Universität Regensburg
Mathematik

Motivic Landweber exactness

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Preprint Nr. 15/2008

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1 Introduction

The Landweber exact functor theorem combined with Brown representability provides an almost unreasonably efficient toolkit for constructing homotopy types out of purely algebraic data. Among the many examples arising this way is the presheaf of elliptic homology theories on the moduli stack of elliptic curves. In this paper we incite the use of such techniques in the algebro-geometric setting of motivic homotopy theory.

In what follows we shall state some of the main results in the paper, comment on the proofs and discuss some of the background and relation to previous works. Throughout we employ a stacky viewpoint of the subject which originates with formulations in stable homotopy theory pioneered by Morava and Hopkins. Let S be a regular noetherian base scheme of finite Krull dimension and $\mathbf{SH}(S)$ the corresponding motivic stable homotopy category. A complex point $\mathsf{Spec}(\mathbf{C}) \to S$ induces a functor $\mathbf{SH}(S) \to \mathbf{SH}$ to the classical stable homotopy category. Much of the work in this paper is guidelined by the popular quest of hoisting results in $\mathbf{SH}$ to the more complicated motivic category.

To set the stage, denote by MGL the algebraic cobordism spectrum introduced by Voevodsky [29]. By computation we show $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$ is a flat Hopf algebroid in Adams graded Abelian groups. (Our standard conventions concerning graded objects are detailed in Section 3. Recall that $\mathsf{MGL}_* \equiv \mathsf{MGL}_{2*,*}$.) The useful fact that MGL gives rise to an algebraic stack $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ in the formulation introduced by the first author comes to bear. (This apparatus is reviewed in Section 2.) By comparing with the complex cobordism spectrum MU we deduce a 2-categorical commutative diagram:

$$\begin{tikzcd} \mathsf{Spec}(\mathsf{MGL}_*) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{MU}_*) \arrow[d] \\ {[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]} \arrow[r] & {[\mathsf{MU}_*/\mathsf{MU}_*\mathsf{MU}]} \end{tikzcd} \tag{1}$$

The right hand part of the diagram is well-known: Milnor's computation of MU_* and Quillen's identification of the canonical formal group law over MU_* with the universal formal group law are early success stories in modern algebraic topology. As a $\mathbf{G}_{\mathfrak{m}}$-stack the lower right hand corner identifies with the moduli stack of strict graded formal groups. Our plan from the get-go was to prove (1) is cartesian and use that description of the algebraic cobordism part of the diagram to deduce motivic analogs of theorems in stable homotopy theory. It turns out this strategy works for general base schemes.

Recall that an MU_*-module M_* is Landweber exact if $v_0^{(p)}, v_1^{(p)}, \ldots$ forms a regular sequence in M_* for every prime p. Here $v_0^{(p)} = p$ and the $v_i^{(p)}$ for $i > 0$ are indecomposable elements of degree $2p^i - 2$ in MU_* with Chern numbers divisible by p. Using the cartesian diagram (1) we show the following result for Landweber exact motivic homology theories, see Theorem 7.3 for a more precise statement.

Theorem: *Suppose* A_* *is a Landweber exact graded* MU_**-algebra. Then*

$$\mathsf{MGL}_{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{A}_*$$

is a bigraded ring homology theory on $\mathbf{SH}(S)$*.*

Using the theorem we deduce that

$$\mathsf{MGL}^{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{A}_*$$

is a ring cohomology theory on the subcategory of strongly dualizable objects of $\mathbf{SH}(S)$. In the case of the Laurent polynomial ring $\mathbf{Z}[\beta, \beta^{-1}]$ on the Bott element, this observation is part of the proof in [26] of the motivic Conner-Floyd isomorphism

$$\mathsf{MGL}^{**}(-) \otimes_{\mathsf{MU}_*} \mathbf{Z}[\beta, \beta^{-1}] \xrightarrow{\cong} \mathsf{KGL}^{**}$$

for the motivic spectrum KGL representing homotopy algebraic K-theory.

Define the category of Tate objects $\mathbf{SH}(S)_{\mathcal{T}}$ as the smallest localizing triangulated subcategory of the motivic stable homotopy category containing the set $\mathcal{T}$ of all mixed motivic spheres

$$S^{p,q} \equiv S_s^{p-q} \wedge \mathbf{G}_{\mathfrak{m}}^q.$$

The Tate objects are precisely the cellular spectra in the terminology of [7]. Our choice of wording is deeply rooted in the theory of motives. Since the inclusion $\mathbf{SH}(S)_{\mathcal{T}} \subseteq \mathbf{SH}(S)$ preserves sums and $\mathbf{SH}(S)$ is compactly generated the inclusion acquires a right adjoint $\mathsf{p}_{\mathbf{SH}(S),\mathcal{T}} \colon \mathbf{SH}(S) \to \mathbf{SH}(S)_{\mathcal{T}}$ called the Tate projection. When E is a Tate object and F a motivic spectrum there is thus an isomorphism

$$\mathsf{E}_{**}(\mathsf{F}) \cong \mathsf{E}_{**}(\mathsf{p}_{\mathbf{SH}(S),\mathcal{T}}\mathsf{F}).$$

As in topology, it follows that the E_{**}-homology of F is determined by the E_{**}-homology of mixed motivic spheres. This observation is a key input in showing $(\mathsf{E}_*, \mathsf{E}_*\mathsf{E})$ is a flat

Hopf algebroid in Adams graded Abelian groups provided one - and hence both - of the canonical maps $\mathsf{E}_{**} \to \mathsf{E}_{**}\mathsf{E}$ is flat and the canonical map $\mathsf{E}_{*}\mathsf{E} \otimes_{\mathsf{E}_*} \mathsf{E}_{**} \to \mathsf{E}_{**}\mathsf{E}$ is an isomorphism. Specializing to the example of algebraic cobordism allows us to form the algebraic stack $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ and (1).

Our motivic analog of Landweber's exact functor theorem takes the following form, see Theorem 8.6.

Theorem: *Suppose* M_* *is an Adams graded Landweber exact* MU_**-module. Then there exists a motivic spectrum* E *in* $\mathbf{SH}(S)_{\mathcal{T}}$ *and a natural isomorphism*

$$\mathsf{E}_{**}(-) \cong \mathsf{MGL}_{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{M}_*$$

of homology theories on $\mathbf{SH}(S)$*.*

In addition, if M_* *is a graded* MU_**-algebra then* E *acquires a quasi-multiplication which represents the ring structure on the corresponding Landweber exact theory.*

When the base scheme is the integers we use motivic Landweber exactness and the fact that $\mathbf{SH}(\mathbf{Z})$ is a Brown category, so that all homology theories are representable, to conclude the proof of the motivic exact functor theorem. For a general base scheme we provide base change results which allow us to reduce to the case of the integers. The derived category of modules over MGL - relative to $\mathbf{Z}$ - turns also out to be a Brown category. This suffices to show the above remains valid when translated verbatim to the setting of highly structured MGL-modules. Recall MGL is a motivic symmetric spectrum and the monoid axiom introduced in [25] holds for the motivic stable structure according to [15, Proposition 4.19]. Thus the modules over MGL acquire a closed symmetric monoidal model structure. Moreover, for every cofibrant replacement of MGL there is an induced Quillen equivalence of modules.

We wish to emphasize the close connection between our results and the classical Landweber exact functor theorem. In particular, if M_* is concentrated in even degrees there exists a commutative ring spectrum $\mathsf{E}^{\mathsf{Top}}$ in $\mathbf{SH}$ which represents the corresponding topological Landweber exact theory. Although E and $\mathsf{E}^{\mathsf{Top}}$ are objects in wildly different categories of spectra, it turns out there is an isomorphism

$$\mathsf{E}_{**}\mathsf{E} \cong \mathsf{E}_{**} \otimes_{\mathsf{E}^{\mathsf{Top}}_*} \mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}}.$$

The last section of the paper describes (co)operations and phantom maps between Landweber exact motivic spectra. We use a spectral sequence argument to show that every MGL-module E gives rise to a surjection

$$\mathsf{E}^{p,q}(\mathsf{M}) \longrightarrow \mathsf{Hom}^{p,q}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{M}, \mathsf{E}_{**}). \tag{2}$$

The kernel of (2) identifies with the Ext-term

$$\mathsf{Ext}^{1,(p-1,q)}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{M}, \mathsf{E}_{**}). \tag{3}$$

Imposing the assumption that $\mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}}$ be a projective $\mathsf{E}^{\mathsf{Top}}_*$-module implies the given Ext-term in (3) vanishes, and hence (2) is an isomorphism. The assumption on $\mathsf{E}^{\mathsf{Top}}$ holds for unitary topological K-theory KU and localizations of Johnson-Wilson theories. By way of example we compute the KGL-cohomology of KGL. That is, with the completed tensor product, there is an isomorphism of KGL^{**}-algebras

$$\mathsf{KGL}^{**}\mathsf{KGL} \xrightarrow{\cong} \mathsf{KGL}^{**}\widehat{\otimes}_{\mathsf{KU}^*}\mathsf{KU}^*\mathsf{KU}.$$

By [2] the group $\mathsf{KU}^1\mathsf{KU}$ is trivial and $\mathsf{KU}^0\mathsf{KU}$ is uncountable [2]. We also show that KGL does not support any nontrivial phantom maps. Adopting the proof to $\mathbf{SH}$ reproves the analogous result for KU. These techniques can also be utilized to show there is a Chern character in $\mathbf{SH}(S)$ from KGL to the periodized rational motivic Eilenberg-MacLane spectrum $\mathsf{PM}\mathbf{Q}$ representing rational motivic cohomology. In the arguments we use (semi)model structures on E_∞-algebras, but these can be skipped when restricted to a smooth base over a field on account of the isomorphism between $\mathsf{M}\mathbf{Q}$ and the Landweber theory for the additive formal group law over the rationals.

Inspired by the results herein we make some rather speculative remarks concerning future works. The all-important chromatic approach to stable homotopy theory acquires deep interplays with the algebraic geometry of formal groups. Landweber exact algebras over Hopf algebroids represent a central theme in this endeavor, leading for example to the bicomplete closed symmetric monoidal abelian category of $\mathsf{BP}_*\mathsf{BP}$-comodules. The techniques in this paper furnish a corresponding Landweber exact motivic Brown-Peterson spectrum MBP equivalent to the constructions in [14] and [28]. The object $\mathsf{MBP}_*\mathsf{MBP}$ and questions in motivic chromatic theory at large can be investigated along the lines of this paper. An exact analog of Bousfield's localization machinery in motivic stable homotopy theory was worked out in [24, Appendix A], cf. also [11] for a discussion of the chromatic viewpoint. In a separate paper we shall dispense with the regularity assumption on S. The results in this paper remain valid for noetherian base schemes of finite Krull dimension. Since this generalization uses arguments which are otherwise independent of the present work, the details will appear elsewhere.

2 Preliminaries on algebraic stacks

By a stack we shall mean a category fibered in groupoids over the site comprised by the category of commutative rings endowed with the fpqc-topology. A stack $\mathfrak{X}$ is algebraic if its diagonal is representable and affine, and there exists an affine scheme U together with a faithfully flat map $U \to \mathfrak{X}$, called a presentation of $\mathfrak{X}$. We refer to [10] and [20] for motivation and basic properties of these notions.

Lemma 2.1: *Suppose there are 2-commutative diagrams of algebraic stacks*

$$\begin{tikzcd} \mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{Z}' \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{X}' \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Y} \arrow[r] \arrow[d, "\pi"'] & \mathfrak{Y}' \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{X}' \end{tikzcd} \tag{4}$$

where π is faithfully flat. Then the left hand diagram in (4) is cartesian if and only if the naturally induced commutative diagram

$$\begin{tikzcd} \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{Z}' \times_{\mathfrak{X}'} \mathfrak{Y}' \arrow[d] \\ \mathfrak{Y} \arrow[r] & \mathfrak{Y}' \end{tikzcd} \tag{5}$$

is cartesian.

Proof. The base change of the canonical 1-morphism $\mathfrak{c} : \mathfrak{Z} \to \mathfrak{Z}' \times_{\mathfrak{X}'} \mathfrak{X}$ over $\mathfrak{X}$ along π identifies with the canonically induced 1-morphism

$$\mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y} \xrightarrow{\mathfrak{c}\times 1} (\mathfrak{Z}' \times_{\mathfrak{X}'} \mathfrak{X}) \times_{\mathfrak{X}} \mathfrak{Y} \cong \mathfrak{Z}' \times_{\mathfrak{X}'} \mathfrak{Y} \cong (\mathfrak{Z}' \times_{\mathfrak{X}'} \mathfrak{Y}') \times_{\mathfrak{Y}'} \mathfrak{Y}.$$

This is an isomorphism provided (5) is cartesian; hence so is $\mathfrak{c} \times 1$. By faithfully flatness of π it follows that $\mathfrak{c}$ is an isomorphism. The reverse implication holds trivially. $\square$

Corollary 2.2: *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ algebraic stacks, $U \to \mathfrak{X}$ and $V \to \mathfrak{Y}$ presentations and there is a 2-commutative diagram:*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd} \tag{6}$$

Then (6) is cartesian if and only if one - and hence both - of the commutative diagrams $(i = 1, 2)$

$$\begin{tikzcd} U \times_{\mathfrak{X}} U \arrow[r] \arrow[d, "pr_i"'] & V \times_{\mathfrak{Y}} V \arrow[d, "pr_i"] \\ U \arrow[r] & V \end{tikzcd} \tag{7}$$

is cartesian.

Proof. Follows from Lemma 2.1 since presentations are faithfully flat. □

A presentation $U \to \mathfrak{X}$ yields a Hopf algebroid or cogroupoid object in commutative rings $(\Gamma(\mathcal{O}_U), \Gamma(\mathcal{O}_{U \times_{\mathfrak{X}} U}))$. Conversely, if (A, B) is a flat Hopf algebroid, denote by $[\mathsf{Spec}(A)/\mathsf{Spec}(B)]$ the associated algebraic stack. We note that by [20, Theorem 8] there is an equivalence of 2-categories between flat Hopf algebroids and rigidified algebraic stacks.

Let $\mathsf{Qc}_{\mathfrak{X}}$ denote the category of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules and $\mathcal{A} \in \mathsf{Qc}_{\mathfrak{X}}$ a monoid, or quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras. If X_0 is a scheme and $\pi : X_0 \to \mathfrak{X}$ faithfully flat, then $\mathcal{A}$ is equivalent to the datum of the $\mathcal{O}_{X_0}$-algebra $\mathcal{A}(X_0) \equiv \pi^*\mathcal{A}$ combined with a descent datum with respect to $X_1 \equiv X_0 \times_{\mathfrak{X}} X_0 \rightrightarrows X_0$. When $X_0 = \mathsf{Spec}(A)$ is affine, $X_1 = \mathsf{Spec}(\Gamma)$ is affine, (A, Γ) a flat Hopf algebroid and $\mathcal{A}(X_0)$ a Γ-comodule algebra.

