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SOME COMPLEMENTS TO THE LAZARD ISOMORPHISM

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Abstract. Lazard showed in his seminal work [L] that for rational coefficients continuous group cohomology of p-adic Lie-groups is isomorphic to Lie-algebra cohomology. We refine this result in two directions: firstly we extend his isomorphism under certain conditions to integral coefficients and secondly, we show that for algebraic groups, his isomorphism can be realized by differentiating locally analytic cochains.

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1. Introduction

One of the main results of Lazard's magnus opus [L] on p-adic Lie-groups is a comparison isomorphism between continuous group cohomology, analytic group cohomology and Lie-algebra cohomology. This comparison isomorphism is an important tool in the cohomological study of Galois representations in arithmetic geometry. More recently, it also appeared in topology and homotopy theory in connection with the formal groups associated to cohomology theories and in particular with topological modular forms.

Lazard's comparison theorem holds for $\mathbb{Q}_p$-vector spaces and the isomorphism between continuous cohomology and Lie-algebra cohomology is obtained from a difficult isomorphism between the saturated group ring and the saturated universal enveloping algebra. For some applications (e.g. the connection with the Bloch-Kato exponential map in [HK]) it is important to have a version for integral coefficients and a better understanding of the map between the cohomology theories.

In this paper we extend and complement the comparison isomorphism in two directions:

The first result is an integral version of the isomorphism assuming technical conditions (Theorem 3.1.1). For uniform pro-p-groups one gets a clean result with only a mild condition on the module (Theorem 3.3.2). To our knowledge and with the notable exception of Totaro's work [T], this is the first progress on a problem which Lazard wrote "reste à faire" more than forty years ago [L, Introduction 7,C)].

The second result concerns the definition of the isomorphism in the case of smooth group schemes. Here one can define directly a map from analytic group cohomology to Lie-algebra cohomology with constant coefficients by differentiating cochains (see Definition 4.2.1). We showed in [HK] (see Proposition 4.2.4) that the resulting map is Lazard's comparison isomorphism modulo the identification of continuous cohomology with analytic cohomology. Serre mentioned to us that this was clear to him at the time Lazard's paper was written, however, it was not included in the published results. Unfortunately, we were so far not able to use this simple map to obtain an independent proof of Lazard's comparison result.

The advantage of this description of the map is not only its simplicity but also that it carries over to K-Lie-groups for finite extensions $K/\mathbb{Q}_p$. In Theorem 4.3.1 we prove that this map is also an isomorphism in the case of K-Lie-groups attached to smooth group schemes with connected generic fiber over the integers of K. This generalizes results in [HK] for GL_n and complements the result of Lazard, who only treats $\mathbb{Q}_p$-analytic groups.

The paper is organized as follows. In Section 2 we give a quick tour through the notions from [L] that we need. We hope that this section also proves to be a useful overwiew of the central notions and results in [L]. In Section 3 we prove our integral refinement of Lazard's isomorphism. Finally

Section 4 considers the isomorphism over a general base in the case of group schemes.

2. Review of some results by Lazard

In this section we recall the basic notions about groups and group rings we need to formulate our main results. As we proceed we illustrate the main notions with the example of separated smooth group schemes. We hope that this section helps to guide through the long and difficult paper by Lazard.

2.1. **Saturated groups.**

Definition 2.1.1. [L, II 1.1, II 1.2.10, III 2.1.2] A *filtration* on a group G is a map

$$\omega : G \to \mathbb{R}_+^* \cup \{\infty\}$$

such that

1) For all $x, y \in G$, $\omega(xy^{-1}) \geq \inf\{\omega(x), \omega(y)\}$
2) For all $x, y \in G$, $\omega(x^{-1}y^{-1}xy) \geq \omega(x) + \omega(y)$.

G is called *p-filtered* if in addition for all $x \in G$

$$\omega(x^p) \geq \inf\{\omega(x) + 1, p\omega(x)\}.$$

G is called *p-valued* if ω satisfies

3) $\omega(x) < \infty$ for $x \neq e$
4) $\omega(x) > (p-1)^{-1}$ for all $x \in G$
5) $\omega(x^p) = \omega(x) + 1$ for all $x \in G$.

G is called *p-divisible* if it is p-valued and

6) For all $x \in G$ with $\omega(x) > 1 + \frac{1}{p-1}$ there exists $y \in G$ with $y^p = x$.

Finally, a p-divisible group G is *saturated*, if

7) G is complete for the topology defined by the filtration.

Note that a filtration satisfying the conditions of a p-valuation is automatically p-filtered.

Recall that a pro-p-group is the inverse limit of finite p-groups. This is the case we are going to work with.

Proposition 2.1.2. [L, II 2.1.3] *A p-filtered group is a pro-p-group if and only if it is compact.*

We denote by $\mathbb{F}_p[\epsilon]$ the polynomial ring with generator ϵ in degree 1.

Definition and Lemma 2.1.3. [L, II 1.1., III 2.1.1, III 2.1.3] Let (G, ω) be filtered.

(1) For every $\nu \in \mathbb{R}_+^*$

$$G_\nu := \{x \in \mathbb{G} \,|\, \omega(x) \geq \nu\}, G_\nu^+ := \{x \in G \,|\, \omega(x) > \nu\} \subseteq G$$

are normal subgroups.

(2)
$$\mathrm{gr}(G) := \bigoplus_{\nu\in\mathbb{R}_+^*} G_\nu/G_\nu^+,$$
$\mathrm{gr}(G)$ is a graded Lie-algebra over $\mathbb{F}_p$, the Lie-bracket being induced by the commutator in G.

(3) If (G,ω) is p-valued, then $\mathrm{gr}(G)$ is even a graded $\mathbb{F}_p[\epsilon]$-Lie-algebra, the action of ϵ being induced by $x \mapsto x^p$ ($x \in G_\nu$, $x^p \in G_{\nu+1}$)

(4) In this case, $\mathrm{gr}(G)$ is free as a graded $\mathbb{F}_p[\epsilon]$-module. The *rank of* G is by definition the rank of the $\mathbb{F}_p[\epsilon]$-module $\mathrm{gr}(G)$.

Example 2.1.4. [L, V 2.2.1] Let (G,ω) be a complete p-valued group of finite rank d.

If $\{x_i\}_{i=1,\dots,d} \subseteq G$ are representatives of an ordered basis of the $\mathbb{F}_p[\epsilon]$-module $\mathrm{gr}(G)$, then every $y \in G$ is uniquely an ordered product $y = \prod_{i=1}^{d} x_i^{\lambda_i}$ with $\lambda_i \in \mathbb{Z}_p$ and
$$\omega(y) = \inf_i(\omega(x_i) + v(\lambda_i)),$$
where the valuation on $\mathbb{Z}_p$ is normalized by $v(p) = 1$. (G,ω) has rank d.

Definition 2.1.5. [L, V 2.2.1, V 2.2.7]

(1) In the situation of the example, the family $\{x_i\}_{i=1,\dots,d}$ is called an *ordered basis of* G.

(2) The p-valued group (G,ω) is called *equi-p-valued* if it there exists an ordered basis $\{x_i\}$ as above such that
$$\omega(x_i) = \omega(x_j) \text{ for all } 1 \le i,j \le d.$$

2.2. **Serre's standard groups as examples.** Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers R. Let $\mathfrak{m}$ be the maximal ideal of R. E is a discretely valued field. We normalize its valuation v by $v(p) = 1$. Let e be the ramification index of $E/\mathbb{Q}_p$.

Any formal group law $F(X,Y)$ in n variables over R defines a group structure G on $\mathfrak{m}^n$. These are the *standard groups* as defined by Serre.

Define for $(x_1,\dots,x_n) \in \mathfrak{m}^n$
$$\omega(x_1,\dots,x_n) := \inf_i\{v(x_i)\}.$$
Then we have for $\lambda \ge 0$
$$G_\lambda := \{x \in G : \omega(x) \ge \lambda\}.$$

Proposition 2.2.1. [S1, LG 4.23 Theorem 1 and Corollary and 4.25 Corollary 2 of Theorem 2] *G is a pro-p-group. For any $\lambda \ge 0$ the group G_λ is a normal subgroup of G. Moreover, the map ω defines a filtration on G in the sense of definition 2.1.1,1)-3).*

One can show that a subgroup of the standard group G is saturated.

Lemma 2.2.2. *Let E and G be as above. Let ρ be the smallest integer with $\rho > \frac{e}{p-1}$ then the subgroup (H, ω),*

$$H := \left\{ x \in G : \omega(x) > \frac{1}{p-1} \right\} = G_{\frac{\rho}{e}}$$

is saturated and of finite rank. It is equi-p-valued if and only if $e = 1$.

Proof. Note first that according to [S1, LG 4.21, Corollary] the power series f_p, which defines the p-power map, is of the form

$$f_p(X) = p(X + \varphi(X)) + \psi(X)$$

with $\mathrm{ord}(\varphi) \geq 2$ and $\mathrm{ord}(\psi) \geq p$. It follows that

$$\omega(x^p) \geq \inf\{\omega(x) + 1, p\omega(x)\},\ x \in G$$

because if x has coordinates x_i, then x^p has coordinates

$$f_{p,i}(x_1, \dots, x_n) = px_i + p\varphi_i(x) + \psi_i(x)$$

and the valuations of the summands are bounded below by $1 + \omega(x)$, $1 + 2\omega(x) > 1 + \omega(x)$ and $p\omega(x)$, respectively.
As $\omega(x) > \frac{1}{p-1}$ is equivalent to $\omega(x) + 1 < p\omega(x)$, this implies on H

$$\omega(x^p) \geq \omega(x) + 1.$$

On the other hand, let x_i be a coordinate of x with $\omega(x) = v(x_i)$. Then

$$\omega(x^p) \leq v(px_i + p\phi_i(x) + \psi_i(x)) = 1 + \omega(x_i) = 1 + \omega(x)$$

and hence

$$\omega(x^p) = \omega(x) + 1$$

for all $x \in H$. This shows that (H, ω) is p-valued. To see that H is saturated, we note that by [S1, LG 4.26, Theorem 4], the p-power map induces an isomorphism

$$H_\lambda \xrightarrow{\cong} H_{\lambda+1}$$

for all $\lambda \in (\frac{1}{p-1}, \infty) \cap v(\mathfrak{m})$. As H is complete, this implies that the group is saturated.

