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Cristiana Bertolin

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HOMOLOGICAL INTERPRETATION OF EXTENSIONS AND BIEXTENSIONS OF COMPLEXES

CRISTIANA BERTOLIN

ABSTRACT. Let $\mathbf{T}$ be a topos. Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1, 2, 3$) be a complex of commutative groups of $\mathbf{T}$ with A_i in degree 1 and B_i in degree 0. We define the geometrical notions of extension of K_1 by K_3 and of biextension of (K_1, K_2) by K_3. These two notions generalize to complexes of the kind K_i the classical notions of extensions and biextensions of commutative groups of $\mathbf{T}$. We then apply the geometrical-homological principle of Grothendieck in order to compute the homological interpretation of extensions and biextensions of complexes.

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INTRODUCTION

Let $\mathbf{T}$ be a topos. Denote by $\mathcal{C}$ the category of commutative groups of $\mathbf{T}$, i.e. the category of $\mathbb{Z}$-modules of $\mathbf{T}$. If I is an object of $\mathcal{C}$, we denote by $\mathbb{Z}[I]$ the free $\mathbb{Z}$-module generated by I. Let $\mathcal{D}(\mathcal{C})$ the derived category of the abelian category $\mathcal{C}$.

The geometrical-homological principle of Grothendieck states the following fact: *if an object A of $\mathcal{C}$ admits an explicit representation in $\mathcal{D}(\mathcal{C})$ by a complex* L. *whose components are direct sums of objects of the kind $\mathbb{Z}[I]$, with I object of $\mathcal{C}$, then the groups* $\mathrm{Ext}^i(A, B)$ *admit an explicit geometrical description for any object B of $\mathcal{C}$.*

A first example of this principle of Grothendieck is furnished by the geometrical notion of extensions of objects of $\mathcal{C}$: in fact if P and G are two objects of $\mathcal{C}$, it is a classical result that the group $\mathrm{Ext}^0(P, G)$ is isomorphic to the group of

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automorphisms of any extension of P by G and the group $\mathrm{Ext}^1(P,G)$ is isomorphic to the group of isomorphism classes of extensions of P by G.

In [SGA7] Exposé VII Corollary 3.6.5 Grothendieck furnishes another example of this principle using the geometrical notion of biextension of objects $\mathcal{C}$: if P, Q and G are three objects of $\mathcal{C}$, he proves that the group $\mathrm{Biext}^0(P,Q;G)$ of automorphisms of any biextension of (P,Q) by G and the group $\mathrm{Biext}^1(P,Q;G)$ of isomorphism classes of biextensions of (P,Q) by G, have the following homological interpretation:

$$\mathrm{Biext}^i(P,Q;G) \cong \mathrm{Ext}^i(P \overset{\mathbb{L}}{\otimes} Q, G) \qquad (i=0,1) \tag{0.1}$$

where $P \overset{\mathbb{L}}{\otimes} Q$ is the derived functor of the functor $Q \to P \otimes Q$ in the derived category $\mathcal{D}(\mathcal{C})$. In other words, the strict Picard stack of biextensions of (P,Q) by G is equivalent to the strict Picard stack associated (by the dictionary of [SGA4] Exposé XVIII Proposition 1.4.14) to the object $\tau_{\leq o}\mathbb{R}\mathrm{Hom}(P \overset{\mathbb{L}}{\otimes} Q, G[1])$:

$$\mathcal{B}iext(P,Q;G) \;\cong\; \mathrm{ch}\,\big(\tau_{\leq o}\mathbb{R}\mathrm{Hom}(P \overset{\mathbb{L}}{\otimes} Q, G[1])\big).$$

Other examples of the geometrical-homological principle of Grothendieck are exposed in [Br]: according to loc.cit. Proposition 8.4 the strict Picard stack of symmetric biextensions of (P,P) by G is equivalent to the strict Picard stack associated to the object $\tau_{\leq o}\mathbb{R}\mathrm{Hom}(\mathbb{L}\mathrm{Sym}^2(P), G[1])$ and according to loc.cit. Theorem 8.9 the strict Picard stack of the 3-tuple (L,E,α) (resp. the 4-tuple (L,E,α,β)) defining a cubic structure (resp. a Σ-structure) on the G-torsor L is equivalent to the strict Picard stack associated to the object $\tau_{\leq o}\mathbb{R}\mathrm{Hom}(\mathbb{L}P_2^+(P), G[1])$ (resp. $\tau_{\leq o}\mathbb{R}\mathrm{Hom}(\mathbb{L}\Gamma_2(P), G[1])$).

Let $K_i = [A_i \overset{u_i}{\to} B_i]$ (for $i=1,2,3$) be a complex with A_i and B_i objects of $\mathcal{C}$ in degree 1 and 0 respectively. In this paper we introduce the geometrical notions of extension of K_1 by K_3 and of biextension of (K_1,K_2) by K_3. These two notions generalize to complexes of the kind K_i the classical notions of extensions and biextensions of objects of $\mathcal{C}$. We then apply the geometrical-homological principle of Grothendieck in order to compute the homological interpretation of these geometrical notions of extensions and biextensions of complexes.

Our main result is:

Theorem 0.1. *Let $K_i = [A_i \overset{u_i}{\to} B_i]$ (for $i=1,2,3$) be a complex of commutative groups of $\boldsymbol{T}$ with A_i in degree 1 and B_i in degree 0. Then we have the following canonical isomorphisms*

$$\mathrm{Biext}^i(K_1,K_2;K_3) \cong \mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) \qquad (i=0,1).$$

In other words, the strict Picard stack of biextensions of (K_1,K_2) by K_3 is equivalent to the strict Picard stack associated (by the dictionary of [SGA4] Exposé XVIII Proposition 1.4.14) to the object $\tau_{\leq o}\mathbb{R}\mathrm{Hom}(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3[1])$:

$$\mathcal{B}iext(K_1,K_2;K_3) \;\cong\; \mathrm{ch}\,\big(\tau_{\leq o}\mathbb{R}\mathrm{Hom}(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3[1])\big).$$

If $A_i = 0$ (for $i=1,2,3$), this theorem coincides with the homological interpretation (0.1) of Grothendieck.

The homological interpretation of extensions of complexes of the kind K_i is a special case of Theorem 0.1: in fact, it is furnished by the statement of this Theorem with $K_2 = [0 \overset{0}{\to} \mathbb{Z}]$, since the category $\mathbf{Biext}(K_1, [0 \overset{0}{\to} \mathbb{Z}]; K_3)$ of biextensions of

$(K_1, [0 \xrightarrow{0} \mathbb{Z}])$ by K_3 is equivalent to the category $\mathbf{Ext}(K_1, K_3)$ of extensions of K_1 by K_3:

$$\mathbf{Biext}(K_1, [0 \xrightarrow{0} \mathbb{Z}]; K_3) \cong \mathbf{Ext}(K_1, K_3),$$

and since in the derived category $\mathcal{D}(\mathcal{C})$ we have that

$$\mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} [0 \xrightarrow{0} \mathbb{Z}], K_3) \cong \mathrm{Ext}^i(K_1, K_3) \qquad (i = 0, 1).$$

The idea of the proof of Theorem 0.1 works as follow: Let $K = [A \xrightarrow{u} B]$ be a complex of commutative groups of $\mathbf{T}$ concentrated in degrees 1 and 0 and let L.. be a bicomplex of commutative groups of $\mathbf{T}$ which satisfies $\mathrm{L}_{ij} = 0$ for $(ij) \neq (00), (01), (02), (10), (11), (10), (20)$. To the complex K and to the bicomplex L.. we associate an additive cofibred category $\Psi_{\mathrm{Tot(L..)}}(K)$ which has the following *homological description*:

$$\Psi^i_{\mathrm{Tot(L..)}}(K) \cong \mathrm{Ext}^i(\mathrm{Tot(L..)}, K) \qquad (i = 0, 1) \tag{0.2}$$

where $\Psi^0_{\mathrm{Tot(L..)}}(K)$ is the group of automorphisms of any object of $\Psi_{\mathrm{Tot(L..)}}(K)$ and $\Psi^1_{\mathrm{Tot(L..)}}(K)$ is the group of isomorphism classes of objects of $\Psi_{\mathrm{Tot(L..)}}(K)$. Then, to any complex of the kind $K = [A \xrightarrow{u} B]$ we associate a canonical flat partial resolution L..(K) whose components are direct sums of objects of the kind $\mathbb{Z}[I]$ with I a commutative group of $\mathbf{T}$. Consider now three complexes $K_i = [A_i \xrightarrow{u_i} B_i]$ (for $i = 1, 2, 3$). The category $\Psi_{\mathrm{Tot(L..}(K_1))}(K_3)$ and $\Psi_{\mathrm{Tot(L..}(K_1)\otimes \mathrm{L..}(K_2))}(K_3)$ admit the following *geometrical description*:

$$\begin{aligned} \Psi_{\mathrm{Tot(L..}(K_1))}(K_3) &\cong \mathbf{Ext}(K_1, K_3) \\ \Psi_{\mathrm{Tot(L..}(K_1)\otimes \mathrm{L..}(K_2))}(K_3) &\cong \mathbf{Biext}(K_1, K_2; K_3) \end{aligned} \tag{0.3}$$

Putting together this geometrical description (0.3) with the homological description (0.2), we get

- the proof of the Theorem 0.1;
- the proof that the group of automorphisms of any extension of K_1 by K_3 is the group $\mathrm{Ext}^0(K_1, K_3)$ and that the group of isomorphism classes of extensions of K_1 by K_3 is the group $\mathrm{Ext}^1(K_1, K_3)$.

Notation

In this paper, $\mathbf{T}$ is a topos and $\mathcal{C}$ is the category of commutative groups of $\mathbf{T}$, i.e. the category of $\mathbb{Z}$-modules of $\mathbf{T}$. Recall that we can identify commutative groups of $\mathbf{T}$ with abelian sheaves over $\mathbf{T}$. If I is an object of $\mathcal{C}$, we denote by $\mathbb{Z}[I]$ the free $\mathbb{Z}$-module generated by I (see [SGA4] Exposé IV 11).

All complexes of objects of $\mathcal{C}$ that we consider in this paper are chain complexes. The truncation $\tau_{\geq n}$L. of a complex L. is the following complex: $(\tau_{\geq n}\mathrm{L.})_i = \mathrm{L}_i$ for $i \geq n$ and $(\tau_{\geq n}\mathrm{L.})_i = 0$ for $i < n$.

If L.. is a bicomplex of objects of $\mathcal{C}$, we denote by Tot(L..) the total complex of L..: it is the chain complex whose component of degree n is $\mathrm{Tot(L..)}_n = \sum_{i+j=n} \mathrm{L}_{ij}$.

