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Periodizable motivic ring spectra

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PERIODIZABLE MOTIVIC RING SPECTRA

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Abstract. We show that the cellular objects in the module category over a motivic E_∞-ring spectrum E can be described as the module category over a graded topological spectrum if E is strongly periodizable in our language. A similar statement is proven for triangulated categories of motives. Since MGL is strongly periodizable we obtain topological incarnations of motivic Landweber spectra. Under some categorical assumptions the unit object of the model category for triangulated motives is as well strongly periodizable giving motivic cochains whose module category models integral triangulated categories of Tate motives.

Contents

1. Introduction

In [6] graded E_∞-algebras have been constructed whose module categories are candidates for triangulated categories of Tate motives over a given field. Since then many approaches to triangulated categories of motives were developed, most notably Voevodsky's approach [20]. Thus the question arises if one can directly construct graded E_∞-algebras from these motivic categories modelling Tate motives. Among other things we give a solution to this problem, modulo standard categorical assumptions. In [5] E_∞-motivic cochains have been constructed (but see [11]) without adressing the comparison of the module category to Tate motives.

In [18] and [16] rational cycle complexes have been constructed whose module categories model rational triangulated categories of Tate motives, for a summary

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see [7, II.5.5.4, Th. 111, II.5.5.5]. In this paper we give generalizations of these constructions to integral triangulated categories of Tate motives.

Our approach emphasizes the notion of a strongly periodizable E_∞-algebra. In the motivic context a strong periodization of an E_∞-ring spectrum E is a graded E_∞-ring spectrum P such that in the stable homotopy category we have an isomorphism

$$P \cong \bigvee_{i \in \mathbf{Z}} \Sigma^{2i,i} E \tag{1}$$

with the obvious multiplication. Here $\Sigma^{p,q}$ is the usual motivic shift functor of simplicial degree p and Tate degree q. Note that in order the right hand side of (1) to be a commutative monoid in the motivic stable homotopy category we need some assumptions on E or the base scheme, namely the map $E \wedge T^2 \to E \wedge T^2$ which is the twist in the second variable should be the identity.

Theorem (4.3) implies that if E admits a strong periodization then the cellular objects in the derived category of E-modules have a description in terms of a module category over a graded topological E_∞-ring spectrum. A similar statement is true for E an E_∞-ring object in the category of motives over a given base.

Thus to find good representations of cellular objects it is necessary to prove strong periodizability of a given E_∞-ring object. In section 6 we do this for the motivic cobordism spectrum MGL and the unit object in the category of motives over a field k of characteristic 0. The construction of the former is a generalization of the construction of a strict commutative ring model of MGL in [13]. The strategy for the latter is as follows: first we construct a semi periodization, i.e. an E_∞-algebra P such that

$$P \cong \bigoplus_{i \leq 0} \mathbf{Z}(i)[2i]$$

as algebra in the triangulated category of motives over k using explicit cycle groups. Then we employ a localization technique (proposition (4.2)) to construct a strong periodization. It is here where we need some assumptions from the theory of ∞-categories. We summrize these in section 2. Under these assumptions we thus prove that there is a graded E_∞-algebra in complexes of abelian groups whose derived category of modules is equivalent as tensor triangulated category to the full subcategory of Tate motives in Voevodsky's category of big motives, see corollary (6.8).

Likewise we obtain a representation theorem for the full subcategory of cellular objects in the derived category of MGL-modules, see corollary (6.2). As another corollary, under our categorical assumptions, we obtain the strong periodizability of the motivic Eilenberg MacLane spectrum over perfect fields due to the work of Voevodsky [22] and Levine [8] on the zero slice of the sphere spectrum, see corollary (6.3).

Since motivic Landweber spectra have incarnations as cellular highly structured MGL-modules [12] we thus obtain topological models of these motivic Landweber spectra.

Here is an overview of the sections. In section 4 we first give general background on E_∞-algebras in model categories. Then we give the definition of being strongly periodizable and prove the abstract representation theorem (4.3). Moreover we show that under our categorical assumptions the existence of a semi periodization implies the existence of a periodization, proposition (4.2). Finally we show that every algebra which receives a map from a strongly periodizable algebra is itself strongly periodizable, proposition (4.4). Section 5 contains the technical part to show that our constructions for MGL and the unit sphere in motives are indeed strong periodizations. Section 6 contains our examples of strongly periodizable algebras and the applications to representation theorems.

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2. Preliminaries

In this text we have to deal with commutative algebras in a homotopical setting. In special cases one can directly work with commutative algebras, e.g. in symmetric spectra with the positive model structure. We have chosen to use the language of $\mathbb{S}$-modules as in [1] or [6] and the general setting of [19] which is adapted to an abstract formulation of the problem we are discussing.

We will freely deal with the language of ∞-categories as introduced in [10]. We will make the assumption that the ∞-categories associated to the model or semi model categories appearing in this text are presentable in the language of [10]. Among other things this enables us to localize these ∞-categories, see [10]. Note also that the theory of presentable ∞-categories is equivalent to the theory of combinatorial model categories [10, Rem. 5.5.1.5].

Also we will assume that the theory of algebras in the (semi) model category setting and the ∞-category setting are compatible. E.g. we will assume that the ∞-category associated to the semi model category of E_∞-algebras in a given suitable symmetric monoidal model category $\mathcal{C}$ is equivalent to the ∞-category of commutative algebras in the symmetric monoidal ∞-category associated to $\mathcal{C}$. The same applies in the relative setting of algebras over a given algebra.

We also suppose that the ∞-categories of modules which appear are finitely generated and stable. In particular the associated triangulated categories will be compactly generated.

We call these assumptions our *categorical assumptions*.

We use them only twice when we localize a semi periodization of a given E_∞-algebra to obtain a periodization and when we talk about the zero slice of MGL as a motivic E_∞-ring spectrum.

3. Conventions

If $\mathcal{C}$ is a category and $A \in \mathcal{C}^{\mathbf{Z}}$ a graded object we will write A_r for the object in degree r. We let $A(r)$ be the shift given by $A(r)_k = A_{k-r}$.

An Ω-*spectrum* will be a spectrum X such that the derived adjoints of the structure maps, $X_n \to \underline{\mathbf{R}\mathrm{Hom}}(K, X_{n+1})$, are equivalences. Here K is the object with which we build the spectra.

When dealing with symmetric spectra we have to be very careful about the symmetric group actions, for the convenience of the reader we refer for that to the Manipulation rules for coordinates, [14, Remark I.1.12].