The valuation on R takes values in $\frac{1}{e}\mathbb{Z}$ where e is the ramification index. By definition $H = H_{\frac{\rho}{e}}$. We get

$$\mathrm{gr}(H) \cong \bigoplus_{\lambda \in \frac{1}{e}\mathbb{N}, \lambda \geq \frac{\rho}{e}} k^n\,.$$

As an $\mathbb{F}_p[\epsilon]$-module $\mathrm{gr}(H)$ is freely generated by an $\mathbb{F}_p$-basis of

$$\bigoplus_{\lambda \in \frac{1}{e}\mathbb{N}, 1 + \frac{\rho}{e} > \lambda \geq \frac{\rho}{e}} k^n$$

This is finite because $[k : \mathbb{F}_p] < \infty$.

If $e = 1$, then only a single λ occurs in the sum, namely ρ. If $e > 1$, then $1 + \frac{\rho}{e} > \frac{\rho+1}{e}$ and the sum has generators in more than one degree. $\square$

An important example of the above construction arises from separated smooth group schemes $\mathbb{G}/R$. The formal completion $\widehat{\mathbb{G}}$ of $\mathbb{G}$ along its unit section is a formal group over R and the associated standard group G is via $g \mapsto 1+g$ isomorphic to

$$G \cong \ker(\mathbb{G}(R) \to \mathbb{G}(k)).$$

Example 2.2.3. To have an even more concrete example, consider $\mathbb{G} = \mathrm{Gl}_n$ over R. Let π be the uniformizer. Then

$$\pi^\rho R = \left\{ x \in R \mid v(x) > \frac{1}{p-1} \right\}.$$

It follows from Lemma 2.2.2 that

$$H := 1 + \pi^\rho M_n(R) \subset \mathrm{Gl}_n(R)$$

is a saturated subgroup with respect to the filtration $\omega(1+(x_{i,j})) = \inf_{i,j}(v(x_{i,j}))$. As

$$\mathrm{gr}\,(H) \cong \bigoplus_{\lambda \geq \frac{\rho}{e}, \lambda \in \frac{1}{e}\mathbb{N}} M_n(k)$$

the rank of H is $n^2[R:\mathbb{Z}_p]$. Note that this is not equi-p-valued for $e > 1$.

However, we can view $\mathrm{Gl}_n(R)$ as the group of $\mathbb{Z}_p$-valued points of the Weil-restriction $\mathbb{G}' = \mathrm{Res}_{R/\mathbb{Z}_p}\mathrm{Gl}_n$ which is a separated smooth group scheme over $\mathbb{Z}_p$ [BLR, 7.6, Proposition 5]. This point of view yields a different valuation on the corresponding standard group, which is

$$G' = \widehat{\mathbb{G}}'(\mathbb{Z}_p) = 1 + pM_n(R) \subseteq \mathrm{Gl}_n(R).$$

By Lemma 2.2.2 (G', ω') is saturated and equi-p-valued if $p > 2$.

As an explicit example, choose $R = \mathbb{Z}_p[\pi]$ with $\pi^2 = p$ (hence $e = 2$) and $n = 1$. Let $p > 3$ for simplicity. $G = 1 + \pi R$ has rank 2 with ordered basis $x_1 = 1 + \pi$, $x_2 = 1 + \pi^2$ with

$$\omega(x_1) = \frac{1}{2}, \omega(x_2) = 1$$

On the other hand, $G' = 1 + pR$ has also rank 2 with $x_1' := 1+p, x_2' := 1+\pi^3$ as an ordered basis. They satisfy

$$\omega'(x_1') = \omega'(x_2') = 1$$

G' is saturated and equi-p-valued. Compare this to

$$\omega(x_1') = 1, \omega(x_2') = \frac{3}{2}$$

under the inclusion $G' \subset G$.

2.3. **Valued rings, modules and the functor** Sat**.**

Definition 2.3.1. [L, I 2.1.1. and I 2.2.1] A *filtered* ring Ω is a ring together with a map

$$v : \Omega \to \mathbb{R}_+ \cup \{\infty\}$$

such that for $\lambda, \mu \in \Omega$

1) $v(\lambda - \mu) \geq \min(v(\lambda), v(\mu))$
2) $v(\lambda\mu) \geq v(\lambda) + v(\mu)$
3) $v(1) = 0$.

Put

$$\Omega_\nu := \{\lambda \in \Omega \mid v(\lambda) \geq \nu\}.$$

Ω is called *valued* if in addition

2') $v(\lambda\mu) \geq v(\lambda) + v(\mu)$
4) The topology defined on Ω by the filtration Ω_ν is separated.

Definition 2.3.2. [L, I 2.1.3 and I 2.2.2] A *filtered* module M over a filtered ring Ω is an Ω-module M together with a map

$$w : M \to \mathbb{R}_+ \cup \{\infty\}$$

such that for $x, y \in M$ and $\lambda \in \Omega$

1) $w(x - y) \geq \min(w(x), w(y))$
2) $v(\lambda x) \geq v(\lambda) + w(x)$

Put

$$M_\nu := \{x \in M \mid w(x) \geq \nu\}.$$

A filtered module over a valued ring Ω is called *valued* if in addition

2') $v(\lambda x) \geq v(\lambda) + w(x)$
3) The topology defined on M by the filtration M_ν is separated.

Let Ω be a commutative valued ring and A be an Ω-algebra (e.g. a Lie-algebra).

Definition 2.3.3. [L, I 2.2.4] An Ω-algebra A over a commutative valued ring Ω is *valued*, if it is valued as a ring and (with the same valuation map) valued as an Ω-module.

The following definition is an important technical tool in Lazard's work.

Definition 2.3.4. [L, I 2.2.7] A valued module M over a commutative valued ring Ω is called *divisible*, if for all $\lambda \in \Omega$ and $x \in M$ with $v(\lambda) \leq w(x)$ there exists $y \in M$ such that $\lambda y = x$. The module M is *saturated* if it is divisible and complete.

Lazard shows in [L, I 2.2.10] that the completion of a divisible module is saturated.

Definition 2.3.5. [L, I 2.2.11] The *saturation* $\mathrm{Sat}M$ of a valued module M over a commutative valued ring Ω is the completion of

$$\mathrm{div}M := \{y \in K \otimes_\Omega M \mid w(y) \geq 0\}\,.$$

Here, K is the fraction field of Ω and the valuation w on M is extended to $K \otimes_\Omega M$ by $w(\lambda^{-1} \otimes m) := w(m) - v(\lambda)$ (which is well-defined, see [L, I 2.2.8]).

The saturation $\mathrm{Sat}M$ satisfies the following universal property ([L, I. 2.2.11.]): For any morphism $f : M \to N$ of M into a saturated Ω-module N there is a unique extension to a map $\mathrm{Sat}M \to N$.

2.4. **Group rings.** In this section we fix $\Omega = \mathbb{Z}_p$ with the standard valuation. All algebras are over $\mathbb{Z}_p$.

For any group G let $\mathbb{Z}_p[G]$ be the group ring with coefficients in $\mathbb{Z}_p$.

Definition 2.4.1. [L, II 2.2.1] Let G be a pro-p-group. The *completed group ring* $\mathbb{Z}_p[[G]]$ is the projective limit

$$\mathbb{Z}_p[[G]] := \varprojlim \mathbb{Z}_p[G/U],$$

where U runs through all open normal subgroups of G and every $\mathbb{Z}_p[G/U]$ carries the p-adic topology.

In [L] this ring is denoted $\mathrm{Al}G$.

Definition 2.4.2. [L, III 2.3.1.2] Let G be a p-filtered group. The *induced filtration* w on $\mathbb{Z}_p[G]$ is the lower bound for all filtrations (as $\mathbb{Z}_p$-algebra) such that

$$w(x-1) \geq \omega(x) \text{ for all } x \in G\,.$$

Proposition 2.4.3. [L, III 2.3.3]*Let G be p-valued. Then the induced filtration w on $\mathbb{Z}_p[G]$ is a valuation (as $\mathbb{Z}_p$-module). If G is compact (or equivalently, pro-p), then $\mathbb{Z}_p[[G]]$ is the completion of $\mathbb{Z}_p[G]$ with respect to the valuation topology.*

Example 2.4.4. [L, V 2.2.1] Let G be p-valued, complete and of finite rank d. Let $\{x_i\}_{i=1,\dots,d} \subset G$ be an ordered basis of G. Then $\mathbb{Z}_p[[G]]$ admits $\{z^\alpha \mid \alpha \in \mathbb{N}^d\} \subseteq \mathbb{Z}_p[[G]]$,

$$z^\alpha := \prod_{i=1}^{d} (x_i - 1)^{\alpha_i}$$

as a topological $\mathbb{Z}_p$-basis satisfying

$$w(z^\alpha) = \sum_{i=1}^{d} \alpha_i \omega(x_i).$$

The associated graded is $U_{\mathbb{F}_p[\epsilon]}\mathrm{gr}(G)$, the universal enveloping algebra of the $\mathbb{F}_p[\epsilon]$-Lie-algebra $\mathrm{gr}(G)$.

Remark 2.4.5. Note that if (G, ω) is saturated and non-trivial, then $\mathbb{Z}_p[[G]]$ is never saturated. Indeed, since $\mathrm{gr}G \neq 0$ is a free $\mathbb{F}_p[\epsilon]$-module, we have $\mathrm{gr}^\nu G \neq 0$ for arbitrarily large ν, in particular there exists $g \in G$ with $\omega(g) \geq 1$. Then $x := g - 1 \in \mathbb{Z}_p[[G]]$ satisfies $w(x) \geq 1 = v(p)$, but x is not divisible by p in $\mathbb{Z}_p[[G]]$.

Lemma 2.4.6. *The inclusion $\mathbb{Z}_p[G] \to \mathbb{Z}_p[[G]]$ induces an isomorphism*

$$\mathrm{Sat}\mathbb{Z}_p[G] \cong \mathrm{Sat}\mathbb{Z}_p[[G]].$$

Proof. By [L, I 2.2.2] the natural map $\mathbb{Z}_p[G] \to \mathbb{Z}_p[[G]]$ is injective. It extends to

$$\mathrm{Sat}\mathbb{Z}_p[G] \to \mathrm{Sat}\mathbb{Z}_p[[G]]$$

On the other hand, $\mathrm{Sat}\mathbb{Z}_p[G]$ is complete, hence there is a natural map $\mathbb{Z}_p[[G]] \to \mathrm{Sat}\mathbb{Z}_p[G]$. As the right hand side is saturated, it extends to

$$\mathrm{Sat}\mathbb{Z}_p[[G]] \to \mathrm{Sat}\mathbb{Z}_p[G] \ .$$

The two maps are inverse to each other. $\square$

2.5. **Enveloping algebras.** Let L be a valued $\mathbb{Z}_p$-Lie-algebra and UL its enveloping algebra over $\mathbb{Z}_p$.

