



Equivariant T-Duality of locally  
compact abelian groups

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# EQUIVARIANT T-DUALITY OF LOCALLY COMPACT ABELIAN GROUPS

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## ABSTRACT

Equivariant T-duality triples of locally compact abelian groups are considered. The motivating example dealing with the group  $\mathbb{R}^n$  containing a lattice  $\mathbb{Z}^n$  comes with an isomorphism in twisted equivariant K-theory.

**Keywords:** T-duality, T-duality triples, twisted K-theory

**MSC:** 46L80

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## 1. INTRODUCTION

In the mathematical study of T-duality the notion of T-duality triples has been introduced in [1]. A key feature of T-duality triples is that they come with an isomorphism in twisted K-theory. In [2] T-duality over orbifolds has been studied, and a corresponding isomorphism in Borel K-theory was proven. However, they did not prove the corresponding isomorphism in K-theory. At least partially, this work fills that gap.

In [10] it has been shown that the approach to T-duality of [1] using T-duality triples is equivalent to the  $C^*$ -algebraic approach of [7]. In the latter Connes' Thom isomorphism [5] turns out to be the key tool to identify the K-theories of T-dual objects. The application in K-theory given in this work is also based on Connes' Thom isomorphism, however, the techniques used herein are quite different from [7].

In this work we study T-duality triples (we will call them topological triples) over the singleton space equipped with an action of a finite group  $\Gamma$ . Let us start with its precise definition. A topological triple is a  $\Gamma$ -equivariant commutative diagram of trivialisable principal fibre bundles

$$\begin{array}{ccccc}
 & & P \times \widehat{E} & \xleftarrow{\cong} & E \times \widehat{P} & & \\
 & & \swarrow & & \searrow & & \\
 P & & & E \times \widehat{E} & & & \widehat{P} \\
 & \searrow & & \swarrow & & \searrow & \\
 & E & & & \widehat{E} & & \\
 & \swarrow & & \searrow & & & \\
 & & G/N & & \widehat{G}/N^\perp & & *
 \end{array}$$

where the structure groups of the bundles are as indicated. Here PU is the projective unitary group of some separable Hilbert space,  $G$  is a locally compact abelian group with discrete and cocompact subgroup  $N$ , and  $\widehat{G}$  is the dual group of  $G$  containing  $N^\perp$ , the annihilator of  $N$ . Moreover, the isomorphism on top of the diagram must satisfy a certain condition: By trivialisability of the diagram, for a trivialisability the top isomorphism defines a function  $G/N \times \widehat{G}/N^\perp \rightarrow \text{PU}$ , thus a class in  $\check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\text{U}(1)})$ . The requirement is that there exists a trivialisability such that this class is contained in the subgroup of  $\check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\text{U}(1)})$  which is generated by the Poincaré class. The Poincaré class is the class of the canonical line bundle over  $G/N \times \widehat{G}/N^\perp$ . If for an integer  $L$  the class of the top isomorphism is  $L$  times the Poincaré class, then we call  $L$  the order of the topological triple.

To a topological triple we can associate two  $C^*$ -algebras  $C^*(E, P)$  and  $C^*(\widehat{E}, \widehat{P})$  which are the crossed product  $C^*$ -algebras of the group  $\Gamma$  with

the  $C^*$ -algebras of sections of the associated  $C^*$ -bundles  $P \times_{\text{PU}} \mathcal{K} \rightarrow E$  and  $\widehat{P} \times_{\text{PU}} \mathcal{K} \rightarrow \widehat{E}$ . Here  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators. It is the principal aim of this paper to understand how the two  $C^*$ -algebras  $C^*(E, P)$  and  $C^*(\widehat{E}, \widehat{P})$  are related to each other. In particular, if the order of the topological triple is  $L = 1$ , we establish an isomorphism in K-theory between  $K_{\Gamma, P}^*(E) := K_*(C^*(E, P))$  and  $K_{\Gamma, \widehat{P}}^*(\widehat{E}) := K_*(C^*(\widehat{E}, \widehat{P}))$ . These groups are the equivariant twisted K-theories of  $E$  and  $\widehat{E}$  with twists given by  $P$  and  $\widehat{P}$ , respectively. Alternatively, these are the twisted K-theories of the transformation groupoids  $\Gamma \ltimes E$  and  $\Gamma \ltimes \widehat{E}$  with twists  $P$  and  $\widehat{P}$ , respectively. Turning from a groupoid to the stack presented by it, one may also take these groups as the twisted K-theories of the quotient stacks  $[E/\Gamma]$  and  $[\widehat{E}/\Gamma]$  with twists  $P$  and  $\widehat{P}$ , respectively.

We give a short overview of this work.

We introduce the notion of pairs in section 3. A pair consists of principal fibre bundle over the singleton space with structure group  $G/N$  and a projective unitary trivialisable principal bundle on its total space, where a finite group  $\Gamma$  acts on all spaces by bundle automorphisms.

If on a pair there is an additional action of  $G$ , we call these data a dynamical triple. A dynamical triple defines in a natural way a class in  $H^2(N, U(1))$  called the Mackey obstruction. If the Mackey obstruction of a dynamical triple vanishes we call the triple dualisable (section 5), and we prove in section 7 a classification theorem for dualisable dynamical triples. This is the key tool for our first important result, namely, the existence of a natural duality theory (section 9) of dualisable dynamical triples (on the level of equivalence classes). The dual objects are given by replacing everything by its dual in the sense of the duality theory of abelian groups (section 8).

In section 10 we show that the duality theory of section 9 is the same duality theory which one obtains if one concerns the associated  $C^*$ -dynamical systems and uses the duality theory of abelian crossed products.

In section 11 we introduce topological triples the objects of our main interest. The second important result of this work, the classification theorem of topological triples (section 12), enables us to point out the relation between dynamical triples and topological triples: there is a natural bijection (section 14) between (the equivalence classes of) dualisable dynamical triples and (the equivalence classes of) regular topological triples of order 1. Regular topological triples (section 13) are a subclass of all topological defined by means of their classification.

The main result of this work is then stated in section 15, where all partial results are puzzled together. If we fix the groups  $G = \mathbb{R}^n$  and  $N = \mathbb{Z}^n$ , then we show first that all topological triples are regular. Thus, there is a natural bijection between topological triples of order 1 and dualisable dynamical triples. By the duality theory of section 9 and Connes' Thom isomorphism, we then obtain the mentioned isomorphism in equivariant

twisted K-theory.

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### 2. NOTATION

1. By  $\Gamma$  we will always denote a finite group. By  $G$  will always denote a second countable, Hausdorff, locally compact abelian group which has a discrete, cocompact subgroup  $N$ , i.e. the quotient  $G/N$  is compact. The dual group  $\text{Hom}(G, \text{U}(1))$  of  $G$  is denoted by  $\widehat{G}$ . It contains the cocompact subgroup  $N^\perp := \{\hat{g} \in \widehat{G} \mid \hat{g}|_N = 1\}$ . Recall the classical isomorphisms of Pontrjagin duality [9]

$$G \cong \widehat{\widehat{G}}, \quad \widehat{N} \cong G/N, \quad \widehat{G/N} \cong N^\perp.$$

For all pairings between an abelian group  $H$  and its dual  $\widehat{H}$  we are going to use the bracket notation

$$\langle \cdot, \cdot \rangle : \widehat{H} \times H \rightarrow \text{U}(1).$$

As an example, take  $g \in G$  which is mapped to  $z \in G/N$  by the quotient map, and let  $n \in N^\perp \subset \widehat{G}$ . Then we have the identity

$$\langle n, g \rangle = \langle n, z \rangle,$$

where the first pairing is between  $\widehat{G}$  and  $G$  and the second is between  $N^\perp \cong \widehat{G/N}$  and  $G/N$ .

2. Once and for all we fix Borel sections  $\sigma : G/N \rightarrow G$  and  $\hat{\sigma} : \widehat{G/N} \rightarrow \widehat{G}$  of the quotient maps  $G \rightarrow G/N$  and  $\widehat{G} \rightarrow \widehat{G/N}$ .

3. If  $M$  is an abelian group and module over the finite group  $\Gamma$ , we have the group cohomological chain complex

$$\dots \xrightarrow{\delta} C^n(\Gamma, M) \xrightarrow{\delta} C^{n+1}(\Gamma, M) \xrightarrow{\delta} \dots$$

where  $C^n(\Gamma, M)$  is the set of all maps from  $\Gamma^n$  to  $M$ . We use the following (non standard) convention for the boundary operator  $\delta$

$$\begin{aligned} (\delta m)(a_0, \dots, a_n) &:= (a_0)^{-1} \cdot m(a_1, \dots, a_n) - m(a_1 a_0, a_2, \dots, a_n) \\ &\quad + m(a_0, a_2 a_1, \dots, a_n) - \dots + \dots \\ &\quad + (-1)^{n-1} m(a_0, \dots, a_n a_{n-1}) + (-1)^n m(a_0, \dots, a_{n-1}) \end{aligned}$$

which fits better in our formulas than the usual one. We will write the group law in  $M$  multiplicative whenever the module is  $U(1)$  (trivial module structure) or a space of  $U(1)$ -valued functions on  $G/N$  or  $\widehat{G}/N^\perp$ . In the latter case the action of  $a \in \Gamma$  on a function  $f$  on  $G/N$  is of the form<sup>1</sup>  $(a \cdot f)(z) := f(-\chi(a) + z)$ , where  $\chi : \Gamma \rightarrow G/N$  is a homomorphism. Dually,  $\Gamma$  acts on a space of functions on  $\widehat{G}/N^\perp$  by a homomorphism  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$ . The corresponding boundary operators are denoted  $\delta_\chi$  or  $\delta_{\hat{\chi}}$ .

4. We will also consider modules over the groups  $G$  and  $\widehat{G}$ , then we use the notation  $d$  and  $\hat{d}$  for the boundary operator which is given by the same algebraic formula as  $\delta$  above. On a space of functions on  $G/N$  the group  $G$  acts in obvious way in the arguments of the functions by (the negative of) the canonical homomorphism  $G \rightarrow G/N$ . Similarly,  $\widehat{G}$  acts on functions on  $\widehat{G}/N^\perp$ .

5.  $U(\mathcal{H})$  is the unitary group of some separable Hilbert space  $\mathcal{H}$ , it is equipped with the strong operator topology. The quotient by its center  $U(1)$  is the projective unitary group  $PU(\mathcal{H})$ . We denote the quotient map by

$$\text{Ad} : U(\mathcal{H}) \rightarrow PU(\mathcal{H}).$$

The action of  $PU(\mathcal{H})$  on  $U(\mathcal{H})$  by conjugation is denoted by

$$\begin{aligned} PU(\mathcal{H}) \times U(\mathcal{H}) &\rightarrow U(\mathcal{H}) \\ (v, U) &\mapsto v[U] := VUV^{-1}, \end{aligned}$$

where  $V \in \text{Ad}^{-1}(v)$  is some pre-image of  $v$ .

6. The action of  $PU(\mathcal{H})$  on the  $C^*$ -algebra of compact operators  $\mathcal{K}(\mathcal{H})$  by conjugation is also denoted with squared brackets:  $v[K]$ , for  $K \in \mathcal{K}(\mathcal{H})$ ,  $v \in PU(\mathcal{H})$ . This defines an isomorphism between  $PU(\mathcal{H})$  and the  $C^*$ -automorphism group of  $\mathcal{K}(\mathcal{H})$ .

7. A  $U(\mathcal{H})$ -valued or  $U(1)$ -valued Borel function on any of the groups  $\Gamma, G, G/N, \dots$  gives rise to a unitary multiplication operator on the corresponding  $L^2$ -space. By abuse of notation, we denote the function and its multiplication operator by same symbol. E.g. consider  $G/N \ni z \mapsto \langle g, \sigma(z) \rangle \in U(1)$ , then

$$\begin{aligned} \langle g, \sigma(-) \rangle : L^2(G/N) &\rightarrow L^2(G/N) \\ F &\mapsto (z \mapsto \langle g, \sigma(z) \rangle F(z)). \end{aligned}$$

---

<sup>1</sup> Observe that the minus sign of the action cancels the minus sign of  $(a_0)^{-1}$  in the boundary operator.

To make the reader more familiar with that notation we note here that by composition with  $\text{Ad}$  we obtain a map

$$\begin{aligned} G &\rightarrow \text{PU}(L^2(G/N)) \\ g &\mapsto \text{Ad}(\langle g, \sigma(-) \rangle). \end{aligned}$$

### 3. PAIRS

Let  $\Gamma$  be any finite group, and let  $\mathcal{H}$  be any separable Hilbert space.

**Definition 3.1** A **pair**  $(P, E)$  (over the one point space  $*$ ) with underlying Hilbert space  $\mathcal{H}$  is a  $\Gamma$ -equivariant sequence

$$\Gamma \curvearrowright \begin{pmatrix} P \circlearrowleft \text{PU}(\mathcal{H}) \\ \downarrow \\ E \circlearrowleft G/N \\ \downarrow \\ * \end{pmatrix}$$

of trivialisable principal fibre bundles, where  $\Gamma$  acts (from the left) by bundle automorphisms.

Due to the triviality condition on  $P$  we have a diagram of bundle isomorphisms

$$(1) \quad \begin{array}{ccc} G/N \times \text{PU}(\mathcal{H}) & \longrightarrow & P \\ \downarrow & & \downarrow \\ G/N & \longrightarrow & E \\ \downarrow & & \downarrow \\ * & \xrightarrow{=} & * \end{array}$$

We call any choice of such bundle isomorphisms a **chart** for the pair. Pull-back of the  $\Gamma$ -actions along the chart induces  $\Gamma$ -actions on  $G/N$  and  $G/N \times \text{PU}(\mathcal{H})$  which are given by a homomorphism

$$(2) \quad \chi : \Gamma \rightarrow G/N$$

and a group cohomological 1-cocycle  $\lambda : \Gamma \rightarrow C(G/N, \text{PU}(\mathcal{H}))$ , i.e.

$$(3) \quad \lambda(ba, z) = \lambda(b, z + \chi(a)) \lambda(a, z),$$

for  $a, b \in \Gamma$  and  $z \in G/N$ . Equivalently, one can consider  $(\chi, \lambda)$  as a single homomorphism

$$\chi \times \lambda : \Gamma \rightarrow G/N \rtimes C(G/N, \text{PU}(\mathcal{H})),$$

where  $G/N \rtimes C(G/N, \text{PU}(\mathcal{H}))$  is the semi-direct product.

It is obvious that any tuple  $(\chi, \lambda)$  with these properties defines the structure of a pair on  $G/N \times \text{PU}(\mathcal{H}) \rightarrow G/N \rightarrow *$ , and in this sense we also call such a tuple  $(\chi, \lambda)$  a **pair**.

A **morphism**  $(\varphi, \vartheta, \theta)$  from a pair  $P \rightarrow E \rightarrow *$  with underlying Hilbert space  $\mathcal{H}$  to a pair  $P' \rightarrow E' \rightarrow *$  with underlying Hilbert space  $\mathcal{H}'$  is a commutative diagram of  $\Gamma$ -equivariant bundle isomorphisms

$$(4) \quad \begin{array}{ccc} P & \xrightarrow{\vartheta} & \varphi^* P' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\theta} & E' \\ \downarrow & & \downarrow \\ * & \xrightarrow{=} & * \end{array}$$

where  $\varphi$  is the mod- $\text{U}(1)$  class of a unitary isomorphism  $\mathcal{H} \rightarrow \mathcal{H}'$ , and  $\varphi^* P'$  is the  $\text{PU}(\mathcal{H})$ -bundle  $P' \times_{\text{PU}(\mathcal{H}')} \text{PU}(\mathcal{H})$ , where  $\text{PU}(\mathcal{H}')$  acts on  $\text{PU}(\mathcal{H})$  via  $\varphi^* : \text{PU}(\mathcal{H}') \rightarrow \text{PU}(\mathcal{H})$ . Pairs and their morphisms form a category; composition of morphisms  $(\varphi, \vartheta, \theta)$  and  $(\varphi', \vartheta', \theta')$  is just component-wise composition  $(\varphi' \circ \varphi, \vartheta' \circ \vartheta, \theta' \circ \theta)$ . The resulting category is a groupoid, i.e. every morphism is an isomorphism.

If  $\mathcal{H}_0$  is a separable Hilbert space and  $P$  is a principal  $\text{PU}(\mathcal{H})$ -bundle, we use the notation

$$(5) \quad \begin{aligned} P_{\mathcal{H}_0} &:= \text{PU}(\mathcal{H}_0) \otimes P \\ &:= P \times_{\text{PU}(\mathcal{H})} \text{PU}(\mathcal{H}_0 \otimes \mathcal{H}). \end{aligned}$$

for the associated (stabilised) bundle. We call two pairs  $(P, E)$  and  $(P', E')$  with underlying Hilbert spaces  $\mathcal{H}$  and  $\mathcal{H}'$  **stably isomorphic** if there exists a separable Hilbert space  $\mathcal{H}_0$  such that the pairs  $(P_{\mathcal{H}_0}, E)$  and  $(P'_{\mathcal{H}_0}, E')$  are isomorphic.

We call two pairs  $(P, E)$  and  $(P', E')$  **outer conjugate** if they are isomorphic up to a unitary cocycle, i.e. if there are morphisms of pairs

$$(6) \quad \begin{array}{ccc} P \longrightarrow G/N \times \text{PU}(\mathcal{H}) \circlearrowleft \lambda & & P' \longrightarrow G/N \times \text{PU}(\mathcal{H}) \circlearrowleft \lambda' \\ \downarrow & & \downarrow \\ E \longrightarrow G/N \circlearrowleft \chi & & E' \longrightarrow G/N \circlearrowleft \chi' \\ \downarrow & & \downarrow \\ * \xrightarrow{=} * & & * \xrightarrow{=} * \end{array}$$

then the induced pairs  $(\chi, \lambda)$  and  $(\chi', \lambda')$  satisfy  $\chi = \chi'$  and  $\lambda' = \lambda \text{Ad}(l)$ , where  $l : \Gamma \times G/N \rightarrow \text{U}(\mathcal{H})$  is a continuous function which satisfies the unitary cocycle condition

$$(7) \quad l(ba, z) = \lambda(a, z)^{-1} [l(b, z + \chi(a))] l(a, z).$$



We call two pairs  $(P, E)$  and  $(P', E')$  **stably outer conjugate** if there exists a separable Hilbert space  $\mathcal{H}_0$  such that the pairs  $(P_{\mathcal{H}_0}, E)$  and  $(P'_{\mathcal{H}_0}, E')$  are outer conjugate.

It is elementary to check that the notions of isomorphism, stable isomorphism, outer conjugation and stable outer conjugation are equivalence relations which may be arranged in a diagram of implications

$$\begin{array}{ccc}
 \text{isomorphism} & \Longrightarrow & \text{outer conjugation} \\
 \text{of pairs} & & \text{of pairs} \\
 \Downarrow & & \Downarrow \\
 \text{stable isomorphism} & \Longrightarrow & \text{stable outer conjugation} \\
 \text{of pairs} & & \text{of pairs}
 \end{array}$$

**Example 3.1** Let  $(\chi, \lambda)$  be any pair with underlying Hilbert space  $\mathcal{H}$ , and let  $\rho : \Gamma \rightarrow \mathbf{U}(L^2(\Gamma))$  be the right regular representation, then  $(\chi, \lambda \otimes \text{Ad}(\rho))$  is a pair with underlying Hilbert space  $\mathcal{H} \otimes L^2(\Gamma)$  and stably outer conjugate to  $(\chi, \lambda)$ .

**Proof :** Firstly,  $(\chi, \lambda)$  and  $(\chi, \lambda \otimes \text{Ad}(\mathbb{1}_{L^2(\Gamma)}))$  are stably isomorphic, for let  $\mathcal{H}_0$  be another infinite dimensional Hilbert space, then we have an isomorphism of hilbert spaces  $L^2(\Gamma) \otimes \mathcal{H}_0 \rightarrow \mathcal{H}_0$  which induces an isomorphism of pairs

$$\left( \chi, (\lambda \otimes \text{Ad}(\mathbb{1}_{L^2(\Gamma)})) \otimes \text{Ad}(\mathbb{1}_{\mathcal{H}_0}) \right) \rightarrow \left( \chi, \lambda \otimes \text{Ad}(\mathbb{1}_{\mathcal{H}_0}) \right)$$

with underlying Hilbert spaces  $L^2(\Gamma) \otimes \mathcal{H}_0$  and  $\mathcal{H}_0$ .

Secondly,  $(\chi, \lambda \otimes \text{Ad}(\mathbb{1}_{L^2(\Gamma)}))$  and  $(\chi, \lambda \otimes \text{Ad}(\rho))$  are outer conjugate, for we can define

$$l(a, z) := \mathbb{1}_{\mathcal{H}} \otimes \rho(a)$$

which clearly satisfies (7). ■

Let us denote by  $\text{Par}$  the set valued contravariant functor that sends a finite group  $\Gamma$  to the set of stable outer conjugation classes of pairs, i.e.

$$(8) \quad \text{Par}(\Gamma) := \{ \text{pairs} \} /_{\text{st.out.conj.}}$$

and if  $f : \Gamma' \rightarrow \Gamma$  is a homomorphism of finite groups, then pullback defines a map  $f^* : \text{Par}(\Gamma) \rightarrow \text{Par}(\Gamma')$ .

Let  $(P, E)$  be a pair. We choose a chart to obtain an induced  $(\chi, \lambda)$  pair. Note that the homomorphism  $\chi : \Gamma \rightarrow G/N$  is independent of the choice of the chart. So we obtain another set valued functor

$$(9) \quad \text{Par}(\Gamma, \chi) := \{ \text{pairs with fixed } \chi \} /_{\text{st.out.conj.}}$$

defined on the category of homomorphisms  $\chi : \Gamma \rightarrow G/N$  whose morphisms are commutative diagrams of group homomorphisms

$$\begin{array}{ccc} \Gamma' & \longrightarrow & \Gamma \\ \downarrow \chi' & & \downarrow \chi \\ G/N & \longleftarrow & G/N. \end{array}$$

The set  $\text{Par}(\Gamma, \chi)$  becomes an abelian group by tensor product and complex conjugation of projective unitary bundles.

Let  $L \in \mathbb{Z}$  be any integer. If  $(L\chi, \lambda)$  is a pair, then it is immediate that also  $(\chi, L^*\lambda)$  is a pair, where  $(L^*\lambda)(a, z) := \lambda(a, Lz)$ . This leads to a natural map

$$(10) \quad L^* : \text{Par}(\Gamma, L\chi) \rightarrow \text{Par}(\Gamma, \chi)$$

which we define by sending  $(L\chi, \lambda)$  to  $(\chi, L^*\lambda)$ .

