and Y,

$$v^{(3)} = i\,(vYYY - YvYY + YYvY - YYYv) \tag{7.18}$$

(for details see equation (B.33) in Appendix B). Thus there is hope that the logarithmic poles can be compensated, provided that we can arrange that the contributions by (7.15) to $\mathcal{R}$ of order m^0, m and m^2 in a mass expansion vanish. The last requirement cannot be met if we consider one Dirac sea, because the term (7.16) does contribute to $\mathcal{R}$. But if we consider several Dirac seas, we have more freedom, as the pseudoscalar differential potential (7.15) can be chosen differently for each Dirac sea. For example, we can multiply the potentials acting on the different Dirac seas by real constants $\mathfrak{g}_\alpha$,

$$(\mathcal{B})^\alpha_\beta = \mathfrak{g}_\alpha\, \delta_{\alpha\beta}\, \gamma^5 \left\{v^j, \partial_j\right\} \qquad \text{with} \qquad \alpha,\beta = 1,\ldots,g\,. \tag{7.19}$$

Using this additional freedom, it is indeed possible to arrange that the contribution (7.16) drops out of $\mathcal{R}$. This consideration explains why we must consider *several generations* of elementary particles.

The critical reader might object that there might be other choices of the operator $\mathcal{B}$ which could make it possible to compensate the logarithmic poles without the need for several generations. However, the following consideration shows that (7.15) is indeed the only useful ansatz. First of all, since we want to compensate a contribution to the fermionic projector at the origin, it seems unavoidable to consider local operators (for nonlocal operators see Chapter 10). The only zero order operator are the chiral potentials (4.19), which were already considered in §6.2. Apart from (7.15), the only first order differential operator involving the vector field v and the pseudoscalar matrix γ^5 is the operator

$$\gamma^5 \left\{v_j \sigma^{jk}, \partial_k\right\}\,,$$

where $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$ are the bilinear covariants. This ansatz can be shown to be useless, basically because the calculations in the continuum limit give rise to contractions with the vector ξ, which vanish (see also §9.2). Differential operators of higher order must involve the wave operator $\Box$, which applied to the Dirac wave functions gives rise to lower order operators. This shows that it is not useful to consider differential operators of order higher than one. We conclude that (7.15) and its generalizations to several generations (like (7.19)) are indeed the only possible ansätze for compensating the logarithmic poles.

We end the discussion by having a closer look at the matrix $v^{(3)}$, (7.18). Note that the ansatz (7.19) is diagonal in the generation index and thus commutes with the mass matrix Y. As a consequence, the matrix $v^{(3)}$ vanishes. This means that for compensating the logarithmic poles, the ansatz (7.19) is not sufficient, but we must allow for non-zero off-diagonal elements in the generation index. Thus we replace the factors b_α in (7.19) by a Hermitian matrix $\mathfrak{g} = (\mathfrak{g}^\alpha_\beta)_{\alpha,\beta=1,\ldots,g}$, the so-called *generation mixing matrix*. Later on, the generation mixing matrix will depend on the space-time point x. This leads us to generalize (7.20) by the ansatz

$$(\mathcal{B})^\alpha_\beta = \gamma^5 \left\{ \mathfrak{g}^\alpha_\beta(x)\, v^j(x), \partial_j \right\} , \tag{7.20}$$

thus allowing that the pseudoscalar differential potential mixes the generations.

7.5. **A Vector Differential Potential.** Modifying the auxiliary Dirac equation (4.5) by a first oder operator (7.15) or (7.20) changes the behavior of its solutions drastically. In particular, it is not clear whether the operator $\mathcal{B}$ can be treated perturbatively (4.9). In order to analyze and resolve this problem, we begin by discussing the case where the potential v in (7.15) is a *constant* vector field, for simplicity for one Dirac sea of mass m. Then taking the Fourier transform, the Dirac equation reduces to the algebraic equation

$$(\not{k} - 2i\gamma^5 v^j k_j - m)\, \hat{\Psi}(k) = 0 . \tag{7.21}$$

Multiplying from the left by the matrix $(\not{k} - 2i\gamma^5 v^j k_j + m)$, we find that the momentum of a plane-wave solution must satisfy the dispersion relation

$$k^2 - 4(v^j k_j)^2 - m^2 = 0 .$$

Rewriting this equation as

$$g^{ij} k_i k_j - m^2 = 0 \qquad \text{with} \qquad g^{ij} := \eta^{ij} - 4 v^i v^j ,$$

where $\eta^{ij} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric, we see that the new dispersion relation is the same as that for the Klein-Gordon equation in a space-time with Lorentzian metric g^{ij}. In particular, the characteristics of the Dirac equation become the null directions of the metric g^{ij}. In other words, the light cone is "deformed" to that of the new metric g^{ij}.

This deformation of the light cone leads to a serious problem when we want to compensate the logarithmic poles, as we now discuss. Suppose that we introduce a pseudoscalar differential potential which according to (7.19) or (7.20) depends on the generation index. In the case (7.19), the Dirac seas feel different dispersion relations. In particular, the singularities of the fermionic projector $P(x, y)$ will no longer be supported on one light cone, but will be distributed on the union of the light cones corresponding to the Lorentzian metrics $g^{ij}_\alpha = \eta^{ij} - 4 v^i_\alpha v^j_\alpha$. The ansatz (7.20) leads to a similar effect of a "dissociation of the light cone." In the EL equations, this would lead to large additional contributions, which are highly singular on the light cone and can certainly not compensate the logarithmic poles.

Our method for bypassing this problem is to introduce another differential potential which transforms the dispersion relation back to that of the Klein-Gordon equation in

Minkowski space. In the case of a constant vector field v and one generation, this can be achieved by choosing matrices G^j which satisfy the anti-commutation relations

$$\big\{G^i, G^j\big\} = 2\eta^{ij} + 8v^i v^j \qquad \text{and} \qquad \big\{\gamma^5, G^i(x)\big\} = 0\,,$$

and by modifying (7.21) to

$$(G^j k_j - 2i\gamma^5 v^j k_j - m)\,\hat{\Psi}(k) = 0\,.$$

This modification of the Dirac matrices can be interpreted as introducing a constant gravitational potential corresponding to the metric $\eta^{ij} + 4v^i_\alpha v^j_\alpha$. This construction is extended to the general case (7.20) as follows. We choose $(4g \times 4g)$-matrices $G^j(x)$ which are symmetric with respect to the inner product $\overline{\Psi}\Phi$ on the Dirac spinors and satisfy the anti-commutation relations

$$\big\{G^i(x), G^j(x)\big\} = 2\eta^{ij} + 8\,\mathfrak{g}(x)^2\, v^i(x)\, v^j(x) \qquad \text{and} \qquad \big\{\gamma^5, G^i(x)\big\} = 0\,. \tag{7.22}$$

In the auxiliary Dirac equation (4.5) we insert the additional operator

$$\mathcal{B} = i\big(G^j(x) - \gamma^j\big)\partial_j + G^j(x)\,E_j(x)\,, \tag{7.23}$$

where the matrices E_j involve the spin connection coefficients and are not of importance here (for details see for example [13, §1.5]). We refer to (7.23) as a *vector differential potential*. In the case (7.19), this construction can be understood as introducing for each Dirac sea a gravitational potential corresponding to the metric $\eta^{ij} + 4\mathfrak{g}^2_\alpha v^i_\alpha v^j_\alpha$, whereas in case (7.20), the interpretation is bit more complicated due to the off-diagonal terms.

7.6. **Recovering the Differential Potentials by a Local Axial Transformation.** By introducing the differential potentials (7.20) and (7.23) with G^j according to (7.22), we inserted differential operators into the auxiliary Dirac equation (4.5). We will now show that the effect of these operators on the solutions of the auxiliary Dirac equation can be described by a local transformation

$$\Psi_{\text{aux}}(x) \rightarrow U(x)\,\Psi_{\text{aux}}(x)\,, \tag{7.24}$$

which is unitary with respect to the inner product (4.6). This simplification also allows us to compute the resulting logarithmic pole in detail. We conclude this chapter by a physical interpretation of the differential potentials and of the transformation (7.24).

Recall that we introduced the vector differential potential (7.23) with the goal of transforming the dispersion relation back to the form in the vacuum. Thus if v is a constant vector field, the combination (7.20)+(7.23) leaves the momenta of plane-wave solutions unchanged. This suggests that the sum (7.20)+(7.23) might merely describe a unitary transformation of the Dirac wave functions. Thus we hope that there might be a unitary matrix $U(x)$ such that

$$U(i\partial\!\!\!/ - mY)U^{-1} = i\partial\!\!\!/ + (7.15) + (7.23)\,.$$

Let us verify whether there really is such a unitary transformation. The natural ansatz for U is an exponential of an axial matrix involving the vector field v and the generation mixing matrix,

$$U(x) = \exp\big(-i\mathfrak{g}(x)\,\gamma^5\gamma^j v_j(x)\big)\,. \tag{7.25}$$

Writing out the exponential series and using that $(\gamma^5\gamma^j v_j)^2 = -v^2$, we obtain

$$U(x) = \cos(\mathfrak{g}\varphi)\,\mathbb{1} - i\,\frac{\sin(\mathfrak{g}\varphi)}{\varphi}\,\gamma^5 v\!\!\!/\,, \qquad U(x)^{-1} = \cos(\mathfrak{g}\varphi)\,\mathbb{1} + i\,\frac{\sin(\mathfrak{g}\varphi)}{\varphi}\,\gamma^5 v\!\!\!/\,, \tag{7.26}$$

where the angle $\varphi := \sqrt{-v^2}$ is real or imaginary (note that (7.26) is well-defined even in the limit $\varphi \to 0$). A short calculation yields

$$U\gamma^j - \gamma^j U = -i\,\frac{\sin(\mathfrak{g}\varphi)}{\varphi}\,\big[\gamma^5 \slashed{v}, \gamma^j\big] = -2i\gamma^5\, v^j\,\frac{\sin(\mathfrak{g}\varphi)}{\varphi}$$

and thus

$$\begin{aligned}
&U(i\slashed{\partial} - mY)\,U^{-1} = iU\gamma^j U^{-1}\partial_j + U\gamma^j\,(i\partial_j U^{-1}) - mUYU^{-1} \\
&\quad = i\slashed{\partial} + 2\gamma^5 v^j\,\frac{\sin(\mathfrak{g}\varphi)}{\varphi}\,U^{-1}\partial_j + U\gamma^j\,(i\partial_j U^{-1}) - mUYU^{-1} \\
&\quad = i\slashed{\partial} + \gamma^5\,\frac{\sin(2\mathfrak{g}\varphi)}{\varphi}\,v^j\partial_j + 2i\,\frac{\sin^2(\mathfrak{g}\varphi)}{\varphi^2}\,\slashed{v}\,v^j\partial_j + U\gamma^j\,(i\partial_j U^{-1}) - mUYU^{-1}\,. \qquad (7.27)
\end{aligned}$$

In order to verify that the resulting Dirac operator allows us to recover both (7.20) and (7.23), we assume that v^2 is so small that $\sin(2\mathfrak{g}\varphi) \approx 2\mathfrak{g}\varphi$ and $\sin^2(\mathfrak{g}\varphi) \approx \mathfrak{g}^2\varphi^2$. Then the second summand in (7.27) reduces precisely to the differential operator in (7.20). The third summand in (7.27) gives precisely the differential operator in (7.23), noting that (7.22) has the solution $G^j = \gamma^j + 2\mathfrak{g}^2 \slashed{v} v^j + \mathcal{O}(v^4)$. Likewise, a direct calculation shows that the multiplication operators in (7.20) and (7.23) are contained in the fourth summand in (7.27). Writing out the fourth and fifth summands in (7.27), one finds a rather complicated combination of additional chiral, scalar, pseudoscalar and even bilinear potentials. These additional potentials do not cause any problems; on the contrary, they guarantee that the total transformation of the Dirac wave functions simply is the local transformation (7.24). We conclude that with (7.27) we have found a Dirac operator which includes the differential potentials in (7.20) and (7.23). It has the nice property that it can easily be treated non-perturbatively by the simple local transformation (7.24). We refer to the transformation (7.24) with U according to (7.25) as the *local axial transformation*.

To avoid confusion, we point out that we shall always perform the transformation (7.24) *after* all the other potentials have been inserted into the auxiliary Dirac equation. For example, to combine the chiral potentials (4.19) with the differential potentials, we consider the Dirac operator $U(i\slashed{\partial} + \chi_L \slashed{A}_R + \chi_R \slashed{A}_L - mY)U^{-1}$. The corresponding auxiliary fermionic projector has the form $U(x)P(x,y)U^{-1}(y)$, where $P(x,y)$ is defined by the perturbation series (4.9) with $\mathcal{B}$ composed only of the chiral potentials. It might seem puzzling that the transformation U also changes the zero order terms in the Dirac operator, which thus no longer seem to be under full control. However, the unitary transformation poses no restriction for the zero order terms in the final Dirac operator, because any undesirable multiplication operator $\mathcal{B}$ in the final Dirac operator can be compensated simply by inserting the operator $-U^{-1}\mathcal{B}U$ into the Dirac operator before the unitary transformation. In view of the exhaustive discussion of different multiplication operators in §6.2 and §9.1–§9.3, the zero order terms in the Dirac equation can be changed arbitrarily, and the effect on the fermionic projector is well-understood.

We next work out how the local axial transformation can be used to compensate the logarithmic poles. For simplicity, we only consider a perturbation expansion to first order in v and remark that the corresponding non-perturbative treatment is given in Appendix C. To first order in v, the transformation (7.25) simplifies to

$$U(x) = \mathbb{1} - i\mathfrak{g}\gamma^5\,\slashed{v}(x) + \mathcal{O}(v^2)\,. \qquad (7.28)$$

Transforming the auxiliary fermionic projector (4.1) by U and taking the partial trace (4.4), we obtain for the perturbation of the fermionic projector the expression

$$P \asymp -i\gamma^5 \not{v}\, \acute{\mathfrak{g}} \grave{P} + i\acute{P} \grave{\mathfrak{g}}\, \gamma^5 \not{v} + \mathcal{O}(v^2)\,, \tag{7.29}$$

where we denoted the partial trace similar to (5.2) by accents. Here we always sum over one index of the generation mixing matrix. Thus it is convenient to introducing real functions c_α and d_α by

$$\sum_{\alpha=1}^{g} \mathfrak{g}^\alpha_\beta = c_\beta + i d_\beta \qquad \text{and} \qquad \sum_{\beta=1}^{g} \mathfrak{g}^\alpha_\beta = c_\alpha - i d_\alpha\,, \tag{7.30}$$

where the last equation is verified by taking the adjoint of the first and using that $\mathfrak{g}$ is Hermitian. Combining these equations with the fact that the auxiliary fermionic projector of the vacuum is diagonal on the generations, we can write (7.29) as

$$P \asymp \sum_{\beta=1}^{g} \Big(-i \left[c_\beta \gamma^5 \not{v}, P_\beta\right] + \left\{d_\beta \gamma^5 \not{v}, P_\beta\right\} \Big) + \mathcal{O}(v^2)\,, \tag{7.31}$$

where the P_β stand for the direct summands in (4.1). The next lemma shows that the functions c_β drop out of the EL equations; the proof is again given in Appendix B.

Lemma 7.5. *The perturbation of the fermionic projector by the functions c_β in* (7.31) *drops out of the EL equations to degree five and four on the light cone.*

Noting that the diagonal elements of $\mathfrak{g}$ only contribute to the functions c_β (as is obvious from (7.30) and the fact that the diagonal elements of $\mathfrak{g}$ are real), this lemma again shows (in analogy to the consideration leading to the ansatz (7.20)) that the mixing of the generations is essential for our constructions to work.

Using the result of the last lemma, it remains to analyze the effect of the functions d_β. Computing the leading degrees on the light cone, we obtain

$$\begin{aligned} P(x,y) &\asymp \sum_{\beta=1}^{g} \left\{ d_\beta \gamma^5 \not{v},\, \frac{i\not{\xi}}{2}\, T^{(-1)} + m_\beta\, T^{(0)} \right\} + (\deg < 1) \\ &= \frac{i}{2}\, \gamma^5 \not{v}(x)\, \not{\xi}\, T^{(-1)} \Big(\sum_{\beta=1}^{g} d_\beta(x) \Big) - \frac{i}{2}\, \gamma^5 \not{\xi}\, \not{v}(y)\, T^{(-1)} \Big(\sum_{\beta=1}^{g} d_\beta(y) \Big) \qquad (7.32) \\ &\quad + \Big(\gamma^5 \not{v}(x) \sum_{\beta=1}^{g} m_\beta\, d_\beta(x) + \gamma^5 \not{v}(y) \sum_{\beta=1}^{g} m_\beta\, d_\beta(y) \Big)\, T^{(0)} + (\deg < 1)\,. \qquad (7.33) \end{aligned}$$

Since $\mathfrak{g}$ is Hermitian, summing the first equation in (7.30) over β gives a real number. Hence

$$\sum_{\beta=1}^{g} d_\beta = 0\,, \tag{7.34}$$

so that (7.32) vanishes. The term in (7.33) has a similar form as the contribution by the axial current to $P(x,y)$ (for details see equation (B.18) in Appendix B), but on the light cone it has instead of a logarithmic pole a stronger singularity $\sim \xi^{-2}$. For general regularizations, this term does contribute to the EL equations even to degree five on the light cone. This is made precise in the next lemma, which is again derived in Appendix B.

Lemma 7.6. *Under the assumptions* (7.34)*, the perturbation of the fermionic projector by the functions* d_β *in* (7.31) *leads to a contribution to* $\mathcal{R}$ *of the form*

$$\mathcal{R} \asymp i\xi_k\, v^k \Big(\sum_{\beta=1}^g m_\beta\, d_\beta \Big) \frac{g^2}{T^{(0)}_{[0]}}\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \left(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} - c.c. \right) + (\deg < 5) + o(|\vec{\xi}|^{-4}) \,.$$

Since we do not want to put any conditions on the regularizations, the EL equations yield the constraint

$$\sum_{\beta=1}^g m_\beta\, d_\beta = 0 \,. \tag{7.35}$$

The remaining contributions to the fermionic projector are of degree at most zero on the light cone. They affect the EL equations to degree four, as specified in the next lemma, which is again proved in Appendix B.

Lemma 7.7. *Under the assumptions* (7.34) *and* (7.35)*, the perturbation of the fermionic projector by the functions* d_β *in* (7.31) *leads to a contribution to* $\mathcal{R}$ *of the form*

$$\mathcal{R} \asymp i\xi_k\, v^k\, N_4 + (\deg < 4) + o(|\vec{\xi}|^{-3}) \,,$$

where

$$N_4 = -\frac{2g}{T^{(0)}_{[0]}}\, m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta \Big[T^{(0)}_{[1]} T^{(0)}_{[2]} \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \Big] \tag{7.36}$$

$$+ \frac{2g^2}{T^{(0)}_{[0]}} \sum_{\beta=1}^g m_\beta^3\, d_\beta \Big[T^{(1)}_{[3]} T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \Big] . \tag{7.37}$$

We point out that the simple fraction N_4 involves factors $T^{(1)}_{[0]}$ and $\overline{T^{(1)}_{[0]}}$, which in view of (7.13) have the desired logarithmic pole on the light cone. This indeed makes it possible to compensate the logarithmic poles of the current and mass terms, as will be explained in the next chapter.

We conclude this chapter by a brief physical discussion. With the previous constructions we recovered the differential potentials of §7.4 and §7.5 by a local axial transformation of the auxiliary Dirac operator and the auxiliary fermionic projector of the form

$$(i\partial\!\!\!/ + \mathcal{B} - mY) \rightarrow U(i\partial\!\!\!/ + \mathcal{B} - mY)U^{-1} \,, \quad P^{\text{aux}}(x,y) \rightarrow U(x)\, P^{\text{aux}}(x,y)\, U(y)^{-1} \,,$$

where $U(x)$ is unitary with respect to the inner product (4.6). Here the operator $\mathcal{B}$ is composed of all bosonic potentials and fields except for the differential potentials. It is important to observe that the local axial transformation is performed *before* the partial trace is taken (4.4). In particular, it *cannot* be described by a local unitary transformation of the fermionic projector

$$P(x,y) \rightarrow U(x)\, P(x,y)\, U(y)^{-1} \qquad \text{with } U \in \mathrm{U}(2,2) \,.$$

Indeed, the last transformation would simply lead to a unitary transformation of the closed chain $A_{xy} \rightarrow U(x) A_{xy} U(x)^{-1}$, leaving the spectrum of the closed chain and thus also the Lagrangian unchanged. The local axial transformation, however, *does* affect the Lagrangian, because the spectrum of the closed chain is computed *after* the partial trace has been taken. In view of this fact, the local axial transformation cannot be interpreted merely as a local gauge transformation, but it really changes the physical system. Nevertheless, due to the local nature of the transformation, the local axial transformation does not change the dynamics of the physical system and is thus not directly observable. The physical picture is that the local axial transformation slightly modifies the fermionic wave functions at every space-time point, thus affecting the local form of the fermionic projector.

We finally remark that the local axial transformation also influences the inner product (2.2) between fermionic states, an effect which could in principle be physically observable. However, since the integration in (2.2) extends over all space-time, it is not clear how the inner product (2.2) could be determined in experiments. It thus seems that the local axial transformation does not lead to any conceivable measurable effects.

8. The Field Equations

Having developed a method for compensating the logarithmic poles on the light cone, we are now in the position to derive and analyze the field equations.

8.1. **The Smooth Contributions to the Fermionic Projector at the Origin.**

We add the contributions from Lemmas 7.3, 7.4 and 7.7 and collect all the terms which involve factors of $T^{(1)}_\circ$ or $\overline{T^{(1)}_\circ}$. Using (7.13), we find that the contribution to $\mathcal{R}$ involving factors of $\log|\xi^2|$ has the form

$$\mathcal{R} \asymp -\frac{i\xi_k}{16\pi^3} \left\{ \frac{j^k_{\rm a}}{6} - m^2\, \acute{Y}\grave{Y}\, A^k_{\rm a} + v^k \sum_{\beta=1}^g m_\beta^3\, d_\beta \right\} \frac{\log|\xi^2|}{\overline{T^{(0)}_{[0]}}}\, g^2\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big(T^{(0)}_{[0]} - \overline{T^{(0)}_{[0]}} \Big) + (\deg < 4) + o(|\vec{\xi}|^{-3})\,. \tag{8.1}$$

As explained after (7.14), this term must vanish. According to our definition (7.25), we are free to multiply $\mathfrak{g}(x)$ by any non-zero number if we divide v by the same number. Using this freedom, we may assume that the coefficients d_β (defined by (7.30)) satisfy the relations

$$\sum_{\beta=1}^g m_\beta^3\, d_\beta = 1\,. \tag{8.2}$$

Then the expression in (8.1) vanishes if and only if we set

$$v = -\frac{j_{\rm a}}{6} + m^2\, \acute{Y}\grave{Y} A_{\rm a}\,. \tag{8.3}$$

With (7.34), (7.35) and (8.2) we have an inhomogeneous system of three linearly independent equations for the unknowns $d_1, \ldots, d_g$. If $g < 3$, this system has no solutions. We thus conclude that the logarithmic poles can be compensated only if we have *at least three generations*. In the case $g \geq 3$, on the other hand, the system has solutions, and in Appendix C it is shown that these solutions can indeed be realized by a suitable mixing matrix $\mathfrak{g}$. Thus from now on, we assume that $g \geq 3$, and we choose $\mathfrak{g}$ as well as v such that the conditions (7.34), (7.35), (8.2) and (8.3) hold. Then the logarithmic poles of $\mathcal{R}$ have disappeared.

Before analyzing the remaining contributions to $\mathcal{R}$, we must have a closer look at the non-causal low- and high energy contributions $P^{\rm le}$ and $P^{\rm he}$ in the light cone expansion (4.17). These smooth contributions to the fermionic projector were disregarded in the formalism of the continuum limit as outlined in Chapter 5. This is justified as long as singular contributions to the fermionic projector are considered. In particular, contributions to $P(x,y)$ involving the functions $T^{(-1)}_\circ$, $T^{(0)}_\circ$ or their complex conjugates have poles on the light cone, and therefore smooth corrections would be of lower degree on the light cone, meaning that the corresponding contributions to the EL equations would be negligible corrections of the form (5.9). However, the factors $T^{(1)}_\circ$ and $\overline{T^{(1)}_\circ}$ only have a logarithmic pole, and after the above cancellations of the logarithmic poles, the remaining leading contributions are bounded functions. Thus smooth corrections become relevant. We conclude that it is necessary to determine the *smooth contributions* to the fermionic projector $P(x,y)$ at the origin $x=y$. This analysis is carried out in Appendix D to first order in the bosonic potentials using a resummation technique. In what follows, we use these results and explain them.

To introduce a convenient notation, we write the factors $T^{(1)}_{[p]}$ in generalization of (7.13) as

$$T^{(1)}_{[p]} = \frac{1}{32\pi^3} \Big(\log|\xi^2| + i\pi\, \Theta(\xi^2)\, \epsilon(\xi^0) \Big) + s_{[p]} \,, \tag{8.4}$$

where the real-valued functions $s_{[p]}$, which may depend on the masses and the bosonic potentials, will be specified below. Taking the complex conjugate of (7.13), we get a similar representation for $\overline{T^{(1)}_{[p]}}$. Substituting these formulas into $\mathcal{R}$, the factors $\log|\xi^2|$ cancel each other as a consequence of (8.2) and (8.3). Moreover, a short calculation shows that the factors $i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0)$ in (7.13) also drop out. Applying Lemma 7.1, the EL equations to degree four yield the vector equation

$$j_a\, N_5 - m^2 A_a\, N_6 = J_a\, N_3 \tag{8.5}$$

with N_3 as in (7.10) and

$$N_5 = \frac{g^3}{6\,\overline{T^{(0)}_{[0]}}} \Big[T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big] \tag{8.6}$$

$$- \frac{g}{3\,\overline{T^{(0)}_{[0]}}} \Big[T^{(0)}_{[2]} T^{(0)}_{[1]} \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big]\, m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta \tag{8.7}$$

$$+ g\,(s_{[0]} - s_{[3]})\, \frac{g^2}{3\,\overline{T^{(0)}_{[0]}}}\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big(T^{(0)}_{[0]} - \overline{T^{(0)}_{[0]}} \Big) \tag{8.8}$$

$$N_6 = -\frac{2g\,\hat{Y}^2}{\overline{T^{(0)}_{[0]}}} \Big[T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} \Big(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} + T^{(0)}_{[0]} \overline{T^{(0)}_{[1]}} \Big) - c.c. \Big] \tag{8.9}$$

$$+ \frac{2g^2\,\acute{Y}\grave{Y}}{\overline{T^{(0)}_{[0]}}} \Big[T^{(0)}_{[2]} T^{(0)}_{[0]} \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big] \tag{8.10}$$

$$- \frac{2g\,\acute{Y}\grave{Y}}{\overline{T^{(0)}_{[0]}}} \Big[T^{(0)}_{[2]} T^{(0)}_{[1]} \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \Big]\, m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta \tag{8.11}$$

$$+ (s_{[2]} - s_{[3]})\, \frac{2g^2\,\acute{Y}\grave{Y}}{\overline{T^{(0)}_{[0]}}}\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big(T^{(0)}_{[0]} - \overline{T^{(0)}_{[0]}} \Big) \,. \tag{8.12}$$

By direct inspection one verifies that the integration-by-parts rules (5.8) do not yield relations between the simple fractions. In other words, the appearing simple fraction are all basic fractions. When evaluating weakly on the light cone (5.7), all basic fractions are of degree four, thus producing the same factor $\varepsilon^{-3}(i|\vec{\xi}|)^{-4}$. Using furthermore that no logarithmic divergences appear, we conclude that (8.5) must hold if the basic fractions are replaced by the corresponding regularization parameters.

In the case of *more than three generations*, the parameters $d_1, \ldots, d_g$ are not uniquely determined by the three conditions (7.34), (7.35) and (8.2). Thus by choosing the generation mixing matrix $\mathfrak{g}$ appropriately, we can give the quantity $\sum_{\beta=1}^g m_\beta^2\, d_\beta$ in (8.7) and (8.11) an arbitrary value. In particular, by making this quantity large, we can arrange that (8.5) holds for an arbitrarily small gauge field. In other words, by a suitable choice of the local axial transformation we can make the coupling constant as small as we like, so that the field equations become trivial.

In the remaining case of *three generations*, the parameters d_β are uniquely determined by the conditions (7.34), (7.35) and (8.2). A short calculation gives

$$m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta = 1\,.$$

Substituting this relation into (8.7) and (8.11), we obtain the condition

$$\Big(c_0 - c_1(s_{[0]} - s_{[3]})\Big) j_a - m^2 \left(c_2 \hat{Y}^2 + c_3 \acute{Y}\grave{Y} - 2c_1(s_{[2]} - s_{[3]}) \acute{Y}\grave{Y}\right) A_a = \frac{c_1}{8\pi}\, J_a\,, \quad (8.13)$$

where the constants $c_0, \ldots, c_3$ are the four regularization parameters corresponding to the following basic fractions:

$$\begin{aligned}
c_0 \;&:\; \frac{1}{6\,\overline{T^{(0)}_{[0]}}} \left[\left(3^3\, T^{(0)}_{[0]} T^{(0)}_{[0]} - 2\cdot 3\, T^{(0)}_{[1]} T^{(0)}_{[2]}\right) \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \right] \\
c_1 \;&:\; -\frac{3^2}{T^{(0)}_{[0]}}\, T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \left(T^{(0)}_{[0]} - \overline{T^{(0)}_{[0]}} \right) \\
c_2 \;&:\; -\frac{2\cdot 3}{T^{(0)}_{[0]}} \left[T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} \left(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} + T^{(0)}_{[0]} \overline{T^{(0)}_{[1]}} \right) - c.c. \right] \\
c_3 \;&:\; \frac{2}{T^{(0)}_{[0]}} \left[\left(3^2\, T^{(0)}_{[0]} - g\, T^{(0)}_{[1]} \right) T^{(0)}_{[2]} \overline{T^{(0)}_{[0]} T^{(-1)}_{[0]}} - c.c. \right].
\end{aligned} \quad (8.14)$$

Since we are free to multiply (8.13) by a non-zero constant, our field equations (8.13) involve *three regularization parameters*. For a given regularization scheme, these parameters can be computed to obtain numerical constants, as will be explored further in §8.6. Alternatively, these three parameters can be regarded as empirical constants which take into account the unknown microscopic structure of space-time. Apart from these constants, all the quantities in (8.13) are objects of macroscopic physics, defined independent of the regularization.

It remains to determine the quantities $s_{[0]}$, $s_{[2]}$ and $s_{[3]}$. As in (8.1), we again consider the leading order at the origin, and thus it suffices to compute the functions $s_{[p]}(x,y)$ at $x = y$. Let us begin with the calculation of $s_{[3]}$. Since the factors $T^{(1)}_{[3]}$ and $\overline{T^{(1)}_{[3]}}$ only appear in (7.37), the function $s_{[3]}$ is obtained by computing the local axial transformation of the fermionic projector of the vacuum. According to (4.13)–(4.16), the relevant contribution to the β^{th} Dirac sea involving the logarithmic pole and the constant term at the origin is

$$P_\beta(x,y) \asymp \frac{m_\beta^3}{32\pi^3} \log(m_\beta^2|\xi^2|) + c_0 \,.$$

Since in (8.13) only the differences of the functions $s_{[p]}$ appear, we may always disregard the constant c_0. Computing the corresponding contribution to (7.31) and comparing with (8.4), we find that $s_{[3]}$ is the constant

$$s_{[3]} = \frac{1}{32\pi^3} \frac{\sum_{\beta=1}^3 d_\beta\, m_\beta^3 \log(m_\beta^2)}{\sum_{\beta=1}^3 d_\beta\, m_\beta^3} = \frac{1}{32\pi^3} \sum_{\beta=1}^3 \frac{m_\beta^3 \log(m_\beta^2)}{m_1+m_2+m_3} \prod_{\alpha\neq\beta} \frac{1}{m_\beta - m_\alpha}\,, \tag{8.15}$$

where in the last step we again used the relations (7.34), (7.35) and (8.2). The functions $s_{[0]}$ and $s_{[2]}$ are more difficult to compute. Therefore, we first state the result and discuss it afterwards.

Lemma 8.1. *The operators $s_{[0]}$ and $s_{[2]}$ appearing in* (8.4) *and* (8.13) *have the form*

$$s_{[0]} j_{\text{a}} = \frac{1}{3{\cdot}32\pi^3} \sum_{\beta=1}^3 \Big(\log(m_\beta^2)\, j_a + f_{[0]}^\beta * j_{\text{a}} \Big) + \mathcal{O}(A_a^2) \tag{8.16}$$

$$s_{[2]} A_{\text{a}} = \frac{1}{32\pi^3\, m^2 \acute{Y}\grave{Y}} \sum_{\beta=1}^3 m_\beta^2 \Big(\log(m_\beta^2)\, A_{\text{a}} + f_{[2]}^\beta * A_{\text{a}} \Big) + \mathcal{O}(A_a^2), \tag{8.17}$$

where the star denotes convolution, i.e.

$$(f_{[p]}^\beta * h)(x) = \int f_{[p]}^\beta(x-y)\, h(y)\, d^4y \,.$$

The convolution kernels are the Fourier transforms of the distributions

$$\hat{f}_{[0]}^\beta(q) = 6 \int_0^1 (\alpha-\alpha^2) \log\left| 1 - (\alpha-\alpha^2)\, \frac{q^2}{m_\beta^2} \right| d\alpha \tag{8.18}$$

$$\hat{f}_{[2]}^\beta(q) = \int_0^1 \log\left| 1 - (\alpha-\alpha^2)\, \frac{q^2}{m_\beta^2} \right| d\alpha \,. \tag{8.19}$$

Postponing the proof of this lemma to Appendix D, we here merely discuss the result. For clarity, we first remark that the convolution operators in (8.16) and (8.17) can also be regarded as multiplication operators in momentum space, defined by

$$f_{[p]}^\beta * e^{-iqx} = \hat{f}_{[p]}^\beta(q)\, e^{-iqx}$$

with the functions $\hat{f}_{[p]}^\beta$ as in (8.18) and (8.19). Next, we note that the integrands in (8.18) and (8.19) only have logarithmic poles, so that the integrals are finite. In Appendix D, these integrals are even computed in closed form (see Lemma D.2 and Figure 7). Next, we point out that these integrals vanish if $q^2 = 0$, because then the logarithm in the integrand is zero. Therefore, the the convolutions by $f_{[p]}^\beta$ can be regarded as higher order corrections in q^2 to the field equations. Thus we can say that $s_{[0]}$ and $s_{[2]}$ are composed of constant terms involving logarithms of the Dirac masses, correction terms $f_{[p]}^\beta$ taking into account the dependence on the momentum q^2

of the bosonic potential, and finally correction terms of higher order in the bosonic potential.