Definition 2.5.1. [L, IV 2.2.1] The *canonical filtration*

$$w : UL \to \mathbb{R}_+ \cup \{\infty\}$$

is the lowest bound of all filtrations on UL turning it into a valued $\mathbb{Z}_p$-algebra such that the canonical map $L \to UL$ is a morphism of valued modules.

Lemma 2.5.2. [L, IV 2.2.5] *UL equipped with the canonical filtration is a valued $\mathbb{Z}_p$-algebra and the natural morphism*

$$U\mathrm{gr}(L) \to \mathrm{gr}(UL)$$

is an isomorphism.

2.6. **Group-like and Lie-algebra-like elements.** Everything in this section applies to $A = \mathrm{Sat}\mathbb{Z}_p[[G]]$ where G is a p-valued pro-p-group. We fix $\Omega = \mathbb{Z}_p$ with its standard valuation.

Definition 2.6.1. [L, IV 1.3.1] Let A be a valued $\mathbb{Z}_p$-algebra with diagonal

$$\Delta : A \to \mathrm{Sat}(A \otimes_{\mathbb{Z}_p} A)$$

([L, IV 1.2.3]) and augmentation ϵ. Then we define $\mathcal{G}$, $\mathcal{L}$, $\mathcal{G}^*$ and $\mathcal{L}^*$ by

$$\mathcal{G} = \{x \in A | \epsilon(x) = 1, \Delta(x) = x \otimes x\} \tag{1}$$

$$\mathcal{G}^* = \{x \in \mathcal{G} | w(x) > (p-1)^{-1}\} \tag{2}$$

$$\mathcal{L} = \{x \in A | \Delta(x) = x \otimes 1 + 1 \otimes x\} \tag{3}$$

$$\mathcal{L}^* = \{x \in \mathcal{L} | w(x) > (p-1)^{-1}\} \tag{4}$$

These subsets have the following structures.

Lemma 2.6.2. [L, IV 1.3.2.1 and 1.3.2.2] *$\mathcal{G}$ and $\mathcal{G}^*$ are monoids with respect to the multiplication of A. If A is complete, $\mathcal{G}^*$ is a group and $\mathcal{L}$ and $\mathcal{L}^*$ are Lie-algebras. Moreover, $\mathcal{L} = \mathrm{div}\mathcal{L}^*$.*

For saturated $\mathbb{Z}_p$-algebras A we know much more:

Theorem 2.6.3. [L, IV 1.3.5] *Let A be a saturated $\mathbb{Z}_p$-algebra with diagonal.*

(1) *The exponential maps $\mathcal{G}^*$ to $\mathcal{L}^*$ and the logarithm maps $\mathcal{L}^*$ to $\mathcal{G}^*$. They are inverse homeomorphisms.*
(2) *The Lie-algebra $\mathcal{L}$ is saturated. It is the saturation of $\mathcal{L}^*$.*
(3) *$\mathcal{G}^*$ is a saturated group for the filtration $\omega(x) = w(x-1)$.*
(4) *The associated graded $\mathrm{gr}\mathcal{L}^*$ and $\mathrm{gr}\mathcal{G}^*$ are canonically isomorphic via the logarithm map.*
(5) *$\mathcal{L}^*$ and $\mathcal{G}^*$ generate the same saturated associative subalgebra of A.*

This has the following consequence for the universal enveloping algebra UL of a valued Lie-algebra L.

Theorem 2.6.4. [L, IV 3.1.2 and IV 3.1.3] *Let L be a valued Lie-algebra over $\mathbb{Z}_p$ and UL its universal enveloping algebra. Then*

$$\mathrm{Sat}UL = \mathrm{Sat}U\mathrm{Sat}L \tag{1}$$

$$\mathcal{L}\mathrm{Sat}UL = \mathrm{Sat}L \tag{2}$$

$$\mathcal{G}\mathrm{Sat}UL = \mathcal{G}^*\mathrm{Sat}UL \tag{3}$$

The next result concerns the saturation of the group ring $\mathbb{Z}_p[G]$ (or equivalently of $\mathbb{Z}_p[[G]]$ by 2.4.6).

Theorem 2.6.5. [L, IV 3.2.5] *Let G be a saturated group and $A = \mathrm{Sat}\mathbb{Z}_p[G]$ then*

$$\mathcal{G}^* = G.$$

Let $U\mathcal{L}$ be the universal enveloping algebra of $\mathcal{L}$, then the canonical map

$$\mathrm{Sat}U\mathcal{L} \to \mathrm{Sat}\mathbb{Z}_p[G]$$

is an isomorphism.

We introduce new terminology.

Definition 2.6.6. Let G be a saturated group. We call

$$\mathcal{L}^*(G) = \mathcal{L}^* \subset \mathrm{Sat}\mathbb{Z}_p[G]$$

the *integral Lazard Lie-algebra* of G.

The last theorem then reads

$$\mathrm{Sat}U\mathcal{L}^*(G) \cong \mathrm{Sat}\mathbb{Z}_p[G].$$

Example 2.6.7. Consider the saturated group $H = 1 + \pi^\rho M_n(R)$ from Example 2.2.3 and the algebra $\mathrm{Sat}\mathbb{Z}_p[H]$. We claim that the Lie-algebra

$\mathcal{L}^* = \mathcal{L}^*(H)$ is $\pi^\rho M_n(R)$ and $\mathcal{L} = M_n(R)$. To see this, note that by Theorem 2.6.5 $H = \mathcal{G}^*$ and that by Theorem 2.6.3 $\mathcal{L}^*$ consists of the logarithms of $\mathcal{G}^*$. By [L, III 1.1.4 and 1.1.5.].

$$\text{Log} : 1 + \pi^\rho M_n(R) \to \pi^\rho M_n(R) \ ; 1 + x \mapsto \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n} \tag{4}$$

$$\exp : \pi^\rho M_n(R) \to 1 + \pi^\rho M_n(R) \ ; x \mapsto \sum_{n\geq 0} \frac{x^n}{n!} \tag{5}$$

are both convergent and inverse to each other. By Theorem 2.6.3, $\mathcal{L}$ is the saturation of $\mathcal{L}^*$, which is by Definition 2.3.5

$$\mathcal{L} = \{x \in K \otimes_R \pi^\rho M_n(R) \mid w(x) \geq 0\} = M_n(R).$$

Example 2.6.8. In general the Lazard Lie-algebra does not coincide with the algebraic Lie-algebra. Let $\mathbb{G}$ be a separated smooth group scheme over $\mathbb{Z}_p$ and $\mathrm{Lie}(\mathbb{G})$ its $\mathbb{Z}_p$-Lie-algebra. If $t_1, \ldots, t_n$ are formal coordinates of $\mathbb{G}$ around e, then $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}$ are a $\mathbb{Z}_p$-basis of $\mathrm{Lie}(\mathbb{G})$.

Let G be the associated standard group over $\mathbb{Z}_p$ as in Section 2.2. Let H be the saturated subgroup of G, see Lemma 2.2.2. We have $H = G = (p\mathbb{Z}_p)^n$ if $p \neq 2$ and $H = (4\mathbb{Z}_2)^n$ if $p = 2$. Let $x_1, \ldots, x_n$ be the standard ordered basis of H. We put

$$\delta_i = \log x_i \in \mathrm{Sat}\mathbb{Z}_p[[H]].$$

By [L, IV 3.3.6] they form a $\mathbb{Z}_p$-basis of $\mathcal{L}^*(H)$. As explained in [HK] Section 4.2 and 4.3 the δ_i can be viewed as derivations on $\mathbb{Z}_p[[t_1, \ldots, t_n]]$, the coordinate ring of $\hat{\mathbb{G}}$. Note, however, that the coordinate λ_i in loc.cit. takes values in all of $\mathbb{Z}_p$ on G. Hence $\lambda_i = pt_i$ for $p \neq 2$. This implies

$$\delta_i = p\frac{\partial}{\partial t_i} \mid_{t=0}$$

Hence under the identification of [HK, Proposition 4.3.1], we have

$$\mathcal{L}^*(H) = p\mathrm{Lie}(\mathbb{G}).$$

For $p = 2$ the argument gives

$$\mathcal{L}^*(H) = 4\mathrm{Lie}(\mathbb{G}).$$

2.7. **Resolutions and Cohomology.**

Definition 2.7.1. [L, I 2.1.16, 2.1.17] Let A be a filtered $\mathbb{Z}_p$-algebra, M a filtered A-module.

(1) A family of A-linearly independent elements $(x_i)_{i\in I}$ of M is called *filtered free* if for every family $(\lambda_i)_{i\in I}$ of elements of A, almost all zero,

$$w\left(\sum_{i\in I} \lambda_i x_i\right) = \inf_i (w(x_i) + v(\lambda_i)).$$

M is called *filtered free* if it is generated by a filtered free family.

(2) Suppose A is complete. M is called *complete free* if it is the completion of the submodule generated by a filtered free family.

If A is complete and M is filtered free of finite rank, then M is also complete free.

Definition 2.7.2. [L, V 1.1.3, 1.1.4, 1.1.7] Let A be a filtered augmented $\mathbb{Z}_p$-algebra, M a filtered A-module.

(1) A *filtered acyclic resolution* $X_\bullet$ is a chain complex of filtered A-modules together with an augmentation $\epsilon : X_\bullet \to M$ such that for all $\nu \in \mathbb{R}_+$ the morphism $\epsilon_\nu : X_{\bullet\nu} \to M_\nu$ is a quasi-isomorphism.

(2) A *split filtered resolution* $X_\bullet$ of M is a morphism $\epsilon : X_\bullet \to M$ of chain complexes of filtered A-modules together with filtered morphisms of $\mathbb{Z}_p$-modules (sic, not A-linear!)

$$\eta : M \to X_0, \qquad s_n : X_n \to X_{n+1}$$

defining a homotopy between id and 0 on the extended complex $X_\bullet \xrightarrow{\epsilon} M$ and such that $s_0\eta = 0$. Note that a split filtered resolution is a filtered acyclic resolution.

(3) Let A be complete. We call *complete free acyclic resolution* of M a filtered acyclic resolution $X_\bullet$ by complete free modules.