#### 4. PAIRS, THEIR $C^*$ -ALGEBRAS AND K-THEORY

For any pair  $(P, E)$  we can consider the  $C^*$ -algebra of continuous sections  $\Gamma(E, F)$  for the associated  $C^*$ -bundle  $F := P \times_{\text{PU}(\mathcal{H})} \mathcal{K}(\mathcal{H})$  with the compacts as fibre. Naturally, this  $C^*$ -algebra together with its inherited action  $\alpha$  of  $\Gamma$  defines a  $C^*$ -dynamical system  $(\Gamma, \alpha, \Gamma(E, F))$ . We denote by  $C^*(E, P) := \Gamma \rtimes \Gamma(E, F)$  the corresponding crossed product  $C^*$ -algebra [8]. As  $\Gamma$  is finite, we know by [6] that the equivariant K-theory of  $\Gamma(E, F)$  is given by the K-theory of its crossed product,

$$K_*^\Gamma(\Gamma(E, F)) \cong K_*(\Gamma \rtimes \Gamma(E, F)).$$

In this paper we take this last expression as definition for twisted equivariant K-theory:

**Definition 4.1** *The twisted equivariant K-theory of a pair  $(P, E)$  is*

$$K_{\Gamma, P}^*(E) := K_*(C^*(E, F)).$$

We also use the notation  $C^*(\chi, \lambda)$  and  $K_{\chi, \lambda}^*(G/N)$  for the crossed product  $C^*$ -algebra and the twisted equivariant K-theory of the pair  $(\chi, \lambda)$ , respectively.

Now, consider two pairs  $(P, E)$  and  $(P', E')$  such that their classes match in  $\text{Par}(\Gamma)$ , i.e. they are stably outer conjugate. Then the  $C^*$ -dynamical system  $(\Gamma, \alpha', \Gamma(E', F'))$  is Morita equivalent to  $(\Gamma, \alpha, \Gamma(E, F))$  [4], and therefore we know in particular that there is an isomorphism in twisted equivariant K-theory

$$K_{\Gamma, P}^*(E) \cong K_{\Gamma, P'}^*(E').$$

The sections  $\Gamma(G/N, G/N \times \mathcal{K}(\mathcal{H}))$  of a pair  $(\chi, \lambda)$  can be identified with the continuous functions  $C(G/N, \mathcal{K}(\mathcal{H}))$ , and for a function  $f : G/N \rightarrow$

$\mathcal{K}(\mathcal{H})$  the induced action of  $a \in \Gamma$  is given by

$$(\alpha_{(\chi,\lambda)}(a)f)(z) = \lambda(a,z)^{-1}[f(z + \chi(a))].$$

Now, let  $L \in \mathbb{Z}$  be any integer, and let  $(L\chi, \lambda)$  be a pair. Then we have a morphism of  $C^*$ -dynamical systems

$$(11) \quad \left( \Gamma, \alpha_{(L\chi,\lambda)}, C(G/N, \mathcal{K}(\mathcal{H})) \right) \rightarrow \left( \Gamma, \alpha_{(\chi, L^*\lambda)}, C(G/N, \mathcal{K}(\mathcal{H})) \right)$$

$$(12) \quad f \mapsto (z \mapsto f(Lz))$$

This map induces maps between the crossed products  $L^* : C^*(L\chi, \lambda) \rightarrow C^*(\chi, L^*\lambda)$  and in K-theory  $L^\# : K_{L\chi,\lambda}^*(G/N) \rightarrow K_{\chi, L^*\lambda}^*(G/N)$ . For later purpose we note that  $L^\#$  fits into the six term exact sequence

$$(13) \quad \begin{array}{ccccc} K_{\chi, L^*\lambda}^1(G/N) & \longrightarrow & K_0(C(L^*)) & \longrightarrow & K_{L\chi,\lambda}^0(G/N) \\ & & & & \downarrow L^\# \\ & & & & K_{\chi, L^*\lambda}^0(G/N) \\ & \uparrow L^\# & & & \downarrow L^\# \\ K_{L\chi,\lambda}^1(G/N) & \longleftarrow & K_1(C(L^*)) & \longleftarrow & K_{\chi, L^*\lambda}^0(G/N) \end{array}$$

which is induced by the exact sequence

$$\mathbb{1} \rightarrow C_0((0,1), C^*(\chi, L^*\lambda)) \rightarrow C(L^*) \rightarrow C^*(L\chi, \lambda) \rightarrow \mathbb{1},$$

where  $C(L^*)$  is the mapping cone of  $L^* : C^*(L\chi, \lambda) \rightarrow C^*(\chi, L^*\lambda)$ .

## 5. DYNAMICAL TRIPLES

Let  $(P, E)$  be a pair. The quotient map  $G \ni g \mapsto gN \in G/N$  induces a right  $G$ -action on  $E$ .

**Definition 5.1** A **decker** is just a continuous (right) action  $\rho$  of  $G$  on  $P$  by bundle automorphisms that lifts the induced  $G$ -action on  $E$  and commutes with the  $\Gamma$ -action of  $P$ .

The existence of deckers can be a very restrictive condition on the bundle  $P \rightarrow E$ . (See e.g. Proposition 5.1 below.) In fact, they need not exist and need not to be unique in general, but they play a central rôle in what follows, therefore we introduced an extra name.

**Definition 5.2** A **dynamical triple**  $(\rho, P, E)$  (over the point) is a pair  $(P, E)$  together with a decker  $\rho$ .

Equivalently, a dynamical triple is a  $\Gamma \times G$ -equivariant sequence

$$\Gamma \times G \curvearrowright \left( \begin{array}{c} P \circlearrowright \text{PU}(\mathcal{H}) \\ \downarrow \\ E \circlearrowright G/N \\ \downarrow \\ * \end{array} \right)$$

of *trivialisable principal fibre bundles* where  $\Gamma \times G$  acts from the left<sup>2</sup> by bundle automorphisms such that the restricted action of  $1 \times G$  on  $E$  is the induced action.

Let  $(\rho, P, E)$  be a dynamical triple. As soon as we have chosen a chart for the pair, we see that the decker is given by a continuous 1-cocycle  $\mu : G \rightarrow C(G/N, \text{PU}(\mathcal{H}))$ , i.e. for  $g, h \in G$  and  $z \in G/N$  we have

$$(14) \quad \mu(g+h, z) = \mu(g, z+hN) \mu(h, z).$$

As the two actions of  $\Gamma$  and  $G$  commute, we directly obtain the relation

$$(15) \quad \lambda(a, z+gN) \mu(g, z) = \mu(g, z+\chi(a)) \lambda(a, z)$$

between  $\mu$  and  $(\chi, \lambda)$ .

Conversely, if we are given a tuple  $(\chi, \lambda, \mu)$  satisfying (14), (2), (3) and (15), we obtain the structure of a dynamical triple on  $G/N \times \text{PU}(\mathcal{H}) \rightarrow G/N \rightarrow *$ . We therefore call such tuples also **dynamical triples**.

Equation (15) can be used to give a complete answer to the existence of deckers in the case of  $N = 0$ , i.e.  $G = G/N$ .

**Proposition 5.1** *Assume  $N = 0$ . Let  $P \rightarrow E \xrightarrow{p} *$  be a pair. Then a decker exists if and only if there is a  $\Gamma$ -equivariant bundle isomorphism  $P \cong p^*P'$  for a  $\Gamma$ -equivariant  $\text{PU}(\mathcal{H})$ -bundle  $P' \rightarrow *$ .*

**Proof :** If  $P \cong p^*P'$  for some  $\text{PU}(\mathcal{H})$ -bundle  $P' \rightarrow *$  we obtain a decker by the  $G$ -action on the first entry of the fibered product  $p^*P' = E \times P'$ .

Conversely, we fix a chart by two isomorphisms  $E \cong G/N$  and  $P \cong G/N \times \text{PU}(\mathcal{H})$  to get an induced triple  $(\mu, \chi, \lambda)$ . Then we define  $\lambda' : \Gamma \rightarrow \text{PU}(\mathcal{H})$  by  $\lambda'(a) := \mu(\chi(a), 0)^{-1} \lambda(a, 0)$ , for  $a \in \Gamma$ . This is well-defined since  $G = G/N$ .

Claim 1 :  $\lambda'$  is a homomorphism, i.e. a projective unitary representation.

Proof : Let  $a, b \in \Gamma$ , then

$$\begin{aligned} & \lambda'(b) \lambda'(a) \\ &= \mu(\chi(b), 0)^{-1} \lambda(b, 0) \mu(\chi(a), 0)^{-1} \lambda(a, 0) \\ &\stackrel{(15)}{=} \underbrace{\mu(\chi(b), 0)^{-1} \mu(\chi(a), \chi(b)+0)^{-1}}_{\stackrel{(14)}{=} \mu(\chi(a)+\chi(b), 0)^{-1}} \underbrace{\lambda(b, 0+\chi(a)) \lambda(a, 0)}_{\stackrel{(3)}{=} \lambda(ba, 0)} \\ &= \lambda'(ba). \end{aligned}$$

Thus,  $P' := \text{PU}(\mathcal{H}) \rightarrow *$  becomes  $\Gamma$ -equivariant.

Claim 2 :  $P \cong p^*P'$ .

Proof : The  $\Gamma$ -action on  $p^*P' = E \times \text{PU}(\mathcal{H}) \cong G/N \times \text{PU}(\mathcal{H})$  is given by the cocycle

$$\Gamma \ni a \mapsto \lambda'(a) \in \text{PU}(\mathcal{H}) \subset C(G/N, \text{PU}(\mathcal{H})).$$

<sup>2</sup> $G$  is commutative so any right action is a left action.

We define an isomorphism  $f : P \rightarrow p^*P'$  by use of the chart by

$$\begin{array}{ccc} P & \xrightarrow{f} & p^*P' \\ \cong \downarrow & & \downarrow \cong \\ G/N \times \text{PU}(\mathcal{H}) & \longrightarrow & G/N \times \text{PU}(\mathcal{H}) \\ (z, U) & \longmapsto & (z, \mu(z, 0)^{-1}U). \end{array}$$

This is in fact a  $\Gamma$ -equivariant isomorphism since from (15) it follows for  $G = G/N$  and  $a \in \Gamma$

$$\begin{aligned} \mu(z, 0)^{-1} &= \lambda(a, 0)^{-1} \mu(z, \chi(a))^{-1} \lambda(a, z) \\ &= \lambda(a, 0)^{-1} \mu(\chi(a), 0) \mu(z + \chi(a), 0)^{-1} \lambda(a, z) \\ &= \lambda'(a)^{-1} \mu(z + \chi(a), 0)^{-1} \lambda(a, z). \end{aligned}$$

Thus the isomorphism  $f$  commutes with the action of  $\Gamma$ . ■

We introduce the notions of (iso-)morphism, stable isomorphism, outer conjugation and stable outer conjugation in the same way as we did for pairs, we just have to replace the group  $\Gamma$  and its actions by  $\Gamma \times G$  everywhere. We then have a similar diagram of implications

$$\begin{array}{ccc} \text{isomorphism} & \Longrightarrow & \text{outer conjugation} \\ \text{of dyn. triples} & & \text{of dyn. triples} \\ \Downarrow & & \Downarrow \\ \text{stable isomorphism} & \Longrightarrow & \text{stable outer conjugation} \\ \text{of dyn. triples} & & \text{of dyn. triples} \end{array}$$

We make the notion of outer conjugation more explicit in the next lemma. Let  $(\chi, \lambda, \mu)$  be a dynamical triple. The joint action of  $\Gamma \times G$  is described by the cocycle  $v(a, g, z) := \lambda(a, z + gN)\mu(g, z)$ , and outer conjugate triples are given by  $v'(a, g, z) := v(a, g, z) \text{Ad}(n(a, g, z))$ , wherein  $n : \Gamma \times G \times G/N \rightarrow \text{U}(\mathcal{H})$  is continuous and satisfies

$$n(yx, z) = v(x, z)^{-1} [n(y, z \cdot x)] n(x, z),$$

for  $x, y \in \Gamma \times G$ , and  $z \cdot x := z + \chi(a) + gN$  if  $x = (a, g)$ . We can characterise outer conjugate triples in terms of the cocycles  $\lambda$  and  $\mu$ :

**Lemma 5.1** *Two dynamical triples  $(\chi, \lambda, \mu)$  and  $(\chi', \lambda', \mu')$  (with same underlying Hilbert space  $\mathcal{H}$ ) are outer conjugate if and only if (up to isomorphism)*

$$\chi = \chi', \quad \lambda' = \lambda \text{Ad}(l), \quad \mu' = \mu \text{Ad}(m),$$

where  $l : \Gamma \times G/N \rightarrow \mathbf{U}(\mathcal{H})$  and  $m : G \times G/N \rightarrow \mathbf{U}(\mathcal{H})$  are continuous, unitary functions such that the (joint) cocycle condition

$$\begin{aligned} l(ba, z) &= \lambda(b, z)^{-1} [l(b, z + \chi(a))] l(a, z) \\ m(g+h, z) &= \mu(g, z)^{-1} [m(h, z + gN)] m(g, z) \\ \lambda(a, z)^{-1} [m(g, z + \chi(a))] l(a, z) &= \mu(g, z)^{-1} [l(a, z + gN)] m(g, z) \end{aligned}$$

is satisfied.

**Proof :** To obtain  $l$  and  $m$  from  $n$  one can define

$$l(a, z) := n(a, 0, z) \quad \text{and} \quad m(g, z) := n(1, g, z).$$

Conversly, if  $l$  and  $m$  are given, one can define

$$n(a, g, z) := \mu(g, z)^{-1} [l(a, z + gN)] m(g, z).$$

It is then easily checked that the functions defined in this way satisfy the corresponding cocycle conditions.  $\blacksquare$

We illustrate our notions by two examples.

**Example 5.1** Let  $(\chi, \lambda, \mu)$  be a dynamical triple, and let  $\lambda_G : G \rightarrow \mathbf{U}(L^2(G))$  be the left regular representation of  $G$ , and let  $\rho : \Gamma \rightarrow \mathbf{U}(L^2(\Gamma))$  be the right regular representation of  $\Gamma$ .

Then the two triples  $(\chi, \lambda, \mu)$  and  $(\chi, \lambda \otimes \text{Ad}(\mathbb{1}_{L^2(G)}) \otimes \text{Ad}(\rho), \mu \otimes \text{Ad}(\lambda_G) \otimes \text{Ad}(\mathbb{1}_{L^2(\Gamma)}))$  are stably outer conjugate.

**Proof :** The argument is the same as in Example 3.1. The triple  $(\chi, \lambda, \mu)$  and its stabilisation  $(\chi, \lambda \otimes \mathbb{1} \otimes \mathbb{1}, \mu \otimes \mathbb{1} \otimes \mathbb{1})$  are stably isomorphic, and the triples  $(\chi, \lambda \otimes \mathbb{1}, \mu \otimes \mathbb{1})$  and  $(\chi, \lambda \otimes \text{Ad}(\mathbb{1}_{L^2(G)}) \otimes \text{Ad}(\rho), \mu \otimes \text{Ad}(\lambda_G) \otimes \text{Ad}(\mathbb{1}_{L^2(\Gamma)}))$  are outer conjugate by

$$l(a, z) := \mathbb{1}_{L^2(G)} \otimes \rho(a), \quad m(g, z) := \lambda_G(g) \otimes \mathbb{1}_{L^2(\Gamma)}$$

which satisfy the conditions of Lemma 5.1.  $\blacksquare$

**Example 5.2** Let  $(\chi, \lambda, \mu)$  be a dynamical triple and let  $(\chi, \lambda \text{Ad}(l))$  be an outer conjugate pair to  $(\chi, \lambda)$  which is also isomorphic to  $(\chi, \lambda)$  by a morphism

$$\begin{array}{ccc} \lambda \text{Ad}(l) \circlearrowleft G/N \times \text{PU}(\mathcal{H}) & \xrightarrow{v} & G/N \times \text{PU}(\mathcal{H}) \circlearrowleft \lambda, \mu \\ \downarrow & & \downarrow \\ G/N & \xrightarrow{=} & G/N \\ \downarrow & & \downarrow \\ * & \xrightarrow{=} & * \end{array}$$

which has vanishing class  $[v] = 0 \in \check{H}_\Gamma^1(G/N, \underline{\mathbf{U}(1)})$  (see Appendix B for the definition of  $[v]$ ).

Then the triples  $(\chi, \lambda, \mu)$  and  $(\chi, \lambda \text{Ad}(l), v^* \mu)$  are outer conjugate.

**Proof :** Because the class of  $v$  vanishes, we can assume without restriction that it is implemented by continuous, unitary functions  $u : G/N \rightarrow U(\mathcal{H})$  such that  $\lambda(a, z)^{-1}[u(z + \chi(a))] = u(z) l(a, z)^{-1}$ . The pulled back cocycle  $v^*\mu$  is given by

$$\begin{aligned} (v^*\mu)(g, z) &= \text{Ad}(u(z + gN)^{-1})\mu(g, z) \text{Ad}(u(z)) \\ &= \mu(g, z) \text{Ad}\left(\underbrace{\mu(g, z)^{-1}[u(z + gN)^{-1}]u(z)}_{=: m(g, z)}\right). \end{aligned}$$

We have to check that  $l$  and  $m$  satisfy the joint cocycle condition of Lemma 5.1. The first equality just involves  $l$ , and therefore holds by assumption. We compute the other two

$$\begin{aligned} &m(g + h, z) \\ &= \mu(g + h, z)^{-1}[u(z + gN + hN)^{-1}] u(z) \\ &= \mu(g, z)^{-1}\left[\mu(h, z + gN)^{-1}[u(z + gN + hN)^{-1}]\right] u(z) \\ &= \mu(g, z)^{-1}\left[\mu(h, z + gN)^{-1}[u(z + gN + hN)^{-1}] u(z + gN)\right] \\ &\quad \mu(g, z)^{-1}[u(z + gN)^{-1}] u(z) \\ &= \mu(g, z)^{-1}[m(h, z + gN) m(g, z)] \end{aligned}$$

which proves the second equality; and

$$\begin{aligned} &\lambda(a, z)^{-1}[m(g, z + \chi(a))]l(a, z) \\ &= \lambda(a, z)^{-1}\left[\mu(g, z + \chi(a))^{-1}[u(z + gN + \chi(a))^{-1}] u(z + \chi(a))\right]l(a, z) \\ &\stackrel{(15)}{=} \mu(g, z)^{-1}\lambda(a, z + gN)^{-1}[u(z + gN + \chi(a))^{-1}] \lambda(a, z)^{-1}[u(z + \chi(a))]l(a, z) \\ &\stackrel{[v]=0}{=} \mu(g, z)^{-1}[l(a, z + gN)u(z + gN)^{-1}]u(z) \\ &= \mu(g, z)^{-1}[l(a, z + gN)]m(g, z) \end{aligned}$$

proves the last equality. ■

By Dyn we denote the set valued functor that sends a finite group  $\Gamma$  to the set of equivalence classes of stably outer conjugate dynamical triples, i.e.

$$(16) \quad \text{Dyn}(\Gamma) := \{ \text{dynamical triples} \} /_{\text{st.out.conj.}}$$

In the same manner as we did for the functor Par we can define a subfunctor of Dyn by

$$(17) \quad \text{Dyn}(\Gamma, \chi) := \{ \text{dynamical triples with fixed } \chi \} /_{\text{st.out.conj.}}$$

This yields a decomposition  $\text{Dyn}(\Gamma) = \coprod_{\chi} \text{Dyn}(\Gamma, \chi)$ . For fixed  $\chi$ , each  $\text{Dyn}(\Gamma, \chi)$  has the structure of an abelian group given by tensor product

and complex conjugation of projective unitary bundles, and by Lemma 5.1 there are well-defined forgetful maps

$$\text{Dyn}(\Gamma, \chi) \rightarrow \text{Par}(\Gamma, \chi), \quad \text{Dyn}(\Gamma) \rightarrow \text{Par}(\Gamma)$$

which extend to natural transformations

$$\text{Dyn} \rightarrow \text{Par}.$$

## 6. DUALISABLE DYNAMICAL TRIPLES

Fix a homomorphism

$$\chi : \Gamma \rightarrow G/N.$$

We consider  $L_\chi := L^\infty(G/N, \mathcal{U}(1))$  as a topological  $\Gamma$ - $G$ -bimodule in a natural fashion: The action is given by shifting the argument of a function by  $-\chi(a)$  or  $-gN$  for  $a \in \Gamma$  or  $g \in G$ . The topology on  $L_\chi$  is the strong (or weak) operator topology, i.e. we consider  $L_\chi$  as a set of unitary multiplication operators on the Hilbert space  $L^2(G/N, \mathcal{H})$ .

Let  $B^{k,l}(\Gamma, G, L_\chi)$  be the group of all Borel functions  $\Gamma^k \times G^l \rightarrow L_\chi$ , then we find a double complex

$$\begin{array}{ccccc} & \vdots & & \vdots & & \vdots & & \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \\ B^{2,0}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{2,1}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{2,2}(\Gamma, G, L_\chi) & \xrightarrow{d} & \cdots & \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \\ B^{1,0}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{1,1}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{1,2}(\Gamma, G, L_\chi) & \xrightarrow{d} & \cdots & \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \\ B^{0,0}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{0,1}(\Gamma, G, L_\chi) & \xrightarrow{d} & B^{0,2}(\Gamma, G, L_\chi) & \xrightarrow{d} & \cdots, & \end{array}$$

wherein  $d$  for the group  $G$  and  $\delta_\chi$  for  $\Gamma$  are the group cohomological boundary operators. We denote the resulting total complex by  $(B^\bullet(\Gamma, G, L_\chi), \partial_\chi)$ , i.e.

$$B^p(\Gamma, G, L_\chi) := \bigoplus_{p=k+l} B^{k,l}(\Gamma, G, L_\chi)$$

and (if we write the group law in  $L$  additive for a moment)

$$\partial_\chi|_{B^{k,l}(\Gamma, G, L_\chi)} := d + (-1)^l \delta_\chi.$$

We denote by  $H_{\text{Bor}}^\bullet(\Gamma, G, L_\chi)$  the corresponding cohomology groups.