4. Periodizable E_∞-algebras

In this section we develop the abstract context in which categories of cellular objects will be modelled by modules over graded E_∞-ring spectra resp. algebras.

Let $\mathcal{C}$ be a cofibrantly generated left proper symmetric monoidal model category. We assume that the domains of the generating sets I and J are small relative to the whole category and that the tensor unit and the domains of the maps in I are cofibrant.

Let $\mathcal{S}$ be the category of symmetric spectra in simplicial sets equipped with the stable projective model structure. Also let $\mathcal{A}$ be the category of (unbounded) chain complexes of abelian groups equipped with the projective model structure. Both categories fulfill the assumptions for $\mathcal{C}$.

In the whole section we will assume that $\mathcal{C}$ either receives a symmetric monoidal left Quillen functor l from $\mathcal{S}$ or from $\mathcal{A}$. We denote by $\mathcal{L}$ the image of the linear isometries operad either in $\mathcal{S}$, $\mathcal{A}$ or $\mathcal{C}$, depending in which category we talk about $\mathcal{L}$-algebras or $\mathbb{S}$-modules.

We set $\mathbb{S} := \mathcal{L}(1)$ which is a monoid. W let $\mathbb{S}\mathcal{S}$, $\mathbb{S}\mathcal{A}$ and $\mathbb{S}\mathcal{C}$ be the categories of $\mathbb{S}$-modules in the respective categories. By [19, Proposition 9.3] these are symmetric monoidal model categories with weak unit. The tensor product is given by $M \boxtimes N = \mathcal{L}(2) \otimes_{\mathbb{S}\otimes\mathbb{S}} (M \otimes N)$. The pseudo tensor unit is $\mathbf{1}_\mathcal{S}$, $\mathbf{1}_\mathcal{A}$, $\mathbf{1}_\mathcal{C}$ resp. equipped with the trivial $\mathbb{S}$-module structure.

For the discussion of commutative algebras we only treat the case of $\mathcal{C}$ since those for $\mathcal{S}$ and $\mathcal{A}$ are special cases thereof. We write $\mathrm{Comm}(\mathcal{C})$ for the category of $\mathcal{L}$-algebras in $\mathcal{C}$. This is the same as the category of commutative monoid objects in the symmetric monoidal category of unital $\mathbb{S}$-modules, see [19, Proposition 9.4]. $\mathrm{Comm}(\mathcal{C})$ is a cofibrantly generated semi model category by [19, Corollary 9.7].

For $A \in \mathrm{Comm}(\mathcal{C})$ we write $\mathrm{Comm}(A)$ for algebras under A. By loc. cit. it is a semi model category for cofibrant A. For a map $f\colon A \to B$ between cofibrant algebras the induced map $\mathrm{Comm}(A) \to \mathrm{Comm}(B)$ is a left Quillen functor which is an equivalence if f is.

We denote by $A-\mathrm{Mod}$ the category of A-modules. It is a symmetric monoidal category with pseudo unit, see [19, after Def. 9.8]. If A is cofibrant then by [19, Proposition 9.10] it is a symmetric monoidal model category with weak unit. By loc. cit. if $f\colon A \to B$ is a map between cofibrant algebras then the push forward f_* is a symmetric monoidal left Quillen functor which is a Quillen equivalence if f is an equivalence.

We set $\mathsf{D}(A) := \mathsf{Ho}(QA-\mathrm{Mod})$ where $QA \to A$ is a cofibrant replacement.

We next introduce graded objects. Again we treat the case for $\mathcal{C}$.

We let $\mathcal{C}^{\mathbf{Z}}$ be the category of $\mathbf{Z}$-graded ojects in $\mathcal{C}$, i.e. the $\mathbf{Z}$-fold product of $\mathcal{C}$ with itself. We employ the symmetric monoidal structure on $\mathcal{C}^{\mathbf{Z}}$ which is given on objects by $((a_i)_{i\in\mathbf{Z}} \otimes ((b_j)_{j\in\mathbf{Z}})_k = \coprod_{i+j=k} a_i \otimes b_j$.

With these definitions $\mathcal{C}^{\mathbf{Z}}$ satisfies the same assumptions as $\mathcal{C}$, in particular it is a cofibrantly generated symmetric monoidal model category. Thus the above discussion for $\mathcal{C}$ applies likewise to $\mathcal{C}^{\mathbf{Z}}$.

From now on we will fix a cofibrant object $K \in \mathcal{C}$ which is $\otimes$-invertible in $\mathsf{Ho}\mathcal{C}$.

Let A be a commutative monoid in $\mathsf{Ho}\mathcal{C}$. We define its periodization $P(A)$ to be the monoid in $\mathsf{Ho}\mathcal{C}$ with underlying object $\coprod_{i \in \mathbf{Z}} A \otimes K^{\otimes i}$ and multiplication induced by the maps $(A \otimes K^{\otimes i}) \otimes (A \otimes K^{\otimes j}) \cong A \otimes A \otimes K^{\otimes (i+j)} \to A \otimes K^{\otimes (i+j)}$. When we consider this periodization we will always assume that $\mathrm{id}_A \otimes \tau \colon A \otimes K^{\otimes 2} \to A \otimes K^{\otimes 2}$, $\tau \colon K^{\otimes 2} \to K^{\otimes 2}$ the twist, is the identity. Thus $P(A)$ becomes a commutative monoid. The periodization $P(A)$ can be viewed as a commutative monoid in $(\mathsf{Ho}\mathcal{C})^{\mathbf{Z}}$. Let $E \in \mathrm{Comm}(\mathcal{C})$. Then we can construct the periodization as monoid in $\mathsf{D}(E)^{\mathbf{Z}}$. We denote this also by $P(E)$.

Next observe that there is a symmetric monoidal left Quillen functor $i \colon \mathcal{C} \to \mathcal{C}^{\mathbf{Z}}$ sending X to the sequence $(\ldots, \emptyset, \emptyset, X, \emptyset, \emptyset, \ldots)$, where X sits in degree 0.

Definition 4.1. *Let $E \in \mathrm{Comm}(\mathcal{C})$. The algebra E is called* strongly periodizable *if there is an algebra $P \in \mathsf{Ho}(\mathrm{Comm}(Qi(E)))$, $Qi(E) \to i(E)$ a cofibrant replacement, such that P becomes isomorphic to $P(E)$ as monoid in $\mathsf{D}(E)^{\mathbf{Z}}$ under the image of $i(E)$. A map $i(E) \to P$ satisfying this assumption will be called a* strong periodization *of E.*

Note that if $f \colon i(E) \to P$ is a strong periodization then f induces an equivalence in degree 0.

