Homological interpretation of extensions and biextensions of complexes

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HOMOLOGICAL INTERPRETATION OF EXTENSIONS AND BIEXTENSIONS OF COMPLEXES

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Abstract. Let $T$ be a topos. Let $K_i = [A_i \xrightarrow{u_i} B_i]$ (for $i = 1, 2, 3$) be a complex of commutative groups of $T$ with $A_i$ in degree 1 and $B_i$ in degree 0. We define the geometrical notions of extension of $K_1$ by $K_3$ and of biextension of $(K_1, K_2)$ by $K_3$. These two notions generalize to complexes of the kind $K_i$ the classical notions of extensions and biextensions of commutative groups of $T$. We then apply the geometrical-homological principle of Grothendieck in order to compute the homological interpretation of extensions and biextensions of complexes.

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Introduction

Let $T$ be a topos. Denote by $C$ the category of commutative groups of $T$, i.e. the category of $\mathbb{Z}$-modules of $T$. If $I$ is an object of $C$, we denote by $\mathbb{Z}[I]$ the free $\mathbb{Z}$-module generated by $I$. Let $D(C)$ the derived category of the abelian category $C$.

The geometrical-homological principle of Grothendieck states the following fact: if an object $A$ of $C$ admits an explicit representation in $D(C)$ by a complex $L$, whose components are direct sums of objects of the kind $\mathbb{Z}[I]$, with $I$ object of $C$, then the groups $\text{Ext}^i(A, B)$ admit an explicit geometrical description for any object $B$ of $C$.

A first example of this principle of Grothendieck is furnished by the geometrical notion of extensions of objects of $C$: in fact if $P$ and $G$ are two objects of $C$, it is a classical result that the group $\text{Ext}^0(P, G)$ is isomorphic to the group of

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automorphisms of any extension of \( P \) by \( G \) and the group \( \text{Ext}^i(P, G) \) is isomorphic to the group of isomorphism classes of extensions of \( P \) by \( G \).

In [SGA7] Exposé VII Corollary 3.6.5 Grothendieck furnishes another example of this principle using the geometrical notion of biextension of objects \( C \): if \( P, Q \) and \( G \) are three objects of \( C \), he proves that the group \( \text{Biext}^i(P, Q; G) \) of automorphisms of any biextension of \( (P, Q) \) by \( G \) and the group \( \text{Biext}^i(P, Q; G) \) of isomorphism classes of biextensions of \( (P, Q) \) by \( G \), have the following homological interpretation:

\[
\text{Biext}^i(P, Q; G) \cong \text{Ext}^i(P \otimes Q, G) \quad (i = 0, 1)
\]

where \( P \otimes Q \) is the derived functor of the functor \( Q \to P \otimes Q \) in the derived category \( D(C) \). In other words, the strict Picard stack of biextensions of \( (P, Q) \) by \( G \) is equivalent to the strict Picard stack associated (by the dictionary of [SGA4] Exposé XVIII Proposition 1.4.14) to the object \( \tau_{\leq 0} \text{RHom}(P \otimes Q, G[1]) \):

\[
\text{Biext}(P, Q; G) \cong \text{ch} (\tau_{\leq 0} \text{RHom}(P \otimes Q, G[1]))
\]

Our main result is:

**Theorem 0.1.** Let \( K_i = [A_i \to B_i] \) (for \( i = 1, 2, 3 \)) be a complex with \( A_i \) and \( B_i \) objects of \( C \) in degree 1 and 0 respectively. In this paper we introduce the geometrical notions of extension of \( K_1 \) by \( K_3 \) and of biextension of \( (K_1, K_2) \) by \( K_3 \). These two notions generalize to complexes of the kind \( K_i \) the classical notions of extensions and biextensions of objects of \( C \). We then apply the geometrical-homological principle of Grothendieck in order to compute the homological interpretation of these geometrical notions of extensions and biextensions of complexes.

Our main result is:

**Theorem 0.1.** Let \( K_i = [A_i \to B_i] \) (for \( i = 1, 2, 3 \)) be a complex of commutative groups of \( T \) with \( A_i \) in degree 1 and \( B_i \) in degree 0. Then we have the following canonical isomorphisms

\[
\text{Biext}^i(K_1, K_2; K_3) \cong \text{Ext}^i(K_1 \otimes K_2, K_3) \quad (i = 0, 1).
\]

In other words, the strict Picard stack of biextensions of \( (K_1, K_2) \) by \( K_3 \) is equivalent to the strict Picard stack associated (by the dictionary of [SGA4] Exposé XVIII Proposition 1.4.14) to the object \( \tau_{\leq 0} \text{RHom}(K_1 \otimes K_2, K_3[1]) \):

\[
\text{Biext}(K_1, K_2; K_3) \cong \text{ch} (\tau_{\leq 0} \text{RHom}(K_1 \otimes K_2, K_3[1]))
\]

If \( A_i = 0 \) (for \( i = 1, 2, 3 \)), this theorem coincides with the homological interpretation \((0.1)\) of Grothendieck.

The homological interpretation of extensions of complexes of the kind \( K_i \) is a special case of Theorem 0.1: in fact, it is furnished by the statement of this Theorem with \( K_2 = [0 \to \mathbb{Z}] \), since the category \( \text{Biext}(K_1, [0 \to \mathbb{Z}]; K_3) \) of biextensions of
(\(K_1, [0 \rightarrow \mathbb{Z}]\)) by \(K_3\) is equivalent to the category \(\text{Ext}(K_1, K_3)\) of extensions of \(K_1\) by \(K_3\):

\[
\text{Biext}(K_1, [0 \rightarrow \mathbb{Z}]; K_3) \cong \text{Ext}(K_1, K_3),
\]

and since in the derived category \(\mathcal{D}(\mathcal{C})\) we have that

\[
\text{Ext}^i(K_1 \bigotimes [0 \rightarrow \mathbb{Z}], K_3) \cong \text{Ext}^i(K_1, K_3) \quad (i = 0, 1).
\]

The idea of the proof of Theorem 0.1 works as follow: Let \(K = [A \xrightarrow{u} B]\) be a complex of commutative groups of \(\mathcal{T}\) concentrated in degrees 1 and 0 and let \(L_\ast\) be a bicomplex of commutative groups of \(\mathcal{T}\) which satisfies \(L_{ij} = 0\) for \((ij) \neq (00), (01), (02), (10), (11), (10), (20)\). To the complex \(K\) and to the bicomplex \(L_\ast\) we associate an additive cofibred category \(\Psi_{\text{Tot}(L_\ast)}(K)\) which has the following homological description:

\[
\Psi_{\text{Tot}(L_\ast)}^0(K) \cong \text{Ext}^i(\text{Tot}(L_\ast), K) \quad (i = 0, 1)
\]

where \(\Psi_{\text{Tot}(L_\ast)}^0(K)\) is the group of automorphisms of any object of \(\Psi_{\text{Tot}(L_\ast)}(K)\) and \(\Psi_{\text{Tot}(L_\ast)}^1(K)\) is the group of isomorphism classes of objects of \(\Psi_{\text{Tot}(L_\ast)}(K)\). Then, to any complex of the kind \(K = [A \xrightarrow{u} B]\) we associate a canonical flat partial resolution \(L_\ast(K)\) whose components are direct sums of objects of the kind \(Z[I]\) with \(I\) a commutative group of \(\mathcal{T}\). Consider now three complexes \(K_i = [A_i \xrightarrow{u_i} B_i]\) (for \(i = 1, 2, 3\)). The category \(\Psi_{\text{Tot}(L_\ast(K_1))}(K_3)\) and \(\Psi_{\text{Tot}(L_\ast(K_1) \otimes L_\ast(K_2))}(K_3)\) admit the following geometrical description:

\[
\Psi_{\text{Tot}(L_\ast(K_1))}(K_3) \cong \text{Ext}(K_1, K_3)
\]

\[
\Psi_{\text{Tot}(L_\ast(K_1) \otimes L_\ast(K_2))}(K_3) \cong \text{Biext}(K_1, K_2; K_3)
\]

Putting together this geometrical description (0.3) with the homological description (0.2), we get

- the proof of the Theorem 0.1;
- the proof that the group of automorphisms of any extension of \(K_1\) by \(K_3\) is the group \(\text{Ext}^0(K_1, K_3)\) and that the group of isomorphism classes of extensions of \(K_1\) by \(K_3\) is the group \(\text{Ext}^1(K_1, K_3)\).

**Notation**

In this paper, \(\mathcal{T}\) is a topos and \(\mathcal{C}\) is the category of commutative groups of \(\mathcal{T}\), i.e. the category of \(\mathbb{Z}\)-modules of \(\mathcal{T}\). Recall that we can identify commutative groups of \(\mathcal{T}\) with abelian sheaves over \(\mathcal{T}\). If \(I\) is an object of \(\mathcal{C}\), we denote by \(Z[I]\) the free \(\mathbb{Z}\)-module generated by \(I\) (see [SGA4] Exposé IV 11).

All complexes of objects of \(\mathcal{C}\) that we consider in this paper are chain complexes. The truncation \(\tau_{\geq n}L_\ast\) of a complex \(L_\ast\) is the following complex: \((\tau_{\geq n}L_\ast)_i = L_i\) for \(i \geq n\) and \((\tau_{\geq n}L_\ast)_i = 0\) for \(i < n\).

If \(L_\ast\) is a bicomplex of objects of \(\mathcal{C}\), we denote by \(\text{Tot}(L_\ast)\) the total complex of \(L_\ast\): it is the chain complex whose component of degree \(n\) is \(\text{Tot}(L_\ast)_n = \sum_{i+j=n} L_{ij}\).