The algebraic structure of a dynamical triple gives rise to a 2-cohomology class as we explain now. Let  $(\chi, \lambda, \mu)$  be a dynamical triple. One should realise at this point that when we suppress the non-commutativity of  $\text{PU}(\mathcal{H})$  for a moment the cocycle conditions for  $\lambda$  and  $\mu$  together with (15) are



equivalent to  $\partial_\chi(\lambda, \mu) = 1$ . Now, we lift the cocycles to unitary valued, Borel functions which will define a 2-cocycle. In detail: Choose for both,  $\mu$  and  $\lambda$ , unitary lifts to Borel functions  $\bar{\mu} : G \rightarrow L^\infty(G/N, U(\mathcal{H}))$  and  $\bar{\lambda} : \Gamma \rightarrow L^\infty(G/N, U(\mathcal{H}))$  such that  $\text{Ad}(\bar{\mu}(g, z)) = \mu(g, z)$  and  $\text{Ad}(\bar{\lambda}(a, z)) = \lambda(a, z)$ . Then we define  $(\psi, \phi, \omega)$  by

$$(18) \quad \begin{aligned} \bar{\lambda}(b, z + \chi(a)) \bar{\lambda}(a, z) &= \bar{\lambda}(ba, z) \psi(a, b, z), \\ \bar{\lambda}(a, z + gN) \bar{\mu}(g, z) &= \bar{\mu}(g, z + \chi(a)) \bar{\lambda}(a, z) \phi(a, g, z), \\ \bar{\mu}(h, z + gN) \bar{\mu}(g, z) &= \bar{\mu}(g + h, z) \omega(g, h, z). \end{aligned}$$

Due to (3), (14) and (15), these three functions  $\psi, \phi$  and  $\omega$  are  $U(1)$ -valued and satisfy the algebraic relations (written multiplicative)

$$(19) \quad \begin{aligned} \delta_\chi \psi &= 1, \\ \delta_\chi \phi &= d\psi, \\ d\phi &= \delta_\chi \omega^{-1}, \\ d\omega &= 1 \end{aligned}$$

which is (again in additive notation) equivalent to

$$(20) \quad \partial_\chi(\psi, \phi, \omega) = 0 \in B^3(\Gamma, G, L_\chi),$$

i.e.  $(\psi, \phi, \omega)$  is a 2-cocycle. Of course, one can verify this by direct computation, but indeed it is implicitly clear, because, informally<sup>3</sup>, we have defined  $(\psi, \phi, \omega) := \partial_\chi(\bar{\lambda}, \bar{\mu})$ .

**Proposition 6.1** *The assignment  $(\chi, \lambda, \mu) \mapsto (\psi, \phi, \omega)$  constructed above defines a homomorphism of groups*

$$\text{Dyn}(\Gamma, \chi) \rightarrow H_{\text{Bor}}^2(\Gamma, G, L_\chi).$$

**Proof :** We must check that the defined total cohomology class is independent of all choices. This is simple to verify for the choice of the chart, and the choice of the lifts of the transition functions and cocycles. It is also clear that stably isomorphic triples define the same total cohomology class. By Lemma 5.1 we can directly calculate the cocycle  $(\psi', \phi', \omega')$  for an outer conjugate triple. This calculation is straight forward, and, in fact, for  $\bar{\mu}' := \bar{\mu}m$  and  $\bar{\lambda}' := \bar{\lambda}l$  we find  $(\psi', \phi', \omega') = (\psi, \phi, \omega)$ . ■

Note that we do not claim anything about the injectivity or surjectivity of the map in Proposition 6.1. We do better in the next section, where we consider dualisable triples only. For the definition of these consider the forgetful map

$$(21) \quad \begin{aligned} H_{\text{Bor}}^2(\Gamma, G, L_\chi) &\rightarrow H_{\text{Bor}}^2(G, L^\infty(G/N, U(1))) \cong H^2(N, U(1)). \\ [(\psi, \phi, \omega)] &\mapsto [\omega] \end{aligned}$$

<sup>3</sup> i.e. up to the non-comutativity of  $U(\mathcal{H})$

The corresponding class in  $H^2(N, U(1))$  is called the **Mackey obstruction** of the dynamical triple. We will have our focus on those triples which have a vanishing Mackey obstruction.

**Definition 6.1** *A dynamical triple is called **dualisable** if its Mackey obstruction vanishes.*

There is the contravariant set valued sub-functor of  $\text{Dyn}$  which sends a finite group to the stable outer conjugation classes of dualisable dynamical triples. It is denoted by  $\text{Dyn}^\dagger$ , so

$$(22) \quad \text{Dyn}^\dagger(\Gamma) \subset \text{Dyn}(\Gamma), \quad \text{Dyn}^\dagger(\Gamma, \chi) \subset \text{Dyn}(\Gamma, \chi).$$

## 7. THE CLASSIFICATION OF DUALISABLE DYNAMICAL TRIPLES

We start similar to the previous section, but we stick to the continuous setting rather than to the Borel setting. For a homomorphism

$$\chi : \Gamma \rightarrow G/N$$

we consider the topological  $\Gamma$ - $G$ -bimodule  $M_\chi := C(G/N, U(1))$ , where  $(a, g) \in \Gamma \times G$  acts by shift with  $-\chi(a) - gN$  in the arguments of the functions. The topology on  $M_\chi$  is the compact-open topology. Let  $C^{k,l}(\Gamma, G, M_\chi)$  be the group of all continuous functions  $\Gamma^k \times G^l \rightarrow M_\chi$ . Again we have a double complex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & \\ C^{2,0}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{2,1}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{2,2}(\Gamma, G, M_\chi) & \xrightarrow{d} & \dots \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & \\ C^{1,0}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{1,1}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{1,2}(\Gamma, G, M_\chi) & \xrightarrow{d} & \dots \\ & \delta_\chi \uparrow & & \delta_\chi \uparrow & & \delta_\chi \uparrow & \\ C^{0,0}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{0,1}(\Gamma, G, M_\chi) & \xrightarrow{d} & C^{0,2}(\Gamma, G, M_\chi) & \xrightarrow{d} & \dots, \end{array}$$

We denote the resulting total cohomology groups by  $H_{\text{cont}}^\bullet(\Gamma, G, M_\chi)$ . For the classification of dualisable dynamical triples we have to consider the kernel of the forgetful map

$$H_{\text{cont}}^2(\Gamma, G, M_\chi) \rightarrow H_{\text{cont}}^2(G, C(G/N, U(1)))$$

which we denote by  $H_{\text{cont}}^{2,\dagger}(\Gamma, G, M_\chi)$ .

Now, we can state the classification theorem of dualisable dynamical triples.

**Theorem 7.1** *There is a natural isomorphism of groups*

$$\text{Dyn}^\dagger(\Gamma, \chi) \xrightarrow{\cong} H_{\text{cont}}^{2,\dagger}(\Gamma, G, M_\chi).$$

The proof of Theorem 7.1 is the content of the remainder of this section.

Consider a dualisable dynamical triple  $(\chi, \lambda, \mu)$ . As its Mackey obstruction vanishes, we can find a unitary Borel lift  $\bar{\mu} : G \rightarrow L^\infty(G/N, \mathbf{U}(\mathcal{H}))$  of  $\mu$  which satisfies again the cocycle condition

$$(23) \quad \bar{\mu}(g+h, z) = \bar{\mu}(h, z+gN) \bar{\mu}(g, z).$$

The function  $\bar{\mu}(g, -) \in L^\infty(G/N, \mathbf{U}(\mathcal{H}))$  is a multiplication operator which acts by  $(\bar{\mu}(g, -)F)(z) := \bar{\mu}(g, z)F(z)$  for  $F \in L^2(G/N) \otimes \mathcal{H}$ . We also may consider  $\bar{\mu}(\cdot, -)$  or  $\bar{\mu}(\cdot, z)$  as multiplication operators on  $L^2(G) \otimes L^2(G/N) \otimes \mathcal{H}$  or  $L^2(G) \otimes \mathcal{H}$  respectively.

As a matter of fact, the cocycle condition (23) implies some useful continuity properties of the functions mentioned. This is the content of the next technical lemmata.

**Lemma 7.1** *The map*

$$\begin{aligned} G &\rightarrow \mathbf{U}(L^2(G/N) \otimes \mathcal{H}) \\ g &\mapsto \bar{\mu}(g, -) \end{aligned}$$

*is continuous.*

**Proof :** We consider  $\mathbb{1} \otimes \bar{\mu}(g, -) \in \mathbf{U}(L^2(G) \otimes L^2(G/N) \otimes \mathcal{H})$ . Using the cocycle condition we have

$$\begin{aligned} \mathbb{1} \otimes \bar{\mu}(g, -) &= \bar{\mu}(\cdot, - + gN)^{-1} \bar{\mu}(g + \cdot, -) \\ &= \lambda_{G/N}(-gN) \bar{\mu}(\cdot, -)^{-1} \lambda_{G/N}(gN) \lambda_G(-g) \bar{\mu}(\cdot, -) \lambda_G(g), \end{aligned}$$

where  $\lambda_{G/N}$  and  $\lambda_G$  are the left regular representations of the groups  $G/N$  and  $G$  respectively. As the left regular representations are (strongly) continuous the assertion follows.  $\blacksquare$

In the next lemma  $\sigma : G/N \rightarrow G$  is a Borel section of the quotient map  $G \rightarrow G/N$ .

**Lemma 7.2** *The maps*

$$\begin{aligned} G/N &\rightarrow \mathbf{PU}(L^2(G) \otimes \mathcal{H}) \\ z &\mapsto \text{Ad}(\bar{\mu}(\cdot, z)) \end{aligned}$$

*and*

$$\begin{aligned} G/N &\rightarrow \mathbf{PU}(L^2(G/N) \otimes \mathcal{H}) \\ z &\mapsto \text{Ad}(\bar{\mu}(\sigma(\cdot), z)) \end{aligned}$$

*are continuous.*

**Proof :** Let  $z_\alpha \rightarrow z$  be a converging net. Let  $x_\alpha := z_\alpha - z$  and choose  $g_\alpha \rightarrow 0 \in G$  such that<sup>4</sup>  $g_\alpha N = x_\alpha$ . Then

$$\begin{aligned} \text{Ad}(\bar{\mu}(\cdot, z_\alpha)) &= \text{Ad}(\bar{\mu}(\cdot, z + x_\alpha)) \\ &= \text{Ad}(\bar{\mu}(\cdot + g_\alpha, z)) \text{Ad}(\bar{\mu}(g_\alpha, z)^{-1}) \\ &= \text{Ad}(\bar{\mu}(\cdot + g_\alpha, z)) \mu(g_\alpha, z)^{-1} \\ &= \text{Ad}(\lambda_G(-g_\alpha) \bar{\mu}(\cdot, z) \lambda_G(g_\alpha)) \mu(g_\alpha, z)^{-1} \\ &\rightarrow \text{Ad}(\bar{\mu}(\cdot, z)) \mu(0, z)^{-1}. \end{aligned}$$

As  $\mu(0, z) = \mathbb{1}$ , the continuity of the first map follows.

To prove the continuity of the second map we show first that

$$\begin{aligned} \sigma^* : L^\infty(G, \text{U}(\mathcal{H})) &\rightarrow L^\infty(G/N, \text{U}(\mathcal{H})) \\ \nu(\cdot) &\mapsto \nu(\sigma(\cdot)) \end{aligned}$$

is continuous. Indeed, let  $f \in L^2(G/N)$  and let  $\varphi$  be the characteristic function of  $\sigma(G/N) \subset G$ , so  $g \mapsto f(gN)\varphi(g)$  is a function in  $L^2(G)$ . Let  $\nu_n(\cdot) \rightarrow \nu(\cdot) \in L^\infty(G, \text{U}(\mathcal{H}))$  be a converging sequence. Then

$$\begin{aligned} \|\nu_n(\sigma(\cdot))f(\cdot) - \nu(\sigma(\cdot))f(\cdot)\|^2 &= \int_{G/N} |\nu_n(\sigma(z))f(z) - \nu(\sigma(z))f(z)|^2 dz \\ &= \int_G |\nu_n(g)f(gN)\varphi(g) - \nu(g)f(gN)\varphi(g)|^2 dg \\ &\rightarrow 0, \text{ for } n \rightarrow \infty. \end{aligned}$$

Because  $\text{Ad} \circ \sigma^*$  is constant along the orbits of  $\text{U}(1)$  the dotted arrow in

$$\begin{array}{ccc} L^\infty(G, \text{U}(\mathcal{H})) & \xrightarrow{\sigma^*} & L^\infty(G/N, \text{U}(\mathcal{H})) \\ \downarrow \text{Ad} & & \downarrow \text{Ad} \\ PL^\infty(G, \text{U}(\mathcal{H})) & \cdots \cdots \xrightarrow{\sigma^*} & PL^\infty(G/N, \text{U}(\mathcal{H})) \end{array}$$

is well-defined and continuous by the universal property of the quotient map.

This proves the Lemma. ■

With these two lemmata at hand we can proof the following proposition.

**Proposition 7.1** *For each dualisable dynamical triple  $(\chi, \lambda, \mu)$  with underlying Hilbert space  $\mathcal{H}$  there exists a stably outer conjugate triple  $(\chi, \lambda', \mu')$  with underlying Hilbert space  $\mathcal{H}' := L^2(G/N) \otimes \mathcal{H}$  such that the cocycles  $\lambda'$  and  $\mu'$  permit continuous lifts*

$$\begin{array}{ccc} & \text{U}(\mathcal{H}') & \\ & \nearrow \bar{\lambda}' & \downarrow \\ \Gamma \times G/N & \xrightarrow{\lambda'} & \text{PU}(\mathcal{H}') \end{array} \quad \text{and} \quad \begin{array}{ccc} & \text{U}(\mathcal{H}') & \\ & \nearrow \bar{\mu}' & \downarrow \\ G \times G/N & \xrightarrow{\mu'} & \text{PU}(\mathcal{H}') \end{array},$$

<sup>4</sup> Such  $g_\alpha$  exist – take a local section of the quotient  $G \rightarrow G/N$ .

where  $\bar{\mu}'$  still satisfies the cocycle condition (23).

**Proof :** The triple  $(\chi, \mathbb{1} \otimes \lambda, \mathbb{1} \otimes \mu)$  with underlying Hilbert space  $L^2(G/N) \otimes \mathcal{H}$  is stably isomorphic to  $(\chi, \lambda, \mu)$ . Let

$$\begin{array}{ccc}
 G/N \times \mathrm{PU}(\mathcal{H}') & \xrightarrow{\theta} & G/N \times \mathrm{PU}(\mathcal{H}') \circlearrowleft \mathbb{1} \otimes \lambda, \mathbb{1} \otimes \mu \\
 \downarrow & & \downarrow \\
 G/N & \xrightarrow{=} & G/N \\
 \downarrow & & \downarrow \\
 * & \xrightarrow{=} & *
 \end{array}$$

be the bundle isomorphism which is given by  $\theta(z, U) := (z, \mathrm{Ad}(\bar{\mu}(\sigma(-), z))^{-1}U)$ . By Lemma 7.2 this is well-defined. We define

$$(\chi, \lambda', \mu') := \theta^*(\chi, \mathbb{1} \otimes \lambda, \mathbb{1} \otimes \mu),$$

i.e.

$$\begin{aligned}
 \mu'(g, z) &:= \mathrm{Ad}(\bar{\mu}(\sigma(-), z + gN)) \mu(g, z) \mathrm{Ad}(\bar{\mu}(\sigma(-), z))^{-1} \\
 &= \mathrm{Ad}(\bar{\mu}(\sigma(-), z + gN) \bar{\mu}(g, z) \bar{\mu}(\sigma(-), z)^{-1}) \\
 &= \mathrm{Ad}(\bar{\mu}(g + \sigma(-), z + gN) \bar{\mu}(\sigma(-), z)^{-1}) \\
 &= \mathrm{Ad}(\bar{\mu}(g, z + -)) \\
 &= \mathrm{Ad}(\underbrace{\lambda_{G/N}(-z) \bar{\mu}(g, -) \lambda_{G/N}(z)}_{=: \bar{\mu}'(g, z)}) \\
 &=: \bar{\mu}'(g, z).
 \end{aligned}$$

By Lemma 7.1  $\bar{\mu}'$  is continuous, it is also clear that it satisfies (23).

It remains to show that  $\lambda'$  also possesses a continuous lift. But this follows from equation (15) which implies for  $z = 0$  that

$$\lambda'(a, gN) = \mu'(g, \chi(a)) \lambda(a, 0) \mu'(g, 0)^{-1};$$

as  $\Gamma$  is discrete, the right-hand side has a continuous lift, hence the left-hand side has.  $\blacksquare$

Proposition 7.1 enables us to define the classification map of Theorem 7.1. In fact, we define  $(\psi, \phi, 1)$  as in equation (18), but with the additional assumption, that  $\bar{\lambda}$  and  $\bar{\mu}$  are continuous. This defines

$$(24) \quad \mathrm{Dyn}^\dagger(\Gamma, \chi) \rightarrow H_{\mathrm{cont}}^{2, \dagger}(\Gamma, G, M_\chi).$$

To check that our definition is well-defined requires the same arguments as in Proposition 6.1.

To show that (24) is an isomorphism we construct a map in opposite direction. To do so, let  $(\psi, \phi, 1)$  represent an element of  $H_{\mathrm{cont}}^{2, \dagger}(\Gamma, G, M_\chi)$ . We

regard  $\psi(a, \cdot, z)$  and  $\phi(\cdot, g, z)$  as unitary multiplication operators on  $L^2(\Gamma)$ . Let  $\rho : \Gamma \rightarrow \mathcal{U}(L^2(\Gamma))$  be the right regular representation, then we define

$$\begin{aligned}\lambda_\psi(a, z) &:= \text{Ad}(\psi(a, \cdot, z)\rho(a)) \\ \mu_\phi(g, z) &:= \text{Ad}(\phi(\cdot, g, z)).\end{aligned}$$

The cocycle condition  $\partial_\chi(\psi, \phi, 1) = 1$  is equivalent to

$$\begin{aligned}\psi(b, \cdot, z + \chi(a)\rho(b)) \psi(a, \cdot, z)\rho(a) &= \psi(ba, \cdot, z)\rho(ba) \underbrace{\psi(a, b, z)}_{\in \mathcal{U}(1)}, \\ \psi(a, \cdot, z + gN)\rho(a) \phi(\cdot, g, z) &= \phi(\cdot, g, z + \chi(a)) \psi(a, \cdot, z)\rho(a) \underbrace{\phi(a, g, z)}_{\in \mathcal{U}(1)}, \\ \phi(\cdot, h, z + gN) \phi(\cdot, g, z) &= \phi(\cdot, g + h, z).\end{aligned}$$

By taking  $\text{Ad} : \mathcal{U}(L^2(\Gamma)) \rightarrow \text{PU}(L^2(\Gamma))$  on both sides of the equalities above, it follows that  $(\chi, \lambda_\psi, \mu_\phi)$  is a dynamical triple which is mapped to  $[\psi, \phi, 1]$  under (24). In this way we obtain a map

$$(25) \quad H_{\text{cont}}^{2,+}(\Gamma, G, M_\chi) \rightarrow \text{Dyn}^+(\Gamma, \chi)$$

such that the composition with (24) is the identity on  $H_{\text{cont}}^{2,+}(\Gamma, G, M_\chi)$ .

To conclude that (24) is an isomorphism we show that the composition of (24) with (25) is the identity on  $\text{Dyn}^+(\Gamma, \chi)$ . So take a triple  $(\chi, \lambda, \mu)$  (with underlying Hilbert space  $\mathcal{H}$ ) which permits continuous lifts of  $\lambda$  and  $\mu$ , then define  $(\psi, \phi, 1)$  as in (18) and  $\lambda_\psi, \mu_\phi$  (with underlying Hilbert space  $L^2(\Gamma)$ ) as above. We claim that  $(\chi, \lambda, \mu)$  and  $(\chi, \lambda_\psi, \mu_\phi)$  are stably outer conjugate. In fact,  $(\chi, \lambda, \mu)$  and  $(\chi, \lambda \text{Ad}(\rho), \mu \otimes \mathbb{1})$  (with underlying Hilbert space  $\mathcal{H} \otimes L^2(\Gamma)$ ) are stably outer conjugate. Now, note that (18) implies

$$\begin{aligned}\lambda(a, z)\text{Ad}(\rho(a)) &= \text{Ad}\left(\bar{\lambda}(\cdot, z + \chi(a))^{-1} \bar{\lambda}(\cdot, a, z) \psi(a, \cdot, z) \rho(a)\right) \\ &= \text{Ad}\left(\bar{\lambda}(\cdot, z + \chi(a))\right)^{-1} \lambda_\psi(a, z) \text{Ad}\left(\bar{\lambda}(\cdot, z)\right)\end{aligned}$$

and

$$\begin{aligned}\mu(g, z) &= \text{Ad}\left(\bar{\lambda}(\cdot, z + gN)^{-1} \bar{\mu}(g, z + \chi(\cdot)) \bar{\lambda}(\cdot, a, z) \phi(\cdot, g, z)\right) \\ &= \text{Ad}\left(\bar{\lambda}(\cdot, z + gN)\right)^{-1} \mu_\phi(g, z) \underbrace{\text{Ad}\left(\bar{\mu}(g, z + \chi(\cdot))\right)}_{=: m(g, z)} \text{Ad}\left(\bar{\lambda}(\cdot, a, z)\right).\end{aligned}$$

Thus, we see that there is an isomorphism of dynamical triples

$$\begin{array}{ccc}
\mathbb{1} \otimes \lambda_\psi, \mu_\phi \text{Ad}(m) \circlearrowleft G/N \times \text{PU}(\mathcal{H}_\Gamma) & \xrightarrow{\theta} & G/N \times \text{PU}(\mathcal{H}_\Gamma) \circlearrowleft \lambda \text{Ad}(\rho), \mu \otimes \mathbb{1} \\
\downarrow & & \downarrow \\
G/N & \xrightarrow{=} & G/N \\
\downarrow & & \downarrow \\
* & \xrightarrow{=} & *,
\end{array}$$

where  $\mathcal{H}_\Gamma := \mathcal{H} \otimes L^2(\Gamma)$  and  $\theta(z, U) := (z, \text{Ad}(\bar{\lambda}(\cdot, z))U)$ . But the triple  $(\chi, \mathbb{1} \otimes \lambda_\psi, \mu_\phi \text{Ad}(m))$  is stably outer conjugate to  $(\chi, \lambda_\psi, \mu_\phi)$ , because  $m$  satisfies the cocycle condition of Lemma 5.1. This shows that the composition of (24) with (25) is the identity on  $\text{Dyn}^+(\Gamma, \chi)$ .

We have just proven Theorem 7.1.

## 8. DUAL PAIRS AND TRIPLES

Of course, in the whole discussion we made so far the group  $G$  and its subgroup  $N$  can be replaced by the dual group  $\widehat{G} := \text{Hom}(G, \text{U}(1))$  and the annihilator  $N^\perp := \{\hat{g} | \hat{g}|_N = 1\} \subset \widehat{G}$  of  $N$  everywhere. This is meaningful as  $N^\perp \cong \widehat{G/N}$  is discrete, and  $\widehat{G}/N^\perp \cong \widehat{N}$  is compact.

Let  $\Gamma$  be a finite group and  $\mathcal{H}$  be a separable Hilbert space.