For a commutative monoid A in $\mathsf{Ho}\mathcal{C}$ we let $P_+(A)$ be the algebra $\coprod_{i \in \mathbf{Z}_{\geq 0}} A \otimes K^{\otimes i}$ in $(\mathsf{Ho}\mathcal{C})^{\mathbf{Z}_{\geq 0}}$ and $P_-(A)$ be the algebra $\coprod_{i \in \mathbf{Z}_{\leq 0}} A \otimes K^{\otimes i}$ in $(\mathsf{Ho}\mathcal{C})^{\mathbf{Z}_{\leq 0}}$ (here again we make implicitely the assumption on the twist).

For $E \in \mathrm{Comm}(\mathcal{C})$ we denote by $P_\pm(E)$ also the corresponding algebras in $\mathsf{D}(E)^{\mathbf{Z}_{\gtreqless 0}}$.

We let $i_\pm \colon \mathcal{C} \to \mathcal{C}^{\mathbf{Z}_{\gtreqless 0}}$ be the canonical symmetric monoidal left Quillen functors.

We say that an $E \in \mathrm{Comm}(\mathcal{C})$ admits a *semi periodization* if there is an algebra $P \in \mathsf{Ho}(\mathrm{Comm}(Qi_\pm(E)))$ such that P becomes isomorphic to $P_\pm(E)$ as monoid in $\mathsf{D}(E)^{\mathbf{Z}_{\gtreqless 0}}$ under the image of $i_\pm(E)$.

Proposition 4.2. *Suppose that $E \in \mathrm{Comm}(\mathcal{C})$ admits a semi periodization. Suppose our categorical assumptions hold. Then E is strongly periodizable.*

Proof. This is the standard technique of inverting elements in E_∞-ring objects. Wlog we handle the case that E has a semi periodization in $\mathrm{Comm}(\mathcal{C}^{\mathbf{Z}_{\geq 0}})$. Let $Qi_+(E) \to P$ be such a semi periodization, $Qi_+(E)$ a cofibrant replacement. We denote by P also the image of P with respect to the functor $\mathrm{Comm}(\mathcal{C}^{\mathbf{Z}_{\geq 0}}) \to \mathrm{Comm}(\mathcal{C}^{\mathbf{Z}})$. Thus we have a map $Qi(E) \to P$ in $\mathrm{Comm}(\mathcal{C}^{\mathbf{Z}})$. The element in P we want to invert is the map $a \colon i(K)(1) \to P$ in $(\mathsf{Ho}\mathcal{C})^{\mathbf{Z}}$ corresponding to the map $K \to A \otimes K$ given by the unit of A and the identity on K.

We let $\kappa \colon i(K)(1) \otimes P \to P$ be the map in $\mathsf{D}(P)$ given by multiplication with a. For a lift $\tilde{\kappa}$ of κ to $QP{-}\mathrm{Mod}$ for a cofibrant replacement $QP \to P$ as a map between cofibrant QP-modules we consider the free QP-algebra map $F_{QP}(\tilde{\kappa})$ on $\tilde{\kappa}$. We denote by L_{alg} the localization functor on the ∞-category associated to $\mathrm{Comm}(QP)$ which inverts $F_{QP}(\tilde{\kappa})$. We claim that a local model of QP with respect to L_{alg} will yield a strong periodization of E.

Therefore we first localize the category of P-modules. We let L denote the localization functor on the ∞-category associated to $QP{-}\mathrm{Mod}$ which inverts κ.

On homotopy categories it has the same effect as localizing $\mathsf{D}(P)$ with respect to the full localizing triangulated subcategory spanned by the (co)fiber of κ. For any $M \in \mathsf{D}(P)$ we denote κ_M the map $\kappa \otimes_P \mathrm{id}_M$ or suitable twists by tensor powers of $i(K)(1)$ thereof. The local objects in $\mathsf{D}(P)$ are exactly the modules M such that κ_M is an isomorphism. Moreover, by adjunction, the local objects in $\mathsf{Ho}(\mathrm{Comm}(QP))$ are the algebras which are local as P-modules.

Since the ∞-category of P-modules is finitely generated it is easily seen that L is given by a (homotopy) colimit

$$M \mapsto LM = \mathrm{colim}(M \stackrel{\kappa_M}{\to} M \otimes i(K)^{-1}(-1) \stackrel{\kappa_M}{\to} M \otimes i(K)^{-2}(-2) \to \cdots) \tag{2}$$

(the transition maps are local equivalences, thus the map from M to the colimit is also a local equivalence, and the colimit is local by a finite generation argument).

Thus we can write $LM = M \otimes_P LP$. It follows that L is compatible with the tensor product on the triangulated categories and in the ∞-categorical setting (for the latter see [9, Def. 1.28, Prop. 1.31] and the discussion in [3, par. 5]).

We let $\mathcal{K} \to \mathcal{L}$ be the symmetric monoidal localization functor from the ∞-category of P-modules to the local objects. We get an induced adjunction

$$F\colon \mathrm{Comm}(\mathcal{K}) \leftrightarrow \mathrm{Comm}(\mathcal{L})\colon G$$

on commutative algebras in $\mathcal{K}$ and $\mathcal{L}$. We claim this is the localization at the morphism $F_{QP}(\tilde{\kappa})$. First G is a full embedding: the counit is an isomorphism since it is so on the underlying modules. We see the image of G is exactly the subcategory of $F_{QP}(\tilde{\kappa})$-local objects which settles the claim.

Thus we get a $F_{QP}(\tilde{\kappa})$-local model of P by the unit $P \to GF(P)$.

We have to detect the algebra structure of $GF(P)$ as algebra in $\mathsf{D}(E)^{\mathbf{Z}}$.

First on the level of modules we have $LP \otimes_P LP \cong LP$. This makes LP into an algebra in $\mathsf{D}(P)$. (This is what is called a P-ring spectrum in [1].) This algebra LP is clearly the image of $GF(P)$ in commutative monoids in $\mathsf{D}(P)$. By forgetting the P-module structure (i.e. applying the lax symmetric monoidal functor $\mathsf{D}(P) \to \mathsf{D}(E)^{\mathbf{Z}}$) we get the algebra we want to know.

Since the localization L is given by the colimit (2) we see that LP as module over $i_>(P_+(E))$, $i_>$ the functor $\mathsf{D}(E)^{\mathbf{Z}_{\geq 0}} \to \mathsf{D}(E)^{\mathbf{Z}}$, has the form $(\ldots, E \otimes K^{-2}, E \otimes K^{-1}, E, E \otimes K, E \otimes K^2, \ldots)$ with the obvious multiplication (note that the maps from the stages of the colimit to the colimit have to be compatible with the multiplication).