There is an evident adjunction between left $\mathcal{A}$-modules in $\mathsf{Qc}_{\mathfrak{X}}$ and left $\mathcal{A}(X_0)$-modules in Qc_{X_0}:

$$\pi^* : \mathcal{A} - \mathsf{mod} \rightleftarrows \mathcal{A}(X_0) - \mathsf{mod} : \pi_*$$

Since π_* has an exact left adjoint π^* it preserves injectives and there are isomorphisms

$$\mathsf{Ext}^n_{\mathcal{A}}(\mathcal{M}, \pi_*\mathcal{N}) \cong \mathsf{Ext}^n_{\mathcal{A}(X_0)}(\pi^*\mathcal{M}, \mathcal{N}) \tag{8}$$

between Ext-groups in the categories of quasi-coherent left $\mathcal{A}$- and $\mathcal{A}(X_0)$-modules.

Now assume $i : \mathcal{U} \hookrightarrow \mathfrak{X}$ is the inclusion of an open algebraic substack. Then [20, Propositions 20, 22] imply $i_* : \mathsf{Qc}_{\mathcal{U}} \hookrightarrow \mathsf{Qc}_{\mathfrak{X}}$ is an embedding of a thick subcategory; see also [20, section 3.4] for a discussion of the functoriality of $\mathsf{Qc}_{\mathfrak{X}}$ with respect to $\mathfrak{X}$. For $\mathcal{F}, \mathcal{G} \in \mathsf{Qc}_{\mathcal{U}}$ the Yoneda description of Ext-groups gives isomorphisms

$$\mathsf{Ext}^n_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} i_*\mathcal{F}, \mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} i_*\mathcal{G}) \cong \mathsf{Ext}^n_{i_*\mathcal{A}}(i^*\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{U}}} \mathcal{F}, i^*\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{U}}} \mathcal{G}). \tag{9}$$

We will need the following result.

Proposition 2.3: *Suppose there is a 2-commutative diagram of algebraic stacks*

$$\begin{tikzcd}
 & X_0 \arrow[dd, "\pi"] & \\
X \arrow[ur, "\alpha"] \arrow[rr, "f"] \arrow[dr, "f_X"] \arrow[dd, "\pi_X"'] & & Y \arrow[dl, "f_Y"'] \arrow[dd, "\pi_Y"] \\
 & \mathfrak{X} & \\
\mathcal{U} \arrow[ur, hook, "i_X"] \arrow[rr, hook, "i"] & & \mathcal{U}' \arrow[ul, hook', "i_Y"']
\end{tikzcd}$$

where X, Y, X_0 are schemes, π, π_X, π_Y faithfully flat, and i_X, i_Y (hence also i) open inclusions of algebraic substacks. If $\pi_Y^*\pi_{Y,*}\mathcal{O}_Y \in \mathsf{Qc}_Y$ is projective then

$$\mathsf{Ext}^n_{\mathcal{A}(X_0)}(\mathcal{A}(X_0) \otimes_{\mathcal{O}_{X_0}} \pi^* f_{Y,*}\mathcal{O}_Y, \mathcal{A}(X_0) \otimes_{\mathcal{O}_{X_0}} \alpha_*\mathcal{O}_X)$$

$$\cong \begin{cases} 0 & n \geq 1, \\ \mathsf{Hom}_{\mathcal{O}_Y}(\pi_Y^*\pi_{Y,*}\mathcal{O}_Y, \mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X) & n = 0. \end{cases}$$

Proof. By (8) the group $\mathsf{Ext}^n_{\mathcal{A}(X_0)}(\pi^*(\mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} f_{Y,*}\mathcal{O}_Y), \mathcal{A}(X_0) \otimes_{\mathcal{O}_{X_0}} \alpha_*\mathcal{O}_X)$ is isomorphic to $\mathsf{Ext}^n_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} f_{Y,*}\mathcal{O}_Y, \pi_*(\pi^*\mathcal{A} \otimes_{\mathcal{O}_{X_0}} \alpha_*\mathcal{O}_X))$, which the projection formula identifies with $\mathsf{Ext}^n_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} i_{Y,*}\pi_{Y,*}\mathcal{O}_Y, \mathcal{A} \otimes_{\mathcal{O}_{\mathfrak{X}}} i_{Y,*}i_*\pi_{X,*}\mathcal{O}_X)$. By (9) the latter Ext-groups is isomorphic to $\mathsf{Ext}^n_{i_Y^*\mathcal{A}}(i_Y^*\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{U}'}} \pi_{Y,*}\mathcal{O}_Y, i_Y^*\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{U}'}} i_*\pi_{X,*}\mathcal{O}_X)$. Replacing $i_*\pi_{X,*}\mathcal{O}_X$ by $\pi_{Y,*}f_*\mathcal{O}_X$ and applying (8) gives an isomorphism to $\mathsf{Ext}^n_{\mathcal{A}(Y)}(\pi_Y^*(i_Y^*\mathcal{A} \otimes_{\mathcal{O}_{\mathcal{U}'}} \pi_{Y,*}\mathcal{O}_Y), \mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X) = \mathsf{Ext}^n_{\mathcal{A}(Y)}(\mathcal{A}(Y) \otimes_{\mathcal{O}_Y} \pi_Y^*\pi_{Y,*}\mathcal{O}_Y, \mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X)$. Now $\mathcal{A}(Y) \otimes_{\mathcal{O}_Y} \pi_Y^*\pi_{Y,*}\mathcal{O}_Y$ is a projective left $\mathcal{A}(Y)$-module by the assumption on $\pi_Y^*\pi_{Y,*}\mathcal{O}_Y$. Hence the Ext-term vanishes in every positive degree, while for $n = 0$,

$$\mathsf{Hom}_{\mathcal{A}(Y)}(\mathcal{A}(Y) \otimes_{\mathcal{O}_Y} \pi_Y^*\pi_{Y,*}\mathcal{O}_Y, \mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X) \cong \mathsf{Hom}_{\mathcal{O}_Y}(\pi_Y^*\pi_{Y,*}\mathcal{O}_Y, \mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X).$$

□

3 Conventions

The category of graded objects in an additive tensor category $\mathcal{A}$ refers to integer-graded objects subject to the Koszul sign rule $x \otimes y = (-1)^{|x||y|} y \otimes x$. However, $\mathcal{A}$ will often have a supplementary graded structure. The category of Adams graded objects in $\mathcal{A}$ refers to integer-graded objects in $\mathcal{A}$, but no sign rule for the tensor product is introduced as a consequence of the Adams grading. It is helpful to think of the Adams grading as being even. We will deal with graded Abelian groups, Adams graded graded Abelian groups, or $\mathbf{Z}^2$-graded Abelian groups with a sign rule in the first but not in the second variable, and Adams graded Abelian groups. For an Adams graded graded Abelian group A_{**}, we define $A_i \equiv A_{2i,i}$ and let A_* denote the corresponding Adams graded Abelian group.

The smash product induces a closed symmetric monoidal structure on $\mathbf{SH}(S)$. We denote the internal function spectrum from E to F by $\underline{\mathsf{Hom}}(\mathsf{E},\mathsf{F})$ and the tensor unit or sphere spectrum by $\mathbf{1}$. The dual of E is by definition $\mathsf{E}^\vee \equiv \underline{\mathsf{Hom}}(\mathsf{E},\mathbf{1})$. Note that E_{**} with the usual indexing is an Adams graded graded Abelian group. Let E_i be short for $\mathsf{E}_{2i,i}$. When E is a ring spectrum, i.e. a commutative monoid in $\mathbf{SH}(S)$, we implicitly assume E_{**} is a commutative monoid in Adams graded graded Abelian groups. This latter holds true for orientable ring spectra [14, Proposition 2.16] in view of [19, Theorem 3.2.23].

It is convenient to view evenly graded MU_*-modules to be Adams graded. In this case we will implicitly divide the grading by 2.

4 Homology and cohomology theories

An object F of $\mathbf{SH}(S)$ is called finite (another term is compact) if $\mathsf{Hom}_{\mathbf{SH}(S)}(\mathsf{F},-)$ respects sums. Using the 5-lemma one shows the subcategory of finite objects $\mathbf{SH}(S)_{\mathsf{f}}$ of $\mathbf{SH}(S)$ is thick [12, Definition 1.4.3(a)]. For a set $\mathcal{R}$ of objects in $\mathbf{SH}(S)_{\mathsf{f}}$ let $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ denote the smallest thick triangulated subcategory of $\mathbf{SH}(S)_{\mathsf{f}}$ containing $\mathcal{R}$ and $\mathbf{SH}(S)_{\mathcal{R}}$ the smallest localizing subcategory of $\mathbf{SH}(S)$ containing $\mathcal{R}$ [12, Definition 1.4.3(b)]. The examples we will deal with are the sets of mixed motivic spheres $\mathcal{T}$, the set of (isomorphism classes of) strongly dualizable objects $\mathcal{D}$ and the set $\mathbf{SH}(S)_{\mathsf{f}}$.

Remark 4.1: *According to [7, Remark 7.4]* $\mathbf{SH}(S)_{\mathcal{T}} \subseteq \mathbf{SH}(S)$ *is the full subcategory of cellular motivic spectra introduced in loc. cit.*

Recall $\mathsf{F} \in \mathbf{SH}(S)$ is strongly dualizable if for every $\mathsf{G} \in \mathbf{SH}(S)$ the canonical map

$$\mathsf{F}^\vee \wedge \mathsf{G} \longrightarrow \underline{\mathsf{Hom}}(\mathsf{F},\mathsf{G})$$

is an isomorphism. A strongly dualizable object is finite since $\mathbf{1}$ is finite.

Lemma 4.2: *$\mathbf{SH}(S)_{\mathcal{D},\mathsf{f}}$ is the full subcategory of $\mathbf{SH}(S)_{\mathsf{f}}$ of strongly dualizable objects of $\mathbf{SH}(S)$.*

Proof. Since $\mathcal{D}$ is stable under cofiber sequences and retracts, every object of $\mathbf{SH}(S)_{\mathcal{D},\mathsf{f}}$ is strongly dualizable. □

Lemma 4.3: *$\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ is the full subcategory of compact objects of $\mathbf{SH}(S)_{\mathcal{R}}$ and the latter is compactly generated.*

Proof. Note $\mathbf{SH}(S)_{\mathcal{R}}$ is compactly generated since $\mathbf{SH}(S)$ is so [21, Theorem 2.1, 2.1.1]. If $(-)^c$ indicates a full subcategory of compact objects [21, Theorem 2.1, 2.1.3] implies

$$\mathbf{SH}(S)^c_{\mathcal{R}} = \mathbf{SH}(S)_{\mathcal{R}} \cap \mathbf{SH}(S)^c = \mathbf{SH}(S)_{\mathcal{R}} \cap \mathbf{SH}(S)_{\mathsf{f}}.$$

Hence it suffices to show $\mathbf{SH}(S)_{\mathcal{R}} \cap \mathbf{SH}(S)_{\mathsf{f}} = \mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$. The inclusion "$\supseteq$" is obvious and to prove "$\subseteq$" let $\mathcal{R}'$ be the smallest set of objects closed under suspension, retract and cofiber sequences containing $\mathcal{R}$. Then $\mathcal{R}' \subseteq \mathbf{SH}(S)_{\mathsf{f}}$ and

$$\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}} = \mathbf{SH}(S)_{\mathcal{R}',\mathsf{f}} \subseteq \mathbf{SH}(S)_{\mathsf{f}}, \mathbf{SH}(S)_{\mathcal{R}} = \mathbf{SH}(S)_{\mathcal{R}'}.$$

By applying [21, Theorem 2.1, 2.1.3] to $\mathcal{R}'$ it follows that

$$\mathbf{SH}(S)_{\mathcal{R}} \cap \mathbf{SH}(S)_{\mathsf{f}} = \mathbf{SH}(S)_{\mathcal{R}'} \cap \mathbf{SH}(S)_{\mathsf{f}} = \mathcal{R}' \subseteq \mathbf{SH}(S)_{\mathcal{R}',\mathsf{f}} = \mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}.$$

□

Corollary 4.4: *If $\mathcal{R} \subseteq \mathcal{R}'$ are as above, the inclusion $\mathbf{SH}(S)_{\mathcal{R}} \subseteq \mathbf{SH}(S)_{\mathcal{R}'}$ has a right adjoint $\mathsf{p}_{\mathcal{R},\mathcal{R}'}$.*

Proof. Since $\mathbf{SH}(S)_{\mathcal{R}}$ is compactly generated and the inclusion preserves sums the claim follows from [21, Theorem 4.1]. □

Definition 4.5: *The Tate projection is the functor*

$$\mathsf{p}_{\mathbf{SH}(S)_f,\mathcal{T}} : \mathbf{SH}(S) \longrightarrow \mathbf{SH}(S)_{\mathcal{T}}.$$

Lemma 4.6: *In the situation of Corollary 4.4, the right adjoint $\mathsf{p}_{\mathcal{R}',\mathcal{R}}$ preserves sums.*

Proof. Using [21, Theorem 5.1] it suffices to show that $\mathbf{SH}(S)_{\mathcal{R}} \subseteq \mathbf{SH}(S)_{\mathcal{R}'}$ preserves compact objects. Hence by Lemma 4.3 we are done provided $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}} \subseteq \mathbf{SH}(S)_{\mathcal{R}',\mathsf{f}}$. Clearly this holds since $\mathcal{R} \subseteq \mathcal{R}'$. □

Lemma 4.7: *Suppose $\mathcal{R}$ as above contains $\mathcal{T}$. Then*

$$\mathsf{p}_{\mathcal{R},\mathcal{T}} : \mathbf{SH}(S)_{\mathcal{R}} \longrightarrow \mathbf{SH}(S)_{\mathcal{T}}$$

is an $\mathbf{SH}(S)_{\mathcal{T}}$-module functor.