- Definition 8.1**
- (1) A **dual pair**  $(\widehat{P}, \widehat{E})$  with underlying Hilbert space  $\mathcal{H}$  is a  $\Gamma$ -equivariant sequence  $\widehat{P} \rightarrow \widehat{E} \rightarrow *$ , wherein  $\widehat{E} \rightarrow *$  and  $\widehat{P} \rightarrow \widehat{E}$  are trivialisable principal fibre bundles with structure groups  $\widehat{G}/N^\perp, \text{PU}(\mathcal{H})$  respectively, and  $\Gamma$  acts by bundle automorphisms.
  - (2) A **dual decker**  $\hat{\rho}$  is an action by bundle automorphisms of  $\widehat{G}$  on  $\widehat{P}$  which lifts the induced  $\widehat{G}$ -action on  $\widehat{E}$  and commutes with the given  $\Gamma$ -action of the pair.
  - (3) A **dual dynamical triple**  $(\hat{\rho}, \widehat{P}, \widehat{E})$  is a dual pair  $(\widehat{P}, \widehat{E})$  together with a dual decker  $\hat{\rho}$ .
  - (4) A **dualisable dual dynamical triple** is a dual dynamical triple whose Mackey obstruction vanishes.

It is clear how to define  $\widehat{\text{Par}}(\Gamma), \widehat{\text{Dyn}}(\Gamma)$  and  $\widehat{\text{Dyn}}^+(\Gamma)$ , and for a homomorphism

$$\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$$

we also define  $\widehat{\text{Par}}(\Gamma, \hat{\chi}), \widehat{\text{Dyn}}(\Gamma, \hat{\chi})$  and  $\widehat{\text{Dyn}}^+(\Gamma, \hat{\chi})$  as before.

All statements we have achieved so far translate to the dual setting in the obvious way. In particular we stress the classification theorem for dualisable dual dynamical triples (cp. Theorem 7.1):

**Theorem 8.1** *There is a natural isomorphism of groups*

$$\widehat{\text{Dyn}}^\dagger(\Gamma, \hat{\chi}) \cong H_{\text{cont}}^{2,\dagger}(\Gamma, \widehat{G}, M_{\hat{\chi}}).$$

Here  $M_{\hat{\chi}} := C(\widehat{G}/N^\perp, \text{U}(1))$  is the dual counterpart of  $M_\chi$  introduced in section 7, i.e. it has the  $\Gamma$ - $\widehat{G}$ -bimodule structure given by shift with  $-\hat{\chi}(a) - \hat{g}N^\perp$  in the arguments of the functions, for  $a \in \Gamma, \hat{g} \in \widehat{G}$ .

### 9. THE DUALITY THEORY OF DYNAMICAL TRIPLES

In this section we show that the two functors  $\text{Dyn}^\dagger$  and  $\widehat{\text{Dyn}}^\dagger$  are naturally isomorphic functors. To do so we construct an assignment  $(\chi, \psi, \phi) \mapsto (\hat{\chi}, \hat{\psi}, \hat{\phi})$  and then use the classification theorems of dynamical and dual dynamical triples, Theorem 7.1 and Theorem 8.1.

Let  $(\chi, \lambda, \mu)$  be a dualisable dynamical triple represented by  $(\psi, \phi, 1)$ . As  $d\phi = 1$  we have

$$\begin{aligned} \phi(a, g, hN) \phi(a, h, 0) &= \phi(a, g + h, 0) \\ &= \phi(a, h + g, 0) \\ &= \phi(a, h, gN) \phi(a, g, 0) \end{aligned}$$

which implies for  $g = n \in N$  that  $\phi(a, n, hN) = \phi(a, n, 0)$  holds, hence  $z \mapsto \phi(a, n, z)$  is a constant. Further,  $N \ni n \mapsto \phi(a, n, 0)$  is a continuous homomorphism. Thus there exists a map  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp (\cong \widehat{N})$  such that

$$(26) \quad \langle \hat{\chi}(a), n \rangle = \phi(a, n, 0)^{-1}$$

**Proposition 9.1**  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$  is a homomorphism of groups.

**Proof :** We have  $\delta_\chi \phi = d\psi$ , explicitly this reads for  $g \in G, a, b \in \Gamma$

$$\phi(b, g, \chi(a)) \phi(ba, g, 0)^{-1} \phi(a, g, 0) = \psi(a, b, gN) \psi(a, b, 0)^{-1},$$

and for  $g = n \in N$  the right-hand side vanishes. ■

We choose a lift  $\bar{\chi}$  of  $\chi$  in

$$\begin{array}{ccc} & G & \\ \bar{\chi} \nearrow & & \downarrow \\ \Gamma & \xrightarrow{\chi} & G/N \end{array}$$

and then we define

$$(27) \quad \begin{aligned} \hat{\psi}(a, b, \hat{z}) &:= \psi(a, b, 0) \phi(b, \bar{\chi}(a), 0)^{-1} \\ &\quad \langle \hat{\chi}(ba) + \hat{z}, \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b) \rangle \end{aligned}$$

and

$$(28) \quad \hat{\phi}(a, \hat{g}, \hat{z}) := \langle \hat{g}, \bar{\chi}(a) \rangle^{-1},$$

wherein the scalar product in the definition of  $\hat{\psi}$  is the paring of  $\widehat{G}/N^\perp$  with  $N$ , and paring in the definition of  $\hat{\phi}$  is between  $\widehat{G}$  and  $G$ .



**Proposition 9.2** (In multiplicative notation) the cocycle condition

$$\partial_{\hat{\chi}}(\hat{\psi}, \hat{\phi}, 1) = 1$$

is satisfied, and the class

$$[\hat{\psi}, \hat{\phi}, 1] \in H_{\text{cont}}^{2,+}(\Gamma, \widehat{G}, M_{\hat{\chi}})$$

is independent of the choices of  $\bar{\chi}$  and the representative  $(\hat{\psi}, \hat{\phi}, 1)$ .

**Proof :** The proof is a straight forward calculation.

Firstly, the equality  $\hat{d}\hat{\phi} = 1$  is satisfied as  $\widehat{G} \ni g \mapsto \hat{\phi}(a, \hat{g}, \hat{z})$  is a homomorphism.

Secondly, we have  $\hat{d}\hat{\psi} = \delta_{\hat{\chi}}\hat{\phi}$  as

$$\begin{aligned} (\hat{d}\hat{\psi})(a, b, \hat{g}, \hat{z}) &= \langle gN, \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b) \rangle \\ &= (\delta_{\hat{\chi}}\hat{\phi})(a, b, \hat{g}, \hat{z}). \end{aligned}$$

Thirdly, we show that  $\delta_{\hat{\chi}}\hat{\psi} = 1$  which is the most lengthy equation to check. The proof requires the cocycle condition  $\delta_{\chi}\psi = 1, d\psi = \delta_{\chi}\phi, d\phi = 1$ . In fact, we have

$$\begin{aligned} &(\delta_{\hat{\chi}}\hat{\psi})(a, b, c, z) \\ &= \hat{\psi}(b, c, \hat{z} + \hat{\chi}(a)) \psi(ba, c, \hat{z})^{-1} \hat{\psi}(a, cb, \hat{z}) \hat{\psi}(a, b, \hat{z})^{-1} \\ &= \psi(b, c, 0) \psi(ba, c, 0)^{-1} \psi(a, cb, 0) \psi(a, b, 0)^{-1} \\ &\quad \phi(c, \bar{\chi}(b), 0)^{-1} \phi(c, \bar{\chi}(ba), 0) \phi(cb, \bar{\chi}(a), 0)^{-1} \phi(b, \bar{\chi}(a), 0) \\ &\quad \langle \hat{\chi}(cb) + \hat{\chi}(a) + \hat{z}, \bar{\chi}(cb) - \bar{\chi}(b) - \bar{\chi}(c) \rangle \\ &\quad \langle \hat{\chi}(cba) + \hat{z}, \bar{\chi}(cba) - \bar{\chi}(ba) - \bar{\chi}(c) \rangle^{-1} \\ &\quad \langle \hat{\chi}(cba) + \hat{z}, \bar{\chi}(cba) - \bar{\chi}(a) - \bar{\chi}(cb) \rangle \\ &\quad \langle \hat{\chi}(ba) + \hat{z}, \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b) \rangle^{-1} \\ &= \phi(c, \bar{\chi}(b), 0)^{-1} \phi(c, \bar{\chi}(ba), 0) \phi(c, \bar{\chi}(a), \chi(b))^{-1} \\ &\quad \langle \hat{\chi}(c), \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b) \rangle \\ &= \phi(c, \bar{\chi}(b), 0)^{-1} \phi(c, \bar{\chi}(ba), 0) \phi(c, \bar{\chi}(a), \chi(b))^{-1} \\ &\quad \phi(c, \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b), 0)^{-1} \\ &= 1. \end{aligned}$$

Thus, we have shown that  $\partial_{\hat{\chi}}(\hat{\psi}, \hat{\phi}, 1) = 1$ .

To check that all choices involved do not change the class  $[\hat{\psi}, \hat{\phi}, 1]$  is tedious and left to the reader.  $\blacksquare$

It follows that the assignment  $(\chi, \psi, \phi) \mapsto (\hat{\chi}, \hat{\psi}, \hat{\phi})$  defines a duality map

$$(29) \quad \hat{\cdot} : \text{Dyn}^+(\Gamma) \rightarrow \widehat{\text{Dyn}}^+(\Gamma).$$

**Proposition 9.3** The duality map (29) is an isomorphism, and its inverse is given by replacing everything by its dual counterpart.

**Proof :** We compute the double dual  $(\hat{\chi}, \hat{\psi}, \hat{\phi})$ , show that  $\hat{\chi} = \chi$ , and that  $(\hat{\psi}, \hat{\phi}, 1)$  defines the the same class in  $H_{\text{cont}}^{2,+}(\Gamma, G, M_\chi)$  as  $(\psi, \phi, 1)$ .

Firstly,  $\hat{\chi} = \chi$  as according to (26)

$$\langle \hat{\chi}(a), n^\perp \rangle := \hat{\phi}(a, n^\perp, z)^{-1} = \langle n^\perp, \bar{\chi}(a) \rangle = \langle n^\perp, \chi(a) \rangle,$$

for  $n^\perp \in N^\perp$ .

Secondly, we compute  $\hat{\phi}$ . So let  $\bar{\chi} : \Gamma \rightarrow \widehat{G}$  be a lift of  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$ , then according to (28)

$$\begin{aligned} \hat{\phi}(a, g, z) &:= \langle \bar{\chi}(a), g \rangle^{-1} \\ &= \langle \bar{\chi}(a), g \rangle^{-1} \phi(a, g, z)^{-1} \phi(a, g, z) \\ &= (dv)(a, g, z) \phi(a, g, z) \end{aligned}$$

where  $v : \Gamma \times G/N \rightarrow \text{U}(1)$  is such that  $(dv)(a, g, z) := v(a, gN + z)v(a, z)^{-1} = \langle \bar{\chi}(a), g \rangle^{-1} \phi(a, g, z)^{-1}$ . Such  $v$  exists as  $g \mapsto \langle \bar{\chi}(a), g \rangle^{-1} \phi(a, g, z)^{-1}$  factors over  $G/N$  by definition of  $\hat{\chi}$ , and hence  $(g, z) \mapsto \langle \bar{\chi}(a), g \rangle^{-1} \phi(a, g, z)^{-1}$  is a boundary.

Thridly, we have to compute  $\hat{\psi}$  according to (27):

$$\begin{aligned} \hat{\psi}(a, b, z) &:= \hat{\psi}(a, b, 0) \hat{\phi}(b, \bar{\chi}(a), 0)^{-1} \\ &\quad \langle \bar{\chi}(ba) - \bar{\chi}(a) - \bar{\chi}(b), \chi(ba) + z \rangle. \end{aligned}$$

The calculation is rather lengthy, but straight forward. It only uses the cocycle condition for  $(\psi, \phi, 1)$  and the definitions of  $\hat{\chi}, \hat{\phi}, \hat{\psi}$ . The details are left to the reader, we just state the result, this is

$$\hat{\psi}(a, b, z) = \psi(a, b, z) (\delta_\chi v_1)(a, b, z),$$

where  $v_1 : \Gamma \times G/N \rightarrow \text{U}(1)$  is defined by

$$v_1(a, z) := v(a, z) v(a, -\chi(a))^{-1} \phi(a, -\bar{\chi}(a), 0)^{-1}.$$

Note that  $dv_1 = dv$ , therefore we have shown that (written multiplicative)

$$(\hat{\psi}, \hat{\phi}, 1) = (\psi, \psi, 1) \cdot \partial_\chi(v_1, 1),$$

and thus the proposition follows. ■

It is easily checked now that the duality map (29) is natural in  $\Gamma$ , and therefore we even have constructed a natural equivalence of set-valued functors

$$\text{Dyn}^+ \cong \widehat{\text{Dyn}}^+.$$

## 10. THE DUALITY THEORY, CROSSED PRODUCTS AND K-THEORY

In this section we show that the duality theory of dynamical triples coincides with the  $C^*$ -algebraic duality theory using crossed product  $C^*$ -algebras.

Let  $(\chi, \lambda, \mu)$  be a dualisable dynamical triple. As in section 7 we have continuous lifts  $\bar{\lambda}, \bar{\mu}$ , where  $\bar{\mu}$  still satisfies the cocycle condition. Canonically, we identify the  $C^*$ -algebra of sections of the associated  $C^*$ -bundle

$$(G/N \times \text{PU}(\mathcal{H})) \times_{\text{PU}(\mathcal{H})} \mathcal{K}(\mathcal{H}) \rightarrow G/N$$

with the functions  $C(G/N, \mathcal{K}(\mathcal{H}))$ . It has an action of  $G$  which is given by

$$(g \cdot f)(z) := \mu(g, z)^{-1} [f(z + gN)].$$

For this action we denote by  $G \ltimes C(G/N, \mathcal{K}(\mathcal{H}))$  the corresponding crossed product  $C^*$ -algebra [8]. In [10, Thm. 3.7] an isomorphism of  $C^*$ -algebras

$$T_{\bar{\mu}} : G \ltimes C(G/N, \mathcal{K}(\mathcal{H})) \rightarrow C(\widehat{G}/N^\perp, \mathcal{K}(L^2(N^\perp) \otimes \mathcal{H}))$$

has been introduced which on the dense subspace of compactly supported functions  $f \in C_c(G \times G/N, \mathcal{K}(\mathcal{H})) \subset G \ltimes C(G/N, \mathcal{K}(\mathcal{H}))$  is given by

$$(T_{\bar{\mu}} f)(\hat{g}N^\perp) := \mathcal{F}_{G/N} \circ \langle -\hat{g}, \sigma(-) \rangle \circ \mathcal{F}_{G/N}^{-1} \circ f^{\bar{\mu}}(\hat{g}) \circ \mathcal{F}_{G/N} \circ \langle \hat{g}, \sigma(-) \rangle \circ \mathcal{F}_{G/N}^{-1},$$

where  $\mathcal{F}_{G/N} : L^2(G/N) \rightarrow L^2(N^\perp)$  is the Fourier transform and  $f^{\bar{\mu}}(\hat{g}) \in \mathcal{K}(L^2(N^\perp) \otimes \mathcal{H})$  is the Hilbert-Schmidt operator whose integral kernel is

$$N^\perp \times N^\perp \ni (m, n) \mapsto \mathcal{F}_{G \times G/N}(f \cdot \bar{\mu})(\hat{g} + n, n - m) \in \mathcal{K}(\mathcal{H}).$$

The crossed product  $G \ltimes C(G/N, \mathcal{K}(\mathcal{H}))$  comes along with its natural action of  $\widehat{G}$ , and because the actions of  $G$  and  $\Gamma$  on  $C(G/N, \mathcal{K}(\mathcal{H}))$  commute, the crossed product inherits also a  $\Gamma$ -action commuting with  $\widehat{G}$ . For  $f \in C_c(G \times G/N, \mathcal{K}(\mathcal{H}))$  these actions are

$$(\hat{g} \cdot f)(g, z) := \langle \hat{g}, g \rangle f(g, z), \quad (a \cdot f)(g, z) := \lambda(a, z)^{-1} [f(g, z + \chi(a))].$$

So, the isomorphism  $T_{\bar{\mu}}$  induces two commuting actions of  $\widehat{G}$  and  $\Gamma$  on  $C(\widehat{G}/N^\perp, \mathcal{K}(L^2(N^\perp) \otimes \mathcal{H}))$ . It is lengthy but straight forward to compute these induced actions, namely, the following identities holds

$$T_{\bar{\mu}}(\hat{g} \cdot f)(\hat{z}) = \hat{\mu}(\hat{g}, \hat{z})^{-1} [T_{\bar{\mu}}(f)(\hat{z} + \hat{g}N^\perp)],$$

where  $\hat{\mu}(\hat{g}, \hat{z}) := \text{Ad}(\mathcal{F}_{G/N} \circ \langle \hat{g}, -\sigma(-) \rangle \circ \mathcal{F}_{G/N}^{-1})$ . For the action of  $\Gamma$  we find

$$T_{\bar{\mu}}(a \cdot f)(\hat{z}) = \hat{\lambda}(a, \hat{z})^{-1} [T_{\bar{\mu}}(f)(\hat{z} + \hat{\chi}(a))],$$

where

$$\begin{aligned} \hat{\lambda}(a, \hat{z}) &:= \text{Ad}\left(\mathcal{F}_{G/N} \circ \langle \hat{\chi}(a) + \hat{z}, \sigma(- + \chi(a)) - \sigma(-) - \sigma(\chi(a)) \rangle \right. \\ &\quad \left. \lambda_{G/N}(-\chi(a)) \bar{\lambda}(a, -) \phi(a, -\sigma(-), 0)^{-1} \circ \mathcal{F}_{G/N}^{-1}\right) \end{aligned}$$

and  $\hat{\chi}$  is from (26). The doubtful reader is advised to have a look at section 3.4 of [10], where essentially the same calculations are done.

We see from the structure of the induced actions, that  $(\hat{\chi}, \hat{\lambda}, \hat{\mu})$  is a dualisable dual dynamical triple with underlying Hilbert space  $L^2(N^\perp) \otimes \mathcal{H}$ . Now, the following lemma links the  $C^*$ -algebraic duality theory to the duality theory of the previous section.

**Lemma 10.1** *The class of  $(\hat{\chi}, \hat{\lambda}, \hat{\mu})$  in  $\widehat{\text{Dyn}}^+(\Gamma, \hat{\chi}) \cong H_{\text{cont}}^{2,+}(\Gamma, \widehat{G}, M_{\hat{\chi}})$  coincides with the dual of the class of  $(\chi, \lambda, \mu)$  given by the duality map (29).*

**Proof :** We have to compute the cocycle  $(\hat{\psi}, \hat{\phi}, 1)$  of  $(\hat{\chi}, \hat{\lambda}, \hat{\mu})$  according to (18) for continuous lifts  $\bar{\lambda}, \bar{\mu}$ . In the definitions of  $\hat{\lambda}$  and  $\hat{\mu}$  the arguments of Ad are continuous functions and we define them to be  $\bar{\lambda}$  and  $\bar{\mu}$ . Then the calculation is again straight forward and leads to

$$\begin{aligned} \bar{\lambda}(a, \hat{z} + \hat{g}N^\perp) \bar{\mu}(\hat{g}, \hat{z}) &= \bar{\mu}(\hat{g}, \hat{z} + \hat{\chi}(a)) \bar{\lambda}(a, \hat{z}) \underbrace{\langle \hat{g}, \sigma(\chi(a)) \rangle^{-1}}_{=: \hat{\phi}(a, \hat{g}, z)}, \end{aligned}$$

and (somewhat more lengthy to compute) we also find

$$\begin{aligned} &\bar{\lambda}(ba, \hat{z}^{-1} \bar{\lambda}(b, \hat{z} + \hat{\chi}(a)) \bar{\lambda}(a, \hat{z})) \\ &= \underbrace{\psi(a, b, 0) \phi(b, \sigma(\chi(a)), 0)^{-1} \langle \hat{\chi}(ba) + \hat{z}, \sigma(\chi(ba)) - \sigma(\chi(b)) - \sigma(\chi(a)) \rangle}_{=: \hat{\psi}(a, b, z)}. \end{aligned}$$

A comparison with (28) and (27) proves the lemma. ■

**Remark 10.1** *Combining Lemma 10.1 and the Takai duality theorem (see [8]) we also obtain that the duality map (29) is an isomorphism. However, we obtained that result completely independent of any theory about  $C^*$ -algebras.*

If we fix the groups  $G := \mathbb{R}^n$ ,  $N := \mathbb{Z}^n$  we have a K-theoretic application of Lemma 10.1. Namely, consider a dualisable dynamical triple  $(\rho, E, P)$  and a dualisable dual dynamical triple  $(\hat{\rho}, \hat{E}, \hat{P})$  such that their classes match under the duality map (29). Then there is an isomorphism in twisted equivariant K-theory

$$(30) \quad K_{\Gamma, P}^{*-n}(E) \cong K_{\Gamma, \hat{P}}^*(\hat{E}).$$

In fact, by Connes' Thom isomorphism  $K_*(\mathbb{R}^n \rtimes A) \cong K_{*-n}(A)$ , for any  $\mathbb{R}^n$ - $C^*$ -algebra  $A$  [5], we have a chain of isomorphisms

$$\begin{aligned} K_{\Gamma, P}^*(E) &\cong K_*(\Gamma \rtimes \Gamma(E, P \times_{\text{PU}(\mathcal{H})} \mathcal{K}(\mathcal{H}))) \\ &\cong K_{*+n}(\mathbb{R}^n \rtimes \Gamma \rtimes \Gamma(E, P \times_{\text{PU}(\mathcal{H})} \mathcal{K}(\mathcal{H}))) \\ &\cong K_{*+n}(\Gamma \rtimes \mathbb{R}^n \rtimes \Gamma(E, P \times_{\text{PU}(\mathcal{H})} \mathcal{K}(\mathcal{H}))) \\ &\cong K_{*+n}(\Gamma \rtimes \Gamma(\hat{E}, \hat{P} \times_{\text{PU}(\hat{\mathcal{H}})} \mathcal{K}(\hat{\mathcal{H}}))) \\ &\cong K_{\Gamma, \hat{P}}^{*+n}(E). \end{aligned}$$

## 11. TOPOLOGICAL TRIPLES

In this section we introduce the main objects of our interest – ( $\Gamma$ -equivariant) topological triples. In case of  $G/N = \mathbb{R}^n/\mathbb{Z}^n = \widehat{G}/N^\perp$  their non-equivariant version has been introduced first in [1] under the name T-duality triples. For general compact abelian groups the name topological triples has been used in [10].

There is a canonical  $U(1)$ -principal fibre bundle over  $G/N \times \widehat{G}/N^\perp$  which is called Poincaré bundle. We shortly recall its definition. Let

$$(31) \quad Q := (G/N \times \widehat{G} \times U(1))/N^\perp,$$

where the action of  $N^\perp$  is defined by  $(z, \hat{g}, t) \cdot n^\perp := (z, \hat{g} + n^\perp, t \langle n^\perp, z \rangle^{-1})$ . Then the obvious map  $Q \rightarrow G/N \times \widehat{G}/N^\perp$  is a  $U(1)$ -principal fibre bundle. Dually, we may exchange  $G$  and  $\widehat{G}$  to find a second  $U(1)$ -bundle

$$R := (G \times \widehat{G}/N^\perp \times U(1))/N \rightarrow G/N \times \widehat{G}/N^\perp.$$

We denote the Čech classes in  $\check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{U(1)})$  defined by these bundles by  $[Q]$  and  $[R]$ .