Since on the level of model categories the tensor product $LP \otimes_P LP$ is gotten from $LP \otimes_{i(E)} LP$ by a coequalizer diagram we exhibit a diagram

$$LP \otimes_{i(E)} P \otimes_{i(E)} LP \rightrightarrows LP \otimes_{i(E)} LP \to LP \otimes_P LP$$

(all tensor products are derived tensor products). On the degree 0 part of the image of $GF(P)$ in $\mathsf{D}(E)^{\mathbf{Z}}$ we know already the multiplication on the whole of the image of $GF(P)$ because of the unitality property.

The above diagram then forces the multiplication on the image of $GF(P)$ to be the one claimed. □

Next the left Quillen functor $l\colon \mathcal{S} \to \mathcal{C}$ resp. $l\colon \mathcal{A} \to \mathcal{C}$ comes into play. We denote ly $l^{\mathbf{Z}}$ the prolonguation of l to $\mathbf{Z}$-graded objects. We denote by r the right adjoint to l, thus $r^{\mathbf{Z}}$ is the right adjoint to $l^{\mathbf{Z}}$.

Given an E_∞-algebra E in $\mathcal{C}^{\mathbf{Z}}$ we can look at its imgae under $r^{\mathbf{Z}}$ and study the module category of this algebra.

Theorem 4.3. *Let $E \in \mathrm{Comm}(\mathcal{C})$ be cofibrant and let $g\colon i(E) \to P$ be a strong periodization. Assume wlog that g is a cofibration and P is fibrant. Then $\mathsf{D}(r^{\mathbf{Z}}(P))$ is canonically equivalent to the localizing full triangulated subcategory of $\mathsf{D}(E)$ spanned by the spheres $E \otimes K^i$, $i \in \mathbf{Z}$, as tensor triangulated category. Moreover this equivalence comes from Quillen functors between model categories.*

Proof. We treat the case where $l\colon \mathcal{S} \to \mathcal{C}$, the case $l\colon \mathcal{A} \to \mathcal{C}$ is analogous.

Let $Qr^{\mathbf{Z}}(P) \to r^{\mathbf{Z}}(P)$ be a cofibrant replacement. Let f be the composition $l^{\mathbf{Z}}(Qr^{\mathbf{Z}}(P)) \to l^{\mathbf{Z}}(r^{\mathbf{Z}}(P)) \to P$. Let $M \in Qr^{\mathbf{Z}}(P)-\mathrm{Mod}$ be cofibrant. Then $f_*(l^{\mathbf{Z}}(M)) \in P-\mathrm{Mod}$. Let further v be the functor which sends a graded object $X \in \mathcal{C}^{\mathbf{Z}}$ to $X_0 \in \mathcal{C}$. The functor v can be made into a lax symmetric monoidal functor, i.e. there are associative, commutative and unital transformations $v(X) \otimes v(Y) \to v(X \otimes Y)$. This also extends to $\mathbb{S}$-modules. In particular v sends a P-module to a P_0-module and via pullback along g_0 to an E-module. Thus $v(f_*(l^{\mathbf{Z}}(M))) \in E-\mathrm{Mod}$. Altogether the assignment $M \mapsto v(f_*(l^{\mathbf{Z}}(M)))$ is lax symmetric monoidal and descends to a lax symmetric monoidal triangulated functor

$$F\colon \mathsf{D}(r^{\mathbf{Z}}(P)) \to \mathsf{D}(E).$$

We claim F is a symmetric monoidal full embedding with image the full localizing triangulated subcategory spanned by the $E \otimes K^i$, $i \in \mathbf{Z}$.

We first equate the map F on generating objects. Let $A := Qr^{\mathbf{Z}}(P)$. Note that $\mathsf{D}(A)$ is generated as triangulated category with sums by the objects $A(i)$, $i \in \mathbf{Z}$. Thus to show that F is a full embedding it is sufficient to show that F induces isomorphisms on the Hom groups between the $A(i)[k]$, $i, k \in \mathbf{Z}$. Now all functors involved are compatible with the shifts $\_[k]$. The functor $l^{\mathbf{Z}}$ is also compatible with the shifts $\_(i)$. Since $\_(i) = \_ \otimes^{\mathbf{L}} \mathbf{1}(i)$ push forward along algebra maps is also compatible with the shifts $\_(i)$. Thus $f_* \circ l^{\mathbf{Z}}$ is compatible with the $\_(i)$. So we have $f_*(l^{\mathbf{Z}}(A(i)[k])) = P(i)[k]$. Finally $(P(i)[k])_0 = E \otimes K^{-i}[k]$ and we get

$$F(A(i)[k]) = E \otimes K^{-i}[k].$$

We set $T := \_(i)[k]$. Let $\varphi \in \mathrm{Hom}_{\mathsf{D}(A)}(A, TA)$ By adjunction φ corresponds to a map in $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{S}^{\mathbf{Z}})}(\mathbf{1}, TA)$, this again corresponds to a map $\psi \in \mathrm{Hom}_{\mathsf{Ho}(\mathcal{C}^{\mathbf{Z}})}(\mathbf{1}, TP)$, which is the same as a map in $\mathrm{Hom}_{\mathsf{Ho}\mathcal{C}}(\mathbf{1}, E \otimes K^{-i}[k])$.

Now the effect of F on φ is as follows. First φ is mapped to a map in

$$\mathrm{Hom}_{\mathsf{D}(l^{\mathbf{Z}}(A))}(l^{\mathbf{Z}}(A), Tl^{\mathbf{Z}}(A)),$$

then by push forward to a map in $\mathrm{Hom}_{\mathsf{D}(P)}(P, TP)$. By naturality and the properties of adjunctions this map corresponds to ψ. We finally see that both of the groups $\mathrm{Hom}_{\mathsf{D}(A)}(A, TA)$ and $\mathrm{Hom}_{\mathsf{D}(E)}(E, E \otimes K^{-i}[k])$ are naturally isomorphic to $\mathrm{Hom}_{\mathsf{Ho}\mathcal{C}}(\mathbf{1}, E \otimes K^{-i}[k])$ and that the map induced by F on these Homs corresponds to the identity via these identifications.

We have proved that F is a full embedding. Since F is compatible with sums we see that its image is closed under sums, thus the statement about the image of F follows.