(4) Let $X_\bullet$ be a complete free acyclic resolution of the trivial A-module $\mathbb{Z}_p$, M a complete A-module with linear topology. We call

$$H^n_c(A, M) = H^n(\mathrm{Hom}_c(X_\bullet, M))$$

(with Hom_c continuous A-linear maps) the *n-th continuous cohomology* of A with coefficients in M.

(5) Let A be an augmented $\mathbb{Z}_p$-algebra, M an A-module. We call

$$H^n(A, M) = \mathrm{Ext}^n_A(\mathbb{Z}_p, M)$$

the *n-th cohomology* of A with coefficients in M.

Let A be an augmented $\mathbb{Z}_p$-algebra. If $X_\bullet$ is a resolution of $\mathbb{Z}_p$ by free A-modules of finite rank, then $\mathrm{Hom}_A(X_\bullet, M)$ is a free resolution of M.

3. An integral version of the Lazard isomorphism

The purpose of this section is to establish that continuous group cohomology and Lie-algebra cohomology agree with integral coefficients, at least under certain technical assumptions. This generalizes Lazard's result for coefficients in $\mathbb{Q}_p$-vector spaces.

3.1. **Results.** We fix a saturated and compact group (G, ω) of finite rank d. In particular, G is a pro-p-group by Proposition 2.1.2. We assume

- (G, ω) is equi-p-valued
- ω takes values in $\frac{1}{e}\mathbb{Z}$.

Recall that the integral Lazard Lie-algebra

$$\mathcal{L}^*(G) = \mathcal{L}^*\mathrm{Sat}\mathbb{Z}_p[[G]],$$

is a finite free $\mathbb{Z}_p$-Lie-algebra.

For technical reasons we fix a totally ramified extension $\mathbb{Q}_p \subseteq K$ of degree e with ring of integers $\mathcal{O} \subseteq K$, uniformizer $\pi \in \mathcal{O}$. The valuation on $\mathcal{O}$ is normalized by $v(p) = 1$.

Let M be a linearly topologized complete $\mathbb{Z}_p$-module with a continuous, $\mathbb{Z}_p$-linear action of G. Thus, M is a $\mathbb{Z}_p[[G]]$-module ([L, II 2.2.6]). We assume that

- the $\mathbb{Z}_p[[G]]$-module structure on M extends to a $\mathrm{Sat}\mathbb{Z}_p[[G]]$-module structure.

M is canonically a $\mathcal{L}^*(G)$-module.

We are going to prove in Section 3.4:

Theorem 3.1.1. *Let (G, ω) and M be as above, then:*

(1) *There is an isomorphism of graded $\mathcal{O}$-modules*

$$\phi_G(M) : H_c^*(G, M) \otimes_{\mathbb{Z}_p} \mathcal{O} \simeq H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} \mathcal{O}.$$

It is natural in M.

(2) *If in addition M is a $\mathbb{Q}_p$-vector space, then this isomorphism agrees with Lazard's in* [L, V 2.4.9.]

(3) *Let H be another group satisfying the assumptions of the Theorem and $f : G \to H$ a group homomorphism filtered for the chosen filtrations. In addition assume that* $\mathrm{gr}(H)$ *is generated in degree $\frac{1}{e}$. Then the isomorphism is natural with respect to f.*

(4) *If* $\mathrm{gr}(G)$ *has generators in degree $\frac{1}{e}$, then the isomorphism is compatible with cup-products as follows:*
Assume that M', M'' satisfy the same assumptions as M does and that

$$\alpha : M \hat{\otimes}_{\mathbb{Z}_p} M' \to M''$$

is $\mathrm{Sat}\mathbb{Z}_p[[G]]$*-linear. Then the diagram*

$$\begin{tikzcd}
\big(H_c^*(G, M) \otimes_{\mathbb{Z}_p} H_c^*(G, M')\big) \otimes \mathcal{O} \arrow[r] \arrow[d, "\phi_G(M) \otimes \phi_G(M')"] & H_c^*(G, M'') \otimes \mathcal{O} \arrow[d, "\phi_G(M'')"] \\
\big(H^*(\mathcal{L}^*(G), M) \otimes_{\mathbb{Z}_p} H^*(\mathcal{L}^*(G), M')\big) \otimes \mathcal{O} \arrow[r] & H_c^*(\mathcal{L}^*(G), M'') \otimes \mathcal{O}
\end{tikzcd}$$

commutes. Here, the horizontal maps are the $\mathcal{O}$-linear extensions of the cup-product defined by α.

Remark 3.1.2. If $H^*(\mathcal{L}^*(G), M)$ is a finitely generated $\mathbb{Z}_p$-module, e.g. when M is of finite type, then this implies by the structure of finitely generated modules over principal ideal domains the existence of an isomorphism of graded $\mathbb{Z}_p$-modules

$$H_c^*(G, M) \simeq H^*(\mathcal{L}^*(G), M).$$

However, it is not clear if this isomorphism is natural or compatible with cup-products.

It is not obvious to see which groups satisfy the assumptions of Theorem 3.1.1. We discuss in section 3.3 standard groups and uniform pro-p-groups, which satisfy the assumptions of Theorem 3.1.1. The next section 3.2 discusses the assumptions with some examples.

3.2. **Some examples concerning the assumptions of Theorem 3.1.1.** Here we illustrate the assumptions of Theorem 3.1.1 by a series of remarks and examples.

The integral Lazard isomorphism may not hold for all topologically finitely generated pro-p-groups without p-torsion. However, the assumptions of the Theorem are too restrictive.

Example 3.2.1. Assume $p \geq 5$ and let $D/\mathbb{Q}_p$ be the quaternion-algebra, $\mathcal{O} \subseteq D$ its maximal order and $\Pi \in \mathcal{O}$ a prime element. By Lemma 2.2.2

$$G := 1 + \Pi\mathcal{O} \subseteq \mathcal{O}^*$$

is p-saturated. From [R, Theorem 6.3.22] or [H, Proposition 7], we know that

$$\dim_{\mathbb{F}_p} H^i_c(G, \mathbb{F}_p) = 1, 3, 4, 3, 1 \ (0 \leq i \leq 4).$$

In particular, $H^*_c(G, \mathbb{F}_p) \neq \Lambda^* H^1_c(G, \mathbb{F}_p)$ and G does not admit an equi-p-valuation by [L, V 2.2.6.3 and 2.2.7.2.] However, one can by direct arguments establish an isomorphism

$$H^*_c(G, \mathbb{F}_p) \simeq H^*(\mathcal{L}^*(G), \mathbb{F}_p)$$

of graded $\mathbb{F}_p$-algebras; c.f. Remark 3.4.10. The proof of [H, Proposition 7] shows that the same result holds for coefficients in $\mathbb{Z}_p$.

Not even saturatedness is necessary.

Example 3.2.2. Let $G = 1 + p^2\mathbb{Z}_p$ for $p \neq 2$. This group is not saturated but we have

$$\mathrm{Sat}(G) = 1 + p\mathbb{Z}_p.$$

Put

$$\mathcal{L}^*(G) \subset \mathcal{L}^*(\mathrm{Sat}(G))$$

the image of G under the logarithm map. We still get an isomorphism

$$H^*(G, \mathbb{Z}_p) \to H^*(\mathcal{L}^*(G), \mathbb{Z}_p)$$

induced by log. It is compatible with the one for $\mathrm{Sat}(G)$.

Remark 3.2.3.
(1) We are unaware of a group-theoretical characterization of those pro-p-groups satisfying the assumption of Theorem 3.1.1, but the remark on page 163 of [ST] suggests that they are closely related to uniform pro-p-groups.

(2) It is in general difficult to decide if a given $\mathbb{Z}_p[[G]]$-module structure extends over $\mathrm{Sat}\mathbb{Z}_p[[G]]$, and we refer to [T, page 200] and especially to the proof of [T, Corollary 9.3] for further discussion and useful sufficient conditions.
(3) In Theorem 3.3.2 we have established a sufficient condition for both problems, which have to be addressed here.

There are examples of groups which are saturated with respect to one filtration but not with respect to another. It can also happen that the group is saturated with respect to two filtrations but only equi-p-valued for one of them.

Example 3.2.4. Let $K/\mathbb{Q}_p$ be a finite extension with ramification index e. Let $\mathcal{O}$ be its ring of integers with uniformizer π. As discussed in Example 2.2.3 the group

$$1 + pM_n(\mathcal{O})$$

carries two natural filtrations ω and ω'. Recall that ρ is the smallest integer bigger than $\frac{e}{p-1}$.

(1) If $p = 5$, $e = 2$, then $\rho = 1$ and hence $\pi^\rho \neq 5$. This implies that $1 + 5M_n(\mathcal{O})$ is saturated with respect to ω' but not with respect to ω.
(2) If $p = 3$, $e = 2$, then $\rho = 2$ and hence $\pi^\rho = 3$. The group $1+3M_n(\mathcal{O})$ is saturated with respect to ω and ω', but but only equi-3-valued with respect to the second.

3.3. **The case of standard groups and uniform pro-p-groups.** We discuss two examples, where the assumptions of Theorem 3.1.1 are satisfied. First we consider standard groups and then uniform pro-p-groups.

Example 3.3.1. Let $\mathbb{G}/\mathbb{Z}_p$ be a separated smooth group scheme, $G = \ker(\mathbb{G}(\mathbb{Z}_p) \to \mathbb{G}(\mathbb{F}_p))$ the associated standard group (see Section 2.2). Its filtration takes values in $\mathbb{Z}$. By Lemma 2.2.2, there is an open subgroup H of G which is saturated and equi-p-valued. If $p \neq 2$, then $H = G$ and the generators have degree 1. If $p = 2$, then the generators have degree 2. H satisfies the assumptions of the Theorem with $e = 1$.

Let $M = \mathbb{Z}_p$ with the trivial operation of H. It also satisfies the assumptions of the Theorem. Hence there is a natural isomorphism of graded $\mathbb{Z}_p$-modules

$$H^*_c(H, \mathbb{Z}_p) \simeq H^*(\mathcal{L}^*(H), \mathbb{Z}_p).$$

For $p \neq 2$ it is compatible with cup-products.

This example generalizes to a larger class of groups. Recall the notion of a *uniform* or *uniformly powerful* pro-p-group from [DDMS, Definition 4.1]. Its Lie-algebra $\mathfrak{g}$ is constructed in [DDMS, §8.2] and coincides with the integral Lazard Lie algebra $\mathcal{L}^*(G)$ by [DDMS, Lemma 8.14].