Proof. Let $\iota : \mathbf{SH}(S)_{\mathcal{T}} \to \mathbf{SH}(S)_{\mathcal{R}}$ be the inclusion and $\mathsf{F} \in \mathbf{SH}(S)_{\mathcal{T}}$, $\mathsf{G} \in \mathbf{SH}(S)_{\mathcal{R}}$. Then the counit of the adjunction between ι and $\mathsf{p}_{\mathcal{R},\mathcal{T}}$ yields the canonical map

$$\iota(\mathsf{F} \wedge \mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G})) \cong \iota(\mathsf{F}) \wedge \iota(\mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G})) \longrightarrow \iota(\mathsf{F}) \wedge \mathsf{G},$$

that is adjoint to

$$\mathsf{F} \wedge \mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G}) \longrightarrow \mathsf{p}_{\mathcal{R},\mathcal{T}}(\iota(\mathsf{F}) \wedge \mathsf{G}). \tag{10}$$

We claim (10) is an isomorphism for all F, G. In effect, the full subcategory of $\mathbf{SH}(S)_{\mathcal{T}}$ generated by the objects F for that (10) is an isomorphism for all $\mathsf{G} \in \mathbf{SH}(S)_{\mathcal{R}}$ is easily seen to be localizing, and hence we may assume $\mathsf{F} = S^{p,q}$ for $p, q \in \mathbf{Z}$. The sphere $S^{p,q}$ is invertible, so $\mathbf{SH}(S)_{\mathcal{T}}(-, \mathsf{p}_{\mathcal{R},\mathcal{T}}(\iota(S^{p,q}) \wedge \mathsf{G})) \cong \mathbf{SH}(S)_{\mathcal{R}}(\iota(-), S^{p,q} \wedge \mathsf{G})$ is isomorphic to $\mathbf{SH}(S)_{\mathcal{R}}(\iota(-) \wedge S^{-p,-q}, \mathsf{G}) \cong \mathbf{SH}(S)_{\mathcal{T}}(- \wedge S^{-p-q}, \mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G})) \cong \mathbf{SH}(S)_{\mathcal{T}}(-, S^{p,q} \wedge \mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G}))$. This shows $\mathsf{p}_{\mathcal{R},\mathcal{T}}(\iota(S^{p,q}) \wedge \mathsf{G})$ and $S^{p,q} \wedge \mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G})$ are isomorphic, as desired. □

Remark 4.8: *(i) For every $\mathsf{G} \in \mathbf{SH}(S)$ the counit $\mathsf{p}_{\mathcal{R},\mathcal{T}}(\mathsf{G}) \to \mathsf{G}$, where ι is omitted from the notation, is an π_{**}-isomorphism. Using $\mathsf{p}_{\mathbf{SH}(S),\mathcal{T}}$ rather than the cellular functor introduced in [7] refines Proposition 7.3 of loc. cit.*

(ii) If $\mathsf{E} \in \mathbf{SH}(S)_{\mathcal{T}}$ and $\mathsf{F} \in \mathbf{SH}(S)$ then $\mathsf{E}_{p,q}(\mathsf{F}) \cong \mathsf{E}_{p,q}(\mathsf{p}_{\mathbf{SH}(S),\mathcal{T}}(\mathsf{F}))$ on account of the isomorphisms between $\mathbf{SH}(S)(S^{p,q}, \mathsf{E} \wedge \mathsf{F})$ and

$$\mathbf{SH}(S)_{\mathcal{T}}(S^{p,q}, \mathsf{p}_{\mathbf{SH}(S),\mathcal{T}}(\mathsf{E} \wedge \mathsf{F})) \cong \mathbf{SH}(S)_{\mathcal{T}}(S^{p,q}, \mathsf{E} \wedge \mathsf{p}_{\mathbf{SH}(S),\mathcal{T}}(\mathsf{F})).$$

In [7] it is argued that most spectra should be non-cellular. On the other hand, the E-homology of F agrees with the E-homology of some cellular spectrum. We note that many conspicuous motivic (co)homology theories are representable by cellular spectra: Landweber exact theories, including algebraic cobordism and homotopy algebraic K-theory, and also motivic (co)homology over fields of characteristic zero according to work of Hopkins and Morel.

Definition 4.9: *A homology theory on a triangulated subcategory T of $\mathbf{SH}(S)$ is a homological functor $\mathsf{T} \to \mathsf{Ab}$ which sends sums to sums. Dually, a cohomology theory on T is a homological functor $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ which sends sums to products.*

Lemma 4.10: *Suppose $\mathcal{R} \subseteq \mathcal{D}$ is closed under duals. Then every homology theory on $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ extends uniquely to a homology theory on $\mathbf{SH}(S)_{\mathcal{R}}$.*

Proof. In view of Lemma 4.3 we can apply [12, Corollary 2.3.11] which we refer to for a more detailed discussion. □

Homology and cohomology theories on $\mathbf{SH}(S)_{\mathcal{D},\mathsf{f}}$ are interchangeable according to the categorical duality equivalence $\mathbf{SH}(S)^{\mathrm{op}}_{\mathcal{D},\mathsf{f}} \cong \mathbf{SH}(S)_{\mathcal{D},\mathsf{f}}$. The same holds for every $\mathcal{R}$ for which $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ is contained in $\mathbf{SH}(S)_{\mathcal{D},\mathsf{f}}$ and closed under duality, e.g. $\mathbf{SH}(S)_{\mathcal{T},\mathsf{f}}$. We shall address the problem of representing homology theories on $\mathbf{SH}(S)$ in Section 8. Cohomology theories are always defined on $\mathbf{SH}(S)_{\mathsf{f}}$ unless specified to the contrary.

Definition 4.11: *Let $\mathsf{T} \subset \mathbf{SH}(S)$ be a triangulated subcategory closed under the smash product. A multiplicative or ring (co)homology theory on T, always understood to be commutative, is a (co)homology theory E on T together with maps $\mathbf{Z} \to E(S^{0,0})$ and $E(\mathsf{F}) \otimes E(\mathsf{G}) \to E(\mathsf{F} \wedge \mathsf{G})$ which are natural in $\mathsf{F}, \mathsf{G} \in \mathsf{T}$. These maps are subject to the usual unitality, associativity and commutativity constraints [27, pg. 269].*

Ring spectra in $\mathbf{SH}(S)$ give rise to ring homology and cohomology theories. We shall use the following bigraded version of (co)homology theories.

Definition 4.12: *Let $\mathsf{T} \subset \mathbf{SH}(S)$ be a triangulated subcategory closed under shifts by all mixed motivic spheres $S^{p,q}$. A bigraded homology theory on T is a homological functor Φ from T to Adams graded graded abelian groups taking sums to sums together with natural isomorphisms*

$$\Phi(X)_{p,q} \cong \Phi(\Sigma^{1,0}X)_{p+1,q}$$

and

$$\Phi(X)_{p,q} \cong \Phi(\Sigma^{0,1}X)_{p,q+1}$$

for all p and q such that the diagram

$$\begin{tikzcd} \Phi(X)_{p,q} \arrow[r] \arrow[d] & \Phi(\Sigma^{1,0}X)_{p+1,q} \arrow[d] \\ \Phi(\Sigma^{0,1}X)_{p,q+1} \arrow[r] & \Phi(\Sigma^{1,1}X)_{p+1,q+1} \end{tikzcd}$$

commutes.

Bigraded cohomology theories are defined likewise.

It is clear that a (co)homology theory on T is the same as a bigraded (co)homology theory on T.

5 Tate objects and flat Hopf algebroids

As in stable homotopy theory, we wish to associate a flat Hopf algebroid with suitable motivic ring spectra. By a Hopf algebroid we shall mean a cogroupoid object in the category of commutative rings over either Abelian groups, Adams graded Abelian groups or Adams graded graded Abelian groups. Throughout this section we assume E is a ring spectrum in $\mathbf{SH}(S)_{\mathcal{T}}$. We call E_{**} flat provided one - and hence both - of the canonical maps $\mathsf{E}_{**} \to \mathsf{E}_{**}\mathsf{E}$ is flat, and similarly for E_{*} and $\mathsf{E}_{*} \to \mathsf{E}_{*}\mathsf{E}$.

Lemma 5.1: *(i) If E_{**} is flat then for every motivic spectrum F the canonical map*

$$\mathsf{E}_{**}\mathsf{E} \otimes_{\mathsf{E}_{**}} \mathsf{E}_{**}\mathsf{F} \longrightarrow (\mathsf{E} \wedge \mathsf{E} \wedge \mathsf{F})_{**}$$

is an isomorphism.

(ii) If E_{*} is flat and the canonical map $\mathsf{E}_{*}\mathsf{E} \otimes_{\mathsf{E}_{*}} \mathsf{E}_{**} \to \mathsf{E}_{**}\mathsf{E}$ is an isomorphism, then for every motivic spectrum F the canonical map

$$\mathsf{E}_{*}\mathsf{E} \otimes_{\mathsf{E}_{*}} \mathsf{E}_{*}\mathsf{F} \longrightarrow (\mathsf{E} \wedge \mathsf{E} \wedge \mathsf{F})_{*}$$

is an isomorphism.

Proof. (i): By Lemma 4.7, replacing F by its Tate projection we may assume that F is a Tate object. The proof follows now along the same lines as in topology by first noting that the statement clearly holds when F is a mixed motivic sphere, and secondly that we are comparing homology theories on $\mathbf{SH}(S)_{\mathcal{T}}$ which respect sums. (ii): The two assumptions imply the assumption of (i), so there is an isomorphism

$$\mathsf{E}_{**}\mathsf{E} \otimes_{\mathsf{E}_{**}} \mathsf{E}_{**}\mathsf{F} \longrightarrow (\mathsf{E} \wedge \mathsf{E} \wedge \mathsf{F})_{**}.$$

By the second assumption the left hand side identifies with

$$(\mathsf{E}_{*}\mathsf{E} \otimes_{\mathsf{E}_{*}} \mathsf{E}_{**}) \otimes_{\mathsf{E}_{**}} \mathsf{E}_{**}\mathsf{F} \cong \mathsf{E}_{*}\mathsf{E} \otimes_{\mathsf{E}_{*}} \mathsf{E}_{**}\mathsf{F}.$$

Restricting to bidegrees which are multiples of $(2,1)$ yields the claimed isomorphism. $\square$

Corollary 5.2: *(i) If E_{**} is flat then $(\mathsf{E}_{**}, \mathsf{E}_{**}\mathsf{E})$ is canonically a flat Hopf algebroid in Adams graded graded Abelian groups and for every $\mathsf{F} \in \mathbf{SH}(S)$ the module $\mathsf{E}_{**}\mathsf{F}$ is an $(\mathsf{E}_{**}, \mathsf{E}_{**}\mathsf{E})$-comodule.*

(ii) If E_* is flat and the canonical map $\mathsf{E}_*\mathsf{E} \otimes_{\mathsf{E}_*} \mathsf{E}_{**} \to \mathsf{E}_{**}\mathsf{E}$ is an isomorphism, then $(\mathsf{E}_*, \mathsf{E}_*\mathsf{E})$ is canonically a flat Hopf algebroid in Adams graded Abelian groups and for every $\mathsf{F} \in \mathbf{SH}(S)$ the modules $\mathsf{E}_{**}\mathsf{F}$ and $\mathsf{E}_*\mathsf{F}$ are $(\mathsf{E}_*, \mathsf{E}_*\mathsf{E})$-comodules.

The second part of Corollary 5.2 is really a statement about Hopf algebroids:

Lemma 5.3: *Suppose (A_{**}, Γ_{**}) is a flat Hopf algebroid in Adams graded graded Abelian groups and the natural map $\Gamma_* \otimes_{A_*} A_{**} \to \Gamma_{**}$ is an isomorphism. Then (A_*, Γ_*) has the natural structure of a flat Hopf algebroid in Adams graded Abelian groups, and for every comodule M_{**} over (A_{**}, Γ_{**}) the modules M_{**} and M_* are (A_*, Γ_*)-comodules.*

6 The stacks of topological and algebraic cobordism

6.1 The algebraic stack of MU

Denote by $\underline{\mathsf{FG}}$ the moduli stack of one-dimensional commutative formal groups [20]. It is algebraic and a presentation is given by the canonical map $\mathsf{FGL} \to \underline{\mathsf{FG}}$, where FGL is the moduli scheme of formal group laws. The stack $\underline{\mathsf{FG}}$ carries a canonical line bundle ω and $[\mathsf{MU}_*/\mathsf{MU}_*\mathsf{MU}]$ is equivalent to the corresponding $\mathbf{G}_\mathsf{m}$-torsor $\underline{\mathsf{FG}}^s$ over $\underline{\mathsf{FG}}$.

6.2 The algebraic stack of MGL

In this section we first study the (co)homology of finite Grassmannians over regular noetherian base schemes of finite Krull dimension. Using this computational input we relate the algebraic stacks of MU and MGL. A key result is the isomorphism

$$\mathsf{MGL}_{**}\mathsf{MGL} \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*\mathsf{MU}.$$

If S is the spectrum of a field this can easily be extracted from [6, Theorem 5]. Since it is crucial for the following, we will give a rather detailed argument for the generalization.

We recall the notion of oriented motivic ring spectra formulated by Morel [18], cf. [14], [23] and [28]: If E is a motivic ring spectrum, the unit map $\mathbf{1} \to \mathsf{E}$ yields a class

$1 \in \mathsf{E}^{0,0}(\mathbf{1})$ and hence by smashing with the projective line a class $\mathsf{c}_1 \in \mathsf{E}^{2,1}(\mathbf{P}^1)$. An orientation on E is a class $\mathsf{c}_\infty \in \mathsf{E}^{2,1}(\mathbf{P}^\infty)$ that restricts to c_1. Note that KGL and MGL are canonically oriented.

