Extensions of profinite duality groups

Alexander Schmidt and Kay Wingberg

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Let $G$ be a profinite group and let $p$ be a prime number. By $\text{Mod}_p(G)$ we denote the category of discrete $p$-primary $G$-modules. For $A \in \text{Mod}_p(G)$ and $i \geq 0$, let

$$D_i(G,A) = \lim_{U} H^i(U,A)^*,$$

where $^*$ is $\text{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$, the direct limit is taken over all open subgroups $U$ of $G$ and the transition maps are the duals of the corestriction maps. $D_i(G,A)$ is a discrete $G$-module in a natural way. Assume that $n = \text{cd}_p G$ is finite. Then the $G$-module

$$I(G) = \lim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^\nu \mathbb{Z})$$

is called the dualizing module of $G$ at $p$. Its importance lies in the functorial isomorphism

$$H^n(G,A)^* \cong \text{Hom}_G(A,I(G))$$

for all $A \in \text{Mod}_p(G)$. This isomorphism is induced by the cup-products ($V \subseteq U$)

$$H^n(G,A)^* \times_{p^\nu A^U} H^n(V, \mathbb{Z}/p^\nu \mathbb{Z})^*, \ (\phi,a) \mapsto (\alpha \mapsto \phi(\text{cor}_G^V(\alpha \cup a)))$$

by passing to the limit over $\nu$ and $V$, and then over $U$. The identity-map of $I(G)$ gives rise to the homomorphism

$$\text{tr} : H^n(G,I(G)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

called the trace map.

The profinite group $G$ is called a duality group at $p$ of dimension $n$ if for all $i \in \mathbb{Z}$ and all finite $G$-modules $A \in \text{Mod}_p(G)$, the cup-product and the trace map

1
Hi(G, Hom(A, I(G))) × H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, I(G)) \xrightarrow{ir} \mathbb{Q}_p/\mathbb{Z}_p

yield an isomorphism

H^i(G, Hom(A, I(G))) \cong H^{n-i}(G, A)^*.

**Remark:** In [Ve], J.-L. Verdier used the name **strict Cohen-Macaulay at** \( p \) for what we call a profinite duality group at \( p \) here. In [Pl], A. Pletch defined \( D^n_p \)-groups (and called them duality groups at \( p \) of dimension \( n \)). The \( D^n_p \)-groups of Pletch are exactly the duality groups at \( p \) (in our sense) which, in addition, satisfy the following finiteness condition:

\[ FC(G, p): \quad H^i(G, A) \text{ is finite for all finite } A \in \text{Mod}_p(G) \text{ and for all } i \geq 0. \]

Since any finite, discrete \( G \)-module is trivialized by an open subgroup \( U \) of \( G \), condition \( FC(G, p) \) can also be rephrased in the form:

\[ FC(G, p): \quad H^i(U, \mathbb{Z}/p\mathbb{Z}) \text{ is finite for all open subgroups } U \text{ of } G \text{ and all } i \geq 0. \]

By a duality theorem due to J. Tate, see [Ta] Thm. 3 or [Ve] Prop. 4.3 or [NSW] (3.4.6), a profinite group \( G \) of cohomological \( p \)-dimension \( n \) is a duality group at \( p \) if and only if

\[ D_i(G, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } 0 \leq i < n. \]

As a consequence we see that every open subgroup of a duality group at \( p \) is a duality group at \( p \) as well (of the same cohomological dimension), and if an open subgroup of \( G \) is a duality group at \( p \) and \( cd_p G < \infty \), then \( G \) is a duality group at \( p \) of the same cohomological dimension (use [NSW] (3.3.5)(ii)). Furthermore, any profinite group of cohomological \( p \)-dimension 1 is a duality group at \( p \).

We call a profinite group \( G \) **virtually a duality group at** \( p \) of (**virtual**) **dimension** \( vcd_p G = n \) if an open subgroup \( U \) of \( G \) is a duality group at \( p \) of dimension \( n \).

The objective of this paper is to give a proof of Theorem 1 below, which states that the class of duality groups is closed under group extensions \( 1 \xrightarrow{} H \xrightarrow{} G \xrightarrow{} G/H \xrightarrow{} 1 \) if the kernel satisfies \( FC(H, p) \). Weaker forms of Theorem 1 were first proved by A. Pletch (for \( D^n_p \)-groups, see [Pl]) and by the second author (for Poincaré groups, see [Wi]).

\[ ^1 \text{The proof given by Pletch in [Pl] is only correct for pro-} p \text{-groups as the author assumes that finitely generated projective modules over the complete group ring } \mathbb{Z}_p[G] \text{ are free.} \]
Theorem 1. Let 
\[ 1 \to H \to G \to G/H \to 1 \]
be an exact sequence of profinite groups such that condition \( FC(H, p) \) is satisfied. Then the following assertions hold:

(i) If \( G \) is a duality group at \( p \), then \( H \) is a duality group at \( p \) and \( G/H \) is virtually a duality group at \( p \).