The constant term in $s_{[0]}$ can be understood from the following simple consideration (the argument for $s_{[2]}$ is similar). The naive approach to determine the constant contribution to the β^{th} Dirac sea is to differentiate (4.13) at $a = m_\beta^2$ to obtain

$$T^{(1)}_{[0]} \asymp \frac{1}{32\pi^3} \left(\log(m_\beta^2) + 1 \right) ,$$

where we again omitted the irrelevant constant c_0. Taking the partial trace and comparing with (8.4), we obtain the contribution

$$s_{[0]} \asymp \frac{1}{3{\cdot}32\pi^3} \sum_{\beta=1}^{3} \left(\log(m_\beta^2) + 1 \right) . \tag{8.20}$$

This naive guess is wrong because there is also a contribution to the fermionic projector of the form $\sim \not{\xi} \xi_k j^k_{\text{a}} T^{(0)}_{[0]}$, which when contracted with $\not{\xi}$ yields another constant term which is not taken into account by the formalism of Chapter 5 (see the term (B.14) in Appendix B). This additional contribution cancels the summand +1 in (8.20), giving the desired constant term in (8.16).

Next, it is instructive to consider the scaling behavior of the functions $s_{[p]}$ in the fermion masses. To this end, we consider a joint scaling $m_\beta \to L m_\beta$ of all masses. Sine the expressions (8.15), (8.16) and (8.17) have the same powers of the masses in the numerator and denominator, our scaling amounts to the replacement $\log(m_\beta^2) \to \log(m_\beta^2) + 2 \log L$. Using the specific form of the operators $s_{[p]}$, one easily verifies that the transformation of the constant terms can be described by the replacement $s_{[p]} \to s_{[p]} + 2/(32\pi^3) \log L$. We conclude that for differences of these operators as appearing in (8.13), the constant terms are indeed scaling invariant. In other words, the constant terms in the expressions $s_{[0]} - s_{[3]}$ and $s_{[2]} - s_{[3]}$ depend only on quotients of the masses m_1, m_2 and m_3.

Before discussing the different correction terms in (8.16) and (8.17), it is convenient to combine all the constant terms in (8.16), (8.17) and (8.13). More precisely, multiplying (8.13) by $96\pi^3/c_0$ gives the following result.

Theorem 8.2. *The EL equations to degree four on the light cone give rise to the condition*

$$(C_0 - f_{[0]}*) j_{\text{a}} - (C_2 - 6 f_{[2]}*) A_{\text{a}} = 12\pi^2 \, J_{\text{a}} + \mathcal{O}(A_{\text{a}}^2) \tag{8.21}$$

involving the axial bosonic potential A_{a}, the corresponding axial current j_{a} and the axial Dirac current J_{a} (see (6.21)*,* (7.4) *and* (7.9)*). Here the convolution kernels are the Fourier transforms of the distributions*

$$\hat{f}_{[0]}(q) = \sum_{\beta=1}^{3} \hat{f}^{\beta}_{[0]}(q) \; = \sum_{\beta=1}^{3} 6 \int_0^1 (\alpha - \alpha^2) \, \log \left| 1 - (\alpha - \alpha^2) \, \frac{q^2}{m_\beta^2} \right| d\alpha$$

$$\hat{f}_{[2]}(q) = \sum_{\beta=1}^{3} m_\beta^2 \, \hat{f}^{\beta}_{[0]}(q) = \sum_{\beta=1}^{3} m_\beta^2 \int_0^1 \log \left| 1 - (\alpha - \alpha^2) \, \frac{q^2}{m_\beta^2} \right| d\alpha$$

(with the functions $\hat{f}^{\beta}_{[p]}$ as defined by (8.18) *and* (8.19)*). The constants C_0 and $m_1^2 C_2$ depend only on the regularization and on the ratios of the masses of the fermions.*

With this theorem, we have derived the desired field equations for the axial potential A_{a}. They form a linear hyperbolic system of equations involving a mass term, with corrections in the momentum squared and of higher order in the potential. It is remarkable that the corrections in the momentum squared are described by explicit convolutions, which do not involve any free constants. In order to make the effect of the convolution terms smaller, one must choose the constants C_0 and C_2 larger, also leading to a smaller coupling of the Dirac current. Thus the effect of the convolution terms decreases for a smaller coupling constant, but it cannot be arranged to vanish completely.

We proceed by explaining and analyzing the above theorem, beginning with the convolution operators $f_{[p]}$ (§8.2) and the higher orders in the potential (§8.3). In §8.4 we explain how the standard loop corrections of quantum field theory appear in our approach. In §8.5 we explain why the Higgs boson does not appear in our framework. Finally, in §8.6 we compute the coupling constants and the bosonic rest mass for a few simple regularizations.

8.2. **Violation of Causality.** In this section we want to clarify the significance of the convolution operators in the field equations (8.21). Our first step is to bring the convolution kernels into a more suitable form. For any $a > 0$, we denote by S_a the following Green's function of the Klein-Gordon equation,

$$S_a(x,y) = \int \frac{d^4k}{(2\pi)^4} \, \frac{\mathrm{PP}}{k^2-a} \, e^{-ik(x-y)} \tag{8.22}$$

$$= -\frac{1}{2\pi}\, \delta(\xi^2) + \frac{a}{4\pi}\, \frac{J_1\big(\sqrt{a\,\xi^2}\big)}{\sqrt{a\xi^2}}\, \Theta(\xi^2)\,, \tag{8.23}$$

where in the last step we again set $\xi = y - x$ and computed the Fourier integral using the Bessel function J_1. This Green's function is obviously causal in the sense that it vanishes for spacelike ξ. Due to the principal part, it is indeed the mean of the advanced and retarded Green's function; this choice has the advantage that S_a is symmetric, meaning that $\overline{S_a(x,y)} = S_a(y,x)$. Expanding the Bessel function in a power series, the square roots drop out, showing that S_a is a power series in a. In view of the explicit and quite convenient formula (8.23), it seems useful to express the convolution kernels in terms of S_a. This is done in the next lemma.

Lemma 8.3. *The distributions $f^\beta_{[p]}$ as defined by* (8.18) *and* (8.19) *can be written as*

$$f^\beta_{[0]}(x-y) = \int_{4m_\beta^2}^\infty \left(S_a(x,y) + \frac{\delta^4(x-y)}{a} \right) \sqrt{a-4m_\beta^2}\, (a+2m_\beta^2)\, \frac{da}{a^{\frac{3}{2}}} \tag{8.24}$$

$$f^\beta_{[2]}(x-y) = \int_{4m_\beta^2}^\infty \left(S_a(x,y) + \frac{\delta^4(x-y)}{a} \right) \sqrt{a-4m_\beta^2}\, \frac{da}{\sqrt{a}}\,. \tag{8.25}$$

Proof. We first compute the Fourier transform of the distribution $\log|1-q^2/b|$ for given $b>0$. Using that $\lim_{a\to\infty} \log|1-q^2/a| = 0$ with convergence as a distribution, we have

$$\log\left|1-\frac{q^2}{b}\right| = -\int_b^\infty \frac{d}{da} \log\left|1-\frac{q^2}{a}\right| da = \int_b^\infty \left(\frac{\mathrm{PP}}{q^2-a} + \frac{1}{a} \right) da\,.$$

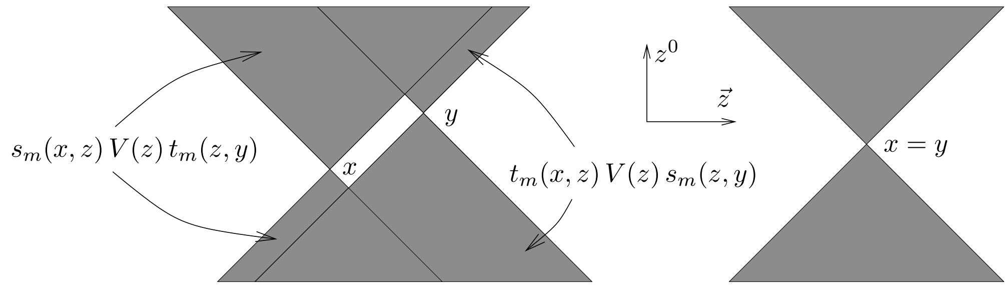

FIGURE 2. The support of the integrand in (8.26).

Now we can compute the Fourier transform with the help of (8.22). Setting $b = m_\beta^2/(\alpha-\alpha^2)$, we obtain

$$\int \frac{d^4q}{(2\pi)^4}\,\log\left|1-(\alpha-\alpha^2)\,\frac{q^2}{m_\beta^2}\right| e^{-iq(x-y)} = \int_{\frac{m_\beta^2}{\alpha-\alpha^2}}^\infty \left(S_a(x,y)+\frac{\delta^4(x-y)}{a}\right) da\,.$$

We finally integrate over α, interchange the orders of integration,

$$\int_0^1 d\alpha\,(\alpha-\alpha^2)^r \int_{\frac{m_\beta^2}{\alpha-\alpha^2}}^\infty da\,(\cdots) = \int_{4m_\beta^2}^\infty da\,(\cdots)\int_0^1 d\alpha\,(\alpha-\alpha^2)^r\,\Theta\Big(a-\frac{m_\beta^2}{\alpha-\alpha^2}\Big),$$

and compute the last integral. □

Qualitatively speaking, this lemma shows that that the distributions $f_{[p]}(x,y)$ can be obtained by integrating the Green's function S_a over the mass parameter a and by subtracting a suitable counter term localized at $\xi=0$. The interesting conclusion is that the convolution kernels $f_{[p]}(x,y)$ in the field equations (8.21) are weakly causal in the sense that they vanish for spacelike ξ. But they are not strictly causal in the sense that the past influences the future and also *the future influences the past*.

Before discussing whether and how such a violation of causality could be observed in experiments, we give a simple consideration which conveys an intuitive understanding for how the non-causal contributions to the field equations come about. For simplicity, we consider the linear perturbation ΔP of a Dirac sea of mass m by a potential $\mathcal{B}$,

$$\Delta P(x,y) = -\int d^4z\,\Big(s_m(x,z)\,V(z)\,t_m(z,y)\;+\;t_m(x,z)\,V(z)\,s_m(z,y)\Big)\,, \qquad (8.26)$$

where s_m is the Dirac Green's function and t_m denotes the Dirac sea of the vacuum, i.e.

$$t_m = (i\partial\!\!\!/ + m)T_{m^2} \qquad \text{and} \qquad s_m = (i\partial\!\!\!/ + m)S_{m^2}\,, \qquad (8.27)$$

and T_{m^2} and S_{m^2} as defined by (4.12) and (8.22) (for details see [11, equations (2.4) and (2.5)]). Let us consider the support of the integrand in (8.26). The Green's function s_m vanishes outside the light cone (see (8.23)), whereas the distribution t_m is non-causal (see (4.13)). Thus in (8.26) we integrate over the union of the double light cone (meaning the interior of the light cones and their boundaries) centered at the points x and y; see the left of Figure 2. In the limit $x\to y$, the integral in (8.26) will diverge, as becomes apparent in the poles of light cone expansion. But after subtracting these divergent contributions, we can take the limit $x\to y$ to obtain a well-defined integral over the double light cone centered at the point $x=y$; see the right of Figure 2.

Indeed, the finite contribution at the origin described by this integral corresponds precisely to the smooth contribution to the fermionic projector as considered in §8.1. This consideration explains why the distributions $f^{\beta}_{[p]}(x-y)$ vanish for spacelike ξ. We even see that the distributions $f^{\beta}_{[p]}(x,y)$ are closely related to the pointwise product in position space of the Bessel functions appearing in the distributions $T_{m^2}(x,y)$ and $S_{m^2}(x,y)$ for timelike ξ. Going into more details, this argument could even be elaborated to an alternative method for computing the convolution kernels. However, for actual computations this alternative method would be less convenient than the resummation technique of Appendix D.

One might object that the above violation of causality occurs simply because in (8.26) we are working with the wrong Green's functions. Indeed, if in (8.26) the first and second factors s_m were replaced by the regarded and advanced Green's function, respectively, the support of the integral would become strictly causal in the sense that z must lie in the causal past of x or y. However, modifying the Green's functions in this way is not admissible, as it would destroy the property that the Dirac sea is composed only of half of the solutions of the Klein-Gordon equation. More generally, the uniqueness of the perturbation expansion of the fermionic projector follows from a causality argument (see [13, §2.2]). Thus there is no freedom in modifying the perturbation expansion, and thus the above violation of causality cannot be avoided.

The violation of causality in the field equations breaks with one of the most fundamental physical principles. The immediate question is whether and how this effect could be verified in experiments. We conclude this section by discussing this question. Before beginning, we point out that the present paper is concerned with a simple fermion system, and one should be careful to draw physical conclusions from this oversimplified physical model. Also, the author has no expertise to address experimental issues. Nevertheless, it seems worth exploring the potential consequences of the causality violation in a few "Gedanken experiments," just to make sure that we do not get immediate contradictions to physical observations. In order to be closer to everyday physics, let us consider what happened if we inserted the nonlocal convolution term into Maxwell's equations. For simplicity, we consider one Dirac wave function Ψ of mass m. Thus dropping the mass term in (8.21) and choosing for convenience the Lorentz gauge, the modified Dirac-Maxwell equations become

$$\boxed{(i\partial\!\!\!/ + A\!\!\!/ - m)\Psi = 0\,, \qquad -\Big[1 - \frac{e^2}{12\pi^2}\, f_{[0]} *\Big]\Box A_k = e^2\, \overline{\Psi}\gamma_k\Psi\,,} \tag{8.28}$$

where chose the constant C_0 such that without the convolution terms, the Maxwell equations take the familiar form $-\Box A_k = e^2\overline{\Psi}\gamma_k\Psi$ (note that we again use the convention where the Dirac equation involves no coupling constants; see also Footnote 1 on page 15). In view of Lemma 8.3, the square bracket is an integral operator which vanishes for spacelike distances. Furthermore, we see from (8.18) and (8.19) (for more details see Lemma D.2) that the functions $\hat{f}^{\beta}_{[0]}(q)$ diverge for large q^2 only logarithmically. Thus in view of the smallness of the fine structure constant $e^2/4\pi \approx 1/137$, for the energy range accessible by experiments the square bracket in (8.28) is an invertible operator. Thus we may write our modified Dirac-Maxwell equations as

$$(i\partial\!\!\!/ + A\!\!\!/ - m)\Psi = 0\,, \qquad -\Box A_k = \Big[1 - \frac{e^2}{12\pi^2}\, f_{[0]} *\Big]^{-1} e^2\, \overline{\Psi}\gamma_k\Psi\,, \tag{8.29}$$

showing that the convolution term can be regarded as a modification of the source term. Alternatively, one may write the Maxwell equation in the standard form

$$-\Box \tilde{A}_k = e^2\, \overline{\Psi} \gamma_k \Psi \tag{8.30}$$

with a so-called *auxiliary potential* $\tilde{A}$ and take the point of view that the convolution term only affects the coupling of the electromagnetic potential to the Dirac equation,

$$(i\partial\!\!\!/ + A\!\!\!/ - m)\Psi = 0 \qquad \text{with} \qquad A := \left[1 - \frac{e^2}{12\pi^2}\, f_{[0]} * \right]^{-1} \tilde{A} \tag{8.31}$$

(note that the wave and convolution operators commute, as they are both multiplication operators in momentum space). Both the "source form" (8.29) and the "coupling form" (8.30) and (8.31) are useful; they give different points of view on the same system of equations. We point out that, as the inverse of a causal operator, the operator on the right of (8.29) and (8.31) is again causal in the sense that its integral kernel vanishes for spacelike distances. Moreover, for large timelike distances the kernel $f_{[0]}$ is oscillatory and decays. More specifically, writing the Green's function S_a in (8.24) with Bessel functions and using their asymptotic expansion for large ξ^2, one finds that

$$f^{\beta}_{[0]}(x-y) \sim m_\beta\, (\xi^2)^{-\frac{3}{2}}\, \cos\left(\sqrt{4m_\beta^2 \xi^2} + \varphi\right) \qquad \text{if } \xi^2 \gg m_\beta^2 \tag{8.32}$$

(where φ is an irrelevant phase).

The formulation (8.30) and (8.31) reveals that our modified Dirac-Maxwell equations are of *variational form*. More precisely, they can be recovered as the EL equations corresponding to the modified Dirac-Maxwell action

$$\mathcal{S}_{\text{DM}} = \int_M \left\{ \overline{\Psi} \Big(i\partial\!\!\!/ + \left[1 - \frac{e^2}{12\pi^2}\, f_{[0]} * \right]^{-1} \tilde{A\!\!\!/} - m \Big) \Psi - \frac{1}{4e^2} \tilde{F}_{ij}\, \tilde{F}^{ij} \right\} d^4x\,,$$

where $\tilde{F}$ is the field tensor corresponding to the auxiliary potential. Hence by applying Noether's theorem, we obtain corresponding conserved quantities, in particular the total *electric charge* and the total *energy* of the system. Thus all conservation laws of the classical Dirac-Maxwell system still hold, but clearly the form of the conserved quantities must be modified by suitable convolution terms.

The simplest idea for detecting the convolution term is to expose an electron to a *laser pulse*. Then the convolution term in the Dirac equation (8.31) might seem to imply that the electron should "feel" the electromagnetic wave at a distance, or could even be influenced by a laser beam flying by in the future, at a time when the electron may already have moved away. However, such obvious violations of causality are impossible for the following reason: An electromagnetic wave satisfies the vacuum Maxwell equations $\Box \tilde{A} = 0$ (see (8.30)). Thus the momentum squared of the electromagnetic wave vanishes, implying that $f_{[0]} * \tilde{A} = 0$, so that the convolution term in (8.31) drops out. In more general terms, the convolution terms are constant if the bosonic field is on-shell. We conclude that the convolution terms can be detected only by *off-shell* bosonic fields, which according to (8.30) occur only at the electromagnetic sources.

Another idea for observing the convolution term is that, according to (8.29), it modifies the way the Dirac current generates an electromagnetic field. Due to the prefactor $e^2/(12\pi^2)$ and in view of the fact that the kernel $f_{[0]}$ decays and has an oscillatory behavior (8.32), this effect will not be large, but it could nevertheless be observable. In particular, one may ask whether the positive and negative charges of protons and electrons still compensate each other in such a way that macroscopic objects appear neutral. If this were not the case, this would have drastic consequences, because then the electromagnetic forces would dominate gravity on the large scale. To analyze this question we consider for example a crystal containing exactly as many positive and negative charges. Then the corresponding auxiliary potential $\tilde{A}$ vanishes outside the crystal (except for dipole effects, which fall off fast with increasing distance). As a consequence, the potential A defined by (8.31) also vanishes outside the crystal, and thus there are no observable electrostatic forces outside the crystal, in agreement with physical observations.

More generally, the above considerations show that the convolution term can lead to observable effects only if the sources of the electromagnetic field and the Dirac particles on which it acts are very close to each other, meaning that the whole interaction must take place on the scale of the Compton length of the electron. One conceivable way of measuring this effect is by considering *electron-electron scattering*. In order to concentrate on the violation of causality, it seems preferable to avoid the noise of the usual electromagnetic interactions by considering two wave packets which stay causally separated, but nevertheless come as close as the Compton length. In this case, an electron in the future could even affect the motion of an electron in the past. However, due to the Heisenberg uncertainty principle, localizing a wave packet on the Compton scale implies that the energy uncertainty is of the order of the rest mass, so that pair creation becomes a relevant effect. Therefore, arranging such wave packets seems a very difficult task.

Another potential method for observing the convolution term is to get a connection to the high-precision measurements of atomic spectra. Thus we conclude the discussion by considering the *static* situation. Integrating the Green's function (8.22) over time, we can compute the remaining spatial Fourier integral with residues to obtain the familiar Yukawa potential,

$$\begin{aligned} V_a(\vec{\xi}) &:= \int_{-\infty}^{\infty} S_a(x,y)\, d\xi^0 = -\int_{\mathbb{R}^3} \frac{d\vec{k}}{(2\pi)^3}\, \frac{e^{-i\vec{k}\vec{\xi}}}{|\vec{k}|^2+a} \\ &= -\frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2 dk}{k^2+a} \int_{-1}^1 e^{-ikr\cos\vartheta} d\cos\vartheta \\ &= \frac{1}{(2\pi)^2\, ir} \int_{-\infty}^\infty \frac{k}{k^2+a}\, e^{-ikr}\, dk = -\frac{1}{4\pi}\, \frac{e^{-\sqrt{a}r}}{r}\,, \end{aligned}$$

where we set $r = |\vec{\xi}|$. Hence in the static case, the convolution operator reduces to the three-dimensional integral

$$(f_{[0]} * h)(\vec{x}) = \int_{\mathbb{R}^3} f_{[0]}(\vec{x}-\vec{y})\, h(\vec{y})\, d\vec{y}$$

involving the kernel

$$f_{[0]}(\vec{\xi}) = \frac{1}{3} \sum_{\beta=1}^3 \int_{4m_\beta^2}^\infty \left[V_a(\vec{\xi}) + \frac{\delta^3(\vec{\xi})}{a} \right] \sqrt{a-4m_\beta^2}\, (a+2m_\beta^2)\, \frac{da}{a^{\frac{3}{2}}}\,. \tag{8.33}$$

We now consider a classical point charge Ze located at the origin. In order to compute the corresponding electric field A_0, we consider the corresponding Maxwell equation (8.29),

$$\Delta A_0(\vec{x}) = \left[1 - \frac{e^2}{12\pi^2}\, f_{[0]}*\right]^{-1} Ze^2\, \delta^3(\vec{x}) = Ze^2\, \delta^3(\vec{x}) + \frac{Ze^4}{12\pi^2}\, f_{[0]}(\vec{x}) + \mathcal{O}(e^6)\,.$$

In order to solve for A_0, we convolute both sides with the Newtonian potential $V_0(\xi) = -1/(4\pi r)$. To compute the resulting convolution of the Newtonian potential with $f_{[0]}$, we first observe that (8.33) involves the Yukawa potential V_a. Since convolution corresponds to multiplication in momentum space, we can use the simple transformation

$$\frac{1}{|\vec{k}|^2}\, \frac{1}{|\vec{k}|^2 + a} = \frac{1}{a} \left(-\frac{1}{|\vec{k}|^2 + a} + \frac{1}{|\vec{k}|^2} \right)$$

to conclude that

$$(V_0 * V_a)(\vec{x}) = \frac{1}{a}\, \big(V_a(\vec{x}) - V_0(\vec{x})\big)\ .$$

We thus obtain

$$A_0(\vec{x}) = -\frac{Ze^2}{4\pi r} + \frac{Ze^4}{12\pi^2} \sum_{\beta=1}^3 \int_{4m_\beta^2}^\infty \left[-\frac{e^{-\sqrt{a}|\xi|}}{4\pi r}\, \frac{1}{a} \right] \sqrt{a - 4m_\beta^2}\, (a + 2m_\beta^2)\, \frac{da}{a^{\frac{3}{2}}}\,.$$

Here the first summand is the Coulomb potential, whereas the second summand is an additional short-range potential. This is very similar to the situation for the relativistic correction described by the Darwin term (a relativistic correction to the Schrödinger equation; see for example [40, Section 3.3]). Concentrating the short range potential at the origin by the replacement

$$\frac{e^{-\sqrt{a}\, r}}{4\pi r} \to \frac{1}{a}\, \delta^3(\vec{x})\,,$$

we can carry out the a-integral to obtain

$$A_0(\vec{x}) = -\frac{Ze^2}{4\pi r} - \frac{Ze^4}{60\pi^2} \sum_{\beta=1}^3 \frac{1}{m_\beta^2}\, \delta^3(\vec{x})\,.$$

We thus end up with a correction to the Dirac Hamiltonian of the form

$$\boxed{H_{\text{noncausal}} = \frac{Ze^4}{60\pi^2} \sum_{\beta=1}^3 \frac{1}{m_\beta^2}\, \delta^3(\vec{x})\,.} \tag{8.34}$$

This correction term has a similar structure as the Darwin term [40, equation (3.87)],

$$H_{\text{Darwin}} = \frac{Ze^2}{8m_e^2}\, \delta^3(\vec{x})\,.$$

However, our correction term is smaller by a factor

$$\frac{2e^2}{15\pi^2} \approx 1.2 \cdot 10^{-3}\,.$$

Thus the effect of our convolution term is very small. Moreover, it acts only near the central charge, where corrections due to the internal structure of the nucleus come into play. Nevertheless, the correction (8.34) seems within reach of future experiments.

Before getting quantitative predictions, one clearly needs to work out the effect for a more realistic physical model.

8.3. **Higher Order Non-Causal Corrections to the Field Equations.** The non-causal convolution terms in the previous section were obtained by computing the non-causal contributions in (4.17) at the origin, considering the first order of the perturbation expansion (4.9). Likewise, the higher orders of this expansion also contribute to P^{le} and P^{he}, giving rise to higher order non-causal corrections to the field equations. In this section we briefly discuss the structure of these correction terms (computing them in detail goes beyond the scope of this paper).

It is natural to distinguish between the low and high energy contributions. The non-causal *low energy contribution* P^{le} in (4.17) can be computed at the origin to every order in $\mathcal{B}$ by extending the resummation technique of Appendix D to higher order (more precisely, according to the residual argument, we again get sums of the form (D.12), but with nested line integrals and multiple series, which are to be carried out iteratively). Similar as explained to first order after Lemma 8.3, the resulting corrections to the field equation are weakly causal in the sense that they can be described by convolutions with integral kernels which vanish for spacelike distances. Thus they have the same mathematical structure, but are clearly much smaller than the convolution terms in §8.2.

The non-causal *high energy contribution* P^{he} in (4.17) is more interesting, because it gives rise to corrections of different type. For simplicity, we explain their mathematical structure only in the case of one generation and only for the leading contribution to P^{he} (see [21] for details)

$$P^{\mathrm{he}} = -\frac{\pi^2}{4} \Big(t_m \,\mathcal{B}\, \overline{t_m} \,\mathcal{B}\, t_m - \overline{t_m} \,\mathcal{B}\, t_m \,\mathcal{B}\, \overline{t_m} \Big) + \mathcal{O}(\mathcal{B}^3) ,$$

where we set

$$t_m = (i\partial\!\!\!/ + m)\, T_{m^2} \qquad \text{and} \qquad \overline{t_m} = (i\partial\!\!\!/ + m)\, \overline{T_{m^2}} ,$$

and $\overline{T_a}$ is the complex conjugate of the distribution T_a, (4.12). Thus the distributions t_m and $\overline{t_m}$ are supported on the lower and upper mass shell, respectively. Evaluating this expression at the origin gives

$$\begin{aligned} P^{\mathrm{he}}(x,x) = -\frac{\pi^2}{4} \int_M d^4z_1 \int_M d^4z_2 \,\Big(& t_m(x,z_1)\, \mathcal{B}(z_1)\, \overline{t_m}(z_1,z_2)\, \mathcal{B}(z_2)\, t_m(z_2,x) \\ & - \overline{t_m}(x,z_1)\, \mathcal{B}(z_1)\, t_m(z_1,z_2)\, \mathcal{B}(z_2)\, \overline{t_m}(z_2,x) \Big) + \mathcal{O}(\mathcal{B}^3) . \end{aligned} \tag{8.35}$$

This is similar to a second order tree diagram, but instead of Green's functions it involves the projectors onto the lower and upper mass shells, which appear in alternating order. The expression is well-defined and finite (see [13, Lemma 2.2.2]). Similar to the correction terms in Theorem 8.2, our expression is a convolution, but now it involves two integrals, each of which contains one factor of $\mathcal{B}$. Consequently, the integral kernel depends on two arguments z_1 and z_2. The interesting point is that this integral kernel does not vanish even if the vectors $z_1 - x$ or $z_2 - x$ are space-like. Thus the corresponding corrections to the field equations violate causality even in the strong sense that in addition to an influence of the future on the past, there are even *interactions for spacelike distances*. This surprising result is in sharp contrast to conventional physical theories. However, since for space-like separation the kernels t_m decay exponentially fast on the Compton scale, the effect is extremely small. In particular, describing this exponential decay by the the Yukawa potential, this effect could be described similar to the correction (8.33) and (8.34). But compared to the latter first order correction, the second order correction by P^{he} would be smaller by a factor e^2. In view of the discussion in §8.2, measuring this correction is at present out of reach. Thus it seems that the only promising approach for detecting an effect of the high energy contribution is to look for an experiment which is sensitive to interactions between regions of space-time with spacelike separation, without being disturbed by any causal interactions.

8.4. **The Standard Quantum Corrections to the Field Equations.** We now explain how the quantum corrections due to the Feynman loop diagrams arise in our model. We will recover all the standard quantum corrections. Moreover, we will obtain quantum corrections of the previously described non-causal terms (see §8.2 and §8.3). For clarity, we proceed in several steps and begin by leaving out the non-causal convolution terms in the field equations (8.21). Furthermore, we consider only one Dirac particle of mass m and disregard the interaction of this particle with the states of the Dirac sea. Under these simplifying assumptions, the interaction is described by the coupled Dirac-Yang/Mills equations

$$(i\partial\!\!\!/ + \gamma^5 A\!\!\!/ - m)\Psi = 0 \,, \qquad \partial_{kl}A^l - \Box A_k - M^2 A_k = e^2\, \overline{\Psi}\gamma^5\gamma_k\Psi \,, \tag{8.36}$$

where A is the axial potential, and the bosonic rest mass M and the coupling constant e are determined from (8.21) by setting $M^2 = C_2/C_0$ and $e^2 = 12\pi^2/C_0$. We point out that the wave function Ψ and the bosonic field A in (8.36) are classical in the sense that no second quantization has been performed.

The equations (8.36) form a coupled system of nonlinear hyperbolic partial differential equations. For such a system, one can get local existence and uniqueness results (see for example [31, Section 5.3] or [43, Chapter 16]), but constructing global solutions is a very difficult task. Therefore, we must rely on a perturbative treatment, giving a connection to Feynman diagrams. Although this connection is quite elementary, it does not seem to be well-known to mathematicians working on partial differential equations. In physics, on the other hand, Feynman diagrams are usually derived from second quantized fields, where the connection to nonlinear partial differential equations is no longer apparent. Therefore, we now explain the procedure schematically from the basics, hopefully bridging a gap between the mathematics and physics communities. In order to be in a simpler and more familiar setting, we consider instead of (8.36) the Dirac-Maxwell equations in the Lorentz gauge, as considered in quantum electrodynamics (see for example [2])[4]

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)\Psi = 0 \,, \qquad -\Box A_k = e\, \overline{\Psi}\gamma_k\Psi \,. \tag{8.37}$$

[4] In order to bring the system (8.36) into a comparable form, one first takes the divergence of the Yang/Mills equation to obtain

$$-M^2\partial_k A^k = e^2\, \overline{\partial_k\Psi}\gamma^5\gamma^k\Psi + e^2\, \overline{\Psi}\gamma^5\gamma^k\partial_k\Psi = -2ie^2 m\, \overline{\Psi}\gamma^5\Psi \,,$$

where in the last step we used the Dirac equation. In particular, the divergence of A in general does not vanish. It seems convenient to subtract from A the gradient of a scalar field Φ,

$$B_k := A_k - \partial_k\Phi \,,$$

in such a way that the new potential B becomes divergence-free. This leads to the system of equations

$$\big(i\partial\!\!\!/ - m + \gamma^5 B\!\!\!/ + \gamma^5(\partial\!\!\!/\Phi)\big)\Psi = 0 \,, \quad -\Box\Phi = -\frac{2ie^2 m}{M^2}\, \overline{\Psi}\gamma^5\Psi \,, \quad (-\Box - M^2)B_k = e^2\, \overline{\Psi}\gamma^5\gamma_k\Psi + M^2\partial_k\Phi \,.$$

This system has the same structure as (8.37), and it can be analyzed with exactly the same methods. For the handling of the factors e see Footnote 1 on page 15.

The natural question in the theory of hyperbolic partial differential equations is the Cauchy problem, where we seek for solutions of (8.37) for given initial values

$$\Psi(t,\vec{x})|_{t=0} = \Psi_0(\vec{x}) \,, \qquad A(t,\vec{x})|_{t=0} = A_0(\vec{x}) \,, \quad \partial_t A(t,\vec{x})|_{t=0} = A_1(\vec{x}) \,. \tag{8.38}$$

In preparation, we formulate the equations as a system which is of first order in time. To this end, we introduce the field Φ with components

$$\Phi = \begin{pmatrix} \Psi \\ A \\ i\partial_t A \end{pmatrix} , \tag{8.39}$$

and write the system in the Hamiltonian form

$$i\partial_t \Phi(t,\vec{x}) = H\big(\Phi(t,\vec{x})\big) := H_0 \Phi + eB(\Phi) \,, \tag{8.40}$$

where in the last step we decomposed the Hamiltonian into its linear and non-linear parts given by

$$H_0 = \begin{pmatrix} -i\gamma^0 \,\vec{\gamma}\vec{\nabla} + \gamma^0 m & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\Delta & 0 \end{pmatrix} , \qquad B(\Phi) = \begin{pmatrix} -\gamma^0 \not{A} \Psi \\ 0 \\ \overline{\Psi} \gamma \Psi \end{pmatrix} . \tag{8.41}$$

In the case $e=0$, we have a linear equation, which is immediately solved formally by exponentiation,

$$\Phi(t) = e^{-itH_0}\, \Phi_0 \,,$$

where we set $\Phi_0 = \Phi|_{t=0}$. This equation is given a rigorous meaning by writing the so-called time evolution operator e^{-itH_0} as a spatial integral operator.