Let \(\mathcal{D}(\mathcal{C})\) be the derived category of the abelian category \(\mathcal{C}\). Denote by \(\mathcal{D}^{1,0}(\mathcal{C})\) the subcategory of \(\mathcal{D}(\mathcal{C})\) of complexes \(K = [A \xrightarrow{u} B]\) with \(A\) concentrated in degree 1 and \(B\) concentrated in degree 0.
1. Extensions and biextensions of complexes

Let $G$ be an object of $\mathcal{C}$. A $G$-torsor is an object of $\mathbf{T}$ endowed with an action of $G$, which is locally isomorphic to $G$ acting on itself by translation.

Let $P, G$ be objects of $\mathcal{C}$. An extension of $P$ by $G$ is an object $E$ of $\mathbf{T}$ such that we have an exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow P \longrightarrow 0.$$ 

By definition the object $E$ is a group. Since in this paper we consider only commutative extensions, $E$ is in fact an object of $\mathcal{C}$. We denote by $\text{Ext}(P, G)$ the category of extensions of $P$ by $G$. It is a classical result that the category $\text{Ext}(P, *)$ of extensions of $P$ by variable objects of $\mathcal{C}$ is an additive cofibred category over $\mathcal{C}$

$$\text{Ext}(P, *) \rightarrow \mathcal{C}$$

$$\text{Ext}(P, G) \rightarrow G$$

Moreover, the Baer sum of extensions defines a group law for the objects of the category $\text{Ext}(P, G)$, which is therefore a Picard category.

Let $P, G$ be objects of $\mathcal{C}$. Denote by $m : P \times P \rightarrow P$ the group law of $P$ and by $\text{pr}_i : P \times P \rightarrow P$ with $i = 1, 2$ the two projections of $P \times P$ in $P$. According [SGA7] Exposé VII 1.1.6 and 1.2, the category of extensions of $P$ by $G$ is equivalent to the category of 4-tuple $(P, G, E, \varphi)$, where $E$ is a $G_P$-torsor over $P$, and $\varphi : \text{pr}_1^*E \text{pr}_2^*E \rightarrow m^*E$ is an isomorphism of torsors over $P \times P$ satisfying some associative and commutative conditions (see [SGA7] Exposé VII diagrams (1.1.4.1) and (1.2.1)):

$$\text{Ext}(P, G) \cong \left\{(P, G, E, \varphi) \mid E = G_P-\text{torsor over } P \text{ and } \varphi : \text{pr}_1^*E \text{pr}_2^*E \cong m^*E \text{ with ass. and comm. conditions}\right\}.$$ 

(1.1)

It will be useful in what follows to look at the isomorphism of torsors $\varphi$ as an associative and commutative group law on the fibres:

$$+ : E_p, E_{p'} \rightarrow E_{p+p'}$$

where $p, p'$ are points of $P(S)$ with $S$ any object of $\mathbf{T}$.

Let $I$ and $G$ be objects of $\mathcal{C}$. Concerning extensions of free commutative groups, in [SGA7] Exposé VII 1.4 Grothendieck proves that there is an equivalence of category between the category of extensions of $\mathbb{Z}[I]$ by $G$ and the category of $G_I$-torsors over $I$:

$$\text{Ext}(\mathbb{Z}[I], G) \cong \text{Tors}(I, G_I)$$

(1.2)

Let $P, Q$ and $G$ be objects of $\mathcal{C}$. A biextension of $(P, Q)$ by $G$ is a $G_{P\times Q}$-torsor $B$ over $P \times Q$, endowed with a structure of commutative extension of $Q_P$ by $G_P$ and a structure of commutative extension of $P_Q$ by $G_Q$, which are compatible one with another (for the definition of compatible extensions see [SGA7] Exposé VII Définition 2.1). If $m_P, p_1, p_2$ (resp. $m_Q, q_1, q_2$) denote the three morphisms $P \times P \rightarrow P \times Q$ (resp. $P \times Q \times Q \rightarrow P \times Q$) deduced from the three morphisms $P \times P \rightarrow P$ (resp. $Q \times Q \rightarrow Q$) group law, first and second projection, the equivalence of categories (1.1) furnishes the following equivalent definition: a biextension of $(P, Q)$ by $G$ is a $G_{P\times Q}$-torsor $B$ over $P \times Q$ endowed with two
isomorphisms of torsors

\[ \varphi : p_1^* E \to p_2^* E \to m_1^* E \quad \psi : q_1^* E \to q_2^* E \to m_2^* E \]

over \( P \times P \times Q \) and \( P \times Q \times Q \) respectively, satisfying some associative, commutative and compatible conditions (see [SGA7] Exposé VII diagrams (2.0.5),(2.0.6),(2.0.8), (2.0.9), (2.1.1)). As for extensions, we will look at the isomorphisms of torsors \( \varphi \) and \( \psi \) as two associative and commutative group laws on the fibres which are compatible with one another:

\[ +_{p,q} : E_{p,q} \to E_{p+q,q} \quad +_{q,p} : E_{p,q} \to E_{p,q+q} \]

where \( p, p' \) (resp. \( q, q' \)) are points of \( P(S) \) (resp. of \( Q(S) \)) with \( S \) any object of \( \mathbf{T} \).

Let \( K_i = [A_i \to B_i] \) (for \( i = 1, 2 \)) be a complex of objects of \( \mathcal{C} \) with \( A_i \) in degree 1 and \( B_i \) in degree 0.

**Definition 1.1.** An extension \((E, \beta)\) of \( K_1 \) by \( K_2 \) consists of

1. an extension \( E \) of \( B_1 \) by \( B_2 \);
2. a trivialization \( \beta \) of the extension \( u_1^* E \) of \( A_1 \) by \( B_2 \) obtained as pull-back of the extension \( E \) via \( u_1 : A_1 \to B_1 \), i.e. an homomorphism \( \beta : A_1 \to B_2 \).

Let \( K_i = [A_i \to B_i] \) and \( K'_i = [A'_i \to B'_i] \) (for \( i = 1, 2 \)) be complexes of objects of \( \mathcal{C} \) concentrated in degrees 1 and 0. Let \((E, \beta)\) be an extension of \( K_1 \) by \( K_2 \) and let \((E', \beta')\) be an extension of \( K'_1 \) by \( K'_2 \).

**Definition 1.2.** A morphism of extensions

\[(E, \Upsilon) : (E, \beta) \to (E', \beta')\]

consists of

1. a morphism \( E = (F, f_1, f_2) : E \to E' \) from the extension \( E \) to the extension \( E' \). In particular,
   \[f_1 : B_1 \to B'_1 \quad f_2 : B_2 \to B'_2\]
   are homomorphisms of commutative groups of \( \mathbf{T} \).
2. a morphism of extensions
   \[\Upsilon = (\Upsilon, g_1, f_2) : u_1^* E \to u'_1^* E'\]
   compatible with the morphism \( E = (F, f_1, f_2) \) and with the trivializations \( \beta \) and \( \beta' \). In particular,
   \[g_1 : A_1 \to A'_1\]
   is an homomorphism of commutative groups of \( \mathbf{T} \).

We denote by \( \mathbf{Ext}(K_1, K_2) \) the category of extensions of \( K_1 \) by \( K_2 \). If the complex \( K_1 \) is fixed, the category \( \mathbf{Ext}(K_1, \ast) \) of extensions of \( K_1 \) by variable complexes \( K_2 \) is an additive cofibred category over \( \mathcal{D}^{[1,0]}(\mathcal{C}) \)

\[
\mathbf{Ext}(K_1, \ast) \to \mathcal{D}^{[1,0]}(\mathcal{C})
\]

\[
\mathbf{Ext}(K_1, K_2) \to K_2
\]

This is an easy consequence of the analogous properties of the category of extensions of objects of \( \mathcal{C} \). Moreover the Baer sum of extensions defines a group law for the objects of the category \( \mathbf{Ext}(K_1, K_2) \), which is therefore a Picard category. The zero object \((E_0, \beta_0)\) of \( \mathbf{Ext}(K_1, K_2) \) with respect to this group law consists of
• the trivial extension \( E_0 = B_1 \times B_2 \) of \( B_1 \) by \( B_2 \), i.e. the zero object of \( \text{Ext}(B_1, B_2) \), and
• the trivialization \( \beta_0 = (id_{A_1}, 0) \) of the extension \( u_1^* E_0 = A_1 \times B_2 \) of \( A_1 \) by \( B_2 \). We can consider \( \beta_0 \) as a lifting \( (u_1, 0) : A_1 \to B_1 \times B_2 \) of \( u_1 : A_1 \to B_1 \).

The group of automorphisms of any object of \( \text{Ext}(K_1, K_2) \) is canonically isomorphic to the group of automorphisms \( \text{Aut}(E_0, \beta_0) \) of the zero object of \( \text{Ext}(K_1, K_2) \).

Explicitly, \( \text{Aut}(E_0, \beta_0) \) consists of the couple \( (f_0, f_1) \) where

• \( f_0 : B_1 \to B_2 \) is an automorphism of the trivial extension \( E_0 \), i.e. \( f_0 \in \text{Aut}(E_0) = \text{Ext}^0(B_1, B_2) \), and
• \( f_1 : A_1 \to A_2 \) is an homomorphism such that the composite \( u_2 \circ f_1 \) is compatible with the pull-back \( u_1^* f_0 \) of the automorphism \( f_0 \) of \( E_0 \), i.e. \( u_2 \circ f_1 = f_0 \circ u_1 \).

We have therefore the canonical isomorphisms

\[
\text{Aut}(E_0, \beta_0) \cong \text{Hom}_{D(C)}(K_1, K_2) = \text{Ext}^0(K_1, K_2).
\]

The group law of the category \( \text{Ext}(K_1, K_2) \) induces a group law on the set of isomorphism classes of objects of \( \text{Ext}(K_1, K_2) \), which is canonically isomorphic to the group \( \text{Ext}^1(K_1, K_2) \), as we will prove in Corollary 4.3.

Let \( K_i = [A_i \xrightarrow{u_i} B_i] \) (for \( i = 1, 2, 3 \)) be a complex of objects of \( C \) with \( A_i \) in degree 1 and \( B_i \) in degree 0.