We have to prove that F is symmetric monoidal, i.e. that the natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$ are isomorphisms. We have to show that v is symmetric monoidal (in a derived sense) on the image of the functor $f_* \circ l^{\mathbf{Z}}$, where on the left hand side we use the tensor product $\otimes_P$ and on the right hand side we use $\otimes_E$. Since the tensor product is triangulated and compatible with sums it suffices for this to show this property for the objects $f_*(l^{\mathbf{Z}}(A(i)[k]))$, $i, k \in \mathbf{Z}$.

For this situation our claim follows from the definition of strong periodization and the following general fact. If $B \in \mathrm{Comm}(\mathcal{C}^{\mathbf{Z}})$ (say cofibrant), $i(E) \to B$ a map of algebras, $X = B(i_1)[k_1]$, $Y = B(i_2)[k_2]$, then the natural map

$$B_{-i_1} \otimes_E B_{-i_2}[k_1 + k_2] = v(X) \otimes_E v(Y) \to v(X \otimes_B Y) = B_{-i_1-i_2}[k_1 + k_2]$$

is given by the multiplication in B. □

Proposition 4.4. *Let $E \in \mathrm{Comm}(\mathcal{C})$ be strongly periodizable and let $E \to E'$ be a map in $\mathrm{Comm}(\mathcal{C})$. Then E' is strongly periodizable.*

Proof. Let $Qi(E) \to i(E)$ be a cofibrant replacement and $Qi(E) \to P$ a strong periodization. Then $i(E') \to i(E') \otimes^{\mathbf{L}}_{Qi(E)} P$ is a strong periodization of E'. □

5. Symmetric spectra and bimorphisms

Our main references for symmetric spectra are [4] and [14]. We let $\mathcal{C}$ be a left proper cellular symmetric monoidal model category and $K \in \mathcal{C}$ a cofibrant object. We let $\mathcal{S}$ be the category of symmetric K-spectra with the stable model structure as defined in [4]. Its underlying category is the category of right $\mathrm{Sym}(K)$-modules in symmetric sequences in $\mathcal{C}$. Since $\mathrm{Sym}(K)$ is commutative it has a tensor product denoted $\wedge$. We also denote by K the image of K in $\mathcal{S}$.

Recall from [4, Def. 8.9] the shift functor s_- with the property $(s_- X)_n = X_{1+n}$. Contrary to what is said in loc. cit. the Σ_n-action on X_{1+n} is via the monomorphism $\Sigma_n \to \Sigma_{1+n}$ which is induced by the strictly monotone embedding $\{1, \ldots, n\} \to \{1, \ldots, n+1\}$ omitting 1 in the target.

Recall from [14, I.3.] that a map $X \wedge Y \to Z$ in $\mathcal{S}$ is a *bimorphism* from (X, Y) to Z, where a bimorphism consists of $\Sigma_p \times \Sigma_q$-equivariant maps

$$X_p \otimes Y_q \to Z_{p+q}$$

such that natural diagrams commute.

We denote by $\chi_{p,q} \in \Sigma_{p+q}$ the block permutation, see [14].

As noted in [14, I.3.] there is a natural morphism

$$(s_- X) \wedge Y \to s_-(X \wedge Y).$$

More generally if we are given a morphism $X \wedge Y \to Z$ with components $\alpha_{p,q} \colon X_p \otimes Y_q \to Z_{p+q}$ we exhibit a natural morphism

$$(s_-^r X) \wedge (s_-^s Y) \to s_-^{r+s} Z \tag{3}$$

having components

$$X_{r+p} \otimes Y_{s+q} \xrightarrow{\alpha_{r+p,s+q}} Z_{r+p+s+q} \xrightarrow{1\times\chi_{p,s}\times 1} Z_{r+s+p+q}\ .$$

Suppose now that we are in the following situation: Let M, N, P, M', N', P' be spectra and let maps $M \to s_-^r M'$, $N \to s_-^s N'$, $P \to s_-^{r+s} P'$, $M \wedge N \to P$ and $M' \wedge N' \to P'$ be given. We say that these maps are *compatible* if the diagram

$$\begin{array}{ccccc}
M_p \otimes N_q & \longrightarrow & P_{p+q} & \longrightarrow & P'_{r+s+p+q} \\
\downarrow & & & \nearrow{\scriptstyle 1\times\chi_{p,s}\times 1} & \\
M'_{r+p} \otimes N'_{s+q} & \longrightarrow & P'_{r+p+s+q} & &
\end{array}$$

commutes. This is the same as saying that the diagram

$$\begin{array}{ccc}
M \wedge N & \longrightarrow & P \\
\downarrow & & \downarrow \\
(s_-^r M') \wedge (s_-^s N') & \longrightarrow & s_-^{r+s} P'
\end{array}$$

commutes, where the bottom horizontal map is the one from (3).

We denote by R and Q the functorial fibrant and cofibrant replacement functors in $\mathcal{S}$.

Note that by [4, Theorem 8.10] we have natural isomorphisms $K^r \wedge (QX) \cong s_-^r(RX)$ in $\mathsf{Ho}\mathcal{S}$. For later reference we note that for the proof of this fact a natural map

(4) $$X \to (s_- X)^K$$

is used in loc. cit. Contrary to what is said in loc. cit. this map involves a non-trivial block permutation.

From the situation above we thus get maps $M \to K^r \wedge M'$, $N \to K^s \wedge N'$ and $P \to K^{r+s} \wedge P'$ in $\mathsf{Ho}\mathcal{S}$, where we use the derived smash product. We form the diagram

(5) $$\begin{array}{ccc}
K^r \wedge M' \wedge K^s \wedge N' & \longleftarrow & M \wedge N \\
\downarrow{\scriptstyle \cong} & & \\
K^r \wedge K^s \wedge M' \wedge N' & & \downarrow \\
\downarrow & & \\
K^{r+s} \wedge P' & \longleftarrow & P
\end{array}$$

in $\mathsf{Ho}\mathcal{S}$.

Lemma 5.1. *Let the situation be as above. Suppose the given maps are compatible. Suppose further that the maps $s_-^r M' \to s_-^r(RM')$, $s_-^s N' \to s_-^s(RN')$ and $s_-^{r+s} P' \to s_-^{r+s}(RP')$ are stable equivalences. Then the diagram (5) commutes in $\mathsf{Ho}\mathcal{S}$.*

Proof. We first note that for any symmetric spectrum X there is a map of symmetric spectra

$$K \wedge X \to s_- X,$$

see [14, Example I.2.18]. Iterating we get maps

$$K^r \wedge X \to s^r_- X. \tag{6}$$

The n-th component is given by

$$K^r \otimes X_n \cong X_n \otimes K^r \to X_{n+r} \xrightarrow{\chi_{n,r}} X_{r+n},$$

where the twist map, the structure map of X and the block permutation are used.