(ii) If \( H \) and \( G/H \) are duality groups at \( p \), then \( G \) is a duality group at \( p \).

Moreover, in both cases we have:
\[ \text{cd}_p G = \text{cd}_p H + \text{vcd}_p G/H, \]
and there is a canonical \( G \)-isomorphism
\[ I(G)^\vee \cong I(H)^\vee \hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee, \]
where \( \vee \) is the Pontryagin dual and \( \hat{\otimes}_{\mathbb{Z}_p} \) is the tensor-product in the category of compact \( \mathbb{Z}_p \)-modules.

Remark: The assumption \( FC(H, p) \) is necessary, as the following examples show:

1. Let \( G \) be the free pro-\( p \)-group on two generators \( x, y \) and let \( H \subset G \) be the normal subgroup generated by \( x \). Then \( H \) is free of infinite rank, \( G/H \) is free of rank one and \( 1 \to H \to G \to G/H \to 1 \) is an exact sequence in which all three groups are duality groups of dimension one.

2. Let \( D \) be a duality group at \( p \) of dimension 2, \( F \) a duality group at \( p \) of dimension 1 and \( G = F \ast D \) their free product. The kernel of the projection \( G \to D \) has cohomological \( p \)-dimension 1, hence is a duality group a \( p \) of dimension 1. The group \( G \) has cohomological \( p \)-dimension 2 but is is not a duality group at \( p \).

In the proof of Theorem 1, we make use of the following

Proposition 2. Let 
\[ 1 \to H \to G \to G/H \to 1 \]
be an exact sequence of profinite groups. Assume that \( FC(H, p) \) holds. Then there is a spectral sequence of homological type
\[ E^2_{i,j} = D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \implies D_{i+j}(G, \mathbb{Z}/p\mathbb{Z}). \]
Proof. Let \( g \) run through the open normal subgroups of \( G \). Then \( gH/H \cong g/g \cap H \) runs through the open normal subgroups of \( G/H \). For a \( G \)-module \( A \in \text{Mod}_p(G) \), we consider the Hochschild-Serre spectral sequence

\[
E(g, g \cap H, A) : E_2^{ij}(g, g \cap H, A) = H^i(g/g \cap H, H^j(g \cap H, A)) \Longrightarrow H^{i+j}(g, A).
\]

If \( g' \subseteq g \) is another open normal subgroup of \( G \), then the corestriction yields a morphism

\[
cor : E(g', g' \cap H, A) \longrightarrow E(g, g \cap H, A)
\]

of spectral sequences. The map

\[
E_2^{ij}(g', g' \cap H, A) \longrightarrow E_2^{ij}(g, g \cap H, A)
\]

is the composite of the maps

\[
H^i\left(g'/g' \cap H, H^j(g' \cap H, A)\right) \xrightarrow{\text{cor}_{g'/g' \cap H}} H^i\left(g/g \cap H, H^j(g \cap H, A)\right)
\]

and the map between the limit terms is the corestriction

\[
cor_{g'} : H^{i+j}(g', A) \longrightarrow H^{i+j}(g, A).
\]

For \( 2 \leq r \leq \infty \) we set

\[
E_2^{ij} = D_{ij}^2(G, H, A) := \lim_{\rightarrow} E_r^{ij}(g, g \cap H, A)^*.
\]

As taking duals and direct limits are exact operations, the terms \( D_{ij}^2(G, H, A) \), \( 2 \leq r \leq \infty \), establish a homological spectral sequence which converges to \( D_n(G, A) \). If \( h \) runs through the open subgroups of \( H \) which are normal in \( G \), then the cohomology groups \( H^j(h, A) \) are \( G \)-modules in a natural way. If \( g \) is open in \( G \) with \( g \cap H \subseteq h \), then these groups are \( g/g \cap H \)-modules. We see that

\[
D_{ij}^2(G, H, A) = \lim_{\rightarrow \leftarrow \subseteq h} \lim_{g \subseteq G} H^i(g/g \cap H, H^j(h, A))^*,
\]

where for both limits the transition maps are (induced by) \( \text{cor}^* \). In order to conclude the proof of the proposition, it remains to construct isomorphisms

\[
D_{ij}^2(G, H, \mathbb{Z}/p\mathbb{Z}) \cong D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z})
\]
for all $i$ and $j$. To this end note that all occurring abelian groups are $\mathbb{F}_p$-vector spaces, so that $^*$ is $\text{Hom}(-, \mathbb{F}_p)$. Further note that for vector spaces $V, W$ over a field $k$ the homomorphism

$$V^* \otimes W^* \longrightarrow (V \otimes W)^*, \ \phi \otimes \psi \longmapsto (v \otimes w \mapsto \phi(v)\psi(w))$$

is an isomorphism provided that $V$ or $W$ is finite-dimensional. Let $h$ be an open subgroup of $H$ which is normal in $G$ and let $g' \subseteq g$ be open subgroups of $G$ such that $g$ acts trivially on the finite group $H_j(h, \mathbb{Z}/p\mathbb{Z})$. Then, by [NSW] (1.5.3)(iv), the diagram