**Definition 1.3.** A biextension \((B, \Psi_1, \Psi_2, \lambda)\) of \((K_1, K_2)\) by \(K_3\) consists of

1. a biextension \(B\) of \((B_1, B_2)\) by \(B_3\);
2. a trivialization \(\Psi_1\) (resp. \(\Psi_2\)) of the biextension \((u_1, id_{B_1})^*B\) of \((A_1, B_2)\) by \(B_3\) (resp. of the biextension \((id_{B_1}, u_2)^*B\) of \((B_1, A_2)\) by \(B_3\)) obtained as pull-back of \(B\) via \((u_1, id_{B_1}) : A_1 \times B_2 \to B_1 \times B_2\) (resp. via \((id_{B_1}, u_2) : B_1 \times A_2 \to B_1 \times B_2\)).
3. an homomorphism \(\lambda : A_1 \otimes A_2 \to A_3\) such that the composite \(A_1 \otimes A_2 \xrightarrow{\lambda} A_3 \xrightarrow{u_3} B_3\) is compatible with the restriction over \((A_1, A_2)\) of the trivializations \(\Psi_1\) and \(\Psi_2\).

Let \(K_i = [A_i \xrightarrow{u_i} B_i]\) and \(K_i' = [A_i' \xrightarrow{u_i'} B_i']\) (for \(i = 1, 2, 3\)) be complexes of objects of \(C\) concentrated in degrees 1 and 0. Let \((B, \Psi_1, \Psi_2, \lambda)\) be a biextension of \((K_1, K_2)\) by \(K_3\) and let \((B', \Psi_1', \Psi_2', \lambda')\) be a biextension of \((K_1', K_2')\) by \(K_3'\).

**Definition 1.4.** A morphism of biextensions

\[(F, \Psi_1, \Psi_2, g_3) : (B, \Psi_1, \Psi_2, \lambda) \longrightarrow (B', \Psi_1', \Psi_2', \lambda')\]

consists of

1. a morphism \(F = (f_1, f_2, f_3) : B \to B'\) from the biextension \(B\) to the biextension \(B'\). In particular,
\[
f_1 : B_1 \longrightarrow B'_1 \quad f_2 : B_2 \longrightarrow B'_2 \quad f_3 : B_3 \longrightarrow B'_3
\]

are homomorphisms of commutative groups of \(T\).
2. a morphism of biextensions
\[
\Psi_1 = (\Psi_1, g_1, f_2, f_3) : (u_1, id_{B_2})^*B \longrightarrow (u_1', id_{B_2'})^*B'
\]
compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations $\Psi_1$ and $\Psi'_1$, and a morphism of biextensions $\Upsilon_2 = (\Upsilon_2, g_1, g_2, f_3) : (id_{B_1}, u_2)^*B \to (id_{B'_1}, u'_2)^*B'$ compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations $\Psi_2$ and $\Psi'_2$. In particular,

$$g_1 : A_1 \to A'_1 \quad g_2 : A_2 \to A'_2$$

are homomorphisms of commutative groups of $T$. By pull-back, the two morphisms $\Upsilon_1 = (\Upsilon_1, g_1, f_2, f_3)$ and $\Upsilon_2 = (\Upsilon_2, f_1, g_2, f_3)$ define a morphism of biextensions $\Upsilon = (\Upsilon_1, g_1, g_2, f_3) : (u_1, u_2)^*B \to (u'_1, u'_2)^*B'$ compatible with the morphism $F = (F, f_1, f_2, f_3)$ and with the trivializations $\Psi$ and $\Psi'$.

(3) an homomorphism $g_3 : A_3 \to A'_3$ of commutative groups of $T$ compatible with $u_3$ and $u'_3$ (i.e. $u'_3 \circ g_3 = f_3 \circ u_3$) and such that

$$\lambda \circ (g_1 \times g_2) = g_3 \circ \lambda$$

We denote by $\text{Biext}(K_1, K_2; K_3)$ the category of biextensions of $(K_1, K_2)$ by $K_3$. If the complexes $K_1$ and $K_2$ are fixed, the category $\text{Biext}(K_1, K_2; *)$ of biextensions of $(K_1, K_2)$ by variable complexes $K_3$ is an additive cofibred category over $D^{[1,0]}(C)$

$$\text{Biext}(K_1, K_2; *) \to D^{[1,0]}(C)$$

$$\text{Biext}(K_1, K_2; K_3) \to K_3$$

This is an easy consequence of the fact that the category of biextensions of objects of $C$ is an additive cofibred category over $C$ (see [SGA7] Exposé VII 2.4). The Baer sum of extensions defines a group law for the objects of the category $\text{Biext}(K_1, K_2; K_3)$ which is therefore a Picard category (see [SGA7] Exposé VII 2.5). The zero object $(B_0, \Psi_{01}, \Psi_{02}, \lambda_0)$ of $\text{Biext}(K_1, K_2; K_3)$ with respect to this group law consists of

- the trivial biextension $B_0 = B_1 \times B_2 \times B_3$ of $(B_1, B_2)$ by $B_3$, i.e. the zero object of $\text{Biext}(B_1, B_2; B_3)$, and
- the trivialization $\Psi_0 = (id_{B_1}, id_{B_2}, 0)$ (resp. $\Psi_{02} = (id_{B_1}, id_{B_2}, 0)$) of the biextension $(u_1, id_{B_2})^*B_0 = A_1 \times B_2 \times B_3$ of $(A_1, B_2)$ by $B_3$ (resp. of the biextension $(id_{B_1}, u_2)^*B_0 = B_1 \times A_2 \times B_3$ of $(B_1 \times A_2)$ by $B_3$). These two trivialization have to coincide over $A_1 \times A_2$.
- the zero homomorphism $\lambda_0 = 0 : A_1 \otimes A_2 \to A_3$.

The group of automorphisms of any object of $\text{Biext}(K_1, K_2; K_3)$ is canonically isomorphic to the group of automorphisms of the zero object $(B_0, \Psi_{01}, \Psi_{02}, \lambda_0)$, that we denote $\text{Biext}^0(K_1, K_2; K_3)$. Explicitly, $\text{Biext}^0(K_1, K_2; K_3)$ consists of the couple $(f_0, f_{11} + f_{12})$ where

- $f_0 : B_1 \otimes B_2 \to B_3$ is an automorphism of the trivial biextension $B_0$, i.e. $f_0 \in \text{Biext}^0(B_1, B_2; B_3) = \text{Hom}(B_1 \otimes B_2, B_3)$, and
- $f_{11} : A_1 \otimes B_2 \to A_3$ (resp. $f_{12} : B_1 \otimes A_2 \to A_3$) is an homomorphism such that the composite $u_3 \circ f_{11}$ (resp. $u_3 \circ f_{12}$) is compatible with the pull-back $(u_1, id_{B_2})^*f_0$ (resp. $(id_{B_1}, u_2)^*f_0$) of the automorphism $f_0$ of $B_0$, i.e. $u_3 \circ (f_{11} + f_{12}) = f_0 \circ (u_1 \otimes id_{B_2} + id_{A_1} \otimes u_2)$.

We have therefore the canonical isomorphisms

$$\text{Biext}^0(K_1, K_2; K_3) \cong \text{Hom}_{D(C)}(K_1 \otimes K_2, K_3) = \text{Ext}^0(K_1 \otimes K_2, K_3).$$
The group law of the category $\text{Biext}(K_1, K_2; K_3)$ induces a group law on the set of isomorphism classes of objects of $\text{Biext}(K_1, K_2; K_3)$, that we denote by $\text{Biext}^1(K_1, K_2; K_3)$.

**Remark 1.5.** According to the above geometrical definitions of extensions and biextensions of complexes, we have the following equivalence of categories

$$\text{Biext}(K_1, [0 \to \mathbb{Z}]; K_3) \cong \text{Ext}(K_1, K_3).$$

Moreover we have also the following isomorphisms

$$\text{Biext}^i(K_1, [\mathbb{Z} \to 0]; K_3) = \begin{cases} \text{Hom}(B_1, A_3), & i = 0; \\ \text{Hom}(K_1, K_3), & i = 1. \end{cases}$$

Remark that we get the same results applying the homological interpretation of biextensions furnished by our main Theorem 0.1.

2. The additive cofibred category $\Psi_{\text{Tot}(L\ldots)}$

Consider the following bicomplex $L\ldots$ of objects of $\mathcal{C}$:

$$\begin{array}{cccc}
L_{-3} & L_{-2} & L_{-1} & L_0 \\
\{ & \{ & \{ & \} \\
0 & \to & L_{02} & \to 0 \\
\downarrow & & \downarrow d_{01} & \\
0 & \to & L_{11} & \to L_{01} & \to 0 \\
\downarrow d_{10} & & \downarrow d_{00} & \\
0 & \to & L_{20} & \to L_{10} & \to L_{00} & \to 0 \\
\downarrow d_{10} & & \downarrow d_{00} & \\
0 & \to & 0 & \to 0 & \\
\end{array}$$

(2.1)

The total complex $\text{Tot}(L\ldots)$ is the complex

$$L_{02} + L_{11} + L_{20} \xrightarrow{D_1} L_{01} + L_{10} \xrightarrow{D_0} L_{00} \to 0$$

where the differential operators $D_1$ and $D_0$ can be computed from the diagram (2.1).

In this section we define an additive cofibred category $\Psi_{\text{Tot}(L\ldots)}$ over $\mathcal{D}^{[1,0]}(\mathcal{C})$.

Let $K = [A \xrightarrow{\alpha} B]$ be an object of $\mathcal{D}^{[1,0]}(\mathcal{C})$.