The identification $K^r \wedge X \cong s^r_-(RX)$ in $\mathsf{Ho}\mathcal{S}$ is induced by the natural map

$$K^r \wedge (QX) \to K^r \wedge X \to s^r_- X \to s^r_-(RX)$$

in $\mathcal{S}$, since the transformations (6), (4) and the unit for the adjunction

$$K \wedge (\_) \leftrightarrow (\_)^K$$

are suitably compatible.

We leave it to the reader to check that the square

$$\begin{array}{ccc} K^r \wedge M' \wedge K^s \wedge N' & \longrightarrow & s^r_- M' \wedge s^s_- N' \\ \downarrow \cong & & \\ K^r \wedge K^s \wedge M' \wedge N' & & \downarrow \\ \downarrow & & \\ K^{r+s} \wedge P' & \longrightarrow & s^{r+s}_- P' \end{array} \tag{7}$$

commutes, where in the horizontal maps the maps (6) are used and the right vertical map is (3).

We build the following diagram:

$$\begin{array}{ccccccc} & & Q(K^r \wedge QM') \wedge Q(K^s \wedge QN') & \longrightarrow & Q(s^r_- M') \wedge Q(s^s_- N') & \longleftarrow & QM \wedge QN \\ & & \downarrow \sim & & \downarrow & & \downarrow \\ K^r \wedge K^s \wedge Q(QM' \wedge QN') & \longrightarrow & K^r \wedge QM' \wedge K^s \wedge QN' & \longrightarrow & s^r_- M' \wedge s^s_- N' & \longleftarrow & M \wedge N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^{r+s} \wedge QP' & \longrightarrow & K^{r+s} P' & \longrightarrow & s^{r+s}_- P' & \longleftarrow & P \end{array}.$$

The lower middle square commutes since (7) commutes. All other squares also commute. In the two top rows the left most horizontal maps are equivalences. The composition of the left most maps in the last row also is an equivalence. Thus viewing the diagram as a diagram in $\mathsf{Ho}\mathcal{S}$ shows the claim. □

6. Examples

6.1. **Motivic cobordism.** In this section we are in the situation where $l\colon \mathcal{S} \to \mathcal{C}$. The category $\mathcal{C}$ will be a model category modelling the stable motivic homotopy category. In order that $\mathcal{C}$ reveives a functor from $\mathcal{S}$ we have to use the following slight modification of the usual versions for that category. Let S be a base scheme, Noetherian of finite Krull dimension. We let Sh_S be the category of simplicial presheaves on Sm/S, the category of smooth schemes over S, endowed with a model structure that is Nisnevich and $\mathbb{A}^1$-local. The category of symmetric S^1_s-spectra $\mathrm{Sp}_s(S)$, S^1_s the simplicial circle, in Sh_S now receives a symmetric monoidal left Quillen functor from $\mathcal{S}$, and we let $\mathcal{C}$ be the category of symmetric T-spectra in $\mathrm{Sp}_s(S)$, $T = \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$ the Tate object. We leave it to the reader to verify that $\mathcal{C}$ hits all of our requirements.

To construct strong periodizations with K the image of T in $\mathcal{C}$ we will nevertheless work in the category of symmetric T-spectra in Sh_S. By transport of structure we will not loose anything.

Recall from [13] the strictly associative and commutative model of the algebraic cobordism spectrum MGL. We will construct a strong periodization of it.

As in [13] we consider for any natural numbers n, m the space $\mathbb{A}^{nm} \cong \mathbb{A}^m \times \cdots \times \mathbb{A}^m$ (n factors) with the Σ_n-action coming from this product decomposition. Instead of only considering n-planes in this space we consider k-planes for all possible k.

We have to define a graded symmetric T-spectrum PMGL.

We define the space $\mathsf{PMGL}_{r,n}$ in grade r and spectrum level n: if $r < -n$ we set $\mathsf{PMGL}_{r,n} = \mathrm{pt}$. Otherwise set $\mathsf{PMGL}_{r,n} = \mathrm{colim}_m \mathsf{Th}(\xi_{n+r,nm})$, where $\xi_{n+r,nm}$ is the tautological vector bundle over the Grassmannian $\mathbf{Gr}(n+r, nm)$ (the colim starts for such m such that $nm \geq n+r$). Having this definition the construction works exactly as in [13].

We have multiplication maps $\mathsf{PMGL}_{r_1,n_1} \wedge \mathsf{PMGL}_{r_2,n_2} \to \mathsf{PMGL}_{r_1+r_2,n_1+n_2}$ which are $\Sigma_{n_1} \times \Sigma_{n_2}$-equivariant, we have the units $\mathrm{pt} \to \mathsf{PMGL}_{0,0}$ and $T \to \mathsf{PMGL}_{0,1}$. This is the data we need to define a ring spectrum, see [14, Def. I.1.3]. The structure maps of the individual spectra PMGL_r are induced by the multiplication maps and the second unit.

Theorem 6.1. *The graded spectrum* PMGL *is a strong periodization of* MGL.

Proof. We first show that the individual spectra PMGL_r have the correct homotopy type.

We note that the spectra PMGL_r are *semistable* in a motivically analogous sense as in [14, Th. 4.44]. In particular the maps $s_-^r \mathsf{PMGL}_k \to s_-^r(R\mathsf{PMGL}_k)$, R a fibrant replacement functor and s_- the shift functor, see section 5, are stable equivalences.

Let $r' \leq r$ and $s = r - r'$. We define maps of spectra $\mathsf{PMGL}_r \to s_-^s \mathsf{PMGL}_{r'}$ as follows.

There are maps $\mathbf{Gr}(n+r, nm) \to \mathbf{Gr}(n+r, (s+n)m)$ induced by the inclusion $\mathbb{A}^{nm} \hookrightarrow \mathbb{A}^{(s+n)m}$. Those are covered by maps of the universal vector bundles inducing maps of Thom spaces. Taking the colimit $m \to \infty$ we get maps $\mathsf{PMGL}_{r,n} \to \mathsf{PMGL}_{r',s+n}$ which are weak equivalences. It is easily seen that these maps assemble to a map of spectra $\mathsf{PMGL}_r \to s_-^s \mathsf{PMGL}_{r'}$ which is a level equivalence. Thus by [4, Theorem 8.10] we get an isomorphism $\mathsf{PMGL}_r \cong K^s \wedge \mathsf{PMGL}_{r'}$ in $\mathsf{Ho}\mathcal{S}$. This shows that the PMGL_r have the correct homotopy type.

