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Abstract
Let M be a topological G_2–manifold. We prove that the space of infinitesimal associative deformations of a compact associative submanifold Y with boundary in a coassociative submanifold X is the solution space of an elliptic problem. For a connected boundary ∂Y of genus g, the index is given by $\int_{\partial Y} c_1(\nu_X) + 1 - g$, where ν_X denotes the orthogonal complement of $T\partial Y$ in $TX_{|\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to its natural complex structure. Further, we exhibit explicit examples of non trivial index.

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1 Introduction

The group G_2 is one of the possible holonomy groups of an irreducible and non–symmetric riemannian manifold. As such, manifolds of holonomy G_2 were an active area of research in riemannian geometry, culminating with Joyce's celebrated construction of compact holonomy G_2–manifolds [23]. As they are necessarily seven dimensional, one refers to G_2 as an *exceptional* holonomy group. In recent years physicists also paid accrued interest to these since the arrival of M–theory.

The deep and rich interplay between geometry and algebra on manifolds with a G_2–structure is reflected in the existence of special submanifolds, namely *associative* ones of dimension 3 and *coassociative* ones of dimension 4. These are particular instances of Harvey's and Lawson's *calibrated submanifolds* [20], a notion which also embraces complex submanifolds of a Kähler manifold or special lagrangian submanifolds of a Calabi–Yau. McLean [28] proved that the infinitesimal coassociative deformations of a coassociative X is an unobstructed elliptic problem. The dimension of the moduli space is $b^2_+(X)$, i.e. the dimension of self–dual harmonic 2–forms on X. For associative submanifolds, the problem, though still elliptic, is more involved. Firstly, existence of smooth deformations is, as in the case of complex submanifolds, in general obstructed. Secondly, the *virtual* dimension, that is, the index of the elliptic equation, always vanishes on dimensional grounds, so that no prediction on the existence of infinitesimal deformations can be made. This result was extended to arbitrary manifolds with topological G_2–structure (whose holonomy is not necessarily contained in G_2) by Akbulut and Salur [3], who also address smoothness and compactness issues of the deformation spaces.

On the other hand, on symplectic manifolds one is naturally led to study the moduli space of (pseudo–)holomorphic curves with boundary in a lagrangian submanifold [17], [19]. In physics, Aganagic and Vafa translated this boundary problem for special lagrangians of a

Calabi–Yau into an open string problem [1], following Witten's use of the moduli space of complex curves in the stringy world [31]. Now taking a Calabi–Yau 3–fold K times a circle yields a natural riemannian manifold $M = K \times S^1$ with holonomy contained in G_2. Moreover, holomorphic curves and special lagrangians times a circle give examples of associative and coassociative submanifolds in M. In this way, the duality of complex versus special lagrangian submanifolds is matched by the duality of associative versus coassociative submanifolds in a holonomy G_2–manifold. On one side, this hints at the existence of a G_2–analogue of Floer theory as conjectured in [14]. Further, it makes (co–)associatives play a key rôle in attempts to set up topological M–theory [4], [13]. It is therefore natural to study deformations of (co–)associatives with boundary. Inspired by the work of Butscher [11], who investigated deformations of special lagrangians with boundary on a symplectic, codimension 2 submanifold inside some compact Calabi–Yau, Kovalev and Lotay investigated in a recent paper the analogous problem for manifolds with closed G_2–structures, where a compact coassociative has its boundary in a fixed, codimension 2 submanifold [25].

In this paper, we consider an associative Y inside topological G_2–manifolds and study the space of infinitesimal associative deformations of Y with boundary inside a fixed coassociative X. We identify this space with the solutions of an elliptic boundary value problem whose index is given for connected boundary ∂Y of genus g by

$$\operatorname{index}(X,Y) = \int_{\partial Y} c_1(\nu_X) + 1 - g.$$

Here, ν_X denotes the orthogonal complement of $T\partial Y$ in $TX_{|\partial Y}$ and $c_1(\nu_X)$ the first Chern class of ν_X with respect to a natural complex structure we are going to define below. Further, we extend this result to 4–dimensional submanifolds X which do not contain any associative. In a sense, this class of submanifolds inside a (topological) G_2–manifolds forms the natural counterpart of totally real submanifolds inside (almost) complex manifolds. Finally, assuming that Y is an embedded 3–disk, we associate with Y an element $\mu_{G_2}(\partial Y) \in \pi_2\big(G_2/SO(4)\big) \cong \mathbb{Z}_2$, which is best thought of as a G_2–analogue of the Maslov index. Under suitable identifications, we show that

$$\mu_{G_2}(X,Y) = \int_{\partial Y} c_1(\nu_X) \bmod 2 = \big(\operatorname{index}(X,Y) + 1\big) \bmod 2.$$

Explicit examples of pairs (X,Y) with non–trivial index will be given in Section 6. In particular, we shall construct compact pairs (X,Y) inside compact holonomy G_2–manifolds using Joyce's method [23].

A further natural issue is to study smooth– and compactness issues in the vein of [3], but we will leave this to another paper. The techniques we use are the standard ones from PDE theory; our reference is [6] whose conventions we shall follow throughout this paper.

2 The group G_2

We start by recalling some classical facts about G_2 (cf. for instance [3], [9], [20] and [23]).

2.1 The octonions

The octonions define a real 8–dimensional, non–associative division algebra $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$ generated by $\langle \mathbf{1}, i, j, k, e, e\cdot i, e\cdot j, e\cdot k\rangle$. Taking these generators as an orthonormal basis induces an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathbb{O}$ compatible with the algebra structure. Further, we obtain a *vector cross product* taking values in the imaginary octonions $\mathrm{Im}\,\mathbb{O} = \langle\mathbf{1}\rangle^\perp \cong \mathbb{R}^7$ defined by

$$u \times v = \mathrm{Im}(\overline{v} \cdot u).$$

Here, $\overline{v}$ is the natural conjugation which sends $v \in \mathrm{Im}\,\mathbb{O}$ to $-v$. The term cross product is justified by the properties $u \times v = -v \times u$ and $|u \times v| = |u \wedge v|$. Over $\mathbb{R}^7$, this yields the 3–form

$$\varphi_0(u, v, w) = \langle u \times v, w\rangle, \tag{1}$$

which can be written with respect to the orthonormal basis $e_1 = i$, $e_2 = k, \ldots, e_7 = e \cdot k$ as[1]

$$\varphi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}). \tag{2}$$

We refer to any basis $\{e_j\}$ such that φ_0 is of the form (2) as a G_2*–frame*, since the stabiliser of φ_0 inside $GL(7)$ is the real algebraic Lie group G_2, which is of dimension 14. Conversely, any G_2–invariant form $\varphi \in \Lambda^3\mathbb{R}^{7*}$ induces a positive definite inner product $\langle\cdot\,,\cdot\rangle_\varphi$ and a cross product $\times_\varphi$ as follows. Firstly, with φ we can associate a volume form μ_φ (which is somehow difficult to write down explicitly, cf. the appendix in [22]). Then we define

$$\langle u, v\rangle_\varphi = \big((u \llcorner \varphi) \wedge (v \llcorner \varphi) \wedge \varphi\big)/6\mu_\varphi, \quad \langle u \times_\varphi v, w\rangle_\varphi = \varphi(u, v, w). \tag{3}$$

Next, we consider the *associator*

$$[u, v, w] = \frac{1}{2}\big((u \cdot v) \cdot w - u \cdot (v \cdot w)\big)$$

which is totally skew–symmetric. The associated $\mathrm{Im}\,\mathbb{O}$–valued 3–form in $\Lambda^3\mathbb{R}^{7*} \otimes \mathbb{R}^7$ will be written χ_0. For this form, we have the important identity

$$\chi_0(u, v, w) = -u \times (v \times w) - \langle u, v\rangle w + \langle u, w\rangle v. \tag{4}$$

In particular, we find $u \times (u \times a) = -|u|^2 \cdot a$ if a is orthogonal to u.

Finally we define a 4–form over $\mathbb{R}^7$ by

$$\psi_0(u, v, w, x) = \frac{1}{2}\langle u, [v, w, x]\rangle.$$

This form actually coincides with the Hodge dual of φ_0, so that in a G_2–frame $\{e_j\}$,

$$\psi_0 = \star\varphi_0 = -e^{12} \wedge (e^{47} + e^{56}) - e^{13} \wedge (e^{46} - e^{57}) + e^{23} \wedge (e^{45} + e^{67}) + e^{4567}.$$

2.2 Associative and coassociative planes

An oriented 3–plane $Y \subset \mathbb{R}^7$ is *associative* if the 3–form φ_0 restricted to Y coincides with the induced Euclidean volume form on Y. By $G_{\varphi_0}(\mathbb{R}^7)$ we denote the subset of associatives inside $G_3(\mathbb{R}^7)$, the grassmannian of oriented 3–planes in $\mathbb{R}^7$. It is diffeomorphic to $G_2/SO(4)$, where the action of G_2 on $\mathrm{Im}\,\mathbb{O}$ restricts to the action of $SO(4)$ on $\mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. Here, $\mathrm{Im}\,\mathbb{H}$ and $\mathbb{H}$ are isomorphic $\big($as $SO(4)$–representations$\big)$ with the space of anti–self–dual forms $\Lambda^2_-\mathbb{R}^{4*}$ and the standard vector representation $\mathbb{R}^4$ respectively. As the name suggests, the

[1] This is the convention adopted in [3], [9] and [23].

associator vanishes on associative planes. In fact, associativity is tantamount to saying that the restriction to Y of χ_0 vanishes. Furthermore, for Y associative any $u \in Y$ of norm 1 induces a hermitian structure $u\times : Y^\perp \to Y^\perp$. It follows that $Y^\perp$ is the irreducible Clifford module of *Cliff*$(Y, \langle\cdot\,,\cdot\rangle_{|Y})$. Also note that $Y^\perp \times Y^\perp \to Y$.

On the other hand, an oriented 4–plane X is said to be *coassociative* if and only if ψ_0 restricted to X is equal to the induced riemannian volume form. This is equivalent to saying that the restriction of φ_0 to X vanishes. As for associatives the set of coassociatives $G_{\psi_0}(\mathbb{R}^7)$ is diffeomorphic to $G_2/SO(4)$.