**Definition 11.1** *The class  $\pi := -[Q]$  constructed above is called the **Poincaré class** of  $G/N \times \widehat{G}/N^\perp$ .*

In fact, there is no proper choice for this definition as the following is true.

**Lemma 11.1**  *$[Q]$  and  $[R]$  are inverses of each other, i.e.*

$$[Q] + [R] = 0 \in \check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{U(1)}).$$

**Proof :** See Lemma 2.2 of [10]. ■

We now turn to the definition of topological triples. Let  $P \rightarrow E \rightarrow *$  be a pair and let  $\widehat{P} \rightarrow \widehat{E} \rightarrow *$  be a dual pair with same underlying Hilbert space  $\mathcal{H}$ . So we have the following diagram in the  $\Gamma$ -equivariant category

$$(32) \quad \begin{array}{ccccc} & & P \times \widehat{E} & & E \times \widehat{P} \\ & \swarrow & \searrow & \swarrow & \searrow \\ P & & E \times \widehat{E} & & \widehat{P} \\ & \searrow & \swarrow & \searrow & \swarrow \\ & E & & \widehat{E} & \\ & \searrow & & \swarrow & \\ & & * & & \end{array}$$

Assume that there is a  $\Gamma$ -equivariant  $PU(\mathcal{H})$ -bundle isomorphism  $\kappa : E \times \widehat{P} \rightarrow P \times \widehat{E}$  (being the identity on the base  $E \times \widehat{E}$ ) which makes diagram (32)

commute. We choose charts for the pair and the dual pair to trivialise (32). This induces an automorphism of the trivial  $\text{PU}(\mathcal{H})$ -bundle over  $G/N \times \widehat{G}/N^\perp$ ,

$$\begin{array}{ccc} G/N \times \widehat{G}/N^\perp \times \text{PU}(\mathcal{H}) & \longleftarrow & G/N \times \widehat{G}/N^\perp \times \text{PU}(\mathcal{H}) \\ & \searrow & \swarrow \\ & G/N \times \widehat{G}/N^\perp & \end{array}$$

which is given by a function (which we also denote by)

$$\kappa : G/N \times \widehat{G}/N^\perp \rightarrow \text{PU}(\mathcal{H}),$$

and this function defines a Čech class  $[\kappa] \in \check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\text{U}(1)})$ .

**Definition 11.2** *Let  $L \in \mathbb{Z}$  be any integer.  $\kappa$  satisfies the **Poincaré condition of order  $L$**  if the equality  $[\kappa] = L \cdot \pi + p_1^*x + p_2^*y$  holds, for the Poincaré class  $\pi$  and some classes  $x \in \check{H}^1(G/N, \underline{\text{U}(1)})$  and  $y \in \check{H}^1(\widehat{G}/N^\perp, \underline{\text{U}(1)})$ . Here  $p_1, p_2$  are the projections from  $G/N \times \widehat{G}/N^\perp$  on the first and second factor.*

Note that in this definition the classes  $x, y$  are just of minor importance. They are manifestations of the freedom to choose another chart. In fact, one can always modify the given charts such that  $x$  and  $y$  vanish.

**Definition 11.3** *Let  $L \in \mathbb{Z}$  be any integer. A  $(\Gamma$ -equivariant) **topological triple of order  $L$**  (over the point),  $(\kappa, (P, E), (\widehat{P}, \widehat{E}))$ , is a commutative diagram*

$$(33) \quad \begin{array}{ccccc} & & P \times \widehat{E} & \xleftarrow{\kappa} & E \times \widehat{P} & & \\ & \swarrow & & \searrow & & \swarrow & \\ & P & & & E \times \widehat{E} & & \widehat{P} \\ & \searrow & & \swarrow & \searrow & \swarrow & \\ & & E & & \widehat{E} & & \\ & & \searrow & & \swarrow & & \\ & & & * & & & \end{array} ,$$

wherein  $(P, E)$  is a pair,  $(\widehat{P}, \widehat{E})$  is a dual pair (with same underlying Hilbert space  $\mathcal{H}$ ), and  $\kappa$  is an isomorphism that satisfies the Poincaré condition of order  $L$ .

We call two topological triples  $(\kappa, (P, E), (\widehat{P}, \widehat{E}))$  and  $(\kappa', (P', E'), (\widehat{P}', \widehat{E}'))$  (of same order, but with underlying Hilbert spaces  $\mathcal{H}, \mathcal{H}'$  respectively) **isomorphic** if there is a morphism of pairs  $(\varphi, \vartheta, \theta)$  from  $(P, E)$  to  $(P', E')$  and

a morphism of dual pairs  $(\hat{\varphi}, \hat{\vartheta}, \hat{\theta})$  from  $(\hat{P}, \hat{E})$  to  $(\hat{P}', \hat{E}')$  such that the induced diagram

$$(34) \quad \begin{array}{ccc} P \times \hat{E} & \xleftarrow{\kappa} & E \times \hat{P} \\ \downarrow \vartheta \times \hat{\theta} & & \downarrow \theta \times \hat{\vartheta} \\ \varphi^* P' \times \hat{E}' & \xleftarrow{\kappa'} & E' \times \hat{\varphi}^* \hat{P}' \end{array}$$

is strictly commutative. The triples are called **stably isomorphic** if the stabilised triples  $(\mathbb{1} \otimes \kappa, (P_{\mathcal{H}_1}, E), (\hat{P}_{\mathcal{H}_1}, \hat{E}))$  and  $(\mathbb{1} \otimes \kappa', (P'_{\mathcal{H}_1}, E'), (\hat{P}'_{\mathcal{H}_1}, \hat{E}'))$  are isomorphic for some separable Hilbert space  $\mathcal{H}_1$ . The meaning of the index  $\mathcal{H}_1$  is stabilisation as in equation (5).

We call two topological triples  $(\kappa, (P, E), (\hat{P}, \hat{E}))$  and  $(\kappa', (P', E'), (\hat{P}', \hat{E}'))$  (of same order, but with underlying Hilbert spaces  $\mathcal{H}, \mathcal{H}'$  respectively) **equiv-  
alent** if the pairs  $(P, E)$  and  $(P', E')$  and the dual pairs  $(\hat{P}, \hat{E})$  and  $(\hat{P}', \hat{E}')$  are outer conjugate and the joint equivariant Čech class (see Appendix C) of the bundle isomorphisms  $\kappa$  and  $\kappa'$  vanishes:

$$[\kappa, \kappa'] = 0 \in \check{H}_1^1(G/N \times \hat{G}/N^\perp, \underline{\mathbf{U}(1)}).$$

The triples are called **stably equivalent** if for a separable Hilbert space  $\mathcal{H}_1$  the stabilised triples  $(\mathbb{1} \otimes \kappa, (P_{\mathcal{H}_1}, E), (\hat{P}_{\mathcal{H}_1}, \hat{E}))$  and  $(\mathbb{1} \otimes \kappa', (P'_{\mathcal{H}_1}, E'), (\hat{P}'_{\mathcal{H}_1}, \hat{E}'))$  are equivalent.

These notions can be arranged in a diagram of implications

$$\begin{array}{ccc} \text{isomorphism} & \implies & \text{equivalence} \\ \text{of top. triples} & & \text{of top. triples} \\ \Downarrow & & \Downarrow \\ \text{stable isomorphism} & \implies & \text{stable equivalence} \\ \text{of top. triples} & & \text{of top. triples} \end{array}.$$

Stable equivalence will be the correct notion of equivalence for us, and we introduce a set valued functor  $\text{Top}_L$  which associates to a finite group  $\Gamma$  the set of stable equivalence classes of topological triples of order  $L$ , i.e.

$$\text{Top}_L(\Gamma) := \{ \text{topological triples of order } L \} /_{\text{st. equiv.}}$$

If we fix a homomorphism  $\chi : \Gamma \rightarrow G/N$ -bundle, we can consider all topological triples for which (after a chart is chosen) the  $\Gamma$ -action on  $E$  is given by  $\chi$ . We define

$$\text{Top}_L(\Gamma, \chi) := \{ \text{topological triples of order } L \text{ with fixed } \chi \} /_{\text{st. equiv.}}$$

so there is a decomposition  $\text{Top}_L(\Gamma) = \coprod_{\chi} \text{Top}_L(\chi, \Gamma)$ . We also let

$$\text{Top}_L(\Gamma, \chi, \hat{\chi}) := \{ \text{topological triples of order } L \text{ with fixed } \chi \text{ and } \hat{\chi} \} /_{\text{st. equiv.}}$$

for fixed  $\chi : \Gamma \rightarrow G/N$  and  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$  in the obvious way. This yields a decomposition  $\text{Top}_L(\Gamma) = \coprod_{\chi, \hat{\chi}} \text{Top}_L(\Gamma, \chi, \hat{\chi})$ .

The notion of stable equivalence on topological triples is made such that we have natural forgetful maps

$$\begin{array}{ccc} & \text{Top}_L(\Gamma) & \\ & \swarrow \quad \searrow & \\ \text{Par}(\Gamma) & & \widehat{\text{Par}}(\Gamma). \end{array}$$

There are also natural maps

$$\begin{aligned} \text{Top}_L(\Gamma, \chi, \hat{\chi}) \times \text{Top}_K(\Gamma, \chi, \hat{\chi}) &\rightarrow \text{Top}_{L+K}(\Gamma, \chi, \hat{\chi}), \\ \text{Top}_L(\Gamma, \chi, \hat{\chi}) &\rightarrow \text{Top}_{-L}(\Gamma, \chi, \hat{\chi}) \end{aligned}$$

induced by tensor product and complex conjugation of projective unitary bundles. These maps turn  $\text{Top}_0(\Gamma, \chi, \hat{\chi})$  into an abelian group, where the class of  $(\mathbb{1}, (\chi, \mathbb{1}), (\hat{\chi}, \mathbb{1}))$  is the unit. Each  $\text{Top}_L(\Gamma, \chi, \hat{\chi})$  becomes a  $\text{Top}_0(\Gamma, \chi, \hat{\chi})$ -torsor, and  $\text{Top}_\bullet(\Gamma, \chi, \hat{\chi}) := \coprod_{L \in \mathbb{Z}} \text{Top}_L(\Gamma, \chi, \hat{\chi})$  becomes an abelian group which fits into the short exact sequence

$$0 \longrightarrow \text{Top}_0(\Gamma, \chi, \hat{\chi}) \longrightarrow \text{Top}_\bullet(\Gamma, \chi, \hat{\chi}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

**Remark 11.1** *Note that the empty set is a torsor for any group, and that the quotient of the empty set by any group is the one point space. Therefore the above short exact sequence is meaningful even if  $\text{Top}_L(\Gamma, \chi, \hat{\chi}) = \emptyset$  for all  $L \neq 0$ .  $\text{Top}_0(\Gamma, \chi, \hat{\chi})$  is never empty, as it contains at least its unit  $[\mathbb{1}, (\chi, \mathbb{1}), (\hat{\chi}, \mathbb{1})]$ .*

By its very definition, the Poincaré class  $\pi$  has a geometric interpretation by the Poincaré bundle (31). However, what we need is an analytical description of  $\pi$  which is more adequate for the analysis we are going to do. So let  $\sigma : G/N \rightarrow G$  and  $\hat{\sigma} : \widehat{G}/N^\perp \rightarrow \widehat{G}$  be Borel sections of the quotient maps, then for each fixed  $(z, \hat{z}) \in G/N \times \widehat{G}/N^\perp$  there are multiplication operators

$$\begin{aligned} \bar{\kappa}^\sigma(z, \hat{z}) &:= \langle \hat{\sigma}(\hat{z}), \sigma(-z) - \sigma(\cdot) \rangle \in L^\infty(G/N, \text{U}(1)) \subset \text{U}(L^2(G/N)), \\ \bar{\kappa}^{\hat{\sigma}}(z, \hat{z}) &:= \langle \hat{\sigma}(\cdot - \hat{z}) - \hat{\sigma}(\cdot), \sigma(z) \rangle \in L^\infty(\widehat{G}/N^\perp, \text{U}(1)) \subset \text{U}(L^2(\widehat{G}/N^\perp)), \end{aligned}$$

wherein for a  $\text{U}(\mathcal{H})$ -valued function  $f$  on  $G/N$ , or  $g$  on  $\widehat{G}/N^\perp$ , we denoted by  $f(\cdot)$  and  $g(\cdot)$ , respectively, the corresponding multiplication operator; for example if  $F \in L^2(G/N)$ , then  $(f(\cdot)F)(x) := f(x)F(x)$ . These operators



give rise to the following projective unitary functions

$$(35) \quad \begin{aligned} \kappa^\sigma : G/N \times \widehat{G}/N^\perp &\rightarrow \text{PU}(L^2(G/N)), \\ (z, \hat{z}) &\mapsto \text{Ad}(\overline{\kappa^\sigma}(z, \hat{z})) \end{aligned}$$

$$(36) \quad \begin{aligned} \hat{\kappa}^{\hat{\sigma}} : G/N \times \widehat{G}/N^\perp &\rightarrow \text{PU}(L^2(\widehat{G}/N^\perp)). \\ (z, \hat{z}) &\mapsto \text{Ad}(\overline{\hat{\kappa}^{\hat{\sigma}}}(z, \hat{z})) \end{aligned}$$

**Lemma 11.2** *The maps  $\kappa^\sigma$  and  $\hat{\kappa}^{\hat{\sigma}}$  are continuous,  $\kappa^\sigma$  is independent of  $\hat{\sigma}$  and  $\hat{\kappa}^{\hat{\sigma}}$  is independent of  $\sigma$ .*

*Moreover, the Čech classes  $[\kappa^\sigma], [\hat{\kappa}^{\hat{\sigma}}] \in \check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\mathbf{U}(1)})$ , of the two maps are independent of  $\sigma$  and  $\hat{\sigma}$  and*

$$\pi = [\kappa^\sigma] = -[\hat{\kappa}^{\hat{\sigma}}].$$

**Proof :** See Lemma 2.3 and Lemma 2.4 of [10]. ■

It is worth mentioning that  $\kappa^\sigma(0, \hat{z}) = \mathbb{1} = \kappa^\sigma(z, 0)$ . The first equality is trivial, the second follows from  $\hat{\sigma}(0) \in N^\perp$ , so  $\langle \hat{\sigma}(0), \sigma(-z) - \sigma(-) \rangle = \langle \hat{\sigma}(0), \sigma(-z) \rangle \in \mathbf{U}(1)$ . Clearly, the same is true for  $\hat{\kappa}^{\hat{\sigma}}$ . By similar reasoning, we can prove the following Lemma.

**Lemma 11.3** *For integers  $K, L \in \mathbb{Z}$  consider the map  $(K, L) : (z, \hat{z}) \mapsto (Kz, L\hat{z})$ . Its induced map*

$$(K, L)^* : \check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\mathbf{U}(1)}) \rightarrow \check{H}^1(G/N \times \widehat{G}/N^\perp, \underline{\mathbf{U}(1)})$$

*maps the Poincaré class  $\pi$  to  $K \cdot L \cdot \pi$ .*

**Proof :** We show that  $\kappa^\sigma(Kz, L\hat{z})$  equals  $\kappa^\sigma(z, \hat{z})^{KL}$  up to a unitary function.

As  $\langle \hat{\sigma}(L\hat{z}), \sigma(z) \rangle \in \mathbf{U}(1)$  and  $\sigma(-z) - \sigma(-) + \sigma(z) \in N$ , we have first

$$\begin{aligned} \kappa^\sigma(z, L\hat{z}) &= \text{Ad}(\langle \hat{\sigma}(L\hat{z}), \sigma(-z) - \sigma(-) + \sigma(z) \rangle) \\ &= \text{Ad}(\langle L\hat{z}, \sigma(-z) - \sigma(-) + \sigma(z) \rangle) \\ &= \text{Ad}(\langle \hat{z}, \sigma(-z) - \sigma(-) + \sigma(z) \rangle)^L. \end{aligned}$$

Second, by use of  $\sigma(-Kz) - K\sigma(-z) + (K-1)\sigma(-) \in N$ , we have

$$\begin{aligned} \kappa^\sigma(Kz, \hat{z}) &= \text{Ad}(\langle \hat{\sigma}(\hat{z}), \sigma(-Kz) - \sigma(-) \rangle) \\ &= \text{Ad}(\langle \hat{\sigma}(\hat{z}), \sigma(-Kz) - K\sigma(-z) + (K-1)\sigma(-) \rangle) \\ &\quad \text{Ad}(\langle \hat{\sigma}(\hat{z}), K\sigma(-z) - K\sigma(-) \rangle) \\ &= \text{Ad}(\langle \hat{z}, \sigma(-Kz) - K\sigma(-z) + (K-1)\sigma(-) \rangle) \\ &\quad \text{Ad}(\langle \hat{\sigma}(\hat{z}), \sigma(-z) - \sigma(-) \rangle)^K. \end{aligned}$$

Finally, note that

$$(z, \hat{z}) \mapsto \langle \hat{z}, \sigma(-Kz) - K\sigma(-z) + (K-1)\sigma(-) \rangle \in \mathbf{U}(L^2(G/N))$$

is continuous. In fact, the map

$$G/N \times \widehat{G} \ni (z, \hat{g}) \mapsto \langle \hat{g}, \sigma(- - Kz) - K\sigma(- - z) + (K - 1)\sigma(-) \rangle$$

factors over  $G/N \times \widehat{G}/N^\perp$  and  $\langle \hat{g}, \sigma(- - z) \rangle = \lambda_{G/N}(z) \langle \hat{g}, \sigma(-) \rangle \lambda_{G/N}(z)^{-1}$  is continuous by the same argument as in the proof of Lemma 7.2. ■

By Lemma 11.3 we can define for any triple of integers  $K, L, M \in \mathbb{Z}$  a natural map

$$(37) \quad \text{Top}_M(\Gamma, K\chi, L\hat{\chi}) \rightarrow \text{Top}_{KLM}(\Gamma, \chi, \hat{\chi}),$$

namely, let  $(\kappa, (K\chi, \lambda), (L\hat{\chi}, \hat{\lambda}))$  be a topological triple of order  $M$ , then the isomorphism  $\kappa$  is given by a continuous function (also denoted by)  $\kappa : G/N \times \widehat{G}/N^\perp \rightarrow \text{PU}(\mathcal{H})$  such that

$$\lambda(a, z)\kappa(z, \hat{z}) = \kappa(z + (K\chi)(a), \hat{z} + (L\hat{\chi})(a))\hat{\lambda}(a, \hat{z})$$

is satisfied. This implies that

$$(K^*\lambda)(a, z)((K, L)^*\kappa)(z, \hat{z}) = ((K, L)^*\kappa)(z + \chi(a), \hat{z} + \hat{\chi}(a))(L^*\hat{\lambda})(a, \hat{z})$$

is also satisfied, where  $(K^*\lambda)(a, z) := \lambda(a, Kz)$ ,  $(L^*\hat{\lambda})(a, \hat{z}) := \hat{\lambda}(a, L\hat{z})$  and  $((K, L)^*\kappa)(z, \hat{z}) := \kappa(Kz, L\hat{z})$ . It is clear that  $(\chi, K^*\lambda)$  is a pair and that  $(\hat{\chi}, L^*\hat{\lambda})$  is a dual pair, and Lemma 11.3 implies that  $(K, L)^*\kappa$  satisfies the Poincaré condition of order  $KLM$ . Hence,  $((K, L)^*\kappa, (\chi, K^*\lambda), (\hat{\chi}, L^*\hat{\lambda}))$  is a topological triple of order  $KLM$ . This construction defines (37).

In section 13 we introduce the notion of regular topological triples, and we will see that the restriction of (37) to the set of regular topological triples is an isomorphism. In particular, we will consider the special cases  $M = K = 1$  and  $M = L = 1$ , then (37) becomes

$$(38) \quad \begin{aligned} \text{Top}_1(\Gamma, \chi, L\hat{\chi}) &\rightarrow \text{Top}_L(\Gamma, \chi, \hat{\chi}), \\ \text{Top}_1(\Gamma, K\chi, \hat{\chi}) &\rightarrow \text{Top}_K(\Gamma, \chi, \hat{\chi}). \end{aligned}$$

This means that at least partial information of higher order triples is encoded in triples of order 1.

It is clear that the maps of (38) respect the structure of the underlying pairs and dual pairs, i.e. we have commutative squares

$$\begin{array}{ccc} \text{Top}_1(\Gamma, \chi, L\hat{\chi}) & \longrightarrow & \text{Top}_L(\Gamma, \chi, \hat{\chi}), & \text{Top}_1(\Gamma, K\chi, \hat{\chi}) & \longrightarrow & \text{Top}_K(\Gamma, \chi, \hat{\chi}). \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \widehat{\text{Par}}(\Gamma, L\hat{\chi}) & \xrightarrow{L^*} & \widehat{\text{Par}}(\Gamma, \hat{\chi}) & \text{Par}(\Gamma, K\chi) & \xrightarrow{K^*} & \text{Par}(\Gamma, \chi) \end{array}$$

Here  $K^*$  and  $L^*$  are from (10), and the vertical maps are the natural forgetful maps.

## 12. THE CLASSIFICATION OF TOPOLOGICAL TRIPLES

In this section we give a description of  $\text{Top}_\bullet(\Gamma, \chi, \hat{\chi})$  in terms of group cohomology.

Fix a pair of homomorphisms

$$\chi : \Gamma \rightarrow G/N, \quad \hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp,$$

and consider the  $\Gamma$ -module  $M_{\chi \times \hat{\chi}} := C(G/N \times \widehat{G}/N^\perp, \mathbf{U}(1))$  and its submodules  $M_\chi := C(G/N, \mathbf{U}(1))$  and  $M_{\hat{\chi}} := C(\widehat{G}/N^\perp, \mathbf{U}(1))$ , where  $\Gamma$  acts by shift with (the negative of)  $\chi \times \hat{\chi}$  in the arguments of functions. So we have a diagram of  $\Gamma$ -modules

$$\begin{array}{ccccc} & & M_\chi & & \\ & \nearrow & \downarrow & \searrow & \\ \mathbf{U}(1) = M_\chi \cap M_{\hat{\chi}} & & & & M_\chi M_{\hat{\chi}} \longrightarrow M_{\chi \times \hat{\chi}} \\ & \searrow & \downarrow & \nearrow & \\ & & M_{\hat{\chi}} & & \end{array}$$

which yield to two long exact sequences in group cohomology (39)

$$\begin{array}{ccccccc} & & & & \vdots & & \\ & & & & \downarrow & & \\ & & & & H^2(\Gamma, M_\chi \cap M_{\hat{\chi}}) & & \\ & & & & \downarrow & & \\ & & & & H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}}) & & \\ & & & & \downarrow \ominus & & \\ \cdots & \longrightarrow & H^1(\Gamma, M_{\chi \times \hat{\chi}}) & \longrightarrow & H^1(\Gamma, M_{\chi \times \hat{\chi}}/M_\chi M_{\hat{\chi}}) & \xrightarrow{B} & H^2(\Gamma, M_\chi M_{\hat{\chi}}) \longrightarrow H^2(\Gamma, M_{\chi \times \hat{\chi}}) \longrightarrow \cdots, \\ & & & & \downarrow c & & \\ & & & & H^3(\Gamma, M_\chi \cap M_{\hat{\chi}}) & & \\ & & & & \downarrow & & \\ & & & & \vdots & & \end{array}$$

where the vertical sequence is the Mayer-Vietoris sequence of Appendix A, and the horizontal sequence is the usual Bockstein long exact sequence.