Theorem 3.3.2. *Let $p \neq 2$ be a prime, G a uniform pro-p-group and M a finite free $\mathbb{Z}_p$-module with a continuous action of G such that the resulting group homomorphism*

$$\varrho : G \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(M)$$

has image in $1 + p\mathrm{End}_{\mathbb{Z}_p}(M)$. Then M is canonically a module for the Lie-algebra $\mathfrak{g}$ of G and there is an isomorphism of graded $\mathbb{Z}_p$-modules

$$H^*_c(G, M) \simeq H^*(\mathfrak{g}, M) \tag{6}$$

which is compatible with cup-products whenever these are defined.

Remark 3.3.3. If G is an arbitrary $\mathbb{Q}_p$-analytic group with $p \neq 2$ acting continuously on the finite free $\mathbb{Z}_p$-module M, then there is always an open subgroup $U \subseteq G$ such that the action of U on M satisfies the assumptions of Theorem 3.3.2.

Proof. We have the following two claims:

(1) G admits a valuation ω for which it is p-saturated of finite rank and equi-p-valued with an ordered basis consisting of elements of filtration 1.

(2) The $\mathbb{Z}_p$-module M admit a valuation w for which it is saturated and such that for all $g \in G, m \in M : w((g-1)m) \geq w(m) + \omega(g)$.

Granting these claims, we see as in [T, pages 200-201] that the $\mathbb{Z}_p[[G]]$-module structure of M extends over $\mathrm{Sat}\mathbb{Z}_p[[G]]$ and hence obtain (6) by applying Theorem 3.1.1,(1) with $\mathcal{O} = \mathbb{Z}_p$ and remarking that $\mathfrak{g} = \mathcal{L}^*((G, \omega))$. The proof of claim 1) is essentially given in [ST, Remark on page 163] but we include details for convenience: The lower p-series

$$G = G_1 \supseteq G_2 \supseteq \ldots$$

[DDMS, Definition 4.1] consists of normal subgroups satisfying $(G_n, G_m) \subseteq G_{n+m}$ and $\cap_{n \geq 1} G_n = \{e\}$ [DDMS, Proposition 1.16] and hence

$$\omega(x) := \sup\{n \in \mathbb{N} \,|\, x \in G_n\}, x \in G$$

defines a filtration of G by [L, II.1.1.2.4.]. Now [DDMS, Lemma 4.10] states that for all $n, k \geq 1$ the p^n-th power map of G is a homeomorphism $G_k \stackrel{\cong}{\to} G_{k+n}$ and induces bijections $G_k/G_{k+l} \stackrel{\cong}{\to} G_{k+n}/G_{k+n+l}$ for all $l \geq 0$.

From this we get all the properties of ω in Definition 2.1.1 we need:

4) trivial since $1 > \frac{1}{p-1}$.
5) If $x \in G$ has filtration $n = \omega(x)$ then $[x^p] \in G_{n+1}/G_{n+2}$ is non-trivial, i.e. $\omega(x^p) = \omega(x) + 1$.
6) If $x \in G$ satisfies $\omega(x) > 1 + \frac{1}{p-1}$ then $\omega(x) \geq 2$, hence $x \in G^p$.

As G is complete, we see that (G, ω) is p-saturated, clearly of finite rank. More precisely, from the above we get that $\mathrm{gr}G$ is $\mathbb{F}_p[\epsilon]$-free on $\mathrm{gr}^1 G$ and thus G is equi-p-valued with an ordered basis consisting of elements of filtration 1.

As for claim 2), we choose a $\mathbb{Z}_p$-basis $\{e_i\} \subseteq M$ and declare it to be a filtered basis with $w(e_i) = 0$, i.e.

$$w\left(\sum_i \lambda_i e_i\right) = \inf_i\{v(\lambda_i)\}, \text{ for } \lambda_i \in \mathbb{Z}_p$$

Clearly, (M, w) is saturated.

We consider the continuous homomorphism of pro-p-groups

$$\varrho : G \longrightarrow 1 + p\mathrm{End}_{\mathbb{Z}_p}(M) =: G'$$

and claim that the lower-p-series of G' is given by $G'_n = 1 + p^n\mathrm{End}_{\mathbb{Z}_p}(M)$, $n \geq 1$. Since G' is powerful, [DDMS, Lemma 2.4] gives $G'_{n+1} = \Phi(G'_n)$, the Frattini subgroup, for all $n \geq 1$ and arguing inductively, it suffices to see that $\Phi(1+p^n\mathrm{End}_{\mathbb{Z}_p}(M)) = 1+p^{n+1}\mathrm{End}_{\mathbb{Z}_p}(M)$. Since the Frattini subgroup is generated by p-th powers and commutators, we have "$\supseteq$" and [DDMS, Proposition 1.16] then gives

$$\begin{aligned}\Phi(1 + p^n\mathrm{End}_{\mathbb{Z}_p}(M))/(1 + p^{n+1}\mathrm{End}_{\mathbb{Z}_p}(M))& \\ &= \Phi((1 + p^n\mathrm{End}_{\mathbb{Z}_p}(M)))/(1 + p^{n+1}\mathrm{End}_{\mathbb{Z}_p}(M)) \\ &= \Phi((\mathbb{F}_p, +)^{n^2}) = 0.\end{aligned}$$

Since ϱ respects the lower p-series, we conclude that

$$\varrho(G_n) \subseteq 1 + p^n\mathrm{End}_{\mathbb{Z}_p}(M)\,,\ n \geq 1$$

which implies that $w((g-1)m) \geq w(m) + \omega(g)$ for all $g \in G, m \in M$.
If M', M'' satisfy the same assumptions as M does and

$$\alpha : M \otimes_{\mathbb{Z}_p} M' \to M''$$

is G-linear defining cup-products in $H^*(G, -)$ then both the source and the target of α are canonically $\mathrm{Sat}\mathbb{Z}_p[[G]]$-modules as seen above and α is $\mathrm{Sat}\mathbb{Z}_p[[G]]$-linear. Hence (6) is compatible with cup-products by Theorem 3.1.1,(4). $\square$

3.4. **Proof of Theorem 3.1.1.** We now describe the set-up for the rest of the section.

We fix a saturated group (G, ω) of finite rank d. Let

$$\mathcal{L}^*(G) = \mathcal{L}^*\mathrm{Sat}\mathbb{Z}_p[[G]]$$

be its integral Lazard Lie-algebra. It is a finite free $\mathbb{Z}_p$-module.

We fix an ordered basis $\{x_1, \ldots, x_d\} \subseteq G$, and put $\omega_i := \omega(x_i)$. For every $0 \leq k \leq n$ let

$$\mathcal{I}_k := \{(i_1, \ldots, i_k) \,|\, 1 \leq i_1 < \ldots < i_k \leq n\}$$

and for $I \in \mathcal{I}_k$ write $|I| := \sum_{s=1}^{k} \omega_s$. For $I \in \mathcal{I}_0 = \emptyset$ we put by abuse of notation $|I| = 0$.

We assume that there exists an integer $e \geq 1$ such that $\omega(G) \subseteq \frac{1}{e}\mathbb{Z}$ and fix a totally ramified extension $\mathbb{Q}_p \subseteq K$ of degree e with ring of integers $\mathcal{O} \subseteq K$, uniformizer $\pi \in \mathcal{O}$. The valuation on $\mathcal{O}$ is normalized by $v(p) = 1$. The artificial introduction of $\mathcal{O}$ is a trick invented by Totaro in [T]. In this section all valued modules and algebras are over $\mathcal{O}$. In particular, the saturation functor is taken in the category of valued $\mathcal{O}$-modules.

The inclusion $\mathbb{Z}_p \subseteq \mathcal{O}$ induces

$$\mathbb{F}_p[\epsilon] = \mathrm{gr}\mathbb{Z}_p \subseteq \mathrm{gr}\mathcal{O} = \mathbb{F}_p[\epsilon_K]$$

where ϵ (resp. ϵ_K) is the leading term of $p \in \mathbb{Z}_p$ (resp. $\pi \in \mathcal{O}$). We have $\epsilon_K^e \in \mathbb{F}_p^* \cdot \epsilon$, in particular the degree of ϵ_K is $\frac{1}{e}$.

If M is a valued $\mathcal{O}$-module, $\mathrm{gr}(M)$ is canonically a $\mathbb{F}_p[\epsilon_K]$-module. As pointed out by Totaro ([T, p. 201]) it follows directly from the definitions that

$$\mathrm{gr}(\mathrm{Sat}(M)) = \left(\mathrm{gr}M \otimes_{\mathbb{F}_p[\epsilon_K]} \mathbb{F}_p[\epsilon_K^{\pm 1}]\right)_{\mathrm{degree} \geq 0} .$$

Let

$$A := \mathcal{O}[[G]] := \varprojlim_{U \subseteq G \text{ open normal}} \mathcal{O}[G/U],$$

and

$$B := U_{\mathcal{O}}(\mathcal{L}^*(G) \otimes_{\mathbb{Z}_p} \mathcal{O})^\wedge = U_{\mathbb{Z}_p}(\mathcal{L}^*(G))^\wedge \otimes_{\mathbb{Z}_p} \mathcal{O},$$

the completion of the universal enveloping algebra with respect to its canonical filtration. (This filtration is easily seen (using Poincaré-Birkhoff-Witt) to be the p-adic filtration, hence the claimed equality because $\mathcal{O}$ is finite free as a $\mathbb{Z}_p$-module.) We finally introduce using Theorem 2.6.5

$$C := \mathrm{Sat}A \cong \mathrm{Sat}B.$$

Lemma 3.4.1. $\mathrm{gr}(A) = \mathrm{gr}(B)$ *inside* $\mathrm{gr}(C)$.

Proof. We have

$$\mathrm{gr}A = \mathrm{gr}(\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p} \mathcal{O}) = U_{\mathbb{F}_p[\epsilon]}(\mathrm{gr}G) \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K] = U_{\mathbb{F}_p[\epsilon_K]}(\mathrm{gr}G \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K]),$$

whereas

$$\mathrm{gr}B = \mathrm{gr}(U_{\mathbb{Z}_p}(\mathcal{L}^*(G)) \otimes_{\mathbb{Z}_p} \mathcal{O}) = U_{\mathbb{F}_p[\epsilon_K]}(\mathrm{gr}\mathcal{L}^*(G) \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K]).$$

Since $\mathrm{gr}\mathcal{L}^*(G) = \mathrm{gr}G$ by Theorem 2.6.5 and Theorem 2.6.3, (4) the claim follows. □

Remark 3.4.2. Moreover, Totaro shows in [T, pages 201-202] that for

$$\mathfrak{t} := (\mathrm{gr}G \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\mathrm{degree} \geq 0},$$

a finite graded free $\mathbb{F}_p[\epsilon_K]$-Lie-algebra with generators in degree zero, we have $\mathrm{gr}C = U_{\mathbb{F}_p[\epsilon_K]}(\mathfrak{t})$.