Lemma 8.4. *For any $t \geq 0$, the operator e^{-itH_0} can be written as*

$$(e^{-itH_0} \Phi)(\vec{x}) = \int_{\mathbb{R}^3} R_t(\vec{x}-\vec{y})\, \Phi(\vec{y})\, d\vec{y} \,, \tag{8.42}$$

where the integral kernel is the distribution

$$R_t(\vec{x}) = \begin{pmatrix} s^\wedge_m(t,\vec{x})\, (i\gamma^0) & 0 & 0 \\ 0 & -\partial_t S^\wedge_0(t,\vec{x}) & iS^\wedge_0(t,\vec{x}) \\ 0 & -i\Delta S^\wedge_0(t,\vec{x}) & -\partial_t S^\wedge_0(t,\vec{x}) \end{pmatrix} , \tag{8.43}$$

which involves the retarded Green's functions defined by

$$S^\wedge_a(x) = \lim_{\varepsilon \searrow 0} \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon k^0} \tag{8.44}$$

$$s^\wedge_m(t,\vec{x}) = (i\not{\partial}_x + m)\, S^\wedge_{m^2}(x) \,. \tag{8.45}$$

Proof. Using that for any $t>0$, the Green's function $S^\wedge_a$ is a solution of the Klein-Gordon equation $(-\Box - a) S_a(x) = 0$, a short calculation using (8.45) shows that (8.42) is a solution of the equation $(i\partial_t - H_0)(e^{-itH_0}\Phi) = 0$. In order to verify the correct initial conditions, we differentiate $S^\wedge_a$ with respect to time and carry out the t-integration with residues to obtain

$$\begin{aligned} \lim_{t\searrow 0} \partial_t^n S^\wedge_a(t,\vec{x}) &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d\vec{k}\, e^{i\vec{k}\vec{x}} \lim_{\varepsilon, t \searrow 0} \int_{-\infty}^{\infty} \frac{(-i\omega)^n}{\omega^2 - |\vec{k}|^2 - m^2 + i\varepsilon\omega}\, e^{-i\omega t}\, d\omega \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} d\vec{k}\, e^{i\vec{k}\vec{x}}\, (-2\pi i)\, \frac{(-i\omega)^n}{2\omega}\Big|_{\omega = \pm\sqrt{|\vec{k}|^2+m^2}} = \left\{ \begin{array}{cl} 0 & \text{if } n=0 \\ -\delta^3(\vec{x}) & \text{if } n=1 \,. \end{array} \right. \end{aligned}$$

Using this result in (8.45) and (8.43) shows that indeed $\lim_{t\searrow 0} R_t(\vec{x}) = \delta^3(\vec{x})$. ☐

In the nonlinear situation $e \neq 0$, it is useful to work in the so-called "interaction picture" (see for example [42, Section 8.5]). We thus employ the ansatz

$$\Phi(t) = e^{-itH_0} \Phi_{\text{int}}(t) , \tag{8.46}$$

giving rise to the nonlinear equation

$$i\partial_t \Phi_{\text{int}} = e B_{\text{int}}(\Phi_{\text{int}}(t)) , \tag{8.47}$$

where

$$B_{\text{int}}(\Phi_{\text{int}}(t)) = e^{itH_0} B\big(e^{-itH_0} \Phi_{\text{int}}(t)\big).$$

We regard (8.47) as an ordinary differential equation in time, which in view of (8.42) is nonlocal in space. From (8.46) one sees that Φ_{int} comes with the initial data $\Phi_{\text{int}}|_{t=0} = \Phi_0$. Taking a power ansatz in e,

$$\Phi_{\text{int}}(t) = \Phi^{(0)}_{\text{int}}(t) + e\, \Phi^{(1)}_{\text{int}}(t) + e^2\, \Phi^{(2)}_{\text{int}}(t) + \cdots ,$$

a formal solution of the Cauchy problem for Φ_{int} is obtained by integrating (8.47) inductively order by order,

$$\begin{aligned} \Phi^{(0)}_{\text{int}}(t) &= \Phi_0 , \qquad \Phi^{(1)}_{\text{int}}(t) = -i \int_0^t B_{\text{int}}\Big(\Phi^{(0)}_{\text{int}}(\tau)\Big)\, d\tau \\ \Phi^{(2)}_{\text{int}}(t) &= -i \int_0^t \nabla B_{\text{int}}\Big(\Phi^{(0)}_{\text{int}}(\tau)\Big) \cdot \Phi^{(1)}_{\text{int}}(\tau)\, d\tau \\ &= (-i)^2 \int_0^t d\tau\, \nabla B_{\text{int}}\Big(\Phi^{(0)}_{\text{int}}(\tau)\Big) \int_0^\tau d\sigma\, B_{\text{int}}\Big(\Phi^{(0)}_{\text{int}}(\sigma)\Big), \quad \ldots \end{aligned}$$

(here ∇B denotes the Jacobi matrix of B, where as in (5.15) we consider the real and imaginary parts of the arguments as independent variables). Substituting these formulas into (8.46), we obtain the desired solution Φ of the original Cauchy problem expressed as a sum of iterated time integrals, involving intermediate factors of the free time evolution operator $e^{-i\tau H_0}$. In particular, we obtain to second order

$$\begin{aligned} \Phi(t) &= e^{-itH_0} \Phi_0 - ie \int_0^t e^{-i(t-\tau)H_0} B\big(e^{-i\tau H_0} \Phi_0\big) \\ &\quad - e^2 \int_0^t d\tau\, e^{-i(t-\tau)H_0}\, \nabla B\big(e^{-i\tau H_0}\Phi_0\big) \int_0^\tau d\sigma\, e^{-i(\tau-\sigma)H_0}\, B\big(e^{-i\sigma H_0}\big) + \mathcal{O}(e^3) . \end{aligned}$$

We remark that in the case when $B(\Phi)$ is linear in Φ, this expansion simplifies to the well-known Dyson series (also referred to as the time-ordered exponential). In view of (8.39), we have derived a unique formal solution of the Cauchy problem (8.37) and (8.38).

Combining the above expansion for $\Phi(t)$ with the formula for the time evolution operator in Lemma 8.4, one can write the above perturbation expansion in a manifestly covariant form. Namely, when multiplying the operators R_t with B (or similarly ∇B or higher derivatives), the factors γ^0 in the first component of (8.43) and in the formula for B in (8.41) cancel each other, giving the Lorentz invariant expression $s^\wedge \not{A}$. Likewise, the Dirac current in (8.41) multiplies the retarded Green's function $S^\wedge_0$. Moreover, we can combine the spatial and time time integrals to integrals over Minkowski space. In this way, we can identify the contributions to the perturbation expansion with the familiar Feynman diagrams. More precisely, every integration variable corresponds to a vertex of the diagram, whereas the bosonic and fermionic Green's functions $S^\wedge_0$ and $s^\wedge_m$ are written as wiggled and straight lines, respectively. Denoting the argument of the

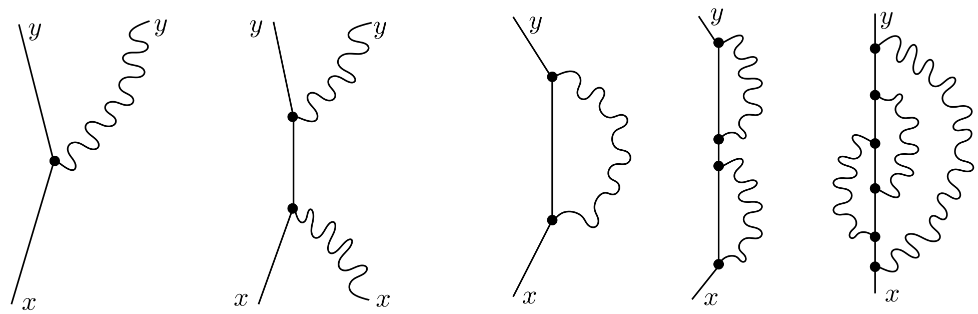

Figure 3. Feynman tree diagrams (left) and bosonic loop diagrams (right).

solution $\Phi(t,\vec{y})$ by y, whereas $x=(0,\vec{x})$ stands for the argument of the initial values, we obtain many diagrams as exemplified in Figure 3. Note that apart from simply-connected tree diagrams, we also obtain diagrams which are not simply connected, referred to as the *bosonic loop diagrams*. We come to the following conclusion:

- All bosonic loop diagrams can be obtained from the nonlinear system of partial differential equations (8.37), working purely with classical fields. For the derivation of the bosonic loop diagrams, there is no need for second quantization.

In order to make the connection to quantum field theory clearer, we point out that in quantum physics one usually does not consider the initial value problem (8.38). Instead, one is interested in the n-point functions, which give information about the correlation of the fields at different space-time points. The *two-point function* is obtained by choosing initial values involving δ^3-distributions. Similarly, all the n-point functions can be recovered once the solution of the Cauchy problem is known. Thus from a conceptual point of view, the only difference between our expansion and the Feynman diagrams in quantum field theory is that, since in quantum physics the Feynman diagrams do not come from an initial value problem, there is a *freedom in choosing the Green's function*. Note that in the setting of the Cauchy problem, one necessarily gets the retarded Green's function (see (8.43)). In contrast, in quantum field theory one is free to work instead with any other Green's function. Indeed, different choices lead to different approaches for handling the perturbation series. The most common choice is the so-called *Feynman propagator* (see for example [2]), which is motivated from the physical picture that the positive frequencies (describing particles) move to the future, whereas the negative frequencies (corresponding to anti-particles) move to the past. In this standard approach, the loop diagrams diverge. This problem is bypassed in the renormalization program by first regularizing the diagrams, and then removing the regularization while simultaneously adjusting the masses and coupling constants (see for example [4]). A quantum field theory is called *renormalizable* if this renormalization procedure works to all orders in perturbation theory, involving only a finite number of free parameters. There are different equivalent renormalization procedures, the most common being dimensional renormalization (see for example [37]). But the Feynman propagator is not a canonical choice, and indeed this choice suffers from the problem of not being invariant under general coordinate transformations (for more details see [13, §2.1]). An alternative method, which seems natural but has not yet been worked out, would be to extend the choice of Green's functions in the causal

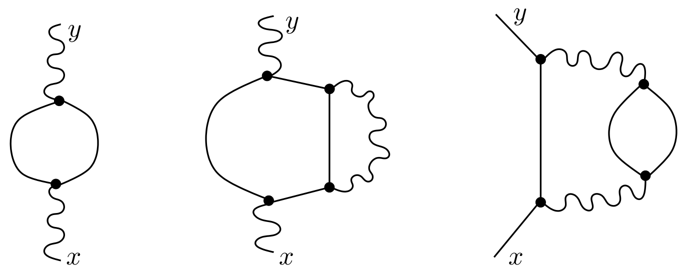

FIGURE 4. Typical Feynman diagrams involving fermion loops.

perturbation expansion (4.9) (see also [21]) to the loop diagrams. Yet another method is the so-called *causal approach* based on ideas of Epstein and Glaser [8], which uses the freedom in choosing the Green's function to avoid the divergences of quantum field theory (see also [41]). We also mention that our above derivation of Feynman diagrams is certainly not the most sophisticated or most elegant method. Maybe the cleanest method for the formal perturbation expansion is obtained in the framework of path integrals (see for example [35, 38]).

Recall that one simplification of the system (8.36) was that we considered only one Dirac particle and disregarded the interaction of this particle with the states of the Dirac sea. In particular, we did not allow for the creation of a particle/anti-particle pair. This shortcoming is reflected in our perturbation expansion in that the *fermionic loop diagrams* are missing (see Figure 4 for a few examples). This problem can easily be cured, because the framework of the fermionic projector does allow for the description of second-quantized fermions (see the discussion in §4.5). More specifically, we can describe pair creation in (4.10) by removing a particle from the Dirac sea and by occupying instead a state of positive energy. In the perturbation expansion, this effect is to be taken into account by additional diagrams which involve closed fermion loops. Since the fermionic projector respects the Pauli exclusion principle (see [13, Chapter 3]), the fermion loops automatically come with the correct relative signs. We conclude that

- The framework of the fermionic projector in the continuum limit yields all the Feynman diagrams of standard quantum field theory, including all bosonic and fermionic loop diagrams.

We again point out that this statement is true although we work only with classical bosonic fields. This raises the question why a quantization of the bosonic fields is at all needed, and what this "quantization" actually means. Here we shall not enter a discussion of this point, but refer the reader to [14, Section 4] and to the constructions in [19].

We next briefly discuss the effect of the non-causal correction terms (which we encountered in §8.2 and §8.3) on the perturbation series. The convolution terms in (8.21) can be taken into account simply by including them into the perturbation operator B in (8.40). In the subsequent perturbation expansion, this gives rise to new types of Feynman diagrams. Similarly, each of the higher order correction terms mentioned in §8.3 yields additional Feynman diagrams. The renormalizability of the resulting perturbation expansion is an open problem. But not matter what the answer to this question will be, one can say that, since the non-causal corrections are already very small, their loop corrections will be even smaller by factors of e^2. Thus it seems at least very difficult to observe them in the laboratory.

To summarize, the considerations in this section show that our approach is in agreement with all high-precision measurements of quantum field theory. The only measurable deviations of our approach from standard quantum field theory are the non-causal correction terms as discussed in §8.2 and §8.3, as well as the loop corrections to these non-causal terms. Naively counting powers of e^2, it seems that the convolution terms in (8.21) as well as the contribution by (8.35) are the most promising corrections for experimental tests.

We finally remark that the described method of first taking the continuum limit, then expanding the resulting equations in terms of Feynman diagrams and renormalizing these diagrams should be considered as preliminary. This method has the great advantage that it gives a simple connection to Feynman diagrams and to the renormalization program, making it easier to compare our approach to standard quantum field theory. But ultimately, a fully convincing theory should work exclusively with the regularized fermionic projector, thereby completely avoiding the ultraviolet divergences of quantum field theory. Before one can attack this program, one needs to have a better understanding of our variational principle in discrete space-time.

8.5. **The Absence of the Higgs Boson.** In this section we compare the mechanism leading to the mass term in the field equations (see (7.8) and (8.21)) with the Higgs mechanism of the standard model. Clearly, our framework is considerably different from that of the standard model, so that no simple comparison is possible. But in order to make the connection between the formalisms as close as possible, we can consider the effective action of the continuum limit and compare it to the action of a corresponding model involving a Higgs field. For simplicity leaving out the non-causal convolution terms and considering only one particle of mass m, the field equations (8.21) coupled to the Dirac equation are recovered as the EL equations corresponding to the action

$$\mathcal{S}_{\text{DYM}} = \int_M \left\{ \overline{\Psi}(i\partial\!\!\!/ + \gamma^5 A\!\!\!/ - m)\Psi - \frac{1}{4e^2}\, F_{ij}F^{ij} + \frac{M^2}{2e^2}\, A_j A^j \right\} d^4x \,, \qquad (8.48)$$

where A denotes the axial potential and $F_{ij} = \partial_i A_j - \partial_j A_i$ is the corresponding field tensor. The coupling constant e and the bosonic mass M are related to the constants in (8.21) by $e^2 = 12\pi^2/C_0$ and $M^2 = C_2/C_0$. We point out that this action is *not* invariant under the axial gauge transformation

$$\Psi(x) \to e^{-i\gamma^5 \Lambda(x)}\Psi(x) \,, \qquad A \to A + \partial\Lambda \,, \qquad (8.49)$$

because both the fermionic mass term $m\overline{\Psi}\Psi$ and the bosonic mass term $M^2 A_j A^j/(2e^2)$ have no axial symmetry. As explained in §6.2, the absence of an axial symmetry can be understood from the fact that the transformation of the wave function in (8.49) is not unitary, and thus it does not correspond to a local symmetry of our functionals in (2.8) (see also (6.33) and (6.34)).

The axial gauge transformation (8.49) can be realized as a local symmetry by adding a Higgs field ϕ, in complete analogy to the procedure in the standard model. More precisely, we introduce ϕ as a complex scalar field which behaves under axial gauge transformations as

$$\phi(x) \to e^{-2i\Lambda(x)}\phi(x) \,. \qquad (8.50)$$

The fermionic mass term can be made gauge invariant by inserting suitable factors of ϕ. Moreover, in view of (8.49), we can introduce a corresponding gauge-covariant derivative D by

$$D_j = \partial_j + 2iA_j \; .$$

Thus the *Dirac-Yang/Mills-Higgs action* defined by

$$\begin{aligned}\mathcal{S}_{\text{DYMH}} = \int_M \Big\{&\overline{\Psi}(i\partial\!\!\!/ + \gamma^5 A\!\!\!/)\Psi - m\overline{\Psi}(\phi\chi_L + \overline{\phi}\chi_R)\Psi \\ &-\frac{1}{4e^2}\, F_{ij}F^{ij} + \frac{M^2}{8e^2}\,(\overline{D_j\phi})(D^j\phi) - V\big(|\phi|^2\big)\Big\}\, d^4x\end{aligned} \tag{8.51}$$

is invariant under the axial gauge transformation (8.49) and (8.50). We now follow the construction of spontaneous symmetry breaking in the standard model. For V we choose a double well potential having its minimum at $|\phi|^2 = 1$. Then the Higgs field ϕ has a non-trivial vacuum with $|\phi| = 1$. Thus choosing an axial gauge where ϕ is real and positive, we can write ϕ as

$$\phi(x) = 1 + h(x)$$

with a real-valued field h. Since h vanishes in the vacuum, we may expand the action in powers of h. Taking the leading orders in h, we obtain the *action after spontaneous symmetry breaking*

$$\mathcal{S}_{\text{DYMH}} = \mathcal{S}_{\text{DYM}} + \mathcal{S}_{\text{Higgs}}$$

with

$$\mathcal{S}_{\text{Higgs}} = \int_M \left\{ -mh\,\overline{\Psi}\Psi + \frac{M^2 h}{e^2}\, A_j A^j + \frac{M^2}{8e^2}\,(\partial_j h)(\partial^j h) - 2V''(1)\, h^2 \right\} d^4x \, . \tag{8.52}$$

We conclude that for the action (8.51), the Higgs mechanism yields an action which reproduces the effective action of the continuum limit (8.48), but gives rise to additional terms involving a real Higgs boson h. The Higgs boson has a rest mass as determined by the free parameter $V''(1)$. It couples to both the wave function Ψ and the axial potential A.

The Higgs field h can also be described in the setting of the fermionic projector, as we know explain. Note that the coupling terms of the Higgs field to Ψ and A can be obtained from (8.48) by varying the masses according to

$$m \to (1 + h(x))\, m \; , \qquad M \to (1 + h(x))\, M \; .$$

Taking into account that in our framework, the bosonic masses are given in terms of the fermion masses (see (8.13)), this variation is described simply by inserting a *scalar potential* into the Dirac equation. Likewise, for a system involving several generations, we must scale all fermion masses by a factor $1+h$. This is implemented in the auxiliary Dirac equation (4.5) by choosing

$$\mathcal{B} = -mh(x)\, Y \; . \tag{8.53}$$

The remaining question is whether scalar perturbations of the form (8.53) occur in the setting of the fermionic projector, and whether our action principle (2.9) reproduces the dynamics of the Higgs field as described by (8.52). A-priori, *any* symmetric perturbation of the Dirac equation is admissible, and thus we can certainly consider the scalar perturbation (8.53). Since (8.53) is even under parity transformations, the corresponding leading perturbations of the eigenvalues λ^L_s and λ^R_s will be the same. In view of (7.1) and the formulas for the unperturbed eigenvalues (6.26), we find that the leading perturbations corresponding to (8.53) drop out of the EL equations. We thus conclude that, although a Higgs field can be described in our framework, the action

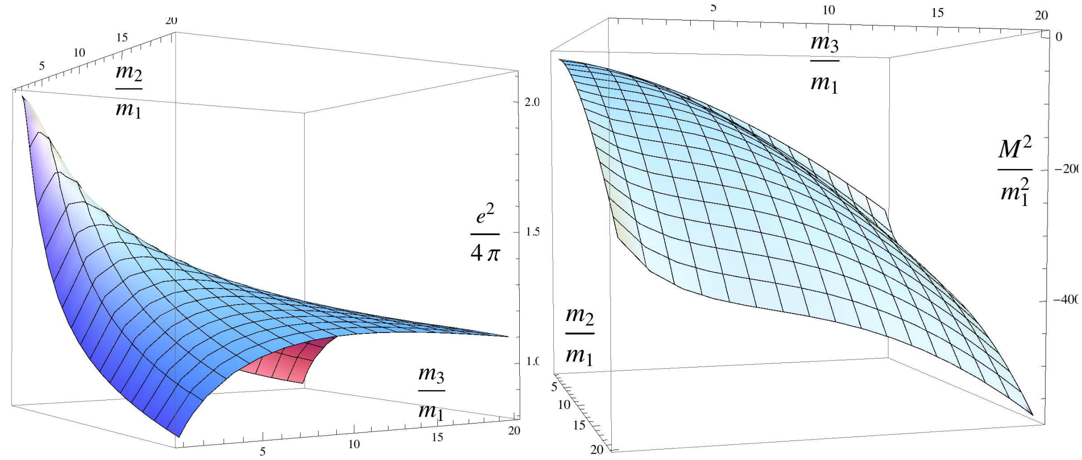

FIGURE 5. The coupling constant and the bosonic mass for the regularization (8.55).

principle (2.9) does not describe a dynamics of this field, but instead predicts that the Higgs field must vanish identically (for more details on scalar perturbations see Lemma B.1).

8.6. **The Coupling Constant and the Bosonic Mass in Examples.** The regularization parameters $c_0, \ldots, c_3$ in the field equations (8.13) are given in terms of simple fractions (8.14). For a given regularization method, we can evaluate these simple fractions and compute the coupling constant and the bosonic rest mass. We now exemplify the procedure by considering the two simplest methods of regularization:

(A) An *exponential factor* in momentum space: We define the distribution $\hat{P}^\varepsilon$ in (3.3) by

$$\hat{P}^\varepsilon(k) = \sum_{\beta=1}^{g} (\not{k} + m_\beta)\, \delta(k^2 - m^2)\, \Theta(-k^0)\; \exp(\varepsilon k^0)\ . \tag{8.54}$$

To the considered leading degree on the light cone, this regularization corresponds to the simple replacements

$$T^{(0)}_{[p]} \to -\frac{1}{8\pi^3} \frac{1}{2r\,(t-r-i\varepsilon)}, \qquad T^{(-1)}_{[p]} \to -\frac{2}{r}\frac{\partial}{\partial t} T^{(0)}_{[p]} = -\frac{1}{8\pi^3\, r^2} \frac{1}{(t-r-i\varepsilon)^2}, \tag{8.55}$$

and similarly for the complex conjugates. Using these formulas in (8.13), the basic fractions all coincide up to constants, giving the equation

$$\Big(-\frac{1}{2} - 96\pi^3\, (s_{[0]} - s_{[3]}) \Big) j_a + 4m^2 \Big(\hat{Y}^2 + \acute{Y}\grave{Y} + 48\pi^3 (s_{[2]} - s_{[3]}) \acute{Y}\grave{Y} \Big)\, A_a = 12\pi^2\, J_a\ .$$

According to Lemma 8.1 and (8.15), the functions $s_{[p]}$ involve the masses of the fermions and also the convolution terms $f^\beta_{[p]}$. For clarity, we here leave out the convolution terms (which are analyzed in detail in §8.2 and Appendix D). Then we can write the field equation in the usual form

$$j_a - M^2 A_a = e^2 J_a\ ,$$

where the coupling constant e and the mass M are given functions of the ratios m_2/m_1 and m_3/m_1 of the fermion masses. In Figure 5, these constants are plotted as functions of the mass ratios. The coupling constant is of the order one; it is largest if the fermion

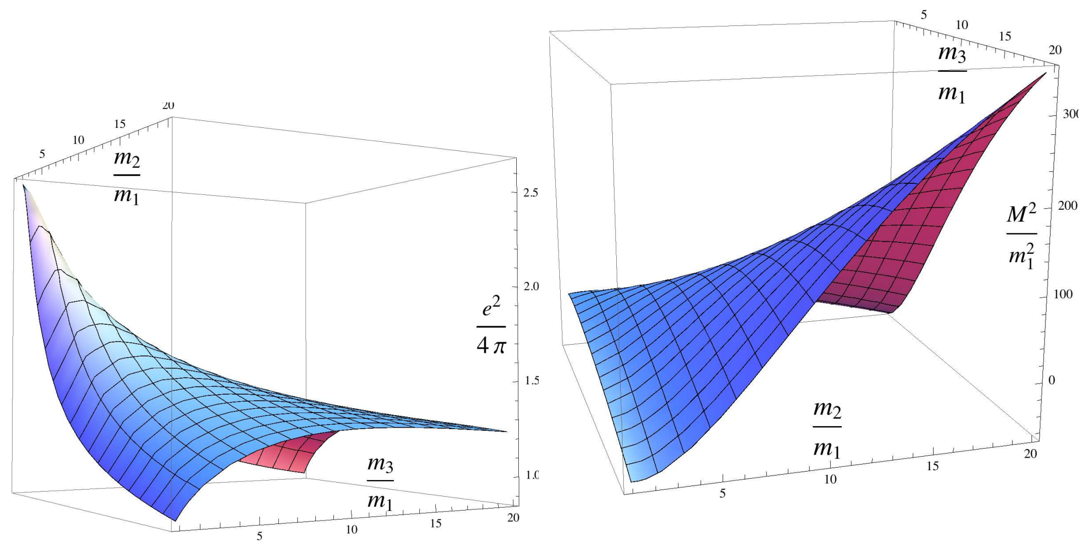

FIGURE 6. The coupling constant and the bosonic mass for the regularization (8.56) and $\varepsilon_1 = 2\varepsilon_0$ and $\varepsilon_2 = 5\varepsilon_1$.

masses are close to each other. The term M^2 is always negative. Thus the mass term has the wrong sign, showing that the regularization (8.55) is not a physically reasonable regularization.

The obvious idea for getting more general regularizations is to choose the parameter ε differently for the three Dirac seas in (8.54). Since the distributions $T^{(n)}_{[p]}$ are formed as linear combinations of contributions from the individual seas with weights m^p_β (see [13, equation (5.3.19)] for details), we can just as well choose the parameters ε in (8.55) as a function of p,

$$T^{(0)}_{[p]} \to -\frac{1}{8\pi^3}\, \frac{1}{2r\,(t-r-i\varepsilon_p)}\,, \qquad T^{(-1)}_{[p]} \to -\frac{1}{8\pi^3\, r^2}\, \frac{1}{(t-r-i\varepsilon_p)^2}\,, \tag{8.56}$$

It turns out that by modifying the additional parameters $\varepsilon_2/\varepsilon_1$ and $\varepsilon_3/\varepsilon_1$, we can indeed give the mass term the correct sign, as is exemplified in Figure 6. The interesting point is that the coupling constant is almost unaffected, indicating that our method of computing the coupling constant is quite robust to regularization details. To avoid confusion, we point out that the coupling constant and the bosonic mass do not depend on the regularization length ε, but only on the form of the regularization and on the fermion masses.

(B) A *cutoff* in momentum space: In **(A)** we considered a regularization which was "soft" in the sense that it was smooth in momentum space. To give a complementary example, we now consider the "hard" regularization obtained by inserting a Heaviside function into the integrand of (3.1). Thus we define the distribution $\hat{P}^\varepsilon$ in (3.3) by

$$\hat{P}^\varepsilon(k) = \sum_{\beta=1}^{g} (\not{k} + m_\beta)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, \Theta(1 + \epsilon k^0)\,.$$

This regularization is described in analogy to (8.55) by

$$T^{(0)}_{[p]} \to -\frac{1}{16\pi^3} \, \frac{1-e^{-\frac{i(t-r)}{\varepsilon}}}{2r\,(t-r)} \,, \qquad T^{(-1)}_{[p]} \to -\frac{2}{r} \, \frac{\partial}{\partial t} T^{(0)}_{[p]} \,.$$

Thus as expected, the cutoff in momentum space gives rise to rapid oscillations in position space. Using these formulas in (8.13), the resulting basic fractions are no longer multiples of each other. But the weak evaluation integrals (5.7) can still be computed in closed form. We thus obtain the field equation

$$\Big(-\frac{3}{4} - 96\pi^3 \, (s_{[0]} - s_{[3]}) \Big) j_a + 6m^2 \left(\hat{Y}^2 + \acute{Y}\grave{Y} + 32\pi^3 (s_{[2]} - s_{[3]}) \acute{Y}\grave{Y} \right) A_a = 12\pi^2 \, J_a \,.$$

This equation can be analyzed exactly as in example **(A)**, giving the same qualitative results. It is remarkable that the constants in the field equations in example **(A)** and **(B)** differ at most by a factor 3/2, again indicating that the results do not depend sensitively on the method of regularization.

9. The Euler-Lagrange Equations to Degree Three and Lower

The wave functions in (4.10) do not only have a vector and axial component as considered in §7.2, but they also have scalar, pseudoscalar and bilinear components. We will now analyze the effect of these contributions on the EL equations (§9.1 and §9.2). Moreover, we will insert further potentials into the Dirac equation and analyze the consequences. More precisely, in §9.2 we consider bilinear potentials, whereas in §9.3 we consider scalar and pseudoscalar potentials and discuss the remaining possibilities in choosing other potentials and fields. We conclude this chapter with a discussion of the structure of the EL equations to degree three and lower (§9.4).

9.1. **Scalar and Pseudoscalar Currents.** In analogy to the Dirac currents in §7.2, we introduce the *scalar Dirac current* J_{s} and the *pseudoscalar Dirac current* J_{p} by

$$J_{\mathrm{s}} = \sum_{k=1}^{n_f} \overline{\Psi_k} \Psi_k - \sum_{l=1}^{n_a} \overline{\Phi_l} \Phi_l \qquad \text{and} \qquad J_{\mathrm{p}} = \sum_{k=1}^{n_f} \overline{\Psi_k} \, i\gamma^5 \Psi_k - \sum_{l=1}^{n_a} \overline{\Phi_l} \, i\gamma^5 \Phi_l \,. \tag{9.1}$$

According to (4.10), these currents lead to a perturbation of the fermionic projector. In view of the fact that the scalar and pseudoscalar currents involve no Dirac matrix which could be contracted with a factor of ξ, one expects that the resulting contribution to the EL equations should be one degree lower than that of the axial current (see Lemma 7.4). Thus to leading order at the origin, one might expect contributions of the form

$$Q(x,y) \asymp J_{\mathrm{p}} \,(\text{monomial of degree three}) + (\deg < 3) \,. \tag{9.2}$$

However, perturbing the fermionic projector of the vacuum by (9.1), one sees that the corresponding contribution to $Q(x,y)$ of degree three on the light cone vanishes (see Lemma B.1, where it also explained how the cancellations come about). Taking into account the axial potentials, we do get contributions to $Q(x,y)$ of degree three on the light cone, which are of the form

$$Q(x,y) \asymp A^j_a \, \xi_j \, J_{\mathrm{s}} \,(\text{monomial of degree three}) + (\deg < 3) \,. \tag{9.3}$$

However, these contribution have the same tensor structure as the contributions of the axial potentials and currents of degree three, which will be discussed in §9.4 below.

9.2. **Bilinear Currents and Potentials.** The particles and anti-particles in (4.10) also have a bilinear component, leading us to introduce the *bilinear Dirac current* $J_{\rm b}$ by

$$J_{\rm b}^{ij} = \sum_{k=1}^{n_f} \overline{\Psi_k} \sigma^{ij} \Psi_k - \sum_{l=1}^{n_a} \overline{\Phi_l} \sigma^{ij} \Phi_l$$

(here $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$ are again the bilinear covariants). Likewise, one may want to insert a *bilinear potential*

$$\mathcal{B} = H_{ij}(x)\, \sigma^{ij}$$

into the auxiliary Dirac equation (4.5) (where H is an anti-symmetric tensor field).

Let us briefly discuss the effect of a bilinear current and a bilinear potential on the EL equations. The bilinear current corresponds to a perturbation of the fermionic projector of the form

$$\Delta P(x,y) = -\frac{1}{8\pi}\, J_{\rm b}^{ij} \sigma_{ij} + o(|\vec{\xi}|^0)\,.$$

The bilinear potential, on the other hand, gives rise to different types of contributions to the light-cone expansion which involve H and its partial derivatives (for details see [11, Appendix A.5]). When computing the perturbations of the eigenvalues $\lambda_\pm^{L/R}$ of the closed chain (cf. [13, Appendix G] or Appendix B), all Dirac matrices in ΔP are contracted with outer factors ξ. As a consequence, the contribution of the bilinear current drops out. Moreover, due to the anti-symmetry of H, all the bilinear contributions of the bilinear potential to the EL equations vanish. What remains are terms involving the divergence $\partial_j H^{ij}$ of the bilinear potential or derivatives of the divergence. All these terms can be interpreted as effective vector or axial potentials or their derivatives. Furthermore, the resulting contributions to the EL equations are of degree three on the light cone. Such contributions will again be discussed in §9.4 below.

9.3. **Further Potentials and Fields.** Having considered many different perturbations of the Dirac operator, we are now in the position to draw a few general conclusions, and to discuss a few potentials and fields which are not covered by the previous analysis. First of all, we point out that in Lemmas 7.3 and 7.4 the vector components dropped out, so that we got no contribution by an electromagnetic field. This cancellation can be understood from the general structure of our action principle, namely from the fact that the Lagrangian (6.28) involves the differences of the absolute values of the left- and right-handed eigenvalues. As a consequence, any perturbation which affects the eigenvalues λ^L_s and λ^R_s in the same way necessarily drops out of the EL equations. In other words, the EL equations are only affected by perturbations of the fermionic projector which have *odd parity*.

The fact that the electromagnetic field does not enter the EL equations does not necessarily imply that the electromagnetic field must vanish. But it means that no electromagnetic fields are generated in the system, so that the only possible electromagnetic field must be radiation coming from infinity. Having isolated systems without incoming radiation in mind, we conclude that our system involves *no electromagnetic field*.

The above consideration for the electromagnetic field also applies to the *gravitational field*, as we now explain. Introducing gravitational fields (see for example [13,

§1.5]) gives rise to contributions to the light-cone expansion which involve the metric, the curvature and the derivatives of curvature (see [11, AppendixB]). The main effect of the gravitational field can be understood as a "deformation" of the light cone corresponding to the fact that the light cone is now generated by the null geodesics. The corresponding contributions to the light-cone expansion drop out of the EL equations if we make the action principle diffeomorphism invariant simply by replacing the measure d^4x in (2.8) by $\sqrt{|\det g_{ij}|}\, d^4x$. In addition, there are terms involving the curvature of space-time, whose singularity on the light cone is of so small degree that the corresponding closed chain can again be treated perturbatively. Since these curvature terms are even under parity transformations, they drop out of the EL equations. We conclude that our model involves no gravitational field.

From the physical point of view, it might seem disappointing that our model involves no electromagnetic and gravitational fields. However, the simple explanation is that our system of one sector is too small to involve these fields. If one considers systems of several sectors, the equation analogous to (6.28) will involve the differences of the eigenvalues λ^c_s and $\lambda^{c'}_{s'}$ in different sectors. Then potentials no longer drop out even if they have even parity, provided that they are not the same in all sectors. Only a detailed analysis of systems involving several sectors will show whether the electromagnetic and gravitational fields will appear in the physically correct way.

Having understood why gravitational fields drop out of the EL equations, one might want to consider instead an *axial gravitational field* as described by a perturbation of the form

$$\mathcal{B} = i\gamma^5 \gamma_i\, h^{ij}\, \partial_j + (\text{lower order terms})\,. \tag{9.4}$$

Such a field cannot occur for the following reason. As just mentioned, a gravitational field describes a deformation of the light cone, which due to the diffeomorphism invariance of our action principle does not enter the EL equations. Similarly, an axial gravitational field (9.4) describes a deformation of the light cone, but now differently for the left- and right-handed components of the Dirac sea. Thus the light cone "splits up" into two light cones, leading to highly singular contributions to the EL equations. Thus in order to satisfy the EL equations, the axial gravitational field must vanish.