**Definition 2.1.** Denote by $\Psi_{\text{Tot}(L\ldots)}(K)$ the category whose objects consist of 4-tuple $(E, \alpha, \beta, \gamma)$ where

1. $E$ is an extension of $L_{00}$ by $B$;
2. $(\alpha, \beta)$ is a trivialization of the extension $D_0^*E$ of $L_{01} + L_{10}$ by $B$ obtained as pull-back of $E$ via $D_0$. Moreover we require that the corresponding trivialization $D_1^*(\alpha, \beta)$ of $D_1^*D_0^*(E)$ is the trivialization arising from the isomorphism of transitivity $D_1^*D_0^*(E) \cong (D_0 \circ D_1)^*(E)$ and the relation $D_0 \circ D_1 = 0$;
   In other words, $(\alpha, \beta)$ is a lifting $L_{01} + L_{10} \to E$ of $D_0 : L_{01} + L_{10} \to L_{00}$ such that $(\alpha, \beta) \circ D_1 = 0$;
3. $\gamma : L_{20} \to A$ is an homomorphism such that the composite $L_{20} \xrightarrow{\gamma} A \xrightarrow{\alpha} B$ is compatible with the restriction $D_1^*(\beta)$ of the trivialization $\beta$ over $L_{20}$. 


A morphism \((F, id, f_B, f_A) : (E, \alpha, \beta, \gamma) \to (E', \alpha', \beta', \gamma')\) between two objects of \(\Psi_{\text{Tot}(L_\cdot)}(K)\) consists of

1. a morphism \((F, id, f_B) : E \to E'\) of extensions inducing the identity on \(L_{00}\) and such that \(F \circ \alpha = \alpha'\) and \(F \circ \beta = \beta'\). In particular,

\[
F : E \longrightarrow E' \quad \text{id} : L_{00} \longrightarrow L_{00} \quad f_B : B \longrightarrow B;
\]

2. an homomorphism \(f_A : A \to A\) such that \(f_A \circ \gamma = \gamma'\).

Remark that the conditions \(u \circ \gamma = D_{10}^\gamma(\beta)\) and \(u \circ \gamma' = D_{10}^\gamma(\beta')\) imply that \(f_B \circ u = u \circ f_A\), i.e. the couple \((f_A, f_B)\) defines a morphism of complexes \(K \to K'\).

The composition of morphisms of \(\Psi_{\text{Tot}(L_\cdot)}(K)\) is defined using the composition of morphisms of extensions and the composition of morphisms of complexes \((f_A, f_B) : [A \xrightarrow{u} B] \to [A \xrightarrow{u} B]\).

We can summarize the data \((E, \alpha, \beta, \gamma)\) in the following diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
B & = & B \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{u} & D_1^\alpha D_0^\gamma E \\
\downarrow & \downarrow & \downarrow \\
L_{02} + L_{11} + L_{20} & \xrightarrow{\alpha, \beta, \gamma} & D_0^\alpha E \longrightarrow E \\
\end{array}
\]

We have a functor

\[
\Pi : \Psi_{\text{Tot}(L_\cdot)} \longrightarrow \mathcal{D}^{[1, 0]}(\mathcal{C})
\]

\[
\Psi_{\text{Tot}(L_\cdot)}(K) \longrightarrow K
\]

which is cofibring. In fact, let \((f_1, f_0) : K = [A \xrightarrow{u} B] \to K' = [A' \xrightarrow{u'} B']\) be a morphism of \(\mathcal{D}^{[1, 0]}(\mathcal{C})\), and let \((E, \alpha, \beta, \gamma)\) be an object of the fibre \(\Psi_{\text{Tot}(L_\cdot)}(K)\) over \(K\). Denote by \((f_0)_* E\) the push-down of \(E\) via the homomorphism \(f_0 : B \to B'\) and by \((F, id, f_0) : E \to E'\) the corresponding morphism of extensions inducing the identity on \(L_{00}\). The object \(((f_0)_* E, F \circ \alpha, F \circ \beta, f_1 \circ \gamma)\) is clearly an object of the fibre \(\Psi_{\text{Tot}(L_\cdot)}(K')\) over \(K'\) and the morphism

\[
(F, id, f_0, f_1) : (E, \alpha, \beta, \gamma) \longrightarrow ((f_0)_* E, F \circ \alpha, F \circ \beta, f_1 \circ \gamma)
\]

is a cocartesian morphism for the functor \(\Pi : \Psi_{\text{Tot}(L_\cdot)} \longrightarrow \mathcal{D}^{[1, 0]}(\mathcal{C})\): it is enough to use the analogue property of the morphism of extensions \((F, id, f_0) : E \to (f_0)_* E\) which is a classical result. Therefore the category \(\Psi_{\text{Tot}(L_\cdot)}\) is a cofibring category over \(\mathcal{D}^{[1, 0]}(\mathcal{C})\).

Finally, the cofibring category \(\Psi_{\text{Tot}(L_\cdot)}\) is additive, i.e. it satisfies the two following conditions:

1. \(\Psi_{\text{Tot}(L_\cdot)}(0)\) is equivalent to the trivial category;
2. \(\Psi_{\text{Tot}(L_\cdot)}(K \times K') \longrightarrow \Psi_{\text{Tot}(L_\cdot)}(K) \times \Psi_{\text{Tot}(L_\cdot)}(K')\) is an equivalence of categories for any object \(K, K'\) of \(\mathcal{D}^{[1, 0]}(\mathcal{C})\).

This is an easy consequence of the fact that the cofibring category \(\text{Ext}(L_{00}, *)\) of extensions of \(L_{00}\) by objects of \(\mathcal{C}\) is additive.
For any object $K = [A \xrightarrow{u} B]$ of $\mathcal{D}^{[1,0]}(\mathcal{C})$, the Baer sum of extensions defines a group law for the objects of the category $\Psi_{\text{Tot}(L_{i})}(K)$. The zero object of $\Psi_{\text{Tot}(L_{i})}(K)$ with respect to this law group is the 4-tuple $(E_0, \alpha_0, \beta_0, \gamma_0)$ where

- $E_0 = L_{00} \times B$ is the trivial extension of $L_{00}$ by $B$, i.e. the zero object of $\text{Ext}(L_{00}, B)$, and
- $\alpha_0$ is the trivialization $(id_{L_{01}}, 0)$ of the extension $d_{00}^0E_0 = L_{01} \times B$ of $L_{01}$ by $B$; $\beta_0$ is the trivialization $(id_{L_{10}}, 0)$ of the extension $D_{00}L_{00}E_0 = L_{10} \times B$ of $L_{10}$ by $B$. We can consider $\alpha_0$ (resp. $\beta_0$) as a lifting $(d_{00}, 0)$ (resp. $(D_{00}, 0)$) of $d_{00} : L_{01} \to L_{00}$ (resp. of $D_{00} : L_{10} \to L_{00}$),
- $\gamma_0 = 0 : L_{20} \to A$ is the zero homomorphism.

The group of automorphisms of any object of $\Psi_{\text{Tot}(L_{i})}(K)$ is canonically isomorphic to the group of automorphisms of the zero object of $\Psi_{\text{Tot}(L_{i})}(K)$, that we denote by $\Psi_{\text{Tot}(L_{i})}(K)$. Explicitly, $\Psi_{\text{Tot}(L_{i})}(K)$ consists of the couple $(f_0, (f_{01}, f_{10}))$ where

- $f_0 : L_{00} \to B$ is an automorphism of the trivial extension $E_0$, i.e. $f_0 \in \text{Aut}(E_0) = \text{Ext}^0(L_{00}, B)$, and
- $f_{01} : L_{01} \to A$ (resp. $f_{10} : L_{10} \to A$) is an isomorphism such that the composite $u \circ f_{01}$ (resp. $u \circ f_{10}$) is compatible with the pull-back $d_{00}^0(f_0)$ (resp. $D_{00}^0(f_0)$) of the automorphism $f_0$ of $E_0$, i.e.

\begin{equation}
(2.2) \quad u \circ f_{01} = f_0 \circ d_{00} \quad \text{(resp.} \quad u \circ f_{10} = f_0 \circ D_{00})
\end{equation}

The group law of the category $\Psi_{\text{Tot}(L_{i})}(K)$ induces a group law on the set of isomorphism classes of objects of $\Psi_{\text{Tot}(L_{i})}(K)$ that we denote by $\Psi_{\text{Tot}(L_{i})}(K)$.

3. Homological description of $\Psi_{\text{Tot}(L_{i})}$

**Theorem 3.1.** Let $K = [A \xrightarrow{u} B]$ be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and 0. We have the following canonical isomorphisms

\[ \Psi_{\text{Tot}(L_{i})}(K) \cong \text{Ext}^0(\text{Tot}(L_{i}), K) = \text{Hom}_{\mathcal{D}(\mathcal{C})}(\text{Tot}(L_{i}), K) \]

\[ \Psi_{\text{Tot}(L_{i})}(K) \cong \text{Ext}^1(\text{Tot}(L_{i}), K) = \text{Hom}_{\mathcal{D}(\mathcal{C})}(\text{Tot}(L_{i}), K[1]) \]

**Proof.** Let $(f_0, (f_{01}, f_{10}))$ be an element of $\Psi_{\text{Tot}(L_{i})}(K)$, i.e. an automorphism of the zero object $(E_0, \alpha_0, \beta_0, \gamma_0)$ of $\Psi_{\text{Tot}(L_{i})}(K)$. We will show that the morphisms $f_0 : L_{00} \to B$ and $(f_{01}, f_{10}) : L_{01} + L_{10} \to A$ define a morphism $\text{Tot}(L_{i}) \to K$ in the derived category $\mathcal{D}(\mathcal{C})$. Consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & L_{02} + L_{11} + L_{20} & \overset{D_{01}}{\to} & L_{01} + L_{10} & \overset{D_{00}}{\to} & L_{00} & \to & 0 \\
0 & \to & A & \xrightarrow{u} & B & & & &
\end{array}
\]

(3.1)