Let $\mathcal{D}(\mathcal{C})$ be the derived category of the abelian category $\mathcal{C}$. Denote by $\mathcal{D}^{[1,0]}(\mathcal{C})$ the subcategory of $\mathcal{D}(\mathcal{C})$ of complexes $K = [A \xrightarrow{u} B]$ with A concentrated in degree 1 and B concentrated in degree 0.

1. Extensions and biextensions of complexes

Let G be an object of $\mathcal{C}$. A **G-torsor** is an object of $\mathbf{T}$ endowed with an action of G, which is locally isomorphic to G acting on itself by translation.

Let P, G be objects of $\mathcal{C}$. An **extension of** P **by** G is an object E of $\mathbf{T}$ such that we have an exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow P \longrightarrow 0.$$

By definition the object E is a group. Since in this paper we consider only commutative extensions, E is in fact an object of $\mathcal{C}$. We denote by $\mathbf{Ext}(P,G)$ the category of extensions of P by G. It is a classical result that the category $\mathbf{Ext}(P,*)$ of extensions of P by variable objects of $\mathcal{C}$ is an additive cofibred category over $\mathcal{C}$

$$\begin{array}{ccc} \mathbf{Ext}(P,*) & \longrightarrow & \mathcal{C} \\ \mathbf{Ext}(P,G) & \mapsto & G \end{array}$$

Moreover, the Baer sum of extensions defines a group law for the objects of the category $\mathbf{Ext}(P,G)$, which is therefore a Picard category.

Let P, G be objects of $\mathcal{C}$. Denote by $m : P \times P \to P$ the group law of P and by $pr_i : P \times P \to P$ with $i = 1,2$ the two projections of $P \times P$ in P. According [SGA7] Exposé VII 1.1.6 and 1.2, the category of extensions of P by G is equivalent to the category of 4-tuple (P,G,E,φ), where E is a G_P-torsor over P, and $\varphi : pr_1^*E\ pr_2^*E \to m^*E$ is an isomorphism of torsors over $P \times P$ satisfying some associative and commutative conditions (see [SGA7] Exposé VII diagrams (1.1.4.1) and (1.2.1)):

$$\begin{aligned} \mathbf{Ext}(P,G) \quad \cong \quad & \Big\{(P,G,E,\varphi) \;\Big|\; E = G_P\text{−torsor over } P \text{ and} \\ & \qquad \varphi : pr_1^*E\ pr_2^*E \simeq m^*E \text{ with ass. and comm. conditions}\Big\}. \end{aligned} \tag{1.1}$$

It will be useful in what follows to look at the isomorphism of torsors φ as an associative and commutative group law on the fibres:

$$+ : E_p\ E_{p'} \longrightarrow E_{p+p'}$$

where p, p' are points of $P(S)$ with S any object of $\mathbf{T}$.

Let I and G be objects of $\mathcal{C}$. Concerning extensions of free commutative groups, in [SGA7] Exposé VII 1.4 Grothendieck proves that there is an equivalence of category between the category of extensions of $\mathbb{Z}[I]$ by G and the category of G_I-torsors over I:

$$\mathbf{Ext}(\mathbb{Z}[I],G) \cong \mathbf{Tors}(I,G_I) \tag{1.2}$$

Let P, Q and G be objects of $\mathcal{C}$. A **biextension of** (P,Q) **by** G is a $G_{P\times Q}$-torsor B over $P \times Q$, endowed with a structure of commutative extension of Q_P by G_P and a structure of commutative extension of P_Q by G_Q, which are compatible one with another (for the definition of compatible extensions see [SGA7] Exposé VII Définition 2.1). If m_P, p_1, p_2 (resp. m_Q, q_1, q_2) denote the three morphisms $P \times P \times Q \to P \times Q$ (resp. $P \times Q \times Q \to P \times Q$) deduced from the three morphisms $P \times P \to P$ (resp. $Q \times Q \to Q$) group law, first and second projection, the equivalence of categories (1.1) furnishes the following equivalent definition: a biextension of (P,Q) by G is a $G_{P\times Q}$-torsor B over $P \times Q$ endowed with two isomorphisms of torsors

$$\varphi : p_1^*E\ p_2^*E \longrightarrow m_P^*E \qquad \psi : q_1^*E\ q_2^*E \longrightarrow m_Q^*E$$

over $P\times P\times Q$ and $P\times Q\times Q$ respectively, satisfying some associative, commutative and compatible conditions (see [SGA7] Exposé VII diagrams (2.0.5),(2.0.6),(2.0.8), (2.0.9), (2.1.1)). As for extensions, we will look at the isomorphisms of torsors φ and ψ as two associative and commutative group laws on the fibres which are compatible with one another:

$$+_{P/Q} : E_{p,q}\ E_{p',q} \longrightarrow E_{p+p',q} \qquad +_{Q/P} : E_{p,q}\ E_{p,q'} \longrightarrow E_{p,q+q'}$$

where p, p' (resp. q, q') are points of $P(S)$ (resp. of $Q(S)$) with S any object of $\mathbf{T}$.

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1, 2$) be a complex of objects of $\mathcal{C}$ with A_i in degree 1 and B_i in degree 0.

Definition 1.1. An **extension** (E, β) **of** K_1 **by** K_2 consists of

(1) an extension E of B_1 by B_2;
(2) a trivialization β of the extension u_1^*E of A_1 by B_2 obtained as pull-back of the extension E via $u_1 : A_1 \to B_1$, i.e. an homomorphism $\beta : A_1 \to B_2$.

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ and $K_i' = [A_i' \stackrel{u_i'}{\to} B_i']$ (for $i = 1, 2$) be complexes of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. Let (E, β) be an extension of K_1 by K_2 and let (E', β') be an extension of K_1' by K_2'.

Definition 1.2. A **morphism of extensions**

$$(\underline{F}, \underline{\Upsilon}) : (E, \beta) \longrightarrow (E', \beta')$$

consists of

(1) a morphism $\underline{F} = (F, f_1, f_2) : E \to E'$ from the extension E to the extension E'. In particular,
$$f_1 : B_1 \longrightarrow B_1' \qquad f_2 : B_2 \longrightarrow B_2'$$
are homomorphisms of commutative groups of $\mathbf{T}$.
(2) a morphism of extensions
$$\underline{\Upsilon} = (\Upsilon, g_1, f_2) : u_1^*E \longrightarrow {u_1'}^*E'$$
compatible with the morphism $\underline{F} = (F, f_1, f_2)$ and with the trivializations β and β'. In particular,
$$g_1 : A_1 \longrightarrow A_1'$$
is an homomorphism of commutative groups of $\mathbf{T}$.

We denote by $\mathbf{Ext}(K_1, K_2)$ the category of extensions of K_1 by K_2. If the complex K_1 is fixed, the category $\mathbf{Ext}(K_1, *)$ of extensions of K_1 by variable complexes K_2 is an additive cofibred category over $\mathcal{D}^{[1,0]}(\mathcal{C})$

$$\begin{array}{rcl} \mathbf{Ext}(K_1, *) & \longrightarrow & \mathcal{D}^{[1,0]}(\mathcal{C}) \\ \mathbf{Ext}(K_1, K_2) & \mapsto & K_2 \end{array}$$

This is an easy consequence of the analogous properties of the category of extensions of objects of $\mathcal{C}$. Moreover the Baer sum of extensions defines a group law for the objects of the category $\mathbf{Ext}(K_1, K_2)$, which is therefore a Picard category. The zero object (E_0, β_0) of $\mathbf{Ext}(K_1, K_2)$ with respect to this group law consists of

- the trivial extension $E_0 = B_1 \times B_2$ of B_1 by B_2, i.e. the zero object of $\mathbf{Ext}(B_1, B_2)$, and
- the trivialization $\beta_0 = (id_{A_1}, 0)$ of the extension $u_1^* E_0 = A_1 \times B_2$ of A_1 by B_2. We can consider β_0 as a lifting $(u_1, 0) : A_1 \to B_1 \times B_2$ of $u_1 : A_1 \to B_1$.

The group of automorphisms of any object of $\mathbf{Ext}(K_1, K_2)$ is canonically isomorphic to the group of automorphisms $\mathrm{Aut}(E_0, \beta_0)$ of the zero object of $\mathbf{Ext}(K_1, K_2)$. Explicitly, $\mathrm{Aut}(E_0, \beta_0)$ consists of the couple (f_0, f_1) where

- $f_0 : B_1 \to B_2$ is an automorphism of the trivial extension E_0, i.e. $f_0 \in \mathrm{Aut}(E_0) = \mathrm{Ext}^0(B_1, B_2)$, and
- $f_1 : A_1 \to A_2$ is an homomorphism such that the composite $u_2 \circ f_1$ is compatible with the pull-back $u_1^* f_0$ of the automorphism f_0 of E_0, i.e. $u_2 \circ f_1 = f_0 \circ u_1$.

We have therefore the canonical isomorphisms

$$\mathrm{Aut}(E_0, \beta_0) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(K_1, K_2) = \mathrm{Ext}^0(K_1, K_2).$$

The group law of the category $\mathbf{Ext}(K_1, K_2)$ induces a group law on the set of isomorphism classes of objects of $\mathbf{Ext}(K_1, K_2)$, which is canonically isomorphic to the group $\mathrm{Ext}^1(K_1, K_2)$, as we will prove in Corollary 4.3.

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1, 2, 3$) be a complex of objects of $\mathcal{C}$ with A_i in degree 1 and B_i in degree 0.

Definition 1.3. A **biextension** $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ **of** (K_1, K_2) **by** K_3 consists of

(1) a biextension $\mathcal{B}$ of (B_1, B_2) by B_3;
(2) a trivialization Ψ_1 (resp. Ψ_2) of the biextension $(u_1, id_{B_2})^*\mathcal{B}$ of (A_1, B_2) by B_3 (resp. of the biextension $(id_{B_1}, u_2)^*\mathcal{B}$ of (B_1, A_2) by B_3) obtained as pull-back of $\mathcal{B}$ via $(u_1, id_{B_2}) : A_1 \times B_2 \to B_1 \times B_2$ (resp. via $(id_{B_1}, u_2) : B_1 \times A_2 \to B_1 \times B_2$). These two trivializations have to coincide over (A_1, A_2);
(3) an homomorphism $\lambda : A_1 \otimes A_2 \to A_3$ such that the composite $A_1 \otimes A_2 \stackrel{\lambda}{\to} A_3 \stackrel{u_3}{\to} B_3$ is compatible with the restriction over (A_1, A_2) of the trivializations Ψ_1 and Ψ_2.