In order to avoid the problem of the axial deformations of the light cone, one may want to consider a so-called *axial conformal field*

$$\mathcal{B} = i\Xi(x)\gamma^5\gamma^j\partial_j + (\text{lower order terms})\,. \tag{9.5}$$

But this field is of no use, as the following consideration shows. Let us consider for a given real function Λ the so-called *axial scaling transformation* $U = e^{\gamma^5 \Lambda(x)}$. This transformation is unitary, and thus it has no effect on the EL equations. Transforming the Dirac operator according to

$$\begin{aligned}(i\partial\!\!\!/ - m) \rightarrow & U(i\partial\!\!\!/ - m)U^{-1} = iU^2 \partial\!\!\!/ - m + i\gamma^5 (\partial\!\!\!/ \Lambda) \\ &= \cosh(2\Lambda(x))\, i\partial\!\!\!/ + \sinh(2\Lambda(x))\, i\gamma^5 \partial\!\!\!/ - m + i\gamma^5(\partial\!\!\!/\Lambda)\,,\end{aligned}$$

the summand $\sinh(2\Lambda(x))\, i\gamma^5 \partial\!\!\!/$ can be identified with the first-order term in (9.5). The summand $\cosh(2\Lambda(x))\, i\partial\!\!\!/$, on the other hand, is a conformal gravitational field, and we already saw that gravitational fields do not enter the EL equations. Thus in total, we are left with a perturbation of the Dirac operator by chiral potentials, as considered earlier in this paper.

Next, we briefly consider a *scalar* or *pseudoscalar potential* (4.21). The leading contributions to the fermionic projector involve the potentials Ξ and Φ, whereas to lower degree on the light cone also derivatives of these potentials appear. Since Φ has even parity, its leading contribution to the EL equations vanishes. For the potential Ξ, the leading contribution cancels in analogy to (9.2) (see also Lemma B.1). But to degree four on the light cone, one gets cross terms similar to (9.3) which also involve the axial potential. In order for these additional terms to vanish, we are led to setting the scalar and pseudoscalar potentials equal to zero. Thus there seems no point in considering scalar or pseudoscalar potentials. Nevertheless, scalar or pseudoscalar perturbations might enter the EL equations to degree three and lower, as will be discussed in §9.4 below.

In the analysis of the axial potential we made one assumption which requires a brief explanation. Namely, when introducing the axial potential $A_{\rm a}$ in §6.2, we assumed that it couples to all generations in the same way (see (6.20)). In view of the constructions of §7.4–§7.6, where it was essential that the potentials were different for each generation, the ansatz (6.20) seems rather special, and one might wonder what would happen if we replaced the potential $A_{\rm a}$ in (6.20) by a matrix potential acting non-trivially on the generations. Indeed, this scenario was already discussed in [13, Remark 6.2.3], and thus here we briefly repeat the main idea. Suppose that the potential $A_{\rm a}$ in (6.20) were a matrix. Then the exponentials in (6.22) would have to be replaced by ordered exponentials of the form

$$\mathrm{Texp}\left(-i\int_x^y A_{\rm a}^j \xi_j\right).$$

This is a unitary matrix whose eigenvalues can be regarded as different phase factors. Thus when taking the partial trace, we do not get a single phase factor, but instead a linear combination of different phases. As a consequence, the relations (6.27) will in general be violated, so that we get a contribution to the EL equations to degree five on the light cone. Going this argument backwards, we can say that the EL equations to degree five imply that the eigenvalues (6.26) should involve only one phase, meaning that the axial potential must in fact be of the form (6.20) with a vector field $A_{\rm a}$.

9.4. **The Non-Dynamical Character of the EL Equations to Lower Degree.** In our analysis of the EL equations we began with the leading degree five on the light cone. The analysis to degree four revealed the field equations and thus described the dynamics. Generally speaking, to degree three and lower on the light cone, we get many more conditions, but on the other hand, we also get much more freedom to modify the fermionic projector. Namely, to degree three the Dirac currents as well as vector and axial potentials give rise to many terms, which in general do not cancel each other, even if the field equations of Chapter 8 are satisfied. For example, the factors $T^{(n)}_{\{p\}}$ involving curly brackets come into play (see (5.5)), and we also get contributions involving higher derivatives of the potentials. To degree two and one on the light cone, we get even more terms, involving the cross terms of different potentials and the terms generated by the mass expansion of the fermionic projector. In order to satisfy the EL equations to lower degree, all these terms must cancel each other. The good news is that we also get more and more free parameters. For example, we can consider scalar and pseudoscalar potentials (4.21), or bilinear potentials, or the other potentials discussed in §9.3. All these potentials can be chosen independently on the generations. Taking into account that to lower degree on the light cone, more and more cross terms between different perturbations come into play, we obtain a very complicated structure involving a large number of free parameters to modify the EL equations to degree three and lower.

In view of this complexity, it is not clear whether the EL equations can be satisfied to every degree on the light cone or not. The analysis becomes so complicated that it seems impossible to answer this question even with more computational effort. A possible philosophy to deal with this situation would be to take a pragmatic point of view that one should simply satisfy the EL equations as far as possible, but stop once the equations can no longer be handled. Since we do not find this point of view convincing, we now go one step further and explain why the analysis of the EL to lower degree (no matter what the results of this analysis would or will be) will have no influence on the dynamics of the system.

The field equations of Chapter 8 were *dynamical* in the sense that they involved partial differential equations of the potentials, and by solving these equations one finds that the potential is non-trivial even away from the sources. This property of the field equations is a consequence of the fact that the leading contributions to the fermionic projector (namely the phase factors in (6.22)) dropped out of the EL equations. This made is possible that the derivative terms became relevant (although they were of lower degree on the light cone), leading to dynamical field equations. The set of perturbations which can lead to dynamical field equations is very limited, because this requires that the potential itself must drop out of the EL equations, meaning that the potential must correspond to a local symmetry of the system. All in this sense dynamical perturbations have been considered in this paper. This implies that all further potentials and fields will be *non-dynamical* in the sense that the potentials themselves (and not their derivatives) will enter the EL equations. This would give rise to algebraic relations between these potentials. In particular, these potentials would vanish away from the sources. Thus they do not describe a dynamical interaction, also making it difficult to observe them experimentally.

A possible idea for avoiding non-dynamical potentials is to choose the potentials differently for each generation, in such a way that the most singular contribution vanishes when taking the partial trace over the generation index. This would open the possibility that the leading contribution to the EL equations might involve derivatives, thereby giving rise to dynamical field equations. We now give an argument which explains why this idea does not seem to work: Suppose that the potential is chosen as a matrix on the generations. Then for the potential to drop out of the EL equations, we must impose that a certain partial trace involving the potential must vanish at every space-time point. But this implies that this partial trace also vanishes if the potential is replaced by its derivatives. In particular, these derivative terms again drop out of the EL equations, making it impossible to get dynamical field equations.

This concludes our analysis of *local* potentials. Our treatment was exhaustive in the sense that we considered all multiplication operators and considered all relevant first order operators. Our results gave a good qualitative picture of how the fermionic projector is affected by different kinds of perturbations of the Dirac operator, and what the resulting contributions to the EL equations are. Clearly, the present paper cannot cover all possible perturbations, and some details still remain unsettled. Nevertheless, as explained above, our analysis covers all perturbations which should be of relevance for the dynamics of our fermion systems.

10. Nonlocal Potentials

So far, we only analyzed the EL equations at the origin, i.e. for the leading contribution in an expansion in the parameter $\xi = y - x$. But of course, the EL equations should also be satisfied away from the origin. In this chapter, we will explore whether this can be accomplished by introducing *nonlocal* potentials into the Dirac equation. We shall see that this method will indeed make it possible to satisfy the EL equations to every order in an expansion around $\xi = 0$ (see Theorem 10.5). But it will not become possible to satisfy the EL equations globally for arbitrary x and y (see §10.4).

In order to introduce the problem, we consider the perturbation of the fermionic projector by a particle wave function Ψ, i.e. in view of (4.10)

$$P(x,y) \asymp -\frac{1}{2\pi}\, \Psi(x)\overline{\Psi(y)}\,. \tag{10.1}$$

In the variable $y-x$, this contribution oscillates on the scale of the Compton wave length, whereas in the variable $y+x$, it varies typically on the larger atomic or macroscopic scale. For this reason, it is appropriate to begin by analyzing *homogeneous* perturbations which depend only on the variable $y-x$. In §10.3 we shall extend our constructions to build in an additional dependence on the variable $y+x$.

10.1. Homogeneous Perturbations of the Fermionic Projector.

In order to explain the basic idea in the homogeneous setting, we again consider a system of one vacuum Dirac sea (5.25) for a given rest mass $m > 0$. This system is composed of states $\Psi_k(x) = \chi(k)\, e^{-ikx}$ which are plane waves of momentum k on the lower mass shell. A perturbation of the system which has not been considered so far is to *vary the momenta* of the states, dropping the mass shell condition. Considering a variation δk of the momentum leads us to replace $\Psi_k(x)$ by the plane wave

$$\chi(k)\, e^{-i(k+\delta k)x}\,. \tag{10.2}$$

Then to first order in δk, the individual states are varied by

$$\delta\Psi_k(x) = -i\,\langle x, \delta k\rangle\; \Psi_k(x) + \mathcal{O}((\delta k)^2)\,,$$

and this yields the following perturbation of the fermionic projector,

$$\delta P(x,y) = i \int \frac{d^4k}{(2\pi)^4} \langle \xi, \delta k\rangle (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{10.3}$$

The vector field δk can be chosen arbitrarily on the lower mass shell, giving us a lot of freedom to vary $P(x,y)$. However, due to the factor ξ inside the inner product, the variation necessarily vanishes at the origin $x = y$ (at least under the reasonable assumption that the function δk has suitable decay properties at infinity). Our idea is to use perturbations of the form (10.3) to compensate for the effect of the fermionic wave functions (10.1) in the EL equations away from the origin. At the origin, where (10.3) has no effect, our previous analysis remains valid, and the EL equations still lead to the field equations as worked out in Chapter 8. But the perturbations (10.3) will justify why it is unnecessary to consider the higher order terms in an expansion around $\xi = 0$.

The aim of this chapter is to work out this idea quantitatively, and to explore in increasing generality what the potential and the limitations of the resulting methods are. First of all, we remind the reader that only the axial component of the currents enters the EL equations to degree four (whereas the scalar, pseudoscalar and bilinear components drop out; see §9.1 and §9.2). But the perturbation (10.3) does not have an axial component, and thus we must generalize (10.3) such as to include a perturbation which is odd under parity transformations. To this end, we choose a vector field q in momentum space with the properties

$$\langle k, q(k) \rangle = 0 \qquad \text{and} \qquad q(k)^2 = -1 \,. \tag{10.4}$$

Then the operators

$$\Pi_\pm(k) := \frac{1}{2} \left(1 \mp \gamma^5 \slashed{q}(k) \right)$$

are projectors which commute with the fermionic projector of the vacuum; they project onto the two spin orientations in direction of q and $-q$, respectively. Thus multiplying the fermionic projector by $\Pi_\pm$,

$$P_\pm(k) = \Pi_\pm(k)\, (\slashed{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0) \,,$$

we decompose the Dirac sea into two "subseas" $P_\pm$, which are still composed of solutions of the free Dirac equation. We remark that this decomposition was already used in [13, §C.1], where it was shown that in a suitable limit $m \searrow 0$, the projectors $P_\pm(k)$ go over to chiral Dirac seas composed of left- or right-handed particles. Here the above decomposition gives us the freedom to vary the momenta of each subsea independently. Thus we generalize (10.3) by

$$\delta P(x,y) = i \sum_{s=\pm} \int \frac{d^4k}{(2\pi)^4} \langle \xi, \delta k_s \rangle P_s(k)\, e^{-ik(x-y)} \,, \tag{10.5}$$

where δk_+ and δk_- are two vector fields on the lower mass shell. For our purposes, it will be sufficient to always assume that these vector fields are smooth and have rapid decay. Thus we can extend them to *Schwartz functions in momentum space*, $\delta k_\pm \in \mathcal{S}(\hat{M})$. As a consequence, the function $\delta P(x,y)$ is smooth (but due to the restriction to the mass shell in (10.5), it will in general not have rapid decay). Clearly, more general functions $\delta k_\pm$ could be realized by approximation.

In the EL equations, $\delta P(x,y)$ is contracted with a factor $\slashed{\xi}$. The corresponding vector and axial components are computed by

$$\frac{1}{4} \operatorname{Tr} \Big(\slashed{\xi}\, \delta P(x,y) \Big) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4}\, \big\langle \xi, \delta k_+ + \delta k_- \big\rangle \, \big\langle \xi, k \big\rangle \, T_{m^2}(k)\, e^{ik\xi} \tag{10.6}$$

$$\frac{1}{4} \operatorname{Tr} \Big(\gamma^5 \slashed{\xi}\, \delta P(x,y) \Big) = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4}\, \big\langle \xi, \delta k_+ - \delta k_- \big\rangle \, \big\langle \xi, mq \big\rangle \, T_{m^2}(k)\, e^{ik\xi} \,, \tag{10.7}$$

where the distribution $T_{m^2}(k) := \delta(k^2 - m^2)\, \Theta(-k^0)$ is supported on the lower mass shell. Collecting the factors of ξ, these expressions can be written in the form $\xi_j \xi_l A^{jl}$ with a symmetric tensor field $A^{jl}(\xi)$. We first want to eliminate the tensor indices, leaving us with scalar Fourier transforms. Thus suppose that for a given smooth tensor field $A^{jl}(\xi)$ we want to find the corresponding vector fields $\delta k_\pm$ and q in (10.6) or (10.7). In the vector component (10.6), we can rewrite the factor $\big\langle \xi, k \big\rangle$ as a ξ-derivative, leading to the equation

$$\xi^j \frac{\partial}{\partial \xi^j} f_l(\xi) = \xi^j A_{jl}(\xi) \,, \tag{10.8}$$

where f_l is the Fourier integral

$$f_l = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\, (\delta k_+ + \delta k_-)_l \, T_{m^2}(k)\, e^{ik\xi} \,.$$

Integrating the ordinary differential equation (10.8) gives the solution

$$f_l(\xi) = \xi^j \int_0^1 A_{jl}(\tau\xi)\, d\tau\,,$$

being a smooth vector field. Thus for every choice of the vector index l we must solve the equation

$$f(\xi) = \int \frac{d^4k}{(2\pi)^4}\, \hat{f}(k)\, T_{m^2}(k)\, e^{ik\xi}\,, \tag{10.9}$$

where we set $f = f_l$ and $\hat{f} = (\delta k_+ + \delta k_-)_l/2$. In this way, we have reduced (10.6) to scalar Fourier integrals of the form (10.9). The same can be accomplished for the axial component (10.7) with the following construction. We write the factor ξ in (10.7), which is contracted with q, as a k-derivative of the factor $e^{ik\xi}$ and integrate by parts. Using the relation

$$q^j \frac{\partial}{\partial k^j} T_{m^2}(k) = q^j \frac{\partial}{\partial k^j} \big(\delta(k^2 - m^2)\, \Theta(-k^0) \big) = 2 q^j k_j\, \delta'(k^2 - m^2)\, \Theta(-k^0) = 0\,,$$

where in the last step we applied the orthogonality relation in (10.4), we obtain the equation

$$\xi^j A_{jl}(\xi) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\frac{\partial}{\partial k^j} \Big((\delta k_+ - \delta k_-)_l\, m q^j \Big) \right] T_{m^2}(k)\, e^{ik\xi}\,.$$

We again fix the index l, but now set $f(\xi) = \xi^j A_{jl}(\xi)$. Furthermore, we introduce the vector field $v^j = (\delta k_+ - \delta k_-)_l\, m q^j$. Suppose that the smooth scalar function f can be represented in the form (10.9) with a suitable Schwartz function $\hat{f}$. Then the remaining task is to satisfy on the lower mass shell the equation

$$\frac{\partial}{\partial k^j} v^j(k) = \hat{f}\,. \tag{10.10}$$

In order to verify that this equation always has a solution, it is useful to rewrite it as a geometric PDE defined intrinsically on the hyperbola $\mathcal{H} = \{k \in \hat{M} \,|\, k^2 = m^2,\ k^0 < 0\}$. Namely, the orthogonality condition in (10.4) implies that the vector v is tangential to $\mathcal{H}$, and the derivatives in (10.10) can be rewritten as the covariant divergence on $\mathcal{H}$,

$$\nabla_j v^j = \hat{f} \in \mathcal{S}(\mathcal{H}) \tag{10.11}$$

(where ∇ is the Levi-Civita connection on $\mathcal{H}$). Conversely, for a given vector field v on $\mathcal{H}$ satisfying (10.11), extending v to a vector field in $\hat{M}$ which is everywhere orthogonal to k gives the desired solution of (10.10). A simple solution of (10.11) is obtained by first solving the Poisson equation $\Delta_{\mathcal{H}} \phi = \hat{f}$ (for example using the explicit form of the Green's function on the hyperbola) and setting $v = \nabla\phi$. However, this solution has the disadvantage that v has no rapid decay at infinity. In order to do better, we must exploit that our function $f(\xi) = \xi^j A_{jl}(\xi)$ vanishes at the origin, and thus the integral of $\hat{f}$ vanishes,

$$\int_{\mathcal{H}} \hat{f}\, d\mu_{\mathcal{H}} = 0\,. \tag{10.12}$$

Combining this fact with the freedom to add to v an arbitrary divergence-free vector field, one can indeed construct a solution v of (10.11) within the Schwartz class, as is shown in the following lemma[5].

[5]I thank Bernd Ammann for the idea of solving the equation on the leaves of a foliation, after subtracting the mean value of f.

Lemma 10.1. *For every function $\hat{f} \in \mathcal{S}(\mathcal{H})$ whose integral vanishes* (10.12)*, there is a vector field $v \in \mathcal{S}(\mathcal{H})$ which satisfies* (10.11)*.*

Proof. We parametrize the hyperbola $\mathcal{H}$ by

$$\Big(-\sqrt{m^2+\rho^2}, \rho\cos\vartheta, \rho\sin\vartheta\cos\varphi, \rho\sin\vartheta\sin\varphi\Big) \in \hat{M},$$

where $\rho := |\vec{k}|$, and (ϑ, φ) are the standard polar coordinates on the 2-sphere. Then the metric on $\mathcal{H}$ is diagonal,

$$g_{ij} = \text{diag}\left(\frac{m^2}{\rho^2+m^2}, \rho^2, \rho^2\sin^2\vartheta\right).$$

For the vector field v we take the ansatz as the sum of a radial part $u^j = (u^\rho, 0, 0)$ and an angular part $w^j = (0, w^\vartheta, w^\varphi)$. We also regard w as a vector field on the sphere. Then the equation (10.11) can be written as

$$\text{div}_{\mathcal{H}}(u) + \text{div}_{S^2}(w) = \hat{f}\,. \tag{10.13}$$

Taking the average of $\hat{f}$ over spheres defines a spherically symmetric Schwartz function $\overline{f}$,

$$\overline{f}(\rho) := \frac{1}{4\pi}\int_{S^2} \hat{f}(\rho,\vartheta,\varphi)\, d\varphi\, d\cos\vartheta \;\in\; \mathcal{S}(\mathcal{H})\,.$$

Writing the integration measure as $d\mu_{\mathcal{H}} = \sqrt{\det g}\, d\rho\, d\vartheta\, d\varphi$ and using Fubini, the condition (10.12) implies that

$$\int_0^\infty \overline{f}(\rho)\, \frac{\rho^2\, d\rho}{\sqrt{\rho^2+m^2}} = 0\,. \tag{10.14}$$

Solving the Poisson equation $\Delta_{\mathcal{H}}\phi = \overline{f}$, the resulting function ϕ is smooth and again spherically symmetric. Setting $u = \nabla\phi$, we obtain a smooth radial vector field, being a solution of the equation

$$\text{div}_{\mathcal{H}}(u) = \overline{f}\,. \tag{10.15}$$

Writing the covariant divergence as

$$\text{div}_{\mathcal{H}}(u) = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\Big(\sqrt{\det g}\, u^j\Big) = \frac{\sqrt{\rho^2+m^2}}{\rho^2}\frac{\partial}{\partial\rho}\left(\frac{\rho^2\, u^\rho}{\sqrt{\rho^2+m^2}}\right), \tag{10.16}$$

the divergence condition (10.15) becomes an ordinary differential equation, having the explicit solution

$$u^\rho(\rho) = \frac{\sqrt{\rho^2+m^2}}{\rho^2}\int_0^\rho \overline{f}(\tau)\, \frac{\tau^2\, d\tau}{\sqrt{\tau^2+m^2}}\,.$$

Using (10.14), one immediately verifies that the vector field u and all its derivatives have rapid decay at infinity. Hence u is in the desired Schwartz class.

Using (10.15) in (10.13), it remains to consider the differential equation

$$\text{div}_{S^2}(w) = \hat{f} - \overline{f}\,. \tag{10.17}$$

Since the function $\hat{f} - \overline{f}$ has mean zero on every sphere, it can be expanded in terms of spherical harmonics starting at $l = 1$,

$$(\hat{f} - \overline{f})(\rho,\vartheta,\varphi) = \sum_{l=1}^\infty \sum_{k=-l}^l c_{lk}(\rho)\, Y_{lk}(\vartheta,\varphi)\,.$$

Since $\hat{f} - \overline{f}$ is a Schwartz function, the coefficients c_{lk} are all smooth in ρ, and these coefficients together with all their ρ-derivatives have rapid decay in both ρ and l, uniformly in k. Hence the Poisson equation $\Delta_{S^2}\phi = \hat{f} - \overline{f}$ can be solved explicitly by

$$\phi(\rho, \vartheta, \varphi) = -\sum_{l=1}^{\infty} \sum_{k=-l}^{l} \frac{c_{lk}(\rho)}{l(l+1)}\, Y_{lk}(\vartheta, \varphi)\,,$$

defining again a Schwartz function on $\mathcal{H}$. Introducing the vector field w by $w^j = \nabla^j_{S^2}\phi = (0, \partial_\vartheta \phi, \cos^{-2}\vartheta\, \partial_\varphi \phi)$ gives the desired solution of (10.17) in the Schwartz class. □

10.2. **The Analysis of Homogeneous Perturbations on the Light Cone.** Following the above arguments, it remains to consider the scalar Fourier transform (10.9). Having a weak evaluation on the light cone (5.7) in mind, we may restrict attention to the light cone $L = \{\xi \,|\, \xi^2 = 0\}$. Thus our task is to analyze the Fourier integral

$$f(\xi) = \int \frac{d^4k}{(2\pi)^4}\, \hat{f}(k)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{ik\xi} \qquad \text{for } \xi \in L\,. \tag{10.18}$$

More precisely, for a given smooth function $f \in C^\infty(M)$ we want to find a Schwartz function $\hat{f} \in \mathcal{S}(\hat{M})$ such that the Fourier integral (10.18) coincides on the light cone with f. The question is for which f such a function $\hat{f}$ exists. We begin the analysis in the simple case that the function $\hat{f}$ when restricted to the mass shell depends only on the variable $\omega = k^0$ (by linearity, we can later realize more general functions $\hat{f}$ by superposition). Then the resulting Fourier integral is spherically symmetric, so that the function $f(\xi)$ will only depend on the time and radial variables $t = \xi^0$ and $r = |\vec{\xi}|$. Restricting attention to the light cone $t = \pm r$, we end up with a one-dimensional problem. More precisely, setting $p = |\vec{k}|$ and denoting the angle between $\vec{\xi}$ and $\vec{k}$ by ϑ, the Fourier integral (10.18) becomes

$$\begin{aligned} f(t,r) &= \frac{1}{8\pi^3} \int_{-\infty}^0 d\omega\, \hat{f}(\omega)\, e^{i\omega t} \int_0^\infty p^2\, dp\, \delta(\omega^2 - p^2 - m^2) \int_{-1}^1 d\cos\vartheta\, e^{-ipr\cos\vartheta} \\ &= \frac{i}{8\pi^3\, r} \int_{-\infty}^0 d\omega\, \hat{f}(\omega)\, e^{i\omega t} \int_0^\infty p\, dp\, \delta(\omega^2 - p^2 - m^2)\, \left(e^{-ipr} - e^{ipr}\right) \\ &= \frac{i}{16\pi^3\, r} \int_{-\infty}^{-m} d\omega\, \hat{f}(\omega)\, e^{i\omega t} \left(e^{-i\sqrt{\omega^2-m^2}\,r} - e^{i\sqrt{\omega^2-m^2}\,r}\right). \end{aligned}$$

Hence on the light cone $t = \pm r$ we obtain the representation

$$it\, f(t) = \frac{1}{16\pi^3} \int_{-\infty}^{-m} d\omega\, \hat{f}(\omega)\, \left(e^{i\omega_+ t} - e^{i\omega_- t}\right), \tag{10.19}$$

where we set

$$\omega_\pm := \omega \pm \sqrt{\omega^2 - m^2}\,.$$

The right side of (10.19) differs from an ordinary Fourier integral in two ways: First, the integrand does not involve one plane wave, but the difference of the two plane waves $e^{i\omega_\pm t}$, whose frequencies are related to each other by $\omega_+ \omega_- = m^2$. Second, in (10.19) only negative frequencies appear. Let us discuss these two differences after each other. The appearance of the combination $(e^{i\omega_+ t} - e^{i\omega_- t})$ in (10.19) means that the coefficients of the plane waves cannot be chosen arbitrarily, but a contribution for a frequency $\omega_- < -m$ always comes with a corresponding contribution of frequency $\omega_+ > -m$. This frequency constraint makes it impossible to represent a general negative-frequency function f; for example, it is impossible to represent a function $tf(t)$ whose frequencies are supported in the interval $[-\infty, -m)$. However, this *frequency constraint* can be regarded as a shortcoming of working with a single Dirac sea. If we considered instead a realistic system of several Dirac seas (3.1), the Fourier integral (10.19) would involve a sum over the generations,

$$it\, f(t) = \frac{1}{16\pi^3} \sum_{\beta=1}^{g} \int_{-\infty}^{-m_\beta} d\omega\; \hat{f}^\beta(\omega) \left(e^{i\omega_+^\beta t} - e^{i\omega_-^\beta t} \right) \tag{10.20}$$

with $\omega_\pm^\beta(\omega) := \omega \pm (\omega^2 - m_\beta^2)^{-\frac{1}{2}}$ and $g > 1$. Then the freedom in choosing g independent functions $\hat{f}^\beta$ would indeed make it possible to approximate any negative-frequency function, as the following lemma shows.

Lemma 10.2. *Assume that the number of generations $g \geq 2$. Assume furthermore that the ordinary Fourier transform $\hat{f}$ of a given Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is supported in the interval $(-\infty, 0)$. Then there is a sequence of Schwartz functions $\hat{f}_n^\beta \in \mathcal{S}(\mathbb{R})$ such that the corresponding functions $f_n(t)$ defined by the Fourier integrals* (10.20) *as well as all their derivatives converge uniformly to $f(t)$,*

$$\sup_{t \in \mathbb{R}} \left| \partial_t^K \big(f_n(t) - f(t)\big) \right| \xrightarrow{n \to \infty} 0 \qquad \textit{for all } K \geq 0\,. \tag{10.21}$$

Proof. Let us try to find functions $\hat{f}^\beta$ in (10.20) such that the right side of (10.20) gives the plane wave $e^{i\Omega t}$ with $\Omega < 0$. Again ordering the masses according to (3.2), we choose ω_1 such that $\omega_+^g(\omega_1) = \Omega$ or $\omega_-^g(\omega_1) = \Omega$, i.e.

$$\omega_1 = \frac{\Omega^2 + m_g^2}{2\Omega}\,.$$

Then choosing $\hat{f}^g(\omega) = \pm\delta(\omega - \omega_1)$, we obtain the desired plane wave $e^{i\Omega t}$, but as an error term we get the plane wave $-e^{i\Omega_1 t}$ with $\Omega_1 = m_g^2/\Omega$. In order to compensate the error, we next choose ω_2 such that $\omega_+^1(\omega_2) = \Omega_1$ or $\omega_-^1(\omega_2) = \Omega_1$. Choosing $\hat{f}^1(\omega) = \delta(\omega - \omega_2)$, the plane wave $-e^{i\Omega_1 t}$ drops out, but we obtain instead the plane wave $e^{i\Omega_2 t}$ with $\Omega_2 = m_1^2/\Omega_1 = m_1^2 \Omega / m_g^2$. We proceed by compensating the plane waves in turns by the last Dirac sea and the first Dirac sea. After n iteration steps, the functions $\hat{f}^1$ and $\hat{f}^g$ take the form

$$\hat{f}_n^1(\omega) = -\sum_{l=1}^{n} \delta\left(\omega - \frac{\Omega_{2n+1}^2 + m_1^2}{2\Omega_{2n+1}} \right), \qquad \hat{f}_n^g(\omega) = \sum_{l=0}^{n-1} \delta\left(\omega - \frac{\Omega_{2n}^2 + m_g^2}{2\Omega_{2n}} \right), \tag{10.22}$$

where

$$\Omega_{2n} = \frac{m_1^{2n}}{m_g^{2n}}\, \Omega \qquad \text{and} \qquad \Omega_{2n+1} = \frac{m_g^{2n+2}}{m_1^{2n}\, \Omega}\,.$$

The Fourier integral (10.20) gives rise to the plane waves

$$e^{i\Omega t} - e^{i\lambda^n \Omega t} \qquad \text{where} \qquad \lambda := \frac{m_1^2}{m_g^2} < 1\,.$$

In order to form superpositions of these plane waves, we next multiply by a Schwartz function $\hat{h}(\Omega)$ and integrate over Ω. Then the Fourier integral (10.20) becomes

$$it f_n(t) = \int_{-\infty}^{0} \hat{h}(\Omega) \left(e^{i\Omega t} - e^{i\lambda^n \Omega t} \right) d\Omega \,.$$

Choosing $\hat{h}(\omega) = -\partial_\omega \hat{f}(\omega)/(2\pi)$, we can extend the integration to the whole real line. After integrating by parts, we can carry out the Fourier integral to obtain

$$f_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\Omega) \left(e^{i\Omega t} - \lambda^n e^{i\Omega\,(\lambda^n t)} \right) d\Omega = f(t) - \lambda^n f(\lambda^n t) \,.$$

From this explicit formula it is obvious that the functions f_n converge in the limit $n \to \infty$ as desired (10.21). ◻

As is immediately verified, the functions f_n as well as all their derivatives also converge in $L^2(\mathbb{R})$. However, we point out that for the functions $\hat{f}^1_n$ and $\hat{f}^g_n$ the convergence is a bit more subtle. Namely, from (10.22) one sees that for large n, these functions involve more and more contributions for large ω. A direct calculation shows that in the limit $n \to \infty$, these functions converge to smooth functions which decays at infinity only $\sim 1/\omega$.

Let us now discuss the consequences of the fact that (10.19) only involves negative frequencies. This restriction is already obvious in the Fourier integral (10.5), before the reduction to scalar Fourier integrals (10.18) or (10.19). Since contractions with factors ξ merely correspond to differentiations in momentum space which preserve the sign of the frequencies, the following considerations apply in the same way before or after the contractions with ξ have been performed. We point out that the contribution by a Dirac wave function (10.1) can be composed of positive frequencies (=particles) or negative frequencies (=anti-particles), and thus in (10.1) we cannot restrict attention to negative frequencies. This raises the question whether a contribution to (10.1) of positive frequency can be compensated by a contribution to (10.18) of negative frequency. The answer to this question is not quite obvious, because the EL equations involve both $P(x,y)$ and its adjoint $P(y,x) = P(x,y)^*$. Since taking the adjoint reverses the sign of the frequencies, a negative-frequency contribution to $P(x,y)$ affects the EL equations by contributions of both positive and negative energy. Thus one might hope that perturbations of $P(x,y)$ of positive and negative energy could compensate each other in the EL equations. However, such a compensation is impossible, as the following lemma shows.

Lemma 10.3. *Assume that $\hat{f}$ and $\hat{g}$ are the Fourier transforms of chiral perturbations of the fermionic projector, such that $\hat{f}$ has a non-vanishing contribution inside the upper mass cone, whereas $\hat{g}$ is supported inside the lower mass cone. Then the linear contributions of f and g to the EL equations to degree four cannot compensate each other for all ξ.*

Proof. By linearity, we may restrict attention to the spherically symmetric situation, so that f and g restricted to the light cone are functions of one variable t. Since the negative-frequency component of f can clearly be compensated by g, we can assume that f and g are composed purely of positive and negative frequencies, respectively. The perturbation g affects the EL equations to degree four by (see Lemma (7.4) and its proof in Appendix B)

$$\mathcal{R} \asymp c \left(M_1 \, g(\xi) + M_2 \, \overline{g(\xi)} \right) + (\deg < 4),$$

where

$$M_1 = T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \qquad \text{and} \qquad M_2 = -T^{(-1)}_{[0]} \overline{T^{(-1)}_{[0]}} \, \frac{T^{(0)}_{[0]}}{\overline{T^{(0)}_{[0]}}} \,, \tag{10.23}$$

and $c > 0$ is an irrelevant constant. Evaluating weakly on the light cone (5.7), we obtain the contribution

$$c_1 \, g(t) + c_2 \, \overline{g(t)} \,, \tag{10.24}$$

where c_1 and c_2 are real regularization parameters (real because the degree is even). Similarly, the perturbation f yields the contribution

$$c_1 \, f(t) + c_2 \, \overline{f(t)} \,. \tag{10.25}$$

In order for (10.25) to compensate (10.24), both the negative and positive frequencies must cancel each other, leading to the conditions

$$c_1 \, g(t) = c_2 \, \overline{f(t)} \qquad \text{and} \qquad c_2 \, \overline{g(t)} = c_1 \, f(t) \,. \tag{10.26}$$

Taking the complex conjugate of the first equation, multiplying it by c_2 and subtracting c_1 times the second equation, we get

$$\left(c_1^2 - c_2^2 \right) f(t) = 0 \,.$$

We thus obtain the condition

$$c_1 = \pm c_2 \,. \tag{10.27}$$

If this condition holds, we can indeed satisfy (10.26) by setting $g(t) = \pm \overline{f(t)}$.