By definition, the morphisms $f_0 : L_{01} \to A$ and $f_{10} : L_{10} \to A$ satisfies the equalities (2.2). Since $D_{00} = (d_{00}, D_{00})$, we get that $f_0 \circ D_{00} = u \circ (f_{01}, f_{10})$, i.e. the second diagram of (3.1) is commutative. Concerning the first diagram of (3.1), we have to prove that $(f_{01}, f_{10}) \circ D_{1} = 0$ with $D_{1} = (d_{01}, 0) + (D_{01}, d_{10}) + (0, D_{10})$. The first equality (2.2) and the condition $d_{00} \circ d_{10} = 0$ imply

\[ u \circ f_{01} \circ d_{10} = f_0 \circ d_{00} \circ d_{01} = 0, \]

which furnishes (since $u$ is arbitrary)

\[ (f_{01}, f_{10}) \circ (d_{01}, 0) = 0. \]
Both equalities (2.2) and the condition $d_{00} \circ D_{01} + D_{00} \circ d_{10} = 0$ give that
\[ u \circ (f_{01}, f_{10}) \circ (D_{01}, d_{10}) = f_0 \circ (d_{00}, D_{00}) \circ (D_{01}, d_{10}) = 0, \]
which implies that
\[ (f_{01}, f_{10}) \circ (D_{01}, d_{10}) = 0 \]
since $u$ is arbitrary. Because of the second equality (2.2) and the condition $D_{00} \circ D_{10} = 0$, we have that
\[ u \circ f_{10} \circ D_{10} = f_0 \circ D_{00} \circ D_{10} = 0, \]
which furnishes (again because $u$ is arbitrary)
\[ (f_{01}, f_{10}) \circ (0, D_{10}) = 0. \]

Therefore also the first diagram of (3.1) is commutative. Hence we have construct a morphism
\[
\Psi_{\text{Tot}(L_{-})}^d(K) \rightarrow \text{Hom}_{\mathcal{D}(C)}(\text{Tot}(L_{-}), K) \\
(f_0, (f_{01}, f_{10})) \rightarrow (f_0, (f_{01}, f_{10}))
\]
which is clearly a canonical isomorphism.

Let $(E, \alpha, \beta, \gamma)$ be an object of $\Psi_{\text{Tot}(L_{-})}^d(K)$. We will show that $(E, \alpha, \beta, \gamma)$ defines a morphism $\text{Tot}(L_{-}) \rightarrow K[1]$ in the derived category $\mathcal{D}(C)$. Recall that $E$ is an extension of $L_{00}$ by $B$. Denote $j : E \rightarrow L_{00}$ the corresponding surjective morphism.

Consider the complex $[E \xrightarrow{j} L_{00}]$, with $E$ in degree 0 and $L_{00}$ in degree -1. It is a resolution of $B$, and so in the derived category $\mathcal{D}(C)$ we have that
\[ B = [E \xrightarrow{j} L_{00}]. \]

Since by definition, the data $(\alpha, \beta)$ can be considered as a lifting $L_{01} + L_{10} \rightarrow E$ of $\mathbb{D}_0 : L_{01} + L_{10} \rightarrow L_{00}$ such that $(\alpha, \beta) \circ \mathbb{D}_1 = 0$, we can construct in $\mathcal{D}(C)$ the following morphism
\[
0 \rightarrow L_{02} + L_{11} + L_{20} \xrightarrow{D_1} L_{01} + L_{10} \xrightarrow{D_2} L_{00} \rightarrow 0 \\
0 \rightarrow E \xrightarrow{j} L_{00} \rightarrow 0
\]
that we denote by
\[ c(E, \alpha, \beta) : \text{Tot}(L_{-}) \rightarrow [E \xrightarrow{j} L_{00}][1] = B[1]. \]

Now we use the homomorphism $\gamma : L_{20} \rightarrow A$ and the above morphism $c(E, \alpha, \beta)$ in order to construct in the derived category $\mathcal{D}(C)$ a morphism $c(E, \alpha, \beta, \gamma) : \text{Tot}(L_{-}) \rightarrow [A \xrightarrow{u} B][1]$. Consider the diagram
\[
0 \rightarrow L_{02} + L_{11} + L_{20} \xrightarrow{(d_{01}, 0)} L_{01} + L_{10} \xrightarrow{D_3} L_{00} \rightarrow 0 \\
0 \rightarrow A \xrightarrow{u} B \rightarrow 0 \rightarrow 0.
\]
Recall that $\mathbb{D}_1 = (d_{01}, 0) + (D_{01}, d_{10}) + (0, D_{10})$. By construction $c(E, \alpha, \beta) \circ (d_{01}, 0) + (D_{01}, d_{10}) = 0$. By definition the homomorphism $\gamma$ satisfies the equality $u \circ \gamma = \beta \circ D_{10}$, and so we have that $c(E, \alpha, \beta)|_{L_{10}} \circ (0, D_{10}) = u \circ \gamma$. Therefore the diagram (3.3) is commutative. Since the morphism $c(E, \alpha, \beta, \gamma)$ depends only
on the isomorphism class of the object \((E, \alpha, \beta, \gamma)\), we have construct a canonical morphism

\[
\begin{align*}
c : \Psi^1_{\text{Tot}(\mathbb{L})}(K) & \longrightarrow \text{Hom}_{\mathcal{D}(\mathbb{C})}(\text{Tot}(\mathbb{L}), K[1]) \\
(E, \alpha, \beta, \gamma) & \mapsto c(E, \alpha, \beta, \gamma).
\end{align*}
\]

Now we have to show that this morphism is an isomorphism.

Injectivity: Let \((E, \alpha, \beta, \gamma)\) be an object of \(\Psi^1_{\text{Tot}(\mathbb{L})}(K)\) such that the morphism \(c(E, \alpha, \beta, \gamma)\) that it defines in \(\mathcal{D}(\mathbb{C})\), is the zero morphism. The corresponding morphism \(c(E, \alpha, \beta) : \text{Tot}(\mathbb{L}) \longrightarrow [E \xrightarrow{j} L_{00}] [1]\) must also be zero in \(\mathcal{D}(\mathbb{C})\).

Now we will show that \(c(E, \alpha, \beta)\) is already zero in the category \(\mathcal{K}(\mathbb{C})\) of complexes modulo homotopy. Recall that the complex \([E \xrightarrow{j} L_{00}]\) is a resolution of \(B\) in \(\mathcal{D}(\mathbb{C})\).

The hypothesis that \(c(E, \alpha, \beta)\) is zero in \(\mathcal{D}(\mathbb{C})\) implies that there is a resolution of \(B\) in \(\mathcal{D}(\mathbb{C})\) of the kind \([C_0 \xrightarrow{i} C_{-1}]\) with \(C_0\) in degree 0 and \(C_{-1}\) in degree -1, and a quasi-isomorphism \((v_0, v_{-1}) : [E \xrightarrow{j} L_{00}] \longrightarrow [C_0 \xrightarrow{i} C_{-1}]\), explicitly

\[
\begin{align*}
0 & \rightarrow E \overset{j}{\rightarrow} L_{00} \overset{v_{-1}}{\rightarrow} C_0 \overset{0}{\rightarrow} 0,
\end{align*}
\]

such that the composite \((v_0, v_{-1}) \circ c(E, \alpha, \beta)\) is homotopic to zero. Since the morphism \((v_0, v_{-1})\) induces the identity on \(B\), it identifies \(E\) with the fibred product \(L_{00} \times_{C_{-1}} C_0\) of \(L_{00}\) and \(C_0\) over \(C_{-1}\). Therefore, the homomorphism \(s : L_{00} \rightarrow C_0\) inducing the homotopy \((v_0, v_{-1}) \circ c(E, \alpha, \beta) \sim 0\), i.e. satisfying \(i \circ s = v_{-1} \circ id_{L_{00}}\), factorizes through an homomorphism

\[
S : L_{00} \longrightarrow E = L_{00} \times_{C_{-1}} C_0
\]

which satisfies \(j \circ S = id_{L_{00}}\). This last equality means that the homomorphism \(S\) splits the extension \(E\) of \(L_{00}\) by \(B\) and so the complex \([E\xrightarrow{j} L_{00}]\) is isomorphic in \(\mathcal{K}(\mathbb{C})\) to \(B\), i.e. to the complex \([B \rightarrow 0]\) with \(B\) in degree 0. But then it is clear that the morphism

\[
\text{Hom}_{\mathcal{K}(\mathbb{C})}(\text{Tot}(\mathbb{L}), B) \longrightarrow \text{Hom}_{\mathcal{D}(\mathbb{C})}(\text{Tot}(\mathbb{L}), B)
\]

is an isomorphism and that \(c(E, \alpha, \beta)\) is already zero in the category \(\mathcal{K}(\mathbb{C})\). There exists therefore an homomorphism \(h : L_{00} \rightarrow E\) such that

\[
j \circ h = id_{L_{00}} \quad h \circ D_0 = (\alpha, \beta),
\]

i.e. \(h\) splits the extension \(E\), which is therefore the trivial extension \(E_0\) of \(L_{00}\) by \(B\), and \(h\) is compatible with the trivializations \((\alpha, \beta)\). Moreover, concerning the data \(\gamma : L_{20} \rightarrow A\) we get that

\[
u \circ \gamma = \beta \circ D_{10} = h \circ D_{00} \circ D_{10} = 0,
\]

and so (since \(u\) is arbitrary) the homomorphism \(\gamma\) is zero. Hence we can conclude that the object \((E, \alpha, \beta, \gamma)\) lies in the isomorphism class of the zero object of \(\Psi^1_{\text{Tot}(\mathbb{L})}(K)\).