For $0 \leq d \leq n$ define the ring

$$\mathsf{R}_{n,d} \equiv \mathbf{Z}[\mathsf{x}_1, \ldots, \mathsf{x}_{n-d}]/(\mathsf{s}_{d+1}, \ldots, \mathsf{s}_n), \tag{11}$$

where s_i is given by

$$1 + \sum_{n=1}^{\infty} \mathsf{s}_n t^n \equiv (1 + \mathsf{x}_1 t + \mathsf{x}_2 t^2 + \ \ldots + \mathsf{x}_{n-d} t^{n-d})^{-1} \text{ in } \mathbf{Z}[\mathsf{x}_1, \ldots, \mathsf{x}_{n-d}][[t]]^\times.$$

By assigning weight i to x_i every $\mathsf{s}_k \in \mathbf{Z}[\mathsf{x}_1, \ldots, \mathsf{x}_k]$ is homogeneous of degree k. In (11), $\mathsf{s}_j = \mathsf{s}_j(\mathsf{x}_1, \ldots, \mathsf{x}_{n-d}, 0, \ldots)$ by convention when $d+1 \leq i \leq n$. We note that $\mathsf{R}_{n,d}$ is a free $\mathbf{Z}$-module of rank $\binom{n}{d}$. For every sequence $\underline{a} = (a_1, \ldots, a_d)$ subject to the inequalities $n - d \geq a_1 \geq a_2 \geq \ldots \geq a_d \geq 0$, we set:

$$\Delta_{\underline{a}} \equiv \det \begin{pmatrix} \mathsf{x}_{a_1} & \mathsf{x}_{a_1+1} & \cdots & \mathsf{x}_{a_1+d-1} \\ \mathsf{x}_{a_2-1} & \mathsf{x}_{a_2} & \cdots & \mathsf{x}_{a_2+d-2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathsf{x}_{a_d-d+1} & \cdots & \cdots & \mathsf{x}_{a_d} \end{pmatrix}$$

Here $\mathsf{x}_0 \equiv 1$ and $\mathsf{x}_i \equiv 0$ for $i < 0$ or $i > n - d$. The Schur polynomials $\{\Delta_{\underline{a}}\}$ form a basis for $\mathsf{R}_{n,d}$ as a $\mathbf{Z}$-module. Let $\pi : \mathsf{R}_{n+1,d+1} \to \mathsf{R}_{n,d+1}$ be the unique surjective ring homomorphism where $\pi(\mathsf{x}_i) = \mathsf{x}_i$ for $1 \leq i \leq n - d - 1$ and $\pi(\mathsf{x}_{n-d}) = 0$. It is easy to see that $\pi(\Delta_{\underline{a}}) = \Delta_{\underline{a}}$ if $a_1 \leq n - d - 1$ and $\pi(\Delta_{\underline{a}}) = 0$ for $a_1 = n - d$. Hence the kernel of π is the principal ideal generated by x_{n-d}. That is,

$$\ker(\pi) = \mathsf{x}_{n-d} \cdot \mathsf{R}_{n+1,d+1}. \tag{12}$$

Moreover, let $\iota : \mathsf{R}_{n,d} \to \mathsf{R}_{n+1,d+1}$ be the unique monomorphism of abelian groups such that for every $\underline{a}$, $\iota(\Delta_{\underline{a}}) = \Delta_{\underline{a}'}$ where $\underline{a}' = (n - d, \underline{a}) \equiv (n - d, a_1, \ldots, a_d)$. Clearly we get

$$\mathrm{im}(\iota) = \ker(\pi). \tag{13}$$

Note that ι is a map of degree $n - d$. We will also need the unique ring homomorphism $f : \mathsf{R}_{n+1,d+1} \to \mathsf{R}_{n,d} = \mathsf{R}_{n+1,d+1}/(\mathsf{s}_{d+1})$ where $f(\mathsf{x}_i) = \mathsf{x}_i$ for all $1 \leq i \leq n-d$. Elementary matrix manipulations establish the equalities

$$f(\Delta_{(a_1,\ldots,a_d,0)}) = \Delta_{(a_1,\ldots,a_d)} \tag{14}$$

and

$$\iota(\Delta_{(a_1,\ldots,a_d)}) = \mathsf{x}_{n-d} \cdot \Delta_{(a_1,\ldots,a_d,0)}. \tag{15}$$

Next we discuss some geometric constructions involving Grassmannians.

For $0 \leq d \leq n$, denote by $\mathbf{Gr}_{n-d}(\mathbb{A}^n)/\mathbf{Z}$ the scheme parameterizing subvector bundles of rank $n-d$ of the trivial rank n bundle such that the inclusion of the subbundle is locally split. Similarly, $\mathsf{G}(n,d)/\mathbf{Z}$ denotes the scheme parameterizing locally free quotients of rank d of the trivial bundle of rank n; clearly $\mathsf{G}(n,d) \cong \mathbf{Gr}_{n-d}(\mathbb{A}^n)$. It is known that $\mathsf{G}(n,d)/\mathbf{Z}$ is smooth of relative dimension $d(n-d)$ and if

$$0 \longrightarrow \mathcal{K}_{n,d} \longrightarrow \mathcal{O}^n_{\mathsf{G}(n,d)} \longrightarrow \mathcal{Q}_{n,d} \longrightarrow 0 \tag{16}$$

is the universal short exact sequence of vector bundles on $\mathsf{G}(n,d)$, letting $\mathcal{K}'_{n,d}$ denote the dual of $\mathcal{K}_{n,d}$, the tangent bundle is given by

$$\mathcal{T}_{\mathsf{G}(n,d)/\mathbf{Z}} \cong \mathcal{Q}_{n,d} \otimes \mathcal{K}'_{n,d}. \tag{17}$$

The map

$$i : \mathsf{G}(n,d) \cong \mathbf{Gr}_{n-d}(\mathbb{A}^n) \hookrightarrow \mathbf{Gr}_{n-d}(\mathbb{A}^{n+1}) \cong \mathsf{G}(n+1,d+1)$$

classifying $\mathcal{K}_{n,d} \subseteq \mathcal{O}^n_{\mathsf{G}(n,d)} \hookrightarrow \mathcal{O}^{n+1}_{\mathsf{G}(n,d)}$ is a closed immersion. From (17) it follows that the normal bundle $\mathcal{N}(i)$ of i identifies with $\mathcal{K}_{n,d}$. Next consider the composition on $\mathsf{G}(n+1,d+1)$

$$\alpha : \mathcal{O}^n_{\mathsf{G}(n+1,d+1)} \hookrightarrow \mathcal{O}^{n+1}_{\mathsf{G}(n+1,d+1)} \longrightarrow \mathcal{Q}_{n+1,d+1}$$

for the inclusion into the first n factors. The complement of the support of $\mathrm{coker}(\alpha)$ is an open subscheme $U \subseteq \mathsf{G}(n+1,d+1)$ and there is a map $\pi : U \to \mathsf{G}(n,d+1)$ classifying $\alpha|_U$. It is easy to see that π is an affine bundle of dimension d, and hence

$$\pi \text{ is a motivic weak equivalence.} \tag{18}$$

An argument with geometric points reveals that $U = \mathsf{G}(n+1,d+1) \setminus i(\mathsf{G}(n,d))$. We summarize the above with a diagram:

$$\mathsf{G}(n,d) \xhookrightarrow{i} \mathsf{G}(n+1,d+1) \hookleftarrow U \xrightarrow{\pi} \mathsf{G}(n,d+1). \tag{19}$$

With these precursors out of the way we are ready to compute the (co)homology of finite Grassmannians with respect to any oriented motivic ring spectrum.

For every $0 \leq d \leq n$ there is a unique morphism of E^{**}-algebras $\varphi_{n,d} : \mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n,d} \to \mathsf{E}^{**}(\mathsf{G}(n,d))$ such that $\varphi_{n,d}(\mathsf{x}_i) = \mathsf{ch}_i(\mathcal{K}_{n,d})$ for $1 \leq i \leq n-d$. This follows from (16) and the standard calculus of Chern classes in E-cohomology. Note that $\varphi_{n,d}$ is bigraded if we assign degree $(2i,i)$ to $\mathsf{x}_i \in \mathsf{R}_{n,d}$.

Proposition 6.1: *For $0 \leq d \leq n$ the map of E^{**}-algebras*

$$\varphi_{n,d} : \mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n,d} \longrightarrow \mathsf{E}^{**}(\mathsf{G}(n,d))$$

is an isomorphism.

Proof. First observe the result holds when $d=0$ and $d=n$ since then $\mathsf{G}(n,d) = S$. By induction it suffices to show that if $\varphi_{n,d}$ and $\varphi_{n,d+1}$ are isomorphisms, then so is $\varphi_{n+1,d+1}$. To that end we contemplate the diagram:

$$\begin{array}{ccccc}
\mathsf{E}^{*-2r,*-r}(\mathsf{G}(n,d)) & \xrightarrow{\alpha} & \mathsf{E}^{**}(\mathsf{G}(n+1,d+1)) & \xrightarrow{\beta} & \mathsf{E}^{**}(\mathsf{G}(n,d+1)) \\
{\scriptstyle \varphi_{n,d}(-2r,-r)}\uparrow\cong & & {\scriptstyle \varphi_{n+1,d+1}}\uparrow & & {\scriptstyle \varphi_{n,d+1}}\uparrow\cong \\
(\mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n,d})(-2r,-r) & \xrightarrow{1\otimes\iota} & \mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n+1,d+1} & \xrightarrow{1\otimes\pi} & \mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n,d+1}
\end{array} \tag{20}$$

Here $r \equiv \mathrm{codim}(i) = n-d$ and $(-2r,-r)$ indicates a shift. The top row is part of the long exact sequence in E-cohomology associated with (19) using the Thom isomorphism $\mathsf{E}^{*+2r,*+r}(\mathsf{Th}(\mathcal{N}(i))) \cong \mathsf{E}^{**}(\mathsf{G}(n,d))$ and the fact that $\mathsf{E}^{**}(U) \cong \mathsf{E}^{**}(\mathsf{G}(n,d+1))$ by (18). The lower sequence is short exact by (13). Since $\mathcal{K}_{n+1,d+1}|_U \cong \pi^*(\mathcal{K}_{n,d+1}) \oplus \mathcal{O}_U$ we get $\beta(\varphi_{n+1,d+1}(\mathsf{x}_i)) = \beta(\mathsf{ch}_i(\mathcal{K}_{n+1,d+1})) = \mathsf{ch}_i(\mathcal{K}_{n+1,d+1}|_U) = \pi^*(\mathsf{ch}_i(\mathcal{K}_{n,d+1})) = \varphi_{n,d+1}(1 \otimes \pi(\mathsf{x}_i))$. Therefore, the right hand square in (20) commutes, β is surjective and the top row in (20) is short exact. Next we study the Gysin map α.

Since $i^*(\mathcal{K}_{n+1,d+1}) = \mathcal{K}_{n,d}$ there is a cartesian square of projective bundles:

$$\begin{array}{ccc}
\mathbf{P}(\mathcal{K}_{n,d} \oplus \mathcal{O}) & \xrightarrow{i'} & \mathbf{P}(\mathcal{K}_{n+1,d+1} \oplus \mathcal{O}) \\
{\scriptstyle p}\downarrow & & \downarrow \\
\mathsf{G}(n,d) & \xrightarrow{i} & \mathsf{G}(n+1,d+1)
\end{array}$$

By the induction hypothesis $\varphi_{n,d}$ is an isomorphism. Thus the projective bundle theorem gives

$$\mathsf{E}^{**}(\mathbf{P}(\mathcal{K}_{n,d} \oplus \mathcal{O})) \cong (\mathsf{E}^{**} \otimes_{\mathbf{Z}} \mathsf{R}_{n,d})[\mathsf{x}]/(\mathsf{x}^{r+1} + \sum_{i=1}^{r} (-1)^i \varphi_{n,d}(\mathsf{x}_i)\mathsf{x}^{r+1-i}),$$

where $\mathsf{x} \equiv \mathsf{ch}_1(\mathcal{O}_{\mathbf{P}(\mathcal{K}_{n,d}\oplus\mathcal{O})}(1)) \in \mathsf{E}^{2,1}(\mathbf{P}(\mathcal{K}_{n,d} \oplus \mathcal{O}))$. Similarly,

$$\mathsf{E}^{**}(\mathbf{P}(\mathcal{K}_{n+1,d+1} \oplus \mathcal{O})) \cong \mathsf{E}^{**}(\mathsf{G}(n+1, d+1))[\mathsf{x}']/(\mathsf{x}'^{r+1} + \sum_{i=1}^{r}(-1)^i \varphi_{n+1,d+1}(\mathsf{x}'_i)\mathsf{x}'^{r+1-i}),$$

where $\mathsf{x}' \equiv \mathsf{ch}_1(\mathcal{O}_{\mathbf{P}(\mathcal{K}_{n+1,d+1}\oplus\mathcal{O})}(1))$ and $\mathsf{x}'_i = \mathsf{ch}_i(\mathcal{K}_{n+1,d+1}) \in \mathsf{R}_{n+1,d+1}$. (We denote the canonical generators of $\mathsf{R}_{n+1,d+1}$ by x'_i in order to distinguish them from $\mathsf{x}_i \in \mathsf{R}_{n,d}$.) Recall the Thom class of $\mathcal{K}_{n,d} \cong \mathcal{N}(i)$ is constructed from

$$\mathsf{th} \equiv \mathsf{ch}_r(p^*(\mathcal{K}_{n,d}) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{K}_{n,d}\oplus\mathcal{O})}(1)) = \mathsf{x}^r + \sum_{i=1}^{r}(-1)^i \varphi_{n,d}(\mathsf{x}_i)\mathsf{x}^{r-i} \in \mathsf{E}^{2r,r}(\mathbf{P}(\mathcal{K}_{n,d} \oplus \mathcal{O})).$$

Using $i'^*(\mathsf{x}') = \mathsf{x}$ and $i^*(\varphi_{n+1,d+1}(\mathsf{x}'_i)) = \varphi_{n,d}(\mathsf{x}_i)$ for $1 \leq i \leq r$, we get that

$$\tilde{\mathsf{th}} \equiv \mathsf{x}'^r + \sum_{i=1}^{r}(-1)^i \varphi_{n+1,d+1}(\mathsf{x}'_i)\mathsf{x}'^{r-i} \in \mathsf{E}^{2r,r}(\mathbf{P}(\mathcal{K}_{n+1,d+1} \oplus \mathcal{O}))$$

satisfies $i'^*(\tilde{\mathsf{th}}) = \mathsf{th}$, and if $z : \mathsf{G}(n+1, d+1) \to \mathbf{P}(\mathcal{K}_{n+1,d+1} \oplus \mathcal{O})$ denotes the zero-section, then

$$z^*(\tilde{\mathsf{th}}) = (-1)^{n-d}\varphi_{n+1,d+1}(\mathsf{x}'_{n-d}) \in \mathsf{E}^{2(n-d),n-d}(\mathsf{G}(n+1, d+1)). \tag{21}$$

Moreover, since $i^*(\mathcal{K}_{n+1,d+1}) = \mathcal{K}_{n,d}$ we conclude that

$$\mathsf{E}^{**}(i) \circ \varphi_{n+1,d+1} = \varphi_{n,d} \circ (1 \otimes f). \tag{22}$$

By inspection of the construction of the Thom isomorphism it follows that

$$\alpha \circ \mathsf{E}^{**}(i) \text{ equals multiplication by } z^*(\tilde{\mathsf{th}}). \tag{23}$$

Now for every partition $\underline{a}$ as above we compute

$$\begin{aligned} \alpha \circ \varphi_{n,d}(\Delta_{\underline{a}}) &\overset{(14)}{=} \alpha \circ \varphi_{n,d} \circ (1 \otimes f)(\Delta_{(\underline{a},0)}) \overset{(22)}{=} \alpha \circ \mathsf{E}^{**}(i) \circ \varphi_{n+1,d+1}(\Delta_{(\underline{a},0)}) \\ &\overset{(23)}{=} z^*(\tilde{\mathsf{th}}) \cdot \varphi_{n+1,d+1}(\Delta_{(\underline{a},0)}) \overset{(21)}{=} \varphi_{n+1,d+1}((-1)^{n-d}\mathsf{x}'_{n-d} \cdot \Delta_{(\underline{a},0)}) \\ &\overset{(15)}{=} (-1)^{n-d} \cdot \varphi_{n+1,d+1}((1 \otimes \iota)(\Delta_{\underline{a}})). \end{aligned}$$

This verifies that the left hand square in (20) commutes up to a sign. Hence, by the 5-lemma, $\varphi_{n+1,d+1}$ is an isomorphism. $\square$

Since $\Sigma^\infty_+ \mathsf{G}(n,d) \in \mathbf{SH}(S)$ is dualizable and E is oriented we see that for all $0 \leq d \leq n$ the Kronecker product

$$\mathsf{E}^{**}(\mathsf{G}(n,d)) \otimes_{\mathsf{E}_{**}} \mathsf{E}_{**}(\mathsf{G}(n,d)) \longrightarrow \mathsf{E}_{**} \tag{24}$$

is a perfect pairing of finite free E_{**}-modules.