We conclude that f and g can compensate each other if and only if we impose the relation (10.27) between the regularization parameters corresponding to the basic fractions M_1 and M_2. Imposing relations between the regularization parameters was not used previously in this paper, and one could simply reject (10.27) by saying that we do not want to restrict the class of admissible regularization by introducing such relations. However, this argumentation would not be fully convincing, as it would not allow for the possibility that the microscopic structure of space-time on the Planck scale corresponds to a regularization which does have the special property (10.27). This possibility is ruled out by the following argument which shows that there are in fact no regularizations which satisfy (10.27): As only the real parts of basic fractions enter (5.7), it suffices to consider the real parts of M_1 and M_2. Using the specific form of these monomials in (10.23), we obtain

$$\frac{\text{Re}(M_1 + M_2)}{2} = \left| \frac{T^{(-1)}_{[0]}}{T^{(0)}_{[0]}} \right|^2 \left(\text{Im} \, T^{(0)}_{[0]} \right)^2 , \qquad \frac{\text{Re}(M_1 - M_2)}{2} = \left| \frac{T^{(-1)}_{[0]}}{T^{(0)}_{[0]}} \right|^2 \left(\text{Re} \, T^{(0)}_{[0]} \right)^2 .$$

Both these expressions are non-negative, and thus there cannot be cancellations between positive and negative contributions, no matter how we regularize. Without a regularization, we know from (4.13)–(4.15) that

$$\text{Re} \, T^{(0)} = -\frac{1}{8\pi^3} \, \frac{\text{PP}}{\xi^2} \qquad \text{and} \qquad \text{Im} \, T^{(0)}_{[0]} = -\frac{i}{8\pi^2} \, \delta(\xi^2) \, \epsilon(\xi^0) \,.$$

Regularizing these terms, we find that for any regularization, both $M_1 + M_2$ and $M_1 - M_2$ are non-zero to degree four on the light cone. Hence (10.27) is violated. □

We come to the definitive conclusion that using perturbations of the form (10.5), it is in general impossible to satisfy the EL equations to degree four globally for all ξ. But, as we will now show, it is possible to satisfy the EL equations *locally* near $\xi = 0$, in the sense that we can compensate all contributions in a Taylor expansion in ξ to an arbitrarily high order. We consider the obvious generalization of (10.5) to several generations

$$\delta P(x,y) = i \sum_{\beta=1}^{g} \sum_{s=\pm} \int \frac{d^4k}{(2\pi)^4} \langle \xi, \delta k_s \rangle \, P_\pm(k)\, e^{-ik(x-y)} \,. \qquad (10.28)$$

Proposition 10.4. *Suppose that the number of generations $g \geq 2$. Then for any given smooth functions $h_{\mathrm{v}}, h_{\mathrm{a}} \in C^\infty(M)$ and every parameter $L > 2$, there are vector fields $\delta k_\pm$ and q in the Schwartz class such that for all multi-indices κ with $2 \leq |\kappa| \leq L$,*

$$\partial_\xi^\kappa \left[\mathrm{Tr}\left(\not{\xi}\, \delta P(x,y) \right) - h_{\mathrm{v}}(\xi) \right]\Big|_{\xi=0} = 0 = \partial_\xi^\kappa \left[\mathrm{Tr}\left(\gamma^5 \not{\xi}\, \delta P(x,y) \right) - h_{\mathrm{a}}(\xi) \right]\Big|_{\xi=0} . \qquad (10.29)$$

Before coming to the proof, we point out that, since the right side of (10.6) and (10.7) involves two factors of ξ, it is in general impossible to satisfy (10.29) in case $|\kappa| < 2$. If the functions h_{v} or h_{a} are formed by taking the vector or axial component of a contribution to the fermionic projector (for example the contribution by a particle (10.1)), then (10.29) is trivially satisfied in the case $|\kappa| = 0$, simply because all functions vanish at the origin $\xi = 0$. However, in this case the linear terms in ξ of h_{v} and h_{a} will in general be non-zero. Indeed, these linear terms are responsible for the field equations as worked out in Chapter 8. Thus (10.29) means that all higher order terms in an expansion in ξ can be compensated by perturbations of the form (10.28).

Proof of Proposition 10.4. Following the arguments after (10.5) and applying Lemma 10.1, it again suffices to analyze scalar Fourier integrals. Furthermore, using the polarization formula for the multi-index κ, we may restrict attention to the spherically symmetric situation (10.20) with Schwartz functions $\hat{f}^\beta \in \mathcal{S}(\mathbb{R})$. Keeping track of the factors ξ in the arguments after (10.5), it remains to show that for every smooth function $h(t)$ there are functions $\hat{f}_\beta \in \mathcal{S}(\hat{M})$ such that the corresponding function $f(t)$ defined by (10.20) satisfies the conditions

$$\frac{d^l}{dt^l} \big(f(t) - h(t) \big)\big|_{t=0} = 0 \qquad \text{for all } l = 0, \ldots, L \,. \qquad (10.30)$$

We choose a test function $\hat{\eta} \in C_0^\infty((-1,1))$ and denote its Fourier transform by η. Furthermore, we choose a parameter $\Omega_0 < -4m_g$ and introduce the function $\hat{\eta}_{\Omega_0} \in C_0^\infty((-5m_g, -3m_g))$ by

$$\hat{\eta}_{\Omega_0}(\Omega) = \frac{1}{|4\Omega_0|}\, \hat{\eta}\Big(\frac{\Omega - \Omega_0}{4\Omega_0} \Big) \,.$$

Thus $\hat{\eta}_{\Omega_0}$ is supported for large negative frequencies. For any such frequency $\Omega \in \operatorname{supp} \hat{\eta}_{\Omega_0}$, we want to construct the plane wave $e^{i\Omega t}$, with an error term which is again of large negative frequency. To this end, we proceed similar as in the proof of Lemma 10.2 by iteratively perturbing the first and last Dirac seas, but now beginning with the first sea. Thus we first construct the plane wave $e^{i\Omega t}$ by perturbing the first sea, and compensate the error term by perturbing the last Dirac sea. This gives in analogy to (10.22)

$$\hat{f}^1(\omega) = \delta\bigg(\omega - \frac{\Omega^2 + m_1^2}{2\Omega} \bigg) \,, \qquad \hat{f}^g(\omega) = -\delta\bigg(\omega - \frac{\Omega\, m_g^2}{2m_1^2} - \frac{m_1^2}{2\Omega} \bigg) \,,$$

giving rise to the plane wave

$$e^{i\Omega t} - e^{i\Omega t\, m_g^2/m_1^2} \,. \tag{10.31}$$

Multiplying by $\hat{\eta}_{\Omega_0}$ and integrating over Ω, we find that the function

$$f(t) := \frac{1}{it}\Big(e^{i\Omega_0 t}\, \eta(4\Omega_0 t) - e^{i\Omega_1 t}\, \eta(4\Omega_1 t) \Big) \qquad \text{with} \qquad \Omega_1 = \frac{m_g^2}{m_1^2}\, \Omega_0$$

has the desired Fourier representation (10.20). Since differentiating (10.20) with respect to t merely generates factors of $\omega_\pm^\beta$, the functions $f^{(l)}(t) := t^{-1} \partial_t^l (t f(t))$, $l = 1, \ldots, L$, can again be represented in the form (10.20). By a suitable choice of the function $\hat{\eta}$, we can clearly arrange that the parameters $f(0), f^{(1)}(0), \ldots, f^{(L)}(0)$ are linearly independent. Thus by adding to f a suitable linear combination of the functions $f^{(l)}$, we can arrange (10.30). □

For clarity, we point out that the function f in (10.30) will in general not be a good global approximation to h. In particular, it is impossible to pass to the limit $L \to \infty$. We also remark that this proposition also holds in the case $g = 1$ of only Dirac sea. However, in this case it would not be possible to compensate the error term, so that instead of (10.31) we would have to work with the combination $e^{i\Omega t} - e^{it\, m^2/\Omega}$. This has the disadvantage that in the limit $\Omega \to -\infty$, we would get contributions of low frequency, making it impossible to generalize the result to the non-homogeneous situation (see the proof of Theorem 10.5 below). This is why Proposition 10.4 was formulated only in the case $g \geq 2$.

10.3. **Nonlocal Potentials, the Quasi-Homogeneous Ansatz.** We now want to extend the previous results to the non-homogeneous situation. Since we only consider perturbations which are diagonal on the generations, all constructions immediately carry over to several generations by taking sums. For notational simplicity, we will write all formulas only for one generation. Our method is to first describe our previous perturbations of the fermionic projector (10.5) by homogeneous perturbations of the Dirac operator. Replacing this perturbation operator by a nonlocal operator will then make it possible to describe the desired non-homogeneous perturbations of the fermionic projector. We begin by noting that the plane wave (10.2) is a solution of the Dirac equation $(i\partial\!\!\!/ - \delta k\!\!\!/ - m)\Psi = \mathcal{O}((\delta k)^2)$. Thus the linear perturbation (10.3) can be described equivalently by working with the perturbed Dirac operator in momentum space

$$k\!\!\!/ - \delta k\!\!\!/(k) - m \,.$$

Similarly, for the perturbation (10.5), we must find a perturbation $\mathfrak{n}$ of the Dirac operator which is symmetric and, when restricted to the image of the operators $P_\pm(k)$, reduces to the operators $-\delta k\!\!\!/_\pm$. In order to determine $\mathfrak{n}$, it is convenient to decompose the vector fields $\delta k_\pm$ as

$$\delta k_\pm = V \pm (\phi\, q + A) \qquad \text{where} \qquad \langle A, q \rangle = 0 \tag{10.32}$$

(thus V is the vector part, whereas ϕ and A describe the components of the axial part parallel and orthogonal to q, respectively). Then using (10.4) together with the relations

$$\gamma^5 q\!\!\!/\, \Pi_\pm = \mp \Pi_\pm \,,$$

we obtain

$$\begin{aligned} -\delta \slashed{k}_\pm P_\pm &= (-\slashed{V} \mp (\phi \, \slashed{q} + \slashed{A})) \, P_\pm = -\slashed{V} P_\pm + (\phi \, \slashed{q} + \slashed{A})(\gamma^5 \slashed{q}) P_\pm \\ &= \left(-\slashed{V} + \phi \gamma^5 + \gamma^5 \slashed{q} \, \slashed{A}\right) P_\pm = \Big[-\slashed{V} + \frac{\phi}{m} \gamma^5 \slashed{k} + \gamma^5 \slashed{q} \, \slashed{A} \Big] P_\pm \, , \end{aligned}$$

where in the last step we used that $(\slashed{k} - m) P_\pm = 0$. The square bracket has the desired properties of $\mathfrak{n}$. Thus the perturbation (10.5) is equivalently described by the Dirac operator in momentum space

$$\slashed{k} + \mathfrak{n} + m \qquad \text{with} \qquad \mathfrak{n}(k) = -\slashed{V}(k) + \frac{\phi(k)}{m} \gamma^5 \slashed{k} + \gamma^5 \slashed{q}(k) \, \slashed{A}(k) \, , \tag{10.33}$$

and the fields V, A and ϕ as defined by (10.32). Note that $\mathfrak{n}$ is composed of three terms, which can be regarded as a vector, an axial and a bilinear perturbation of the Dirac operator. The perturbations in (10.32) are all homogeneous. The reader interested in the perturbation expansion for the corresponding Dirac solutions is referred to [13, §C.1].

In order to generalize to non-homogeneous perturbations, we write the operator $\mathfrak{n}$ as the convolution operator in position space

$$(\mathfrak{n} \, \Psi)(x) = \int_M \mathfrak{n}(x,y) \, \psi(y) \, d^4y \tag{10.34}$$

with the integral kernel

$$\mathfrak{n}(x,y) = \int \frac{d^4k}{(2\pi)^4} \, \hat{\mathfrak{n}}(k) \, e^{ik\xi} \, .$$

Now we can replace $\mathfrak{n}(x,y)$ by a general nonlocal kernel. We refer to the operator $\mathfrak{n}$ as a *nonlocal potential* in the Dirac equation. For technical convenience, it seems appropriate to make suitable decay assumptions at infinity, for example by demanding that

$$\mathfrak{n}(x,y) \in \mathcal{S}(M \times M) \, . \tag{10.35}$$

Then the corresponding fermionic projector can be introduced perturbatively exactly as outlined in §4.4. Before we can make use of the general ansatz (10.34) and (10.35), we must specify $\mathfrak{n}(x,y)$. To this end, we fix the variable $\zeta := y + x$ and consider the fermionic projector as a function of the variable ξ only. Then we are again in the homogeneous setting, and we can choose the operator $\mathfrak{n}$ as in (10.33). In order to clarify the dependence on the parameter ζ, we denote this operator by $\mathfrak{n}(k, \zeta)$. As explained at the beginning of this chapter, we consider $y+x$ as a macroscopic variable, whereas k is the momentum of the quantum mechanical oscillations. This motivates us to introduce the kernel $\mathfrak{n}(x,y)$ similar to the transformation from the Wigner function to the statistical operator by

$$\mathfrak{n}(x,y) = \int \frac{d^4k}{(2\pi)^4} \, \mathfrak{n}\big(k, y+x\big) \, e^{ik(y-x)} \, . \tag{10.36}$$

Although this so-called *quasi-homogeneous ansatz* is quite simple, it makes it possible to satisfy the EL equations to degree four locally around every space-time point, as is made precise in the following theorem.

Theorem 10.5. *Suppose that the number of generations $g \geq 2$, and that we are given an integer $L \geq 2$ and a parameter $\delta > 0$. Then for any given smooth functions $h_{\mathrm{v}}, h_{\mathrm{a}} \in$*

$\mathcal{S}(M \times M)$, there is a nonlocal potential $\mathfrak{n}$ of the form (10.34) with a kernel $\mathfrak{n}(x,y) \in \mathcal{S}(M \times M)$ of the form (10.36) such that for all multi-indices κ with $2 \leq |\kappa| \leq L$,

$$\left| \partial_\xi^\kappa \Big[\mathrm{Tr}\left(\not{q}\, \Delta P(x,y) \right) - h_{\mathrm{v}}(x,y) \Big] \Big|_{x=y} \right| + \left| \partial_\xi^\kappa \left[\mathrm{Tr}\left(\gamma^5 \not{q}\, \Delta P(x,y) \right) - h_{\mathrm{a}}(x,y) \right] \Big|_{x=y} \right| < \delta \,.$$

Here ΔP denotes the perturbation of the fermionic projector to first order in the nonlocal potential $\mathfrak{n}$.

Proof. For a fixed choice of the parameter Ω_0 and for any given ζ, we choose the homogeneous perturbation δP as in the proof of Proposition 10.4 and rewrite it according to (10.33) as a homogeneous perturbation $\mathfrak{n}(k,\zeta)$ of the Dirac equation. Introducing the nonlocal potential by (10.34) and (10.36), the rapid decay of the functions $h_{\mathrm{v}}, h_{\mathrm{a}}$ implies that the kernel $\mathfrak{n}(x,y)$ has rapid decay also in ζ (the rapid decay in ξ is obvious because $\mathfrak{n}(.,\zeta) \in \mathcal{S}(\hat{M})$). Since the same is true for all derivatives, we conclude that $\mathfrak{n}(x,y) \in \mathcal{S}(M \times M)$.

The corresponding perturbation of the fermionic projector ΔP can be analyzed with the methods introduced in [11]. We first pull out the Dirac matrices to obtain

$$\Delta P(x,y) = (i\partial\!\!\!/_x + m) \left(-i \frac{\partial}{\partial y^k} + m \right) \Delta T_{m^2}[\mathfrak{n}](x,y)\, \gamma^k \,, \tag{10.37}$$

where ΔT_{m^2} is the perturbation of the corresponding solution of the inhomogeneous Klein-Gordon equation (see [11, equations (2.4) and (2.5)])

$$\begin{aligned} \Delta T_{m^2}[\mathfrak{n}](x,y) &= -\int_M d^4z_1 \int_M d^4z_2 \\ &\quad\times \Big(S_{m^2}(x,z_1)\, \mathfrak{n}(z_1,z_2)\, T_{m^2}(z_2,y) \;+\; T_{m^2}(x,z_1)\, \mathfrak{n}(z_1,z_2)\, S_{m^2}(z_2,y) \Big) \end{aligned}$$

(where T_a and S_a are again given by (4.12) and (8.22)). We next transform to momentum space. Setting

$$\mathfrak{n}(p,q) = \int_M \mathfrak{n}(p,\zeta)\, e^{\frac{iq\zeta}{2}}\, d^4\zeta$$

with the "macroscopic" momentum vector q, the above formula for ΔT_{m^2} becomes (see [11, equations (3.8) and (3.9)])

$$\begin{aligned} &\Delta T_{m^2}[\mathfrak{n}]\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big) \\ &\quad = -S_{m^2}\Big(p+\frac{q}{2}\Big)\, \mathfrak{n}(p,q)\, T_{m^2}\Big(p-\frac{q}{2}\Big) - T_{m^2}\Big(p+\frac{q}{2}\Big)\, \mathfrak{n}(p,q)\, S_{m^2}\Big(p-\frac{q}{2}\Big) \,. \end{aligned}$$

Now we can perform the light-cone expansion exactly as in [11, Section 3]. This gives (cf. [11, equation (3.21)])

$$\begin{aligned} &\Delta T_{m^2}[\mathfrak{n}]\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big) = -\mathfrak{n}(p,q) \\ &\quad\times \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\frac{q^2}{4}\right)^n \sum_{k=0}^\infty \frac{1}{(2k+1)!} \sum_{l=0}^k \begin{bmatrix} 2k \\ l \end{bmatrix} \left(\frac{q^2}{2}\right)^l \left(\frac{q^j}{2} \frac{\partial}{\partial p^j} \right)^{2k-2l} T^{(n+1+l)}_{m^2}(p) \,, \end{aligned}$$

where the curly brackets are combinatorial factors whose detailed form is not needed here (see [11, equation (3.13)]).

Let us discuss how the Fourier transform of this expansion behaves in the limit $\Omega_0 \to -\infty$. According to the construction of the functions $\hat{f}_\beta$ in the proof of Proposition 10.4, the function $\mathfrak{n}(p,q)$ is supported in the region $p^2 \sim \Omega_0^2$. Moreover, the p-derivatives of $\mathfrak{n}$ scale in powers of $1/\Omega_0$. This implies that every derivative of the factor T_{m^2} gives a scaling factor of Ω_0^{-2}. Since every such derivative comes with factor q^2, we obtain a scaling factor $(q^2/\Omega_0^2)^{n+l}$. Thus in the limit $\Omega_0 \to -\infty$, it suffices to consider the lowest summand in $n+l$,

$$\Delta T_{m^2}[\mathfrak{n}] = -\mathfrak{n}(p,q) \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{q^j}{2} \frac{\partial}{\partial p^j} \right)^{2k} T^{(1)}_{m^2}(p) \left[1 + \mathcal{O}\Big(\frac{q^2}{\Omega_0^2} \Big) \right].$$

When transforming to position space, the p-derivatives can be integrated by parts. If they act on the function $\mathfrak{n}(p,q)$, this generates scale factors of the order $\mathcal{O}(|q|/|\Omega_0|)$ which again tend to zero as $\Omega_0 \to -\infty$. Thus it remains to consider the case when these derivatives act on the plane wave $e^{ip\xi}$. We thus obtain

$$\Delta T_{m^2}[\mathfrak{n}](x,y) = -\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \mathfrak{n}(p,q) \sum_{k=0}^{\infty} \frac{(-q\xi)^{2k}}{2^k(2k+1)!}\, T^{(1)}_{m^2}(p)\, e^{ip\xi - \frac{iq\zeta}{2}} + \mathcal{O}\Big(\frac{1}{\Omega_0} \Big).$$

We now consider a Taylor expansion in ξ around $\xi = 0$ up to the given order L. This amounts to replacing the factor $e^{e^{ip\xi}}$ by its power series and collecting the powers of ξ. The remaining task is to compare the factors $p\xi$ with $q\xi$. This is a subtle point, because the fact that $p^2 \sim \Omega_0^2$ does not imply that the inner product $p\xi$ is large. Indeed, this effect was responsible for the appearance of low frequencies in the Fourier integral (10.19). However, in the proof of Proposition 10.4 we arranged by a suitable choice of $\hat{f}_1$ and $\hat{f}_g$ that these low-frequency contributions cancel, so that we were working only with the high-frequency terms (10.31). Restating this fact in the present context, we can say that for the leading contribution to ΔT_{m^2}, the factor $p\xi$ is larger than $q\xi$ by a factor of the order $\mathcal{O}(|q|/|\Omega_0|)$. Thus it suffices the consider the summand $k=0$,

$$\Delta T_{m^2}[\mathfrak{n}](x,y) = -\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \mathfrak{n}(p,q)\, T^{(1)}_{m^2}(p)\, e^{ip\xi - \frac{iq\zeta}{2}} + \mathcal{O}\Big(\frac{1}{\Omega_0} \Big).$$

Now we can carry out the q-integration to obtain

$$\Delta T_{m^2}[\mathfrak{n}](x,y) = -16 \int \frac{d^4q}{(2\pi)^4}\, \mathfrak{n}(p, y+x)\, T^{(1)}_{m^2}(p)\, e^{ip\xi} + \mathcal{O}\Big(\frac{1}{\Omega_0} \Big).$$

Using this result in (10.37), we can carry out the derivatives to recover precisely the homogeneous perturbation (10.5) for fixed ζ. ☐

The scaling argument used in the last proof can be understood non-technically as follows. It clearly suffices to consider the region where y lies in a small neighborhood of x. Thus we may perform the rescaling $\xi \to \xi/\lambda$ with a scale factor $\lambda \gg 1$, leaving ζ unchanged. This corresponds to changing the momentum scale by $\Omega_0 \to \lambda\Omega_0$. In the limiting case $\lambda \to \infty$, the fermionic projector depends on ξ on a smaller and smaller scale. On this scale, the dependence on the variable $x+y$ drops out, so that the quasi-homogeneous ansatz (10.36) becomes exact. For this argument to work, one must ensure that in the homogeneous setting all frequencies scale like Ω_0, as was arranged in (10.31).

10.4. **Concluding Remarks.** We conclude this chapter with a brief discussion of our results. We first point out that, since the orders $|\kappa| = 0$ and $|\kappa| = 1$ are excluded in Theorem 10.5, the results of the previous chapters are not affected (see also the paragraph after the statement of Proposition 10.4). In particular, the field equations of Theorem 8.2 remain valid. The main conclusion from Theorem 10.5 is that the higher orders in ξ of an expansion of the EL equations to degree four can always be satisfied by suitable nonlocal potentials. These nonlocal potentials are non-dynamical similar as explained in §9.4 for local potentials. Furthermore, in view of the freedom in the constructions of Proposition 10.4, these nonlocal potentials are not uniquely determined, so that we do not get any physical predictions. Our point of view is that the nonlocal potentials give us a lot of freedom to perturb the higher orders in ξ, explaining why we do not get dynamical field equations for tensor fields of higher rank.

A surprising conclusion of our analysis is that, although the EL equations can be satisfied locally to every order in ξ, it is in general impossible to satisfy them globally for large ξ (see Lemma 10.3 and the paragraph before this lemma). At first sight, this might seem to imply that the EL equations are overdetermined and cannot be solved. On the other hand, the general compactness results in [18] indicate that our action principle does have non-trivial minimizers, so that the EL equations are expected to admit solutions. Thus there should be a way to compensate the above nonlocal error terms. A possible method is to modify the wave functions globally in space-time. Whether and how in detail this is supposed to work is a difficult question which we cannot answer here. Instead, we explain what this situation means physically: Suppose that a physical system is described by a minimizer of our action principle. Then the corresponding EL equations to degree four do not only yield the field equations, but they give rise to additional conditions which are nonlocal and can therefore not be specified by a local observer. In [14, Section 7] such so-called *nonlocal quantum conditions* were proposed to explain phenomena which in ordinary quantum mechanics are probabilistic.

More specifically, the fact that the EL equations cannot in general be satisfied globally might explain the tendency for quantum mechanical wave functions to be localized, as we now outline. Suppose that the fermionic projector is perturbed by a fermionic wave function (10.1). At the origin $\xi = 0$, this perturbation leads to the field equations as worked out in Chapter 8. The higher orders in ξ can be compensated by nonlocal potentials. But the contribution for large ξ cannot be compensated, thereby increasing our action. Thus seeking for minimizers, our action principle should try to arrange that the contribution (10.1) vanishes for large ξ. This might explain why quantum mechanical wave functions are usually not spread out over large distances, but are as much as possible localized, even behaving as point particles. This idea is elaborated further in [19].

Appendix A. Testing on Null Lines

In this appendix we justify the EL equations in the continuum limit (5.29) by specifying the wave functions Ψ_1 and Ψ_2 used for testing (5.24) in the setting with a general interaction and for systems involving several generations. Our method is to adapt the causal perturbation expansion (4.9) to obtain corresponding expansions for Ψ_1 and Ψ_2. We then consider the scaling of these terms to every order in perturbation theory. We rely on results from [13] and [21], also using the same notation.

We begin with the Dirac equation for the auxiliary fermionic projector of the general form (4.5), where we assume that $\mathcal{B}$ is a multiplication operator which is smooth and decays so fast at infinity that

$$\int_M \left| x^I \partial_x^J \mathcal{B}(x) \right| d^4x < \infty \qquad \text{for all multi-indices } I \text{ and } J \text{ with } |I| \leq 2 \,. \tag{A.1}$$

Under this assumption, every Feynman diagram of the causal perturbation expansion (4.9) is well-defined and finite (see [13, Lemma 2.2.2]). As in [13, §2.6], we denote the spectral projectors of the operator $(i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1})$ by $\tilde{p}_{+\mu}$. In contrast to (5.26), where we cut out an ω-strip around the mass shell, it is here more convenient to remove neighboring mass shells by setting

$$\Psi_1 = \eta - \int_{-\Delta m}^{\Delta m} \widehat{\tilde{p}}_{+\mu} \, \eta \, d\mu \,, \tag{A.2}$$

where the tilde again denotes the partial trace over the generations. When taking the product $P\Psi_1$, we get cross terms involving different generations. However, as in the proof of [13, Theorem 2.6.1] one sees that these cross terms vanish in the infinite volume limit. Thus Ψ_1 indeed lies in the kernel of the Dirac operator. Moreover, by choosing Δm sufficiently small, we can make the difference $\Psi_1 - \eta$ as small as we like.

The construction of Ψ_2 is a bit more involved. In order to get into the framework involving several generations, we first extend the wave packet in (5.27) to an object with $4g$ components,

$$\psi := (i\partial\!\!\!/ + mY)\,\theta \quad \text{with} \quad \theta = \Big(e^{-i\Omega(t+x)}\, \phi(t+x-\ell, y, z) \Big) \oplus \underbrace{0 \oplus \cdots \oplus 0}_{g-1 \text{ summands}} \,. \tag{A.3}$$

For Ψ_2 we make an ansatz involving a partial trace over the generation index,

$$\Psi_2 = \sum_{\beta=1}^{g} \left(\psi_\beta + \Delta\psi_\beta^{\mathrm{D}} + \Delta\psi_\beta^{\mathrm{E}} \right), \tag{A.4}$$

where the corrections ψ_β^{D} and $\Delta\psi_\beta^{\mathrm{E}}$ should take into account that the auxiliary Dirac equation must hold and that the generalized energy must be negative, respectively. In order to specify $\Delta\psi_\beta^{\mathrm{D}}$, we first apply the free Dirac operator to ψ,

$$\begin{aligned} (i\partial\!\!\!/ - mY)\,\psi &= (i\partial\!\!\!/ - mY)(i\partial\!\!\!/ + mY) \Big(e^{-i\Omega(t+x)}\, \phi(t+x-\ell, y, z) \oplus 0 \oplus \cdots \oplus 0 \Big) \\ &= -(\Box + m^2) \Big(e^{-i\Omega(t+x)}\, \phi(t+x-\ell, y, z) \Big) \oplus 0 \oplus \cdots \oplus 0 = \left(\partial_y^2 + \partial_z^2 - m^2 \right) \theta \,, \end{aligned}$$

where $m \equiv m_1$ is the mass of the first generation. We point out that the obtained expression does not involve the "large" parameter $|\Omega|$; in this sense it is a small error term. In order for $\psi + \Delta\psi_\beta^{\mathrm{D}}$ to satisfy the auxiliary Dirac equation (4.5), the wave function $\Delta\psi_\beta^{\mathrm{Dirac}}$ must be a solution of the inhomogeneous Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\,\Delta\psi^{\mathrm{D}} = -\left(\partial_y^2 + \partial_z^2 - m^2 \right) \theta - \mathcal{B}\psi \,. \tag{A.5}$$

Solutions of this equation could be constructed rigorously with energy estimates (see for example [31]). But here we are here content with a perturbative treatment. Denoting the Green's function of the zero mass free Dirac operator by s, i.e.

$$i\partial\!\!\!/ \, s = 1 \,,$$

we can solve for $\Delta\psi^{\rm D}$ in terms of the perturbation series

$$\Delta\psi^{\rm D} = -\sum_{k=0}^{\infty} \left(-s(\mathcal{B} - mY)\right)^k s \left[\left(\partial_y^2 + \partial_z^2 - m^2\right) f + \mathcal{B}\psi\right] . \tag{A.6}$$

We point out that $\Delta\psi^{\rm D}$ is not uniquely determined, and this non-uniqueness is reflected by the fact that there is the freedom in choosing different Green's functions, like the advanced or retarded Green's functions or the Feynman propagator (for details on the above operator expansions and the different Green's functions see [13] or [21]). For our purpose, it is most preferable to work with the retarded Green's function $s^\wedge$, whose kernel $s^\wedge(x,y)$ is given explicitly by (see [13, §2.5])

$$s^\wedge(x,y) = -\frac{1}{2\pi}\, i\partial\!\!\!/_x \delta(\xi^2)\, \Theta(-\xi^0) . \tag{A.7}$$

This has the advantage that the support of $\Delta\psi^D$ lies in the future of ψ, and thus it is disjoint from the support of Ψ_1 (see Figure 1 on page 25).

The function $\psi + \Delta\psi^{\rm D}$ solves the auxiliary Dirac equation, but it will in general have a component of generalized positive energy. This positive-energy contribution must be subtracted in order to obtain a vector in the image of P. Formally, this can be achieved by setting

$$\Delta\psi^{\rm E} = -(1-P)\left(\psi + \Delta\psi^{\rm D}\right) . \tag{A.8}$$

In order to give this equation a meaning, one must keep in mind that the normalization of the fermionic projector involves a δ-distribution in the mass parameters, i.e. $P_{+\mu}P_{+\mu'} = \delta(\mu-\mu')P_{+\mu}$. Thus using the formalism introduced in [21, Section 2], we can make sense of (A.8) as an operator product simply by omitting the resulting δ-distributions. With (A.4) as well as (A.3), (A.6) and (A.8), we have introduced Ψ_2 by a well-defined perturbation series.

We now estimate $\Delta\Psi^{\rm E}$ for large $|\Omega|$, with a similar method as previously used in [12, Theorem 3.4] for the estimate of the non-causal high energy contribution.

Lemma A.1. *To very order n in perturbation theory and for every $\nu \in \mathbb{N}$, there is a constant $C(n,\nu)$ such that the wave function $\Delta\psi^E$ as defined by* (A.8) *satisfies the inequality*

$$\sup_{x\in M} \left|(\Delta\psi^E)^{(n)}(x)\right| \le \frac{C(n,\nu)}{|\Omega|^\nu} . \tag{A.9}$$

Proof. The $n^{\rm th}$ order contribution $(\Delta\psi^{\rm E})^{(n)}$ can be written as a finite number of terms of the form

$$g := C_n\, \mathcal{B}\, C_{n-1}\, \mathcal{B}\, \cdots\, \mathcal{B}\, C_0 \phi , \tag{A.10}$$

where every factor C_l is a linear combination of the operators p, k, and s. Here ϕ stands either for the wave function ψ in (A.3) or for the square bracket in (A.6). In either case, ψ is given explicitly and involves the free parameter Ω. It is preferable to proceed in momentum space. The regularity and decay assumption (A.1) implies that

$$\sup_{k\in\hat{M}} \left|k^J \partial^I \hat{\mathcal{B}}(k)\right| < \infty \qquad \text{for all multi-indices } I \text{ and } J \text{ with } |I| \le 2 . \tag{A.11}$$

Setting $F_0 = \hat{\phi}$ and

$$F_l(k) = \int \frac{d^4q}{(2\pi)^4}\, \hat{\mathcal{B}}(k-q)\, C_l(q)\, F_{l-1}(q) \qquad (\text{where } 1 \le l \le n) , \tag{A.12}$$

we can write the Fourier transform of g as

$$\hat{g}(k) = C_n(k)\, F_n(k)\,. \tag{A.13}$$

It clearly suffices to prove the lemma for ν an even number. Let us show inductively that the functions F_l satisfy the bounds

$$\sup_{(\omega,\vec{k})\in\hat{M}} (\omega-\Omega)^\nu \Big(|F_l(\omega,\vec{k})| + \sum_{i=0}^{3} |\partial_i F_l(\omega,\vec{k})| \Big) < C(l,\nu) \qquad \text{uniformly in } \Omega\,. \tag{A.14}$$

In the case $l=0$, the claim follows immediately from the explicit form of ψ. To prove the induction step, we use the inequality

$$(\omega-\Omega)^\nu \leq c(\nu)\,\big((\omega-\omega')^\nu + (\omega'-\Omega)^\nu\big)$$

to obtain the estimate

$$\begin{aligned}
&\big|(\omega-\Omega)^\nu\, F_l(\omega,\vec{k})\big| \\
&\leq c(\nu) \left| \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{\mathbb{R}^3} \frac{d\vec{k}'}{(2\pi)^3} \left[(\omega-\omega')^\nu\, \hat{\mathcal{B}}(\omega-\omega',\vec{k}-\vec{k}')\right] C_l(\omega',\vec{k}')\, F_{l-1}(\omega',\vec{k}') \right| \\
&\quad + c(\nu) \left| \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{\mathbb{R}^3} \frac{d\vec{k}'}{(2\pi)^3}\, \hat{\mathcal{B}}(\omega-\omega',\vec{k}-\vec{k}')\, C_l(\omega',\vec{k}') \left[(\omega'-\Omega)^\nu F_{l-1}(\omega',\vec{k}')\right] \right|.
\end{aligned}$$

Furthermore, the factors C_l involve at most first derivatives; more precisely, they are bounded in terms of Schwartz norms by (see for example [13, Proof of Lemma 2.2.2])

$$|C_l(f)| \leq \text{const}\, \|f\|_{4,1} \qquad \text{for all } f \in \mathcal{S}\,.$$

Combining these inequalities, we can use the induction hypothesis together with (A.11) to bound the expression $|(\omega-\Omega)^\nu\, F_l(\omega,\vec{k})|$ uniformly in Ω and $(\omega,\vec{k})$. The expression $|(\omega-\Omega)^\nu\, \partial_i F_l(\omega,\vec{k})|$ can be estimated in exactly the same way if one keeps in mind that if we differentiate (A.12) with respect to k, the derivative acts only on the potential $\hat{\mathcal{B}}$, but not on the factor F_{l-1}. This proves (A.14).

We next consider the operators in (A.10) in more detail. The factor $(1-P)$ in (A.8) can be regarded as a projector onto the generalized positive-energy solutions of the Dirac equation. The perturbation expansion of the fermionic projector can be arranged in such a way that each operator product involves at least one factor $p-k$ which projects onto the negative-energy solutions (for details see [21]). Similarly, the factor $(1-P)$ in (A.8) implies that we can arrange the operator products such that every contribution (A.10) involves at least one factor $p+k$, being supported on the upper mass cone. Thus in the corresponding induction step, we may replace an arbitrary even number of factors $(\omega-\Omega)$ by factors of $|\Omega|$. In the following induction steps we proceed as in (A.14). At the end, we apply (A.13) to obtain the result. □

We can now prove the main result of this appendix.

Proposition A.2. *Consider a fermion system in Minkowski space with an interaction $\mathcal{B}$ which is a multiplication operator satisfying the regularity and decay assumptions* (A.1). *Assume furthermore that the pole of Q is of order $o(|\vec{\xi}|^{-4})$ at the origin (see Definition 7.2). Then the EL equations* (5.20) *imply that the operator Q vanishes identically in the continuum limit* (5.29).