Surjectivity: Consider a morphism \(((f_{01}, f_{10}), (f_{02}, f_{11}, f_{20})) : \text{Tot}(\mathbb{L}) \rightarrow [A\xrightarrow{u}]

commutativity of the above diagram implies that
(i.e. from an homomorphism $L$
this is enough for our goal since only the groups $\text{Ext}$
groups of the partial resolution and of
"partial resolution" means that we have an isomorphism between the homology
$D$
category
(3.5)
where
$f$
which satisfies
$j$
1
obtained as pull-back of
$C$
Since
$0$
is an extension of
$1$
Moreover the condition
$E$
α, β
extension
(3.6)
Consider now the morphism
$E = F_0C_0$
that we prefer: here we assume $f_{02} = f_{11} = 0$ and therefore we get the equality
$u \circ f_{20} = f_{10} \circ D_{10}$.

Consider now a resolution of $B$ in $D(\mathcal{C})$ of the kind $[C_0 \rightarrow C_{-1}]$ with $C_0$ in degree
0 and $C_{-1}$ in degree -1. We can then assume that the morphism $(f_{01}, f_{10})$ comes from the following morphism Tot$(L_{-1}) \rightarrow [C_0 \rightarrow C_{-1}][1]$ of $D(\mathcal{C})$
$0 \rightarrow L_{02} + L_{11} + L_{20} \xrightarrow{\underline{D}_1} L_{01} + L_{10} \xrightarrow{\underline{D}_0} L_{00} \rightarrow 0$
(3.6)
$0 \rightarrow C_0 \xrightarrow{i} C_{-1} \rightarrow 0$.

Since $C_0$ is an extension of $C_{-1}$ by $B$, we can consider the extension
$E = F_0C_0$
obtained as pull-back of $C_0$ via $F_{00} : L_{00} \rightarrow C_{-1}$. The condition $F_{00} \circ D_0 = i \circ (F_{01}, F_{10})$ implies that $(F_{01}, F_{10}) : L_{01} + L_{10} \rightarrow C_0$ factories through an homomorphism
$(\alpha, \beta) : L_{01} + L_{10} \rightarrow E$
which satisfies $j \circ (\alpha, \beta) = D_0$, with $j : E \rightarrow L_{00}$ the canonical surjection of the extension $E$. Moreover the condition $(F_{01}, F_{10}) \circ D_1 = 0$ furnishes the equality
$(\alpha, \beta) \circ D_1 = 0$. Therefore the data $(E, \alpha, \beta, f_{20})$ is an object of the category
$\Psi_{\text{Tot}(L_{-1})}(K)$. Consider now the morphism $c(E, \alpha, \beta, f_{20}) : \text{Tot}(L_{-1}) \rightarrow K[1]$ associated to $(E, \alpha, \beta, f_{20})$. By construction, the morphism (3.6) is the composite of the morphism (3.2) deduced from $c(E, \alpha, \beta, f_{20})$ with the morphism
$(F, F_{00}) : [E \xleftarrow{j} L_{00}] \rightarrow [C_0 \xrightarrow{i} C_{-1}]$
where $F$ is the canonical morphism $E = F_{00}C_0 \rightarrow C_0$. Since this last morphism $(F, F_{00})$ is a morphism of resolutions of $B$, we can conclude that in the derived category $D(\mathcal{C})$ the morphism $((f_{01}, f_{10}), (f_{02}, f_{11}, f_{20})) : \text{Tot}(L_{-1}) \rightarrow [A \xrightarrow{u} B][1]$ is the morphism $c(E, \alpha, \beta, f_{20})$.

4. A CANONICAL FLAT PARTIAL RESOLUTION FOR A COMPLEX CONCENTRATED IN TWO CONSECUTIVE DEGREES

Let $K = [A \xrightarrow{u} B]$ be a complex of objects of $\mathcal{C}$ concentrated in degrees 1 and
0. First we construct a canonical flat partial resolution of the complex $K$. Here “partial resolution” means that we have an isomorphism between the homology groups of the partial resolution and of $K$ only in degree 1 and 0. Remark that this is enough for our goal since only the groups $\text{Ext}^1$ and $\text{Ext}^0$ are involved in the statement of the main Theorem 0.1.
Consider the following bicomplex $L.(K)$ which satisfies $L_{ij}(K) = 0$ for $(ij) \neq (00), (01), (02), (10)$, which is endowed with an augmentation map $\epsilon_0 : L_{00}(K) \to B$, $\epsilon_1 : L_{10}(K) \to A$, and which depends functorially on $K$:

\[
\begin{array}{ccc}
L_{22}(K) & L_{11}(K) & L_{00}(K) \\
\downarrow & \downarrow & \downarrow \\
L_{21}(K) & L_{10}(K) & L_{00}(K) \\
\end{array}
\]

The non trivial components of $L.(K)$ are explained in the above diagram. In order to define the differential operators $D_.,$ and $d.,$ and the augmentation map $\epsilon,$ we introduce the following notation: If $P$ is an object of $C$, we denote by $[p]$ the point of $Z[P](S)$ defined by the point $p$ of $P(S)$ with $S$ an object of $T$. In an analogous way, if $p,q$ and $r$ are points of $P(S)$ we denote by $[p,q], [p,q,r]$ the elements of $Z[P \times P](S)$ and $Z[P \times P \times P](S)$ respectively. For any object $S$ of $T$ and for any $a \in A(S), b_1, b_2, b_3 \in B(S)$, we set

\[
\begin{align*}
\epsilon_0[b] &= b \\
\epsilon_1[a] &= a \\
d_{00}[b_1,b_2] &= [b_1 + b_2] - [b_1] - [b_2] \\
d_{01}[b_1,b_2] &= [b_1] - [b_2, b_1] \\
d_{00}[b_1,b_2,b_3] &= [b_1, b_2, b_3] - [b_1, b_2 + b_3] + [b_1, b_2] - [b_2, b_3] \\
D_{00}[a] &= [a(a)]
\end{align*}
\]

These morphisms of commutative groups define a bicomplex $L.(K)$ endowed with an augmentation map $\epsilon : L_{00}(K) \to K$. Remark that the relation $\epsilon_0 \circ d_{00} = 0$ is just the group law on $B$, and the relation $d_{00} \circ d_{01} = 0$ decomposes in two relations which express the commutativity and the associativity of the group law on $B$. This augmented bicomplex $L.(K)$ depends functorially on $K$: in fact, any morphism $f : K \to K'$ of complexes of objects of $C$ concentrated in degrees 1 and 0, furnishes a commutative diagram

\[
\begin{array}{ccc}
L.(K) & \xrightarrow{L.(f)} & L.(K') \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
K & \xrightarrow{f} & K'
\end{array}
\]

Moreover the components of the bicomplex $L.(K)$ are flat since they are free $\mathbb{Z}$-modules. In order to conclude that $L.(K)$ is a canonical flat partial resolution of the complex $K$ we need the following Lemma:

**Lemma 4.1.** The additive cofibred category $\text{Ext}(K, *)$ of extensions of $K$ by a variable object of $D^{[1, \infty]}(C)$ is equivalent to the additive cofibred category $\Psi_{\text{Tot}(L.(K))}$:

\[
\text{Ext}(K; *) \cong \Psi_{\text{Tot}(L.(K))}
\]
Proof. The total complex $\text{Tot}(L\ldots(K))$ is

$$Z[B \times B] + Z[B] \xrightarrow{D_0} Z[B \times B] + Z[A] \xrightarrow{D_1} Z[B] \rightarrow 0$$

where $D_0 = D_{00} + d_{00}$ and $D_1 = d_{01}$. If $K' = [A' \xrightarrow{\nu'} B']$ is an object of $D^{[1,0]}(\mathcal{C})$, in order to describe explicitly the objects of the category $\Psi_{\text{Tot}(L\ldots(K))}(K')$ we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of $Z[B]$ by $B'$ is a $(B')_B$-torsor,
- an extension of $Z[B \times B] + Z[A]$ by $B$ consists of a couple of a $(B')_B \times B$-torsor and a $(B')_A$-torsor, and finally
- an extension of $Z[B \times B] + Z[B \times B \times B]$ by $B'$ consists of a couple of a $(B')_B \times B$-torsor and a $(B')_B \times B \times B$-torsor.

According to these considerations an object $(E, \alpha, \beta, \gamma)$ of $\Psi_{\text{Tot}(L\ldots(K))}(K')$ consists of

1. a $B'$-torsor $E$ over $B$
2. a couple of two trivializations $\alpha$ and $\beta$ of the couple of two $B'$-torsors over $B \times B$ and $A$, which are the pull-back of $E$ via $D_0$. More precisely:
   - a trivialization of the $\alpha$ of the $B'$-torsor over $B \times B$ and $B \times B \times B$ which is the pull-back of $E$ via $d_{00} : Z[B \times B] \rightarrow Z[B]$. This trivialization can be interpreted as a group law on the fibres of the $B'$-torsors over $B \times B$:
     $$+ : E_{b_1} \rightarrow E_{b_1+b_2}$$
   - where $b_1, b_2$ are points of $B(S)$ with $S$ any object of $\mathcal{T}$.
   - a trivializations of the $\beta$ of the $B'$-torsor $(D_{00})^* E$ over $A$ which is the pull-back of $E$ via $D_0 : Z[A] \rightarrow Z[B]$. The compatibility of $\alpha$ and $\beta$ with the relation $D_0 \circ D_1 = 0$ involves only $\alpha$ and it imposes on the data $(E, +)$ two relations through the two torsors over $B \times B$ and $B \times B \times B$. These two relations are the relations of commutativity and of associativity of the group law $+$, which mean that $+$ defines over $E$ a structure of commutative extension of $B$ by $B'$;
3. $\gamma$ is the zero homomorphism since $L_{20}(K) = 0$.

The object $(E, +, \beta)$ of $\Psi_{\text{Tot}(L\ldots(K))}(K')$ is an extension of $K$ by $K'$ and so we can conclude that the category the category $\Psi_{\text{Tot}(L\ldots(K))}(K')$ is equivalent to the category $\text{Ext}(K, K')$. The proof that we have in fact an equivalence of additive cofibrant categories is left to the lector.