2.3 G_2–manifolds

Next, consider a 7–dimensional manifold M. If the structure group $GL(7)$ reduces to G_2, we say that M carries a *G_2–structure* and refer to M as a *topological G_2–manifold* or simply as a *G_2–manifold* for sake of brevity. In this case, there exists a 3–form φ on M such that the associated G_2–principal frame bundle consists of isomorphisms between (T_xM, φ_x) and $(\mathbb{R}^7, \varphi_0)$ for $x \in M$. By an *abus de langage*, we refer to the defining 3–form φ itself as the G_2–structure. Then there exists a vector cross product $\times = \times_\varphi$ on TM, inducing the structure of $\operatorname{Im}\mathbb{O}$ on any tangent space T_xM. Moreover, formula (3) gives rise to a globally defined riemannian metric $g = g_\varphi$ with Levi–Civita connection ∇^g. Similarly, there are global counterparts $\psi = \star_\varphi\varphi \in \Omega^4(M)$ and $\chi \in C^\infty(M, \Lambda^3T^*M \otimes TM)$ of ψ_0 and χ_0 respectively.

A G_2–manifold is said to be *torsion–free* if $\nabla^g\varphi = 0$. This is tantamount to saying that there exist coordinates around each point such that $\varphi(x) = \varphi_0 + O(|x|^2)$. In this case, the holonomy of g is contained in G_2. In the sequel, we say that a torsion–free G_2–manifold is a *holonomy G_2–manifold*, if the holonomy actually equals[2] G_2.

An oriented 3–dimensional submanifold Y is called *associative* if the pull–back of φ to Y is equal to the induced riemannian volume form. Equivalently, the pull–back of χ to Y is identically zero. Furthermore, an oriented 4–dimensional submanifold X is called *coassociative* if the pull–back of ψ to X is equal to the induced riemannian volume form. Equivalently, the pull–back of φ to X is identically zero. As shown by Harvey and Lawson, (co–)associative manifolds have the important property of being homologically volume minimising if the form φ (respectively ψ) is closed. In particular, following their language, φ and ψ define *calibrations*, and (co–)associative submanifolds are *calibrated*. By a result of Fernandez and Gray [16], $d\varphi = d\psi = 0$ is equivalent to $\nabla^g\varphi = 0$ so that (M, φ) is torsion–free. We shall see examples of torsion–free G_2–manifolds, associatives and coassociatives in Section 6.

3 The geometry of the deformation problem

Let M be a G_2–manifold and $Y \subset M$ a compact associative whose boundary ∂Y is contained in a fixed 4–submanifold $X \subset M$. We wish to describe the space of infinitesimal deformations of Y in the class of associatives with boundary in X, that is, the *Zariski tangent space* $T^{\mathrm{Zar}}\mathfrak{M}_{X,Y}$ of

$$\mathcal{M}_{X,Y} = \{Y' \,|\, Y' \text{ associative isotopic to } Y \text{ with } \partial Y' \subset X\}.$$

[2]Note that some authors do not make this distinction and refer to any torsion–free G_2–manifold as a holonomy G_2–manifold or even simply as a G_2–manifold.

3.1 The closed case

Our starting point is the closed case $\partial Y = \emptyset$, that is we first discuss the Zariski tangent space $T^{\mathrm{Zar}}\mathfrak{M}_Y$ of $\mathcal{M}_Y = \{Y' \,|\, Y' \text{ associative isotopic to } Y \text{ with } \partial Y' = \emptyset\}$. In the case of associatives inside *torsion–free* G_2–manifolds, this problem was settled by McLean [28]. Later on, his result was generalised by Akbulut and Salur [2], [3] to arbitrary G_2–manifolds. We outline these results following the approach of [7].

Let (M, φ, ∇) be a G_2–manifold endowed with a compatible connection ∇, i.e. $\nabla\varphi \equiv 0$. Such connections always exist (cf. for instance [9]), but are not necessarily unique. In particular, ∇ preserves the metric and therefore induces a connection on the normal bundle $\nu \to Y$, which we also denote by ∇. Since Y is compact, any deformation $Y \to Y_t$, $t \in (-\epsilon, \epsilon)$ can be assumed to be normal: Reparametrising by a time–dependent diffeomorphism if necessary, one can achieve that $\partial Y_t(p)/\partial t \in \nu_p$. Consequently, nearby (i.e. C^1–close) Y_t can be identified with sections $s_t \in C^\infty(Y, \nu)$ via the exponential map. The associative Y then corresponds to s_0 = zero–section of ν. Pulling TY_t back to $TM_{|Y}$ through parallel transport with respect to ∇ along the curves $\lambda \mapsto \exp_p\big(\lambda s_t(p)\big)$, $p \in Y$, we obtain a curve in the fibre $G_3(TM)_p = G_3(T_pM)$. For $t = 0$, it passes through $E_p := T_pY$, which is an element in $G_\varphi(TM)_p$, the subset of calibrated 3–planes in $G_3(TM)_p$. The derivative of this curve at 0 can be identified with $\nabla\dot{s}_0(p)$, and the linearised condition is thus $\nabla\dot{s}_0(p) \in T_{E_p}G_\varphi(M) \subset T_{E_p}G_3(M) \cong E_p^* \otimes \nu_p$. Moreover, the vector cross product gives rise to an exact sequence

$$0 \to T_{E_p}G_\varphi(TM) \to T_{E_p}G_3(TM) \cong E_p^* \otimes \nu_p \xrightarrow{\times} \nu_p \to 0.$$

Picking an orthonormal basis e_1, e_2, e_3 of E_p we obtain the condition $\nabla\dot{s}_0(p) \in T_{E_p}G_3^\varphi(M)$ if and only if $\sum e_i \times \nabla_{e_i}\dot{s}_0(p) = 0$. An invariant formulation of this equation can be given in terms of a *Dirac operator*[3]. The fibre bundles TY and ν are associated with an $SO(4)$–representation. Further, since Y is spinnable the principal $SO(4)$–bundle can be lifted to a principal $Spin(4)$–bundle. Then ν is associated with the tensor product of a spin representation and some other representation of $Spin(4)$. As a result we may regard ν as a *twisted spinor bundle*. Under this identification, the operator $\sum e_i \times \nabla$ becomes the Dirac operator of (ν, ∇). Summing up, we arrive at a generalised version of McLean's

Theorem 3.1 ([3], [28]) *Let (M, φ) be a G_2–manifold and $Y \subset M$ an associative. Then $T^{\mathrm{Zar}}\mathfrak{M}_Y$ can be identified with the kernel of a twisted Dirac operator $\mathbf{D}^\nabla : C^\infty(Y, \nu) \to C^\infty(Y, \nu)$ taken with respect to a connection ∇ induced by a compatible connection of (M, φ). In particular, we obtain the natural Dirac operator on ν with respect to ∇^g if (M, φ) is torsion–free.*

In the sequel, we denote the Dirac operator $\mathbf{D}^\nabla$ of Y simply by $\mathbf{D}$.

Remark: The *Dirac equation* $\mathbf{D}s = 0$ is elliptic, and as a consequence, the *virtual dimension* of $\mathfrak{M}_Y$, that is, $\dim T^{\mathrm{Zar}}\mathfrak{M}_Y$, is finite. Furthermore, Y being odd–dimensional, we have $\mathrm{index}\,(\mathbf{D}) = \dim\ker(\mathbf{D}) - \dim\mathrm{coker}(\mathbf{D}) = 0$. In generic situations where one expects the cokernel to vanish, we would get $\dim T^{\mathrm{Zar}}\mathfrak{M}_Y = 0$, and associatives would be rigid objects. However, we have no a priori control on the virtual dimension in terms of topological datum. This stands in sharp contrast to the deformations of a coassociative X, where McLean showed that the (actual) dimension of the (smooth) moduli space $\mathfrak{M}_X$ is $b^2_+(X)$, the dimension of harmonic self–dual 2–forms [28].

[3]For sake of brevity, we refer to any operator $\mathbf{D}$ as a Dirac operator if it is of *Dirac type*, that is, the principal symbol of $\mathbf{D}^2$ satisfies $\sigma(\mathbf{D}^2)(p, \xi) = ||\xi||^2$, cf. also [6], Section 3.

3.2 The geometry on the boundary

Next assume that the associative Y has a non–empty boundary ∂Y inside a fixed coassociative X. We first need to understand the geometry on the boundary of Y.

Fix a collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ of ∂Y and let u denote the inward pointing unit vector field defined on $\mathcal{C}$. As before, $\nu \to Y$ denotes the normal bundle, as well as its restriction to ∂Y. In virtue of Section 2.2, $\nu_{|\mathcal{C}}$ carries a hermitian structure near the boundary induced by u, namely

$$G : \nu \to \nu, \quad G(x) = u \times x.$$

This acts indeed as an isometry with respect to g, as

$$g(Ga, Gb) = \varphi(u, a, u \times b) = -g\big(u \times (u \times b), a\big) = g(a, b)$$

for any a, $b \in \nu_{|\mathcal{C}}$. Let $\nu_X \subset TX_{|\partial Y}$ denote the orthogonal complement of $T\partial Y$ in $TX_{|\partial Y}$.

Lemma 3.2 *For the bundle $\nu \to \partial Y$ the following holds:*

1. *The bundle ν_X is contained in ν and is stable under G.*
2. *The orthogonal complement μ_X of ν_X in ν is also stable under G.*
3. *Viewing $T\partial Y$, ν_X and μ_X as G–complex bundles, we have*

$$\overline{\mu}_X \cong \nu_X \otimes_{\mathbb{C}} T\partial Y,$$

that is $\mu_X^{0,1} \cong \nu_X^{1,0} \otimes T^{1,0}\partial Y \cong \nu_X^{1,0} \otimes \overline{K}_{\partial Y}$, where $K_{\partial Y}$ is the canonical line bundle over ∂Y.