Proposition 6.2: *(i)* $\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}_d) = \mathsf{E}^{**}[[\mathsf{c}_1, \ldots, \mathsf{c}_d]]$ *where* $\mathsf{c}_i \in \mathsf{E}^{2i,i}(\mathrm{B}\mathbf{GL}_d)$ *is the ith Chern class of the tautological rank d vector bundle.*

(ii) *a)* $\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}) = \mathsf{E}^{**}[[\mathsf{c}_1, \mathsf{c}_2, \ldots]]$ *where* c_i *is the ith Chern class of the universal bundle.*

b) $\mathsf{E}_{**}(\mathrm{B}\mathbf{GL}) = \mathsf{E}_{**}[\beta_0, \beta_1, \ldots]/(\beta_0 = 1)$ *as* E_{**}*-algebras where* $\beta_i \in \mathsf{E}_{2i,i}(\mathrm{B}\mathbf{GL})$ *is the image of the dual of* $\mathsf{c}_1^i \in \mathsf{E}^{2i,i}(\mathrm{B}\mathbf{GL}_1)$*.*

(iii) *There are Thom isomorphisms* E^{**}*-modules*

$$\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}) \longrightarrow \mathsf{E}^{**}(\mathsf{MGL})$$

and of E_{**}*-algebras*

$$\mathsf{E}_{**}(\mathsf{MGL}) \longrightarrow \mathsf{E}_{**}(\mathrm{B}\mathbf{GL}).$$

Proof. Parts (i) and $(ii)a)$ are clear from the above. From (24) we conclude there are canonical isomorphisms

$$\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}_d) \longrightarrow \mathsf{Hom}_{\mathsf{E}_{**}}(\mathsf{E}_{**}(\mathrm{B}\mathbf{GL}_d), \mathsf{E}_{**}),$$

$$\mathsf{E}_{**}(\mathrm{B}\mathbf{GL}_d) \longrightarrow \mathsf{Hom}_{\mathsf{E}_{**},c}(\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}_d), \mathsf{E}_{**}).$$

The notation $\mathsf{Hom}_{\mathsf{E}_{**},c}$ refers to continuous E_{**}-linear maps with respect to the inverse limit topology on $\mathsf{E}^{**}(\mathrm{B}\mathbf{GL}_d)$ and the discrete topology on E_{**}. From this, the proofs of parts $(ii)b)$ and (iii) carry over verbatim from topology. $\square$

Corollary 6.3: *(i) The tuple* $(\mathsf{MGL}_{**}, \mathsf{MGL}_{**}\mathsf{MGL})$ *is a flat Hopf algebroid in Adams graded graded Abelian groups. For every motivic spectrum* F *the module* $\mathsf{MGL}_{**}\mathsf{F}$ *is an* $(\mathsf{MGL}_{**}, \mathsf{MGL}_{**}\mathsf{MGL})$*-comodule.*

(ii) By restriction of structure the tuple $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$ *is a flat Hopf algebroid in Adams graded Abelian groups. For every motivic spectrum* F *the modules* $\mathsf{MGL}_{**}\mathsf{F}$ *and* $\mathsf{MGL}_*\mathsf{F}$ *are* $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$*-comodules.*

Proof. (i): We note MGL is a Tate object by [7, Theorem 6.4], Remark 4.1 and MGL_{**} is flat by Proposition 6.2(iii) with $\mathsf{E} = \mathsf{MGL}$. Hence the statement follows from Corollary 5.2(i). (ii): The bidegrees of the generators β_i in Proposition 6.2 are multiples of $(2, 1)$. This implies the assumptions in Corollary 5.1(ii) hold, and the statement follows. □

The flat Hopf algebroid $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$ gives rise to the algebraic stack

$$[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}].$$

Although the grading is not required for the definition, it defines a $\mathbf{G}_\mathfrak{m}$-action on the stack and we may therefore form the quotient stack $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_\mathfrak{m}$. For $\mathsf{F} \in \mathbf{SH}(S)$, let $\mathfrak{F}(\mathsf{F})$ be the $\mathbf{G}_\mathfrak{m}$-equivariant quasi-coherent sheaf on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ associated with the comodule structure on $\mathsf{MGL}_*\mathsf{F}$ furnished by Corollary 6.3(ii). Denote by $\mathfrak{F}/\mathbf{G}_\mathfrak{m}(\mathsf{F})$ the descended quasi-coherent sheaf on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_\mathfrak{m}$.

Lemma 6.4: *(i)* $\mathsf{MGL}_{**}\mathsf{MGL} \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*\mathsf{MU} \cong \mathsf{MGL}_{**}[b_0, b_1, \ldots]/(b_0 = 1)$.

(ii) Let x, x' *be the images of the orientation on* MGL *with respect to the two natural maps* $\mathsf{MGL}_* \to \mathsf{MGL}_*\mathsf{MGL}$*. Then* $\mathsf{x}' = \sum_{i \geq 0} b_i \mathsf{x}^{i+1}$ *(where* $b_0 = 1$*).*

Proof. Here b_i is the image under the Thom isomorphism of β_i in Proposition 6.2. Part (i) follows by comparing the familiar computation of $\mathsf{MU}_*\mathsf{MU}$ with our computation of $\mathsf{MGL}_{**}\mathsf{MGL}$. For part (ii), the computations leading up to [1, Corollary 6.8] carry over unchanged. □

6.3 Formal groups and stacks

A graded formal group over an evenly graded ring A_* or more generally over an algebraic $\mathbf{G}_\mathfrak{m}$-stack is a group object in formal schemes over the base with a compatible $\mathbf{G}_\mathfrak{m}$-action such that locally in the Zariski topology it looks like $\mathrm{Spf}(R_*[\![\mathsf{x}]\!])$, as a formal scheme with $\mathbf{G}_\mathfrak{m}$-action, where x has weight -1. (Note that every algebraic $\mathbf{G}_\mathfrak{m}$-stack can be covered by affine $\mathbf{G}_\mathfrak{m}$-stacks.) This is equivalent to demanding that x has weight 0 (or any other fixed weight) by looking at the base change $R \to R[\mathsf{y}, \mathsf{y}^{-1}]$, y of weight 1. A strict graded formal group is a graded formal group together with a trivialization of the line bundle of invariant vector fields with the trivial line bundle of weight 1. The strict graded formal group associated with the formal group law over MU_* inherits a coaction of $\mathsf{MU}_*\mathsf{MU}$ compatible with the grading and the trivialization; thus it descends to a strict graded formal group over $\underline{\mathsf{FG}}^s$. As a stack, $\underline{\mathsf{FG}}^s$ is the moduli stack of formal groups with a trivialization of the line bundle of invariant vector fields, while as a $\mathbf{G}_\mathfrak{m}$-stack it is the moduli stack of strict graded formal groups. It follows that $\underline{\mathsf{FG}}$ (with trivial $\mathbf{G}_\mathfrak{m}$-action) is the moduli stack of graded formal groups. For a $\mathbf{G}_\mathfrak{m}$-stack $\mathfrak{X}$ the space of $\mathbf{G}_\mathfrak{m}$-maps to $\underline{\mathsf{FG}}$ is the space of maps from the stack quotient $\mathfrak{X}/\mathbf{G}_\mathfrak{m}$ to $\underline{\mathsf{FG}}$. Hence a graded formal group is tantamount to a formal group over $\mathfrak{X}/\mathbf{G}_\mathfrak{m}$.

An orientable theory gives rise to a strict graded formal group over the coefficients:

Lemma 6.5: *If* $\mathsf{E} \in \mathbf{SH}(S)$ *is an oriented ring spectrum satisfying the assumptions in Corollary 5.2(ii) then the corresponding strict graded formal group over* E_* *inherits a compatible* $\mathsf{E}_*\mathsf{E}$*-coaction and there is a descended strict graded formal group over the stack* $[\mathsf{E}_*/\mathsf{E}_*\mathsf{E}]$*. In particular, the flat Hopf algebroid* $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$ *acquires a well defined strict graded formal group,* $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ *a strict graded formal group and the quotient stack* $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_\mathfrak{m}$ *a formal group.*

Proof. Functoriality of $\mathsf{E}^*(\mathsf{F})$ in E and F ensures the formal group over E_* inherits an $\mathsf{E}_*\mathsf{E}$-coaction. For example, compatibility with the comultiplication of the formal group amounts to commutativity of the diagram:

$$\begin{tikzcd}
(\mathsf{E}\wedge\mathsf{E})^*(\mathbb{P}^\infty) \arrow[r] \arrow[d] & (\mathsf{E}\wedge\mathsf{E}\wedge\mathsf{E})^*(\mathbb{P}^\infty) \arrow[d] \\
(\mathsf{E}\wedge\mathsf{E})^*(\mathbb{P}^\infty\times\mathbb{P}^\infty) \arrow[r] & (\mathsf{E}\wedge\mathsf{E}\wedge\mathsf{E})^*(\mathbb{P}^\infty\times\mathbb{P}^\infty)
\end{tikzcd}$$

All maps respect gradings, so there is a graded formal group over the Hopf algebroid. Different orientations yield formal group laws which differ by a strict isomorphism, so there is an enhanced strict graded formal group over the Hopf algebroid. It induces a strict graded formal group over the $\mathbf{G}_\mathfrak{m}$-stack $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ and quotienting out by the $\mathbf{G}_\mathfrak{m}$-action yields a formal group over the quotient stack. □

For oriented motivic ring spectra E, F denote by $\varphi(\mathsf{E}, \mathsf{F})$ the strict isomorphism of formal group laws over $(\mathsf{E}\wedge\mathsf{F})_*$ from the pushforward of the formal group law over E_* to the one of the formal group law over F_* given by the orientations on $\mathsf{E}\wedge\mathsf{F}$ induced by E and F.

Lemma 6.6: *Suppose* $\mathsf{E}, \mathsf{F}, \mathsf{G}$ *are oriented spectra and let* $p\colon (\mathsf{E}\wedge\mathsf{F})_* \to (\mathsf{E}\wedge\mathsf{F}\wedge\mathsf{G})_*$*,* $q\colon (\mathsf{F}\wedge\mathsf{G})_* \to (\mathsf{E}\wedge\mathsf{F}\wedge\mathsf{G})_*$ *and* $r\colon (\mathsf{E}\wedge\mathsf{G})_* \to (\mathsf{E}\wedge\mathsf{F}\wedge\mathsf{G})_*$ *denote the natural maps. Then* $r_*\varphi(\mathsf{E}, \mathsf{G}) = \mathsf{p}_*\varphi(\mathsf{E}, \mathsf{F}) \circ q_*\varphi(\mathsf{F}, \mathsf{G})$*.*

Corollary 6.7: *If* $\mathsf{E} \in \mathbf{SH}(S)$ *is an oriented ring spectrum and satisfies the assumptions in Corollary 5.2*(i)*, there is a map of Hopf algebroids* $(\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU}) \to (\mathsf{E}_{**}, \mathsf{E}_{**}\mathsf{E})$ *such that* $\mathsf{MU}_* \to \mathsf{E}_{**}$ *classifies the formal group law on* E_{**} *and* $\mathsf{MU}_*\mathsf{MU} \to \mathsf{E}_{**}\mathsf{E}$ *the strict isomorphism* $\varphi(\mathsf{E}, \mathsf{E})$*. If* E *satisfies the assumptions in Corollary 5.2*(ii) *then this map factors through a map of Hopf algebroids* $(\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU}) \to (\mathsf{E}_*, \mathsf{E}_*\mathsf{E})$*. The induced map of stacks classifies the strict graded formal group on* $[\mathsf{E}_*/\mathsf{E}_*\mathsf{E}]$*.*

6.4 A map of stacks

Corollary 6.7 and the orientation of MGL furnish a map of flat Hopf algebroids

$$(\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU}) \longrightarrow (\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$$

such that the induced map of $\mathbf{G}_{\mathfrak{m}}$-stacks $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}] \to \underline{\mathsf{FG}}^s$ classifies the strict graded formal group on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$. Thus there is a 2-commutative diagram:

$$\begin{tikzcd} \mathsf{Spec}(\mathsf{MGL}_*) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{MU}_*) \arrow[d] \\ {[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]} \arrow[r] & \underline{\mathsf{FG}}^s \end{tikzcd} \tag{25}$$

Quotienting out by the $\mathbf{G}_{\mathfrak{m}}$-action yields a map of stacks $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_{\mathfrak{m}} \to \underline{\mathsf{FG}}$ which classifies the formal group on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_{\mathfrak{m}}$.

Proposition 6.8: *The diagram (25) is cartesian.*

Proof. Combine Corollary 2.2 and Lemma 6.4. Part (ii) of the lemma is needed to ensure that the left and right units of $(\mathsf{MU}_*, \mathsf{MU}_*\mathsf{MU})$ and $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$ are suitably compatible. □

Corollary 6.9: *The diagram*

$$\begin{tikzcd} \mathsf{Spec}(\mathsf{MGL}_*) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathsf{MU}_*) \arrow[d] \\ {[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_{\mathfrak{m}}} \arrow[r] & \underline{\mathsf{FG}} \end{tikzcd} \tag{26}$$

is cartesian.

7 Landweber exact theories

Recall the Lazard ring L is isomorphic to MU_*. For a prime p we fix a regular sequence

$$p = v_0^{(p)}, v_1^{(p)}, \ldots \in \mathsf{MU}_*$$

where $v_n^{(p)}$ has degree $2(p^n - 1)$ as explained in the introduction. An (ungraded) L-module M is Landweber exact if $(v_0^{(p)}, v_1^{(p)}, \ldots)$ is a regular sequence on M for every p. An Adams graded MU_*-module M_* is Landweber exact if the underlying ungraded module is Landweber exact as an L-module [13, Definition 2.6]. In stacks this translates as follows: An L-module M gives rise to a quasi-coherent sheaf $\mathsf{M}^\sim$ on $\mathsf{Spec}(\mathsf{L})$ and M is Landweber exact if and only if $\mathsf{M}^\sim$ is flat over $\underline{\mathsf{FG}}$ with respect to $\mathsf{Spec}(\mathsf{L}) \to \underline{\mathsf{FG}}$, see [20, Proposition 7].