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ and $K'_i = [A'_i \stackrel{u'_i}{\to} B'_i]$ (for $i = 1, 2, 3$) be complexes of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. Let $(\mathcal{B}, \Psi_1, \Psi_2, \lambda)$ be a biextension of (K_1, K_2) by K_3 and let $(\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$ be a biextension of (K'_1, K'_2) by K'_3.

Definition 1.4. A **morphism of biextensions**

$$(\underline{F}, \underline{\Upsilon}_1, \underline{\Upsilon}_2, g_3) : (\mathcal{B}, \Psi_1, \Psi_2, \lambda) \longrightarrow (\mathcal{B}', \Psi'_1, \Psi'_2, \lambda')$$

consists of

(1) a morphism $\underline{F} = (F, f_1, f_2, f_3) : \mathcal{B} \to \mathcal{B}'$ from the biextension $\mathcal{B}$ to the biextension $\mathcal{B}'$. In particular,

$$f_1 : B_1 \longrightarrow B'_1 \qquad f_2 : B_2 \longrightarrow B'_2 \qquad f_3 : B_3 \longrightarrow B'_3$$

are homomorphisms of commutative groups of $\mathbf{T}$.
(2) a morphism of biextensions

$$\underline{\Upsilon}_1 = (\Upsilon_1, g_1, f_2, f_3) : (u_1, id_{B_2})^*\mathcal{B} \longrightarrow (u'_1, id_{B'_2})^*\mathcal{B}'$$

compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_1 and Ψ'_1, and a morphism of biextensions

$$\underline{\Upsilon}_2 = (\Upsilon_2, f_1, g_2, f_3) : (id_{B_1}, u_2)^*\mathcal{B} \longrightarrow (id_{B'_1}, u'_2)^*\mathcal{B}'$$

compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ_2 and Ψ'_2. In particular,

$$g_1 : A_1 \longrightarrow A'_1 \qquad g_2 : A_2 \longrightarrow A'_2$$

are homomorphisms of commutative groups of $\mathbf{T}$. By pull-back, the two morphisms $\underline{\Upsilon}_1 = (\Upsilon_1, g_1, f_2, f_3)$ and $\underline{\Upsilon}_2 = (\Upsilon_2, f_1, g_2, f_3)$ define a morphism of biextensions $\underline{\Upsilon} = (\Upsilon, g_1, g_2, f_3) : (u_1, u_2)^*\mathcal{B} \to (u'_1, u'_2)^*\mathcal{B}'$ compatible with the morphism $\underline{F} = (F, f_1, f_2, f_3)$ and with the trivializations Ψ and Ψ'.

(3) an homomorphism $g_3 : A_3 \to A'_3$ of commutative groups of $\mathbf{T}$ compatible with u_3 and u'_3 (i.e. $u'_3 \circ g_3 = f_3 \circ u_3$) and such that

$$\lambda' \circ (g_1 \times g_2) = g_3 \circ \lambda$$

We denote by $\mathbf{Biext}(K_1, K_2; K_3)$ the category of biextensions of (K_1, K_2) by K_3. If the complexes K_1 and K_2 are fixed, the category $\mathbf{Biext}(K_1, K_2; *)$ of biextensions of (K_1, K_2) by variable complexes K_3 is an additive cofibred category over $\mathcal{D}^{[1,0]}(\mathcal{C})$

$$\begin{array}{ccc} \mathbf{Biext}(K_1, K_2; *) & \longrightarrow & \mathcal{D}^{[1,0]}(\mathcal{C}) \\ \mathbf{Biext}(K_1, K_2; K_3) & \mapsto & K_3 \end{array}$$

This is an easy consequence of the fact that the category of biextensions of objects of $\mathcal{C}$ is an additive cofibred category over $\mathcal{C}$ (see [SGA7] Exposé VII 2.4). The Baer sum of extensions defines a group law for the objects of the category $\mathbf{Biext}(K_1, K_2; K_3)$ which is therefore a Picard category (see [SGA7] Exposé VII 2.5). The zero object $(\mathcal{B}_0, \Psi_{0,1}, \Psi_{0,2}, \lambda_0)$ of $\mathbf{Biext}(K_1, K_2; K_3)$ with respect to this group law consists of

- the trivial biextension $\mathcal{B}_0 = B_1 \times B_2 \times B_3$ of (B_1, B_2) by B_3, i.e. the zero object of $\mathbf{Biext}(B_1, B_2; B_3)$, and
- the trivialization $\Psi_{01} = (id_{A_1}, id_{B_2}, 0)$ (resp. $\Psi_{02} = (id_{B_1}, id_{A_2}, 0)$) of the biextension $(u_1, id_{B_2})^*\mathcal{B}_0 = A_1 \times B_2 \times B_3$ of (A_1, B_2) by B_3 (resp. of the biextension $(id_{B_1}, u_2)^*\mathcal{B}_0 = B_1 \times A_2 \times B_3$ of $(B_1 \times A_2)$ by B_3). These two trivialization have to coincide over $A_1 \times A_2$,
- the zero homomorphism $\lambda_0 = 0 : A_1 \otimes A_2 \to A_3$.

The group of automorphisms of any object of $\mathbf{Biext}(K_1, K_2; K_3)$ is canonically isomorphic to the group of automorphisms of the zero object $(\mathcal{B}_0, \Psi_{01}, \Psi_{02}, \lambda_0)$, that we denote $\mathrm{Biext}^0(K_1, K_2; K_3)$. Explicitly, $\mathrm{Biext}^0(K_1, K_2; K_3)$ consists of the couple $(f_0, f_{11} + f_{12})$ where

- $f_0 : B_1 \otimes B_2 \to B_3$ is an automorphism of the trivial biextension $\mathcal{B}_0$, i.e. $f_0 \in \mathrm{Biext}^0(B_1, B_2; B_3) = \mathrm{Hom}(B_1 \otimes B_2, B_3)$, and
- $f_{11} : A_1 \otimes B_2 \to A_3$ (resp. $f_{12} : B_1 \otimes A_2 \to A_3$) is an homomorphism such that the composite $u_3 \circ f_{11}$ (resp. $u_3 \circ f_{12}$) is compatible with the pull-back $(u_1, id_{B_2})^* f_0$ (resp. $(id_{B_1}, u_2)^* f_0$) of the automorphism f_0 of $\mathcal{B}_0$, i.e. $u_3 \circ (f_{11} + f_{12}) = f_0 \circ (u_1 \otimes id_{B_2} + id_{B_1} \otimes u_2)$.

We have therefore the canonical isomorphisms

$$\mathrm{Biext}^0(K_1, K_2; K_3) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) = \mathrm{Ext}^0(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3).$$

The group law of the category $\mathbf{Biext}(K_1, K_2; K_3)$ induces a group law on the set of isomorphism classes of objects of $\mathbf{Biext}(K_1, K_2; K_3)$, that we denote by $\mathrm{Biext}^1(K_1, K_2; K_3)$.

Remark 1.5. According to the above geometrical definitions of extensions and biextensions of complexes, we have the following equivalence of categories

$$\mathbf{Biext}(K_1, [0 \to \mathbb{Z}]; K_3) \cong \mathbf{Ext}(K_1, K_3).$$

Moreover we have also the following isomorphisms

$$\mathrm{Biext}^i(K_1, [\mathbb{Z} \to 0]; K_3) = \begin{cases} \mathrm{Hom}(B_1, A_3), & i = 0; \\ \mathrm{Hom}(K_1, K_3), & i = 1. \end{cases}$$

Remark that we get the same results applying the homological interpretation of biextensions furnished by our main Theorem 0.1.

2. The additive cofibred category $\Psi_{\mathrm{Tot(L..)}}$

Consider the following bicomplex L.. of objects of $\mathcal{C}$:

$$\begin{array}{llccccccccc}
 & & \overbrace{\quad}^{\mathrm{L}_{3*}} & & \overbrace{\quad}^{\mathrm{L}_{2*}} & & \overbrace{\quad}^{\mathrm{L}_{1*}} & & \overbrace{\quad}^{\mathrm{L}_{0*}} & & \\
\mathrm{L}_{*3} & \{ & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
\mathrm{L}_{*2} & \{ & & & & & 0 & \to & \mathrm{L}_{02} & \to & 0 \\
 & & & & & & \downarrow & & \downarrow d_{01} & & \\
\mathrm{L}_{*1} & \{ & & & 0 & \to & \mathrm{L}_{11} & \stackrel{D_{01}}{\to} & \mathrm{L}_{01} & \to & 0 \\
 & & & & \downarrow & & \downarrow d_{10} & & \downarrow d_{00} & & \\
\mathrm{L}_{*0}(K) & \{ & 0 & \to & \mathrm{L}_{20} & \stackrel{D_{10}}{\to} & \mathrm{L}_{10} & \stackrel{D_{00}}{\to} & \mathrm{L}_{00} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & \to & 0 & \to & 0 & &
\end{array} \qquad (2.1)$$

The total complex Tot(L..) is the complex

$$\mathrm{L}_{02} + \mathrm{L}_{11} + \mathrm{L}_{20} \xrightarrow{\mathbb{D}_1} \mathrm{L}_{01} + \mathrm{L}_{10} \xrightarrow{\mathbb{D}_0} \mathrm{L}_{00} \longrightarrow 0$$

where the differential operators $\mathbb{D}_1$ and $\mathbb{D}_0$ can be computed from the diagram (2.1). In this section we define an additive cofibred category $\Psi_{\mathrm{Tot(L..)}}$ over $\mathcal{D}^{[1,0]}(\mathcal{C})$.
Let $K = [A \stackrel{u}{\to} B]$ be an object of $\mathcal{D}^{[1,0]}(\mathcal{C})$.

Definition 2.1. Denote by

$$\Psi_{\mathrm{Tot(L..)}}(K)$$

the category whose **objects** consist of 4-tuple $(E, \alpha, \beta, \gamma)$ where

(1) E is an extension of L_{00} by B;
(2) (α, β) is a trivialization of the extension $\mathbb{D}_0^* E$ of $\mathrm{L}_{01} + \mathrm{L}_{10}$ by B obtained as pull-back of E via $\mathbb{D}_0$. Moreover we require that the corresponding trivialization $\mathbb{D}_1^*(\alpha, \beta)$ of $\mathbb{D}_1^*\mathbb{D}_0^*(E)$ is the trivialization arising from the isomorphism of transitivity $\mathbb{D}_1^*\mathbb{D}_0^*(E) \cong (\mathbb{D}_0 \circ \mathbb{D}_1)^*(E)$ and the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$; In other words, (α, β) is a lifting $\mathrm{L}_{01} + \mathrm{L}_{10} \to E$ of $\mathbb{D}_0 : \mathrm{L}_{01} + \mathrm{L}_{10} \to \mathrm{L}_{00}$ such that $(\alpha, \beta) \circ \mathbb{D}_1 = 0$.
(3) $\gamma : \mathrm{L}_{20} \to A$ is an homomorphism such that the composite $\mathrm{L}_{20} \stackrel{\gamma}{\to} A \stackrel{u}{\to} B$ is compatible with the restriction $D_{10}^*(\beta)$ of the trivialization β over L_{20}.