Proof: Let us fix a local orthonormal basis (u, v, w) on the boundary by choosing a local unit vector field $v \in T\partial Y$. We then set $w = u \times v$, which lies in $T\partial Y$ in virtue of the associativity of Y. If $a \in \nu_X$, then $g(a, u) = 0$, for $v \times w = u$ and $\varphi(v, w, a) = 0$, X being coassociative. Clearly, the vectors $a \times v$ and $a \times w$ are orthogonal to v and w as well as to u, since

$$g(a \times v, u) = \varphi(a, v, u) = -g(u \times v, a) = -g(w, a) = 0,$$

and similarly for $a \times w$. Hence $a \times v$, $a \times w \in \nu$. Further, these vectors are orthogonal to TX, for a, v, $w \in TX$ and X is coassociative, so that for $n \in \nu_X$ we find $g(a \times v, n) = \varphi(a, v, n) = 0$ etc.. Hence $a \times v$ and $a \times w$ span μ_X. As a consequence, $u \times a \in \nu$ is orthogonal to μ_X $\big($for $g(u \times a, a \times v) = \varphi(u, a, a \times v)$ etc.$\big)$, so that ν_X is spanned by a and $u \times a = Ga$. This shows that ν_X is stable under G. On the other hand, $g\big(u \times (a \times v), a\big) = \varphi\big(u, (a \times v), a\big) = 0$ and similarly $g\big(u \times (a \times v), u \times a\big) = 0$, hence $u \times (a \times v) \in \mu_X$ which shows that μ_X is also stable under G.

The Riemann surface structure on ∂Y is induced by the hermitian structure $G = u\times$, for $Gv = u \times v = w$ and $Gw = u \times w = -v$ (to keep notation tight we abuse notation and also write G for the endomorphism on $T\partial Y$ induced by $u\times$). The map

$$a \otimes y \in \nu_X \otimes_{\mathbb{C}} T\partial Y \mapsto a \times y \in \overline{\mu}_X,$$

where we now view ν_X and $\overline{\mu}_X$ as complex line bundles via G, is well–defined and a *real* bundle isomorphism. It remains to see that it is complex–linear, i.e.

$$Ga \times y = a \times Gy = -G(a \times y).$$

This is equivalent to $(u \times a) \times y$ and $a \times (u \times y)$ being equal to $-u \times (a \times y)$. But this follows from (4) and the skew–symmetry of χ. ∎

Let $\mathbf{B} : C^\infty(\partial Y, \nu) \to C^\infty(\partial Y, \mu_X) \subset C^\infty(\partial Y, \nu)$ be the orthogonal projector[4] taking smooth sections of $\nu = \nu_X \oplus \mu_X$ to μ_X. As a corollary to the generalised version of McLean's theorem as given in Theorem 3.1 and Lemma 3.2, we obtain:

Corollary 3.3 *The Zariski tangent space of $\mathcal{M}_{X,Y}$ can be identified with solutions of the system*

$$\mathbf{D}s = 0\,, \quad \mathbf{B}(s_{|\partial Y}) = 0, \quad s \in C^\infty(Y, \nu). \tag{5}$$

4 Ellipticity and index

In view of applying the standard machinery of index theory for manifolds with boundary, we consider the complexification of equation (5), namely

$$(\mathbf{D}^{\mathbb{C}} \oplus \mathbf{B}^{\mathbb{C}})(\sigma \oplus \sigma_{|\partial Y}) = 0, \quad s \in C^\infty(Y, \nu^{\mathbb{C}}) \tag{6}$$

with operators $\mathbf{D}^{\mathbb{C}}$ and $\mathbf{B}^{\mathbb{C}}$ extended to the complexified bundles $\nu^{\mathbb{C}} = \nu \otimes \mathbb{C}$ and $\mu_X^{\mathbb{C}} = \mu_X \otimes \mathbb{C}$. However, we shall not distinguish between the original operators and their complexification in the sequel for sake of keeping notation tight. To show that the kernel is finite–dimensional requires a suitable ellipticity condition. We first introduce the *Calderón projector* $\mathbf{Q_D}$ associated with a Dirac operator $\mathbf{D}$ (cf. [6], Thm. 12.4). This is an order 0 pseudo–differential operator

$$\mathbf{Q_D} : C^\infty(\partial Y, \nu) \to \mathcal{H}(\mathbf{D}) = \{s_{|\partial Y} \,|\, s \in C^\infty(Y, \nu),\, \mathbf{D}s = 0\} \subset C^\infty(\partial Y, \nu)$$

mapping smooth sections of ν over ∂Y to the space of Cauchy data[5] of $\mathbf{D}$.

Definition 4.1 (cf. [6], Def. 18.1) Let Y be an arbitrary smooth manifold with boundary, $\nu \to Y$ a (twisted) spinor bundle and $\mu \to \partial Y$ a vector bundle. A pseudo–differential operator $\mathbf{B} : C^\infty(\partial Y, \nu) \to C^\infty(\partial Y, \mu)$ of order 0 is said to define an *elliptic boundary condition* (abbreviated e.b.c.) if and only if

1. the extension $\mathbf{B}^{(s)} : H^s(\partial Y, \nu) \to H^s(\partial Y, \mu)$ to the chain of Sobolev spaces $H^s(\partial Y, \nu)$ and $H^s(\partial Y, \mu)$, $s \geq 0$, has closed range.
2. the restriction of the principal symbol $b = \sigma(\mathbf{B})_{|\mathrm{range}\left(\sigma(\mathbf{Q_D})\right)} : \mathrm{range}\big(\sigma(\mathbf{Q_D})\big) \to \mathrm{range}(b)$ is an isomorphism.

Furthermore, an e.b.c. is said to be *local*, if in addition $\mathrm{range}(p, \xi) = \nu_p$ holds for all $p \in \partial Y$ (in this case, 1. is automatically satisfied, cf. [6], Rem. 18.2).

If $\mathbf{B}$ defines an e.b.c., then regularity holds ([6], Thm. 19.1), that is, $s \in H^s(Y, \nu) \cap \ker(\mathbf{D} \oplus \mathbf{B})$ implies $s \in C^\infty(Y, \nu)$. Note that for even–dimensional manifolds the existence of local e.b.c. is topologically obstructed (e.g. [5], Section II.7.B), while on odd–dimensional manifolds, the orthogonal projector $\mathbf{P}^\pm$ onto the space of positive and negative half–spinors $\nu^\pm$ over the (even–dimensional) boundary always defines a local e.b.c. with vanishing index ([6], Thm. 21.5). Furthermore, we have the

[4]By an *orthogonal projector* we understand an operator $\mathbf{P}$ of order 0 satisfying $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^*$.

[5]We are glossing over some technical details such as the passing to the "closed double" $M = Y \cup_{\partial Y} Y$, cf. Chapters 9, 11 and 12 in [6].

Theorem 4.2 ([6], Thm. 20.12, Thm. 20.13 and Rem. 22.25) *If* $\mathbf{B}$ *defines a local e.b.c., then*

1. *the operator* $\mathbf{D} \oplus \mathbf{B}$ *is Fredholm with index*

$$\text{index}\,(\mathbf{D} \oplus \mathbf{B}) = \text{index}\,\big(\mathbf{B}\mathbf{Q}_{\mathbf{D}} : \mathcal{H}(\mathbf{D}) \to C^\infty(\partial Y, \nu)\big).$$

2. $\text{index}\,(\mathbf{D} \oplus \mathbf{B})$ *depends only on the homotopy type of the principal symbols involved.*

The following result is a valuable tool for explicit index computations.

Proposition 4.3 ([6], Thm. 21.2) *Let* $\mathbf{D} : C^\infty(Y, \nu) \to C^\infty(Y, \nu)$ *be a Dirac operator on some (twisted) spinor bundle* ν *over an odd–dimensional manifold* Y *with boundary. Further, consider two orthogonal projectors onto subbundles* $\nu_{1,2}$ *of* $\nu_{|\partial Y}$, $\mathbf{B}_1 : C^\infty(\partial Y, \nu) \to C^\infty(\partial Y, \nu_1)$ *and* $\mathbf{B}_2 : C^\infty(\partial Y, \nu) \to C^\infty(\partial Y, \nu_2)$, *and suppose they define a local e.b.c.. Then*

$$\text{index}\,(\mathbf{D} \oplus \mathbf{B}_2) - \text{index}\,(\mathbf{D} \oplus \mathbf{B}_1) = \text{index}\,\big(\mathbf{B}_2\mathbf{Q}_{\mathbf{D}}\mathbf{B}_1^* : C^\infty(\partial Y, \nu_1) \to C^\infty(\partial Y, \nu_2)\big).$$

Coming back to equation (6), we shall write $\text{index}\,(X, Y)$ for $\text{index}\,(\mathbf{D} \oplus \mathbf{B})$. Let $\overline{\partial}_{\nu_X}$ denote the Cauchy–Riemann operator associated with the natural complex structure (or equivalently, the natural orientation) of ν_X (cf. Lemma 3.2). Finally, we are in a position to prove the central theorem of this paper.

Theorem 4.4 *The operator* $\mathbf{B}$ *in (6) defines a local e.b.c. with index*

$$\text{index}\,(X, Y) = \text{index}\,(\overline{\partial}_{\nu_X}).$$

Proof: Let us fix some collar neighbourhood $\mathcal{C} \cong \partial Y \times [0, \epsilon)$ of ∂Y for which we may assume the riemannian structure to be a product (possibly after homotopically deforming the metric). Further, we complete the inward pointing coordinate vector u to a local orthonormal basis $\big(v(y,t),\, w(y,t)\big)$ of $T_y\partial Y \times \{t\}$ such that $u \times v = w$. Near the boundary, we have the decomposition $\mathbf{D} = u \times (\nabla_u + \mathbf{R})$ with

$$\mathbf{R} = w \times \nabla_v - v \times \nabla_w, \tag{7}$$

as follows from $(a \times b) \times c = -a \times (b \times c)$ valid whenever $\{a, b, c\}$ is an orthogonal family, cf. (4). Note that the bundles of positive and negative half–spinors $\nu^\pm$ inside $\nu^{\mathbb{C}}$ are just the eigenspaces of $G = u\times$.