Proof. We introduce the wave functions Ψ_1 and Ψ_2 perturbatively by (A.2) and (A.4) with ψ, $\Delta\psi^{\mathrm{D}}$ and $\Delta\psi^{\mathrm{E}}$ according to (A.3), (A.6) and (A.8). Evaluating the commutator $[P,Q]$ as in (5.23) gives the condition (5.24). Following the arguments in §5.2, the leading terms give (5.29), and thus it remains to consider all correction terms. The corrections of Ψ_1 can be made arbitrarily small by choosing the parameter Δm in (A.2) sufficiently small. Working in (A.6) with the retarded Green's function (A.7), the support of $\Delta\psi^{\mathrm{D}}$ does not intersect the support of η, so that the corresponding contribution to (5.24) is well-defined in the continuum limit.

The wave function $\Delta\psi^{\mathrm{E}}$ is more problematic, because it will in general not vanish on the support of η. But according to Lemma A.1, we can make $\Delta\psi^{\mathrm{E}}$ arbitrarily small by choosing $|\Omega|$ sufficiently large. This is not quite good enough for two reasons: First, the integrand in (5.24) becomes more and more oscillatory as $|\Omega|$ is increased, so that the leading contribution to (5.24) will also become small as $|\Omega|$ gets large. And secondly, even if $\Delta\psi^{\mathrm{E}}$ is small, it gives rise to a contribution at $x=y$ where $Q(x,y)$ is ill-defined. The first problem can be treated by noting that the oscillations in the integrand of (5.24) will give rise to a polynomial decay in Ω (typically a $1/\Omega$ behavior), whereas according to (A.9), the wave function $\Delta\psi^{\mathrm{E}}$ decays in Ω even rapidly. Thus we can indeed arrange that (A.9) is arbitrarily small compared to the leading contribution in (5.24). For the second problem we need to use that the pole of Q is of order $o(|\vec{\xi}|^{-4})$ at the origin: Due to this assumption, the integrand in (5.24) will be at most logarithmically divergent at $x=y$. By modifying Ψ_2 by a suitable negative-energy solution of the Dirac equation (for example a wave packet of negative energy, whose amplitude is fine-tuned), one can arrange that this logarithmic divergence drops out. Then the integrals in (5.24) become finite, and by choosing $|\Omega|$ sufficiently large, we can arrange that the contribution of $\Delta\psi^{\mathrm{E}}$ to (5.24) is much smaller than the leading contribution which yields (5.29). □

Before discussing the result of this proposition, we estimate an operator product which is similar to (A.10) but involves a nonlocal potential as considered in §10.3.

Lemma A.3. *We consider the expression*

$$g = C_n\, B_n\, C_{n-1}\, B_{n-1}\, \cdots\, B_1\, C_0 \phi\,,$$

where every factor C_l stands for one of the operators p, k, or s. As in the proof of Lemma A.1, the function ϕ is either the wave function ψ in (A.3) *or the square bracket in* (A.6)*. Each factor B_l either stands for the multiplication operator $\mathcal{B}$ satisfying* (A.1)*, or else it is a nonlocal operator $\mathfrak{n}$ of the Schwartz class* (10.35)*. We assume that at least one factor B_l is a nonlocal operator. Then for every integer ν there is a constant $C(\nu)$ such that*

$$\sup_{x\in M} |g(x)| \leq \frac{C(\nu)}{|\Omega|^\nu}\,. \tag{A.15}$$

Proof. As in the proof of Lemma A.1, we proceed inductively in momentum space. Suppose that p is the smallest index such that $B_p = \mathfrak{n}$. Then for all $l < p$, only the potential $\mathcal{B}$ is involved, and the functions F_l defined by (A.12) again satisfy the inequalities (A.14). In the p^{th} induction step, we must replace (A.12) by

$$F_p(k) = \int \frac{d^4q}{(2\pi)^4}\, \hat{n}(k,q)\, C_l(q)\, F_{l-1}(q)\,,$$

where $\hat{n} \in \mathcal{S}(\hat{M} \times \hat{M})$ denotes the Fourier transform of $n(x,y)$. Using the induction hypothesis (A.14) together with the rapid decay of $\hat{n}(p,q)$ in the variable q, we obtain a factor $|\Omega|^{-\nu}$. In the remaining induction steps, we can use the simpler method of [13, Lemma 2.2.2] to obtain the result. □

The setting of Proposition A.2 is too special for our applications, making it necessary to consider the following extensions:

- Taking into account the *wave functions* of the particles and anti-particles: We first note that, being solutions of the Dirac equation, the wave functions of the particles and anti-particles in (4.10) are orthogonal to the wave function Ψ_1 (as is obvious from (A.2)). Furthermore, by choosing $|\Omega|$ much larger than the energies of all particle and anti-particle wave functions, we can arrange that these wave functions are also orthogonal to Ψ_2. Then all the wave functions drop out of (5.24), so that we are back in the setting of Proposition A.2.
- Handling the *local axial transformation*: Following the constructions in §7.6, we must apply the local axial transformation (7.25) to the fermionic projector before taking the partial trace. Likewise, we here apply this transformation to the wave functions Ψ_1 and Ψ_2 before taking the partial trace. This additional transformation preserves all local estimates, so that all our arguments still go through.
- Arranging the right order of the pole of Q at the origin: As we saw in Chapter 6, the operator Q vanishes identically to degree five on the light cone. Thus the leading contribution to Q is of degree four on the light cone. Since Q always involves a factor $\not{\xi}$ (see (7.2)), the pole of Q at the origin is indeed of the required order $o(|\vec{\xi}|^{-4})$.

  One might object that the near the origin $x=y$, where the continuum limit of $Q(x,y)$ is not well-defined, the arguments of Chapter 6 do not apply, and thus there might be a non-zero contribution to Q which scales like (6.7), thus having a pole $\sim |\vec{\xi}|^{-4}$. However, as explained after (6.8), we may assume that in the vacuum, the operator Q vanishes identically, even at the origin where the formalism of the continuum limit does not apply. Since to degree five, an interaction only leads to phase transformations (see (6.26)), the operator Q will then again vanish identically. As a consequence, the pole of Q will indeed scale like $|\vec{\xi}|^{-3}$, even without relying on the formalism of the continuum limit.
- Handling *nonlocal potentials*: Proposition A.2 does not apply to nonlocal potentials as introduced in §10.3. Another difficulty is that the support argument used for $\Delta\psi^{\mathrm{D}}$ will no longer apply. But as shown in Lemma A.3, any contribution to $\Delta\psi^{\mathrm{D}}$ or $\Delta\psi^{\mathrm{E}}$ which involves a nonlocal potential satisfies the inequality (A.15) and can thus be made arbitrarily small by choosing $|\Omega|$ sufficiently large. Following the arguments in the proof of Proposition A.2, this gives us control of all error terms due to the nonlocal potentials, to every order in perturbation theory.

We conclude that with the help of Proposition A.2, we can justify the EL equations of the continuum limit (5.29) for all fermion systems considered in this paper.

We finally analyze the scalings in a universe of finite life time.

Remark A.4. *(A universe of finite life-time)* Suppose that instead of Minkowski space, we are considering a more realistic universe of finite life time $t_{\max}$, like a cosmology with a "big bang" and a "big crunch." In this case, the Fourier integral (5.25) still gives a good local description of a Dirac sea (this is made precise in the example of a closed FRW geometry in [24, Theorem 5.1]). However, one can no longer expect a continuum of states, and therefore the condition $P\Psi_1 = 0$ can no longer be satisfied by removing an arbitrarily thin strip around the mass shell. More precisely, the width $\Delta\omega$ of the strip in (5.26) should be at least as large as the "coarseness" of the states in momentum space. This gives rise to the scaling (for details in the example of the closed FRW geometry see [24, Section 5])

$$\Delta\omega \sim \frac{1}{t_{\max}} \,.$$

The corresponding contribution to the Fourier integral (5.26) scales as follows,

$$\Delta\Psi_1 := \Psi_1(x) - \eta(x) \sim \sup|\hat{\eta}|\, \frac{\Delta\omega}{\delta^3} \sim \Delta\omega\, \delta\, |\eta(0)| \sim \frac{\delta}{t_{\max}}\, |\Psi_1(0)| \,.$$

As a consequence, the wave function Ψ_1 no longer vanishes on $\mathfrak{L}$, but

$$\Psi_1|_{\mathfrak{L}} \sim \frac{\delta}{t_{\max}}\, \sup|\Psi_1| \,.$$

Furthermore, since $\Delta\Psi_1$ is supported near the lower mass shell in momentum space, it decays in position space at infinity like the fundamental solution $(p_m - k_m)(0, y)$, smeared out on the scale δ. Combining these statements, we find that the corresponding contribution to the expectation value (5.24) scales like

$$<\!\Delta\Psi_1|Q\,\Psi_2\!> \sim \sup|\Psi_1|\, \sup|\Psi_2|\, \delta^4\, \frac{\delta}{t_{\max}}\, \frac{1}{\varepsilon^{L-1}}\, \varepsilon^{3-p} \,.$$

where p denotes the order of the pole at the origin being defined as the smallest integer p such that

$$\limsup_{x \to y}\, (|\xi^0| + |\vec{\xi}|)^{L-p}\, |\eta(x,y)| < \infty \,.$$

In comparison, the main contribution on the light cone around the origin scales like

$$<\!\Psi_1|Q\,\Psi_2\!> \sim \sup|\Psi_1|\, \sup|\Psi_2|\, \delta^4\, \frac{1}{\varepsilon^{L-1}}\, \ell^{-p}\, \frac{\delta^2}{|\Omega|} \,,$$

and thus

$$\frac{<\!\Delta\Psi_1|Q\,\Psi_2\!>}{<\!\Psi_1|Q\,\Psi_2\!>} \sim \frac{\varepsilon^{3-p}\, \ell^p\, |\Omega|}{t_{\max}\, \delta} = \left(\frac{\varepsilon}{\ell}\right)^{3-p} \varepsilon\, |\Omega|\, \frac{\ell^3}{\varepsilon l_{\max}\, \delta} \,. \tag{A.16}$$

This equation involves the fundamental length scale $\sqrt{\varepsilon l_{\max}}$. The time since the big bang is estimated to about 13 billion years, which is the same order of magnitude as the size of the visible universe, estimated to 28 billion parsec. Thus it seems reasonable to assume that

$$t_{\max} > 10^{10}\,\text{years} \sim 10^{26}\,\text{meters} \,.$$

Taking for ε the Planck length $\varepsilon \sim 10^{-35}$ meters, we obtain

$$\sqrt{\varepsilon\, l_{\max}} \sim 10^{-4}\,\text{meters} \,.$$

It is remarkable that this is about the length scale of macroscopic physics. Thus by choosing $\varepsilon|\Omega|$ sufficiently small, we can make the quotient (A.16) arbitrarily small without violating the scalings (5.28). We conclude that even if the life time of our universe is finite, this would have no effect on the statement of Proposition A.2. ◊

Appendix B. Spectral Analysis of the Closed Chain

In this appendix we analyze how different contributions to the fermionic projector effect the EL equations. In particular, we shall give the proofs of Lemmas 7.3, 7.4, 7.5, 7.6 and 7.7. Furthermore, we will analyze a pseudoscalar differential potential (see (B.32) and (B.33)) and a scalar/pseudoscalar potential (Lemma B.1).

We first explain the methods and the general procedure. The behavior of the fermionic projector near the light cone is described by the light-cone expansion (4.17). We concentrate on the singular behavior on the light cone as described by the series in (4.17), disregarding the smooth non-causal contributions P^{le} and P^{he} (for the smooth contributions see Appendix D and §8.3; also cf. the end of §5.1). The terms of this series can be computed as described in [13, §2.5] (for more details see [11] and [12]). The main task is to calculate the corresponding perturbation of the eigenvalues $\lambda^{L/R}_{\pm}$, because then the effect on the EL equations is given by Lemma 7.1. In principle, the perturbation of the eigenvalues can be determined in a straightforward manner by substituting the summands of the light-cone expansion into the closed chain A_{xy} (2.5), and by performing a standard perturbation calculation for the eigenvalues of the (4×4)-matrix A_{xy}. However, the combinatorics of the tensor contractions inside the closed chain makes this direct approach so complicated that it is preferable to use a more efficient method developed in [13, Appendix G]. We now outline this method, giving at the same time a somewhat different viewpoint.

Basically, the method is to work in a special basis of the spinors in which the calculations become as simple as possible. More precisely, we choose a *double null spinor frame* $(\mathfrak{f}^{L/R}_{\pm})$, being an eigenvector basis of the closed chain A_{xy} of the vacuum to degree three. Thus following (6.2), we introduce the matrix

$$A^0_{xy} = \frac{g^2}{4}\, (\not{\xi} T^{(-1)}_{[0]}) \overline{(\not{\xi} T^{(-1)}_{[0]})}\,.$$

According to (6.5), the corresponding spectral projectors in the formalism of the continuum limit are given by

$$F^0_{\pm} = \frac{1}{2}\Big(\mathbb{1} \pm \frac{[\not{\xi}, \overline{\not{\xi}}]}{z - \overline{z}} \Big), \tag{B.1}$$

and they satisfy the relations

$$F^0_+ \, \not{\xi} \overline{\not{\xi}} = z\, F^0_+\,, \qquad \text{and} \qquad F^0_- \, \not{\xi} \overline{\not{\xi}} = \overline{z}\, F^0_-. \tag{B.2}$$

Furthermore, A^0 is invariant on the left- and right-handed components, and thus we may choose choose joint eigenvectors of the matrices A_0 and γ^5. This leads us to introduce the four eigenvectors $\mathfrak{f}^{L/R}_{\pm}$ by the relations

$$\boxed{\chi_c F^0_s \, \mathfrak{f}^c_s = \mathfrak{f}^c_s} \tag{B.3}$$

with $c \in \{L, R\}$ and $s \in \{+, -\}$, which defines each of these vectors up to a constant. For clarity in notation, we write the inner product on Dirac spinors $\overline{\Psi}\Phi \equiv \Psi^\dagger \gamma^0 \Phi$ as $\prec \Phi | \Psi \succ$, and refer to it as the *spin scalar product*. Then the calculation

$$\prec \mathfrak{f}^L_+ \,|\, \mathfrak{f}^L_+ \succ \,=\, \prec \chi_L \mathfrak{f}^L_+ \,|\, \chi_L \mathfrak{f}^L_+ \succ \,=\, \prec \mathfrak{f}^L_+ \,|\, \chi_R \chi_L \mathfrak{f}^L_+ \succ \,=\, 0$$

(and similarly for the other eigenvectors) shows that these vectors are indeed all null with respect to the spin scalar product. Moreover, taking the adjoint of (B.1) with respect to the spin scalar product, one sees that

$$(F_+^0)^* = F_-^0 \,. \tag{B.4}$$

As a consequence, the inner products vanish unless the lower indices are different, for example

$$\prec \mathfrak{f}_+^L \,|\, \mathfrak{f}_+^R \succ \;=\; \prec F_+^0 \mathfrak{f}_+^L \,|\, F_+^0 \mathfrak{f}_+^R \succ \;=\; \prec \mathfrak{f}_+^L \,|\, F_-^0 F_+^0 \mathfrak{f}_+^R \succ \;=\; 0 \,.$$

We conclude that all inner products between the basis vectors vanish except for the inner products $\prec \mathfrak{f}_+^L | \mathfrak{f}_-^R \succ$, $\prec \mathfrak{f}_+^R | \mathfrak{f}_-^L \succ$ as well as their complex conjugates $\prec \mathfrak{f}_-^R | \mathfrak{f}_+^L \succ$ and $\prec \mathfrak{f}_-^L | \mathfrak{f}_+^R \succ$. We assume that all the non-vanishing inner products are equal to one,

$$\boxed{\,|\prec \mathfrak{f}_+^L \,|\, \mathfrak{f}_-^R \succ| = 1 = |\prec \mathfrak{f}_+^R \,|\, \mathfrak{f}_-^L \succ| \,.\,} \tag{B.5}$$

In order to specify the phases and relative scalings of the basis vectors, we introduce a space-like unit vector u which is orthogonal to ξ and $\overline{\xi}$. Then the imaginary vector $v = iu$ satisfies the relations

$$\langle v, \xi \rangle = 0 = \langle v, \overline{\xi} \rangle \,, \quad \langle v, v \rangle \;=\; 1 \quad \text{and} \quad \overline{v} \;=\; -v \,. \tag{B.6}$$

As a consequence, the operator $\not v$ commutes with F_+^0 and F_-^0, and since it flips parity, we may set $\mathfrak{f}_+^R = \not v \, \mathfrak{f}_+^L$. Next, from (B.1) one derives the identities

$$F_-^0 \, \not\xi = \not\xi \, F_+^0 \qquad \text{and} \qquad F_-^0 \, \overline{\not\xi} = \overline{\not\xi} \, F_+^0 \,, \tag{B.7}$$

which can be used as follows. The first of these identities implies that the vectors $\not\xi \, \mathfrak{f}_+^L$ and $\mathfrak{f}_-^R$ are linearly dependent. The calculation

$$\prec \mathfrak{f}_+^L \,|\, \not\xi \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \not\xi \, \frac{\not\xi \overline{\not\xi}}{z} \, \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \frac{\not\xi \not\xi}{z} \, \overline{\not\xi} \, \mathfrak{f}_+^L \succ \;=\; \prec \mathfrak{f}_+^L \,|\, \overline{\not\xi} \mathfrak{f}_+^L \succ$$

(where we used (B.3) and (B.1)) shows that the vector $\not\xi \, \mathfrak{f}_+^L$ is indeed a real multiple of $\mathfrak{f}_-^R$. Hence by normalizing $\mathfrak{f}_+^L$ appropriately, we can arrange[6] that $\mathfrak{f}_-^R = \not\xi \, \mathfrak{f}_+^L$. Using the second identity in (B.7), we also find that $\mathfrak{f}_-^R = \overline{\not\xi} \, \mathfrak{f}_+^L$. Similarly, we may also set $\mathfrak{f}_-^L = \not\xi \, \mathfrak{f}_+^R = \overline{\not\xi} \, \mathfrak{f}_+^R$. The resulting relations between our basis vectors are summarized in the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{f}_+^L & \xrightarrow{\;\;\not v\;\;} & \mathfrak{f}_+^R \\ \not\xi \big\downarrow \overline{\not\xi} & & \not\xi \big\downarrow \overline{\not\xi} \\ \mathfrak{f}_-^R & \xrightarrow{\;-\not v\;} & \mathfrak{f}_-^L \end{array} \tag{B.8}$$

With (B.3), (B.5) and (B.8) we have introduced the double null spinor frame $(\mathfrak{f}_\pm^{L/R})$. The construction involves the freedom in choosing the operator $\not v$ according to (B.6); for given $\not v$ the basis vectors are unique up to an irrelevant common phase.

[6] Let us explain why we do not consider the opposite sign $\mathfrak{f}_-^R = -\not\xi \, \mathfrak{f}_+^L$. To this end, we must show that $\prec \mathfrak{f}_+^L | \not\xi \mathfrak{f}_+^L \succ > 0$. Since for any given positive or definite spinor ζ, the vector $\chi_L F_+ \zeta$ is a multiple of $\mathfrak{f}_+^L$, it suffices to compute instead the sign of the combination $\prec \chi_L F_+ \zeta | \not\xi \chi_L F_+ \zeta \succ$. Applying (B.4) and (B.7), this inner product simplifies to $\prec \zeta | \chi_R F_- \not\xi \zeta \succ$. With the help of (6.19) and (6.1), we can treat the factor $\not\xi$ as an outer factor. Then our inner product simplifies to the expectation value $\prec \zeta | \chi_R \not\xi \zeta \succ$. This expectation value is positive if we follow the convention introduced before (5.7) that $\xi^0 > 0$.

We next explain how we can represent a given linear operator B on the spinors in the double null frame $(\mathfrak{f}^{L/R}_\pm)$. Following the notation in [13, Appendix G], we denote the matrix entry in the column (c,s) and row (c',s') by $F^{cc'}_{ss'}(B)$. These matrix entries are obtained by acting with B on the vector $\mathfrak{f}^{c'}_{s'}$ and taking the inner product with the basis vector which is conjugate to $\mathfrak{f}^c_s$, i.e.

$$F^{cc'}_{ss'}(B) = \prec \mathfrak{f}^{\overline{c}}_{\overline{s}} \,|\, B\, \mathfrak{f}^{c'}_{s'} \succ , \tag{B.9}$$

where the conjugation flips the indices according to $L \leftrightarrow R$ and $+ \leftrightarrow -$. Similarly, we can also express the projectors $\chi_c F_s$ in terms of the basis vectors, for example

$$\chi_L F^0_+ = |\mathfrak{f}^L_+ \succ \prec \mathfrak{f}^R_-| \,. \tag{B.10}$$

For computing (B.9), we use the relations in (B.8) to express the vector $\mathfrak{f}^{c'}_{s'}$ in terms of $\mathfrak{f}^L_+$, choosing the relations which do not involve factors of $\overline{\not{\xi}}$. Similarly, we express the vector $\mathfrak{f}^{\overline{c}}_{\overline{s}}$ in terms of $\mathfrak{f}^R_-$, avoiding factors of $\not{\xi}$. Applying (B.10), we can then rewrite the inner product as a trace involving the operator F^0_+. More precisely, a straightforward calculation yields

$$\left. \begin{array}{rclcrcl} F^{LL}_{++}(B) &=& \mathrm{Tr}(F^0_+\, \chi_L\, B) &,& F^{LR}_{++}(B) &=& \mathrm{Tr}(F^0_+\, \overline{\not{\xi}}\, \chi_L\, B) \\ F^{LL}_{+-}(B) &=& \mathrm{Tr}(\not{\xi}\, F^0_+\, \overline{\not{\xi}}\, \chi_L\, B) &,& F^{LR}_{+-}(B) &=& \mathrm{Tr}(\not{\xi}\, F^0_+\, \chi_L\, B) \\ F^{LL}_{-+}(B) &=& \dfrac{1}{z}\, \mathrm{Tr}(F^0_+\, \overline{\not{\xi}}\, \not{\xi} \chi_L\, B) &,& F^{LR}_{-+}(B) &=& \dfrac{1}{z}\, \mathrm{Tr}(F^0_+\, \not{\xi}\, \chi_L\, B) \\ F^{LL}_{--}(B) &=& \dfrac{1}{z}\, \mathrm{Tr}(\not{\xi}\, F^0_+\, \not{\xi}\, \chi_L\, B) \,,&& F^{LR}_{--}(B) &=& \dfrac{1}{z}\, \mathrm{Tr}(\not{\xi}\, F^0_+\, \overline{\not{\xi}}\, \not{\xi}\, \chi_L\, B) \end{array} \right\} \tag{B.11}$$

(see also [13, equation G.19], where these relations are derived with a different method). Indeed, it suffices to compute the given eight matrix elements, because the other eight matrix elements are obtained by the replacements $L \leftrightarrow R$. Moreover, the matrix elements of the adjoint (with respect to the spin scalar product) are obtained by

$$F^{cc'}_{ss'}(B^*) = \prec \mathfrak{f}^{\overline{c}}_{\overline{s}} \,|\, B^*\, \mathfrak{f}^{c'}_{s'} \succ = \overline{\prec \mathfrak{f}^{c'}_{s'} \,|\, B\, \mathfrak{f}^{\overline{c}}_{\overline{s}} \succ} = \overline{F^{\overline{c'}\,\overline{c}}_{\overline{s'}\,\overline{s}}(B)} \,.$$

Now we can describe our general procedure for the spectral analysis of the closed chain. We first perform the light-cone expansion of the fermionic projector. Then we calculate the matrix elements of the fermionic projector according to (B.11). Transforming to the double null spinor frame at such an early stage has the advantage that the contractions of the tensor indices (which come up by taking traces of products of Dirac matrices) are relatively easy to compute. After forming the closed chain A_{xy} in our double-null spinor basis, we can compute the eigenvalues of A_{xy} with a standard perturbation calculation (see [13, Appendix G.1] for a formulation with contour integrals). As the unperturbed operator we choose the closed chain (6.24) which involves the axial phases. This is particularly convenient because the unperturbed operator is diagonal in our double null spinor basis, and moreover the unperturbed eigenvalues are non-degenerate according to (6.26). Thus it suffices to use simple perturbation theory without degeneracies. Next, it is useful that the unperturbed eigenvalues (6.26) form two complex conjugate pairs. This will remain true if perturbations of lower degree are taken into account, so that $\overline{\lambda^c_s} = \lambda^{\overline{c}}_{\overline{s}}$. Therefore, it suffices to consider the eigenvalue λ^L_+. The eigenvalue λ^R_+ is then obtained by the replacement $L \leftrightarrow R$, whereas the eigenvalues $\lambda^{L/R}_-$ are obtained by complex conjugation. Expressing the perturbation calculation for λ^L_+ in terms of the traces (B.11), one finds that to the considered degree on the light cone, the vector v drops out.

To avoid calculation errors, the light-cone expansion was carried out with the help of the C++ program class_commute, which was originally developed for the calculations in [11] and [12]. The traces in (B.11), which involve the contractions of tensor indices, are also computed with the help of class_commute. The resulting matrix elements of the fermionic projector are exported to the computer algebra program Mathematica (from this moment on, the tensor indices are simply treated as fixed text strings). The perturbation calculation as well as the expansions around the origin are then carried out by an algorithm implemented in Mathematica. This also has the advantage that the standard simplification algorithms of Mathematica and the comfortable front end are available. The C++ program class_commute and its computational output as well as the Mathematica worksheets are available from the author on request.

We now list the results of these calculations, also giving some intermediate steps. We note that some of these results are already obtained in [13, Appendix G.3], however without using the algorithm implemented in Mathematica.

Proof of Lemma 7.3. Using for line integrals as in [12] the short notation (6.23) and

$$\int_x^y [p, q \,|\, r]\, f := \int_0^1 \alpha^p \,(1-\alpha)^q \,(\alpha - \alpha^2)^r \; f(\alpha y + (1-\alpha) x)\; d\alpha\;, \tag{B.12}$$

the relevant contributions to the light-cone expansion can be written as (cf. [13, Appendix G.3] and [12, Appendix A])

$$\begin{aligned}
\chi_L P(x,y) &= \frac{i}{2}\, \chi_L\, e^{-i\Lambda^{xy}_L}\, \xi\!\!\!/\, T^{(-1)} && \text{(B.13)}\\
&- \frac{1}{2}\, \chi_L\, \xi\!\!\!/\, \xi_i \int_x^y [0,0\,|\,1]\, j^i_L\, T^{(0)} && \text{(B.14)}\\
&+ \frac{1}{4}\, \chi_L\, \xi\!\!\!/ \int_x^y F^{ij}_L\, \gamma_i \gamma_j\, T^{(0)} && \text{(B.15)}\\
&- \chi_L\, \xi_i \int_x^y [0,1\,|\,0]\, F^{ij}_L\, \gamma_j\, T^{(0)} && \text{(B.16)}\\
&- \chi_L\, \xi_i \int_x^y [0,1\,|\,1]\, \partial\!\!\!/ j^i_L\, T^{(1)} && \text{(B.17)}\\
&- \chi_L \int_x^y [0,2\,|\,0]\, j^i_L\, \gamma_i\, T^{(1)} && \text{(B.18)}\\
&- i\, \chi_L\, \xi_i \int_x^y Y A^i_R\, T^{(0)} && \text{(B.19)}\\
&+ \frac{im}{2}\, \chi_L\, \xi\!\!\!/ \int_x^y (Y A\!\!\!/_R - A\!\!\!/_L Y)\, T^{(0)} && \text{(B.20)}\\
&+ im\, \chi_L\, \xi_i \int_x^y [0,0\,|\,1]\, Y\, j^i_R\, T^{(1)} && \text{(B.21)}\\
&- \frac{im}{2}\, \chi_L \int_x^y [1,0\,|\,0]\, Y F^{ij}_R \gamma_i \gamma_j\, T^{(1)} && \text{(B.22)}\\
&- \frac{im}{2}\, \chi_L \int_x^y [0,1\,|\,0]\, F^{ij}_L \gamma_i \gamma_j\, Y\, T^{(1)} && \text{(B.23)}
\end{aligned}$$

$$+\, im\, \chi_L \int_x^y [0,1\,|\,0] \Big(Y(\partial_j A^j_R) - (\partial_j A^j_L)\, Y \Big)\, T^{(1)} \tag{B.24}$$

$$+\, \frac{m^2}{2}\, \chi_L\, \xi\!\!/\, \xi_i \int_x^y [1,0\,|\,0]\, YY A^i_L\, T^{(0)} \tag{B.25}$$

$$+\, \frac{m^2}{2}\, \chi_L\, \xi\!\!/\, \xi_i \int_x^y [0,1\,|\,0]\, A^i_L YY\, T^{(0)} \tag{B.26}$$

$$-\, m^2\, \chi_L\, \xi_i \int_x^y [0,0\,|\,1]\, YY F^{ij}_L \gamma_j\, T^{(1)} \tag{B.27}$$

$$-\, m^2\, \chi_L\, \xi_i \int_x^y [0,2\,|\,0]\, F^{ij}_L \gamma_j\, YY\, T^{(1)} \tag{B.28}$$

$$+\, m^2\, \chi_L \int_x^y [1,0\,|\,0]\, YY A\!\!\!/_L\, T^{(1)} \tag{B.29}$$

$$-\, m^2\, \chi_L \int_x^y [0,0\,|\,0]\, Y A\!\!\!/_R\, Y\, T^{(1)} \tag{B.30}$$

$$+\, m^2\, \chi_L \int_x^y [0,1\,|\,0]\, A\!\!\!/_L\, YY\, T^{(1)} \tag{B.31}$$

$$+\, \xi\!\!/\, (\deg < 1) + (\deg < 0) + \mathcal{O}(A^2_{L/R})\,,$$

where $F^{jk}_c = \partial^j A^k_c - \partial^k A^j_c$ is the chiral field tensor and $j^k_c = \partial^k_j A^j_c - \Box A^k_c$ is the corresponding chiral current. The term (B.13) is our unperturbed fermionic projector (6.22); all other summands are our perturbation. We take the partial trace, expand in powers of ξ and compute the matrix elements (B.11). In these computations, the error terms of the form $\xi\!\!/\, (\deg < 1)$ are contracted with another factor $\xi\!\!/$ or $\overline{\xi\!\!/}$, giving rise to a term of lower degree. Likewise, the higher orders in A either give rise to terms either of lower degree on the light cone or of higher order in the expansion around the origin. Computing $\Delta\lambda^L_1$ by a first order perturbation calculation gives

$$\begin{aligned}
\Delta\lambda^L_1 =& \frac{ig^2}{3}\, j^i_L\, \xi_i\, T^{(1)}_{[0]} \overline{T^{(-1)}_{[0]}} - \frac{ig^2}{6}\, j^i_R\, \xi_i\, T^{(0)}_{[0]} \overline{T^{(0)}_{[0]}} \\
&- 2igm^2\, A^i_{\mathrm{a}}\, \xi_i\, \acute{Y}\grave{Y} \left(T^{(1)}_{[2]} \overline{T^{(-1)}_{[0]}} + T^{(0)}_{[0]} \overline{T^{(0)}_{[2]}} \right) \\
&- 2im^2\, A^i_{\mathrm{a}}\, \xi_i\, \hat{Y}^2\, \frac{T^{(0)}_{[1]} \overline{T^{(-1)}_{[0]}} \Big(T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} + c.c. \Big) - c.c.}{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}} + (\deg < 2) + o(|\vec{\xi}|^{-1})\,.
\end{aligned}$$

The other eigenvalues are obtained by the replacement $L \leftrightarrow R$ and by complex conjugation. Substituting the resulting formulas into (7.1) gives the result. ☐

Proof of Lemma 7.4. According to (7.9) and (4.10), the perturbation $\Delta P(x,y)$ by the Dirac current satisfies the relations

$$\mathrm{Tr}\left(\gamma^j\, \Delta P(x,y)\right) = -\frac{1}{2\pi}\, J^j_{\mathrm{v}}\,, \qquad \mathrm{Tr}\left(\gamma^5 \gamma^j\, \Delta P(x,y)\right) = -\frac{1}{2\pi}\, J^j_{\mathrm{a}}$$

and thus

$$\Delta P(x,x) = -\frac{1}{8\pi}\, \gamma_j J^j_{\mathrm{v}} + \frac{1}{8\pi}\, \gamma^5 \gamma_j J^j_{\mathrm{a}}\,.$$

The corresponding perturbation of the eigenvalue $\Delta\lambda_1^L$ is computed to be

$$\Delta\lambda_1^L \asymp \frac{ig}{8\pi}\, J^i_L\, \xi_i\, \overline{T^{(-1)}_{[0]}}\, .$$

The formula for $\mathcal{R}$ follows by direct calculation. □

We now come to the pseudoscalar differential potential (7.15). The contribution to the fermionic projector linear in v has the form

$$P(x,y) \asymp \frac{1}{2}\, \gamma^5 \not{\xi}\, \xi_i \int_x^y \not{\partial} v^i\, T^{(-1)} + \gamma^5\, \xi_i\, v^i(x)\, T^{(-1)} + (\deg < 2)\, . \tag{B.32}$$

After applying the relation $2\not{\xi}\not{\partial} v^i = 2\xi^j \partial_j v^i + [\not{\xi}, \not{\partial} v^i]$, we can in the first term integrate by parts to obtain (7.16). The light-cone expansion to lower degree involves many terms, which we do not want to give here. To higher order in the mass, the contributions become less singular on the light cone. In particular, the leading term cubic in the mass takes the form

$$\begin{aligned} P(x,y) \asymp\; & -\frac{im^3}{2}\, \gamma^5 \not{\xi}\, \xi_i \int_x^y [0,1\,|\,0] \left(Y v^i Y Y - v^i Y Y Y \right) T^{(0)} \\ & -\frac{im^3}{2}\, \gamma^5 \not{\xi}\, \xi_i \int_x^y [1,0\,|\,0] \left(Y Y Y v^i - Y Y v^i Y \right) T^{(0)} \\ & + im^3 \gamma^5 \int_x^y [0,1\,|\,0] \left(Y \not{v} Y Y - \not{v} Y Y Y \right) T^{(1)} \\ & + im^3 \gamma^5 \int_x^y [1,0\,|\,0] \left(Y Y Y \not{v} - Y Y \not{v} Y \right) T^{(1)} \\ & + \not{\xi}\, (\deg < 1) + (\deg < 0)\, . \end{aligned} \tag{B.33}$$

Expanding in powers of ξ, we obtain (7.17) and (7.18).

We point out that for the scalar differential potential, the higher orders in perturbation theory are difficult to handle because they are *not* of lower degree on the light cone. Moreover, a resummation procedure similar to that for chiral potential does not seem to work. For a constant potential, this problem corresponds to the effect of the "deformation of the light cone" as discussed after (7.21). In the more general setting here, this problem means that the scalar differential potential cannot be treated perturbatively in a convincing way. This serious difficulty was our original motivation for introducing the vector differential potential (7.23), and to rewrite the combination of these potentials by the local axial transformation (7.25).