Lemma 7.1: *Let M_* be an Adams graded MU_*-module and $\mathsf{M}_*^\sim$ the associated quasi-coherent sheaf on $\mathsf{Spec}(\mathsf{MU}_*)$. Then M_* is Landweber exact if and only if $\mathsf{M}_*^\sim$ is flat over $\underline{\mathsf{FG}}^s$ with respect to $\mathsf{Spec}(\mathsf{MU}_*) \to \underline{\mathsf{FG}}^s$.*

Proof. We need to prove the "only if" implication. Assume M_* is Landweber exact so that $\mathsf{M}^\sim$ has a compatible $\mathbf{G}_\mathfrak{m}$-action. Let $\mathsf{q}\colon \mathsf{Spec}(\mathsf{MU}_*) \to [\mathsf{Spec}(\mathsf{MU}_*)]/\mathbf{G}_\mathfrak{m}$ be the quotient map and $\mathsf{N}_*^\sim$ the descended quasi-coherent sheaf of $\mathsf{M}_*^\sim$ on $[\mathsf{Spec}(\mathsf{MU}_*)/\mathbf{G}_\mathfrak{m}]$. There is a canonical map $\mathsf{N}_*^\sim \to \mathsf{q}_*\mathsf{M}_*^\sim$, which is the inclusion of the weight zero part of the $\mathbf{G}_\mathfrak{m}$-action. By assumption, $\mathsf{M}_*^\sim$ is flat over $\underline{\mathsf{FG}}$, i.e. $\mathsf{q}_*\mathsf{M}_*^\sim$ is flat over $\underline{\mathsf{FG}}$. Since $\mathsf{N}_*^\sim$ is a direct summand of $\mathsf{q}_*\mathsf{M}_*^\sim$ it is flat over $\underline{\mathsf{FG}}$. Hence $\mathsf{M}_*^\sim$ is flat over $\underline{\mathsf{FG}}^s$ since there is a cartesian diagram:

$$\begin{array}{ccc} \mathsf{Spec}(\mathsf{MU}_*) & \longrightarrow & \underline{\mathsf{FG}}^s \\ \downarrow & & \downarrow \\ [\mathsf{Spec}(\mathsf{MU}_*)]/\mathbf{G}_\mathfrak{m} & \longrightarrow & \underline{\mathsf{FG}} \end{array}$$

□

Remark 7.2: *Lemma 7.1 does not hold for (ungraded) L-modules: The map $\mathsf{Spec}(\mathbf{Z}) \to \underline{\mathsf{FG}}^s$ classifying the strict formal multiplicative group over the integers is not flat, whereas the corresponding L-module $\mathbf{Z}$ is Landweber exact.*

In the following statements we view Adams graded Abelian groups as Adams graded graded Abelian groups via the line $\mathbf{Z}(2,1)$. For example an MU_*-module structure on an

Adams graded graded Abelian group M_{**} is an MU_*-module in this way. Thus $\mathsf{MGL}_{**}\mathsf{F}$ is an MU_*-module for every motivic spectrum F.

Theorem 7.3: *Suppose A_* is a Landweber exact MU_*-algebra, i.e. there is a map of commutative algebras $\mathsf{MU}_* \to \mathsf{A}_*$ in Adams graded Abelian groups such that A_* viewed as an MU_*-module is Landweber exact. Then the functor $\mathsf{MGL}_{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{A}_*$ is a bigraded ring homology theory on $\mathbf{SH}(S)$.*

Proof. By Corollary 6.8 there is a projection p from

$$\mathsf{Spec}(\mathsf{A}_*) \times_{\underline{\mathsf{FG}}^s} [\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}] \cong \mathsf{Spec}(\mathsf{A}_*) \times_{\mathsf{Spec}(\mathsf{MU}_*)} \mathsf{Spec}(\mathsf{MGL}_*)$$

to $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ such that

$$\mathsf{MGL}_*\mathsf{F} \otimes_{\mathsf{MU}_*} \mathsf{A}_* \cong \Gamma(\mathsf{Spec}(\mathsf{A}_*) \times_{\underline{\mathsf{FG}}^s} [\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}], \mathsf{p}^*\mathfrak{F}(\mathsf{F})). \tag{27}$$

(This is an isomorphism of Adams graded Abelian groups, but we won't use that fact.) The assignment $\mathsf{F} \mapsto \mathfrak{F}(\mathsf{F})$ is a homological functor since $\mathsf{F} \mapsto \mathsf{MGL}_*\mathsf{F}$ is a homological functor, and p is flat since it is the pullback of $\mathsf{Spec}(\mathsf{A}_*) \to \underline{\mathsf{FG}}^s$ which is flat by Lemma 7.1. Thus p^* is exact. Taking global sections over an affine scheme is an exact functor. Therefore, $\mathsf{F} \mapsto \Gamma(\mathsf{Spec}(\mathsf{A}_*) \times_{\underline{\mathsf{FG}}^s} [\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}], \mathsf{p}^*\mathfrak{F}(\mathsf{F}))$ is a homological functor on $\mathbf{SH}(S)$, so that by (27) $\mathsf{F} \mapsto \mathsf{MGL}_*\mathsf{F} \otimes_{\mathsf{MU}_*} \mathsf{A}_*$ is a homological functor with values in Adams graded Abelian groups. It follows that $\mathsf{F} \mapsto (\mathsf{MGL}_*\mathsf{F} \otimes_{\mathsf{MU}_*} \mathsf{A}_*)_0$, the degree zero part in the Adams graded Abelian group, is a homological functor, and it commutes with sums. Hence it is a homology theory on $\mathbf{SH}(S)$. The associated bigraded homology theory is clearly the one formulated in the theorem. Finally, the ring structure is induced by the ring structures on the homology theory represented by MGL and on A_*. $\square$

We note the proof works using $\mathfrak{F}/\mathbf{G}_{\mathsf{m}}(\mathsf{F})$ instead of $\mathfrak{F}(\mathsf{F})$; this makes the reference to Lemma 7.1 superfluous since neglecting the grading does not affect the proof.

Corollary 7.4: *The functor $\mathsf{MGL}^{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{A}_*$ is a ring cohomology theory on strongly dualizable motivic spectra.*

Proof. Applying the functor in Theorem 7.3 to the Spanier-Whitehead duals of strongly dualizable motivic spectra yields the cohomology theory on display. Its ring structure is induced by the ring structure on A_*. $\square$

Proposition 7.5: *The maps* $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}] \to \underline{\mathsf{FG}}^s$ *and* $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]/\mathbf{G}_\mathfrak{m} \to \underline{\mathsf{FG}}$ *are affine.*

Proof. Use Proposition 6.8, Corollary 6.9 and the fact that being an affine morphism can be tested after faithfully flat base change. □

Remark 7.6: *We can formulate the above reasoning in more sheaf theoretic terms: Namely, denoting by* $i\colon [\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}] \to \underline{\mathsf{FG}}^s$ *the canonical map, the Landweber exact theory is given by taking sections of* $i_*\mathfrak{F}(\mathsf{F})$ *over* $\mathsf{Spec}(\mathsf{A}_*) \to \underline{\mathsf{FG}}^s$*. It is a homology theory by Proposition 7.5 since* $\mathsf{Spec}(\mathsf{A}_*) \to \underline{\mathsf{FG}}^s$ *is flat.*

Next we give the versions of the above theorems for MU_*-modules.

Proposition 7.7: *Suppose* M_* *is an Adams graded Landweber exact* MU_**-module. Then* $\mathsf{MGL}_{**}(-)\otimes_{\mathsf{MU}_*}\mathsf{M}_*$ *is a homology theory on* $\mathbf{SH}(S)$ *and* $\mathsf{MGL}^{**}(-)\otimes_{\mathsf{MU}_*}\mathsf{M}_*$ *a cohomology theory on strongly dualizable spectra.*

Proof. The map $i\colon [\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}] \to \underline{\mathsf{FG}}^s$ is affine according to Proposition 7.5. With $p\colon \mathsf{Spec}(\mathsf{MU}_*) \to \underline{\mathsf{FG}}^s$ the canonical map, the first functor in the proposition is given by

$$\mathsf{F} \longmapsto \Gamma(\mathsf{Spec}(\mathsf{MU}_*), \mathsf{M}_* \otimes_{\mathsf{MU}_*} p^* i_* \mathfrak{F}(\mathsf{F})),$$

which is exact by assumption.

The second statement is proven by taking Spanier-Whitehead duals. □

A Landweber exact theory refers to a homology or cohomology theory constructed as in Proposition 7.7. There are periodic versions of the previous results:

Proposition 7.8: *Suppose* M *is a Landweber exact* L*-module. Then* $\mathsf{MGL}_*(-)\otimes_{\mathsf{L}}\mathsf{M}$ *is a* $(2,1)$*-periodic homology theory on* $\mathbf{SH}(S)$ *with values in ungraded Abelian groups. The same statement holds for cohomology of strongly dualizable objects. These are ring theories if* M *is a commutative* L*-algebra.*

Next we formulate the corresponding results for (highly structured) MGL-modules. This viewpoint goes back to [16] and plays an important role in our treatment, cf. sec. 9.

Proposition 7.9: *Suppose* M_* *is a Landweber exact Adams graded* MU_**-module. Then* $\mathsf{F} \mapsto \mathsf{F}_{**} \otimes_{\mathsf{MU}_*} \mathsf{M}_*$ *is a bigraded homology theory on the derived category* $\mathcal{D}_{\mathsf{MGL}}$ *of* MGL*-modules.*

Proof. The proof proceeds along a now familiar route. What follows reviews the main steps. We wish to construct a homological functor from $\mathcal{D}_{\mathsf{MGL}}$ to quasi-coherent sheaves on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$. Our first claim is that for every $\mathsf{F} \in \mathcal{D}_{\mathsf{MGL}}$ the Adams graded MGL_*-module F_* is an $(\mathsf{MGL}_*, \mathsf{MGL}_*\mathsf{MGL})$-comodule. As in Lemma 5.1,

$$\mathsf{MGL}_{**}\mathsf{MGL} \otimes_{\mathsf{MGL}_{**}} \mathsf{F}_{**} \longrightarrow (\mathsf{MGL} \wedge \mathsf{F})_{**}$$

is an isomorphism restricting to an isomorphism

$$\mathsf{MGL}_*\mathsf{MGL} \otimes_{\mathsf{MGL}_*} \mathsf{F}_* \longrightarrow (\mathsf{MGL} \wedge \mathsf{F})_*.$$

This is proven by observing it holds for "spheres" $\Sigma^{p,q}\mathsf{MGL}$, both sides are homological functors and commute with sums. This establishes the required comodule structure. Next, the proof of Proposition 7.7 using flatness of M_* viewed as a quasi-coherent sheaf on $[\mathsf{MGL}_*/\mathsf{MGL}_*\mathsf{MGL}]$ shows the functor in question is a homology theory. The remaining parts are clear. $\square$

Remark 7.10: *We shall leave the formulations of the cohomology, algebra and periodic versions of Proposition 7.9 to the reader.*

8 Representability and base change

Here we deal with the question when a motivic (co)homology theory is representable. Let $\mathcal{R}$ be a subset of $\mathbf{SH}(S)_{\mathsf{f}}$ such that $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ consists of strongly dualizable objects and is closed under smash products and duals.

First, recall the notions of unital algebraic stable homotopy categories and Brown categories from [12, Definition 1.1.4 and next paragraph]: A stable homotopy category is a triangulated category equipped with sums, a compatible closed tensor product, a set $\mathcal{G}$ of strongly dualizable objects generating the triangulated category as a localizing subcategory, and such that every cohomological functor is representable. It is unital algebraic if the tensor unit is finite (thus the objects of $\mathcal{G}$ are finite) and a Brown category if homology functors and natural transformations between them are representable.

A map between objects in a stable homotopy category is phantom if the induced map between the corresponding cohomology functors on the full subcategory of finite objects is the zero map. In case the category is unital algebraic this holds if and only if the map between the induced homology theories is the zero map.

Lemma 8.1: *The category $\mathbf{SH}(S)_{\mathcal{R}}$ is a unital algebraic stable homotopy category. The set $\mathcal{G}$ can be chosen to be (representatives of) the objects of $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$.*

Proof. The nontrivial part is to verify that every cohomological functor on $\mathbf{SH}(S)_{\mathcal{R}}$ is representable. This follows from the generalized Brown representability theorem [21]. $\square$

Lemma 8.2: *Suppose S can be covered by affines which are spectra of countable rings. Then $\mathbf{SH}(S)_{\mathcal{R}}$ is a Brown category and the category of homology functors on $\mathbf{SH}(S)_{\mathcal{R}}$ is naturally equivalent to $\mathbf{SH}(S)_{\mathcal{R}}$ modulo phantom maps.*

Proof. The first part follows by combining [12, Theorem 4.1.5] and [29, Proposition 5.5] and the second part by the definition of a Brown category. $\square$

Suppose $\mathcal{R}, \mathcal{R}'$ are as above and $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}} \subset \mathbf{SH}(S)_{\mathcal{R}',\mathsf{f}}$. Then a cohomology theory on $\mathbf{SH}(S)_{\mathcal{R}',\mathsf{f}}$ represented by F restricts to a cohomology theory on $\mathbf{SH}(S)_{\mathcal{R},\mathsf{f}}$ represented by $\mathsf{p}_{\mathcal{R}',\mathcal{R}}(\mathsf{F})$. For Landweber exact theories the following holds:

Proposition 8.3: *Suppose a Landweber exact homology theory restricted to $\mathbf{SH}(S)_{\mathcal{T},\mathsf{f}}$ is represented by a Tate spectrum E. Then E represents the theory on $\mathbf{SH}(S)$.*

Proof. Let M_* be a Landweber exact Adams graded MU_*-module affording the homology theory under consideration. By assumption there is an isomorphism on $\mathbf{SH}(S)_{\mathcal{T},\mathsf{f}}$

$$\mathsf{E}_{**}(-) \cong \mathsf{MGL}_{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{M}_*.$$

By Lemma 4.10 the isomorphism extends to $\mathbf{SH}(S)_{\mathcal{T}}$. Since MGL is cellular, an argument as in Remark 4.8 shows that both sides of the isomorphism remain unchanged when replacing a motivic spectrum by its Tate projection. $\square$

Next we consider a map $f\colon S' \to S$ of base schemes. The derived functor $\mathbf{L}f^*$, see [22, Proposition A.7.4], sends the class of compact generators $\Sigma^{p,q}\Sigma^\infty X_+$ of $\mathbf{SH}(S)$ - X a smooth S-scheme - to compact objects of $\mathbf{SH}(S')$. Hence [21, Theorem 5.1] implies $\mathbf{R}f_*$

preserves sums, and the same result shows $\mathbf{L}f^*$ preserves compact objects in general. A modification of the proof of Lemma 4.7 shows $\mathbf{R}f_*$ is an $\mathbf{SH}(S)_{\mathcal{T}}$-module functor, i.e. there is an isomorphism

$$\mathbf{R}f_*(\mathsf{F}' \wedge \mathbf{L}f^*\mathsf{G}) \cong \mathbf{R}f_*(\mathsf{F}') \wedge \mathsf{G} \tag{28}$$

in $\mathbf{SH}(S)$, which is natural in $\mathsf{F}' \in \mathbf{SH}(S')$, $\mathsf{G} \in \mathbf{SH}(S)_{\mathcal{T}}$.