Locally, we will work with the following basis of $\nu^{\mathbb{C}}$: Choose a nowhere vanishing local section $a \in C^\infty(\partial Y, \nu_X)$ so that $\nu_X^{\mathbb{C}}$ is spanned by $\alpha = a - iGa$ and $\overline{\alpha} = a + iGa$ respectively, cf. Lemma 3.2. Consider then the sections $\beta = -v \times \overline{\alpha}$ and $\overline{\beta} = -v \times \alpha$. Again, the lemma implies that these span $\mu_X^{\mathbb{C}}$ locally. Further, α and β span ν^+ while $\overline{\alpha}$ and $\overline{\beta}$ span ν^-. As an example, take $G\alpha = Ga + ia = i\alpha$ and $G\overline{\beta} = v \times (u \times \alpha) = -i\overline{\beta}$ etc.. For any subsequent matrix representation over $\nu^{\mathbb{C}}$, the ordered basis $\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$ shall be used.

Checking that $\mathbf{B}$ defines a local e.b.c. requires the principal symbol $q = \sigma(\mathbf{Q}_{\mathbf{D}})$ of the associated Calderón operator. By the Calderón–Seeley theorem (cf. Thm. 12.4 in [6]), q is the projector onto the eigenspace of $\sigma(\mathbf{R})$ corresponding to eigenvalues with positive real part. With respect to our fixed local ordered basis of $\nu^{\mathbb{C}}$ around $p \in \partial Y$, v and w act as

$$v\times = \begin{pmatrix} \mathbf{0} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \mathbf{0} \end{pmatrix}, \quad w\times = \begin{pmatrix} \mathbf{0} & \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \\ \begin{matrix} 0 & i \\ -i & 0 \end{matrix} & \mathbf{0} \end{pmatrix}.$$

This follows from $v\times\alpha=-\overline{\beta}$, $w\times\alpha=-u\times(v\times\alpha)=-i\overline{\beta}$ etc.. For $(\eta_v,\eta_w)\in T_p^*\partial Y\backslash\{0\}$ of unit norm, we deduce from (7) (with $\eta=\eta_v+i\eta_w$) that

$$\sigma(\mathbf{R})(p,\eta)=i(\eta_v\cdot w\times-\eta_w\cdot v\times)=\left(\begin{array}{cc}\mathbf{0} & \begin{matrix}0 & \overline{\eta}\\ -\overline{\eta} & 0\end{matrix}\\ \begin{matrix}0 & -\eta\\ \eta & 0\end{matrix} & \mathbf{0}\end{array}\right)=\left(\begin{array}{cc}\mathbf{0} & r_-(p,\eta)\\ r_+(x,\eta) & \mathbf{0}\end{array}\right).$$

Now $r_+(p,\eta)^*=r_-(p,\eta)$ and $r_+(p,\eta)=r_-(p,\eta)^{-1}$, so that $\sigma(\mathbf{R})(p,\eta)^*=\sigma(\mathbf{R})(p,\eta)=\sigma(\mathbf{R})(p,\eta)^{-1}$. Consequently, the eigenvalues are ±1, and the projector on the eigenspace associated with 1 is given by

$$q(p,\eta)=\frac{1}{2}\left(\begin{array}{cc}\mathrm{Id}_2 & r_-(p,\eta)\\ r_+(p,\eta) & \mathrm{Id}_2\end{array}\right).$$

On the other hand, $\mathbf{B}$ is the orthogonal projector onto $\mu_X^{\mathbb{C}}$, so that its principal symbol is the matrix (taken with respect to the fixed basis of $\nu^{\mathbb{C}}$ and $\{\beta,\overline{\beta}\}$ of $\mu_X^{\mathbb{C}}$)

$$\sigma(\mathbf{B})(p,\eta)=\left(\begin{array}{cccc}0&1&0&0\\0&0&0&1\end{array}\right),$$

which is of full rank. Since

$$\sigma(\mathbf{B})\circ q(p,\eta)=\frac{1}{2}\left(\begin{array}{cccc}0&1&-\overline{\eta}&0\\ \eta&0&0&1\end{array}\right)$$

is also of full rank, the boundary condition defined by $\mathbf{B}$ is local elliptic according to Definition 4.1.

It remains to compute the index. Let $\mathbf{P}^+$ denote the orthogonal projector onto ν^+. In virtue of Theorem 4.3 and the established local ellipticity of $\mathbf{B}$,

$$\operatorname{index}(X,Y)=\operatorname{index}(\mathbf{D}\oplus\mathbf{B})=\operatorname{index}\big(\mathbf{B}\mathbf{Q}_{\mathbf{D}}\mathbf{P}^+:C^\infty(\partial Y,\nu^+)\to C^\infty(\partial Y,\mu_X^{\mathbb{C}})\big).$$

But the symbol of $\mathbf{B}\mathbf{Q}_{\mathbf{D}}\mathbf{P}^+$ is just

$$\begin{aligned}\sigma(\mathbf{B}\mathbf{Q}_{\mathbf{D}}\mathbf{P}^+)(p,\eta) &= \sigma(\mathbf{B})\circ q\circ\sigma(\mathbf{P}^+)(p,\eta)\\ &= \frac{1}{2}\left(\begin{array}{cccc}0&1&-\overline{\eta}&0\\ \eta&0&0&1\end{array}\right)\left(\begin{array}{cc}1&0\\0&1\\0&0\\0&0\end{array}\right)\\ &= \frac{1}{2}\left(\begin{array}{cc}0&1\\ \eta&0\end{array}\right):\nu_p^+\to(\mu_X^{\mathbb{C}})_p,\end{aligned}$$

where the matrix is taken with respect to the basis $\{\alpha,\beta\}$ of S^+ and $\{\beta,\overline{\beta}\}$ of $\mu_X^{\mathbb{C}}$. In particular, the symbol sends β to β and therefore acts as the identity on $\mu_X^{1,0}=\nu^+\cap\mu_X^{\mathbb{C}}$. On the other hand, the induced map $\nu_X^{1,0}=\nu^+\cap\nu_X^{\mathbb{C}}\to\mu_X^{0,1}=\nu^-\cap\mu_X^{\mathbb{C}}$ is up to $-i$ the symbol of the Cauchy–Riemann operator $\overline{\partial}_{\nu_X}$ on $\nu_X^{1,0}$ after the identification $\mu_X^{0,1}\cong\nu_X^{1,0}\otimes\overline{K}_{\partial Y}$ (cf. Lemma 3.2). Indeed, on a trivialisation of $\nu_X^{\mathbb{C}}$ it acts as $\overline{\partial}_{\nu_X}=(\partial_1+i\partial_2)/2$, where (x_1,x_2) are coordinates such that $\partial_1(p)=v(p)$ and $\partial_2(p)=w(p)$. Hence $\sigma(\overline{\partial}_{\nu_X})(p,\eta)=i(\eta_v+i\eta_w)/2$ which is what we wanted. ■

As a consequence of Riemann–Roch, we obtain the

Corollary 4.5 *Let Σ_{g_j} be the connected components of ∂Y of genus g_j, and $c_1(\nu_{X|\Sigma_{g_j}})$ be the first Chern classes of $\nu_{X|\Sigma_{g_j}}$. Then*

$$\text{index}\,(X,Y) = \sum_j \int_{\Sigma_{g_j}} c_1(\nu_{X|\Sigma_{g_j}}) + 1 - g_j.$$

5 ψ–positive boundary conditions

For the deformation problem (6) the bundle ν_X is the only non–intrinsic piece of datum attached to Y, and its properties were derived using the coassociativity of X. In view of the previous theorem, we may relax the coassociativity condition as follows:

Definition 5.1 Let (M,φ) be a G_2–manifold and $\psi = \star\varphi$. An orientable 4–submanifold of M is said to be *ψ–positive at $p \in X$* if and only if its tangent space at p is ψ–positive for some orientation, i.e. $\psi_{p|T_pX} > 0$. Further, X is called *ψ–positive* if for a suitable orientation X is ψ–positive at all points $p \in X$.

Obvious examples of ψ–positive submanifolds are coassociatives. An alternative characterisation of ψ–positivity is the notion of φ–freeness going back to recent work of Harvey and Lawson [21].

Lemma 5.2 *A suitably oriented 4–plane $F \in G_4(\mathbb{R}^7)$ is ψ–positive at $p \in X$ if and only if F is φ–*free*, i.e. F contains no associative 3–plane.*

Proof: If $F \in G_4(T_xM)$ is not φ–free, then F contains an associative subplane E. Hence $\varphi_{|E^\perp} \equiv 0$ and in particular $\varphi_{|F^\perp} = \psi_{|F} \equiv 0$, which proves necessity. Conversely, assume that $\psi_{|F} \equiv 0$. We write $F^\perp = u \wedge v \wedge w$ and choose a vector a in the orthogonal complement to the linear span of u, v, w, $u \times v$, $u \times w$, $v \times w$, which at most is six dimensional. As a result, the 4–plane $E^\perp$ spanned by $F^\perp$ and a is coassociative, for $\varphi_{|E^\perp} \equiv 0$. Hence $E \subset F$ is associative. ■

Put differently, for any a, $b \in T_pX$, we have $a \times b \notin T_pX$ if X is ψ–positive at p. From this point of view the class of ψ–positives inside G_2–manifolds naturally matches the class of totally real submanifolds inside Kähler manifolds (which, in particular, comprises lagrangians).