Proposition 4.2. We have that $H_2(\text{Tot}(L\ldots(K))) = 0$ and the augmentation map $\epsilon : L_0(K) \rightarrow K$ induces the isomorphisms $H_1(\text{Tot}(L\ldots(K))) \simeq H_1(K)$ and $H_0(\text{Tot}(L\ldots(K))) \simeq H_0(K)$.

Proof. In order to prove this Lemma, we apply [SGA7] Exposé VII Proposition 3.5.3 to the augmentation map $\epsilon : L_0(K) \rightarrow K$, i.e. we have to prove that for any complex $K' = [A' \xrightarrow{\nu'} B']$ of $D^{[1,0]}(\mathcal{C})$ the functor

$$\epsilon^* : \Psi_{\text{Tot}(K)}(K') \rightarrow \Psi_{\text{Tot}(L\ldots(K))}(K')$$

is an equivalence of category. According to our definition 2.1, it is clear that the category $\Psi_{\text{Tot}(K)}(K')$ is equivalent to the category $\text{Ext}(K, K')$ of extensions of $K$ by $K'$. On the other hand, by the Lemma 4.1 also the category $\Psi_{\text{Tot}(L\ldots(K))}(K')$ is equivalent to the category $\text{Ext}(K, K')$. \qed
Corollary 4.3. Let $K$ and $K'$ two objects $D^{[1,0]}(C)$. Then the group of automorphisms of any extension of $K$ by $K'$ is isomorphic to the group $\text{Ext}^0(K, K')$, and the group of isomorphism classes of extensions of $K$ by $K'$ is isomorphic to the group $\text{Ext}^1(K, K')$.

Proof. According to the above proposition, it exists an arbitrary flat resolution $L'\cdot(K)$ of $K$ such that the groups $\text{Tot}(L\cdot(K))_j$ and $\text{Tot}(L'\cdot(K))_j$ are isomorphic for $j = 0, 1, 2$. We have therefore a canonical homomorphism

\[ L\cdot(K) \rightarrow L'\cdot(K) \]

inducing a canonical homomorphism

\[ \text{Tot}(L\cdot(K)) \rightarrow \text{Tot}(L'\cdot(K)) \]

which is an isomorphism in degrees 0, 1 and 2. According to SGA 7 Proposition 3.4.7 we have the following equivalence of additive cofibred categories

\[ \Psi_{\text{Tot}(L\cdot(K))}(K') \cong \Psi_{\tau\geq 2L'\cdot(K)}(K'). \]

Hence in order to get the statement of this corollary we have to put together

- the geometrical description of the category $\Psi_{\text{Tot}(L\cdot(K))}(K')$ furnished by the Lemma 4.1:
  \[ \text{Ext}(K, K') \cong \Psi_{\text{Tot}(L\cdot(K))}(K'), \]

- the homological description of the groups $\Psi_{\tau\geq 2\text{Tot}(L'\cdot(K))}(K')$ for $i = 0, 1$ furnished by the Theorem 3.1:
  \[ \Psi_{\tau\geq 2\text{Tot}(L'\cdot(K))}(K') \cong \text{Ext}^i(L'\cdot(K), K) \cong \text{Ext}^i(K, K'). \]

\[ \square \]

5. Geometrical description of $\Psi_{\text{Tot}(L\cdot)}$

Let $K = [A_1 \xrightarrow{\alpha_i} B_i]$ (for $i = 1, 2$) be a complex of objects of $C$ concentrated in degrees 1 and 0 and let $L\cdot(K_i)$ be its canonical flat partial resolution. Denote by $L.(K_1, K_2)$ the complex $\text{Tot}(L\cdot(K_1) \otimes L\cdot(K_2))$. In this section we prove the following geometrical description of the category $\Psi_{\tau\geq 2L.(K_1, K_2)}$:

Theorem 5.1. The additive cofibred category $\text{Biext}(K_1, K_2; \ast)$ of biextensions of $(K_1, K_2)$ by a variable object of $D^{[1,0]}(C)$ is equivalent to the additive cofibred category $\Psi_{\tau\geq 2L.(K_1, K_2)}$:

\[ (5.1) \quad \text{Biext}(K_1, K_2; \ast) \cong \Psi_{\tau\geq 2L.(K_1, K_2)} \]

Proof. Denote by $L.(K_1, K_2)$ the bicomplex $L\cdot(K_1) \otimes L\cdot(K_2)$: explicitly, $L_{ij}(K_1, K_2) = 0$ for $(ij) \neq (00), (01), (02), (03), (04), (10), (11), (12), (20)$ and its
non trivial components are
\[ L_{00}(K_1, K_2) = L_{00}(K_1) \otimes L_{00}(K_2) \]
\[ = [B_1 \times B_2] \]
\[ L_{01}(K_1, K_2) = L_{01}(K_1) \otimes L_{01}(K_2) + L_{01}(K_1) \otimes L_{00}(K_2) \]
\[ = Z[B_1 \times B_2] + [B_1 \times B_1 \times B_2] \]
\[ L_{02}(K_1, K_2) = L_{02}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{00}(K_2) + L_{01}(K_1) \otimes L_{01}(K_2) \]
\[ = Z[B_1 \times B_2] + Z[B_1 \times B_2 \times B_2] \]
\[ L_{03}(K_1, K_2) = L_{03}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{01}(K_2) \]
\[ L_{04}(K_1, K_2) = L_{02}(K_1) \otimes L_{02}(K_2) \]
\[ L_{10}(K_1, K_2) = L_{10}(K_1) \otimes L_{00}(K_2) + L_{00}(K_1) \otimes L_{10}(K_2) \]
\[ = Z[A_1 \times B_2] + Z[B_1 	imes A_2] \]
\[ L_{11}(K_1, K_2) = L_{10}(K_1) \otimes L_{01}(K_2) + L_{01}(K_1) \otimes L_{10}(K_2) \]
\[ = Z[A_1 \times B_2] + Z[B_1 \times B_1 \times A_2] \]
\[ L_{12}(K_1, K_2) = L_{10}(K_1) \otimes L_{02}(K_2) + L_{02}(K_1) \otimes L_{10}(K_2) \]
\[ L_{20}(K_1, K_2) = L_{10}(K_1) \otimes L_{10}(K_2) \]
\[ = Z[A_1 \times A_2] \]

The truncation \( \tau_{\geq 2} L_-(K_1, K_2) \) is the complex
\[ L_{02}(K_1, K_2)+L_{11}(K_1, K_2)+L_{20}(K_1, K_2) \overset{d_1}{\rightarrow} L_{01}(K_1, K_2)+L_{10}(K_1, K_2) \overset{d_0}{\rightarrow} L_{00}(K_1, K_2) \rightarrow 0 \]
where the differential operators \( D_0 \) and \( D_1 \) can be computed from the below diagram, where we’ve don’t have written the identity homomorphisms in order to avoid too heavy notation (for example instead of \((id \times D_{00}^{K_2} D_{00}^{K_1} \times id)\) we have written just \((D_{00}^{K_2}, D_{00}^{K_1})\)): 

\[ \begin{array}{c}
L_{01}(K_1, K_2) \\
L_{11}(K_1, K_2) \\
L_{20}(K_1, K_2)
\end{array} \]

\[ \downarrow \begin{array}{l}
d_{02}^{K_2} + d_{01}^{K_1} \\
d_{01}^{K_2} + d_{00}^{K_1} \\
d_{00}^{K_2}
\end{array} \]

\[ \begin{array}{c}
L_{02}(K_1, K_2) \\
L_{10}(K_1, K_2) \\
L_{00}(K_1, K_2)
\end{array} \]

Explicitly the condition \( D_0 \circ D_1 = 0 \) means:
- the following sequences are exact:

\[ \begin{align*}
\mathbb{Z}[B_1 \times B_2] + \mathbb{Z}[B_1 \times B_2 \times B_2] & \overset{d_{00}^{K_2}}{\longrightarrow} \mathbb{Z}[B_1 \times B_2] \\
\mathbb{Z}[B_1 \times B_1 \times B_2] & \overset{d_{00}^{K_1}}{\longrightarrow} \mathbb{Z}[B_1 \times B_2] \\
\mathbb{Z}[B_1 \times B_2] & \overset{d_{00}^{K_2}}{\longrightarrow} \mathbb{Z}[B_1 \times B_2]
\end{align*} \]

- the following diagrams are anticommutative:

\[ \begin{align*}
\mathbb{Z}[B_1 \times B_1 \times B_2] & \overset{d_{01}^{K_2}}{\longrightarrow} \mathbb{Z}[B_1 \times B_1 \times B_2] \\
\mathbb{Z}[B_1 \times B_2] & \overset{d_{00}^{K_2}}{\longrightarrow} \mathbb{Z}[B_1 \times B_2]
\end{align*} \]
In order to describe explicitly the objects of the fibre \( \Psi_{\tau_2L(K_1,K_2)}(K_3) \) of the cofibred category \( \Psi_{\tau_2L(K_1,K_2)} \) we use the description (1.2) of extensions of free commutative groups in terms of torsors:

- an extension of \( (\tau_2L(K_1,K_2))_0 \) by \( B_3 \) is a \( (B_3)_{B_1\times B_2} \)-torsor,
- an extension of \( (\tau_2L(K_1,K_2))_1 \) by \( B_3 \) consists of a \( (B_3)_{B_1\times B_2} \)-torsor, a \( (B_3)_{B_1\times B_2} \)-torsor, and a \( (B_3)_{B_1\times A_2} \)-torsor, and finally
- an extension of \( (\tau_2L(K_1,K_2))_2 \) by \( B_3 \) consists of a system of 8 torsors under the groups deduced from \( B_3 \) by base change over the bases \( B_1\times B_2\times B_2, \ B_1\times B_2\times B_2, \ B_1\times B_1\times B_1\times B_2, \ B_1\times B_1\times B_1\times B_2, \ B_1\times B_1\times B_2\times B_2, \ A_1\times B_2\times B_2, \ B_1\times B_1\times A_2, \ A_1\times A_2 \) respectively.