Lemma 3.4.3. (1) *Let X be a filtered free A-module with A-basis $e_1, \ldots, e_r$. Then* $\mathrm{Sat}X$ *is a filtered free C-module on generators*

$$e_i' = \pi^{-ew(e_i)} e_i, \ i = 1, \ldots, r$$

(2) *Let Y be a filtered free B-module with B-basis $f_1, \dots, f_s$. Then* $\mathrm{Sat}Y$ *is a filtered free C-module on generators*
$$f'_j = \pi^{-ew(f_j)} f_j,\ j = 1, \dots, s$$

Proof. It suffices to consider the case of the algebra A. The argument for B is the same. Without loss of generality $r = 1$. By construction (and because X is torsion-free), there are embeddings

$$\begin{tikzcd} & \mathrm{div}X \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & K \otimes_{\mathcal{O}} X \\ & (\mathrm{div}A) \otimes_A X \arrow[ur] & \end{tikzcd}$$

By assumption, any element x of $K \otimes X$ can be written in the form
$$x = \pi^v a e_1 = \pi^{v+ew(e_1)} a e'_1, \in \mathbb{Z}, a \in A$$
It is in $\mathrm{div}X$ if and only if
$$w(x) = \frac{v}{e} + w(a) + w(e_1) \geq 0$$
This is equivalent to $\pi^{v+ew(e_1)} a \in \mathrm{div}A$ and hence $x \in \mathrm{div}(A)e'_1$. Hence $\mathrm{div}X = (\mathrm{div}A) \otimes_A X$.

Finally, apply the completion functor to the equality. $\square$

Remark 3.4.4. This is the step where we make use of the coefficient extension to $\mathcal{O}$.

Both A and B are canonically subrings of C, and our first aim is to compare the cohomology of the (abstract) rings A and B with that of C. Both A and B are augmented $\mathcal{O}$-algebras, hence we have an A- (resp. B-) module structure on $\mathcal{O}$ which we will refer to as trivial.

Proposition 3.4.5. (1) *The trivial A-module $\mathcal{O}$ admits a resolution $X_\bullet$ such that X_k is filtered free of rank $\binom{d}{k}$ over A on generators $\{e_I \mid I \in \mathcal{I}_k\}$ of filtration $w(e_I) = |I|$.*

(2) *The trivial B-module $\mathcal{O}$ admits a resolution $Y_\bullet$ such that Y_k is filtered free of rank $\binom{d}{k}$ over B on generators $\{f_I \mid I \in \mathcal{I}_k\}$ of filtration $w(f_I) = |I|$.*

(3) *Furthermore, $X_\bullet$ and $Y_\bullet$ can be chosen such that* $\mathrm{gr}X_\bullet = \mathrm{gr}Y_\bullet$ *as complexes of* $\mathrm{gr}A = \mathrm{gr}B$*-modules.*

Proof. (1) The base extension from $\mathbb{Z}_p$ to $\mathcal{O}$ of the quasi-minimal complex of G has the desired properties [L, V 2.2.2.]. To see that the generators have the indicated filtration, remember that the quasi-minimal complex is obtained by lifting the standard complex $\overline{X}_\bullet$

of the $\mathbb{F}_p[\epsilon]$-Lie-algebra $\mathrm{gr}(G)$ which has $\overline{X}_k = \Lambda^k_{\mathbb{F}_p[\epsilon]}(\mathrm{gr}(G))$ finite graded free on
$$\{x_{i_1}G^+_{\omega_1} \wedge \ldots \wedge x_{i_k}G^+_{\omega_k}\}.$$
(2) The Lie-algebra $\mathcal{L}^*(G)$ is $\mathbb{Z}_p$-free on generators $\mathrm{Log}(x_i)$ of filtration ω_i. Hence the standard complex of $\mathcal{L}^*(G) \otimes_{\mathbb{Z}_p} \mathcal{O}$ is as desired.
(3) The equality $\mathrm{gr}X_\bullet = \mathrm{gr}Y_\bullet$ follows by construction from $\mathrm{gr}(G) \cong \mathrm{gr}\mathcal{L}^*(G)$.

□

Example 3.4.6. If G is equi-p-valued, i.e, $\omega_i = \omega_j$ for all i, j, then $X_\bullet$ and $Y_\bullet$ are *minimal* in the sense of [L, V 2.2.5], i.e., $X_\bullet \otimes \mathbb{F}_p$ and $Y_\bullet \otimes \mathbb{F}_p$ have zero differentials.

For the following, we fix complexes $X_\bullet$ and $Y_\bullet$ satisfying the conclusion of Proposition 3.4.5. Note that C is an augmented $\mathcal{O}$-algebra with augmentation extending both the one of A and the one of B.

Lemma 3.4.7. *Both* $\mathrm{Sat}X_\bullet$ *and* $\mathrm{Sat}Y_\bullet$ *are finite filtered resolutions of the trivial C-module $\mathcal{O}$, the modules $SatX_k$ (resp.* $\mathrm{Sat}Y_k$*) being filtered free on generators $\{\pi^{-e|I|}e_I \,|\, I \in \mathcal{I}_k\}$ (resp. $\{\pi^{-e|I|}f_I \,|\, I \in \mathcal{I}_k\}$) of filtration zero over C.*

Proof. Clearly, $\mathrm{Sat}X_\bullet$ and $\mathrm{Sat}Y_\bullet$ are canonically complexes of C-modules. Since both $X_\bullet$ and $Y_\bullet$ admit the structure of a *split* resolution, and this structure is preserved by the additive functor Sat, both $\mathrm{Sat}X_\bullet$ and $\mathrm{Sat}Y_\bullet$ are resolutions of $\mathrm{Sat}\mathcal{O} = \mathcal{O}$.

The statement on generators follows directly from Lemma 3.4.3. □

For $0 \le k \le n$ and $I \in \mathcal{I}_k$ denote by $e'_I \in \mathrm{Sat}X_k$ (resp. $f'_I \in \mathrm{Sat}Y_k$) the C-generators found above, i.e. $e'_I := \pi^{-e|I|}e_I, f'_I := \pi^{-e|I|}f_I$.
We see that the canonical morphisms of complexes over C
$$C \otimes_A X_\bullet \hookrightarrow \mathrm{Sat}X_\bullet$$
and
$$C \otimes_B Y_\bullet \hookrightarrow \mathrm{Sat}Y_\bullet$$
are injective.

We pause to remark that, evidently, the above injections are isomorphisms rationally, a key input in Lazard's comparison isomorphism for rational coefficients. Similarly, an integral version of this comparison isomorphism is essentially equivalent to $C \otimes_A X_\bullet$ being isomorphic to $C \otimes_B Y_\bullet$ and we proceed to prove this in a special case as follows.

Proposition 3.4.8. *There exists an isomorphism*
$$\phi : \mathrm{Sat}X_\bullet \to \mathrm{Sat}Y_\bullet$$
of filtered complexes over C such that $\mathrm{gr}\phi = \mathrm{id}$ *and $H^0(\phi)$ is the identity of $\mathcal{O}$. Any two such ϕ are chain homotopic where the homotopy h can be chosen such that* $\mathrm{gr}(h) = 0$.

Proof. In order to construct the isomorphism it suffices using [L, V 2.1.5] (applicable by Proposition 3.4.7) to canonically identify the complexes $\mathrm{grSat}X_\bullet$ and $\mathrm{grSat}Y_\bullet$ of $\mathrm{gr}C$-modules. Recall from Proposition 3.4.5 that

$$\mathrm{gr}X_\bullet = \mathrm{gr}Y_\bullet$$

This implies

$$\begin{aligned}\mathrm{grSat}X_\bullet &= (\mathrm{gr}X_\bullet \otimes_{\mathbb{F}_p[\epsilon_K]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\mathrm{degree}\geq 0} \\ &= (\mathrm{gr}Y_\bullet \otimes_{\mathbb{F}_p[\epsilon_K]} \mathbb{F}_p[\epsilon_K^{\pm 1}])_{\mathrm{degree}\geq 0} = \mathrm{grSat}Y_\bullet.\end{aligned}$$

Now, $H^0(\phi)$ is an $\mathcal{O}$-linear automorphism of $\mathcal{O}$, hence given by multiplication with a unit $\alpha \in \mathcal{O}^*$. Using that its associated graded is the identity, one easily obtains $\alpha = 1$, as claimed.

We turn to the construction of the homotopy. Let ϕ, ϕ' be isomorphisms as above. Let $e'_I \in \mathrm{Sat}(X_0)$ be a basis element. We need to define $h_0(e_I) \in \mathrm{Sat}(Y_0)$ such that

$$dh_0(e'_I) = (\phi - \phi')(e'_I) =: y_I$$

By assumption $\mathrm{gr}(\phi - \phi') = 0$, and hence $y_I \in \mathrm{Sat}(Y_0)_{\frac{1}{e}}$. As ϕ and ϕ' are isomorphisms of resolution of $\mathcal{O}$, we have $\epsilon(y_I) = 0$. Recall that $\mathrm{Sat}X_\bullet$ and $\mathrm{Sat}Y_\bullet$ are *filtered* resolutions. Hence y_I has a preimage $\tilde{y}_I \in \mathrm{Sat}(Y_1)_{\frac{1}{e}}$. Put

$$h_0(e'_I) = \tilde{y}_I$$

By C-linearity, this defines h_0. It satisfies $\mathrm{gr}(h_0) = 0$. As usual, the same argument can be used inductively to define h_i for all $i \geq 0$. ☐

Proposition 3.4.9. *If, in the situation of Proposition 3.4.8, (G, ω) is assumed to be equi-p-valued, then ϕ restricts to an isomorphism*

$$\psi : C \otimes_A X_\bullet \to C \otimes_B Y_\bullet$$

of complexes over C. If moreover $\mathrm{gr}(G)$ is generated in degree $\frac{1}{e}$, then any two such isomorphisms are homotopic.