Next we wish to investigate the Zariski tangent space of $\mathfrak{M}_{X,Y}$ under the assumption that X is merely ψ–positive. To that end, let $N_X \to \partial Y$ be the orthogonal complement of $T\partial Y$ in $TX_{|\partial Y}$, and define ν_X to be the image of N_X under the orthogonal projection $\mathbf{P} : TM_{|Y} \to \nu_{|Y}$. As before, $\mu_X \to \partial Y$ denotes the orthogonal complement of ν_X in $\nu_{|\partial Y}$. The geometry on the boundary is specified by the following

Lemma 5.3 *If X is ψ–positive, then*

1. *the restriction of $\mathbf{P}$ to N_X defines an isomorphism onto $\nu_X = \mathbf{P}(N_X)$.*
2. *for any non–zero $b \in \mu_X$, $Gb = u \times b \notin \nu_X$, that is for the orthogonal projection $\mathbf{p}_{\mu_X}$ on μ_X we have $\mathbf{p}_{\mu_X}(Gb) \neq 0$.*

Proof: The kernel of $\mathbf{P}_{|TM_{|\partial Y}}$ is $TY_{|\partial Y}$. Since TX is ψ–positive, $N_X \cap \ker\pi = \{0\}$ according to the previous proposition, whence the first assertion. Next, suppose there is a $b_0 \in \mu_X$ of unit norm such that $Gb_0 \in \nu_X$. Let $b_1 \in \mu_X$ be a vector orthogonal to b_0. Then Gb_1 lies in ν_X, for it is orthogonal to b_1 and $g(b_0, Gb_1) = -g(Gb_0, b_1) = 0$. By 1. there exist uniquely determined $n_{0,1} \in N_X$ with $\mathbf{P}(n_{0,1}) = Gb_{0,1}$. Since $N_X \perp T\partial Y$, we have in fact $n_{0,1} = Gb_{0,1} + \lambda_{0,1}u$ for $\lambda_{0,1} \in \mathbb{R}$ (where u denotes again the inward pointing normal vector on the boundary). It is straightforward to check that the orthogonal complement of $T_{\pi(b_0)}X$ in $T_{\pi(b_0)}M$ is spanned by $A = b_0$, $B = b_1$ and $C = u - \lambda_0 Gb_0 - \lambda_1 Gb_1$. Moreover, $v = A \times B \in TY_{|\partial Y}$ belongs in fact to $T\partial Y$, for $g(u, v) = \varphi(b_0, b_1, u) = -g(Gb_1, b_0) = 0$. Consequently, $\varphi(A, B, C) = g(A \times B, C) = 0$ which contradicts the ψ–positiveness of TX. ■

Consequently, $N = N_X \oplus \mu_X$ and $\nu_X \oplus \mu_X$ are isomorphic subbundles of $TM_{|\partial Y}$. Further, we can extend $N \to \partial Y$ to a new subbundle $N \subset TM_{|Y}$ transversal to TY, and adapt Corollary 3.3 to this more general situation: If N_X is orthogonal to $TY_{|\partial Y}$ (this does not imply the fibres of N to be coassociative), we can take $N = \nu$ and identify $T^{\mathrm{Zar}}\mathcal{M}_{X,Y}$ with the space of solutions of (6). Otherwise first extend N_X trivially on some collar neighbourhood. Since being isomorphic is a stable property, we can homotope N_X over $\mathcal{C}$ into a new bundle (still written N_X) with fibres in ν_X sufficiently far away from the boundary. Then extend $N = N_X \oplus \mu_X$ by ν. A section $s \in C^\infty(Y, N)$ induces a family of submanifolds Y_t with boundary in X if $s_{|\partial Y}$ lies in the kernel of $\mathbf{B} : C^\infty(\partial Y, N) \to C^\infty(\partial Y, \mu_X)$, the orthogonal projection onto μ_X in $N_{|\partial Y}$. Further, for Y_t to be associative we need the normal component $\mathbf{P}(s)$ of s to be harmonic. Thus $T^{\mathrm{Zar}}\mathcal{M}_{X,Y}$ can be identified with the solution space of

$$\mathbf{DP}(s) = 0, \quad \mathbf{B}(s_{|\partial Y}) = 0, \quad s \in C^\infty(Y, N). \tag{8}$$

Proposition 5.4 *Let $Y \subset M$ be an associative with boundary inside a ψ–positive submanifold X. Then*

$$\mathrm{index}\,(X, Y) = \mathrm{index}\,(\overline{\partial}_{\nu_X}).$$

Remark: Here, the Cauchy–Riemann operator on ν_X is defined by the complex structure coming from the induced orientation on $\nu_X \cong N_X$.

Proof: For simplicity we assume $\mathbf{P} = \mathrm{Id}$, i.e. N_X is orthogonal to $TY_{|\partial Y}$. We choose a nowhere vanishing local section a of ν_X and extend $b = -v \times a$ to a local trivialisation $\{b, \widetilde{b}\}$ of μ_X. By the previous lemma we may take $\widetilde{b}$ to be the orthogonal projection of Gb to μ_X. Let $(0, 1, s, t)$ be the coordinates of $\widetilde{b}$ with respect to the local basis $\{b, Gb, v \times b, w \times b\}$ of ν, where v is a nowhere vanishing local section of $T\partial Y$ and $w = Gv = u \times v$. The latter basis gives rise to the basis $\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\}$ of $\nu^{\mathbb{C}}$ as given in the proof of Theorem 4.4 with respect to which the matrix of $\sigma(\mathbf{B}^{\mathbb{C}})$ can be written as

$$\sigma(\mathbf{B}^{\mathbb{C}})(x, \eta) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ z & -i & \overline{z} & i \end{pmatrix},$$

where $z = s + it$. For $\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+$, we therefore find

$$\sigma(\mathbf{B}^{\mathbb{C}}\mathbf{QP}^+)(x, \eta) = \frac{1}{2}\begin{pmatrix} \eta & 1 \\ z + i\eta & -i - z\eta \end{pmatrix} : S^+_x \to (\mu^{\mathbb{C}}_X)_x. \tag{9}$$

The determinant of this matrix is $(-2i\eta - z - \overline{z}\eta^2)/4$, and multiplication with $\overline{\eta}$ shows this to vanish only if $\mathrm{Re}(\overline{T}\eta) = -i$. Hence the system (8) is still elliptic. Furthermore,

$F(Gb,t) = t\mathbf{p}_{\nu_X}(Gb) \oplus \mathbf{p}_{\mu_X}(Gb)$ is a global homotopy deforming $G\mu_X$ into μ_X and which in particular deforms Gb into $\widetilde{b}$. Consequently, the symbol (9) is homotoped into the symbol of $\overline{\partial}_{\nu_X}$. By homotopy invariance we recover the same index as before. ■

6 Examples

In this section we construct explicit examples of pairs (X, Y) with $\operatorname{index}(X, Y) \neq 0$. Throughout this section we denote by Σ_g a surface diffeomorphic to a compact oriented Riemann surface of genus g.

Local coassociative submanifolds. The first example will be local in nature, that is, the boundary of the compact associative will be contained in a local coassociative submanifold. Existence will be established by using Cartan–Kähler theory which requires all geometric objects involved (manifolds, boundaries, maps etc.) to be real analytic. Note that a torsion–free G_2–manifold (M, g) is Ricci–flat so that the underlying riemannian metric is real analytic in harmonic coordinates [12]. Since φ is harmonic with respect to g (cf. Section 2.3), it will be real analytic in these coordinates, too. Consider then an associative Y with real analytic boundary ∂Y, and a nowhere vanishing real analytic section $a \in C^\infty(\partial Y, \nu)$. The geodesic flow $\gamma_a : \partial Y \times (-\epsilon, \epsilon) \to M$ induced by a is also real analytic in harmonic coordinates and therefore generates an analytical submanifold N of dimension 3. Further, $\varphi(v, w, a) = 0$ for $v, w \in T\partial Y$. We conclude that the pull–back of φ to N vanishes identically, for $\nabla\varphi = 0$. A Cartan–Kähler type argument as invoked by Harvey and Lawson [20] (see also [8]) shows that N is contained in a local coassociative X. Further, ν_X is generated by a and $u \times a$, where u denotes again the inward pointing normal vector field of ∂Y. Thus $c_1(\nu_X) = 0$, whence $\operatorname{index}(X, Y) = 1 - g$. For example, taking $M = \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$ and $\partial Y = \Sigma_g \subset \operatorname{Im}\mathbb{H}$, where Σ_g is a g–handle body in $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ and Y the relatively compact interior bounded by Σ_g, yields examples of arbitrary negative index. An actual deformation is given by moving Y along the straight line determined by a. In fact, taking for Σ_0 the standard 2–sphere bounding the unit ball, one obtains a smooth moduli space which is thus of actual dimension 1 (cf. [18]).

Local associative submanifolds. In the same vein, consider a real analytic surface Σ_g inside some real analytic coassociative X in M. Again, Cartan–Kähler theory yields the existence of a local associative Y containing Σ_g [20], [29]. Using a collar neighbourhood of Σ_g inside Y we can construct an associative of the form $\Sigma_g \times [0, 1]$ which we keep on denoting by Y for simplicity. Further, we can translate X into a submanifold X' containing $\Sigma_g \times \{1\}$ by a suitable diffeomorphism C^1–close enough to the identity. Of course, there is no reason for this diffeomorphism to preserve the G_2–structure, so that X' will not be coassociative in general. However, as ψ–positivity is a pointwise open condition, X' will be still ψ–positive for suitable choices, and the generalised deformation theorem applies. To compute the index we note that $N_{X'} \to \Sigma_g \times \{1\}$ is obtained by translating $\nu_X \to \Sigma_g \times \{0\}$, but the orientation flips as u which points inward at $\Sigma \times \{0\}$ points outward at $\Sigma_g \times \{1\}$. Hence $c_1(\nu_X) = -c_1(\nu_{X'})$ and consequently $\operatorname{index}(X \cup X', Y) = 2(1 - g)$.

The Bryant–Salamon construction. In [10] Bryant and Salamon constructed holonomy G_2–metrics on (an open set of) the total space of the spinor bundle $S \to M^3$, where M is a three–dimensional space form. In particular, when M is taken to be the round sphere $\mathcal{S}^3$, there exists a *complete* holonomy G_2–metric on the total space $S \cong \mathcal{S}^3 \times \mathbb{H}$ such that the fibres are orthogonal to the horizontal distribution of the canonical spin connection induced by ∇^g. Furthermore, the zero section defines an associative. Any 3–ball Y inside $\mathcal{S}^3$ is therefore associative with boundary $\partial Y = \mathcal{S}^2 = \Sigma_0$. Let u be again the inward pointing normal vector field near the boundary. Assume that $S_{|\Sigma_0}$ has a subbundle which is a $G = u\times$–complex line bundle over Σ_0 of degree n. If X_n denotes the total space, then X_n is coassociative at ∂Y (cf. Lemma 3.2), hence ψ–positive near ∂Y. Therefore, the index formula applies and yields $\operatorname{index}(X_n, Y) = n + 1$. For $n = 0$, such a line bundle can be constructed by taking a nowhere vanishing section $a \in \Gamma(S_{|\Sigma_0})$. This gives the trivial complex line bundle spanned by a and $u \times a$, whence $\operatorname{index}(X_0, Y) = 1$.