Let us introduce two classes

$$\begin{aligned} \chi \cup \hat{\chi} &\in H^3(\Gamma, \mathbf{U}(1)), \\ \chi \sqcup \hat{\chi} &\in H^2(\Gamma, M_\chi M_{\hat{\chi}}), \end{aligned}$$

namely, we define a map  $\cup$  by the composition of either two consecutive maps in the following commutative diagram

$$(40) \quad \begin{array}{ccc} H^1(\Gamma, G/N) \times H^1(\Gamma, \widehat{G}/N^\perp) & \longrightarrow & H^2(\Gamma, N) \times H^1(\Gamma, \widehat{G}/N^\perp) \\ \downarrow & & \downarrow \\ H^1(\Gamma, G/N) \times H^2(\Gamma, N^\perp) & \longrightarrow & H^3(\Gamma, U(1)). \end{array}$$

Therein the upper horizontal and the left vertical maps are given by the corresponding Bockstein homomorphisms in one factor and the identity in the other. The two maps to the right lower corner are given by the canonical pairings of dual groups, e.g. a class  $([\omega], \hat{\chi}) \in H^2(\Gamma, N) \times H^1(\Gamma, \widehat{G}/N^\perp)$  is mapped to the class of the 3-cocycle  $(a, b, c) \mapsto \langle \hat{\chi}(a), \omega(b, c) \rangle$ . The commutativity of diagram (40) can be explicitly calculated; we give the argument in Corollary 12.1 below.

For the definition of  $\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi}$  we first define a Borel function

$$(41) \quad \beta : \Gamma \times G/N \times \widehat{G}/N^\perp \rightarrow U(1)$$

by the explicit formula

$$\beta(a, z, \hat{z}) := \langle \hat{\sigma}(\hat{z} + \hat{\chi}(a)) - \hat{\sigma}(\hat{z}) - \hat{\sigma}(\hat{\chi}(a)), \sigma(z + \chi(a)) \rangle \langle \hat{\sigma}(\hat{z}), \sigma(\chi(a)) \rangle,$$

where  $\sigma : G/N \rightarrow G$  and  $\hat{\sigma} : \widehat{G}/N^\perp \rightarrow \widehat{G}$  are chosen Borel sections of the quotient maps. Let us also define

$$\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi} := \delta_{\chi \times \hat{\chi}} \beta.$$

Although  $\beta$  fails to be continuous, the following statement for  $\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi}$  is true.

**Lemma 12.1** *The map  $\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi} : \Gamma^2 \times G/N \times \widehat{G}/N^\perp \rightarrow U(1)$  is continuous, and we have a decomposition*

$$(\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi})(a, b, z, \hat{z}) = \gamma(a, b) \langle \hat{\eta}(a, b), z \rangle^{-1} \langle \hat{z}, \eta(a, b) \rangle,$$

wherein

$$\gamma(a, b) := \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(\chi(b)) \rangle \langle \hat{\eta}(a, b), \chi(ba) \rangle^{-1},$$

and

$$\eta(a, b) := \sigma(\chi(b)) - \sigma(\chi(ba)) + \sigma(\chi(a)) \in N \cong \widehat{\widehat{G}/N^\perp},$$

and

$$\hat{\eta}(a, b) := \hat{\sigma}(\hat{\chi}(b)) - \hat{\sigma}(\hat{\chi}(ba)) + \hat{\sigma}(\hat{\chi}(a)) \in N^\perp \cong \widehat{\widehat{G}/N}.$$

**Proof :** One computes

$$(\delta_{\chi \times \hat{\chi}} \beta)(a, b, z, \hat{z}) = \beta(b, z + \chi(a), \hat{z} + \hat{\chi}(a)) \beta(ba, z, \hat{z})^{-1} \beta(a, z, \hat{z})$$

right from the definition. The resulting formula is

$$\begin{aligned}
(\delta_{\chi \times \hat{\chi}} \beta)(a, b, z, \hat{z}) &= \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(\chi(b)) \rangle \\
&\quad \langle \hat{\sigma}(\hat{\chi}(ba)) - \hat{\sigma}(\hat{\chi}(a)) - \hat{\sigma}(\hat{\chi}(b)), \sigma(\chi(ba)) \rangle \\
&\quad \langle \hat{\sigma}(\hat{\chi}(ba)) - \hat{\sigma}(\hat{\chi}(a)) - \hat{\sigma}(\hat{\chi}(b)), \sigma(z) \rangle \\
&\quad \langle \hat{\sigma}(\hat{z}), \sigma(\chi(b)) + \sigma(\chi(a)) - \sigma(\chi(ba)) \rangle
\end{aligned}$$

which directly implies the statements above.  $\blacksquare$

By its definition  $\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi}$  is a Borel boundary, but by the above lemma, it is even a continuous cocycle. So there is a well-defined class

$$(42) \quad \chi \sqcup \hat{\chi} := [\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi}] \in H^2(\Gamma, M_\chi M_{\hat{\chi}}),$$

in fact, it is not difficult to check that the class  $\chi \sqcup \hat{\chi}$  is independent of the choice of the sections  $\sigma$  and  $\hat{\sigma}$ .

**Corollary 12.1** (1) *Diagram (40) commutes, so  $\chi \sqcup \hat{\chi}$  is well-defined, and*  
(2) *the connecting homomorphism  $C$  in the vertical long exact sequence of (39) maps the class  $\chi \sqcup \hat{\chi}$  to  $\chi \cup \hat{\chi}$ .*

**Proof :** 1. By the lemma above we have

$$1 = (\delta_{\chi \times \hat{\chi}} \chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi})(a, b, c, z, \hat{z}) = (\delta_{\chi \times \hat{\chi}} \gamma)(a, b, c) \langle \hat{\eta}(b, c), \chi(a) \rangle^{-1} \langle \hat{\chi}(a), \eta(b, a) \rangle.$$

Thus, we see that the 3-cocycles

$$(a, b, c) \mapsto \langle \hat{\chi}(a), \eta(b, a) \rangle \quad \text{and} \quad (a, b, c) \mapsto \langle \hat{\eta}(b, c), \chi(a) \rangle$$

are cohomologous, but they represent precisely the two compositions in diagram (40).

2. The second statement follows from the definition of the connecting homomorphism (see Appendix A) in the vertical long exact sequence of (39).  $\blacksquare$

For any integer  $L \in \mathbb{Z}$ , let

$$\begin{aligned}
T_L(\Gamma, \chi, \hat{\chi}) &:= \left\{ ([w], [\psi], [\hat{\psi}]) \mid B([w]) + L \cdot (\chi \sqcup \hat{\chi}) = [\psi] \ominus [\hat{\psi}] \right\} \\
&\subset H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}}) \times H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}}).
\end{aligned}$$

The set  $T_L(\Gamma, \chi, \hat{\chi})$  fits into diagram (39), for  $L = 0$  only it is a pullback:  
(43)

$$\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \downarrow & & \\
 & & & & H^2(\Gamma, \mathbf{U}(1)) & & \\
 & & & & \downarrow & & \\
 & & T_L(\Gamma, \chi, \hat{\chi}) & \longrightarrow & H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}}) & & \\
 & & \downarrow & & \downarrow \ominus & & \\
 \cdots & \longrightarrow & H^1(\Gamma, M_{\chi \times \hat{\chi}}) & \longrightarrow & H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}}) & \xrightarrow{B} & H^2(\Gamma, M_\chi M_{\hat{\chi}}) & \longrightarrow & H^2(\Gamma, M_{\chi \times \hat{\chi}}) & \longrightarrow & \cdots \\
 & & & & \downarrow c & & \\
 & & & & H^3(\Gamma, \mathbf{U}(1)) & & \\
 & & & & \downarrow & & \\
 & & & & \vdots & & 
 \end{array}$$

The obvious maps

$$\begin{aligned}
 T_L(\Gamma, \chi, \hat{\chi}) \times T_K(\Gamma, \chi, \hat{\chi}) &\rightarrow T_{L+K}(\Gamma, \chi, \hat{\chi}), \\
 T_L(\Gamma, \chi, \hat{\chi}) &\rightarrow T_{-L}(\Gamma, \chi, \hat{\chi})
 \end{aligned}$$

turn  $T_0(\Gamma, \chi, \hat{\chi})$  into an abelian group, and each  $T_L(\Gamma, \chi, \hat{\chi})$  into a  $T_0(\Gamma, \chi, \hat{\chi})$ -torsor. Moreover,  $T_\bullet(\Gamma, \chi, \hat{\chi}) := \coprod_{L \in \mathbb{Z}} T_L(\Gamma, \chi, \hat{\chi})$  becomes an abelian group which fits into the short exact sequence

$$0 \longrightarrow T_0(\Gamma, \chi, \hat{\chi}) \longrightarrow T_\bullet(\Gamma, \chi, \hat{\chi}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

There is another short exact sequence induced by the previous,

$$0 \longrightarrow T_0(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}) \longrightarrow T_\bullet(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where  $H^1(\Gamma, M_{\chi \times \hat{\chi}})$  acts in the first factor of each  $T_L(\Gamma, \chi, \hat{\chi})$  via the map  $H^1(\Gamma, M_{\chi \times \hat{\chi}}) \rightarrow H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}})$ .

Now, we state the classification theorem of topological triples.

**Theorem 12.1** *There is a natural isomorphism of short exact sequences*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Top}_0(\Gamma, \chi, \hat{\chi}) & \longrightarrow & \text{Top}_\bullet(\Gamma, \chi, \hat{\chi}) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow = \\
 0 & \longrightarrow & T_0(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}) & \longrightarrow & T_\bullet(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}) & \longrightarrow & \mathbb{Z} \longrightarrow 0.
 \end{array}$$

**Remark 12.1** *As we already noted in Remark 11.1, the empty set is a torsor for any group, and the quotient of the empty set by any group is the one point space. Thus, Theorem 12.1 makes perfectly sense even if  $T_L(\Gamma, \chi, \hat{\chi}) = \emptyset$  for all  $L \neq 0$ .*

The proof of Theorem 12.1 is the content of the remainder of this section.

We first show how one constructs the classes

$$\begin{aligned} [w] &\in H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}), \\ ([\psi], [\hat{\psi}]) &\in H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}}) \end{aligned}$$

of a topological triple.

Let  $(\kappa, (P, E), (\hat{P}, \hat{E}))$  be a topological triple of order  $L$ . We fix a chart which defines the corresponding trivialised data  $(\chi, \lambda')$  and  $(\hat{\chi}, \hat{\lambda}')$  for the pairs  $(P, E)$  and  $(\hat{P}, \hat{E})$ . Without restriction we can assume that  $\hat{\mathcal{H}} = L^2(G/N) \otimes \mathcal{H}$  is the underlying Hilbert space of the triple. The isomorphism  $\kappa$  is given by a continuous map which we also denote by  $\kappa : G/N \times \hat{G}/N^\perp \rightarrow \text{PU}(\hat{\mathcal{H}})$ . It satisfies

$$\lambda'(a, z) \kappa(z, \hat{z}) = \kappa(\chi(a) + z, \hat{\chi}(a) + \hat{z}) \hat{\lambda}'(a, \hat{z}).$$

Due to the Poincaré condition the map  $\kappa$  is of the form

$$\kappa(z, \hat{z}) = \kappa^x(z) \kappa^\sigma(z, \hat{z})^L \text{Ad}(u(z, \hat{z})) \kappa^y(\hat{z}),$$

wherein  $\kappa^\sigma(z, \hat{z}) = \text{Ad}(\langle \hat{\sigma}(\hat{z}), \sigma(- - z) - \sigma(-) \rangle)$  is from Lemma 11.2,  $\kappa^x, \kappa^y$  are some continuous projective unitary functions, and  $u : G/N \times \hat{G}/N^\perp \rightarrow \text{U}(\hat{\mathcal{H}})$  is a continuous unitary function. We introduce some short hands to get rid of  $\kappa^x$  and  $\kappa^y$ . We let

$$\lambda(a, z) := \kappa^x(\chi(a) + z)^{-1} \lambda'(a, z) \kappa^x(z),$$

and analogously

$$\hat{\lambda}(a, \hat{z}) := \kappa^y(\chi(a) + z) \hat{\lambda}'(a, z) \kappa^y(z)^{-1}.$$

Thus we have

$$(44) \quad \begin{aligned} \lambda(a, z) &= \kappa^\sigma(\chi(a) + z, \hat{\chi}(a) + \hat{z})^L \text{Ad}(u(\chi(a) + z, \hat{\chi}(a) + \hat{z})) \\ &\hat{\lambda}(a, \hat{z}) \text{Ad}(u(z, \hat{z}))^{-1} \kappa^\sigma(z, \hat{z})^{-L}. \end{aligned}$$

**Lemma 12.2** *There exist continuous lifts*

$$\begin{array}{ccc} & \text{U}(\hat{\mathcal{H}}) & \text{and} & \text{U}(\hat{\mathcal{H}}) \\ & \nearrow \bar{\lambda} & & \nearrow \bar{\hat{\lambda}} \\ \Gamma \times G/N & \xrightarrow{\lambda} & \text{PU}(\hat{\mathcal{H}}) & \Gamma \times \hat{G}/N^\perp \xrightarrow{\hat{\lambda}} \text{PU}(\hat{\mathcal{H}}) \\ & & \downarrow & & \downarrow \end{array}$$

**Proof :** Take  $\hat{z} = 0$  in (44) which yields

$$\lambda(a, z) = \kappa^\sigma(\chi(a) + z, \hat{\chi}(a))^L \text{Ad}(u(\chi(a) + z, \hat{\chi}(a))) \hat{\lambda}(a, 0) \text{Ad}(u(z, 0))^{-1}.$$

Each single factor of the right hand side has a continuous lift, so  $\lambda$  has. By taking  $z = 0$  we conclude that  $\hat{\lambda}$  has a continuous lift.  $\blacksquare$

By use of this lemma, we define

$$(45) \quad [\psi] \in H^2(\Gamma, M_\chi), \quad [\hat{\psi}] \in H^2(\Gamma, M_{\hat{\chi}})$$

by

$$\bar{\lambda}(b, \hat{\chi}(a) + z) \bar{\lambda}(a, z) = \bar{\lambda}(ba, z) \psi(a, b, z)$$

and

$$\bar{\lambda}(b, \hat{\chi}(a) + z) \bar{\lambda}(a, \hat{z}) = \bar{\lambda}(ba, \hat{z}) \psi(a, b, z).$$

Of course,  $\kappa^\sigma$  does not have a continuous unitary lift, but we take the Borel lift

$$(z, \hat{z}) \mapsto \bar{\kappa}^\sigma(z, \hat{z}) := \langle \hat{\sigma}(\hat{z}), \sigma(-z) - \sigma(-) \rangle$$

to define

$$\alpha : \Gamma \rightarrow L^\infty(G/N \times \widehat{G}/N^\perp, \mathbf{U}(1))$$

to be the  $\mathbf{U}(1)$ -valued perturbation in

$$(46) \quad \begin{aligned} \bar{\lambda}(a, z) &= \bar{\kappa}^\sigma(z + \chi(a), \hat{z} + \hat{\chi}(a))^L u(z + \chi(a), \hat{z} + \hat{\chi}(a)) \\ \bar{\lambda}(a, \hat{z}) &u(z, \hat{z})^{-1} \bar{\kappa}^\sigma(z, \hat{z})^{-L} \alpha(a, z, \hat{z}). \end{aligned}$$

By direct computation, it follows that

$$(47) \quad (\delta_{\chi \times \hat{\chi}} \alpha)(b, a, z, \hat{z}) = \psi(b, a, z) \hat{\psi}(b, a, \hat{z})^{-1},$$

for  $\psi$  and  $\hat{\psi}$  as above.

The next lemma is the defining lemma of  $w$ .

**Lemma 12.3** *There is a continuous function  $w : \Gamma \times G/N \times \widehat{G}/N^\perp \rightarrow \mathbf{U}(1)$  such that*

$$\alpha = \beta^L w,$$

where  $\beta$  is from (41).

**Proof :** Equation (46) is equivalent to

$$\begin{aligned} \alpha(a, z, \hat{z}) \bar{\kappa}^\sigma(z, \hat{z})^{-L} \bar{\kappa}^\sigma(z + \chi(a), \hat{z} + \hat{\chi}(a))^L &= \kappa^\sigma(z, \hat{z})^{-L} [\bar{\lambda}(a, z)] u(z, \hat{z}) \\ &\bar{\lambda}(a, \hat{z})^{-1} u(z + \chi(a), \hat{z} + \hat{\chi}(a))^{-1}, \end{aligned}$$

but we observe that the right hand side consists of continuous terms only.

To explore the continuity properties of  $\alpha$  we compute straightforwardly

$$\begin{aligned} &\bar{\kappa}^\sigma(z, \hat{z})^{-1} \bar{\kappa}^\sigma(z + \chi(a), \hat{z} + \hat{\chi}(a)) \\ &= \langle \hat{\sigma}(\hat{z}), -\sigma(-z) + \sigma(-) \rangle \langle \hat{\sigma}(\hat{z} + \hat{\chi}(a)), \sigma(-z - \chi(a)) - \sigma(-) \rangle \\ &= \langle \hat{\sigma}(\hat{z} + \hat{\chi}(a)) - \hat{\sigma}(\hat{z}) - \hat{\sigma}(\hat{\chi}(a)), -\sigma(z + \chi(a)) \rangle \langle \hat{\sigma}(\hat{z}), -\sigma(\chi(a)) \rangle \\ &\quad \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(-z - \chi(a)) - \sigma(-) \rangle \langle \hat{\sigma}(\hat{z}), \sigma(\chi(a) - \sigma(- + \chi(a)) - \sigma(-) \rangle \\ &\quad \langle \hat{\sigma}(\hat{z}), -\sigma(-z) + \sigma(-) + \sigma(-z - \chi(a)) - \sigma(- - \chi(a)) \rangle, \end{aligned}$$

and we see that the factor in the first line is precisely  $\beta(a, z, \hat{z})^{-1} \in \mathbf{U}(1)$ , and the factors in the second and third line are all continuous functions  $\Gamma \times G/N \times \widehat{G}/N^\perp \rightarrow L^\infty(G/N, \mathbf{U}(1))$ . Therefore it follows that the product map  $(a, z, \hat{z}) \mapsto \alpha(a, z, \hat{z}) \beta(a, z, \hat{z})^{-L}$  is continuous.  $\blacksquare$



Note that  $w$  of the previous lemma defines a class  $[w] \in H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}})$ . In fact, we have  $\delta_{\chi \times \hat{\chi}} w(a, b, z, \hat{z}) \in M_\chi M_{\hat{\chi}}$  as

$$\begin{aligned} \delta_{\chi \times \hat{\chi}} w(a, b, z, \hat{z}) &= \delta_{\chi \times \hat{\chi}} \alpha(a, b, z, \hat{z}) \delta_{\chi \times \hat{\chi}} \beta^{-L}(a, b, z, \hat{z}) \\ &= \psi(a, b, z) \hat{\psi}(a, b, \hat{z})^{-1} (\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi})(a, b, z, \hat{z})^{-L} \end{aligned}$$

This equation also states that  $B([w]) + L \cdot \chi \sqcup \hat{\chi} = [\psi] \ominus [\hat{\psi}]$ . Thus we have defined a class

$$([w], [\psi], [\hat{\psi}]) \in T_L(\Gamma, \chi, \hat{\chi}).$$

It is clear that this class is not changed if the topological triple is replaced by a stable isomorphic one. However, if we replace the triple by an equivalent one, we see that the cocycles  $\psi, \hat{\psi}$  remain unchanged whereas  $\alpha$ , hence  $w$ , may be changed by 1-cocycle  $\Gamma \rightarrow M_{\chi \times \hat{\chi}}$ . This means that the construction of the three classes leads us to a well-defined map

$$(48) \quad \text{Top}_L(\Gamma, \chi, \hat{\chi}) \rightarrow T_L(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}),$$

and we show how to construct an inverse.

Let  $([w], [\psi], [\hat{\psi}]) \in T_L(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}})$  be represented by the  $U(1)$ -valued functions

$$w : \Gamma \rightarrow M_{\chi \times \hat{\chi}}, \quad \psi : \Gamma^2 \rightarrow M_\chi, \quad \hat{\psi} : \Gamma^2 \rightarrow M_{\hat{\chi}}.$$

By re-normalising  $\psi$  and  $\hat{\psi}$  we can assume without restriction that  $\delta_{\chi \times \hat{\chi}} w \delta_{\chi \times \hat{\chi}} \beta^L = \psi \hat{\psi}^{-1}$ . We make the following definitions

$$\begin{aligned} \lambda_\psi(a, z) &:= \text{Ad}(\psi(a, \cdot, z) \rho(a)) \in \text{PU}(L^2(\Gamma)), \\ \hat{\lambda}_{\hat{\psi}}(a, \hat{z}) &:= \text{Ad}(\hat{\psi}(a, \cdot, \hat{z}) \rho(a)) \in \text{PU}(L^2(\Gamma)), \\ \kappa_w(z, \hat{z}) &:= \text{Ad}\left(\bar{\kappa}^\sigma(z + \chi(\cdot), \hat{z} + \hat{\chi}(\cdot))^L \beta(\cdot, z, \hat{z})^L w(\cdot, z, \hat{z})\right) \\ &\in \text{PU}(L^2(\Gamma) \otimes L^2(G/N)), \end{aligned}$$

where  $\rho : \Gamma \rightarrow U(L^2(\Gamma))$  is the right regular representation. We claim that  $(\kappa_w, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$  is a topological triple which is mapped to  $([w], [\psi], [\hat{\psi}])$  under (48).