Proposition 8.4: *Suppose a Landweber exact homology theory over S determined by the Adams graded MU_*-module M_* is representable by $\mathsf{E} \in \mathbf{SH}(S)_{\mathcal{T}}$. Then $\mathbf{L}f^*\mathsf{E} \in \mathbf{SH}(S')_{\mathcal{T}}$ represents the Landweber exact homology theory over S' determined by M_*.*

Proof. For an object F' of $\mathbf{SH}(S')$, adjointness, the assumption on E and (28) imply $(\mathbf{L}f^*\mathsf{E})_{**}(\mathsf{F}') = \pi_{**}(\mathsf{F}' \wedge \mathbf{L}f^*\mathsf{E})$ is isomorphic to

$$\pi_{**}(\mathbf{R}f_*(\mathsf{F}' \wedge \mathbf{L}f^*\mathsf{E})) \cong \pi_{**}(\mathbf{R}f_*\mathsf{F}' \wedge \mathsf{E}) \cong \pi_{**}(\mathsf{MGL} \wedge \mathbf{R}f_*\mathsf{F}') \otimes_{\mathsf{MU}_*} \mathsf{M}_*.$$

Again by adjointness and (28) there is an isomorphism with

$$\pi_{**}(\mathsf{MGL}_{S'} \wedge \mathsf{F}') \otimes_{\mathsf{MU}_*} \mathsf{M}_* = \mathsf{MGL}_{S',**}\mathsf{F}' \otimes_{\mathsf{MU}_*} \mathsf{M}_*.$$

□

In the next lemma we show the pullback from Proposition 8.4 respects multiplicative structures. In general one cannot expect that ring structures on the homology theory lift to commutative monoid structures on representing spectra. Instead we will consider quasi-multiplications on spectra, by which we mean maps $\mathsf{E} \wedge \mathsf{E} \to \mathsf{E}$ rendering the relevant diagrams commutative up to phantom maps.

Lemma 8.5: *Suppose a Landweber exact homology theory afforded by the Adams graded MU_*-algebra A_* is represented by a Tate object $\mathsf{E} \in \mathbf{SH}(S)_{\mathcal{T}}$ with quasi-multiplication $m \colon \mathsf{E} \wedge \mathsf{E} \to \mathsf{E}$. Then $\mathbf{L}f^*m \colon \mathbf{L}f^*\mathsf{E} \wedge \mathbf{L}f^*\mathsf{E} \to \mathbf{L}f^*\mathsf{E}$ is a quasi-multiplication and represents the ring structure on the Landweber exact homology theory determined by A_* over S'.*

Proof. Let $\phi \colon \mathsf{F}_1 \wedge \mathsf{F}_2 \to \mathsf{F}_3$ be a map in $\mathbf{SH}(S)_{\mathcal{T}}$. Let F'_i be the base change of F_i to S'. If $\mathsf{F}', \mathsf{G}' \in \mathbf{SH}(S')$ there are isomorphisms $\mathsf{F}'_{i,**}\mathsf{F}' \cong \mathsf{F}_{i,**}\mathbf{R}f_*\mathsf{F}'$ employed in the proof of

Proposition 8.4, and likewise for G'. These isomorphisms are compatible with ϕ in the sense provided by the commutative diagram:

$$\begin{tikzcd}
\mathsf{F}'_{1,**}\mathsf{F}' \otimes \mathsf{F}'_{2,**}\mathsf{G}' \arrow[r] & \mathsf{F}'_{3,**}(\mathsf{F}' \wedge \mathsf{G}') \\
 & \mathsf{F}_{3,**}(\mathbf{R}f_*(\mathsf{F}' \wedge \mathsf{G}')) \arrow[u, "\cong"'] \\
\mathsf{F}_{1,**}\mathbf{R}f_*\mathsf{F}' \otimes \mathsf{F}_{2,**}\mathbf{R}f_*\mathsf{G}' \arrow[r] \arrow[uu, "\cong"'] & \mathsf{F}_{3,**}(\mathbf{R}f_*\mathsf{F}' \wedge \mathbf{R}f_*\mathsf{G}') \arrow[u]
\end{tikzcd}$$

Applying the above to the quasi-multiplication m implies $\mathbf{L}f^*m$ represents the ring structure on the Landweber theory over S'. Hence $\mathbf{L}f^*m$ is a quasi-multiplication since the commutative diagrams exist for the homology theories, i.e. up to phantom maps. $\square$

We are ready to prove the motivic analog of Landweber's exact functor theorem.

Theorem 8.6: *Suppose* M_* *is an Adams graded Landweber exact* MU_**-module. Then there exists a Tate object* $\mathsf{E} \in \mathbf{SH}(S)_{\mathcal{T}}$ *and an isomorphism of homology theories on* $\mathbf{SH}(S)$

$$\mathsf{E}_{**}(-) \cong \mathsf{MGL}_{**}(-) \otimes_{\mathsf{MU}_*} \mathsf{M}_*.$$

In addition, if M_* *is a graded* MU_**-algebra then* E *acquires a quasi-multiplication which represents the ring structure on the Landweber exact theory.*

Proof. First, let $S = \mathsf{Spec}(\mathbf{Z})$. By Landweber exactness, see Proposition 7.7, the right hand side of the claimed isomorphism is a homology theory on $\mathbf{SH}(\mathbf{Z})$. Its restriction to $\mathbf{SH}(\mathbf{Z})_{\mathcal{T},\mathsf{f}}$ is represented by some $\mathsf{E} \in \mathbf{SH}(\mathbf{Z})_{\mathcal{T}}$ since $\mathbf{SH}(\mathbf{Z})_{\mathcal{T}}$ is a Brown category by Lemma 8.2. We may conclude in this case using Proposition 8.3. The general case follows from Proposition 8.4 since $\mathbf{L}f^*(\mathbf{SH}(\mathbf{Z})_{\mathcal{T}}) \subseteq \mathbf{SH}(S)_{\mathcal{T}}$ for $f : S \to \mathsf{Spec}(\mathbf{Z})$.

Now assume M_* is a graded MU_*-algebra. We claim that the representing spectrum $\mathsf{E} \in \mathbf{SH}(\mathbf{Z})_{\mathcal{T}}$ has a quasi-multiplication representing the ring structure on the Landweber theory: The corresponding ring cohomology theory on $\mathbf{SH}(\mathbf{Z})_{\mathcal{T},\mathsf{f}}$ can be extended to ind-representable presheaves on $\mathbf{SH}(\mathbf{Z})_{\mathcal{T},\mathsf{f}}$. Evaluating $\mathsf{E}(\mathsf{F}) \otimes \mathsf{E}(\mathsf{G}) \to \mathsf{E}(\mathsf{F} \wedge \mathsf{G})$ with $\mathsf{F} = \mathsf{G}$ the ind-representable presheaf given by E on $\mathsf{id}_{\mathsf{E}} \otimes \mathsf{id}_{\mathsf{E}}$ gives a map $(\mathsf{E}\wedge\mathsf{E})_0(-) \to \mathsf{E}_0(-)$ of homology theories. Since $\mathbf{SH}(\mathbf{Z})_{\mathcal{T}}$ is a Brown category this map lifts to a map $\mathsf{E}\wedge\mathsf{E} \to \mathsf{E}$ of spectra which is a quasi-multiplication since it represents the multiplication of the underlying homology theory. The general case follows from Lemma 8.5. $\square$

Remark 8.7: *A complex point* $\mathsf{Spec}(\mathbf{C}) \to S$ *induces a sum preserving* $\mathbf{SH}(S)_{\mathcal{T}}$*-module realization functor* $r\colon \mathbf{SH}(S) \to \mathbf{SH}$ *to the stable homotopy category. By the proof of Proposition 8.4 it follows that the topological realization of a Landweber exact theory is the corresponding topological Landweber exact theory, as one would expect.*

Proposition 8.8: *Suppose* M_* *is an Adams graded Landweber exact* MU_**-module. Then there exists an* MGL*-module* E *and an isomorphism of homology theories on* $\mathcal{D}_{\mathsf{MGL}}$

$$(\mathsf{E} \wedge_{\mathsf{MGL}} -)_{**} \cong (-)_{**} \otimes_{\mathsf{MU}_*} \mathsf{M}_*.$$

In addition, if M_* *is a graded* MU_**-algebra then* E *acquires a quasi-multiplication in* $\mathcal{D}_{\mathsf{MGL}}$ *which represents the ring structure on the Landweber exact theory.*

Proof. We indicate the proof. By Proposition 7.9 it suffices to show that the homology theory given by the right hand side of the isomorphism is representable. When the base scheme is $\mathsf{Spec}(\mathbf{Z})$ we claim that $\mathcal{D}_{\mathsf{MGL},\mathcal{T}}$ is a Brown category. In effect, $\mathbf{SH}(S)_{\mathsf{f}}$ is countable [29, Proposition 5.5] and MGL is a countable direct homotopy limit of finite spectra, so it follows that $\mathcal{D}_{\mathsf{MGL},\mathcal{T},\mathsf{f}}$ is also countable. So by [12, Theorem 4.1.5] $\mathcal{D}_{\mathsf{MGL},\mathcal{T}}$ is a Brown category. Thus there is a an object of $\mathcal{D}_{\mathsf{MGL},\mathcal{T}}$ representing the Landweber exact theory over $\mathsf{Spec}(\mathbf{Z})$. Now let $f\colon S \to \mathsf{Spec}(\mathbf{Z})$ be the unique map and $\mathbf{L}f^*_{\mathsf{MGL}}\colon \mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}}} \to \mathcal{D}_{\mathsf{MGL}_S}$ the pullback functor between MGL-modules. It has a right adjoint $\mathbf{R}f_{\mathsf{MGL},*}$. As prior to Proposition 8.4, we conclude $\mathbf{R}f_{\mathsf{MGL},*}$ preserves sums and is a $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}},\mathcal{T}}$-module functor. The proof of Proposition 8.4 shows $\mathbf{L}f^*_{\mathsf{MGL}}$ represents the Landweber theory over S.

By inferring the analog of Lemma 8.5 our claim about the quasi-multiplication is proven along the lines of the corresponding statement in Theorem 8.6. □

9 Operations and cooperations

Let A_* be a Landweber exact Adams graded MU_*-algebra and E a motivic spectrum with a quasi-multiplication which represents the corresponding Landweber exact theory. Denote by $\mathsf{E}^{\mathsf{Top}}$ the ring spectrum representing the corresponding topological Landweber exact theory. Then $\mathsf{E}^{\mathsf{Top}}_* \cong \mathsf{A}_*$, $\mathsf{E}^{\mathsf{Top}}$ is a commutative monoid in the stable homotopy category and there are no even degree nontrivial phantom maps between such topological spectra [13, Section 2.1].

Proposition 9.1: *In the above situation the following hold.*

(i) $\mathsf{E}_{**}\mathsf{E} \cong \mathsf{E}_{**} \otimes_{\mathsf{E}_*^{\mathsf{Top}}} \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}$.

(ii) E *satisfies the assumption of Corollary 5.2(ii).*

(iii) *The flat Hopf algebroid* $(\mathsf{E}_{**}, \mathsf{E}_{**}\mathsf{E})$ *is induced from* $(\mathsf{MGL}_{**}, \mathsf{MGL}_{**}\mathsf{MGL})$ *via the map* $\mathsf{MGL}_{**} \to \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{A}_* \cong \mathsf{E}_{**}$.

Proof. The isomorphism $\mathsf{E}_{**}\mathsf{F} \cong \mathsf{MGL}_{**}\mathsf{F} \otimes_{\mathsf{MU}_*} \mathsf{A}_*$ can be recasted as

$$\mathsf{E}_{**}\mathsf{F} \cong \mathsf{MGL}_{**}\mathsf{F} \otimes_{\mathsf{MGL}_*} \mathsf{MGL}_* \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}} \cong \mathsf{MGL}_{**}\mathsf{F} \otimes_{\mathsf{MGL}_*} \mathsf{E}_*$$

and

$$\mathsf{E}_{**}\mathsf{F} \cong \mathsf{MGL}_{**}\mathsf{F} \otimes_{\mathsf{MGL}_{**}} \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}} \cong \mathsf{MGL}_{**}\mathsf{F} \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**}.$$

In particular, $\mathsf{E}_{**}\mathsf{E} \cong \mathsf{MGL}_{**}\mathsf{E} \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**} \cong \mathsf{E}_{**}\mathsf{MGL} \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**}$ is isomorphic to

$$(\mathsf{MGL}_{**}\mathsf{MGL} \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**}) \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**} \cong \mathsf{E}_{**} \otimes_{\mathsf{MGL}_{**}} \mathsf{MGL}_{**}\mathsf{MGL} \otimes_{\mathsf{MGL}_{**}} \mathsf{E}_{**}. \tag{29}$$

Moreover, since $\mathsf{MGL}_{**}\mathsf{MGL} \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*\mathsf{MU}$,

$$\mathsf{E}_*^{\mathsf{Top}} \otimes_{\mathsf{MU}_*} \mathsf{MGL}_{**}\mathsf{MGL} \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}} \cong \mathsf{E}_*^{\mathsf{Top}} \otimes_{\mathsf{MU}_*} \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*\mathsf{MU} \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}}$$

is isomorphic to

$$\mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}} \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{E}_*^{\mathsf{Top}} \otimes_{\mathsf{E}_*^{\mathsf{Top}}} \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}} \cong \mathsf{E}_{**} \otimes_{\mathsf{E}_*^{\mathsf{Top}}} \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}.$$

This proves the first part of the proposition. In particular,

$$\mathsf{E}_*\mathsf{E} \cong \mathsf{E}_* \otimes_{\mathsf{E}_*^{\mathsf{Top}}} \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}} \tag{30}$$

and

$$\mathsf{E}_{**}\mathsf{E} \cong \mathsf{E}_{**} \otimes_{\mathsf{E}_*} \mathsf{E}_*\mathsf{E}. \tag{31}$$