The Calabi–Yau extension. Let (K, ω, Ω) be a Calabi–Yau 3–fold with Kähler 2–form ω and holomorphic volume form Ω. Then $\omega^l/l!$ and $\cos(\theta)\operatorname{Re}\Omega + \sin(\theta)\operatorname{Im}\Omega$ define calibrations on K which calibrate complex submanifolds (of *real* dimension $2l$) and *special lagrangians of phase* $e^{i\theta}$ respectively. On $M = K \times \mathcal{S}^1$ we can define a torsion–free G_2–structure by

$$\varphi = \operatorname{Re}\Omega + \omega \wedge dt.$$

Further, if $C \subset K$ is a complex curve or $L \subset K$ a special lagrangian of phase 1, then $C \times \mathcal{S}^1$ and $L \times \{pt\}$ are associative. If $S \subset K$ is a complex surface or $L \subset K$ is a special lagrangian of phase i, then $S \times \{pt\}$ and $L \times \mathcal{S}^1$ are coassociative. Therefore, let L be a special lagrangian of phase 1 with boundary $\partial L = \Sigma_g$ inside a complex surface S. The normal bundle of L is just $J(TL)$, for L is lagrangian. On the other hand, S is complex, so that $J(T\partial L) \subset TS$. Lifting S and L to the coassociative $X = S \times \{pt\}$ and the associative $Y = L \times \{pt\}$ inside M, we find that the underlying real rank 2 bundle of ν_X coincides with $J(T\partial L) \oplus 0$. Now the normal bundle $\nu \to Y$ is generated by $J(TL)$ and ∂_t. Consequently, ∂_t is a section of μ_X, the orthogonal complement of ν_X inside ν. Lemma 3.2 then implies $\nu_X \cong \overline{T\partial L} = K_{\partial L}$ as complex line bundles so that $c_1(\nu_X) = c_1(K_{\partial L}) = 2(1-g)$. We finally obtain $\operatorname{index}(X, Y) = 3(1-g)$ from the index formula. These considerations apply for instance to real, 3–dimensional submanifolds contained in the real part of a smooth quintic inside $\mathbb{CP}^4$, and whose boundaries are real analytic. The real part condition guarantees that such submanifolds are special lagrangian, while the boundary condition ensures the boundary to be contained in a complex surface S.

Flat compact examples. The space $\mathbb{R}^7$ together with its standard coordinates $x_1, \ldots, x_7$ defines a trivial example of a flat G_2–manifold with $\varphi = \varphi_0$ as in (1) (with dx_{ijk} in place of e^{ijk}). Since it is translation invariant, the G_2–structure descends to the torus $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ where it defines a compact torsion–free G_2–manifold. Further, the isometric involutions

$$\begin{aligned} \sigma_0(x_1, \ldots, x_7) &= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) \\ \tau_0(x_1, \ldots, x_7) &= (-x_1, x_2, x_3, x_4, x_5, -x_6, -x_7) \end{aligned}$$

satisfy $\sigma_0^*\varphi_0 = \varphi_0$ and $\tau_0^*\varphi_0 = -\varphi_0$. The fixed point loci $Y = \{(x_1, x_2, x_3, 0, 0, 0, 0) \,|\, x_i \in \mathbb{R}\} \cong \mathbb{R}^3$ and $X = \{(0, x_2, x_3, x_4, x_5, 0, 0) \,|\, x_i \in \mathbb{R}\} \cong \mathbb{R}^4$ of σ_0 and τ_0 define an associative and a coassociative respectively, as can be seen by direct inspection. Furthermore, σ_0 and τ_0 commute with the action of $\mathbb{Z}^7$ and therefore descend to the torus, where their fixed point locus consists of the 16 associative 3–tori $\{(x_1, x_2, x_3, a_4, a_5, a_6, a_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z}\} \cong T^3$ and the eight coassociative 4–tori $\{(a_1, x_2, x_3, x_4, x_5, a_6, a_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z}\} \cong T^4$ ($a_i \in \frac{1}{2}\mathbb{Z}$). Then $Y = \{(t, x_2, x_3, 0, 0, 0, 0) \,|\, 0 \leq t \leq 1/2,\ x_i \in \mathbb{R}/\mathbb{Z}\} \cong I \times \Sigma_1$ is an associative whose boundary lies in the coassociative $X = \{(0, x_2, x_3, x_4, x_5, 0, 0) \,|\, x_i \in \mathbb{R}/\mathbb{Z}\}$. Since ν_X is trivial, we find $\operatorname{index}(X, Y) = 0$. Of course, there exists actual deformations of Y along X induced by the flow of ∂_{x_4} and ∂_{x_5}.

Joyce manifolds. For the remaining examples we first invoke the following result which generalises the previous example.

Proposition 6.1 ([23], Sec. 10.8) *Let (M,φ) be a torsion–free G_2–manifold.*
(i) If $\sigma : M \to M$ is a nontrivial isometric involution with $\sigma^*\varphi = \varphi$, then the fixed point locus of σ defines an associative submanifold.
(ii) If $\tau : M \to M$ is a nontrivial isometric involution with $\tau^*\varphi = -\varphi$, then each connected component of the fixed point locus of τ is either a coassociative 4–fold or a single point.

A method for the construction of compact holonomy G_2–manifolds is also due to Joyce [23] (see [24] for a quick introduction). In essence, one considers quotients T^7/Γ, where Γ is a discrete group acting as a subgroup of φ_0–preserving isometries of T^7. Hence, φ_0 descends to a G_2–form φ on the quotient, but dividing by the action of Γ will produce singularities – the image under the quotient map of the fixed points of $\Gamma\backslash\{\mathrm{Id}\}$ in T^7. Concretely, we choose the group $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^3$ given in 12.2 of [23], which is generated by

$$\begin{array}{rcl}\alpha(x_1,\ldots,x_7) & = & (x_1,x_2,x_3,-x_4,-x_5,-x_6,-x_7)\\ \beta(x_1,\ldots,x_7) & = & (x_1,-x_2,-x_3,x_4,x_5,\frac{1}{2}-x_6,-x_7)\\ \gamma(x_1,\ldots,x_7) & = & (-x_1,x_2,-x_3,x_4,\frac{1}{2}-x_5,x_6,\frac{1}{2}-x_7).\end{array}$$

There are no fixed points of $\delta \in \Gamma\backslash\{\mathrm{Id}\}$ unless $\delta = \alpha$, β or γ. For instance, α fixes the 3–tori $\alpha T^3_{a_4a_5a_6a_7} = \{(x_1,x_2,x_3,a_4,a_5,a_6,a_7)\,|\,x_i \in \mathbb{R}/\mathbb{Z}\}$ with $a_i \in \frac{1}{2}\mathbb{Z}$; we denote their image in the quotient by $\alpha T^3_{[a_2a_3b_6a_7]}$. The subgroup $\langle\beta,\gamma\rangle$ acts freely on this set of 16 tori – for example, β maps αT^3_{0000} to $\alpha T^3_{00\frac{1}{2}0}$ (whence $\alpha T^3_{[0000]} = \alpha T^3_{[00\frac{1}{2}0]}$), so that α contributes four copies of T^3 to the singular locus of T^7/Γ. A similar analysis applies for β and γ. In total, the singular locus consists of 12 copies of T^3 which do not intersect. Near any singular 3–torus we can fix a neighbourhood homeomorphic to $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$, for instance by mapping $[x_1,x_2,x_3,a_4+y_4,a_5+y_5,a_6+y_6,a_7+y_7]$ near $\alpha T^3_{[a_4a_5a_6a_7]}$ to $(x_1,x_4,x_5,\{z_1 = y_2+iy_3, z_2 = y_6+iy_7\})$, where $\{z_1,z_2\} = \{-z_1,-z_2\}$ denotes the coset of (z_1,z_2) inside $\mathbb{C}^2/\{\pm 1\}$. We then blow up the singular complex manifold $\mathbb{C}^2/\{\pm 1\}$, that is, we glue in an exceptional divisor $\mathbb{CP}^1$ at the origin to obtain

$$Y = \{(\{z_1,z_2\},[w_1:w_2])\,|\,z_1w_2 = z_2w_1\} \subset \mathbb{C}^2/\{\pm 1\} \times \mathbb{CP}^1.$$

Furthermore, $T^3 \times Y$ carries a natural family $\{\varphi_t\,|\,t \in (0,\epsilon)\}$ of torsion–free G_2–structures. Proceeding this way with the remaining singular tori, we end up with a smooth, simply–connected resolution $\pi : \widetilde{T^7/\Gamma} \to T^7/\Gamma$ on which we can define a family of G_2–structures $\widetilde{\varphi}_t$ by using a partition of unity. Since this family has "small" torsion, Joyce's deformation theorem ensures the existence of a torsion–free G_2–structure $\widetilde{\varphi}$ which defines a holonomy G_2–structure on topological grounds.