A **morphism** $(F, id, f_B, f_A) : (E, \alpha, \beta, \gamma) \to (E', \alpha', \beta', \gamma')$ between two objects of $\Psi_{\mathrm{Tot(L..)}}(K)$ consists of

(1) a morphism $(F, id, f_B) : E \to E'$ of extensions inducing the identity on L_{00} and such that $F \circ \alpha = \alpha'$ and $F \circ \beta = \beta'$. In particular,
$$F : E \longrightarrow E' \qquad id : \mathrm{L}_{00} \longrightarrow \mathrm{L}_{00} \qquad f_B : B \longrightarrow B;$$

(2) an homomorphism $f_A : A \to A$ such that $f_A \circ \gamma = \gamma'$.

Remark that the conditions $u \circ \gamma = D^*_{10}(\beta)$ and $u \circ \gamma' = D^*_{10}(\beta')$ imply that $f_B \circ u = u \circ f_A$, i.e. the couple (f_A, f_B) defines a morphism of complexes $K \to K$. The composition of morphisms of $\Psi_{\mathrm{Tot(L..)}}(K)$ is defined using the composition of morphisms of extensions and the composition of morphisms of complexes $(f_A, f_B) : [A \stackrel{u}{\to} B] \to [A \stackrel{u}{\to} B]$.

We can summarize the data $(E, \alpha, \beta, \gamma)$ in the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & B & = & B & = & B & & \\
 & {}^{u}\nearrow & \downarrow & & \downarrow & & \downarrow & & \\
A & & \mathbb{D}_1^*\mathbb{D}_0^*E & \to & \mathbb{D}_0^*E & \to & E & & \\
 & {}^{\gamma}\nwarrow & \downarrow & & {\scriptstyle(\alpha,\beta)}\uparrow\downarrow & & \downarrow & & \\
0 & \to & \mathrm{L}_{02}+\mathrm{L}_{11}+\mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01}+\mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have a functor
$$\begin{array}{rcl}\Pi : \Psi_{\mathrm{Tot(L..)}} & \longrightarrow & \mathcal{D}^{[1,0]}(\mathcal{C}) \\ \Psi_{\mathrm{Tot(L..)}}(K) & \mapsto & K\end{array}$$

which is cofibring. In fact, let $(f_1, f_0) : K = [A \stackrel{u}{\to} B] \to K' = [A' \stackrel{u'}{\to} B']$ be a morphism of $\mathcal{D}^{[1,0]}(\mathcal{C})$, and let $(E, \alpha, \beta, \gamma)$ be an object of the fibre $\Psi_{\mathrm{Tot(L..)}}(K)$ over K. Denote by $(f_0)_*E$ the push-down of E via the homomorphism $f_0 : B \to B'$ and by $(F, id, f_0) : E \to E'$ the corresponding morphism of extensions inducing the identity on L_{00}. The object $((f_0)_*E, F \circ \alpha, F \circ \beta, f_1 \circ \gamma)$ is clearly an object of the fibre $\Psi_{\mathrm{Tot(L..)}}(K')$ over K' and the morphism
$$(F, id, f_0, f_1) : (E, \alpha, \beta, \gamma) \longrightarrow ((f_0)_*E, F \circ \alpha, F \circ \beta, f_1 \circ \gamma)$$
is a cocartesian morphism for the functor $\Pi : \Psi_{\mathrm{Tot(L..)}} \to \mathcal{D}^{[1,0]}(\mathcal{C})$: it is enough to use the analogue property of the morphism of extensions $(F, id, f_0) : E \to (f_0)_*E$ which is a classical result. Therefore the category $\Psi_{\mathrm{Tot(L..)}}$ is a cofibred category over $\mathcal{D}^{[1,0]}(\mathcal{C})$.

Finally, the cofibred category $\Psi_{\mathrm{Tot(L..)}}$ is additive, i.e. it satisfies the two following conditions:

(1) $\Psi_{\mathrm{Tot(L..)}}(0)$ is equivalent to the trivial category;
(2) $\Psi_{\mathrm{Tot(L..)}}(K \times K') \longrightarrow \Psi_{\mathrm{Tot(L..)}}(K) \times \Psi_{\mathrm{Tot(L..)}}(K')$ is an equivalence of category for any object K, K' of $\mathcal{D}^{[1,0]}(\mathcal{C})$.

This is an easy consequence of the fact that the cofibred category $\mathbf{Ext}(\mathrm{L}_{00}, *)$ of extensions of L_{00} by objects of $\mathcal{C}$ is additive.

For any object $K = [A \stackrel{u}{\to} B]$ of $\mathcal{D}^{[1,0]}(\mathcal{C})$, the Baer sum of extensions defines a group law for the objects of the category $\Psi_{\mathrm{Tot(L..)}}(K)$. The zero object of $\Psi_{\mathrm{Tot(L..)}}(K)$ with respect to this law group is the 4-tuple $(E_0, \alpha_0, \beta_0, \gamma_0)$ where

- $E_0 = \mathrm{L}_{00} \times B$ is the trivial extension of L_{00} by B, i.e. the zero object of $\mathbf{Ext}(\mathrm{L}_{00}, B)$, and
- α_0 is the trivialization $(id_{\mathrm{L}_{01}}, 0)$ of the extension $d^*_{00}E_0 = \mathrm{L}_{01} \times B$ of L_{01} by B; β_0 is the trivialization $(id_{\mathrm{L}_{10}}, 0)$ of the extension $D^*_{00}E_0 = \mathrm{L}_{10} \times B$ of L_{10} by B. We can consider α_0 (resp. β_0) as a lifting $(d_{00}, 0)$ (resp. $(D_{00}, 0)$) of $d_{00} : \mathrm{L}_{01} \to \mathrm{L}_{00}$ (resp. of $D_{00} : \mathrm{L}_{10} \to \mathrm{L}_{00}$),
- $\gamma_0 = 0 : \mathrm{L}_{20} \to A$ is the zero homomorphism.

The group of automorphisms of any object of $\Psi_{\mathrm{Tot(L..)}}(K)$ is canonically isomorphic to the group of automorphisms of the zero object of $\Psi_{\mathrm{Tot(L..)}}(K)$, that we denote by $\Psi^0_{\mathrm{Tot(L..)}}(K)$. Explicitly, $\Psi^0_{\mathrm{Tot(L..)}}(K)$ consists of the couple $(f_0, (f_{01}, f_{10}))$ where

- $f_0 : \mathrm{L}_{00} \to B$ is an automorphism of the trivial extension E_0, i.e. $f_0 \in \mathrm{Aut}(E_0) = \mathrm{Ext}^0(\mathrm{L}_{00}, B)$, and
- $f_{01} : \mathrm{L}_{01} \to A$ (resp. $f_{10} : \mathrm{L}_{10} \to A$) is an homomorphism such that the composite $u \circ f_{01}$ (resp. $u \circ f_{10}$) is compatible with the pull-back $d^*_{00}(f_0)$ (resp. $D^*_{00}(f_0)$) of the automorphism f_0 of E_0, i.e.

$$\begin{array}{lrcl} & u \circ f_{01} & = & f_0 \circ d_{00} \\ (\text{resp.} & u \circ f_{10} & = & f_0 \circ D_{00}\ ) \end{array} \tag{2.2}$$

The group law of the category $\Psi_{\mathrm{Tot(L..)}}(K)$ induces a group law on the set of isomorphism classes of objects of $\Psi_{\mathrm{Tot(L..)}}(K)$ that we denote by $\Psi^1_{\mathrm{Tot(L..)}}(K)$.

3. Homological description of $\Psi_{\mathrm{Tot(L..)}}$

Theorem 3.1. *Let $K = [A \stackrel{u}{\to} B]$ be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. We have the following canonical isomorphisms*

$$\begin{array}{rcl} \Psi^0_{\mathrm{Tot(L..)}}(K) & \cong & \mathrm{Ext}^0(\mathrm{Tot(L..)}, K) = \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathrm{Tot(L..)}, K) \\ \Psi^1_{\mathrm{Tot(L..)}}(K) & \cong & \mathrm{Ext}^1(\mathrm{Tot(L..)}, K) = \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathrm{Tot(L..)}, K[1]). \end{array}$$

Proof. Let $(f_0, (f_{01}, f_{10}))$ be an element of $\Psi^0_{\mathrm{Tot(L..)}}(K)$, i.e. an automorphism of the zero object $(E_0, \alpha_0, \beta_0, \gamma_0)$ of $\Psi_{\mathrm{Tot(L..)}}(K)$. We will show that the morphisms $f_0 : \mathrm{L}_{00} \to B$ and $(f_{01}, f_{10}) : \mathrm{L}_{01} + \mathrm{L}_{10} \to A$ define a morphism $\mathrm{Tot(L..)} \to K$ in the derived category $\mathcal{D}(\mathcal{C})$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{L}_{02} + \mathrm{L}_{11} + \mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01} + \mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0 \\ & & \downarrow & & {\scriptstyle (f_{01}, f_{10})}\downarrow & & \downarrow{\scriptstyle f_0} & & \\ & & 0 & \to & A & \stackrel{u}{\to} & B & \to & 0. \end{array} \tag{3.1}$$