Proof. We have the solid diagram of complexes over C

$$\begin{tikzcd} C \otimes_A X_\bullet \arrow[r, hook, "\iota_1"] \arrow[d, dotted, "\psi"] & \mathrm{Sat}X_\bullet \arrow[d, "\phi", "\simeq"'] \\ C \otimes_B Y_\bullet \arrow[r, hook, "\iota_2"] & \mathrm{Sat}Y_\bullet \end{tikzcd}$$

Since the horizontal maps are injective, ϕ factors as a chain-map if for every $0 \leq k \leq n$ we have

$$\phi_k(C \otimes_A X_k) \subseteq C \otimes_B Y_k. \tag{*}$$

If ψ exists, it is necessarily an isomorphism by completeness and the fact that its associated graded is the identity. Alternatively observe that the following argument applies likewise to ϕ^{-1} to produce an inverse of ψ.

To see what $(*)$ means, fix $0 \le k \le n$ and remember the C-generators $e_I \in C \otimes_A X_k, e'_I \in \mathrm{Sat}X_k, f_I \in C \otimes_B Y_k$ and $f'_I \in \mathrm{Sat}Y_k$ $(I \in \mathcal{I}_k)$ satisfying $\iota_1(e_I) = \pi^{e|I|}e'_I$ and $\iota_2(f_I) = \pi^{e|I|}f'_I$. Expanding

$$\phi_k(e'_I) = \sum_{J \in \mathcal{I}_k} c_{I,J} f'_J \, , \, c_{I,J} \in C$$

we see, using that C is saturated, that $(*)$ for our fixed k is equivalent to

$$(**) \qquad \forall I, J \in \mathcal{I}_k : w(c_{I,J}) \ge |J| - |I|,$$

w denoting the filtration of C. If (G, ω) is equi-p-valued, all the differences on the right-hand-side of $(**)$ are zero, so that $(**)$ is trivially true.

By Proposition 3.4.8 any two such ϕ are chain homotopic via a homotopy $h : \mathrm{Sat}X_\bullet \to \mathrm{Sat}Y_\bullet$ such that $\mathrm{gr}(h) = 0$. It remains to check that it restricts to a homotopy $h : C \otimes_A X_\bullet \to C \otimes_B Y_\bullet$. We use the same generators as before. The additional assumption that $\mathrm{gr}(G)$ is generated in degree $\frac{1}{e}$ implies $|I| = \frac{k}{e}$ for $I \in \mathcal{I}_k$.

Consider e'_I for $I \in \mathcal{I}_k$. Then $h_k(e'_I) \in \mathrm{Sat}Y_{k+1}$ and expands as

$$h_k(e'_I) = \sum_{J \in \mathcal{I}_{k+1}} d_{I,J} f'_J \, , \, d_{I,J} \in C$$

$\mathrm{gr}(h) = 0$, hence $\pi | d_{I,J}$ for all I, J. As $e'_I = \pi^{-k} e_I$ and $f'_J = \pi^{-(k+1)} f_J$ this implies

$$h_k(e_I) = \sum_{J \in \mathcal{I}_{k+1}} d_{I,J} \pi^{-1} f_J \ .$$

with $d_{I,J}\pi^{-1} \in C$ as required. □

Remark 3.4.10. It seems difficult to directly relate the complexes $C \otimes_A X_\bullet$ and $C \otimes_B Y_\bullet$ using the filtration techniques successfully employed for example in [ST] and [T], essentially because these complexes do not satisfy any reasonable exactness properties.

In fact, we have $H_*(C \otimes_A X_\bullet) = \mathrm{Tor}^A_*(C, \mathcal{O})$ and $H_*(C \otimes_B Y_\bullet) = \mathrm{Tor}^B_*(C, \mathcal{O})$ and one can check that, unless $G = \{e\}$, the algebra C is not flat over neither A nor B.

We have examples of saturated but not equi-p-valued groups and an isomorphism ϕ as above which does not restrict as in Proposition 3.4.9, but in all these examples it was possible by inspection to modify ϕ suitably.

It thus remains a tantalizing open problem to decide whether the assumption "equi-p-valued" is superfluous in Proposition 3.4.9. Of course, a positive answer would greatly extend the range of applicability of our integral Lazard comparison isomorphism.

Proof of Theorem 3.1.1. There is a filtration ω of G such that (G, ω) is p-saturated, equi-p-valued of finite rank and $\omega(G) \subseteq \frac{1}{e}\mathbb{Z}$ for some integer $e \ge 1$, hence we are in the situation studied in this subsection and in particular recall $\mathcal{O}, A, B, C, X_\bullet$ and $Y_\bullet$ from above. The continuous group cohomology

$H_c^*(G,M)$ is defined using continuous cochains and the Bar-differential as in [L, V 2.3.1.]. By [L, V 1.2.6 and 2.2.3.1] we have

$$H_c^*(G,M) \simeq H_c^*(\mathbb{Z}_p[[G]],M) \simeq \mathrm{Ext}^*_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p,M)$$

and analogously

$$H_c^*(G,M\otimes_{\mathbb{Z}_p}\mathcal{O}) \simeq \mathrm{Ext}^*_A(\mathcal{O},M\otimes_{\mathbb{Z}_p}\mathcal{O})$$

by the flatness of $\mathcal{O}$ over $\mathbb{Z}_p$. Introduce $N := M\otimes_{Z_p}\mathcal{O}$. Since $X_\bullet$ is a finite free resolution of $\mathcal{O}$ over A, we obtain

$$\mathrm{Ext}^*_A(\mathcal{O},N) = H^*\mathrm{Hom}_A(X_\bullet,N) = H^*\mathrm{Hom}_C(C\otimes_A X_\bullet,N)$$

using that the A-module structure on N extends to a C-module structure, and then

$$\begin{aligned} ... \overset{Prop.3.4.9}{\simeq} \;& H^*\mathrm{Hom}_C(C\otimes_B Y_\bullet,N) = H^*\mathrm{Hom}_B(Y_\bullet,N) \\ & \simeq H^*(\mathcal{L}^*(G)\otimes_{\mathbb{Z}_p}\mathcal{O},N) \simeq H^*(\mathcal{L}^*(G),M)\otimes_{\mathbb{Z}_p}\mathcal{O}, \end{aligned}$$

the last but one isomorphism by [T, Lemma 9.2]. Summing up we have

$$H_c^*(G,M)\otimes_{\mathbb{Z}_p}\mathcal{O} \simeq H^*(\mathcal{L}^*(G),M)\otimes_{\mathbb{Z}_p}\mathcal{O} \tag{7}$$

of $\mathcal{O}$-modules.

We now turn to functoriality. Let $f : G\to H$ be filtered group homomorphism. This induces a commutative diagram

$$\begin{array}{ccc} \mathrm{gr}(G) & \xrightarrow{\mathrm{gr}(f)} & \mathrm{gr}(H) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{gr}(\mathcal{L}^*(H)) & \longrightarrow & \mathrm{gr}(\mathcal{L}^*(H)) \end{array}$$

As in Proposition 3.4.8 it lifts to a diagram of filtered complexes of $\mathrm{Sat}A(G)$-modules

$$\begin{array}{ccc} \mathrm{Sat}X_\bullet(G) & \longrightarrow & \mathrm{Sat}X_\bullet(H) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{Sat}Y_\bullet(G) & \longrightarrow & \mathrm{Sat}Y_\bullet(H) \end{array}$$

which commutes up to homotopy and such that taking gradeds gives back the previous diagram, and such that taking gradeds of the homotopy is 0. As in Proposition 3.4.9 it restricts to a diagram of filtered complexes of $A(G)$-modules

$$\begin{array}{ccc} C(G)\otimes X_\bullet(G) & \longrightarrow & C(G)\otimes X_\bullet(H) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ C(G)\otimes Y_\bullet(G) & \longrightarrow & C(G)\otimes Y_\bullet(H) \end{array}$$

which commutes up to homotopy.

Compatibility with cup-products is the case $\Delta : G \to G \times G$. Note that the generators of $G \times G$ are of the form $(x, 1)$ and $(1, y)$ for generators x, y of G. Their filtration is the same as that of x, y. $\square$

4. The Lazard isomorphism for algebraic group schemes

In this section we give, in the case of p-adic Lie-goups arising from algebraic groups, a direct description of a map from analytic group cohomology to Lie-algebra cohomology by differentiating the cochains. In 4.2.4 we show that this coincides with Lazard's isomorphism.

4.1. **Group schemes.** Let p be a prime number, K be a finite extension of $\mathbb{Q}_p$, let R be its ring of integers with prime element π. Throughout $\mathbb{G}$ will be a separated smooth group scheme over R and $\mathfrak{g}$ its R-Lie-algebra in the following sense:

$$\mathfrak{g} = \mathrm{Lie}(\mathbb{G}) = \mathrm{Der}_R(\mathcal{O}_{\mathbb{G},e}, R)\ .$$

Then $\mathfrak{g}_K := \mathfrak{g} \otimes_R K = \mathrm{Lie}(\mathbb{G}_K)$ is its Lie-algebra as a K-manifold.

Note that this category is stable under base change and Weil restriction for finite flat ring extensions $R \to S$. If $A \to B$ is a ring extension with B finite and locally free over A and X an A-scheme, we write $X_B = X \times_A \mathrm{Spec}B$. If Y is a B-scheme, we write $\mathrm{Res}_{B/A}X$ for the Weil restriction, i.e., $\mathrm{Res}_{B/A}X(T) = X(T_B)$ for all A-schemes T. See [BLR, §7.6] for properties of the Weil restriction. In particular, if $\mathbb{G}$ is a group scheme over a discrete valuation ring R, then $\mathbb{G}$ is quasi-projective by [BLR, §6.4, Theorem 1]. This suffices to guarantee that $\mathrm{Res}_{S/R}(\mathbb{G})$ exists for finite extensions S/R.

The following bit of algebraic geometry will be needed in the proofs.

Lemma 4.1.1. *Let L/K be finite extension, S the ring of integers of L. Consider a separated smooth group scheme $\mathbb{G}$ over S. Then $\mathbb{G}$ is a direct factor of $\mathrm{Res}_{S/R}(\mathbb{G})_S$.*

Proof. Let X be an S-scheme. For all S-schemes T we describe T-valued points of $\mathrm{Res}_{S/R}(X)_S$:

$$\begin{aligned}\mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S) = \mathrm{Mor}_R(T, \mathrm{Res}_{S/R}(X)) &= \mathrm{Mor}_S(T \times_R \mathrm{Spec}S, X)\\ &= \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X)\end{aligned}$$

The natural map $\iota : S \to S \otimes_R S$ which maps $s \mapsto s \otimes 1$ induces the transformation of functors

$$\mathrm{Mor}_S(T \times_S \mathrm{Spec}S, X) \xrightarrow{\iota} \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X) = \mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S)$$

and hence a morphism

$$\iota : X \to \mathrm{Res}_{S/R}(X)_S$$

This is nothing but the adjunction morphism.