To show that the multiplication is the correct one we use lemma (5.1) as follows: let $m' \leq m$, $n' \leq n$, $p' = m' + n'$, $p = m+n$, $r = m - m'$, $s = n - n'$, $M = \mathsf{PMGL}_m$, $N = \mathsf{PMGL}_n$, $P = \mathsf{PMGL}_{m+n}$, $M' = \mathsf{PMGL}_{m'}$, $N' = \mathsf{PMGL}_{n'}$, $P' = \mathsf{PMGL}_{m'+n'}$, $M \wedge N \to P$, $M' \wedge N' \to P'$ the multiplication maps, $M \to s_-^r M'$, $N \to s_-^s N'$,

$P \to s_-^{r+s} P'$ the maps defined above. Then it is easily checked that these maps are compatible in the sense of section 5. Lemma (5.1) now shows that the multiplication is the correct one. □

Recall that the cellular objects $\mathsf{D}(\mathsf{MGL})_{\mathcal{T}} \subset \mathsf{D}(\mathsf{MGL})$ comprise the full localizing triangulated subcategory spanned by the Tate spheres $\mathsf{MGL} \wedge K^i$, $i \in \mathbf{Z}$.

Corollary 6.2. *There is a graded E_∞-ring spectrum A such that $\mathsf{D}(A)$ is equivalent as tensor triangulated category to $\mathsf{D}(\mathsf{MGL})_{\mathcal{T}}$.*

Proof. This follows from theorem (6.1) and theorem (4.3). □

For any motivic spectrum X we denote the slices by $s_i(X)$, see [21]. Since MGL is effective ([15]) there is a map of ring spectra in the stable motivic homotopy category $\mathsf{MGL} \to s_0\mathsf{MGL}$. As noticed in [17, Remark 7.2] this map can be realized as a map of motivic E_∞-ring spectra when our categorical assumptions hold. Moreover by [8] (and [22] for fields of characteristic 0) and [15, Cor. 3.3] this map is the map from MGL to the motivic Eilenberg MacLane spectrum $\mathsf{M}\mathbf{Z}$ when S is the spectrum of a perfect field.

Let $\mathsf{D}(\mathsf{M}\mathbf{Z})_{\mathcal{T}} \subset \mathsf{D}(\mathsf{M}\mathbf{Z})$ be the full localizing triangulated subcategory spanned by the Tate spheres $\mathsf{M}\mathbf{Z} \wedge K^i$, $i \in \mathbf{Z}$.

Corollary 6.3. *Suppose S is the spectrum of a perfect field and our categorical assumptions hold. Then $\mathsf{M}\mathbf{Z}$ is strongly periodizable. In particular there is a graded E_∞-ring spectrum A such that $\mathsf{D}(A)$ is equivalent as tensor triangulated category to $\mathsf{D}(\mathsf{M}\mathbf{Z})_{\mathcal{T}}$.*

Proof. By the remarks above this follows from proposition (4.4), theorem (6.1) and theorem (4.3). □

Remark 6.4. *Here we suppose our categorical assumptions hold. Then by the above discussion $s_0\mathsf{MGL}$ is strongly periodizable, in particular $(s_0\mathsf{MGL})_{\mathbb{Q}}$ is strongly periodizable. Thus if S is regular* [15, cor. 6.4] *states that the Landweber theory $\mathsf{L}\mathbf{Q} \cong (s_0\mathsf{MGL})_{\mathbb{Q}}$ has a strongly periodizable E_∞-structure. Since these spectra are rational it follows from our representation theorem (4.3) that there is a graded rational cdga A such that $\mathsf{D}(A) \simeq \mathsf{D}(\mathsf{L}\mathbf{Q})_{\mathcal{T}}$, the latter category being defined similarly as above. We note that $\mathsf{D}(\mathsf{L}\mathbf{Q})_{\mathcal{T}}$ is a good model for rational Tate motives over any regular base.*

Remark 6.5. *There is a map of motivic E_∞-ring spectra $\mathsf{MGL} \to \mathsf{KGL}$, see* [3, Prop. 5.10]. *It thus follows from proposition (4.4) and theorem (6.1) that KGL is strongly periodizable.*

Remark 6.6. *Let A be the graded topological spectrum of corollary (6.2) and let a complex point of S be given. Then topological realization provides us with a map of graded E_∞-ring spectra $\varphi\colon A \to \mathsf{PMU}$. The topological realization functor $\mathsf{D}(A) \simeq \mathsf{D}(\mathsf{MGL})_{\mathcal{T}} \to \mathsf{D}(\mathsf{MU})$ can be modelled by push forward along φ and taking the zeroth component of the resulting graded PMU-module.*

6.2. **Motivic cohomology.** In this section we will be in the situation where $l\colon \mathcal{A} \to \mathcal{C}$. Let k be a field. We will assume that k is of characteristic 0. We first explain what $\mathcal{C}$ is. We let $\mathsf{Shv}_{Nis}(\mathsf{SmCor}(k))$ be the category of Nisnevich sheaves with transfers on the category of smooth schemes over k, see [20]. The category of complexes $\mathsf{Cpx}(\mathsf{Shv}_{Nis}(\mathsf{SmCor}(k)))$ has an $\mathbb{A}^1$- and Nisnevich local symmetric monoidal model structure such that the canonical functor from $\mathcal{A}$ is symmetric monoidal left Quillen and such that $\mathbb{T} := S^0\mathbf{Z}_{tr}(\mathbf{P}^1, \{\infty\})$ is cofibrant (S^0X denotes the complex where X sits in degree 0). The category $\mathcal{C}$ is defined to be the category of symmetric $\mathbb{T}$-spectra in $\mathsf{Cpx}(\mathsf{Shv}_{Nis}(\mathsf{SmCor}(k)))$ with the stable model structure defined in [4]. The object K is defined to be the image of $\mathbb{T}$ in $\mathcal{C}$.