For the local axial transformation (7.25), the fermionic projector can easily be computed non-perturbatively. To first order in v, we obtain the contribution (7.31) (for a non-perturbative treatment see Appendix C).

Proof of Lemma 7.5. Since the commutator in (7.31) vanishes if P_β is a multiple of the identity matrix, the contribution by the parameters c_β in (7.31) is the term

$$P(x,y) \asymp \gamma^5\, v_j \xi^j \sum_{\beta=1}^{g} c_\beta\, \left(T^{(-1)}_{[0]} + m_\beta^2\, T^{(0)}_{[2]} \right) + (\deg < 1)\, . \tag{B.34}$$

Denoting this term by $\Delta P(x,y)$, we must compute the corresponding first order contribution to the eigenvalues $\lambda^c_\pm$. For the unperturbed fermionic projector $P_0(x,y)$ we again choose the fermionic projector of the vacuum modified by axial phases (B.13).

The first order perturbation calculation involves traces of products of the matrices $P(x,y)$, $P(x,y)^* = P(y,x)$, $\Delta P(x,y)$ and $(\Delta P(x,y)^*$. More precisely, omitting the arguments (x,y), to first order in ΔP we obtain terms of the form

$$\mathrm{Tr}((\Delta P) P^* \cdots P P^*) \qquad \text{and} \qquad \mathrm{Tr}((\Delta P)^* P \cdots P^* P)\,, \tag{B.35}$$

which involve one factors ΔP or $(\Delta P)^*$, and where the matrices and their adjoints alternate. According to (B.34), the matrix ΔP (and similarly (ΔP^*) is a multiple of γ^5. But P_0 and P_0^* only involve the Dirac matrices $\not\xi$ and $\overline{\not\xi}$. As a consequence, the traces in (B.35) vanish. □

Proof of Lemma 7.6. In view of (7.34), the contribution of the parameters d_β in (7.31) to the fermionic projector is given by

$$P(x,y) \asymp 2 \sum_{\beta=1}^g m_\beta\, d_\beta\, \gamma^5 \not v\, T^{(0)} + \not\xi\, (\deg < 2) + (\deg < 1)\,.$$

The corresponding perturbation of the eigenvalue $\Delta\lambda_1^L$ is computed by

$$\Delta\lambda_1^L \asymp 2ig\, v_j \xi^j \sum_{\beta=1}^g m_\beta\, d_\beta\, T^{(0)}_{[1]} \overline{T^{(-1)}_{[0]}} + (\deg < 3)\,.$$

The other eigenvalues are again obtained by the replacement $L \leftrightarrow R$ and by complex conjugation. Substituting the resulting formulas into (7.1) gives the result. □

Proof of Lemma 7.7. In view of (7.34) and (7.35), the contribution of the parameters d_β in (7.31) to the fermionic projector is

$$P(x,y) \asymp \frac{i}{2} \sum_{\beta=1}^g m_\beta^2\, d_\beta\, \gamma^5 [\not v, \not\xi]\, T^{(0)} + 2 \sum_{\beta=1}^g m_\beta^3\, d_\beta\, \gamma^5 \not v\, T^{(1)} + (\deg < 0)\,.$$

The corresponding perturbation of the eigenvalue $\Delta\lambda_1^L$ is given by

$$\begin{aligned}
\Delta\lambda_1^L \asymp{}& 2ig\, v_j \xi^j \sum_{\beta=1}^g m_\beta^3\, d_\beta\, T^{(1)}_{[3]} \overline{T^{(-1)}_{[0]}} \\
&+ 2i\, v_j \xi^j\, m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta\, \frac{T^{(-1)}_{[0]} T^{(0)}_{[0]} \overline{T^{(0)}_{[1]} T^{(0)}_{[2]}} - c.c.}{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}} \\
&+ 2i\, v_j \xi^j\, m\hat{Y} \sum_{\beta=1}^g m_\beta^2\, d_\beta\, \frac{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \Big(T^{(0)}_{[1]} \overline{T^{(0)}_{[2]}} - T^{(0)}_{[2]} \overline{T^{(0)}_{[1]}} \Big)}{T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}}} + (\deg < 2)\,.
\end{aligned}$$

The other eigenvalues are again obtained by the replacement $L \leftrightarrow R$ and by complex conjugation. Substituting the resulting formulas into (7.1) gives the result. □

We finally consider the perturbation of the fermionic projector by the scalar and pseudoscalar Dirac current (9.1). According to (4.10), the corresponding perturbation of the fermionic projector is given by

$$\Delta P(x,y) = -\frac{1}{8\pi} \big(J_\mathrm{s} + i\gamma^5\, J_\mathrm{a} \big) + o(|\vec{\xi}|^0)\,. \tag{B.36}$$

Lemma B.1. *The first order contribution of the perturbation* (B.36) *to the operator* $Q(x,y)$ *is of degree two on the light cone.*

Proof. A first order perturbation calculation yields

$$\Delta\lambda_1^L = \frac{J_{\mathrm{s}}}{4\pi}\, m\hat{Y}\; \frac{T^{(0)}_{[0]}\Big(T^{(-1)}_{[0]}\overline{T^{(0)}_{[1]}} - c.c.\Big) + \overline{T^{(-1)}_{[0]}}\Big(T^{(0)}_{[1]}\overline{T^{(0)}_{[0]}} - c.c.\Big)}{T^{(0)}_{[0]}\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\overline{T^{(0)}_{[0]}}} + (\deg < 1)\,. \quad \text{(B.37)}$$

Note that the pseudoscalar current dropped out. This cancellation can be understood exactly as in the proof of Lemma 7.5 by considering the traces (B.35) and noting that the corresponding ΔP is again a multiple of the matrix γ^5.

The first order contribution to the EL equations is obtained by considering first variations of the Lagrangian in the from (6.28) and substituting the formulas for $\Delta\lambda^c_s$. Counting degrees, this contribution is expected to be of degree three on the light cone. However, since (B.37) has even parity, whereas the first variation of the Lagrangian involves factors $(\lambda^L_s - |\lambda^R_s)$ with odd parity, this expected contribution of degree three vanishes. □

Appendix C. The Local Axial Transformation to Higher Order

We shall now analyze the local axial transformation of the fermionic projector of the vacuum, taking into account the higher orders in v which were disregarded after (7.28). In order to understand why these higher orders in v are a potential problem, one should recall that in §7.6 we satisfied the EL equations to degree five by a special choice of the local axial transformation (see (7.35)). Then the local axial transformation still affected the EL equations to degree four, making it possible to compensate the logarithmic poles on the light cone (see Lemma 7.7). If the higher orders in v gave contributions to the EL equations to degree five, these contributions would be by an order ε^{-1} *larger* than the contribution computed in Lemma 7.7, and would thus *not* be negligible (even if the expansion parameter $\mathfrak{g}v$ were very small).

In order to rule out this potential problem, we must show that the EL equations to degree five can be respected by the local axial transformation even non-perturbatively. This is done in the following theorem.

Theorem C.1. *Suppose that the number of generations* $g \geq 3$. *Then for any vector field* $u(x)$ *satisfying the condition*

$$\langle u(x), u(x)\rangle \geq \; -\frac{g^2}{256} \max_{\alpha\in\{2,\ldots,g-1\}} (m_g - m_\alpha)^2 (m_\alpha - m_1)^2 (m_1 + m_\alpha + m_g)^2 \qquad \text{(C.1)}$$

(where the masses are in increasing order (3.2)*) there is a local axial transformation of the form* (7.25) *such that at every space-time point* x *and for all Minkowski vectors* ξ *the following relations hold:*

$$\acute{U}\,\not{\xi}\,\grave{U}^{-1} = c_0\,\not{\xi} \qquad \text{(C.2)}$$

$$m\,\acute{U} Y \grave{U}^{-1} = c_1\, m\hat{Y} \qquad \text{(C.3)}$$

$$m^3\,\acute{U} Y^3 \grave{U}^{-1} = c_3\, m^3 \acute{Y} Y \grave{Y} + i\gamma^5 \not{u}(x) \qquad \text{(C.4)}$$

(where $\grave{U}^{-1}$ *denotes similar to* (5.2) *the partial trace of the matrix* U^{-1}*). Here the constants* c_1 *and* c_3 *may depend on* x*, and* c_0 *may depend on* x *and* ξ*.*

The relations (C.2) and (C.3) imply that the local axial transformation does not affect the EL equations to degree five. The contribution to the fermionic projector corresponding to (C.4) makes it possible to compensate logarithmic poles on the light cone, (exactly as explained in §8.1). The inequality (C.1) implies that the vector field appearing in the logarithmic pole must not be spacelike and too large. However, the constraint (C.1) does not seem to be physically relevant because under realistic conditions, the vector fields $m^2 A_a$ and $j_{\rm a}$ are very small on the scale $\sim m^3$. For notational convenience, the matrix $m^2 \acute{U} \not{\xi} Y^2 \grave{U}^{-1}$, which is disregarded in the above theorem although it enters the EL equations to degree four (cf. the contribution (7.36)), will be analyzed separately in Proposition C.6 below.

The remainder of this appendix is devoted to the proof of this theorem and to the statement and proof of Proposition C.6. First, since all constructions will be performed locally at a given space-time point x, we may fix x throughout and omit the space-time dependence. Moreover, we always choose the vector v in (7.25) as a multiple of the vector u which appears in the statement of the theorem. Since in (7.25) we may multiply $\mathfrak{g}$ by a positive number and divide v by the same number, it suffices to consider the three cases $v^2 = 0$, $v^2 = 1$ and $v^2 = -1$, where the vector v is null, timelike or spacelike, respectively. In the case $v^2 = 0$, the higher orders in v vanish, so that (7.28) and the constructions thereafter become exact. Thus it remains to consider the cases $v^2 = 1$ and $v^2 = -1$. According to (7.26), the local axial transformation in these two cases involves hyperbolic and trigonometric functions, respectively,

$$U = \cosh \mathfrak{g} - i\gamma^5 \not{v} \sinh \mathfrak{g} \,, \qquad U^{-1} = \cosh \mathfrak{g} + i\gamma^5 \not{v} \sinh \mathfrak{g} \qquad \text{if } v^2 = 1 \tag{C.5}$$

$$U = \cos \mathfrak{g} - i\gamma^5 \not{v} \sin \mathfrak{g} \,, \qquad U^{-1} = \cos \mathfrak{g} + i\gamma^5 \not{v} \sin \mathfrak{g} \qquad \text{if } v^2 = -1 \,. \tag{C.6}$$

In view of this difference, it is preferable to treat the cases $v^2 = 1$ and $v^2 = -1$ separately.

First case: $v^2 = 1$. The first step is to rewrite the conditions (C.2) and (C.3) in terms of $g \times g$-matrices acting on the generations. The condition (C.2) should hold for any ξ, and by linearity it suffices to consider the two cases $\langle \xi, v \rangle = 0$ and $\xi = v$. In the first case, the factor $\not{\xi}$ commutes with U. Thus $U \not{\xi} U^{-1} = \not{\xi}$, and (C.2) is trivially satisfied. In the second case, the factors $\not{\xi}$ and $\gamma^5 \not{v}$ anti-commute, and using (C.5) we find that $U \not{\xi} U^{-1} = U^2 \not{\xi}$. Again using (C.5), the condition (C.2) can be written as

$$\acute{U} \grave{U} = \acute{U}^{-1} \grave{U}^{-1} \,. \tag{C.7}$$

Similarly, the relations (C.3) and (C.4) can be restated as

$$\acute{U} Y \grave{U}^{-1} - \acute{U}^{-1} Y \grave{U} = 0 \tag{C.8}$$

$$m^3 \left(\acute{U} Y^3 \grave{U}^{-1} - \acute{U}^{-1} Y^3 \grave{U} \right) = 2i\gamma^5 \not{u} \,. \tag{C.9}$$

Our procedure is to first satisfy (C.7) and (C.8); then we will analyze for which vectors u we can arrange (C.9). The main advantage of (C.7) and (C.8) over the equivalent conditions (C.2) and (C.3) is that in (C.7) and (C.8) only the factor $i\gamma^5 \not{v}$ in U and U^{-1} involves Dirac spinors. Thus diagonalizing this factor and restricting attention to the respective eigenspaces, we obtain matrix conditions on $\mathbb{C}^g$. More precisely, using the relation $(i\gamma^5 \not{v})^2 = \mathbb{1}$, one readily sees that the matrix $i\gamma^5 \not{v}$ has eigenvalues ± 1. We can thus rewrite (C.7) and (C.8) in the equivalent form

$$\langle \mathfrak{l} \,|\, V^2 \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, V^{-2} \mathfrak{l} \rangle \,, \qquad \langle \mathfrak{l} \,|\, V Y V^{-1} \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, V^{-1} Y V \mathfrak{l} \rangle \,, \tag{C.10}$$

where we introduced the matrix V and the vector $\mathfrak{l}$ by

$$V = \cosh \mathfrak{g} + \sinh \mathfrak{g} = e^{\mathfrak{g}} \in \text{Mat}(\mathbb{C}^g) , \qquad \mathfrak{l} = (1, \ldots, 1) \in \mathbb{C}^g , \tag{C.11}$$

and $\langle . | . \rangle$ denotes the standard scalar product on $\mathbb{C}^g$. Note that the matrix V is *positive definite*. In what follows, we can work only with this property of V, because the representation (C.11) can then be obtained by setting $\mathfrak{g} = \log V$. Next, we want to exploit the fact that only certain combinations of the matrix elements of V and V^{-1} enter (C.10). This can be achieved by setting

$$\mathfrak{m} = V^{-1} \mathfrak{l} \qquad \text{and} \qquad \mathfrak{n} = V \mathfrak{l} , \tag{C.12}$$

giving the equivalent conditions

$$\boxed{\langle \mathfrak{n} \,|\, \mathfrak{n} \rangle = \langle \mathfrak{m} \,|\, \mathfrak{m} \rangle , \qquad \langle \mathfrak{n} \,|\, Y \mathfrak{m} \rangle = \langle \mathfrak{m} \,|\, Y \mathfrak{n} \rangle ,} \tag{C.13}$$

which instead of matrices merely involve the two unknown vectors $\mathfrak{m}, \mathfrak{n} \in \mathbb{C}^g$.

Before we can work exclusively with the vectors $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{n}$, we need to work out geometric conditions which allow the representation (C.12). In order to avoid trivialities, we assume that the vectors $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{n}$ are linearly independent.

Lemma C.2. *For given linearly independent vectors* $\mathfrak{l}, \mathfrak{m}, \mathfrak{n} \in \mathbb{C}^g$ *there exists a positive definite matrix* V *satisfying* (C.12) *if and only if the following relations hold:*

$$\langle \mathfrak{m} \,|\, \mathfrak{n} \rangle = \|\mathfrak{l}\|^2 \tag{C.14}$$

$$\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle > 0 , \quad \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle > 0 \tag{C.15}$$

$$\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \, \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle > \|\mathfrak{l}\|^4 . \tag{C.16}$$

Proof. We first verify that the relations (C.14)–(C.16) are necessary. Thus assume that (C.12) holds for a positive definite matrix V. Then (C.14) follows immediately from the calculation

$$\|\mathfrak{l}\|^2 = \langle \mathfrak{l} \,|\, V V^{-1} \mathfrak{l} \rangle = \langle V \mathfrak{l} \,|\, V^{-1} \mathfrak{l} \rangle = \langle \mathfrak{n} \,|\, \mathfrak{m} \rangle .$$

The inequalities (C.15) are readily obtained from the estimates

$$0 < \langle \mathfrak{l} \,|\, V \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle \qquad \text{and} \qquad 0 < \langle \mathfrak{l} \,|\, V^{-1} \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{m} \rangle .$$

We now form the matrix of expectation values of V with respect to the vectors $\mathfrak{l}$ and $\mathfrak{m}$,

$$\begin{pmatrix} \langle \mathfrak{l} \,|\, V \mathfrak{l} \rangle & \langle \mathfrak{l} \,|\, V \mathfrak{m} \rangle \\ \langle \mathfrak{m} \,|\, V \mathfrak{l} \rangle & \langle \mathfrak{m} \,|\, V \mathfrak{m} \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle & \langle \mathfrak{l} \,|\, \mathfrak{l} \rangle \\ \langle \mathfrak{l} \,|\, \mathfrak{l} \rangle & \langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \end{pmatrix} .$$

Since V is positive definite, the determinant of the last matrix must be positive, giving (C.16).

It remains to show that the relations (C.14)–(C.16) are sufficient. Thus assuming that these relations hold, we must construct a corresponding positive definite matrix V satisfying (C.12). It suffices to construct a positive definite matrix $\tilde{V}$ on the subspace spanned by $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{n}$, because extending $\tilde{V}$ on the orthogonal complement of this subspace by the identity gives the desired matrix V. Since we want to satisfy (C.12), the corresponding matrix of expectation values must be of the form

$$\begin{pmatrix} \langle \mathfrak{l} \,|\, V \mathfrak{l} \rangle & \langle \mathfrak{l} \,|\, V \mathfrak{m} \rangle & \langle \mathfrak{l} \,|\, V \mathfrak{n} \rangle \\ \langle \mathfrak{m} \,|\, V \mathfrak{l} \rangle & \langle \mathfrak{m} \,|\, V \mathfrak{m} \rangle & \langle \mathfrak{m} \,|\, V \mathfrak{n} \rangle \\ \langle \mathfrak{n} \,|\, V \mathfrak{l} \rangle & \langle \mathfrak{n} \,|\, V \mathfrak{m} \rangle & \langle \mathfrak{n} \,|\, V \mathfrak{n} \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle & \langle \mathfrak{l} \,|\, \mathfrak{l} \rangle & \langle \mathfrak{n} \,|\, \mathfrak{n} \rangle \\ \langle \mathfrak{m} \,|\, \mathfrak{n} \rangle & \langle \mathfrak{m} \,|\, \mathfrak{l} \rangle & \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle \\ \langle \mathfrak{n} \,|\, \mathfrak{n} \rangle & \langle \mathfrak{n} \,|\, \mathfrak{l} \rangle & C \end{pmatrix} , \tag{C.17}$$

involving only one unknown parameter $C := \langle \mathfrak{n} \,|\, V \mathfrak{n} \rangle > 0$. According to (C.15) and (C.16), the upper left 2×2 submatrix is positive definite. Thus by choosing C sufficiently large, we can arrange that the whole matrix in (C.17) is positive. Then (C.17) defines a positive definite matrix $\tilde{V}$ with the required properties. □

It remains to construct vectors $\mathfrak{m}$ and $\mathfrak{n}$ satisfying the relations (C.13)–(C.16). We first note that the first relation in (C.13) can be arranged by the scale transformation

$$\mathfrak{m} \to \lambda \mathfrak{m} \,, \quad \mathfrak{n} \to \frac{1}{\lambda}\, \mathfrak{n} \qquad \text{with } \lambda > 0 \tag{C.18}$$

without affecting all the other relations. Thus the first relation in (C.13) can be disregarded. Moreover, the first inequality in (C.15) can be arranged by the phase transformation

$$\mathfrak{m} \to e^{i\varphi}\, \mathfrak{m} \,, \quad \mathfrak{n} \to e^{i\varphi}\, \mathfrak{n} \qquad \text{with } \varphi \in [0, 2\pi) \,, \tag{C.19}$$

leaving all the relations except for (C.15) unchanged. If the first relation in (C.15) holds, the second follows immediately from (C.16). Thus we may also disregard (C.15), leaving us with the set of conditions

$$\langle \mathfrak{n} \,|\, Y \mathfrak{m} \rangle = \langle \mathfrak{m} \,|\, Y \mathfrak{n} \rangle \,, \qquad \langle \mathfrak{m} \,|\, \mathfrak{n} \rangle = g \tag{C.20}$$

$$\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \, \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle > g^2 \,. \tag{C.21}$$

Introducing the real parameters τ_β and d_β by

$$\overline{\mathfrak{n}_\beta}\, \mathfrak{m}_\beta = \tau_\beta + 2 i d_\beta \,, \tag{C.22}$$

we can write the relations (C.20) as

$$\sum_{\beta=1}^g \tau_\beta = g \,, \qquad \sum_{\beta=1}^g d_\beta = 0 = \sum_{\beta=1}^g m_\beta\, d_\beta \,. \tag{C.23}$$

This notation gives a close connection to the analysis in §7.6, because in a first order expansion in v one easily verifies that $\tau_\beta = 1$, whereas d_β coincides with the corresponding parameters in (7.30). Hence the last two equations in (C.23) correspond to (7.34) and (7.35), whereas the first equation in (C.23) gives an additional condition to be fulfilled to higher order in v.

We now give a procedure to satisfy the condition (C.21).

Lemma C.3. *For a given solution of* (C.23)*, there are vectors* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which satisfy both* (C.20) *and* (C.21)*.*

Proof. For given parameters τ_β and d_β satisfying (C.23), we can fulfill the conditions (C.20) by choosing any complex solutions $\mathfrak{m}_\beta$ and $\mathfrak{n}_\beta$ of (C.22). This choice involves the freedom of rescaling the parameters $\mathfrak{m}_\beta$ and $\mathfrak{n}_\beta$ by

$$\mathfrak{m}_\beta \to z_\beta\, \mathfrak{m}_\beta \,, \quad \mathfrak{n}_\beta \to (\overline{z_\beta})^{-1}\, \mathfrak{n}_\beta \,, \qquad \text{with } z_\beta \in \mathbb{C} \setminus \{0\} \,. \tag{C.24}$$

We want to use this freedom to arrange (C.21).

We begin with the case that one of the components of $\mathfrak{m}$ or $\mathfrak{n}$ vanishes; by symmetry we can assume that $\mathfrak{m}_1 = 0$. Furthermore, we may assume that $\mathfrak{n}_1 \neq 0$, because otherwise we can remove the first component and repeat the proof for g replaced by $g-1$. Then by a suitable choice of the parameter z_1 in (C.24) close to zero, we can make the component $\mathfrak{n}_1$ positive and arbitrarily large. Thus $\langle \mathfrak{l} \,|\, \mathfrak{n} \rangle$ can be made arbitrarily large without affecting $\langle \mathfrak{l} \,|\, \mathfrak{m} \rangle$. Next, by a suitable choice of the phases of the remaining parameters $z_2, \ldots, z_g$, we can arrange that the components of $\mathfrak{m}$ are all positive, so that $\langle \mathfrak{l} \,|\, \mathfrak{m} \rangle > 0$. In this way, we make the product $\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle$ as large as we like, thus satisfying (C.21).

It remains to consider the case that the components of $\mathfrak{m}$ and $\mathfrak{n}$ are all non-zero. In this case, we can use the transformation (C.24) to arrange that $\mathfrak{m}_\beta = 1$ for all $\beta = 1, \ldots, g$. Then $\mathfrak{m} = \mathfrak{l}$ and thus

$$\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle = g \, \langle \mathfrak{m} \,|\, \mathfrak{n} \rangle = g^2 \,,$$

where in the last step we used the relation on the right of (C.20). This is not quite good enough, because in (C.21) the strict inequality appears. Therefore, we now use the transformation (C.24) to vary $\mathfrak{m}$. Considering first order variations, we have

$$\mathfrak{m}_\beta = 1 + \delta \mathfrak{m}_\beta \,, \qquad \delta \mathfrak{n}_\beta = -\overline{\delta \mathfrak{m}_\beta} \, \mathfrak{n}_\beta$$

and thus

$$\delta \Big(\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle \Big) = \sum_{\beta=1}^{g} \overline{\delta \mathfrak{m}_\beta} \left[\sum_{\alpha=1}^{g} \mathfrak{n}_\alpha - g \, \mathfrak{n}_\beta \right]. \tag{C.25}$$

The square bracket vanishes identically only if the components $\mathfrak{n}_\beta$ are all equal. In view of (C.22) and (C.23), this corresponds to the trivial case

$$\tau_\beta = 1 \quad \text{and} \quad d_\beta = 0 \qquad \text{for all } \beta = 1, \ldots, g \,,$$

which can clearly be arranged by choosing $V = \mathbb{1}$. In all other cases, we can choose $\delta \mathfrak{m}_\beta$ such that (C.25) becomes positive. Having shown that (C.21) is satisfied by a suitable linear perturbation, this condition clearly also holds for a corresponding nonlinear perturbation, provided that this perturbation is sufficiently small. ☐

We have thus reduced the problem to the analysis of the relations (C.23). Clearly, these equations have non-trivial solutions, and by a suitable choice of the parameters d_β, we can give the combination

$$m^3 \Big(\langle \mathfrak{n} \,|\, Y^3 \mathfrak{m} \rangle - \langle \mathfrak{m} \,|\, Y^3 \mathfrak{n} \rangle \Big) = 4i \sum_{\beta=1}^{g} m_\beta^3 \, d_\beta \tag{C.26}$$

an arbitrary imaginary value. Using this fact in (C.9), we see that the vector u can be an arbitrary multiple of v. This concludes the proof of Theorem C.1 in the case $v^2 = 1$.

Second case: $v^2 = -1$. Proceeding as in the first case, we can rewrite the conditions (C.2) and (C.3) again in the form (C.10), where V now is the *unitary* matrix

$$V = \cos \mathfrak{g} + i \sin \mathfrak{g} = e^{i \mathfrak{g}} \,. \tag{C.27}$$

Again introducing the vectors $\mathfrak{l} = (1, \ldots, 1)$ and $\mathfrak{m}$, $\mathfrak{n}$ by (C.12), we can write the conditions (C.10) in analogy to (C.13) as

$$\boxed{\langle \mathfrak{m} \,|\, \mathfrak{n} \rangle = \langle \mathfrak{n} \,|\, \mathfrak{m} \rangle \,, \qquad \langle \mathfrak{m} \,|\, Y \mathfrak{m} \rangle = \langle \mathfrak{n} \,|\, Y \mathfrak{n} \rangle \,.} \tag{C.28}$$

The next lemma is the analog of Lemma C.2 in the case that V is unitary.

Lemma C.4. *For given linearly independent vectors* $\mathfrak{l}, \mathfrak{m}, \mathfrak{n} \in \mathbb{C}^g$ *there exists a unitary matrix* V *satisfying* (C.12) *if and only if the following relations hold:*

$$\|\mathfrak{l}\| = \|\mathfrak{m}\| = \|\mathfrak{n}\| \qquad \textit{and} \qquad \langle \mathfrak{m} \,|\, \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle \,. \tag{C.29}$$

Proof. The necessity of the relations (C.29) is obvious from the calculations

$$\langle \mathfrak{m} \,|\, \mathfrak{m} \rangle = \langle V^{-1} \mathfrak{l} \,|\, V^{-1} \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{l} \rangle = \langle V \mathfrak{l} \,|\, V \mathfrak{l} \rangle = \langle \mathfrak{n} \,|\, \mathfrak{n} \rangle$$
$$\langle \mathfrak{m} \,|\, \mathfrak{l} \rangle = \langle U^{-1} \mathfrak{l} \,|\, \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, U \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{n} \rangle \,.$$

In order to show that these conditions are also sufficient, we make the ansatz

$$V = V_2 V_1$$

with two unitary transformations V_1 and V_2. We choose V_1 such that it maps $\mathfrak{m}$ to $\mathfrak{l}$ (this is possible, because according to (C.29) both vectors have the same norm). Then the unitary transformation V_2 should map the vector $\tilde{\mathfrak{l}} := V_1 \mathfrak{l}$ to $\mathfrak{n}$, while leaving the vector $\mathfrak{l}$ invariant. Using the right equation in (C.29), we find

$$\langle \tilde{\mathfrak{l}} \,|\, \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, V_1^{-1} \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, \mathfrak{m} \rangle = \langle \mathfrak{n} \,|\, \mathfrak{l} \rangle \,,$$

showing that the the vectors $\tilde{\mathfrak{l}}$ and $\mathfrak{n}$ have the same projection on $\mathfrak{l}$. According to the left equation in (C.29), they also have the same norm. Thus we can choose V_2 as a unitary transformation of the orthogonal complement of $\mathfrak{l}$ which maps $\tilde{\mathfrak{l}}$ to $\mathfrak{n}$. □

The remaining task is to construct vectors $\mathfrak{m}$ and $\mathfrak{n}$ satisfying the conditions (C.28) and (C.29). The relation on the left of (C.28) can be arranged by the phase transformation

$$\mathfrak{m} \to e^{i\varphi}\, \mathfrak{m} \,, \quad \mathfrak{n} \to e^{-i\varphi}\, \mathfrak{n} \qquad \text{with } \varphi \in [0, 2\pi) \,, \tag{C.30}$$

leaving all other conditions unchanged. Introducing the parameters τ_β and d_β by

$$\tau_\beta = \frac{1}{2} \left(|\mathfrak{n}_\beta|^2 + |\mathfrak{m}_\beta|^2 \right) \qquad \text{and} \qquad d_\beta = \frac{1}{4} \left(|\mathfrak{n}_\beta|^2 - |\mathfrak{m}_\beta|^2 \right) , \tag{C.31}$$

the conditions on the right of (C.28) and the left of (C.29) again take the form (C.23), but now with the obvious additional constraint

$$|d_\beta| \leq \frac{\tau_\beta}{2} \tag{C.32}$$

(again, our definition of d_β agrees with (7.30) to first order in v). The condition on the right of (C.29) can be arranged by the following construction.

Lemma C.5. *Suppose that the parameters τ_β and d_β satisfy the relations* (C.23) *and* (C.32). *Then there are vectors $\mathfrak{m}$ and $\mathfrak{n}$ which fulfill all conditions in* (C.28) *and* (C.29).

Proof. For given parameters τ_β and d_β satisfying (C.23) and (C.32), we choose solutions $\mathfrak{m}_\beta$ and $\mathfrak{n}_\beta$ of (C.31). This choice involves the freedom to change the phases according to

$$\mathfrak{m}_\beta \to e^{i\varphi_\beta}\, \mathfrak{m}_\beta \,, \quad \mathfrak{n}_\beta \to e^{i\vartheta_\beta}\, \mathfrak{n}_\beta \qquad \text{with } \varphi_\beta, \vartheta_\beta \in [0, 2\pi) \,. \tag{C.33}$$

Our goal is to show that by choosing these phases appropriately, we can satisfy the condition on the right of (C.29). In order to get a more convenient notation, we take the absolute values of the components $|\mathfrak{m}_\beta|$ and $|\mathfrak{n}_\beta|$ and bring them in increasing order to obtain the parameters $\rho_1, \ldots, \rho_{2f}$. Then our task is to show that there are angles $\phi_1, \ldots, \phi_{2f}$ such that

$$\sum_{\beta=1}^{2f} e^{i\phi_\beta}\, \rho_\beta = 0 \,. \tag{C.34}$$

The relations on the left of (C.29) imply that $\sum_{\beta=1}^{2f} \rho_\beta^2 = 2g$, and moreover we know that there is a partition of the ρ_β into two subsets such that the sum of each subset equals g. This implies that

$$\rho_{2f}^2 \leq \rho_1^2 + \cdots \rho_{2f-1}^2 \,.$$

Taking the square root and inductively applying the inequality $\sqrt{ab} \leq \sqrt{a} + \sqrt{b}$ (with $a, b \geq 0$), we find that

$$\rho_{2f} - \rho_{2f-1} \leq \rho_1 + \cdots + \rho_{2f-2} \,. \tag{C.35}$$

We claim that this inequality allows us to choose sign factors $s_\beta \in \{\pm 1\}$, $\beta = 1, \ldots, 2f-2$, such that

$$z := s_1 \rho_1 + \cdots + s_{2f-2}\, \rho_{2f-2} \in \big[\rho_{2f} - \rho_{2f-1},\, \rho_{2f} + \rho_{2f-1} \big] \,. \tag{C.36}$$

This claim can be verified inductively as follows. We choose $s_1 = 1$, $s_2 = 1$, etc. until the partial sum $S(k) := s_1 \rho_1 + \cdots + s_k \rho_k$ exceeds $\rho_{2f} - \rho_{2f-1}$ (which necessarily happens according to (C.35)). Then due to the increasing ordering of the ρ_β, the partial sum S_k lies in the required interval in (C.36). Moreover, we can choose the following sign factors $s_{k+1}, \ldots, s_{2f-2}$ inductively such that the partial sums stay inside this interval. This gives the claim.

Using (C.36), it remains to show that there are angles φ_{2f} and φ_{2f-1} such that

$$e^{i\varphi_{2f}} \rho_{2f} + e^{i\varphi_{2f-1}} \rho_{2f-1} = -z \,.$$

This verified by an elementary consideration, proving (C.34). □

It remains to analyze the conditions (C.23) under the constraint (C.32). Obviously, there are non-trivial solutions. We must show that in (C.4) we can realize every space-like vector u which satisfies (C.1). Evaluating (C.9) on the eigenspaces of the operator $i\gamma^5 \slashed{v}$, we find that

$$u = -m^3 \left(\langle \mathfrak{l} \,|\, V Y^2 V^{-1} \mathfrak{l} \rangle - \langle \mathfrak{l} \,|\, V^{-1} Y^2 V \mathfrak{l} \rangle \right) v \,.$$

Using (C.12) together with the fact that V is unitary, we conclude that

$$u = \frac{1}{4}\, \mathfrak{S}\, v \,, \tag{C.37}$$

where

$$\mathfrak{S} := -4m^3 \left(\langle \mathfrak{m} \,|\, Y^3 \mathfrak{m} \rangle - \langle \mathfrak{n} \,|\, Y^3 \mathfrak{n} \rangle \right) = \sum_{\beta=1}^{g} m_\beta^3\, d_\beta \,,$$

and in the last step we applied (C.31). We need to determine the possible values of the functional $\mathfrak{S}$. Since multiplying the parameters d_β by a number of modulus smaller than one preserves the conditions (C.23) and (C.32), it is obvious that the possible values of $\mathfrak{S}$ form an interval $[-\mathfrak{S}_{\max}, \mathfrak{S}_{\max}]$, and thus it suffices to compute the maximal value $\mathfrak{S}_{\max}$. We first note that we can eliminate the parameters τ_β by rewriting the constraint (C.32) as

$$\sum_{\beta=1}^{g} |d_\beta| \leq \frac{g}{2} \,. \tag{C.38}$$

Moreover, since scaling the parameters d_β by

$$d_\beta \to \frac{d_\beta}{\rho} \qquad \text{with} \qquad \rho = \frac{2}{g} \sum_{\alpha=1}^{g} |d_\beta| \leq 1$$

changes the inequality in (C.38) to an equality and at most increases $\mathfrak{S}$, we see that for finding the maximum of $\mathfrak{S}$ we can assume that equality holds in (C.38). Thus our task is to maximize $\mathfrak{S}$ under the constraints

$$\sum_{\beta=1}^{g} d_\beta = 0\,, \qquad \sum_{\beta=1}^{g} m_\beta\, d_\beta = 0 \qquad \text{and} \qquad \sum_{\beta=1}^{g} |d_\beta| = \frac{g}{2}\,. \tag{C.39}$$

Treating the constraints with Lagrange multipliers λ, μ and κ, we are led to searching for critical points of the functional

$$\sum_{\beta=1}^{g} \left(m_\beta^3 - \lambda\, m_\beta - \mu\right) d_\beta - \kappa \left[\sum_{\beta=1}^{g} |d_\beta| - \frac{g}{2}\right].$$

Taking the distributional derivative with respect to d_α, we obtain the conditions

$$\begin{cases} m_\alpha^3 - \lambda\, m_\alpha - \mu = \kappa & \text{if } d_\alpha > 0 \\ m_\alpha^3 - \lambda\, m_\alpha - \mu = -\kappa & \text{if } d_\alpha < 0 \\ \left|m_\alpha^3 - \lambda\, m_\alpha - \mu\right| \leq |\kappa| & \text{if } d_\alpha = 0\,. \end{cases} \tag{C.40}$$

In order to satisfy the first two conditions in (C.39), there must be indices $\alpha_1 < \alpha_2 < \alpha_3$ such that d_{α_1} and d_{α_3} have the opposite sign as d_{α_2}. This is compatible with (C.40) only if $\lambda > 0$, so that m_{α_1} is on the decreasing and m_{α_3} on the increasing branch of the function $m^3 - \lambda m$. The last condition in (C.40) implies that

$$d_1, d_g > 0 \qquad \text{and} \qquad d_2, \ldots, d_{g-1} \leq 0\,.$$

Comparing the first equation in (C.40) for m_1 and m_g allows us to determine λ,

$$\lambda = \frac{m_g^3 + m_1^3}{m_g - m_1}\,. \tag{C.41}$$

Next, we multiply the equations in (C.40) by d_α and sum over α. Using the constraints (C.39), we obtain

$$\mathfrak{S} = \frac{\kappa g}{2}\,. \tag{C.42}$$

Thus to maximize $\mathfrak{S}$, we must make κ as large as possible. Since at least one of the parameters $d_2, \ldots, d_{f-1}$ must be negative, we find from (C.40) that

$$2\kappa \leq \max_{\alpha \in \{2,\ldots,g-1\}} \Big((m_1^3 - \lambda m_1) - (m_\alpha^3 - \lambda m_\alpha) \Big). \tag{C.43}$$

This upper bound of κ can indeed be realized by solving (C.39) for a configuration where only three parameters d_β are non-zero: $d_1, d_g > 0$ and $d_\alpha < 0$ for an α where the maximum in (C.43) is attained. A short calculation using (C.41) and (C.42) shows that

$$\mathfrak{S}_{\max} = \frac{g}{4} \max_{\alpha \in \{2,\ldots,g-1\}} (m_g - m_\alpha)(m_\alpha - m_1)(m_1 + m_\alpha + m_g)\,.$$

Substituting this result into (C.37) gives (C.1). This concludes the proof of Theorem C.1.