By definition, the morphisms $f_{01} : \mathrm{L}_{01} \to A$ and $f_{10} : \mathrm{L}_{10} \to A$ satisfies the equalities (2.2). Since $\mathbb{D}_0 = (d_{00}, D_{00})$, we get that $f_0 \circ \mathbb{D}_0 = u \circ (f_{01}, f_{10})$, i.e. the second diagram of (3.1) is commutative. Concerning the first diagram of (3.1), we have to prove that $(f_{01}, f_{10}) \circ \mathbb{D}_1 = 0$ with $\mathbb{D}_1 = (d_{01}, 0) + (D_{01}, d_{10}) + (0, D_{10})$. The first equality (2.2) and the condition $d_{00} \circ d_{10} = 0$ imply

$$u \circ f_{01} \circ d_{01} = f_0 \circ d_{00} \circ d_{01} = 0,$$

which furnishes (since u is arbitrary)

$$(f_{01}, f_{10}) \circ (d_{01}, 0) = 0.$$

Both equalities (2.2) and the condition $d_{00} \circ D_{01} + D_{00} \circ d_{10} = 0$ give that

$$u \circ (f_{01}, f_{10}) \circ (D_{01}, d_{10}) = f_0 \circ (d_{00}, D_{00}) \circ (D_{01}, d_{10}) = 0,$$

which implies that

$$(f_{01}, f_{10}) \circ (D_{01}, d_{10}) = 0$$

since u is arbitrary. Because of the second equality (2.2) and the condition $D_{00} \circ D_{10} = 0$, we have that

$$u \circ f_{10} \circ D_{10} = f_0 \circ D_{00} \circ D_{10} = 0,$$

which furnishes (again because u is arbitrary)

$$(f_{01}, f_{10}) \circ (0, D_{10}) = 0.$$

Therefore also the first diagram of (3.1) is commutative. Hence we have construct a morphism

$$\begin{array}{ccc} \Psi^0_{\mathrm{Tot(L..)}}(K) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathrm{Tot(L..)}, K) \\ (f_0, (f_{01}, f_{10})) & \mapsto & (f_0, (f_{01}, f_{10})) \end{array}$$

which is clearly a canonical isomorphism.
Let $(E, \alpha, \beta, \gamma)$ be an object of $\Psi_{\mathrm{Tot(L..)}}(K)$. We will show that $(E, \alpha, \beta, \gamma)$ defines a morphism $\mathrm{Tot(L..)} \to K[1]$ in the derived category $\mathcal{D}(\mathcal{C})$. Recall that E is an extension of L_{00} by B. Denote $j : E \to \mathrm{L}_{00}$ the corresponding surjective morphism. Consider the complex $[E \stackrel{j}{\to} \mathrm{L}_{00}]$, with E in degree 0 and L_{00} in degree -1. It is a resolution of B, and so in the derived category $\mathcal{D}(\mathcal{C})$ we have that

$$B = [E \stackrel{j}{\to} \mathrm{L}_{00}].$$

Since by definition, the data (α, β) can be considered as a lifting $\mathrm{L}_{01} + \mathrm{L}_{10} \to E$ of $\mathbb{D}_0 : \mathrm{L}_{01} + \mathrm{L}_{10} \to \mathrm{L}_{00}$ such that $(\alpha, \beta) \circ \mathbb{D}_1 = 0$, we can construct in $\mathcal{D}(\mathcal{C})$ the following morphism

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{L}_{02} + \mathrm{L}_{11} + \mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01} + \mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0 \\ & & \downarrow & & {\scriptstyle (\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle id} & & \\ & & 0 & \to & E & \stackrel{j}{\to} & \mathrm{L}_{00} & \to & 0 \end{array} \tag{3.2}$$

that we denote by

$$c(E, \alpha, \beta) : \mathrm{Tot(L..)} \longrightarrow [E \stackrel{j}{\to} \mathrm{L}_{00}][1] = B[1].$$

Now we use the homomorphism $\gamma : \mathrm{L}_{20} \to A$ and the above morphism $c(E, \alpha, \beta)$ in order to construct in the derived category $\mathcal{D}(\mathcal{C})$ a morphism $c(E, \alpha, \beta, \gamma)$: $\mathrm{Tot(L..)} \to [A \stackrel{u}{\to} B][1]$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{L}_{02} + \mathrm{L}_{11} + \mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01} + \mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0 \\ & & {\scriptstyle (0,0,\gamma)}\downarrow & & \downarrow{\scriptstyle c(E,\alpha,\beta)} & & \downarrow & & \\ 0 & \to & A & \stackrel{u}{\to} & B & \to & 0 & \to & 0. \end{array} \tag{3.3}$$

Recall that $\mathbb{D}_1 = (d_{01}, 0) + (D_{01}, d_{10}) + (0, D_{10})$. By construction $c(E, \alpha, \beta) \circ (d_{01}, 0) + (D_{01}, d_{10}) = 0$. By definition the homomorphism γ satisfies the equality $u \circ \gamma = \beta \circ D_{10}$, and so we have that $c(E, \alpha, \beta)_{|\mathrm{L}_{10}} \circ (0, D_{10}) = u \circ \gamma$. Therefore the diagram (3.3) is commutative. Since the morphism $c(E, \alpha, \beta, \gamma)$ depends only on the isomorphism class of the object (E,α,β,γ), we have construct a canonical morphism

$$\begin{array}{rcl} c:\Psi^1_{\mathrm{Tot(L..)}}(K) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathrm{Tot(L..)},K[1]) \\ (E,\alpha,\beta,\gamma) & \mapsto & c(E,\alpha,\beta,\gamma). \end{array}$$

Now we have to show that this morphism is an isomorphism.
Injectivity: Let (E,α,β,γ) be an object of $\Psi_{\mathrm{Tot(L..)}}(K)$ such that the morphism $c(E,\alpha,\beta,\gamma)$ that it defines in $\mathcal{D}(\mathcal{C})$, is the zero morphism. The corresponding morphism $c(E,\alpha,\beta):\mathrm{Tot(L..)}\longrightarrow [E\stackrel{j}{\rightarrow}\mathrm{L}_{00}][1]$ (3.2) must also be zero in $\mathcal{D}(\mathcal{C})$. Now we will show that $c(E,\alpha,\beta)$ is already zero in the category $\mathcal{K}(\mathcal{C})$ of complexes modulo homotopy. Recall that the complex $[E\stackrel{j}{\rightarrow}\mathrm{L}_{00}]$ is a resolution of B in $\mathcal{D}(\mathcal{C})$. The hypothesis that $c(E,\alpha,\beta)$ is zero in $\mathcal{D}(\mathcal{C})$ implies that there is a resolution of B in $\mathcal{D}(\mathcal{C})$ of the kind $[C_0\stackrel{i}{\rightarrow}C_{-1}]$ with C_0 in degree 0 and C_{-1} in degree -1, and a quasi-isomorphism $(v_0,v_{-1}):[E\stackrel{j}{\rightarrow}\mathrm{L}_{00}]\rightarrow[C_0\rightarrow C_{-1}]$, explicitly

$$\begin{array}{ccccccc} 0 & \rightarrow & E & \stackrel{j}{\rightarrow} & \mathrm{L}_{00} & \rightarrow & 0 \\ & & {\scriptstyle v_0}\downarrow & & \downarrow{\scriptstyle v_{-1}} & & \\ 0 & \rightarrow & C_0 & \stackrel{i}{\rightarrow} & C_{-1} & \rightarrow & 0, \end{array} \tag{3.4}$$

such that the composite $(v_0,v_{-1})\circ c(E,\alpha,\beta)$ is homotopic to zero. Since the morphism (v_0,v_{-1}) induces the identity on B, it identifies E with the fibred product $\mathrm{L}_{00}\times_{C_{-1}}C_0$ of L_{00} and C_0 over C_{-1}. Therefore, the homomorphism $s:\mathrm{L}_{00}\rightarrow C_0$ inducing the homotopy $(v_0,v_{-1})\circ c(E,\alpha,\beta)\sim 0$, i.e. satisfying $i\circ s=v_{-1}\circ id_{\mathrm{L}_{00}}$, factorizes through an homomorphism

$$S:\mathrm{L}_{00}\longrightarrow E=\mathrm{L}_{00}\times_{C_{-1}}C_0$$

which satisfies $j\circ S=id_{\mathrm{L}_{00}}$. This last equality means that the homomorphism S splits the extension E of L_{00} by B and so the complex $[E\stackrel{j}{\rightarrow}\mathrm{L}_{00}]$ is isomorphic in $\mathcal{K}(\mathcal{C})$ to B, i.e. to the complex $[B\rightarrow 0]$ with B in degree 0. But then it is clear that the morphism

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{C})}(\mathrm{Tot(L..)},B)\longrightarrow \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\mathrm{Tot(L..)},B)$$

is an isomorphism and that $c(E,\alpha,\beta)$ is already zero in the category $\mathcal{K}(\mathcal{C})$. There exists therefore an homomorphism $h:\mathrm{L}_{00}\rightarrow E$ such that

$$j\circ h=id_{\mathrm{L}_{00}} \qquad\qquad h\circ\mathbb{D}_0=(\alpha,\beta),$$

i.e. h splits the extension E, which is therefore the trivial extension E_0 of L_{00} by B, and h is compatible with the trivializations (α,β). Moreover, concerning the data $\gamma:\mathrm{L}_{20}\rightarrow A$ we get that

$$u\circ\gamma=\beta\circ D_{10}=h\circ D_{00}\circ D_{10}=0,$$

and so (since u is arbitrary) the homomorphism γ is zero. Hence we can conclude that the object (E,α,β,γ) lies in the isomorphism class of the zero object of $\Psi_{\mathrm{Tot(L..)}}(K)$.
Surjectivity: Consider a morphism $((f_{01},f_{10}),(f_{02},f_{11},f_{20})):\mathrm{Tot(L..)}\rightarrow[A\stackrel{u}{\rightarrow}$

$B][1]$ in the derived category $\mathcal{D}(\mathcal{C})$:

$$(3.5)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{L}_{02}+\mathrm{L}_{11}+\mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01}+\mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0 \\ & & {\scriptstyle (f_{02},f_{11},f_{20})}\downarrow & & \downarrow{\scriptstyle (f_{01},f_{10})} & & \downarrow & & \\ 0 & \to & A & \stackrel{u}{\to} & B & \to & 0 & \to & 0. \end{array}$$

Since the homomorphism (f_{01}, f_{10}) comes from a morphism $\mathrm{Tot}(\mathrm{L..}) \to B[1]$ of $\mathcal{D}(\mathcal{C})$ (i.e. from an homomorphism $\mathrm{L}_{01}+\mathrm{L}_{10} \to B$ whose composite with $\mathbb{D}_1$ is zero), the commutativity of the above diagram implies that $f_{02} = f_{11} = 0$ or $f_{11} = f_{20} = 0$. Because of the symmetry of the bicomplex L.. (2.1), we can choose arbitrary the condition that we prefer: here we assume $f_{02} = f_{11} = 0$ and therefore we get the equality

$$u \circ f_{20} = f_{10} \circ D_{10}.$$

Consider now a resolution of B in $\mathcal{D}(\mathcal{C})$ of the kind $[C_0 \to C_{-1}]$ with C_0 in degree 0 and C_{-1} in degree -1. We can then assume that the morphism (f_{01}, f_{10}) comes from the following morphism $\mathrm{Tot}(\mathrm{L..}) \to [C_0 \stackrel{i}{\to} C_{-1}][1]$ of $\mathcal{D}(\mathcal{C})$

$$(3.6)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{L}_{02}+\mathrm{L}_{11}+\mathrm{L}_{20} & \stackrel{\mathbb{D}_1}{\to} & \mathrm{L}_{01}+\mathrm{L}_{10} & \stackrel{\mathbb{D}_0}{\to} & \mathrm{L}_{00} & \to & 0 \\ & & \downarrow & & {\scriptstyle (F_{01},F_{10})}\downarrow & & \downarrow{\scriptstyle F_{00}} & & \\ & & 0 & \to & C_0 & \stackrel{i}{\to} & C_{-1} & \to & 0. \end{array}$$