Theorem 6.7. *Suppose our cartegorical assumptions hold. Then the unit sphere in $\mathcal{C}$ has a strong periodization.*

Proof. For any $X, U \in \mathsf{Sm}/k$ we let $\mathrm{z}_{equi}(X, r)(U)$ the free abelian group generated by closed integral subschemes of $X \times_k U$ which are equidimensional of relative dimension r over U, see [2]. The assignment $U \mapsto \mathrm{z}_{equi}(X, r)(U)$ has the structure of a Nisnevich sheaf with transfers on Sm/k. Note that we have natural bilinear maps $\mathrm{z}_{equi}(X, r)(U) \times \mathrm{z}_{equi}(Y, s)(U) \to \mathrm{z}_{equi}(X \times_k Y, r+s)(U)$ which are functorial for finite correspondences. Thus we get maps

$$\mathrm{z}_{equi}(X, r) \otimes \mathrm{z}_{equi}(Y, s) \to \mathrm{z}_{equi}(X \times_k Y, r + s),$$

where the tensor product in $\mathsf{Shv}_{Nis}(\mathsf{SmCor}(k))$ is used.

We are going to define a ring object P in $\mathcal{C}^{\mathbf{Z}_{\leq 0}}$. For non-negative integers r we let the spectra P_{-r} be given by $\mathsf{P}_{-r,n} = S^0\mathrm{z}_{equi}(\mathbb{A}^n, r)$ with the obvious action of Σ_n and with multiplication maps

$$\mathsf{P}_{-r_1,n_1} \otimes \mathsf{P}_{-r_2,n_2} \to \mathsf{P}_{-r_1-r_2,n_1+n_2}$$

given by the above multiplication of cycles. These are $\Sigma_{n_1} \times \Sigma_{n_2}$-equivariant. The two units $\mathbf{Z} \to \mathsf{P}_{0,0}$ and $\mathbb{T} = S^0(\mathrm{z}_{equi}(\mathbf{P}^1, 0)/\mathrm{z}_{equi}(\{\infty\}, 0)) \to \mathsf{P}_{0,1} = S^0\mathrm{z}_{equi}(\mathbb{A}^1, 0)$ are the natural ones. One checks easily that we get a commutative monoid in $\mathcal{C}^{\mathbf{Z}_{\leq 0}}$. Moreover the unit map is an equivalence in degree 0 by [20, Prop. 4.1.5].

We claim that P is a semi periodization of the unit. We first show that the individual spectra P_{-r} have the correct homotopy type.

We claim $s_-^r \mathsf{P}_{-r} \simeq \mathsf{P}_0$, where s_- is the shift functor, see section 5. Indeed, flat pullback of cycles along the projections $\mathbb{A}_k^{r+n} \cong \mathbb{A}_k^r \times_k \mathbb{A}_k^n \to \mathbb{A}_k^n$ gives maps

$$j_n\colon \mathrm{z}_{equi}(\mathbb{A}^n, 0) \to \mathrm{z}_{equi}(\mathbb{A}^{r+n}, r).$$

We claim that these maps assemble to a map of spectra $\mathsf{P}_0 \to s_-^r \mathsf{P}_{-r}$. We have to show that the j_n are compatible with the structure maps

$$S^0\mathrm{z}_{equi}(\mathbb{A}^n, 0) \otimes \mathbb{T} \to S^0\mathrm{z}_{equi}(\mathbb{A}^{n+1}, 0)$$

and

$$S^0\mathrm{z}_{equi}(\mathbb{A}^{r+n}, r) \otimes \mathbb{T} \to S^0\mathrm{z}_{equi}(\mathbb{A}^{r+n+1}, r).$$

This follows since the structure maps are given by mutliplication of cycles from the right and we use flat pullback on the left.

We claim the map $j\colon \mathsf{P}_0 \to s_-^r \mathsf{P}_{-r}$ is a level equivalence. Since P_0 is an Ω-spectrum it follows then that P_{-r} is also an Ω-spectrum and $s_-^r \mathsf{P}_{-r} \simeq (Rs_-)^r \mathsf{P}_{-r}$.

It follows from [4, Theorem 8.10] that $\mathsf{P}_{-r} \wedge^{\mathbf{L}} K^r \simeq \mathsf{P}_0$. Thus P_{-r} will have the correct homotopy type.

We prove that the $S^0 j_n$ are equivalences. First note that by [2, Prop. 5.7 2.] the presheaves $z_{equi}(X, r)$ are pretheories in the sense of [2, sec. 5]. For any presheaf F on Sm/k with values in abelian groups denote by $\underline{C}_* F$ the complex associated to the simplicial presheaf $U \mapsto F(\Delta^\bullet \times U)$.

The proof of [2, Prop. 5.5 1.] shows that for any pretheory F and $U \in \mathsf{Sm}/k$ we have isomorphisms

$$\mathbb{H}^i_{Nis}(U, (\underline{C}_* F)_{Nis}) \cong \mathbb{H}^i_{cdh}(U, (\underline{C}_* F)_{cdh}). \tag{8}$$

Now to show that the $S^0 j_n$ are equivalences it is sufficient to show that the $\underline{C}_* j_n$ are Nisnevich-local equivalences.

This follows from (8), [2, Prop. 8.3 1.] and the definition of the bivariant cycle cohomology [2, Def. 4.3].

To show that P is a semi periodization we are left to show that the multiplication is the correct one. We apply lemma (5.1) with $M = N = P = \mathsf{P}_0$, $M' = \mathsf{P}_{-r}$, $N' = \mathsf{P}_{-s}$, $P' = \mathsf{P}_{-r-s}$. The maps $M \to s^r_- M'$, $N \to s^s_- N'$, $P \to s^{r+s}_- P'$ are the maps j constructed above. The maps $M \wedge N \to P$ and $M' \wedge N' \to P'$ are the multiplication maps. By inspection these maps are compatible in the sense of paragraph 5. Moreover the maps $s^r_- M' \to s^r_-(RM')$, $s^s_- N' \to s^s_-(RN')$ and $s^{r+s}_- P' \to s^{r+s}_-(RP')$ are stable equivalences since all appearing spectra are Ω-spectra. Now lemma (5.1) indeed says that the multiplication is the correct one.

Having constructed a semi periodization it follows from proposition (4.2) that under the categorical assumptions the unit in $\mathcal{C}$ has a periodization. □

We let $\mathsf{DM}(k) := \mathsf{Ho}\mathcal{C}$ and $\mathsf{DM}(k)_{\mathcal{T}}$ be the full localizing triangulated subcategory of $\mathsf{DM}(k)$ generated by the $\mathbf{Z}(i)$ where $\mathbf{Z}(i) = K^i$.

Corollary 6.8. *Suppose our categorical assumptions hold. Then there is an E_∞-algebra A in $\mathcal{A}^{\mathbf{Z}}$ such that $\mathsf{D}(A)$ is equivalent as tensor triangulated category to $\mathsf{DM}(k)_{\mathcal{T}}$.*

Proof. This follows from theorem (6.7) and theorem (4.3). □

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