We note that our method of proof is constructive in the sense that for a given vector u in (C.4) we could compute the corresponding transformation U explicitly. More precisely, if u is timelike, we set $v = u/\sqrt{\langle u, u\rangle}$ to be in the first case $v^2 = 1$. We choose parameters τ_β and d_β which satisfy (C.23). Then the construction of Lemma C.3 gives us parameters $\mathfrak{m}_\beta$ and $\mathfrak{n}_\beta$ which fulfill (C.20) and (C.21). Next, the transformations (C.18) and (C.19) allow us to satisfy the conditions (C.13)–(C.16). The corresponding positive definite operator V can then be chosen according to (C.17) for a sufficiently large parameter C. According to the definition of V, (C.11), we know that $\cosh \mathfrak{g} = (V+V^{-1})/2$ and $\sinh \mathfrak{g} = (V-V^{-1})/2$, and substituting these formulas into (C.5) gives U. Likewise, if u is spacelike, we set $v = u/\sqrt{|\langle u \,|\, u\rangle|}$ to be in the second case $v^2 = -1$. We choose parameters τ_β and d_β which satisfy (C.23) as well as the constraint (C.32). The construction of Lemma C.5 gives us solutions of (C.28) and (C.29). Then the unitary transformation V can be constructed as in Lemma C.4. According to (C.27), we know that $\cos \mathfrak{g} = (V+V^{-1})/2$ and $\sin \mathfrak{g} = (V-V^{-1})/(2i)$, and using these relations in (C.6) gives U.

We finally analyze the matrix $\acute{U} \not{\xi} Y^2 \grave{U}^{-1}$.

Proposition C.6. *Using the above notation,*

$$m^2 \acute{U} \not{\xi} Y^2 \grave{U}^{-1} = c_2 \not{\xi}\, m^2 \acute{Y} \grave{Y} + C_2\, i\gamma^5 \,\langle \xi, v\rangle - \gamma^5\, [\not{\xi}, \not{v}] \sum_{\beta=1}^{g} m_\beta^2\, d_\beta \,, \tag{C.44}$$

where the constants c_2 and C_2 depend on x and ξ.

Proof. Since the term involving the inner product $\langle \xi, v\rangle$ in (C.44) involves a free constant, we may assume that v is orthogonal to ξ. Then we can rewrite the claim (C.44) similar to (C.8) and (C.9) as

$$m^2 \left(\acute{U} \not{\xi} Y^2 \grave{U}^{-1} - \acute{U}^{-1} \not{\xi} Y^2 \grave{U} \right) = 2\, \{ \not{\xi}, \gamma^5 \not{v} \} \sum_{\beta=1}^{g} m_\beta^2\, d_\beta \,.$$

Both sides of this equation commute with the operator $i\gamma^5 \not{v}$, and thus we may again restrict attention to the respective eigenspaces. We thus obtain the equivalent relation

$$m^2 \Big(\langle \mathfrak{l} \,|\, V Y^2 V^{-1} \mathfrak{l} \rangle - \langle \mathfrak{l} \,|\, V^{-1} Y^2 V \mathfrak{l} \rangle \Big) = \sum_{\beta=1}^{g} m_\beta^2\, d_\beta \times \left\{ \begin{array}{cl} 4i & \text{if } v^2 = 1 \\ -4 & \text{if } v^2 = -1 \,, \end{array} \right.$$

where we again compared (C.5) and (C.6) with (C.11) and (C.27). This relation is verified in the case $v^2 = 1$ by using that V is Hermitian and applying (C.12) and (C.22), whereas in the case $v^2 = -1$ we use that V is unitary and apply (C.12) as well as (C.31). ☐

The point of interest is that the contribution which enters the field equation can again be expressed in terms of the parameters d_β. We conclude that the analysis in §8.1 remains valid without changes even if the local axial transformation is treated non-perturbatively. In particular, we learn that, in contrast to what one might have expected naively, the higher orders in $\mathfrak{g} v$ do *not* give rise to higher order corrections to the field equations.

Appendix D. Resummation of the Current and Mass Terms at the Origin

As pointed out in §4.4, the distribution T_a is *not* a power series in a, and thus it cannot be expanded in a Taylor series around $a = 0$ (see (4.13) and the explanation thereafter). The method of subtracting suitable counter terms (4.14) has the shortcoming that the subsequent calculations are valid only *modulo smooth contributions* on the light cone. This method is suitable for analyzing the singularities on the light cone, but it is not sufficient when smooth contributions to the fermionic projector become important (cf. the discussion after (5.11) and the beginning of §8.1). We now present a convenient method for computing the smooth contributions to the fermionic projector. Our method is based on the resummation technique developed in [11, Section 4] and is outlined as follows. We first perform the mass expansion not around zero mass, but around a given mass parameter $a > 0$. Then according to (4.13), the distribution T_a is smooth in a, and we may set

$$T_a^{(n)} = \left(\frac{d}{da} \right)^n T_a \,. \tag{D.1}$$

Adapting the method of the light-cone expansion, we can express any Feynman tree diagram as a sum of terms of the form

$$P^{\text{sea}}(x,y) = \sum_{n=-1}^{\infty} \sum_k m^{p_k} (\text{phase-inserted nested line integrals}) \times T_a^{(n)}(x,y) \,, \tag{D.2}$$

where for each n, the k-sum is finite, whereas the n-sum is to be understood as a formal power series. Note that, in contrast to the series in (4.17), the infinite sum in (D.2) is *not* a light-cone expansion in the sense of Definition 4.1, because the distributions $T_a^{(n)}$ all involve smooth contributions and are thus only of the order $\mathcal{O}((y-x)^0)$. We proceed by partially carrying out the series in (D.2), giving rise to explicit smooth contributions on the light cone. After this resummation has been performed, we recover (4.17), but now with an explicit formula for $P^{\text{le}}(x,y)$.

For simplicity, we develop the method only for the contribution to the fermionic projector needed here: the vector and axial components of the fermionic projector perturbed by chiral potentials to first order. But the method generalizes in a straightforward way to arbitrary Feynman tree diagrams. Furthermore, we begin by considering a single Dirac sea (the generalization to several generalizations will then be straightforward; see the proof of Lemma 8.1 below). We thus consider the contribution to the fermionic projector

$$\Delta P = -s_m(\chi_L \slashed{A}_R + \chi_R \slashed{A}_L) t_m - t_m (\chi_L \slashed{A}_R + \chi_R \slashed{A}_L) s_m \tag{D.3}$$

with the spectral projector t_m and the Green's function s_m as in (8.27). In order to concentrate on the vector and axial components, we want to consider the expression $\text{Tr}(\slashed{\xi} \chi_L \, \Delta P(x,y))$, being a well-defined distribution. The singular part of this distribution on the light cone can be computed by inserting the formulas of the light-cone expansion (B.13)–(B.31) and using the contraction rule

$$\xi^2 \, T^{(n)}(x,y) = -4n T^{(n+1)}(x,y) + (\text{smooth contribution}) \,, \qquad n \in \{-1, 0\} \,. \tag{D.4}$$

(which is immediately verified from the explicit formulas (4.13)–(4.16)). We thus obtain

$$\frac{1}{2} \, \text{Tr} \left(\slashed{\xi} \, \chi_L \, \Delta P(x,y) \right) = 2 \int_x^y \xi_k \, A_L^k \, T_{[0]}^{(0)}(x,y) \tag{D.5}$$

$$- 2 \int_x^y (\alpha - \alpha^2) \, \xi_k \, j_L^k \, T_{[0]}^{(1)}(x,y) + m^2 \int_x^y \xi_k \left(A_L^k - A_R^k \right) \, T_{[2]}^{(1)}(x,y) \tag{D.6}$$

$$+ \, \xi^k \, f_k(x,y) + (\deg < 0) \,, \tag{D.7}$$

where we added subscripts [.] in order to indicate how these factors are to be regularized (although we do not need a regularization at this point), and $f(x,y)$ is a yet undetermined smooth function (clearly, the summand in (D.5) is the gauge term as discussed in §6.2, whereas the summands in (D.6) correspond to the current and mass terms considered in §7.1). Our goal is to compute the functions $f_k(x,y)$ at the origin $x=y$.

Our first step is to perform a mass expansion of the Feynman diagram (D.3) around a given $a \neq 0$. To this end, we need suitable calculation rules which are derived in the next lemma.

Lemma D.1. *The distributions* $T_a^{(n)}$, (D.1) *satisfy for all* $n \in \mathbb{N}_0$ *the calculation rules*

$$(-\Box_x - a)\, T_a^{(n)}(x,y) = n\, T_a^{(n-1)}(x,y) \tag{D.8}$$

$$\frac{\partial}{\partial x^k}\, T_a^{(n+1)}(x,y) = \frac{1}{2}\, \xi_k\, T_a^{(n)}(x,y) \tag{D.9}$$

$$\xi^2\, T_a^{(n)}(x,y) = -4n T_a^{(n+1)}(x,y) - 4a T_a^{(n+2)}(x,y)\,. \tag{D.10}$$

In the case $n=-1$, *the rule* (D.9) *can be used to define the distribution* $\xi_k T_a^{(-1)}$. *Using this definition, the rule* (D.10) *also holds in the case* $n=-1$.

Proof. The relations (D.8) and (D.9) were already derived in [12] (see [12, equations (3.5) and (3.6)]). For self-consistency we here repeat the proof. Clearly, T_a is a distributional solution of the Klein-Gordon equation,

$$(-\Box_x - a)\, T_a(x,y) = 0\,.$$

Differentiating n times with respect to a gives (D.8). Next, we differentiate the identity in momentum space

$$T_a(p) \;=\; \delta(p^2 - a)\, \Theta(-p^0)$$

with respect to p_k to obtain

$$\frac{\partial}{\partial p^k} T_a(p) = 2p_k\, T_a^{(1)}(p)\,.$$

Using that differentiation in momentum space corresponds to multiplication in position space and vice versa, we find

$$\xi_k\, T_a(x,y) = 2 \frac{\partial}{\partial x^k} T_a^{(1)}(x,y)\,.$$

Differentiating n times with respect to a gives (D.9).

To derive (D.10), we first combine (D.9) with the product rule to obtain

$$\Box_x T_a^{(1)} = \partial_x^k \left(\frac{1}{2}\, \xi_k T_a^{(0)} \right) = -2 T_a^{(0)} + \frac{1}{2}\, \xi_k\, \partial_x^k T_a^{(0)} = -2 T_a^{(0)} + \frac{1}{4}\, \xi^2\, T_a^{(-1)}\,.$$

On the other hand, we know from (D.8) that

$$\Box_x T_a^{(1)} = -T^{(0)} - a T^{(1)}\,.$$

Solving for $\xi^2\, T_a^{(-1)}$, we obtain

$$\xi^2\, T_a^{(-1)} = 4 T_a^{(0)} - 4a T^{(1)}\,.$$

We finally differentiate this relation $n+1$ times with respect to a, giving (D.10). □

Alternatively, this lemma could be proved by manipulating the series representation (4.13). We also remark that in the case $n=-1$, the rule (D.9) is consistent with our earlier definition (4.16).

Using the relations (D.8) and (D.9), the mass expansion of the first order Feynman diagram (D.3) was first performed in [11] (see [11, Theorem 3.3], where the mass expansion is referred to as the "formal light-cone expansion"). More generally, for the advanced and retarded Green's function, we have the expansion

$$
\begin{aligned}
&(S_a^{(l)}\, V\, S_a^{(r)})(x,y) \\
&\quad = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\, S_a^{(n+l+r+1)}(x,y)\;,
\end{aligned}
\tag{D.11}
$$

which is proved exactly as in the case $a=0$ (see [12, Lemma 2.1] or [13, Lemma 2.5.2]). The residual argument (cf. [12, Section 3.1]) also generalizes immediately to the case $a>0$, making it possible to extend (D.11) to the so-called *residual fermionic projector* (the non-residual part of the fermionic projector is precisely the non-causal high energy contribution, which can be analyzed with different methods as indicated in §8.3). Applied to our problem, we obtain the expansion

$$
(S_a\, V\, T_a + T_a\, V\, S_a)(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y (\alpha-\alpha^2)^n\, (\Box^n V)\, d\alpha\, T_a^{(n+1)}(x,y)\;, \tag{D.12}
$$

where for the line integrals we again used the short notation (B.12), and S_a is the symmetric Green's function (8.22). The mass expansion of (D.3) is now readily obtained by applying the differential operators $(i\partial\!\!\!/ + m)$ and simplifying the Dirac matrices using the rules (D.9) and (D.10). Multiplying by $\xi\!\!\!/ \chi_L$ and taking the trace, a straightforward calculation using again (D.10) yields

$$
\frac{1}{2}\, \mathrm{Tr}\left(\xi\!\!\!/\, \chi_L P(x,y)\right) \asymp 2\sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y (\alpha-\alpha^2)^n\, \xi_k \left(\Box^n A_L^k\right) T_{m^2}^{(n)}(x,y) \tag{D.13}
$$

$$
- 2\sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y (2\alpha-1)\, (\alpha-\alpha^2)^n \left(\Box^n \partial_i A_L^i\right) T_{m^2}^{(n+1)}(x,y) \tag{D.14}
$$

$$
- m^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y (\alpha-\alpha^2)^n\, \xi_k \left(\Box^n A_L^k + \Box^n A_R^k\right) T_{m^2}^{(n+1)}(x,y) \tag{D.15}
$$

$$
+ 2m^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^y (2\alpha-1)\, (\alpha-\alpha^2)^n \left(\Box^n \partial_i A_L^i\right) T_{m^2}^{(n+2)}(x,y) \tag{D.16}
$$

(this result was again obtained with the help of class_commute; see page 94). Integrating the line integrals by parts,

$$
\int_x^y (2\alpha-1)\, (\alpha-\alpha^2)^n \left(\Box^n \partial_i A_L^i\right) = \frac{1}{n+1} \int_x^y (\alpha-\alpha^2)^{n+1}\, \xi^k \left(\Box^n \partial_{ik} A_L^i\right),
$$

the divergence terms can be rewritten to recover the chiral currents. In particular, in the case $A_L = A_R$ of a vector potential, one immediately verifies that (D.13)–(D.16) has the correct behavior under gauge transformations. Furthermore, one readily sees that the expansion (D.13)–(D.16) is compatible with (D.5)–(D.7) in the sense that the singularities on the light cone coincide. We now subtract (D.13)–(D.16) from (D.5)–(D.7) and solve for $\xi^k f_k(x,y)$. In order to compute $f_k(x,x)$, it suffices to take into account the constant counter term in (4.14), as can be done by the replacement

$$
T_a(x,y) \longrightarrow T_a^{\mathrm{reg}} + N(a) \qquad \text{with} \qquad N(a) := \frac{1}{32\pi^3}\, a \log|a|\;, \tag{D.17}
$$

and similarly for the a-derivatives. Moreover, in the line integrals we may set $x=y$. In order to keep the formulas simple, we also specialize to the situation where *only the axial potential* in (6.21) is present. We thus obtain

$$\begin{aligned}f^k(x,x) &= 2\sum_{n=0}^\infty \frac{1}{n!}\int_0^1 (\alpha-\alpha^2)^n \left(\Box^n A^k_{\mathrm{a}}(x)\right) N^{(n)}(m^2)\, d\alpha \\ &\quad - 2\sum_{n=0}^\infty \frac{1}{(n+1)!}\int_0^1 (\alpha-\alpha^2)^{n+1}\left(\Box^n \partial^k{}_i A^i_{\mathrm{a}}(x)\right) \left(N^{(n+1)}(m^2) - m^2 N^{(n+2)}(m^2)\right) d\alpha\,.\end{aligned}$$

By linearity, it suffices to consider the case that A_{a} is a plane wave of momentum q,

$$A^k_{\mathrm{a}}(z) = \hat{A}^k_{\mathrm{a}}\, e^{-iq(z-x)}\,.$$

Then the above sums are recognized as Taylor series,

$$\left.\begin{aligned}\sum_{n=0}^\infty \frac{\lambda^n}{n!}\, N^{(n)}(m^2) &= N^{(0)}(m^2+\lambda) \\ \sum_{n=0}^\infty \frac{\lambda^n}{(n+1)!}\, N^{(n+\ell)}(m^2) &= \frac{N^{(\ell-1)}(m^2+\lambda) - N^{(\ell-1)}(m^2)}{\lambda}\,,\end{aligned}\right. \tag{D.18}$$

where we introduced the abbreviation $\lambda = -(\alpha-\alpha^2)\, q^2$ (the second equation in (D.18) can be derived from the first by integration over λ). We thus obtain

$$\begin{aligned}f^k(x,x) &= 2\int_0^1 A^k_{\mathrm{a}}(x)\, N\Big(m^2+\lambda\Big) d\alpha \\ &\quad - \frac{2}{\lambda}\int_0^1 (\alpha-\alpha^2)\, \partial^k{}_i A^i_{\mathrm{a}}(x)\ \Big(N^{(0)} - m^2 N^{(1)}\Big)(m^2+\nu)\Big|_{\nu=0}^{\nu=\lambda}\, d\alpha \\ &= \frac{1}{16\pi^3}\int_0^1 A^k_{\mathrm{a}}(x)\ \big(m^2+\lambda\big)\ \log\big|m^2+\lambda\big|\, d\alpha \\ &\quad - \frac{1}{16\pi^3}\int_0^1 (\alpha-\alpha^2)\, \partial^k{}_i A^i_{\mathrm{a}}(x)\, \log\big|m^2+\lambda\big|\, d\alpha\,,\end{aligned}$$

where in the last step we substituted the explicit formula for $N(a)$ in (D.17). In order to rewrite the last result in terms of the axial current, we use the identity $\lambda A^k_{\mathrm{a}} = (\alpha-\alpha^2)\Box A^k_{\mathrm{a}}$ to conclude

$$\begin{aligned}f^k(x,x) &= \frac{m^2}{16\pi^3}\, A^k_{\mathrm{a}}(x)\int_0^1 \log\Big|m^2-(\alpha-\alpha^2)q^2\Big|\, d\alpha \\ &\quad - \frac{1}{16\pi^3}\, j^k_{\mathrm{a}}(x)\int_0^1 (\alpha-\alpha^2)\ \log\Big|m^2-(\alpha-\alpha^2)q^2\Big|\, d\alpha\,.\end{aligned}$$

Substituting this result into the light-cone expansion (D.5)–(D.7) evaluated at the origin and using that $\int_0^1(\alpha-\alpha^2) = 1/6$, one sees that the term $\xi_k f^k(x,x)$ can be incorporated into the formulas of the light-cone expansion by the replacements

$$\left.\begin{aligned}T^{(1)}_{[0]} &\to T^{(1)}_{[0]} + \frac{\log(m^2)}{32\pi^3} + \frac{6}{32\pi^3}\int_0^1 (\alpha-\alpha^2)\ \log\Big|1-(\alpha-\alpha^2)\,\frac{q^2}{m^2}\Big|\, d\alpha \\ T^{(1)}_{[2]} &\to T^{(1)}_{[2]} + \frac{\log(m^2)}{32\pi^3} + \frac{1}{32\pi^3}\int_0^1 \log\Big|1-(\alpha-\alpha^2)\,\frac{q^2}{m^2}\Big|\, d\alpha\,.\end{aligned}\right\} \tag{D.19}$$

To clarify the above construction, we point out that the radius of convergence of the Taylor series in (D.18) is $|\lambda| = m^2$. Thus in the case $|\lambda| > m^2$, these series do *not* converge absolutely, so that (D.18) can be understood only on the level of formal Taylor series. For the reader who feels uncomfortable with formal power series, we remark that all formal expansions could be avoided by regularizing the distribution T_a according to (4.14) *before* performing the light-cone expansion, making a later resummation unnecessary. However, this method seems technically complicated and has not yet been carried out (see also the discussions in [12, Section 3.3] and after (8.26)). In this paper, we will be content with the formal character of (D.18).

We are now ready to prove the main result of this appendix.

Proof of Lemma 8.1. We return to the situation with three generations and a general axial potential $A_\text{a}(z)$. As the axial potential is diagonal on the generation index, the auxiliary fermionic projector splits into the direct sum of three fermionic projectors, corresponding to the Dirac seas of masses m_1, m_2, and m_3. Thus the partial trace (4.4) reduces to a sum over the generation index. Decomposing A_a into Fourier modes,

$$A_\text{a}(z) = \int_M \frac{d^4z}{(2\pi)^4}\, \hat{A}_\text{a}(q)\, e^{-iq(z-x)}\,,$$

for every $\hat{A}_\text{a}(q)$ and for every generation we may apply the replacement rules (D.19). Rewriting the multiplication in momentum space by a convolution in position space gives the formulas (8.16)–(8.19).

In order to check the prefactors, it is convenient to verify whether the arguments of the logarithms can be combined to give dimensionless quantities. This is indeed the case with the expressions

$$\log|\xi^2| + \frac{1}{3}\sum_{\beta=1}^{3} \log(m_\beta^2) = \frac{1}{3}\sum_{\beta=1}^{3} \log\left|m_\beta^2 \xi^2\right|$$

$$\log|\xi^2| + \frac{1}{m^2 \acute{Y}\grave{Y}}\sum_{\beta=1}^{3} m_\beta^2 \log(m_\beta^2) = \frac{1}{m^2 \acute{Y}\grave{Y}}\sum_{\beta=1}^{3} m_\beta^2 \log|m_\beta^2 \xi^2|\,,$$

explaining the prefactors in (8.16) and (8.17) relative to those in (8.4).

We finally need to verify that the smooth contributions which were disregarded in the formalism of §5.1 really enter the EL equations according to the simple replacement rules (D.19). The subtle point is that the contraction rule in the continuum limit (5.5) is not the same as the corresponding distributional identity (D.4), and this might give rise to additional terms which are not captured by (D.19). Fortunately, such additional terms do not appear, as the following consideration shows: To degree four on the light cone, the smooth contributions to $P(x, y)$ enter the EL equations only if the smooth term is contracted with a factor $\not\xi$ without generating a factor ξ^2 (the contributions involving ξ^2 are of degree three on the light cone). Thus for the smooth contributions, the contraction rule (5.5) is not applied, and therefore it could here be replaced by the simpler distributional identities (D.4) and (D.10). □

We finally carry out the α-integrals in (D.19) in closed form and discuss the result. This result will not be used in this paper. But it is nevertheless worth stating, because it gives more explicit information on the structure of the non-causal correction terms.

Lemma D.2. *The functions $\hat{f}^\beta_{[p]}$ defined by* (8.18) *and* (8.19) *can be written as*

$$\hat{f}^\beta_{[p]}(q) = \lim_{\varepsilon \searrow 0} g_{[p]}\Big(\frac{q^2 + i\varepsilon}{4m_\beta^2} \Big), \tag{D.20}$$

where the functions $g_{[p]}(z)$ are defined in the upper half plane by

$$\begin{aligned} g_{[0]}(z) &= -\frac{3+5z}{3z} + \frac{1+z-2z^2}{2z\sqrt{z(z-1)}} \Big[\log\Big(1-2z+\sqrt{z(z-1)}\Big) - i\pi\,\Theta(z-1) \Big] \\ g_{[2]}(z) &= -2 - \frac{\sqrt{z(z-1)}}{z} \Big[\log\Big(1-2z+\sqrt{z(z-1)}\Big) - i\pi\,\Theta(z-1) \Big] , \end{aligned}$$

where the logarithm in the complex plane is as usual cut along the ray $-i\mathbb{R}^+$ (and Θ is the Heaviside function, extended continuously to the upper half plane).

Proof. Writing the logarithm of the absolute value for any $x \in \mathbb{R}$ as

$$\log|1-x| = \lim_{\delta \searrow 0} \Big(\log\big(1-(x+i\delta)\big) + i\pi\,\Theta\big((x+i\delta)-1\big) \Big) ,$$

we obtain the representation (D.20) with

$$\begin{aligned} g_{[0]}(z) &= 6 \int_0^1 (\alpha-\alpha^2) \Big(\log\big(1-4(\alpha-\alpha^2)z\big) + i\pi\,\Theta\big(4(\alpha-\alpha^2)z-1\big) \Big)\, d\alpha \\ g_{[2]}(z) &= \int_0^1 \Big(\log\big(1-4(\alpha-\alpha^2)z\big) + i\pi\,\Theta\big(4(\alpha-\alpha^2)z-1\big) \Big)\, d\alpha \,. \end{aligned}$$

It remains to calculate these integrals for z in the upper half plane, thus avoiding the singularities on the real line. The term involving the Heaviside function is readily computed in closed form. Thus it remains to consider for $\ell = 0, 1$ the integrals

$$\int_0^1 (\alpha-\alpha^2)^\ell \,\log\Big(1-4(\alpha-\alpha^2)\,z\Big)\, d\alpha = \frac{1}{2} \int_0^1 \log\Big(1-xz\Big) \left\{ \Big(\frac{x}{4}\Big)^\ell \,\frac{1}{\sqrt{1-x}} \right\} dx \,,$$

where in the last step we transformed to the integration variable $x := 4(\alpha-\alpha^2)$. After computing the indefinite integral of the expression inside the curly brackets, we can integrate by parts. Then the logarithm in the integrand disappears, and the calculation of the integral becomes elementary. □

In Figure 7 the functions $\hat{f}^\beta_{[0]}$ and $\hat{f}^\beta_{[2]}$ are plotted. One sees that these functions attain their minimum if $q^2 = 4m_\beta^2$, and for this value of q^2 the function has a cusp. The asymptotics for large $|q^2|$ is obtained by dropping the summand one in the argument of the logarithm in (8.18) and (8.19),

$$\begin{aligned} \hat{f}^\beta_{[0]}(q) &\sim 6 \int_0^1 (\alpha-\alpha^2)\, \log\left|(\alpha-\alpha^2)\,\frac{q^2}{m_\beta^2}\right| d\alpha = -\frac{5}{3} + \log\Big(\frac{q^2}{m_\beta^2}\Big) \\ \hat{f}^\beta_{[2]}(q) &\sim \int_0^1 \log\left|(\alpha-\alpha^2)\,\frac{q^2}{m_\beta^2}\right| d\alpha = -2 + \log\Big(\frac{q^2}{m_\beta^2}\Big) \,, \end{aligned}$$

revealing a logarithmic divergence as $q^2 \to \pm\infty$. For small momenta, the functions have the asymptotics

$$\hat{f}^\beta_{[0]}(q) = -\frac{q^2}{5m_\beta^2} + \mathcal{O}(q^4) \,, \qquad \hat{f}^\beta_{[2]}(q) = -\frac{q^2}{6m_\beta^2} + \mathcal{O}(q^4) \,,$$

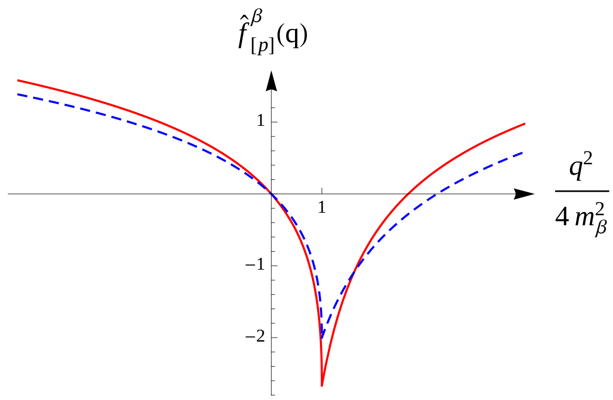

FIGURE 7. The functions $\hat{f}^\beta_{[0]}$ (red, solid) and $\hat{f}^\beta_{[2]}$ (blue, dashed).

describing a non-trivial low energy effect.

APPENDIX E. THE WEIGHT FACTORS ρ_β

In [17] the ansatz for the vacuum (3.1) was generalized by introducing so-called *weight factors* ρ_β for the Dirac seas,

$$P(x,y) = \sum_{\beta=1}^{g} \rho_\beta \int \frac{d^4k}{(2\pi)^4} \,(\not{k} + m_\beta)\, \delta(k^2 - m_\beta^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \,.$$

This generalization turns out to be useful when considering an action principle for the masses of Dirac particles [22]; for a physical discussion see [17, Appendix A]. All the constructions of the present paper could immediately be extended to the setting with weight factors, as we now explain.

The weight factors are introduced into the auxiliary fermionic projector of the vacuum (4.1) by the replacement

$$\bigoplus_{\beta=1}^{g} \;\rightarrow\; \bigoplus_{\beta=1}^{g} \rho_\beta \,.$$

For our systems, the causality compatibility condition (see [17, equation (A.1)]) does not cause problems, because all our potentials are either diagonal in the generation index, or else they can be described by a local axial transformation (see §7.6), in which case the causality compatibility condition is irrelevant. We conclude that the causal perturbation series as well as the light-cone expansion remain well-defined. The weight factors are taken into account simply by inserting them into the resulting formulas. More precisely, the number of generations is to be replaced by the sum of the weights,

$$g \;\rightarrow\; \sum_{\beta=1}^{g} \rho_\beta \,.$$

Moreover, the weights must be introduced into the partial traces by the replacements

$$\hat{Y} \rightarrow \sum_{\alpha,\beta=1}^{g} \rho_\alpha Y^\alpha_\beta \,, \qquad \underbrace{\acute{Y} Y \cdots \grave{Y}}_{p \text{ factors } Y} \rightarrow \sum_{\alpha,\beta,\gamma_1,\ldots,\gamma_{p-1}=1}^{g} \rho_\alpha \, Y^\alpha_{\gamma_1} \cdots Y^{\gamma_1}_{\gamma_2} \cdots Y^{\gamma_{p-1}}_\beta \,, \qquad \text{(E.1)}$$

or more generally using the rule

$$\acute{B} \to \sum_{\beta=1}^{g} \rho_\beta \, B^\beta_{\cdot}$$

(to avoid confusion, we note that due to the causality compatibility condition, the weight factors could just as well be inserted at the last instead of the first summation index). When considering the axial transformation, one must be careful to first multiply by the weight factors, then one performs the local axial transformation, and finally one takes the partial trace.

The only place where the modifications caused by the introduction of the weight factors are not quite obvious is the construction of the local axial transformation in §7.6 and Appendix C. More precisely, in (7.32) the factors $\not{\xi}$ must be replaced by the matrix $X\not{\xi}$, where $X = \mathrm{diag}(\rho_1, \ldots, \rho_g)$ is the asymmetry matrix of the weight factors. As a consequence, the argument leading to (7.34) no longer applies, making it necessary to arrange by the the additional condition

$$\sum_{\beta=1}^{g} \rho_\beta \, d_\beta = 0$$

that (7.32) vanishes. Then the subsequent analysis goes through if in (7.36) and (7.37) we apply (E.1) and insert factors of ρ_β into the sum. Likewise, in the non-perturbative treatment of Appendix C, we must insert the asymmetry matrix X into (C.2)

$$U \not{\xi} X U^{-1} = c_0 \not{\xi} \,,$$

and we must change the definition of powers of the mass matrix to

$$Y^p = \bigoplus_{\beta=1}^{g} \rho_\beta \, m_\beta^p \,. \tag{E.2}$$

The subsequent analysis goes through with the following minor modifications. The first equation in (C.10) changes to $\langle \mathfrak{l} \,|\, VXV \mathfrak{l} \rangle = \langle \mathfrak{l} \,|\, V^{-1} X V^{-1} \mathfrak{l} \rangle$. Likewise, the first equations in (C.13) and (C.28) must be replaced by the conditions $\langle \mathfrak{n} \,|\, X \mathfrak{n} \rangle = \langle \mathfrak{m} \,|\, X \mathfrak{n} \rangle$ and $\langle \mathfrak{m} \,|\, X \mathfrak{n} \rangle = \langle \mathfrak{n} \,|\, X \mathfrak{m} \rangle$, respectively. These two conditions can be satisfied by the transformations (C.18) resp. (C.30). Then all results up to (C.37) remain true. In particular, Theorem C.1 still holds for any vector field u which satisfies the condition $\langle u(x), u(x) \rangle \geq -\varepsilon$ for some $\varepsilon > 0$. In order to determine the maximal value of ε, one must again find the maximum of the functional $\mathfrak{S}$. Here the weight factors make the analysis of the system (C.40) a bit more complicated, and we leave the details to the reader. Proposition C.6 also remains true if we use (E.2) and insert factors ρ_β into the sum.

After these modifications, all our formulas and results remain valid. It seems a promising strategy for the construction of realistic physical models to choose the fermion masses and the weight factors according to state stable vacuum configurations as exemplified in [22].

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NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: `Felix.Finster@mathematik.uni-regensburg.de`