Since C_0 is an extension of C_{-1} by B, we can consider the extension

$$E = F_{00}^{*}C_0$$

obtained as pull-back of C_0 via $F_{00} : \mathrm{L}_{00} \to C_{-1}$. The condition $F_{00} \circ \mathbb{D}_0 = i \circ (F_{01}, F_{10})$ implies that $(F_{01}, F_{10}) : \mathrm{L}_{01} + \mathrm{L}_{10} \to C_0$ factories through an homomorphism

$$(\alpha, \beta) : \mathrm{L}_{01} + \mathrm{L}_{10} \to E$$

which satisfies $j \circ (\alpha, \beta) = \mathbb{D}_0$, with $j : E \to \mathrm{L}_{00}$ the canonical surjection of the extension E. Moreover the condition $(F_{01}, F_{10}) \circ \mathbb{D}_1 = 0$ furnishes the equality $(\alpha, \beta) \circ \mathbb{D}_1 = 0$. Therefore the data $(E, \alpha, \beta, f_{20})$ is an object of the category $\Psi_{\mathrm{Tot}(\mathrm{L..})}(K)$. Consider now the morphism $c(E, \alpha, \beta, f_{20}) : \mathrm{Tot}(\mathrm{L..}) \to K[1]$ associated to $(E, \alpha, \beta, f_{20})$. By construction, the morphism (3.6) is the composite of the morphism (3.2) deduced from $c(E, \alpha, \beta, f_{20})$ with the morphism

$$(F, F_{00}) : [E \stackrel{j}{\to} \mathrm{L}_{00}] \longrightarrow [C_0 \stackrel{i}{\to} C_{-1}]$$

where F is the canonical morphism $E = F_{00}^{*}C_0 \to C_0$. Since this last morphism (F, F_{00}) is a morphism of resolutions of B, we can conclude that in the derived category $\mathcal{D}(\mathcal{C})$ the morphism $((f_{01}, f_{10}), (f_{02}, f_{11}, f_{20})) : \mathrm{Tot}(\mathrm{L..}) \to [A \stackrel{u}{\to} B][1]$ is the morphism $c(E, \alpha, \beta, f_{20})$. □

4. A canonical flat partial resolution for a complex concentrated in two consecutive degrees

Let $K = [A \stackrel{u}{\to} B]$ be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. First we construct *a canonical flat partial resolution of the complex* K. Here "partial resolution" means that we have an isomorphism between the homology groups of the partial resolution and of K only in degree 1 and 0. Remark that this is enough for our goal since only the groups Ext^1 and Ext^0 are involved in the statement of the main Theorem 0.1.

Consider the following bicomplex $\mathrm{L}..(K)$ which satisfies $\mathrm{L}_{ij}(K) = 0$ for $(ij) \neq (00), (01), (02), (10)$, which is endowed with an augmentation map $\epsilon_0 : \mathrm{L}_{00}(K) \to B, \epsilon_1 : \mathrm{L}_{10}(K) \to A$, and which depends functorially on K:

$$\begin{array}{lccccccc}
 & & \overbrace{}^{\mathrm{L}_{2*}(K)} & & \overbrace{}^{\mathrm{L}_{1*}(K)} & & \overbrace{}^{\mathrm{L}_{0*}(K)} & & \\
\mathrm{L}_{*3}(K) & \{ & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
\mathrm{L}_{*2}(K) & \{ & 0 & \to & 0 & \to & \mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B] & \to & 0 \\
 & & & & \downarrow & & \downarrow d_{01} & & \\
\mathrm{L}_{*1}(K) & \{ & 0 & \to & 0 & \to & \mathbb{Z}[B \times B] & \to & 0 \\
 & & & & \downarrow & & \downarrow d_{00} & & \\
\mathrm{L}_{*0}(K) & \{ & 0 & \to & \mathbb{Z}[A] & \stackrel{D_{00}}{\to} & \mathbb{Z}[B] & \to & 0 \\
 & & & & \downarrow \epsilon_1 & & \downarrow \epsilon_0 & & \\
K & \{ & 0 & \to & A & \stackrel{u}{\to} & B & \to & 0
\end{array}$$

The non trivial components of $\mathrm{L}..(K)$ are explained in the above diagram. In order to define the differential operators $D..$ and $d..$ and the augmentation map $\epsilon.$ we introduce the following notation: If P is an object of $\mathcal{C}$, we denote by $[p]$ the point of $\mathbb{Z}[P](S)$ defined by the point p of $P(S)$ with S an object of $\mathbf{T}$. In an analogous way, if p, q and r are points of $P(S)$ we denote by $[p, q]$, $[p, q, r]$ the elements of $\mathbb{Z}[P \times P](S)$ and $\mathbb{Z}[P \times P \times P](S)$ respectively. For any object S of $\mathbf{T}$ and for any $a \in A(S), b_1, b_2, b_3 \in B(S)$, we set

$$\begin{array}{rcl}
\epsilon_0[b] & = & b \\
\epsilon_1[a] & = & a \\
d_{00}[b_1, b_2] & = & [b_1 + b_2] - [b_1] - [b_2] \\
d_{01}[b_1, b_2] & = & [b_1, b_2] - [b_2, b_1] \\
d_{01}[b_1, b_2, b_3] & = & [b_1 + b_2, b_3] - [b_1, b_2 + b_3] + [b_1, b_2] - [b_2, b_3] \\
D_{00}[a] & = & [u(a)]
\end{array} \tag{4.1}$$

These morphisms of commutative groups define a bicomplex $\mathrm{L}..(K)$ endowed with an augmentation map $\epsilon. : \mathrm{L}._0(K) \to K$. Remark that the relation $\epsilon_0 \circ d_{00} = 0$ is just the group law on B, and the relation $d_{00} \circ d_{01} = 0$ decomposes in two relations which expresse the commutativity and the associativity of the group law on B. This augmented bicomplex $\mathrm{L}..(K)$ depends functorially on K: in fact, any morphism $f : K \to K'$ of complexes of objects of $\mathcal{C}$ concentrated in degrees 1 and 0, furnishes a commutative diagram

$$\begin{array}{ccc}
\mathrm{L}..(K) & \stackrel{\mathrm{L}..(f)}{\longrightarrow} & \mathrm{L}..(K') \\
\epsilon. \downarrow & & \downarrow \epsilon. \\
K & \stackrel{f}{\longrightarrow} & K'
\end{array}$$

Moreover the components of the bicomplex $\mathrm{L}..(K)$ are flat since they are free $\mathbb{Z}$-modules. In order to conclude that $\mathrm{L}..(K)$ is a canonical flat partial resolution of the complex K we need the following Lemma:

Lemma 4.1. *The additive cofibred category* $\mathbf{Ext}(K, *)$ *of extensions of* K *by a variable object of* $\mathcal{D}^{[1,0]}(\mathcal{C})$ *is equivalent to the additive cofibred category* $\Psi_{\mathrm{Tot}(\mathrm{L}..(K))}$:

$$\mathbf{Ext}(K; *) \cong \Psi_{\mathrm{Tot}(\mathrm{L}..(K))} \tag{4.2}$$

Proof. The total complex $\mathrm{Tot}(\mathrm{L..}(K))$ is

$$\mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B] \xrightarrow{\mathbb{D}_1} \mathbb{Z}[B \times B] + \mathbb{Z}[A] \xrightarrow{\mathbb{D}_0} \mathbb{Z}[B] \longrightarrow 0$$

where $\mathbb{D}_0 = D_{00} + d_{00}$ and $\mathbb{D}_1 = d_{01}$. If $K' = [A' \stackrel{u'}{\rightarrow} B']$ is an object of $\mathcal{D}^{[1,0]}(\mathcal{C})$, in order to describe explicitly the objects of the category $\Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $\mathbb{Z}[B]$ by B' is a $(B')_B$-torsor,
- an extension of $\mathbb{Z}[B \times B] + \mathbb{Z}[A]$ by B consists of a couple of a $(B')_{B\times B}$-torsor and a $(B')_A$-torsor, and finally
- an extension of $\mathbb{Z}[B \times B] + \mathbb{Z}[B \times B \times B]$ by B' consists of a couple of a $(B')_{B\times B}$-torsor and a $(B')_{B\times B\times B}$-torsor.

According to these considerations an object $(E, \alpha, \beta, \gamma)$ of $\Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$ consists of

(1) a B'-torsor E over B
(2) a couple of two trivializations α and β of the couple of two B'-torsors over $B \times B$ and A, which are the pull-back of E via $\mathbb{D}_0$. More precisely:
   - a trivialization α of the B'-torsor over $B \times B$ and $B \times B \times B$ which is the pull-back of E via $d_{00} : \mathbb{Z}[B \times B] \rightarrow \mathbb{Z}[B]$. This trivialization can be interpreted as a group law on the fibres of the B'-torsors over $B \times B$:
   $$+ : E_{b_1}\ E_{b_2} \longrightarrow E_{b_1+b_2}$$
   where b_1, b_2 are points of $B(S)$ with S any object of $\mathbf{T}$.
   - a trivializations β of the B'-torsor $(D_{00})^*E$ over A which is the pull-back of E via $D_{00} : \mathbb{Z}[A] \rightarrow \mathbb{Z}[B]$.

   The compatibility of α and β with the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ involves only α and it imposes on the data $(E, +)$ two relations through the two torsors over $B \times B$ and $B \times B \times B$. These two relations are the relations of commutativity and of associativity of the group law $+$, which mean that $+$ defines over E a structure of commutative extension of B by B';
(3) γ is the zero homomorphism since $\mathrm{L}_{20}(K) = 0$.

The object $(E, +, \beta)$ of $\Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$ is an extension of K by K' and so we can conclude that the category the category $\Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$. The proof that we have in fact an equivalence of additive cofibred categories is left to the lector. □

Proposition 4.2. *We have that* $\mathrm{H}_2(\mathrm{Tot}(\mathrm{L..}(K))) = 0$ *and the augmentation map* $\epsilon. : \mathrm{L.}_0(K) \rightarrow K$ *induces the isomorphisms* $\mathrm{H}_1(\mathrm{Tot}(\mathrm{L..}(K))) \simeq \mathrm{H}_1(K)$ *and* $\mathrm{H}_0(\mathrm{Tot}(\mathrm{L..}(K))) \simeq \mathrm{H}_0(K)$.