Our first example is induced from the isometric involutions

$$\begin{array}{rcl}\sigma_0(x_1,\ldots,x_7) & = & (x_1,\frac{1}{2}-x_2,\frac{1}{2}-x_3,x_4,x_5,-x_6,\frac{1}{2}-x_7)\\ \tau_0(x_1,\ldots,x_7) & = & (x_1,x_2,\frac{1}{2}-x_3,\frac{1}{2}-x_4,x_5,x_6,\frac{1}{2}-x_7).\end{array}$$

on (T^7,φ_0). Clearly, $\sigma_0^*\varphi_0 = \varphi_0$ and σ_0 commutes with Γ, so it descends to an involution σ on T^7/Γ. Its fixed point set is given by the 16 tori $\mathrm{Fix}(\sigma_0) = \{(x_1,b_2,b_3,x_4,x_5,a_6,b_7)\,|\,x_i \in \mathbb{R}/\mathbb{Z},\ a_6 \in \frac{1}{2}\mathbb{Z},\ b_j \in \frac{1}{4}\mathbb{Z}\}$. Furthermore, $\sigma_0 \circ \delta$ has no fixed points for $\delta \in \Gamma$ unless $\delta = \mathrm{Id}$. As Γ, of order 8, acts transitively on the set of 16 tori, the fixed point set of σ inside T^7/Γ consists of two 3–tori T^3 given by $\{[x_1,\frac{1}{4},\frac{1}{4},x_4,x_5,0,\frac{1}{4}]\,|\,x_i \in \mathbb{R}/\mathbb{Z}\}$ and $\{[x_1,\frac{1}{4},\frac{1}{4},x_4,x_5,\frac{1}{2},\frac{1}{4}]\,|\,x_i \in \mathbb{R}/\mathbb{Z}\}$. These do not hit the singular locus of T^7/Γ. Similarly, we see that $\tau_0^*\varphi_0 = -\varphi_0$, and $\tau_0 \circ \delta$ has no fixed points unless $\delta = \mathrm{Id}$ (in which case $\mathrm{Fix}(\tau_0)$ consists of the eight 4–tori

$\{(x_1, x_2, b_3, b_4, x_5, x_6, b_7) \,|\, x_i \in \mathcal{S}^1,\, b_i \in \frac{1}{4}\mathbb{Z}\})$, or $\delta = \beta\gamma$ (in which case $\mathrm{Fix}(\tau_0\beta\gamma)$ consists of the 128 fixed points $(a_1, a_2, b_3, b_4, b_5, b_6, b_7)$ with $a_i \in \frac{1}{2}\mathbb{Z}$ and $b_i \in \frac{1}{4}\mathbb{Z}$). As Γ acts freely on the 8 tori as well as on the 128 points, the fixed point locus of the induced involution $\tau : T^7/\Gamma \to T^7/\Gamma$ consists of the coassociative 4–torus $X = \{[x_1, x_2, \frac{1}{4}, \frac{1}{4}, x_5, x_6, \frac{1}{4}] \,|\, x_i \in \mathbb{R}/\mathbb{Z}\}$ and of eight isolated points. Again, $\mathrm{Fix}(\tau)$ does not intersect the singular locus of T^7/Γ. Furthermore, $Y = \{[x_1, \frac{1}{4}, \frac{1}{4}, t, x_5, 0, \frac{1}{4}] \,|\, x_{1,5} \in \mathbb{R}/\mathbb{Z},\, \frac{1}{4} \leq t \leq \frac{3}{4}\}$ is associative and diffeomorphic to $\Sigma_1 \times I$. It intersects X in the two 2–tori $\{[x_1, \frac{1}{4}, \frac{1}{4}, b_4, x_5, 0, \frac{1}{4}] \,|\, x_i \in \mathbb{R}/\mathbb{Z}\}$, $b_4 \in \frac{1}{4}\mathbb{Z}\}$. The involutions σ and τ lift to involutions of $\widetilde{\sigma}$ and $\widetilde{\tau}$ on the resolution $\widetilde{T^7/\Gamma}$. For instance, σ maps $T^3_{[a_2a_3b_6a_7]}$ into itself and acts on a neighbourhood homeomorphic to $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$ by

$$(x_1, x_4, x_5, \{z_1, z_2\}) \overset{\sigma}{\mapsto} (x_1, -x_4, -x_5, \{z_1, -z_2\}),$$

so that we can lift σ to the blow up via $(\{z_1, z_2\}, [w_1 : w_2]) \mapsto (\{z_1, -z_2\}, [w_1 : -w_2])$. Further, $\widetilde{\sigma}^*\widetilde{\varphi} = \widetilde{\varphi}$ and $\widetilde{\tau}^*\widetilde{\varphi} = -\widetilde{\varphi}$ as follows from Joyce's construction of $\widetilde{\varphi}$, that is, we can resolve $(T^7/\Gamma, \varphi)$ in a σ, τ*–equivariant* way. Since X and Y do not hit the singular locus in T^7/Γ, we yield an associative $Y \cong \Sigma_1 \times I$, whose boundary is contained in a coassociative 4–torus X. We find $\mathrm{index}\,(X, Y) = 2(1 - g) = 0$, for the contributions of $\int c_1(\nu_X)$ from the two boundary components cancel out as in the associative case.

For the second example we start with the isometric involutions

$$\begin{aligned} \sigma_0(x_1, \ldots, x_7) &= (x_1, \tfrac{1}{2} - x_2, \tfrac{1}{2} - x_3, x_4, x_5, -x_6, -x_7) \\ \tau_0(x_1, \ldots, x_7) &= (\tfrac{1}{2} - x_1, x_2, x_3, x_4, x_5, -x_6, -x_7) \end{aligned}$$

which also satisfy $\sigma_0^*\varphi_0 = \varphi_0$ and $\tau_0^*\varphi_0 = -\varphi_0$ respectively. Now $\sigma_0 \circ \delta$ has no fixed points for $\delta \in \Gamma$ unless $\delta = \mathrm{Id}$ or α. The fixed point locus of σ is therefore the image of the 16 tori $\{(x_1, b_2, b_3, x_4, x_5, a_6, a_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z},\, a_i \in \frac{1}{2}\mathbb{Z},\, b_j \in \frac{1}{4}\mathbb{Z}\}$ fixed by σ_0 and the 16 tori $\{(x_1, b_2, b_3, a_4, a_5, x_6, x_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z},\, a_i \in \frac{1}{2}\mathbb{Z},\, b_j \in \frac{1}{4}\mathbb{Z}\}$ fixed by $\sigma_0\alpha$. Similarly, $\tau_0 \circ \delta$ has no fixed points unless $\delta = \mathrm{Id}$ (with $\mathrm{Fix}(\tau_0) = \{(b_1, x_2, x_3, x_4, x_5, a_6, a_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z},\, a_i \in \frac{1}{2}\mathbb{Z},\, b_1 \in \frac{1}{4}\mathbb{Z}\} \cong 16$ tori), $\delta = \alpha$ (with $\mathrm{Fix}(\tau_0 \circ \alpha) = \{(b_1, x_2, x_3, a_4, a_5, x_6, x_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z},\, a_i \in \frac{1}{2}\mathbb{Z},\, b_1 \in \frac{1}{4}\mathbb{Z}\} \cong 16$ tori), or $\delta = \alpha \circ \beta$ (with $\mathrm{Fix}(\tau_0 \circ \alpha \circ \beta) = \{(b_1, a_2, a_3, a_4, a_5, b_6, a_7) \,|\, a_i \in \frac{1}{2}\mathbb{Z},\, b_i \in \frac{1}{4}\mathbb{Z}\} \cong 128$ points). Consequently, both $\mathrm{Fix}(\sigma)$ and $\mathrm{Fix}(\tau)$ hit the singular α–tori of T^7/Γ. To resolve T^7/Γ in a σ, τ–equivariant way and to compute their fixed point locus, we therefore proceed in two steps (cf. also 12.6.3 in [23]). Firstly, we resolve $T^7/\langle\alpha\rangle \cong T^3 \times (T^4/\langle\pm 1\rangle)$ in a σ, τ–equivariant way. The resulting resolution is diffeomorphic to $T^3 \times K3$, where $K3$ denotes the $K3$ surface. To resolve T^7/Γ completely, we lift the actions of β and γ to $T^3 \times K3$ and finally resolve the orbifold $(T^3 \times K3)/\langle\beta, \gamma\rangle$ in a σ, τ–equivariant way. Since the fixed point locus of the lifts of σ and τ does not hit the fixed points of β and γ on $T^3 \times K3$ it will be enough to determine the fixed point set of σ and τ on $(T^3 \times K3)/\langle\beta, \gamma\rangle$. To that end, we remark that the fixed point set of σ inside $T^7/\langle\alpha\rangle$ consists of the union of $T^1_{b_2b_3} \times S \cong \{(x_1, b_2, b_3) \,|\, x_1 \in \mathbb{R}/\mathbb{Z}\} \times S$ ($b_i \in \frac{1}{4}\mathbb{Z}$). Here, S is the singular connected surface given by the union of $T^2_{a_6a_7}/\langle\pm 1\rangle \cong \{(x_4, x_5, a_6, a_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z}\}/\langle\alpha\rangle$ with $T^2_{a_4a_5}/\langle\pm 1\rangle \cong \{(a_4, a_5, x_6, x_7) \,|\, x_i \in \mathbb{R}/\mathbb{Z}\}/\langle\alpha\rangle$ ($a_j \in \frac{1}{2}\mathbb{Z}$) which intersect at the 16 points $(a_4, \ldots, a_7)$. Similarly, the fixed point set of τ inside T^7/Γ is $T^2_{b_1} \times S = \{(b_1, x_2, x_3)\} \times S$ ($b_1 \in \frac{1}{4}\mathbb{Z}$). To smooth out the singularities coming from α, we blow up $T^4/\langle\alpha\rangle$ at these 16 points. The lift of σ and τ to this resolution yields $T^1_{b_2b_3} \times \Sigma$ and $T^2_{b_1} \times \Sigma$ for some compact oriented Riemann surface Σ as the corresponding fixed point set. Working out the zeroth and first Betti number of Σ as in 12.5.2 of [23] shows that $b_0(\Sigma) = 8$ and $b_1(\Sigma) = 0$, that is, Σ consists eight copies of the 2–sphere $\Sigma_0 = \mathcal{S}^2$: In essence, every $T^2_{a_4a_5}/\langle\pm 1\rangle$ or $T^2_{a_6a_7}/\langle\pm 1\rangle$