The first thing to note is that  $\kappa_w$  is continuous and satisfies the Poincaré condition. In fact, we have

$$\begin{aligned}
\kappa_w(z, \hat{z}) &:= \text{Ad} \left( \bar{\kappa}^\sigma(z + \chi(\cdot), \hat{z} + \hat{\chi}(\cdot))^L \beta(\cdot, z, \hat{z})^L w(\cdot, z, \hat{z}) \right) \\
&= \text{Ad} \left( \langle \hat{\sigma}(\hat{z} + \hat{\chi}(\cdot)), \sigma(- - z + \chi(\cdot)) - \sigma(-) \rangle^L \right. \\
&\quad \langle \hat{\sigma}(\hat{z} + \hat{\chi}(\cdot)) - \hat{\sigma}(\hat{\chi}(\cdot)), \sigma(z) + \sigma(\chi(\cdot)) \rangle^L \\
&\quad \left. w(\cdot, z, \hat{z}) \right) \\
&= \text{Ad} \left( \langle \hat{\sigma}(\hat{z}), \sigma(- - z) - \sigma(-) \rangle^L \right. \\
&\quad \langle \hat{z}, \sigma(- - z + \chi(\cdot)) - \sigma(\chi(\cdot)) - \sigma(- - z) \rangle^L \\
&\quad \langle \hat{\sigma}(\hat{\chi}(\cdot)), \sigma(- - z + \chi(\cdot)) - \sigma(-) \rangle^L \\
&\quad \left. w(\cdot, z, \hat{z}) \right),
\end{aligned}$$

where the first factor guaranties the Poincaré condition (cp. Lemma 11.2), and the remaining factors are already continuous as unitary functions.

The cocycle conditions for  $\lambda_\psi, \hat{\lambda}_{\hat{\psi}}$  follow from the cocycle conditions for  $\psi, \hat{\psi}$  which also imply that the classes defined in (45) are given by  $[\psi], [\hat{\psi}]$ . In fact,  $(\delta_\chi \psi)(a, b, \cdot, z) = 1$  is equivalent to

$$\underbrace{\psi(b, \cdot, z + \chi(a)) \rho(b)}_{\bar{\lambda}_\psi(b, z + \chi(a))} \underbrace{\psi(a, \cdot, z) \rho(a)}_{\bar{\lambda}_\psi(a, z)} = \underbrace{\psi(ba, \cdot, z) \rho(ba)}_{\bar{\lambda}_\psi(ba, z)} \underbrace{\psi(a, b, z)}_{\text{U}(1)} \in \text{U}(L^2(\Gamma)),$$

and the application of  $\text{Ad} : \text{U}(L^2(\Gamma)) \rightarrow \text{PU}(L^2(\Gamma))$  on both sides yields the cocycle condition.

By similar means, the equality

$$(49) \quad (\delta_{\chi \times \hat{\chi}} w)(a, \cdot, z, \hat{z}) (\delta_{\chi \times \hat{\chi}} \beta^L)(a, \cdot, z, \hat{z}) = \psi(a, \cdot, z) \hat{\psi}(a, \cdot, \hat{z})^{-1}$$

leads directly to the equality

$$\begin{aligned}
\lambda_\psi(a, z) \kappa_w(z, \hat{z}) &= \kappa_w(z + \chi(a), \hat{z} + \hat{\chi}(a)) \hat{\lambda}_{\hat{\psi}}(a, \hat{z}) \\
&\in \text{PU}(L^2(\Gamma) \otimes L^2(G/N))
\end{aligned}$$

and also reproduces that  $\beta^L w$  is the perturbation  $\alpha$  in (46).

This proves that  $(\kappa_w, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$  is a topological triple and is mapped to  $([w], [\psi], [\hat{\psi}])$  by (48). If we choose another representative of  $([w], [\psi], [\hat{\psi}])$ , the equivalence class of the constructed triple is not changed. Thus, our construction defines a map

$$(50) \quad T_L(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}) \rightarrow \text{Top}_L(\Gamma, \chi, \hat{\chi})$$

such that the composition with (48) is the identity on  $T_L(\Gamma, \chi, \hat{\chi})$ .

It remains to check that the composition of (48) with (50) is the identity on  $\text{Top}_L(\Gamma, \chi, \hat{\chi})$ . So start with a triple of order  $L$   $(\kappa^{\sigma^L} \text{Ad}(u), (\chi, \lambda), (\hat{\chi}, \hat{\lambda}))$ ,

then take its class  $([w], [\psi], [\hat{\psi}])$  and construct  $(\kappa_w, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$  as above.

We claim that the two triples are stably equivalent. In fact,  $(\kappa^{\sigma^L} \text{Ad}(u), (\chi, \lambda), (\hat{\chi}, \hat{\lambda}))$ , is stably equivalent to  $(\kappa^{\sigma^L} \text{Ad}(u) \otimes \mathbb{1}, (\chi, \lambda \otimes \text{Ad}(\rho)), (\hat{\chi}, \hat{\lambda} \otimes \text{Ad}(\rho)))$ , but the definitions of  $\psi$  and  $\hat{\psi}$  lead to

$$\begin{aligned} \lambda(a, z) \text{Ad}(\rho(a)) &= \underbrace{\text{Ad}(\bar{\lambda}(\cdot, z + \chi(a)))^{-1}}_{=: \theta(z + \chi(a))^{-1}} \underbrace{\text{Ad}(\psi(a, \cdot, z) \rho(a))}_{=: \lambda_\psi(a, z)} \underbrace{\text{Ad}(\bar{\lambda}(\cdot, z))}_{=: \theta(z)}, \end{aligned}$$

and

$$\begin{aligned} \hat{\lambda}(a, z) \text{Ad}(\rho(a)) &= \underbrace{\text{Ad}(\bar{\hat{\lambda}}(\cdot, \hat{z} + \hat{\chi}(a)))^{-1}}_{=: \hat{\theta}(\hat{z} + \hat{\chi}(a))^{-1}} \underbrace{\text{Ad}(\hat{\psi}(a, \cdot, \hat{z}) \rho(a))}_{=: \hat{\lambda}_{\hat{\psi}}(a, z)} \underbrace{\text{Ad}(\bar{\hat{\lambda}}(\cdot, \hat{z}))}_{=: \hat{\theta}(\hat{z})}. \end{aligned}$$

Thus the latter triple is isomorphic to  $(\theta(\kappa^{\sigma^L} \text{Ad}(u)) \hat{\theta}^{-1}, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$ . Finally, the isomorphism  $\theta(\kappa^{\sigma^L} \text{Ad}(u)) \hat{\theta}^{-1}$  can be computed using equation (46) which leads to

$$\theta(z) \kappa^\sigma(z, \hat{z})^L \text{Ad}(u(z, \hat{z})) \hat{\theta}^{-1}(\hat{z}) = \kappa_w(z, \hat{z}) \text{Ad}(u(z + \chi(\cdot), \hat{z} + \chi(\cdot))).$$

This shows that the triple  $(\theta(\kappa^{\sigma^L} \text{Ad}(u)) \hat{\theta}^{-1}, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$  is equivalent to  $(\kappa_w, (\chi, \lambda_\psi), (\hat{\chi}, \hat{\lambda}_{\hat{\psi}}))$ .

This proves Theorem 12.1.

### 13. REGULAR TOPOLOGICAL TRIPLES

Having the classification of the previous section at hand we introduce the notion of regularity of a topological triple. The motivation for considering regular topological triples comes from the fact that the underlying pair and the underlying dual pair of a regular triple (of order 1) admit natural extensions to dynamical triples in duality.

We will see in section 15 that in the important case of  $G = \mathbb{R}^n$ ,  $N = \mathbb{Z}^n$  all topological triples are regular.

**Definition 13.1** *A topological triple of order  $L$  is called **regular** if the first component of its class  $([w], [\psi], [\hat{\psi}]) \in T_L(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}})$  is zero:*

$$[w] = 0 \in H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}}) / H^1(\Gamma, M_{\chi \times \hat{\chi}}).$$

For a finite group  $\Gamma$  we denote by

$$\text{Top}_L^\dagger(\Gamma) \subset \text{Top}_L(\Gamma)$$

the set of all (equivalence classes of) regular topological triples, and similarly,

$$\text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi}) \subset \text{Top}_L(\Gamma, \chi, \hat{\chi})$$

is the set of all regular topological triples with fixed homomorphisms  $\chi, \hat{\chi}$ .

**Corollary 13.1** 1. Let  $(\kappa, (\chi, \lambda), (\hat{\chi}, \hat{\lambda}))$  be a regular topological triple of order  $L \in \mathbb{Z}$ , then

$$L \cdot (\chi \sqcup \hat{\chi}) = 0 \in H^3(\Gamma, \mathbb{U}(1)).$$

2. If, conversely, for any two homomorphisms  $\chi$  and  $\hat{\chi}$  we have that  $L \cdot (\chi \sqcup \hat{\chi}) = 0$ , then there exists a regular topological triple of order  $L$  extending these two homomorphisms.

3. Moreover, the set of regular topological triples of order  $L$ ,  $\text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi})$ , is a  $H^2(\Gamma, \mathbb{U}(1))$ -torsor.

**Proof :** As  $C : \chi \sqcup \hat{\chi} \mapsto \chi \cup \hat{\chi}$  in (43), all three statements of the corollary follow immediately from the exactness of the vertical sequence in (43) and the classification of topological triples. ■

Now, take a regular topological triple in form of a class  $([w], [\psi], [\hat{\psi}]) \in T_L(\Gamma, \chi, \hat{\chi})$ . Choose representatives  $(w, \psi, \hat{\psi})$  satisfying equation (49). The condition  $[w] = 0 \in H^1(\Gamma, M_{\chi \times \hat{\chi}} / M_\chi M_{\hat{\chi}}) / H^1(\Gamma, M_{\chi \times \hat{\chi}})$  means for the function  $w : \Gamma \rightarrow M_{\chi \times \hat{\chi}}$  that it has a decomposition

$$w(a, z, \hat{z}) = v(a, z, \hat{z}) v_1(z) v_2(\hat{z}),$$

where  $\delta_{\chi \times \hat{\chi}} v = 1$ . Thus, if we re-normalise  $\psi$  to  $\psi(a, b, z) v_1(z + \chi(a))^{-1}$  and  $\hat{\psi}$  to  $\hat{\psi}(a, b, \hat{z}) v_2(\hat{z} + \hat{\chi}(a))$ , then equation (49) turns into

$$(51) \quad (\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi})(a, b, z, \hat{z})^L = \psi(a, b, z) \hat{\psi}(a, b, \hat{z})^{-1}.$$

This equality together with the structure of  $\chi \sqcup_{\sigma, \hat{\sigma}} \hat{\chi}$  as stated in Lemma 12.1 give rise to a natural map

$$(52) \quad \hat{L}_* : \text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi}) \rightarrow \text{Top}_1^\dagger(\Gamma, \chi, L\hat{\chi})$$

which we describe next. Putting  $z = 0$  in (51) we see that

$$\hat{\psi}(a, b, \hat{z}) = \gamma(a, b)^{-L} \langle L\hat{z}, \eta(a, b) \rangle^{-1} \psi(a, b, 0)$$

Thus we can define

$$(\hat{L}_* \hat{\psi})(a, b, \hat{z}) := \gamma(a, b)^{-L} \langle \hat{z}, \eta(a, b) \rangle^{-1} \psi(a, b, 0).$$

It follows from  $\delta_{\hat{\chi}} \hat{\psi} = 1$  that  $\delta_{L\hat{\chi}} (\hat{L}_* \hat{\psi}) = 1$ . We leave it to the reader to verify the identity

$$(53) \quad \psi(a, b, z) (\hat{L}_* \hat{\psi})(a, b, \hat{z})^{-1} (\delta_\chi v)(a, b, z) = (\chi \sqcup_{\sigma, \hat{\sigma}} L\hat{\chi})(a, b, z, \hat{z}),$$

wherein  $v(a, z) := \langle L\hat{\sigma}(\hat{\chi}(a)) - \hat{\sigma}(L\hat{\chi}(a)), z + \chi(a) \rangle$  and  $\chi \sqcup_{\sigma, \hat{\sigma}} L\hat{\chi}$  is as in Lemma 12.1 but with  $\hat{\chi}$  replaced by  $L\hat{\chi}$ . Taking the cohomology on both sides of (53) we find

$$[\psi] \ominus [\hat{L}_* \hat{\psi}] = \chi \sqcup (L\hat{\chi}) \in H^2(\Gamma, M_\chi M_{L\hat{\chi}}).$$

In other words  $(1, [\psi], [\hat{L}_* \hat{\psi}])$  is the class of a topological triple of order 1. This defines (52).

Similarly, one defines a natural map

$$(54) \quad L_* : \text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi}) \rightarrow \text{Top}_1^\dagger(\Gamma, L\chi, \hat{\chi}),$$

and it is straight forward to check that the maps defined are bijections whose inverses are the restrictions of the two maps in (38) to regular topological triples. In fact, one can easily generalise this construction to give an inverse of the restriction of (37) to regular triples.

We give another application of (51). Namely, applying therein the boundary operator  $d$  we get

$$(55) \quad \begin{aligned} (d\psi)(a, b, g, z) &= \psi(a, b, gN + z)\psi(a, b, z)^{-1} \\ &= \langle \hat{\eta}(a, b), gN \rangle^{-L} \\ &= \phi(b, g, z + \chi(a))\phi(ba, g, z)^{-1}\phi(a, g, z) \\ &= (\delta_\chi \phi)(a, b, g, z), \end{aligned}$$

where  $\phi(a, g, z) := \langle \hat{\sigma}(\hat{\chi}(a)), g \rangle^{-L}$ , and if we define  $\hat{\phi}(a, \hat{g}, \hat{z}) := \langle \hat{g}, \sigma(\chi(a)) \rangle^{-L}$ , we obtain similarly

$$(56) \quad (\hat{d}\hat{\psi})(a, b, \hat{g}, \hat{z}) = (\delta_{\hat{\chi}} \hat{\phi})(a, b, \hat{g}, \hat{z}).$$

These two equations determine two classes

$$[\psi, \phi, 1] \in H^{2,+}(\Gamma, G, M_\chi), \quad [\hat{\psi}, \hat{\phi}, 1] \in H^{2,+}(\Gamma, \hat{G}, M_{\hat{\chi}}),$$

and hence, by the classification of dynamical triples, we obtain two maps

$$(57) \quad \begin{array}{ccc} & \text{Top}_L^\dagger(\Gamma) & \\ \epsilon_L(\Gamma) \swarrow & & \searrow \hat{\epsilon}_L(\Gamma) \\ \text{Dyn}^\dagger(\Gamma) & & \widehat{\text{Dyn}}^\dagger(\Gamma) \end{array}$$

which are natural in  $\Gamma$ .

The maps of (57), (52) and (54) behave well to each other:

**Lemma 13.1** *The diagrams*

$$\begin{array}{ccc} & \text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi}) & \\ & \swarrow \quad \downarrow & \\ \text{Dyn}^\dagger(\Gamma) & \longleftarrow \text{Top}_1^\dagger(\Gamma, \chi, L\hat{\chi}) & \\ & & \end{array} \quad \begin{array}{ccc} & \text{Top}_L^\dagger(\Gamma, \chi, \hat{\chi}) & \\ & \downarrow \quad \searrow & \\ \text{Top}_1^\dagger(\Gamma, L\chi, \hat{\chi}) & \longrightarrow & \widehat{\text{Dyn}}^\dagger(\Gamma) \end{array}$$

commute.

**Proof :** To prove the commutativity of the first diagram note that the boundary operator  $d$  applied to (53) leads to

$$\begin{aligned} d(\psi\delta_\chi v)(a, b, g, z) &= \langle \hat{\eta}_L(a, b), gN \rangle^{-1} \\ &= \langle \hat{\eta}(a, b), gN \rangle^{-L} (d\delta_\chi v)(a, g, z) \\ &= \delta_\chi(\phi dv)(a, b, g, z), \end{aligned}$$

with  $\phi$  as in (55). So the cocycle  $(\psi\delta_\chi v, \phi dv, 1) = (\psi, \phi, 1) \cdot \partial_\chi(v, 1)$  which represents the composition in the first diagram is cohomologous to  $(\psi, \phi, 1)$  which represents the diagonal map.

The commutativity of the second diagram is proven by an analogous argument.  $\blacksquare$

If we stick the constructed maps for different homomorphisms  $\chi, \hat{\chi}$  together, then we obtain commutative diagrams

$$\begin{array}{ccc} & \text{Top}_L^\dagger(\Gamma) & \\ \epsilon_L(\Gamma) \swarrow & \downarrow \hat{L}_* & \searrow \\ \text{Dyn}^\dagger(\Gamma) & \text{Top}_1^\dagger(\Gamma) & \\ \epsilon_1(\Gamma) \swarrow & \downarrow & \searrow \\ & \text{Par}(\Gamma) & \end{array} \quad \begin{array}{ccc} & \text{Top}_L^\dagger(\Gamma) & \\ L_* \downarrow & \hat{\epsilon}_L(\Gamma) \searrow & \\ \text{Top}_1^\dagger(\Gamma) & \xrightarrow{\hat{\epsilon}_1(\Gamma)} & \widehat{\text{Dyn}}^\dagger(\Gamma), \\ \downarrow & & \searrow \\ \widehat{\text{Par}}(\Gamma) & & \end{array}$$

wherein the maps ending at  $\text{Par}(\Gamma)$  or at  $\widehat{\text{Par}}(\Gamma)$  are the natural forgetful maps. Note that in contrast to (52) and (54) the assembled maps  $L_*, \hat{L}_* : \text{Top}_L^\dagger(\Gamma) \rightarrow \text{Top}_1^\dagger(\Gamma)$  are no isomorphisms in general, as they fail to be surjective.

In the next section we show that  $\epsilon_1(\Gamma)$  and  $\hat{\epsilon}_1(\Gamma)$  are bijections and that the composition  $\epsilon_1(\Gamma)^{-1} \circ \hat{\epsilon}_1(\Gamma)$  is the duality map  $\hat{\phantom{x}}$  introduced in (29).

#### 14. FROM DYNAMICAL TO TOPOLOGICAL TRIPLES

Consider a dualisable dynamical triple which is represented by a class  $[\psi, \phi, 1] \in H_{\text{cont}}^{2,+}(\Gamma, G, M_\chi)$ . Let  $[\hat{\psi}, \hat{\phi}, 1] \in H_{\text{cont}}^{2,+}(\Gamma, \hat{G}, M_{\hat{\chi}})$  be its dual according to the duality map (29). By forgetting  $\phi$  and  $\hat{\phi}$  we obtain a class  $([\psi], [\hat{\psi}]) \in H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}})$ .

**Proposition 14.1** *In the Mayer-Vietoris sequence (39) the class  $([\psi], [\hat{\psi}])$  is mapped to  $\chi \sqcup \hat{\chi}$  by the subtraction map  $\ominus$ .*

**Proof :** The class  $[\psi] \ominus [\hat{\psi}]$  is represented by  $\psi\hat{\psi}^{-1}$  which we can compute using the definition of  $\hat{\psi}$  in (27). Let us denote by  $\sigma : G/N \rightarrow G$  and  $\hat{\sigma} : \hat{G}/N^\perp \rightarrow \hat{G}$  Borel sections of the quotient maps  $G \rightarrow G/N, \hat{G} \rightarrow \hat{G}/N^\perp$ . We also let  $\varphi(a, z) := \phi(a, \sigma(z), 0) \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(z) \rangle$  which is a well-defined

continuous function due to the definition of  $\hat{\chi}$ . We have

$$\begin{aligned} \psi(a, b, z) \hat{\psi}(a, b, \hat{z})^{-1} &= \psi(a, b, z) \psi(a, b, 0)^{-1} \phi(b, \sigma(\chi(a)), 0) \\ &\quad \langle \hat{\chi}(ba) + \hat{z}, \hat{\sigma}(\chi(b)) + \hat{\sigma}(\chi(a)) - \hat{\sigma}(\chi(ba)) \rangle. \end{aligned}$$

Using the cocycle conditions  $d\psi = \delta_\chi \phi$ ,  $d\phi = 1$  and the definition of  $\phi$  this can finally be transformed to

$$\begin{aligned} \psi(a, b, z) \hat{\psi}(a, b, \hat{z})^{-1} &= (\delta_\chi \phi)(a, b, z) (\delta_\chi s)(a, b) \\ &\quad \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(\chi(b)) \rangle \\ &\quad \langle \hat{\sigma}(\hat{\chi}(ba)) - \hat{\sigma}(\hat{\chi}(a)) - \hat{\sigma}(\hat{\chi}(b)), \sigma(\chi(ba)) \rangle \\ &\quad \langle \hat{\sigma}(\hat{\chi}(ba)) - \hat{\sigma}(\hat{\chi}(a)) - \hat{\sigma}(\hat{\chi}(b)), \sigma(z) \rangle \\ &\quad \langle \hat{z}, \sigma(\chi(b)) + \sigma(\chi(a)) - \sigma(\chi(ba)) \rangle, \end{aligned}$$

where  $s(a) := \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(\chi(a)) \rangle$ . If we compare this with the results of Lemma 12.1, the proposition follows.  $\blacksquare$

By this proposition, the class  $([\psi], [\hat{\psi}]) \in H^2(\Gamma, M_\chi) \oplus H^2(\Gamma, M_{\hat{\chi}})$  may also be considered as a class  $(1, [\psi], [\hat{\psi}]) \in T_1(\Gamma, \chi, \hat{\chi}) / H^1(\Gamma, M_{\chi \times \hat{\chi}})$ , and therefore we obtain (the class of) a regular topological triple by the classification of topological triples. Hence, the assignment  $(\chi, [\psi, \phi, 1]) \mapsto (1, [\psi], [\hat{\psi}])$  defines natural maps

$$(58) \quad \begin{aligned} \delta(\Gamma, \chi) : \text{Dyn}^\dagger(\Gamma, \chi) &\rightarrow \text{Top}_1^\dagger(\Gamma, \chi), \\ \delta(\Gamma) : \text{Dyn}^\dagger(\Gamma) &\rightarrow \text{Top}_1^\dagger(\Gamma). \end{aligned}$$

In the next three lemmata we point out the relation between  $\delta(\Gamma, \chi)$  and the maps  $\epsilon_1(\Gamma, \chi)$ ,  $\hat{\epsilon}_1(\Gamma, \chi)$  constructed in the previous section.