Proof. In order to prove this Lemma, we apply [SGA7] Exposé VII Proposition 3.5.3 to the the augmentation map $\epsilon. : \mathrm{L.}_0(K) \rightarrow K$, i.e. we have to prove that for any complex $K' = [A' \stackrel{u'}{\rightarrow} B']$ of $\mathcal{D}^{[1,0]}(\mathcal{C})$ the functor

$$\epsilon.^* : \Psi_{\mathrm{Tot}(K)}(K') \rightarrow \Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$$

is an equivalence of category. According to our definition 2.1, it is clear that the category $\Psi_{\mathrm{Tot}(K)}(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$ of extensions of K by K'. On the other hand, by the Lemma 4.1 also the category $\Psi_{\mathrm{Tot}(\mathrm{L..}(K))}(K')$ is equivalent to the category $\mathbf{Ext}(K, K')$. □

Corollary 4.3. *Let K and K' two objects $\mathcal{D}^{[1,0]}(\mathcal{C})$. Then the group of automorphisms of any extension of K by K' is isomorphic to the group* $\mathrm{Ext}^0(K,K')$*, and the group of isomorphism classes of extensions of K by K' is isomorphic to the group* $\mathrm{Ext}^1(K,K')$.

Proof. According to the above proposition, it exists an arbitrary flat resolution $\mathrm{L}'..(K)$ of K such that the groups $\mathrm{Tot}(\mathrm{L}..(K))_j$ and $\mathrm{Tot}(\mathrm{L}'..(K))_j$ are isomorphic for $j = 0,1,2$. We have therefore a canonical homomorphism

$$\mathrm{L}..(K) \longrightarrow \mathrm{L}'..(K)$$

inducing a canonical homomorphism

$$\mathrm{Tot}(\mathrm{L}..(K)) \longrightarrow \mathrm{Tot}(\mathrm{L}'..(K))$$

which is an isomorphism in degrees 0,1 and 2. According to SGA 7 Proposition 3.4.7 we have the following equivalence of additive cofibred categories

$$\Psi_{\mathrm{Tot}(\mathrm{L}..(K))}(K') \cong \Psi_{\tau_{\geq 2}\mathrm{L}'..(K)}(K').$$

Hence in order to get the statement of this corollary we have to put together

- the geometrical description of the category $\Psi_{\mathrm{Tot}(\mathrm{L}..(K))}(K')$ furnished by the Lemma 4.1:

$$\mathbf{Ext}(K,K') \cong \Psi_{\mathrm{Tot}(\mathrm{L}..(K))}(K'),$$

- the homological description of the groups $\Psi_{\tau_{\geq 2}\mathrm{Tot}(\mathrm{L}'..(K))}(K')$ for $i = 0,1$ furnished by the Theorem 3.1:

$$\Psi^i_{\tau_{\geq 2}\mathrm{Tot}(\mathrm{L}''.(K))}(K') \cong \mathrm{Ext}^i(\mathrm{L}'..(K),K) \cong \mathrm{Ext}^i(K,K').$$

□

5. Geometrical description of $\Psi_{\mathrm{Tot}(\mathrm{L}..)}$

Let $K_i = [A_i \stackrel{u_i}{\to} B_i]$ (for $i = 1,2$) be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and 0 and let $\mathrm{L}..(K_i)$ be its canonical flat partial resolution. Denote by $\mathrm{L}.(K_1,K_2)$ the complex $\mathrm{Tot}(\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2))$. In this section we prove the following geometrical description of the category $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}$:

Theorem 5.1. *The additive cofibred category* $\mathbf{Biext}(K_1,K_2;*)$ *of biextensions of* (K_1,K_2) *by a variable object of* $\mathcal{D}^{[1,0]}(\mathcal{C})$ *is equivalent to the additive cofibred category* $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}$:

$$\mathbf{Biext}(K_1,K_2;*) \cong \Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)} \tag{5.1}$$

Proof. Denote by $\mathrm{L}..(K_1,K_2)$ the bicomplex $\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2)$: explicitly, $\mathrm{L}_{ij}(K_1,K_2) = 0$ for $(ij) \neq (00),(01),(02),(03),(04),(10),(11),(12),(20)$ and its non trivial components are

$$\begin{aligned}
\mathrm{L}_{00}(K_1,K_2) &= \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{00}(K_2)\\
&= \mathbb{Z}[B_1\times B_2]\\
\mathrm{L}_{01}(K_1,K_2) &= \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{01}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{00}(K_2)\\
&= \mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_1\times B_2]\\
\mathrm{L}_{02}(K_1,K_2) &= \mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{00}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{01}(K_2)\\
&= \mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_2\times B_2\times B_2]+\\
&= \mathbb{Z}[B_1\times B_1\times B_2]+\mathbb{Z}[B_1\times B_1\times B_1\times B_2]+\\
&= \mathbb{Z}[B_1\times B_1\times B_2\times B_2]\\
\mathrm{L}_{03}(K_1,K_2) &= \mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{01}(K_2)\\
\mathrm{L}_{04}(K_1,K_2) &= \mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{02}(K_2)\\
\mathrm{L}_{10}(K_1,K_2) &= \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{00}(K_2)+\mathrm{L}_{00}(K_1)\otimes \mathrm{L}_{10}(K_2)\\
&= \mathbb{Z}[A_1\times B_2]+\mathbb{Z}[B_1\times A_2]\\
\mathrm{L}_{11}(K_1,K_2) &= \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{01}(K_2)+\mathrm{L}_{01}(K_1)\otimes \mathrm{L}_{10}(K_2)\\
&= \mathbb{Z}[A_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_1\times A_2]\\
\mathrm{L}_{12}(K_1,K_2) &= \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{02}(K_2)+\mathrm{L}_{02}(K_1)\otimes \mathrm{L}_{10}(K_2)\\
\mathrm{L}_{20}(K_1,K_2) &= \mathrm{L}_{10}(K_1)\otimes \mathrm{L}_{10}(K_2)\\
&= \mathbb{Z}[A_1\times A_2]
\end{aligned}$$

The truncation $\tau_{\geq 2}\mathrm{L.}(K_1,K_2)$ is the complex

$$\mathrm{L}_{02}(K_1,K_2)+\mathrm{L}_{11}(K_1,K_2)+\mathrm{L}_{20}(K_1,K_2)\stackrel{\mathbb{D}_1}{\to}\mathrm{L}_{01}(K_1,K_2)+\mathrm{L}_{10}(K_1,K_2)\stackrel{\mathbb{D}_0}{\to}\mathrm{L}_{00}(K_1,K_2)\to 0$$

where the differential operators $\mathbb{D}_0$ and $\mathbb{D}_1$ can be computed from the below diagram, where we don't have written the identity homomorphisms in order to avoid too heavy notation(for example instead of $(id\times D_{00}^{K_2},D_{00}^{K_1}\times id)$ we have written just $(D_{00}^{K_2},D_{00}^{K_1})$):

$$(5.2)\qquad \begin{array}{ccccc}
 & & & & \mathrm{L}_{02}(K_1,K_2)\\
 & & & & \downarrow d_{01}^{K_2}+d_{01}^{K_1}+(d_{00}^{K_1},d_{00}^{K_2})\\
 & & \mathrm{L}_{11}(K_1,K_2) & \stackrel{D_{00}^{K_1}+D_{00}^{K_2}}{\longrightarrow} & \mathrm{L}_{01}(K_1,K_2)\\
 & & \downarrow d_{00}^{K_2}+d_{00}^{K_1} & & \downarrow d_{00}^{K_2}+d_{00}^{K_1}\\
\mathrm{L}_{20}(K_1,K_2) & \stackrel{(D_{00}^{K_2},D_{00}^{K_1})}{\longrightarrow} & \mathrm{L}_{10}(K_1,K_2) & \stackrel{D_{00}^{K_1}+D_{00}^{K_2}}{\longrightarrow} & \mathrm{L}_{00}(K_1,K_2).
\end{array}$$

Explicitly the condition $\mathbb{D}_0\circ\mathbb{D}_1=0$ means:

- the following sequences are exact:

$$(5.3)\quad \mathbb{Z}[B_1\times B_2\times B_2]+\mathbb{Z}[B_1\times B_2\times B_2\times B_2]\stackrel{d_{01}^{K_2}}{\longrightarrow}\mathbb{Z}[B_1\times B_2\times B_2]\stackrel{d_{00}^{K_2}}{\longrightarrow}\mathbb{Z}[B_1\times B_2]$$

$$(5.4)\quad \mathbb{Z}[B_1\times B_1\times B_2]+\mathbb{Z}[B_1\times B_1\times B_1\times B_2]\stackrel{d_{01}^{K_1}}{\longrightarrow}\mathbb{Z}[B_1\times B_1\times B_2]\stackrel{d_{00}^{K_1}}{\longrightarrow}\mathbb{Z}[B_1\times B_2]$$

- the following diagrams are anticommutative:

$$(5.5)\qquad \begin{array}{ccc}
\mathbb{Z}[B_1\times B_1\times B_2\times B_2] & \stackrel{d_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1\times B_1\times B_2]\\
d_{00}^{K_1}\downarrow & & \downarrow d_{00}^{K_1}\\
\mathbb{Z}[B_1\times B_2\times B_2] & \stackrel{d_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1\times B_2]
\end{array}$$

$$\begin{array}{ccc} \mathbb{Z}[A_1 \times B_2 \times B_2] & \stackrel{D_{00}^{K_1}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2 \times B_2] \\ d_{00}^{K_2}\downarrow & & \downarrow d_{00}^{K_2} \\ \mathbb{Z}[A_1 \times B_2] & \stackrel{D_{00}^{K_1}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.6}$$

$$\begin{array}{ccc} \mathbb{Z}[B_1 \times B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_1 \times B_2] \\ d_{00}^{K_1}\downarrow & & \downarrow d_{00}^{K_1} \\ \mathbb{Z}[B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.7}$$

$$\begin{array}{ccc} \mathbb{Z}[A_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[A_1 \times B_2] \\ D_{00}^{K_1}\downarrow & & \downarrow D_{00}^{K_1} \\ \mathbb{Z}[B_1 \times A_2] & \stackrel{D_{00}^{K_2}}{\longrightarrow} & \mathbb{Z}[B_1 \times B_2] \end{array} \tag{5.8}$$

In order to describe explicitly the objects of the fibre $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3)$ of the cofibred category $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $(\tau_{\geq 2}\mathrm{L}.(K_1,K_2))_0$ by B_3 is a $(B_3)_{B_1 \times B_2}$-torsor,
- an extension of $(\tau_{\geq 2}\mathrm{L}.(K_1,K_2))_1$ by B_3 consists of a $(B_3)_{B_1 \times B_2 \times B_2}$-torsor, a $(B_3)_{B_1 \times B_1 \times B_2}$-torsor, a $(B_3)_{A_1 \times B_2}$-torsor and a $(B_3)_{B_1 \times A_2}$-torsor, and finally
- an extension of $(\tau_{\geq 2}\mathrm{L}.(K_1,K_2))_2$ by B_3 consists of a system of 8 torsors under the groups deduced from B_3 by base change over the bases $B_1 \times B_2 \times B_2,\ B_1 \times B_2 \times B_2 \times B_2,\ B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_2 \times B_2,\ A_1 \times B_2 \times B_2,\ B_1 \times B_1 \times A_2,\ A_1 \times A_2$ respectively.