contributes one 2–sphere on the resolution which we label accordingly by $\mathcal{S}^2_{a_4a_5}$ and $\mathcal{S}^2_{a_6a_7}$. We need to work out the action of β and γ on these sets next. Firstly, β and γ act freely on the set of circles $T^1_{b_2b_3}$ so that they act freely on the fixed 32 copies of $T^1\times\mathcal{S}^2$ resulting in eight distinct copies inside $(T^3\times K3)/\langle\beta,\gamma\rangle$. For the image of $T^2_{b_1}\times\Sigma$ we find four distinct copies of $(T^2\times\mathcal{S}^2)/\langle\beta\rangle$ and two distinct copies of $T^2\times\mathcal{S}^2$, for γ acts freely on the set of 2–tori $T^2_{b_1}$, while β acts trivially and maps for instance $\mathcal{S}^2_{a_4a_5}$ into itself, but $\mathcal{S}^2_{a_6a_7}$ to $\mathcal{S}^2_{\bar{a}_6a_7}$ etc.. Consequently, the boundary of the associative $Y=\{[t,b_2,b_3]\}\times\mathcal{S}^2_{[a_6a_7]}$ inside $(T^3\times K3)/\langle\beta,\gamma\rangle$ consists of two 2–spheres inside the coassociative $X=T^2_{[b_1]}\times\mathcal{S}^2_{[a_6a_7]}\cup T^2_{[\bar{b}_1]}\times\mathcal{S}^2_{[a_6a_7]}$. As a result, we obtain an associative diffeomorphic to $\Sigma_0\times I$ with boundary inside a coassociative $T^2\times\mathcal{S}^2$. By symmetry, $c_1(\nu_{X_1})=-c_1(\nu_{X_2})$ as in the local coassociative case discussed at the beginning of this section. Consequently, we find $\operatorname{index}(X,Y)=2$. In particular, there exists at least a 2–parameter family of infinitesimal deformations.

7 A G_2 analogue of the Maslov index

Finally, we wish to introduce a G_2 analogue of the Maslov index. This index plays an important rôle in symplectic geometry, in particular for Floer homology, where it arises as a sort of relative Morse index [15], [27], [30].

Let us briefly recall its construction. We consider an almost complex manifold (M^{2m},J) with an embedded (not necessarily holomorphic) 2–disk D with boundary in a totally real oriented submanifold X^m (i.e for all $p\in X$, T_pX does not contain any J–complex line). Since D is contractible, we can trivialise $TM_{|D}$ and regard the subbundle $TX_{|\partial D}$ as a closed curve in the set of totally real oriented m–planes in $\mathbb{C}^m$. On the other hand, this set is parametrised by[6] $GL_m(\mathbb{C})/GL^+_m(\mathbb{R})$ and is homotopy equivalent to $U(m)/SO(m)$ – the set of oriented lagrangian m–planes inside $\mathbb{C}^m$. By the exact homotopy sequence for fibrations $\pi_1\big(U(m)/SO(m)\big)\cong\mathbb{Z}$. The *Maslov index* $\mu(X,D)$ of D is the integer corresponding to the homotopy class induced by $TX_{|\partial D}$.

The natural counterpart in the G_2–setting should be the following. Let Y be an embedded (not necessarily associative) 3–disk inside a topological G_2–manifold M such that $\partial Y\cong\mathcal{S}^2$ lies in some ψ–positive submanifold X. Trivialising $TM_{|Y}$ yields thus a map from $\mathcal{S}^2$ to the set $\mathcal{P}_+$ of ψ–positive planes in $\mathbb{R}^7$.

Proposition 7.1 *The set of ψ–positive planes $\mathcal{P}_+\subset G_4(\mathbb{R}^7)$ is homotopy equivalent to $G_{\psi_0}(\mathbb{R}^7)\cong G_2/SO(4)$, the set of coassociatives. In particular, $\pi_2(\mathcal{P}_+)\cong\mathbb{Z}_2$.*

Proof: Instead of $\mathcal{P}_+$ we shall consider the dual set $\mathcal{P}^\perp_+\subset G_3(\mathbb{R}^7)$. Restricting φ_0 to $E\in G_3(\mathbb{R}^7)$ yields a multiple of the volume form induced by $g_{0|E}$, and thus a real number (recall that we always consider the grassmannian of *oriented* subplanes). Since φ_0 is a calibration, we may regard φ_0 as a map $G_3(\mathbb{R}^7)\to[-1,1]$. By convention we shall orient the orthogonal complements $F^\perp$, $F\in\mathcal{P}_+$, in such a way that $\varphi_0(F^\perp)<0$, whence $\mathcal{P}^\perp_+=\varphi_0^{-1}\big([-1,0)\big)$. Any fibre $\varphi^{-1}(\cos\alpha)$ is acted on transitively by G_2 and contains an element of the form

$$E_\alpha=e_1\wedge e_2\wedge(\cos\alpha\,e_3+\sin\alpha\,e_4)$$

with respect to the fixed G_2 frame $e_1,\ldots,e_7$ of Section 2. To see this, write $E_\alpha=x\wedge y\wedge z$ for some unit vectors x, y, $z\in\mathbb{R}^7$. Since G_2 acts transitively on ordered orthonormal pairs

[6] We denote by $GL^+_m(\mathbb{R})$ the set of invertible $m\times m$–matrices with positive determinant.

with stabiliser $SU(2)$ [20], we may assume that $E_\alpha = e_1 \wedge e_2 \wedge z$ upon transformation by a suitable element in G_2. The $SU(2)$–action induced by the inclusion into G_2 gives rise to a decomposition $\mathbb{R}^7 = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$, where $SU(2)$ acts trivially on $\mathrm{Im}\,\mathbb{H}$ and $\mathbb{H} = \mathbb{C}^2$ becomes the standard vector representation. We are still free to modify E_α without changing e_1 and e_2 by an element in $SU(2)$. Since this group acts transitively on the unit sphere in $\mathbb{C}^2$, we may transform the unit vector $z = \sum_{i=3}^7 z^i e_i$ into $\cos\alpha e_3 + \sin\alpha e_4$ with $\cos\alpha = z^3 = \varphi_0(E_\alpha)$. From this one easily deduces that (a) ± 1 are the only critical values and (b) that the hessian of φ_0 is non–degenerate in directions transverse to the orbits $\varphi_0^{-1}(\pm 1)$. Consequently, φ_0 defines a G_2–invariant Morse function in the sense of [26]. By a theorem in the same paper, one can conclude – in analogy with classical Morse theory – that $\varphi_0^{-1}\big((-\infty, 0]\big)$ is (equivariantly) homotopy equivalent to the disk bundle $G_2 \times_{SO(4)} D^4$ (inside the normal bundle) over $\varphi^{-1}\big((-\infty,-1]\big) \cong G_2/SO(4)$ attached. Hence $\mathcal{P}_+^\perp$ is homotopy equivalent to the open disk bundle which can be retracted to the base. In particular, $\mathcal{P}_+$ is of the same homotopy type as $G_2/SO(4)$. Since $\pi_k(G_2) = 0$ for $k = 1$, 2 and $\pi_1\big(SO(4)\big) = \mathbb{Z}_2$, the exact homotopy sequence for fibrations yields the asserted homotopy group. ■

Definition 7.2 Let D be an embedded associative 3–disk in some topological G_2–manifold. We refer to the natural class of $TX_{|\mathcal{S}^2}$ in $\pi_2(\mathcal{P}_+) \cong \mathbb{Z}_2$ as the *G_2–Maslov index* of D and denote it by $\mu_{G_2}(X, D)$.

Proposition 7.3 *If $D \subset M$ is an embedded associative 3–disk with boundary in some coassociative X, then*

$$\mu_{G_2}(X, D) = \int_{\mathcal{S}^2} c_1(\nu_X) \bmod 2 = \big(\mathrm{index}\,(X, D) + 1\big) \bmod 2.$$

Proof: The natural complex structure of $\mathcal{S}^2$ gives a natural identification with $\mathbb{CP}^1$ as a complex manifold. Consequently, any complex line bundle is of the form $\mathcal{O}(n)$ where $n = \int_{\mathbb{CP}^1} c_1\big(\mathcal{O}(n)\big)$. For instance $T\mathcal{S}^2 = \mathcal{O}(2)$, and in particular, we find $TX_{|\mathcal{S}^2} = \mathcal{O}(2) \oplus \mathcal{O}(n)$. By assumption, there is a map

$$f : p \in \mathcal{S}^2 \mapsto F_p \in G^{\psi_0}(\mathbb{R}^7) \cong G_2/SO(4).$$

The homotopy class $[f]$ generates $\pi_2\big(G_2/SO(4)\big)$ if and only if its boundary $\partial[f]$ generates $\pi_1\big(SO(4)\big) \cong \mathbb{Z}_2$. If $K : (D^2, \mathcal{S}^1) \to (\mathcal{S}^2, N)$ denotes the collapsing map sending S^1 to the north pole N of $\mathcal{S}^2$, a representative of $\partial[f]$ is obtained by restricting the lift $\widetilde{f} : (D^2, S^1) \to G_2$ of the composition $f \circ K$ to $\mathcal{S}^1$. We can think of $\widetilde{f}_{|\mathcal{S}^1} \in SO(4)$ as taking values in the set of bases of F_N. Its action on F_N can thus be homotoped into the action of the maximal torus $T^2 = \mathcal{S}^1 \times \mathcal{S}^1$ of $SO(4)$ (seen as structure group of $TX_{|\mathcal{S}^2}$), namely

$$t \in \mathcal{S}^1 \mapsto \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{nit} \end{pmatrix}.$$

On the other hand, the inclusion of the torus induces an epimorphism $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \to \pi_1\big(SO(4)\big) = \mathbb{Z}_2$ which is mod 2 reduction of the second summand, whence $[\partial f] = n \bmod 2$. ■

Example: We consider again the coassociative germ example of Section 6, where Y is the 3–disk D inside $\mathrm{Im}\,\mathbb{H} \subset \mathrm{Im}\,\mathbb{O}$. Then $\mathrm{index}\,(X, D) = 1$. On the other hand, $TX_{|S^2} = \mathcal{O}(2) \oplus \mathcal{O}(0)$ from which we conclude $\mu_{G_2}(X, D) = 0$ in accordance with the proposition.

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