$$\begin{array}{ccc}
H^i(g'/g' \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^i(g'/g' \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})) \\
\downarrow \text{cor} \otimes \text{id} & & \downarrow \text{cor} \\
H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z}))
\end{array}$$

commutes. For fixed $h$, we therefore obtain isomorphisms

$$D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^*$$

$$\cong \left( \lim_{\rightarrow} H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \right) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^*$$

$$\cong \lim_{\rightarrow} H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^*$$

$$\cong \lim_{\rightarrow} \left( H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) \right)^*$$

$$\cong \lim_{\rightarrow} \left( H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z}) \right)^*.$$

Passing to the limit over $h$, we obtain the required isomorphism

$$D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \cong D_{ij}(G, H, \mathbb{Z}/p\mathbb{Z}).$$

\[\square\]

**Corollary 3.** Under the assumptions of Proposition 2, let $i_0$ and $j_0$ be the smallest integers such that $D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \neq 0$ and $D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0$, respectively. Then $D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \neq 0$.

**Proof.** The spectral sequence constructed in Proposition 2 induces an isomorphism

$$D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \cong D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$  

\[\square\]
Proof of Theorem 1. Assume that $G$ is a duality group at $p$ of dimension $d$. Let $\text{cd}_p H = m$ and $n = d - m$. Then there exists an open subgroup $H_1$ of $H$ such that $H^m(H_1, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let $G_1$ be an open subgroup of $G$ such that $H_1 = G_1 \cap H$. Then $G_1$ is a duality group at $p$ of dimension $d$, $\text{cd}_p H_1 = m$ and $G_1/H_1$ is an open subgroup of $G/H$. We consider the exact sequence

$$1 \longrightarrow H_1 \longrightarrow G_1 \longrightarrow G_1/H_1 \longrightarrow 1.$$ 

As $H^m(H_1, \mathbb{Z}/p\mathbb{Z})$ is finite and nonzero, we have $\text{vcd}_p G_1/H_1 = n$, see [NSW] (3.3.9). Furthermore, $D_i(G_1, \mathbb{Z}/p\mathbb{Z}) = 0$, $i < n + m$. Using Corollary 3, we see that $D_i(G_1/H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i < n$ and $D_j(H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $j < m$. Thus $G_1/H_1$, hence $G/H$, is virtually a duality group at $p$ of dimension $n$, and $H_1$, and so $H$, is a duality group at $p$ of dimension $m$. This shows (i).

Assume now that $H$ and $G/H$ are duality groups at $p$ of dimension $m$ and $n$. Then, $\text{cd}_p G = n + m$ by [NSW] (3.3.8), and in the spectral sequence of Proposition 2 we have $E^2_{ij} = 0$ for $(i, j) \neq (n, m)$. Hence $D_r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \neq n + m$ showing that $G$ is a duality group at $p$ of dimension $n + m$.

In order to prove the assertion about the dualizing modules, let $h$ run through all open subgroups of $H$ which are normal in $G$ and $g$ runs through the open subgroups of $G$. Since $m = \text{cd}_p H$, the Hochschild-Serre spectral sequence induces isomorphisms

$$H^{m+n}(g, \mathbb{Z}/p^r\mathbb{Z}) \cong H^{n}(g/g \cap H, H^m(g \cap H, \mathbb{Z}/p^r\mathbb{Z})),$$

and we obtain

$$I(G) \cong \lim_{\nu} \lim_{g} H^{m+n}(g, \mathbb{Z}/p^r\mathbb{Z})^*$$

$$\cong \lim_{\nu} \lim_{g} \lim_{h} H^m(g/g \cap H, H^m(h, \mathbb{Z}/p^r\mathbb{Z}))^*$$

$$\cong \lim_{\nu} \lim_{h} \lim_{g, \text{res}} H^0(g/g \cap H, \text{Hom}(H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H)))$$

$$\cong \lim_{\nu} \lim_{h} \text{Hom}(H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H))$$

$$\cong \text{Hom}_{cts}(\lim_{\nu} \lim_{h} H^m(h, \mathbb{Z}/p^r\mathbb{Z}), I(G/H))$$

$$\cong \text{Hom}_{cts}(\lim_{\nu} \lim_{h} H^m(h, \mathbb{Z}/p^r\mathbb{Z})^* \cap I(G/H))$$

$$\cong \text{Hom}_{cts}(I(H)^*, I(G/H)) \cong (I(H)^* \otimes_{\mathbb{Z}_p} I(G/H)^*)^\vee$$

(see [NSW] (5.2.9) for the last isomorphism). This completes the proof of the theorem. \qed
References


Alexander Schmidt, NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Deutschland. email: alexander.schmidt@mathematik.uni-regensburg.de

Kay Wingberg, Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Deutschland. email: wingberg@mathi.uni-heidelberg.de