According to these considerations an object $(E, \alpha, \beta, \gamma)$ of $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3)$ consists of

(1) a B_3-torsor E over $B_1 \times B_2$
(2) a couple of two trivializations $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ of the couple of two B_3-torsors over $B_1 \times B_2 \times B_2 + B_1 \times B_1 \times B_2$ and $A_1 \times B_2 + B_1 \times A_2$, which are the pull-back of E via $\mathbb{D}_0$. More precisely:
   - a couple of trivializations $\alpha = (\alpha_1, \alpha_2)$ of the couple of B_3-torsors over $B_1 \times B_2 \times B_2$ and $B_1 \times B_1 \times B_2$ which are the pull-back of E via $id \times d_{00}^{K_2} + d_{00}^{K_1} \times id : \mathbb{Z}[B_1 \times B_2 \times B_2] + \mathbb{Z}[B_1 \times B_1 \times B_2] \to \mathbb{Z}[B_1 \times B_2]$. The trivializations (α_1, α_2) can be interpreted as two group laws on the fibres of the couple of B_3-torsors over $B_1 \times B_2 \times B_2$ and $B_1 \times B_2 \times B_2$:

$$+_2 : E_{b_2,b_1}\ E_{b_2',b_1} \longrightarrow E_{b_2+b_2',b_1} \qquad\qquad +_1 : E_{b_2,b_1}\ E_{b_2,b_1'} \longrightarrow E_{b_2,b_1+b_1'}$$

   where b_2, b_2' (resp. b_1, b_1') are points of $B_2(S)$ (resp. of $B_1(S)$) with S any object of $\mathbf{T}$.
   - a couple of trivializations $\beta = (\beta_1, \beta_2)$ of the couple of B_3-torsors $((D_{00}^{K_1} \times id)^* E, (id \times D_{00}^{K_2})^* E)$ over $A_1 \times B_2$ and $B_1 \times A_2$ respectively, which are the pull-back of E via $D_{00}^{K_1} \times id + id \times D_{00}^{K_2} : \mathbb{Z}[A_1 \times B_2] + \mathbb{Z}[B_1 \times A_2] \to \mathbb{Z}[B_1 \times B_2]$.

(3) the compatibility of α and β with the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ imposes, on the data $(E, +_1, +_2)$ and $((D_{00}^{K_1} \times id)^* E, (id \times D_{00}^{K_2})^* E, \beta_1, \beta_2)$, 8 relations of compatibility through the system of 8 torsors over $B_1 \times B_2 \times B_2,\ B_1 \times B_2 \times B_2 \times B_2,\ B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_1 \times B_2,\ B_1 \times B_1 \times B_2 \times B_2\ A_1 \times B_2 \times$

$B_2,\ B_1 \times B_1 \times A_2,\ A_1 \times A_2$. For α the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ imposes the following 5 relations of compatibility on the data $(E, +_1, +_2)$ through the 5 torsors arising from the factor $\mathrm{L}_{02}(K_1, K_2)$ of $\tau_{\geq 2}\mathrm{L}.(K_1, K_2)$:

- the exact sequence (5.3) furnishes the two relations of commutativity and of associativity of the group law $+_2$, which mean that $+_2$ defines over E a structure of commutative extension of $(B_2)_{B_1}$ by $(B_3)_{B_1}$;
- the exact sequence (5.4) expresses the two relations of commutativity and of associativity of the group law $+_1$, which mean that $+_1$ defines over E a structure of commutative extension of $(B_1)_{B_2}$ by $(B_3)_{B_2}$;
- the anticommutative diagram (5.5) means that these two group laws are compatible.

Therefore these 5 conditions implies that the torsor E is endowed with a structure of biextension of (B_1, B_2) by B_3.
For β the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ imposes the following 3 relations of compatibility on the data $((D_{00}^{K_1} \times id)^*E, (id \times D_{00}^{K_2})^*E, \beta_1, \beta_2)$ through the 3 torsors arising from the factors $\mathrm{L}_{11}(K_1, K_2) + \mathrm{L}_{20}(K_1, K_2)$ of $\tau_{\geq 2}\mathrm{L}.(K_1, K_2)$:

- the anticommutative diagram (5.6) furnishes a relation of compatibility between the group law $+_2$ of E and the trivialization β_1 of the pull-back $(D_{00}^{K_1} \times id)^*E$ of E over $A_1 \times B_2$, which means that β_1 is a trivialization of biextension;
- the anticommutative diagram (5.7) furnishes a relation of compatibility between the group law $+_1$ of E and the trivialization β_2 of the pull-back $(id \times D_{00}^{K_2})^*E$ of E over $B_1 \times A_2$, which means that also β_2 is a trivialization of biextension;
- the anticommutative diagram (5.8) means that the two trivializations β_1 and β_2 have to coincide over $A_1 \times A_2$.

(4) $\gamma : \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \to A_3$ is an homomorphism such that the composite $\mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \xrightarrow{\gamma} \mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2] \xrightarrow{u_3} B_3$ is compatible with the restriction of the trivializations β_1, β_2 over $\mathbb{Z}[A_1] \otimes \mathbb{Z}[A_2]$.

The object $(E, \alpha, \beta, \gamma)$ of $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3)$ is therefore a biextension $(E, \beta_1, \beta_2, \gamma)$ of (K_1, K_2) by K_3. The diligent lector can check that the above arguments furnish the equivalence of additive cofibred categories (5.1). □

6. Proof of the main theorem

Let $K_i = [A_i \xrightarrow{u_i} B_i]$ (for $i = 1, 2, 3$) be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. Denote by $\mathrm{L}..(K_i)$ (for $i = 1, 2$) the canonical flat partial resolution of K_i introduced at the beginning of §3. According to Proposition 4.2, it exists an arbitrary flat resolution $\mathrm{L}'..(K_i)$ (for $i = 1, 2$) of K_i such that the groups $\mathrm{Tot}(\mathrm{L}..(K_i))_j$ and $\mathrm{Tot}(\mathrm{L}'..(K_i))_j$ are isomorphic for $j = 0, 1, 2$. We have therefore canonical homomorphisms

$$\mathrm{L}..(K_1) \longrightarrow \mathrm{L}'..(K_1) \qquad \mathrm{L}..(K_2) \longrightarrow \mathrm{L}'..(K_2)$$

inducing a canonical homomorphism

$$\mathrm{Tot}(\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2)) \longrightarrow \mathrm{Tot}(\mathrm{L}'..(K_1) \otimes \mathrm{L}'..(K_2))$$

which is an isomorphism in degrees 0,1 and 2. Denote by $\mathrm{L}.(K_1, K_2)$ (resp. $\mathrm{L}'.(K_1, K_2)$) the complex $\mathrm{Tot}(\mathrm{L}..(K_1) \otimes \mathrm{L}..(K_2))$ (resp. $\mathrm{Tot}(\mathrm{L}'..(K_1) \otimes \mathrm{L}'..(K_2))$). Remark that

$\mathrm{L}'.(K_1,K_2)$ represents $K_1 \overset{\mathbb{L}}{\otimes} K_2$ in the derived category $\mathcal{D}(\mathcal{C})$:

$$\mathrm{L}'.(K_1,K_2) \cong K_1 \overset{\mathbb{L}}{\otimes} K_2.$$

By construction, according to SGA 7 Proposition 3.4.7 we have the following equivalence of additive cofibred categories

$$\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3) \cong \Psi_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(K_3).$$

Hence applying the Theorem 5.1, which furnishes the following geometrical description of the category $\Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3)$:

$$\mathbf{Biext}(K_1,K_2;K_3) \cong \Psi_{\tau_{\geq 2}\mathrm{L}.(K_1,K_2)}(K_3),$$

and applying the Theorem 3.1, which furnishes the following homological description of the groups $\Psi^i_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(K_3)$ for $i = 0, 1$:

$$\Psi^i_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(K_3) \cong \mathrm{Ext}^i(\mathrm{L}'.(K_1,K_2),K_3) \cong \mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3),$$

we get the main Theorem 0.1, i.e.

$$\mathrm{Biext}^i(K_1,K_2;K_3) \cong \mathrm{Ext}^i(K_1 \overset{\mathbb{L}}{\otimes} K_2, K_3) \qquad (i = 0, 1).$$

Remark 6.1. From the exact sequence $0 \to A_3[1] \to K_3 \to B_3 \to 0$ we get the long exact sequence

$$0 \to \Psi^0_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(A_3[1]) \to \Psi^0_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(K_3) \to \Psi^0_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(B_3)$$
$$\to \Psi^1_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(A_3[1]) \to \Psi^1_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(K_3) \to \Psi^i_{\tau_{\geq 2}\mathrm{L}'.(K_1,K_2)}(B_3).$$

The homological interpretation of this long exact sequence is

$$0 \to \mathrm{Hom}(\mathrm{L}'.(K_1,K_2),A_3[1]) \to \mathrm{Hom}(\mathrm{L}'.(K_1,K_2),K_3) \to \mathrm{Hom}(\mathrm{L}'.(K_1,K_2),B_3)$$
$$\to \mathrm{Ext}^1(\mathrm{L}'.(K_1,K_2),A_3[1]) \to \mathrm{Ext}^1(\mathrm{L}'.(K_1,K_2),K_3) \to \mathrm{Ext}^1(\mathrm{L}'.(K_1,K_2),B_3),$$

and its geometrical interpretation is

$$0 \to \mathrm{Hom}(A_1 \otimes B_2 + B_1 \otimes A_2, A_3) \to \mathrm{Hom}(\mathrm{L}'.(K_1,K_2),K_3) \to \mathrm{Hom}(B_1 \otimes B_2, B_3)$$
$$\to \mathrm{Hom}(A_1 \otimes A_2, A_3) \to \mathrm{Biext}^1(K_1,K_2;K_3) \to \mathrm{Biext}^1(K_1,K_2;B_3).$$

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NWF-I Mathematik, Universität Regensburg, D-93040 Regensburg
E-mail address: cristiana.bertolin@mathematik.uni-regensburg.de