**Lemma 14.1** *The composition  $\epsilon_1(\Gamma, \chi) \circ \delta(\Gamma, \chi)$  is the identity on  $\text{Dyn}^\dagger(\Gamma, \chi)$ .*

**Proof :** Start with a cocycle  $(\chi, (\psi, \phi, 1))$  representing a dynamical triple. In the course of the proof of Proposition 14.1 we calculated  $\psi(a, b, z) \hat{\psi}(a, b, \hat{z})^{-1}$ . If we apply here the boundary operator  $d$ , we get after some steps

$$\begin{aligned} d(\psi \delta_\chi \phi^{-1})(a, b, g, z) &= \langle \hat{\sigma}(\hat{\chi}(b)) - \hat{\sigma}(\hat{\chi}(ba)) - \hat{\sigma}(\hat{\chi}(a)), gN \rangle^{-1} \\ &= \delta_\chi(\phi d \phi^{-1})(a, b, g, z). \end{aligned}$$

Thus the cocycle  $(\psi \delta_\chi \phi^{-1}, \phi d \phi^{-1}, 1) = (\psi, \phi, 1) \cdot \partial_\chi(\phi^{-1}, 1)$  which represents the composition is cohomologous to  $(\psi, \phi, 1)$ .  $\blacksquare$

**Lemma 14.2** *The composition  $\delta(\Gamma, \chi) \circ \epsilon_1(\Gamma, \chi)$  is the identity on  $\text{Top}_1^\dagger(\Gamma, \chi)$ .*

**Proof :** Let  $(\psi, \hat{\psi})$  satisfy (51) for  $L = 1$ , taking therein  $z = 0$  we find

$$\hat{\psi}(a, b, \hat{z}) = \gamma(a, b)^{-1} \langle \hat{z}, \eta(a, b) \rangle^{-1} \psi(a, b, 0).$$

Now, let  $\phi$  be defined according to (55), and then let  $\hat{\psi}'$  be defined by (27), so

$$\begin{aligned} \hat{\psi}'(a, b, \hat{z}) &= \psi(a, b, 0) \langle \hat{\sigma}(\hat{\chi}(b)), \sigma(\chi(a)) \rangle \\ &\quad \langle \hat{\chi}(ba) + \hat{z}, \sigma(\chi(ba)) - \sigma(\chi(b)) - \sigma(\chi(a)) \rangle. \end{aligned}$$

This may be transformed to

$$\hat{\psi}'(a, b, \hat{z}) = \hat{\psi}(a, b, \hat{z})(\delta_\chi \nu)(a, b, z)$$

where  $\nu(a, z) := \langle \hat{\sigma}(\hat{\chi}(a)), \sigma(\chi(a)) \rangle^{-1}$  (independent of  $z$ ). Thus, we have the equality  $[\psi, \hat{\psi}] = [\psi, \hat{\psi}'] \in H^2(\Gamma, M_\chi M_{\hat{\chi}})$  which proves the lemma. ■

We have just proven that  $\delta(\Gamma, \chi)$  is the inverse of  $\epsilon_1(\Gamma, \chi)$ . Even more is true:

**Lemma 14.3** *The composition  $\hat{\epsilon}_1(\Gamma) \circ \delta(\Gamma)$  is the duality map  $\hat{\phantom{x}}$  introduced in (29).*

**Proof :** Let  $(\chi, (\psi, \phi, 1))$  represent a dualisable dynamical triple. Its dual is represented by  $(\hat{\chi}, (\hat{\psi}, \hat{\phi}, 1))$  according to (26), (27) and (28).

Now, forget  $\hat{\phi}$ , and compute  $\psi \hat{\psi}^{-1}$  as in the course of the proof of Proposition 14.1. Then apply the boundary operator  $\hat{d}$  which defines  $\hat{\phi}'$  according to (56). We even have  $\hat{\phi}' = \hat{\phi}$ , so the cocycles  $(\hat{\psi}, \hat{\phi}', 1)$  and  $(\hat{\psi}, \hat{\phi}, 1)$  which represent the composition  $\hat{\epsilon}_1(\Gamma) \circ \delta(\Gamma)$  and the duality map  $\hat{\phantom{x}}$  coincide. ■

Dually to the construction of  $\delta(\Gamma)$ , we have a natural map

$$\hat{\delta}(\Gamma) : \widehat{\text{Dyn}}^\dagger(\Gamma) \rightarrow \text{Top}^\dagger(\Gamma)$$

with analogous properties.

To summarise this section, we have a commutative diagram of natural isomorphisms

$$\begin{array}{ccc} & \text{Top}^\dagger(\Gamma) & \\ \delta(\Gamma)=\epsilon(\Gamma)^{-1} \nearrow & & \nwarrow \hat{\delta}(\Gamma)=\hat{\epsilon}(\Gamma)^{-1} \\ \text{Dyn}^\dagger(\Gamma) & \xrightarrow{\hat{\phantom{x}}} & \widehat{\text{Dyn}}^\dagger(\Gamma) \end{array}$$

## 15. TOPOLOGICAL TRIPLES AND K-THEORY

In this section we fix a natural number  $n \in \{1, 2, 3, \dots\}$  and we let

$$G := \mathbb{R}^n, \quad N := \mathbb{Z}^n.$$

We identify the dual group of  $\mathbb{R}$  with  $2\pi\mathbb{R}$  containing the dual lattice  $2\pi\mathbb{Z}$ . The  $n$ -torus is  $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ , and the dual  $n$ -torus is  $\hat{\mathbb{T}}^n := 2\pi\mathbb{R}/2\pi\mathbb{Z}$  which should not be mixed up with the dual group  $\hat{\mathbb{T}}^n = 2\pi\mathbb{Z}$  of  $\mathbb{T}^n$ .

Recall the classification of topological triples from section 12. There we introduced the  $\Gamma$ -module  $M := C(\mathbb{T}^n \times \hat{\mathbb{T}}^n, \text{U}(1))$  with its submodules  $M_\chi := C(\mathbb{T}^n, \text{U}(1))$  and  $M_{\hat{\chi}} := C(\hat{\mathbb{T}}^n, \text{U}(1))$  on which  $\Gamma$  acts with (the negative of) homomorphisms  $\chi : \Gamma \rightarrow \mathbb{T}^n$  and  $\hat{\chi} : \Gamma \rightarrow \hat{\mathbb{T}}^n$ . In the classification of topological triples the group  $H^n(\Gamma, M_{\chi \times \hat{\chi}}/M_\chi M_{\hat{\chi}})/H^1(\Gamma, M_{\chi \times \hat{\chi}})$  occurred, and as we saw in section 13, this group has an interpretation as an obstruction group.



However, in the case of  $G = \mathbb{R}^n$ ,  $N = \mathbb{Z}^n$ , the next lemma simplifies the discussion essentially.

**Lemma 15.1** *If  $G = \mathbb{R}^n$  and  $N = \mathbb{Z}^n$ , then  $H^k(\Gamma, M_{\chi \times \hat{\chi}} / M_{\chi} M_{\hat{\chi}}) = 0$  for all  $k > 0$ .*

**Proof :** Let  $V := C_{\text{null}}(\mathbb{T}^n \times \hat{\mathbb{T}}^n, \mathbb{R})$  denote the vector space of real valued, null-homotopic and base point preserving functions on the product of the torus and the dual torus. Let  $V_1 := C_{\text{null}}(\mathbb{T}^n, \mathbb{R})$  and  $V_2 := C_{\text{null}}(\hat{\mathbb{T}}^n, \mathbb{R})$  be the subspaces of those functions of  $V$  which only depend on one variable. The group  $V \times \text{U}(1)$ , respectively  $V_1 \times V_2 \times \text{U}(1)$ , can be identified with the subgroup of  $M_{\chi \times \hat{\chi}}$ , respectively  $M_{\chi} M_{\hat{\chi}}$ , consisting of all null-homotopic functions. The crucial point is that the respective quotients, the homotopy classes of maps, are isomorphic, namely

$$\begin{aligned} [\mathbb{T}^n, \text{U}(1)] \times [\hat{\mathbb{T}}^n, \text{U}(1)] &\xrightarrow{\cong} [\mathbb{T}^n \times \hat{\mathbb{T}}^n, \text{U}(1)]. \\ ([f], [g]) &\longmapsto [f(\cdot)g(\cdot)] \end{aligned}$$

This leads us to a diagram of short exact sequences of  $\Gamma$ -modules

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V_1 \times V_2 \times \text{U}(1) & \longrightarrow & M_{\chi} M_{\hat{\chi}} & \longrightarrow & [\mathbb{T}^n, \text{U}(1)] \times [\hat{\mathbb{T}}^n, \text{U}(1)] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & V \times \text{U}(1) & \longrightarrow & M_{\chi \times \hat{\chi}} & \longrightarrow & [\mathbb{T}^n \times \hat{\mathbb{T}}^n, \text{U}(1)] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V/V_1 \times V_2 & \longrightarrow & M_{\chi \times \hat{\chi}} / M_{\chi} M_{\hat{\chi}} & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

It follows that  $M_{\chi \times \hat{\chi}} / M_{\chi} M_{\hat{\chi}}$  is  $\Gamma$ -isomorphic to a vector space, and this proves the lemma, because the cohomology of a finite group with values in a vector space always vanishes.  $\blacksquare$

By definition of regularity (Def. 13.1), we have:

**Corollary 15.1** *If  $G = \mathbb{R}^n$  and  $N = \mathbb{Z}^n$ , then every topological triple is regular.*

In particular, if we focus our attention to topological triples of order 1, then we have isomorphisms

$$(59) \quad \text{Dyn}^{\dagger}(\Gamma) \cong \text{Top}_1(\Gamma) \cong \widehat{\text{Dyn}}^{\dagger}(\Gamma).$$

Now, let  $(\kappa, (E, P), (\hat{E}, \hat{P}))$  be a topological triple of order  $L$ , and let us denote by  $(L_*\kappa, (L_*E, L_*P), (\hat{E}, \hat{P}))$  (a representative of) the image of the triple  $(\kappa, (E, P), (\hat{E}, \hat{P}))$  under  $L_*$  (s. (54)). Let  $C(L^*)$  denote the mapping cone of the pullback  $L^* : C^*(L_*E, L_*P) \rightarrow C^*(E, P)$  (cf. (11), p. 10). As the topological triple  $(L_*\kappa, L_*(E, P), (\hat{E}, \hat{P}))$  has order 1, we have by (59) an

isomorphism  $K_{\Gamma, L_* P}^*(L_* E) \cong K_{\Gamma, \widehat{P}}^{*+n}(\widehat{E})$  as explained in (30). So if we replace  $K_{\Gamma, L_* P}^*(L_* E)$  by  $K_{\Gamma, \widehat{P}}^{*+n}(\widehat{E})$  in the six term exact sequence (13), we find an exact sequence

$$\begin{array}{ccccc} K_{\Gamma, \widehat{P}}^n(\widehat{E}) & \longrightarrow & K_{\Gamma, P}^0(E) & \longrightarrow & K_1(C(L^*)) \\ \uparrow & & & & \downarrow \\ K_0(C(L^*)) & \longleftarrow & K_{\Gamma, P}^1(E) & \longleftarrow & K_{\Gamma, \widehat{P}}^{1+n}(\widehat{E}). \end{array}$$

Dually, we also have an exact sequence

$$\begin{array}{ccccc} K_{\Gamma, P}^n(E) & \longrightarrow & K_{\Gamma, \widehat{P}}^0(\widehat{E}) & \longrightarrow & K_1(C(\widehat{L}^*)) \\ \uparrow & & & & \downarrow \\ K_0(C(\widehat{L}^*)) & \longleftarrow & K_{\Gamma, \widehat{P}}^1(\widehat{E}) & \longleftarrow & K_{\Gamma, P}^{1+n}(E), \end{array}$$

where  $C(\widehat{L}^*)$  is the mapping cone of the pullback  $\widehat{L}^* : C^*(\widehat{L}_* \widehat{E}, \widehat{L}_* \widehat{P}) \rightarrow C^*(\widehat{E}, \widehat{P})$ . Here  $(L_* \kappa, (E, P), (\widehat{L}_* \widehat{E}, \widehat{L}_* \widehat{P}))$  is (a representative of) the image of  $(\kappa, (E, P), (\widehat{E}, \widehat{P}))$  under  $\widehat{L}_*$  (s. (52)).

In the special case of  $L = 1$  these sequences reduce to the isomorphism (30), namely:

**Corollary 15.2** *Let  $G = \mathbb{R}^n$  and  $N = \mathbb{Z}^n$ , and let  $(\kappa, (E, P), (\widehat{E}, \widehat{P}))$  be a topological triple of order 1. Then there is an isomorphism in twisted equivariant K-theory*

$$K_{\Gamma, P}^*(E) \cong K_{\Gamma, \widehat{P}}^{*+n}(\widehat{E}).$$

**Proof :** For  $L = 1$  we have  $1^* = \text{id}_{C^*(E, P)}$  and  $\widehat{1}^* = \text{id}_{C^*(\widehat{E}, \widehat{P})}$ . Thus, the mapping cones  $C(1^*)$  and  $C(\widehat{1}^*)$  above are contractible and their K-theories vanish.  $\blacksquare$

## 16. CONCLUSION AND OUTLOOK

In this work we considered topological (T-duality) triples over the singleton space  $*$  with an action of a finite group  $\Gamma$ . In more learned terms, this is a topological triple over the proper étale groupoid  $\Gamma \rightrightarrows *$ . Topological (T-duality) triples over a space  $B$  as considered in [1, 3, 10] may be understood as topological triples over the proper étale groupoid  $B \rightrightarrows B$ . In a future work we are going to combine the methods of [10] and this work to deal with topological (T-duality) triples over any proper étale groupoid  $\Gamma_1 \rightrightarrows \Gamma_0$ , that is a topological (T-duality) triple over  $\Gamma_0$  in the sense of [1, 3, 10] equipped with an action of  $\Gamma_1$ . We then will also derive the corresponding isomorphism in K-theory (cf. Corollary 15.2), and, after proving Morita invariance of the set-up, we also obtain the corresponding statement



where  $C^{k,l}(\Gamma, V.)$  is the set of all continuous functions  $\Gamma^l \times \coprod_{\alpha \in I^{k+1}} V_\alpha \rightarrow \underline{\mathbf{U}}(1)$ , for  $V_\alpha := V_{i_0} \cap \dots \cap V_{i_k}$  if  $\alpha = (i_0, \dots, i_k)$ . The boundary operators  $\delta_\chi$  and  $\check{\delta}$  are the usual boundary operators of group and Čech cohomology. The corresponding total cohomology of this complex is denoted by  $\check{H}_\Gamma^\bullet(V., \underline{\mathbf{U}}(1))$ , and we define the colimit over all  $\Gamma$ -equivariant coverings

$$\check{H}_\Gamma^\bullet(G/N, \underline{\mathbf{U}}(1)) := \operatorname{colim}_V \check{H}_\Gamma^\bullet(V., \underline{\mathbf{U}}(1)).$$

For an equivariant bundle isomorphism

$$\begin{array}{ccc} \lambda \operatorname{Ad}(l) \circlearrowleft G/N \times \operatorname{PU}(\mathcal{H}) & \xrightarrow{v} & G/N \times \operatorname{PU}(\mathcal{H}) \circlearrowleft \lambda \\ \downarrow & & \downarrow \\ \chi \circlearrowleft G/N & \xrightarrow{=} & G/N \circlearrowleft \chi \\ \downarrow & & \downarrow \\ * & \xrightarrow{=} & * \end{array}$$

we are going to define its class  $[v] \in \check{H}_\Gamma^1(G/N, \underline{\mathbf{U}}(1))$ .

Let  $\{V_i | i \in I\}$  be a  $\Gamma$ -equivariant covering of  $G/N$  such that the isomorphism  $v$  is given by functions  $v_i : V_i \rightarrow \operatorname{PU}(\mathcal{H})$  which permit unitary lifts:

$$\begin{array}{ccc} & \operatorname{U}(\mathcal{H}) & \\ & \nearrow \bar{v}_i & \downarrow \\ V_i & \xrightarrow{v_i} & \operatorname{PU}(\mathcal{H}). \end{array}$$

We define  $c_{ij} : V_i \cap V_j \rightarrow \underline{\mathbf{U}}(1)$  by

$$c_{ij}(z, \hat{z}) := v_i(z)v_j(z)^{-1}.$$

Now, for the cocycles  $\lambda, \lambda \operatorname{Ad}(l) : \Gamma \times G/N \rightarrow \operatorname{PU}(\mathcal{H})$  we have

$$\lambda(a, z)v_i(z) = v_i(z + \chi(a))\hat{\lambda}(a, z)\operatorname{Ad}(l(a, z)),$$

for all  $i \in I$ . This equality can be used to define functions  $\gamma_i : \Gamma \times V_i \rightarrow \underline{\mathbf{U}}(1)$  by

$$v_i(z) = \lambda(a, z)^{-1}[v_i(z + \chi(a))]l(a, z)\gamma_i(a, z, \hat{z}).$$

A short calculation shows that  $(c_{..}, \gamma_{..})$  is a 1-cocycle in (60), i.e.

$$\check{\delta}c_{..} = 1, \quad \delta_{\chi \times \hat{\chi}}c_{..} = \check{\delta}\gamma_{..}, \quad \delta_{\chi \times \hat{\chi}}\gamma_{..} = 1.$$

The class of  $(c_{..}, \gamma_{..})$  in  $\check{H}_\Gamma^1(G/N, \underline{\mathbf{U}}(1))$  is denoted by  $[v]$ . We leave it to the reader to verify that the class is independent of the choices of  $v_i, \bar{v}_i$ .

APPENDIX C. THE EQUIVARIANT ČECH CLASS  $[\kappa, \kappa']$ 

The definition of  $[\kappa, \kappa']$  is essentially the same as the definition of  $[v]$ , it just concerns more free parameters.

For two homomorphisms  $\chi : \Gamma \rightarrow G/N$  and  $\hat{\chi} : \Gamma \rightarrow \widehat{G}/N^\perp$  we consider a  $\Gamma$ -equivariant covering  $W := \{W_i | i \in I\}$  of the space  $G/N \times \widehat{G}/N^\perp$ . This yields a double complex

$$(61) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & \\ C^{2,0}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{2,1}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{2,2}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & \dots \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & \\ C^{1,0}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{1,1}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{1,2}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & \dots \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & \\ C^{0,0}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{0,1}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & C^{0,2}(\Gamma, W.) & \xrightarrow{\delta_{\chi \times \hat{\chi}}} & \dots, \end{array}$$

where  $C^{k,l}(\Gamma, W.)$  is the set of all continuous functions  $\Gamma^l \times \prod_{\alpha \in I^{k+1}} W_\alpha \rightarrow \mathbf{U}(1)$ , for  $W_\alpha := W_{i_0} \cap \dots \cap W_{i_k}$  if  $\alpha = (i_0, \dots, i_k)$ . The boundary operators  $\delta_{\chi \times \hat{\chi}}$  and  $\delta$  are the usual boundary operators of group and Čech cohomology. The corresponding total cohomology of this complex is denoted by  $\check{H}_\Gamma^\bullet(W., \mathbf{U}(1))$ , and we define the colimit over all  $\Gamma$ -equivariant coverings

$$\check{H}_\Gamma^\bullet(G/N \times \widehat{G}/N^\perp, \mathbf{U}(1)) := \operatorname{colim}_W \check{H}_\Gamma^\bullet(W., \mathbf{U}(1)).$$

Consider two topological triples  $(\kappa, (P, E), (\widehat{P}, \widehat{E}))$  and  $(\kappa', (P', E'), (\widehat{P}', \widehat{E}'))$  (with underlying Hilbert spaces  $\mathcal{H}, \mathcal{H}'$  respectively) such that the underlying pairs  $(P, E)$  and  $(P', E')$  and the dual pairs  $(\widehat{P}, \widehat{E})$  and  $(\widehat{P}', \widehat{E}')$  are outer conjugate. We are going to define the class

$$[\kappa, \kappa'] \in \check{H}_\Gamma^1(G/N \times \widehat{G}/N^\perp, \mathbf{U}(1)).$$

Let  $\{W_i | i \in I\}$  be a  $\Gamma$ -equivariant covering of  $G/N \times \widehat{G}/N^\perp$  such that the isomorphisms  $\kappa, \kappa'$  are given by functions  $\kappa_i, \kappa'_i : W_i \rightarrow \mathbf{PU}(\mathcal{H})$  which permit unitary lifts:

$$\begin{array}{ccc} & & \mathbf{U}(\mathcal{H}) \\ & \nearrow \bar{\kappa}_i, \bar{\kappa}'_i & \downarrow \\ W_i & \xrightarrow{\kappa_i, \kappa'_i} & \mathbf{PU}(\mathcal{H}). \end{array}$$

We define  $k_i(z, \hat{z}) := \bar{\kappa}_i(z, \hat{z}) \bar{\kappa}'_i(z, \hat{z})^{-1}$  and  $c_{ij} : W_i \cap W_j \rightarrow \mathbf{U}(1)$  by

$$c_{ij}(z, \hat{z}) := k_i(z, \hat{z}) k_j(z, \hat{z})^{-1}.$$

Now, if  $\lambda, \lambda' : \Gamma \times G/N \rightarrow \text{PU}(\mathcal{H})$  and  $\hat{\lambda}, \hat{\lambda}' : \Gamma \times \widehat{G}/N^\perp \rightarrow \text{PU}(\mathcal{H})$  are the cocycles implementing the  $\Gamma$ -actions, we have

$$\begin{aligned}\lambda(a, z)\kappa_i(z, \hat{z}) &= \kappa_i(z + \chi(a), \hat{z} + \hat{\chi}(a))\hat{\lambda}(a, \hat{z}), \\ \lambda'(a, z)\kappa'_i(z, \hat{z}) &= \kappa'_i(z + \chi(a), \hat{z} + \hat{\chi}(a))\hat{\lambda}'(a, \hat{z}),\end{aligned}$$

for all  $i \in I$ . By use of the outer equivalence of the actions,  $\lambda' = \lambda \text{Ad}(l)$ ,  $\hat{\lambda}' = \hat{\lambda} \text{Ad}(\hat{l})$ , we eliminate  $\hat{\lambda}$  and obtain

$$\begin{aligned}\kappa_i(z, \hat{z})\kappa'_i(z, \hat{z})^{-1} &= \lambda(a, z)^{-1}\kappa_i(z + \chi(a), \hat{z} + \hat{\chi}(a))\kappa'_i(z + \chi(a), \hat{z} + \hat{\chi}(a))^{-1}\lambda(a, z) \\ &\quad \text{Ad}(l(a, z))\kappa'_i(z, \hat{z})\text{Ad}(\hat{l}(a, \hat{z}))^{-1}\kappa'_i(z, \hat{z})^{-1}.\end{aligned}$$

This equality can be used to define functions  $\gamma_i : \Gamma \times W_i \rightarrow \text{U}(1)$  by

$$k_i(z, \hat{z}) = \lambda(a, z)^{-1}[k_i(z + \chi(a), \hat{z} + \hat{\chi}(a))]l(a, z)\kappa'_i(z, \hat{z})[\hat{l}(a, \hat{z})]^{-1}\gamma_i(a, z, \hat{z}).$$

A short calculation shows that  $(c_{..}, \gamma_{..})$  is a 1-cocycle in (61), i.e.

$$\check{\delta}c_{..} = 1, \quad \delta_{\chi \times \hat{\chi}}c_{..} = \check{\delta}\gamma_{..}, \quad \delta_{\chi \times \hat{\chi}}\gamma_{..} = 1.$$

The class of  $(c_{..}, \gamma_{..})$  in  $\check{H}_\Gamma^1(G/N \times \widehat{G}/N^\perp, \text{U}(1))$  is denoted by  $[\kappa, \kappa']$ , and we leave it to the reader to check that it is independent of the chosen charts and the chosen lifts.

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