According to these considerations an object \((E,\alpha,\beta,\gamma)\) of \( \Psi_{\tau_2L(K_1,K_2)}(K_3) \) consists of

1. a \( B_3 \)-torsor \( E \) over \( B_1\times B_2 \)
2. a couple of two trivializations \( \alpha = (\alpha_1,\alpha_2) \) and \( \beta = (\beta_1,\beta_2) \) of the couple of two \( B_3 \)-torsors over \( B_1\times B_2\times B_2 + B_1\times B_1\times B_2 \) and \( A_1\times B_2 + B_1\times A_2 \), which are the pull-back of \( E \) via \( \mathbb{D}_0 \). More precisely:
   - a couple of trivializations \( \alpha = (\alpha_1,\alpha_2) \) of the couple of \( B_3 \)-torsors over \( B_1\times B_2\times B_2 \) and \( B_1\times B_1\times B_2 \) which are the pull-back of \( E \) via \( \text{id} \times d_{\alpha_0}^{K_1} + d_{\alpha_0}^{K_1} \times \text{id} : Z[B_1\times B_2\times B_2] = Z[B_1\times B_1\times B_2] \rightarrow Z[B_1\times B_2] \).

The trivializations \( (\alpha_1,\alpha_2) \) can be interpreted as two group laws on the fibres of the couple of \( B_3 \)-torsors over \( B_1\times B_2\times B_2 \) and \( B_1\times B_1\times B_2 \) as

\[
\begin{align*}
+2 : E_{b_2,b_1} &\rightarrow E_{b_2+b_2',b_1} \\
+1 : E_{b_2,b_1} &\rightarrow E_{b_2,b_1'} \\
\end{align*}
\]

where \( b_2, b_2' \) (resp. \( b_1, b_1' \)) are points of \( B_2(S) \) (resp. of \( B_1(S) \)) with \( S \) any object of \( T \).

- a couple of trivializations \( \beta = (\beta_1,\beta_2) \) of the couple of \( B_3 \)-torsors \( ((D_{\alpha_0}^{K_1} \times \text{id})^*E, (\text{id} \times D_{\alpha_0}^{K_1})^*E) \) over \( A_1\times B_2 \) and \( B_1\times A_2 \) respectively, which are the pull-back of \( E \) via \( D_{\alpha_0}^{K_1} \times \text{id} + \text{id} \times D_{\alpha_0}^{K_1} : Z[A_1\times B_2] = Z[B_1\times A_2] \rightarrow Z[B_1\times B_2] \).

The compatibility of \( \alpha \) and \( \beta \) with the relation \( \mathbb{D}_0 \circ \mathbb{D}_1 = 0 \) imposes, on the data \((E,+_1,+_2)\) and \(((D_{\alpha_0}^{K_1} \times \text{id})^*E, (\text{id} \times D_{\alpha_0}^{K_1})^*E, \beta_1, \beta_2)\), 8 relations of compatibility through the system of 8 torsors over \( B_1\times B_2\times B_2, \ B_1\times B_2\times B_2, \ B_1\times B_1\times B_1\times B_2, \ B_1\times B_1\times B_1\times B_2, \ B_1\times B_1\times B_2\times B_2, \ B_1\times B_1\times B_2\times B_2, \ B_1\times B_1\times B_1\times B_2, \ B_1\times B_1\times B_2\times B_2 \times B_1\times B_1\times A_2 \times A_2 \times
$B_2$, $B_1 \times B_1 \times A_2$, $A_1 \times A_2$. For $\alpha$ the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ imposes the following 5 relations of compatibility on the data $(E, +1, +2)$ through the 5 torsors arising from the factor $L_{02}(K_1, K_2)$ of $\tau_{22}L.(K_1, K_2)$:

- The exact sequence (5.3) furnishes the two relations of commutativity and of associativity of the group law $+2$, which mean that $+2$ defines over $E$ a structure of commutative extension of $(B_2)_{B_2}$ by $(B_3)_{B_2}$;
- The exact sequence (5.4) expresses the two relations of commutativity and of associativity of the group law $+1$, which mean that $+1$ defines over $E$ a structure of commutative extension of $(B_1)_{B_2}$ by $(B_3)_{B_2}$;
- The anticommutative diagram (5.5) furnishes that these two group laws are compatible.

Therefore these 5 conditions implies that the torsor $E$ is endowed with a structure of biextension of $(B_1, B_2)$ by $B_3$.

For $\beta$ the relation $\mathbb{D}_0 \circ \mathbb{D}_1 = 0$ imposes the following 3 relations of compatibility on the data $((D^K_0 \times id)^*E, (id \times D^K_0)^*E, \beta_1, \beta_2)$ through the 3 torsors arising from the factors $L_{11}(K_1, K_2) + L_{20}(K_1, K_2)$ of $\tau_{22}L.(K_1, K_2)$:

- The anticommutative diagram (5.6) furnishes a relation of compatibility between the group law $+2$ of $E$ and the trivialization $\beta_1$ of the pullback $(D^K_0 \times id)^*E$ of $E$ over $A_1 \times B_2$, which means that $\beta_1$ is a trivialization of biextension;
- The anticommutative diagram (5.7) furnishes a relation of compatibility between the group law $+1$ of $E$ and the trivialization $\beta_2$ of the pullback $(id \times D^K_0)^*E$ of $E$ over $B_1 \times A_2$, which means that also $\beta_2$ is a trivialization of biextension;
- The anticommutative diagram (5.8) means that the two trivializations $\beta_1$ and $\beta_2$ have to coincide over $A_1 \times A_2$.

(4) $\gamma : Z[A_1] \otimes Z[A_2] \rightarrow A_3$ is an homomorphism such that the composite $Z[A_1] \otimes Z[A_2] \xrightarrow{\gamma} Z[A_1] \otimes Z[A_2] \xrightarrow{u} B_3$ is compatible with the restriction of the trivializations $\beta_1, \beta_2$ over $Z[A_1] \otimes Z[A_2]$.

The object $(E, \alpha, \beta, \gamma)$ of $\Psi_{\tau_{22}L.(K_1, K_2)}(K_3)$ is therefore a biextension $(E, \beta_1, \beta_2, \gamma)$ of $(K_1, K_2)$ by $K_3$. The diligent lector can check that the above arguments furnish the equivalence of additive cofibred categories (5.1).

6. Proof of the main theorem

Let $K_i = [A_i \rightarrow B_i]$ (for $i = 1, 2, 3$) be a complex of objects of $C$ concentrated in degrees 1 and 0. Denote by $L_\cdot(K_i)$ (for $i = 1, 2$) the canonical flat partial resolution of $K_i$ introduced at the beginning of §3. According to Proposition 4.2, it exists an arbitrary flat resolution $L'_\cdot(K_i)$ (for $i = 1, 2$) of $K_i$ such that the groups $\text{Tot}(L_\cdot(K_i))_j$ and $\text{Tot}(L'_\cdot(K_i))_j$ are isomorphic for $j = 0, 1, 2$. We have therefore canonical homomorphisms

$$L_\cdot(K_1) \longrightarrow L'_\cdot(K_1) \quad L_\cdot(K_2) \longrightarrow L'_\cdot(K_2)$$

inducing a canonical homomorphism

$$\text{Tot}(L_\cdot(K_1) \otimes L_\cdot(K_2)) \longrightarrow \text{Tot}(L'_\cdot(K_1) \otimes L'_\cdot(K_2))$$

which is an isomorphism in degrees 0, 1 and 2. Denote by $L_\cdot(K_1, K_2)$ (resp. $L'_\cdot(K_1, K_2)$) the complex $\text{Tot}(L_\cdot(K_1) \otimes L_\cdot(K_2))$ (resp. $\text{Tot}(L'_\cdot(K_1) \otimes L'_\cdot(K_2))$). Remark that
From the exact sequence $\Psi_\tau\Psi_{\tau 2} L.(K_1, K_2) \cong \Psi_{\tau 2} L.(K_1, K_2)(K_3)$.

By construction, according to SGA 7 Proposition 3.4.7 we have the following equivalence of additive cofibred categories

$$\Psi_{\tau 2} L.(K_1, K_2)(K_3) \cong \Psi_{\tau 2} L.(K_1, K_2)(K_3).$$

Hence applying the Theorem 5.1, which furnishes the following geometrical description of the category $\Psi_{\tau 2} L.(K_1, K_2)(K_3)$:

$$\text{Biext}(K_1, K_2; K_3) \cong \Psi_{\tau 2} L.(K_1, K_2)(K_3),$$

and applying the Theorem 3.1, which furnishes the following homological description of the groups $\Psi_{\tau 2} L.(K_1, K_2)(K_3)$ for $i = 0, 1$:

$$\psi_i \cong \text{Hom}(L(K_1, K_2), K_3) \cong \text{Ext}(K_1, K_2, K_3),$$

we get the main Theorem 0.1, i.e.

$$\psi_i \cong \text{Hom}(K_1, K_2; K_3) \cong \text{Ext}(K_1, K_2, K_3) \quad (i = 0, 1).$$

Remark 6.1. From the exact sequence $0 \to A_1[1] \to K_3 \to B_3 \to 0$ we get the long exact sequence

$$0 \to \psi^0 \to \psi^1 \to \psi^2 \to \psi^3 \to \psi^4 \to 0,$$

and applying the Theorem 3.1, which furnishes the following homological description of the groups $\Psi_{\tau 2} L.(K_1, K_2)(K_3)$ for $i = 0, 1$:

$$\psi_i \cong \text{Hom}(K_1, K_2; K_3) \cong \text{Ext}(K_1, K_2, K_3),$$

we get the main Theorem 0.1, i.e.

$$\psi_i \cong \text{Hom}(K_1, K_2; K_3) \cong \text{Ext}(K_1, K_2, K_3) \quad (i = 0, 1).$$

References


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