The multiplication $\mu_S : S \otimes_R S \to S$ is a section of ι. This again induces a transformation of functors

$$\mathrm{Mor}_S(T, \mathrm{Res}_{S/R}(X)_S) = \mathrm{Mor}_S(T \times_S \mathrm{Spec}(S \otimes_R S), X) \xrightarrow{\mu_S} \mathrm{Mor}_S(T \times_S S, X)$$

and hence a morphism

$$\mu_S : \mathrm{Res}_{S/R}(X)_S \to X \ .$$

(Put $T = \mathrm{Res}_{S/R}(X)_S$ and the identity on the left.) By construction μ_S is a section of ι. Both are natural in X, hence $\mathbb{G}$ is a direct factor of $\mathrm{Res}_{S/R}(\mathbb{G})_S$ as group schemes. □

Remark 4.1.2. If L/K is galois of degree d, then $\mathrm{Res}_{L/K}(\mathbb{G})_L \cong \mathbb{G}^d$. This carries over to the integral case if the extension is unramified. The assertion becomes false for ramified covers. Note, however, that the weaker statement of the lemma remains true.

4.2. **Analytic description of the Lazard morphism.** Let $\mathbb{G}$ be a smooth connected group scheme over R with Lie-algebra $\mathfrak{g}$. Let $\mathcal{G} \subset \mathbb{G}(R)$ be an open sub-Lie-group.

We denote by $\mathcal{O}_{\mathrm{la}}(\mathcal{G})$ (locally) analytic functions on $\mathcal{G}$, i.e. those that can be locally written as a converging power series with coefficients in K. We denote by $H^i_{\mathrm{la}}(\mathcal{G}, K)$ (locally) analytic group cohomology, i.e., cohomology of the bar complex $\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_{n\geq 0}$ with the usual differential. We denote $H^i(\mathfrak{g}, K)$ Lie-algebra cohomology, i.e. cohomology of the complex $\Lambda^*(\mathfrak{g}_K^\vee)$ with differential induced by the dual of the Lie-bracket.

Definition 4.2.1. The *Lazard morphism* is the map

$$\Phi : H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i(\mathfrak{g}, K)$$

induced by the morphism of complexes

$$\begin{aligned} \mathcal{O}_{\mathrm{la}}(\mathcal{G}^n) \to (\mathfrak{g}_K^n)^\vee &\to \Lambda^n \mathfrak{g}_K^\vee \\ f \mapsto df_e \end{aligned}$$

Remark 4.2.2. It is not completely obvious that Φ is a morphism of complexes. See [HK, Section 4.6. and Section 4.7.].

Remark 4.2.3. Φ is compatible with the multiplicative structure.

Recall from Lemma 2.2.2 that in the case $K = \mathbb{Q}_p$, the kernel G of $\mathbb{G}(\mathbb{Z}_p) \to \mathbb{G}(\mathbb{F}_p)$ is filtered and has a subgroup $\mathcal{G}$ of finite index which is saturated and equi-p-valued. Indeed for $p \neq 2$, we have $G = \mathcal{G}$.

Let $\mathcal{L}^* = \mathcal{L}^*(\mathcal{G})$ be its integral Lazard Lie-algebra (see Definition 2.6.6). As reviewed in Example 2.6.8 there is a natural isomorphism

$$\mathfrak{g} \otimes \mathbb{Q}_p \cong \mathcal{L}^* \otimes \mathbb{Q}_p \ .$$

Proposition 4.2.4. [HK, Theorem 4.7.1] *For $K = \mathbb{Q}_p$ and $\mathcal{G}$ saturated, the Lazard morphism Φ (see Definition 4.2.1) agrees under the identification of Lie-algebras in Example 2.6.8 with the isomorphism defined by Lazard (*[L, V 2.4.9, V 2.4.10]*)*

In particular, Φ is an isomorphism in this case.

Remark 4.2.5. This is a case where our integral version of the result (Theorem 3.1.1) can be applied. As shown there this is again the same isomorphism.

4.3. **The isomorphism over a general base.**

Theorem 4.3.1. *Let $\mathbb{G}$ be a smooth group scheme over R with connected generic fiber and $\mathcal{G} \subset \mathbb{G}(R)$ an open subgroup. Then the Lazard morphism Φ (see Definition 4.2.1) is an isomorphism.*

The proof will take the rest of this note.

Remark 4.3.2. Let us sketch the argument. We are first going to show injectivity. For this we can restrict to smaller and smaller subgroups $\mathcal{G}$ and even to their limit. In the limit, the statement follows by base change from Lazard's result for $R = \mathbb{Z}_p$. We then show surjectivity. Finite dimensionality of Lie-algebra cohomology implies that the morphism is surjective for small enough $\mathcal{G}$. Algebraicity then implies surjectivity also for the maximal $\mathcal{G}$.

By construction, the Lazard morphism Φ depends only on an infinitesimal neighborhood of e in $\mathcal{G}$. Hence it factors through the Lazard morphism for all open sub-Lie-groups of $\mathcal{G}$ and even through its limit

$$\Phi_\infty : \varinjlim_{\mathcal{G}' \subset \mathcal{G}} H^i_{\mathrm{la}}(\mathcal{G}', K) \to H^i(\mathfrak{g}, K)$$

Lemma 4.3.3. *The limit morphism Φ_∞ is an isomorphism.*

Proof. For $K = \mathbb{Q}_p$ this holds by Proposition 4.2.4 and the work of Lazard [L, V 2.4.9 and V 2.4.10].

The system of open sub-Lie-groups of $\mathcal{G}$ is filtered, hence

$$\varinjlim_{\mathcal{G}' \subset \mathcal{G}} H^i_{\mathrm{la}}(\mathcal{G}', K) = H^i(\mathcal{O}_{\mathrm{la}}(\mathcal{G}^\bullet)_e)$$

where $\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_e$ is the ring of germs of locally analytic functions in e. Note that $\mathbb{G}(\mathbb{Z}_p)$ also carries the structure of a rigid analytic variety and germs of locally analytic functions are nothing but germs or rigid analytic functions. Hence they can be identified with a limit of Tate algebras.

First suppose that $\mathbb{G} = \mathbb{H}_K$ for a smooth group scheme $\mathbb{H}$ over $\mathbb{Q}_p$. Then

$$\mathcal{O}_{\mathrm{la}}(\mathcal{G}^n)_e \cong \mathcal{O}_{\mathrm{la}}(\mathbb{H}(\mathbb{Z}_p)^n) \otimes K$$

because Tate algebras are well-behaved under base change (see [BGR, Chapter 6.1, Corollary 8]). Moreover, Φ_∞ is compatible with base change. As it is an isomorphism for $\mathbb{H}$ it is also an isomorphism for G.

Now consider general $\mathbb{G}$. By Lemma 4.1.1 $\mathbb{G}$ is direct factor of some group of the form $\mathbb{H}_K$ with $\mathbb{H}$ a group over $\mathbb{Z}_p$. Indeed, $H = \mathrm{Res}_{R/\mathbb{Z}_p}(\mathbb{G})$. By naturality, $\Phi_{\infty,\mathbb{G}}$ is a direct factor of $\Phi_{\infty,\mathbb{H}_R}$ and hence by the special case an isomorphism. $\square$

Corollary 4.3.4. Φ *is injective.*

Proof. $\mathbb{G}(R)$ is compact, hence all open sub-Lie-groups are of finite index. If $\mathcal{G}' \subset \mathcal{G}$ is an open normal subgroup, we have

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \cong H^i_{\mathrm{la}}(\mathcal{G}', K)^{\mathcal{G}/\mathcal{G}'}$$

Hence the restriction maps

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i_{\mathrm{la}}(\mathcal{G}', K)$$

are injective. As the system of open normal subgroups is filtered, this also implies that

$$H^i_{\mathrm{la}}(\mathcal{G}, K) \to \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$$

is injective. The injectivity of Φ follows from the injectivity of Φ_∞. □

Lemma 4.3.5. *Let $\mathcal{G} \subset \mathbb{G}(R)$ be an open subgroup. Then there is an open subgroup $\mathcal{H} \subset \mathcal{G}$ such that the Lazard morphism for $\mathcal{H}$ is bijective.*

Remark 4.3.6. Note that this is precisely what Lazard proves over $\mathbb{Q}_p$ with $\mathcal{H}$ the saturated subgroup of $\mathcal{G} = \mathbb{G}(\mathbb{Z}_p)$.

Proof. As Φ_∞ is bijective and Φ injective, it suffices to show that there is $\mathcal{H}$ such that the restriction map

$$H^i_{\mathrm{la}}(\mathcal{H}, K) \to \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$$

is surjective. Let α be a cocycle with class $[\alpha] \in \varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$. By definition it is represented by a cochain on some $\mathcal{G}'$. It is a cocycle (possibly on some smaller $\mathcal{G}'$). Hence $[\alpha]$ is in the image of the restriction map for $\mathcal{G}'$.

Lie-algebra cohomology is finite dimensional by definition, hence this is also true for $\varinjlim_{\mathcal{G}'} H^i_{\mathrm{la}}(\mathcal{G}', K)$. By intersecting the $\mathcal{G}'$ for a basis we get the group $\mathcal{H}$ we wanted to construct. □

Proof of Theorem 4.3.1. Injectivity has already been proved in Corollary 4.3.4. We use an argument of Casselman-Wigner [CW, §3] to conclude. The operation of $\mathbb{G}_K$ on $H^i(\mathfrak{g}, K)$ is algebraic. Hence the stabilizer $\mathbb{S}_K$ is a closed subgroup of $\mathbb{G}_K$. On the other hand it contains some open subgroup of $\mathbb{G}(R)$. This implies that $\mathbb{S}_K = \mathbb{G}_K$ because $\mathbb{G}_K$ is connected. Hence $\mathcal{G} \subset \mathbb{G}(R)$ operates trivially and thus

$$\Phi : H^i_{\mathrm{la}}(\mathcal{G}, K) \to H^i(\mathfrak{g}, K)$$

is surjective. □

Remark 4.3.7. The argument also works for cohomology with coefficients in a finite dimensional algebraic representation of the group.

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