We note that $\mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}$ is flat over $\mathsf{E}_*^{\mathsf{Top}}$ by the topological analog of (29) (this equation shows $\mathsf{Spec}(\mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}) = \mathsf{Spec}(\mathsf{E}_*^{\mathsf{Top}}) \times_{\underline{\mathsf{FG}}^s} \mathsf{Spec}(\mathsf{E}_*^{\mathsf{Top}})$). Hence by (30) $\mathsf{E}_*\mathsf{E}$ is flat over E_*. Together with (31) this is Part *(ii)* of the proposition. Part *(iii)* follows from (29). $\square$

Remark 9.2: *Let* $\mathsf{E}^{\mathsf{Top}}$ *and* $\mathsf{F}^{\mathsf{Top}}$ *be evenly graded topological Landweber exact spectra,* E *and* F *the corresponding motivic spectra. Then* $\mathsf{E} \wedge \mathsf{F}$ *is Landweber exact corresponding to the* MU_**-module* $(\mathsf{E}^{\mathsf{Top}} \wedge \mathsf{F}^{\mathsf{Top}})_*$ *(with either* MU_**-module structure).*

Theorem 9.3: *(i) The map afforded by the Kronecker product*

$$\mathsf{KGL}^{**}\mathsf{KGL} \longrightarrow \mathsf{Hom}_{\mathsf{KGL}_{**}}(\mathsf{KGL}_{**}\mathsf{KGL}, \mathsf{KGL}_{**})$$

is an isomorphism of KGL^{**}*-algebras.*

(ii) With the completed tensor product there is an isomorphism of KGL^{**}*-algebras*

$$\mathsf{KGL}^{**}\mathsf{KGL} \cong \mathsf{KGL}^{**}\widehat{\otimes}_{\mathsf{KU}^*}\mathsf{KU}^*\mathsf{KU}$$

Item (i) *and the module part of* (ii) *generalize to* $\mathsf{KGL}^{**}(\mathsf{KGL}^{\wedge i})$ *for* $i > 1$*.*

Proof. Recall $\mathsf{KU}_*\mathsf{KU}$ is free over KU_* [2] and KGL is the Landweber theory determined by the MU_*-algebra $\mathsf{MU}_* \to \mathbf{Z}[\beta, \beta^{-1}]$ which classifies the multiplicative formal group law $\mathsf{x}+\mathsf{y}-\beta\mathsf{xy}$ over $\mathbf{Z}[\beta, \beta^{-1}]$ with $|\beta| = 2$ [26, Theorem 1.2]. The corresponding topological Landweber exact theory is KU by the Conner-Floyd theorem. Thus by Proposition 9.1 (i) $\mathsf{KGL}_{**}\mathsf{KGL}$ is free over KGL_{**}. Moreover, KGL has the structure of an E_∞-motivic ring spectrum, see [9], [26], so the universal coefficient spectral sequence [7, Proposition 7.7] can be applied to the KGL-modules $\mathsf{KGL} \wedge \mathsf{KGL}$ and KGL; it converges conditionally [5], [17], and the abutment is $\mathsf{Hom}^{**}_{\mathsf{KGL-mod}}(\mathsf{KGL} \wedge \mathsf{KGL}, \mathsf{KGL}) = \mathsf{Hom}^{**}_{\mathbf{SH}(S)}(\mathsf{KGL}, \mathsf{KGL})$. But the spectral sequence degenerates since $\mathsf{KGL}_{**}\mathsf{KGL}$ is a free KGL_{**}-module, hence (i) and (ii).

The more general statement is proved along the same lines by noting the isomorphism

$$\mathsf{E}^{\mathsf{Top}}_*((\mathsf{E}^{\mathsf{Top}})^{\wedge i}) \cong \mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}} \otimes_{\mathsf{E}^{\mathsf{Top}}_*} \cdots \otimes_{\mathsf{E}^{\mathsf{Top}}_*} \mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}},$$

and similarly for the Adams graded and Adams graded graded motivic versions. □

In stable homotopy theory there is a universal coefficient spectral sequence for every Landweber exact ring theory [13, Proposition 2.21]. It appears there is no direct motivic analog: While there is a reasonable notion of evenly generated motivic spectrum as in [13, Definition 2.10] and one can show that a motivic spectrum representing a Landweber exact theory is evenly generated as in [13, Proposition 2.12], this does not have as strong consequences as in topology because the coefficient ring MGL_* is not concentrated in even degrees as MU_*, but see Theorem 9.7 below. We aim to extend the above results on homotopy algebraic K-theory to more general Landweber exact motivic spectra.

Proposition 9.4: *Suppose* M *is a Tate object and* E *an* MGL*-module. Then there is a trigraded conditionally convergent right half-plane cohomological spectral sequence*

$$\mathsf{E}_2^{a,(p,q)} = \mathsf{Ext}^{a,(p,q)}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{M}, \mathsf{E}_{**}) \Rightarrow \mathsf{E}^{a+p,q}\mathsf{M}.$$

Proof. $\mathsf{MGL} \wedge \mathsf{M}$ is a cellular MGL-module so this follows from [7, Proposition 7.10]. □

The differentials in this spectral sequence go

$$d_r\colon \mathsf{E}_r^{a,(p,q)} \longrightarrow \mathsf{E}_r^{a+r,(p-r+1,q)}.$$

Theorem 9.5: *Suppose* M_* *is a Landweber exact graded* MU_**-module concentrated in even degrees and* $\mathsf{M} \in \mathbf{SH}(S)_{\mathcal{T}}$ *represents the corresponding motivic cohomology theory. Then for* $p, q \in \mathbf{Z}$ *and* N *an* MGL*-module spectrum there is a short exact sequence*

$$0 \longrightarrow \mathsf{Ext}^{1,(p-1,q)}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{M}, \mathsf{N}_{**}) \longrightarrow \mathsf{N}^{p,q}\mathsf{M} \xrightarrow{\pi} \mathsf{Hom}^{p,q}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{M}, \mathsf{N}_{**}) \longrightarrow 0.$$

Proof. Let $\mathsf{M}^{\mathsf{Top}}$ be the topological spectrum associated with M_*. Then $\mathsf{MU}_*\mathsf{M}^{\mathsf{Top}}$ is a flat MU_*-module of projective dimension at most one [13, Propositions 2.12 and 2.16]. Hence $\mathsf{MGL}_{**}\mathsf{M} = \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*\mathsf{M}^{\mathsf{Top}}$ is a MGL_{**}-module of projective dimension at most one and consequently the spectral sequence of Proposition 9.4 degenerates at its E_2-page. This implies the derived $\lim^1$-term $\lim^1 \mathsf{E}_r^{***}$ of the spectral sequence is zero; hence it converges strongly. The assertion follows because $\mathsf{E}_\infty^{p,**} = 0$ for all $p \neq 0, 1$. □

Remark 9.6: *(i) For* $p, q \in \mathbf{Z}$*, the group of phantom maps* $\mathsf{Ph}^{p,q}(\mathsf{M}, \mathsf{N}) \subseteq \mathsf{N}^{p,q}\mathsf{M}$ *is defined as* $\{S^{p,q} \wedge \mathsf{M} \xrightarrow{\varphi} \mathsf{N} \mid$ *for all* $\mathsf{E} \in \mathbf{SH}(S)_{\mathcal{T},\mathsf{f}}$ *and* $\mathsf{E} \xrightarrow{\nu} S^{p,q} \wedge \mathsf{M} : \varphi\nu = 0\}$*. It is clear that* $\mathsf{Ph}^{p,q}(\mathsf{M}, \mathsf{N}) \subseteq \ker(\pi)$*.*

(ii) The following topological example due to Strickland shows a nontrivial Ext^1*-term. The canonical map* $\mathsf{KU}_{(p)} \to \mathsf{KU}_p$ *from* p*-local to* p*-complete unitary topological* K*-theory yields a cofiber sequence*

$$\mathsf{KU}_{(p)} \longrightarrow \mathsf{KU}_p \longrightarrow \mathsf{E} \xrightarrow{\delta} \Sigma\mathsf{KU}_{(p)}.$$

Here E *is rational and thus Landweber exact. Thus* δ *is a degree* 1 *map between even Landweber spectra.*

However, δ *is a nonzero phantom map.*

Over fields embeddable into $\mathbf{C}$ *the corresponding boundary map for the motivic Landweber spectra is likewise phantom and non-zero. Using the notion of heights for Landweber exact algebras from [20, Section 5], observe that* E *has height zero while* $\Sigma\mathsf{KU}_{(p)}$ *has height one, compare with the assumptions in Theorem 9.7 below.*

Now fix Landweber exact MU_*-algebras E_* and F_* concentrated in even degrees and a 2-commutative diagram

$$\begin{tikzcd} \mathsf{Spec}(\mathsf{F}_*) \arrow[rr, "f"] \arrow[dr, "f_\mathsf{F}"'] & & \mathsf{Spec}(\mathsf{E}_*) \arrow[dl, "f_\mathsf{E}"] \\ & \mathfrak{X} & \end{tikzcd} \tag{32}$$

where $\mathfrak{X}$ is the stack of formal groups and f_F (resp. f_E) the map classifying the formal group G_F (resp. G_E) canonically associated with the complex orientable cohomology theory corresponding to F_* (resp. E_*). This entails an isomorphism $f^*G_\mathsf{E} \cong G_\mathsf{F}$ of formal groups over $\mathsf{Spec}(\mathsf{F}_*)$. Hence the height of F_* is less or equal to the height of E_*. Let $\mathsf{E}^{\mathsf{Top}}, \mathsf{F}^{\mathsf{Top}}$ (resp. $\mathsf{E}, \mathsf{F} \in \mathbf{SH}(S)_\mathcal{T}$) be the topological (resp. motivic) spectra representing the indicated Landweber exact cohomology theory.

Theorem 9.7: *With the notation above assume* $\mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}}$ *is a projective* $\mathsf{E}^{\mathsf{Top}}_*$*-module.*

(i) The map from Theorem 9.5

$$\pi : \mathsf{F}^{**}\mathsf{E} \longrightarrow \mathsf{Hom}^{**}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{E}, \mathsf{F}_{**}) \cong \mathsf{Hom}_{\mathsf{E}^{\mathsf{Top}}_*}(\mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}}, \mathsf{F}_{**})$$

is an isomorphism.

(ii) Under the isomorphism in (i)*, the bidegree* $(0,0)$ *maps* $S^{*,*} \wedge \mathsf{E} \to \mathsf{F}$ *which respect the quasi-multiplication correspond bijectively to maps of* $\mathsf{E}^{\mathsf{Top}}_*$*-algebras*

$$\mathsf{Hom}_{\mathsf{E}^{\mathsf{Top}}_*-\mathsf{alg}}(\mathsf{E}^{\mathsf{Top}}_*\mathsf{E}^{\mathsf{Top}}, \mathsf{F}_{**}).$$

Remark 9.8: *(i) The assumptions in Theorem 9.7 hold when* $\mathsf{E}^{\mathsf{Top}} = \mathsf{KU}$ *and for certain localizations of Johnson-Wilson theories according to [2] respectively [3]. Theorem 9.7 recovers Theorem 9.3 with no mention of an* E_∞*-structure on* KGL.

(ii) The theorem applies to the quasi-multiplication $(\mathsf{E}\wedge\mathsf{E} \to \mathsf{E}) \in \mathsf{E}^{00}(\mathsf{E}\wedge\mathsf{E})$ *and shows that this is a commutative monoid structure which lifts uniquely the multiplication*

on the homology theory. For example, there is a unique structure of commutative monoid on $\mathsf{KGL}_S \in \mathbf{SH}(S)$ *representing the familiar multiplicative structure of homotopy* K*-theory, see [22] for a detailed account and an independent proof in case* $S = \mathsf{Spec}(\mathbf{Z})$.

(iii) The composite map $\alpha : \mathsf{E}_* \xrightarrow{f} \mathsf{F}_* \to \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{F}_* = \mathsf{F}_{**}$ *yields a canonical bijection between the sets* $\mathsf{Hom}_{\mathsf{E}_*^{\mathsf{Top}}-\mathsf{alg}}(\mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}, \mathsf{F}_{**})$ *and* $\{(\alpha', \varphi)\}$*, where* $\alpha' \colon \mathsf{E}_* \to \mathsf{F}_{**}$ *is a ring homomorphism and* $\varphi\colon \alpha_* G_{\mathsf{E}} \to \alpha'_* G_{\mathsf{E}}$ *a strict isomorphism of strict formal groups.*

(iv) Taking $\mathsf{F} = \mathsf{E}$ *in Theorem 9.7 and using Remark 9.6(*i*) implies that* $\mathsf{Ph}^{**}(\mathsf{E}, \mathsf{E}) = 0$*. For example, there are no nontrivial phantom maps* $\mathsf{KGL} \to \mathsf{KGL}$ *of any bidegree.*

Proof. (of Theorem 9.7): We shall apply Proposition 2.3 with $X_0 \equiv \mathsf{Spec}(\mathsf{MU}_*)$, $X \equiv \mathsf{Spec}(\mathsf{F}_*)$, $Y \equiv \mathsf{Spec}(\mathsf{E}_*)$, $f_X \equiv f_{\mathsf{F}}$ and $f_Y \equiv f_{\mathsf{F}}$, $\pi : \mathsf{Spec}(\mathsf{MU}_*) \to \mathfrak{X}$ the map classifying the universal formal group, f as given by (32) and $\alpha : X = \mathsf{Spec}(\mathsf{F}_*) \to X_0 = \mathsf{Spec}(\mathsf{MU}_*)$ corresponding to the MU_*-algebra structure $\mathsf{MU}_* \to \mathsf{F}_*$. Now by [20, Theorem 26], f_X (resp. f_Y) factors as $f_X = i_X \circ \pi_X$ (resp. $f_Y = i_Y \circ \pi_Y$) with π_X and π_Y faithfully flat and i_X and i_Y inclusions of open substacks. The map i in Proposition 2.3 is induced by f. Finally, MGL_{**} is canonically an $\mathsf{MU}_*\mathsf{MU}$-comodule algebra and the $\mathcal{O}_{\mathfrak{X}}$-algebra $\mathcal{A}$ in Proposition 2.3 corresponds to MGL_{**}, i.e. $\mathcal{A}(X_0) = \mathsf{MGL}_{**}$ and $\pi_Y^* \pi_{Y,*}\mathcal{O}_Y \in \mathsf{Qc}_Y$ to the projective $\mathsf{E}_*^{\mathsf{Top}}$-module $\mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}$. Taking into account the isomorphisms

$$\mathcal{A}(X_0) \otimes_{\mathcal{O}_{X_0}} \pi^* f_{Y,*}\mathcal{O}_Y \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{MU}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}} \cong \mathsf{MGL}_{**}\mathsf{E}$$

$$\mathcal{A}(X_0) \otimes_{\mathcal{O}_{X_0}} \alpha_*\mathcal{O}_X \cong \mathsf{MGL}_{**} \otimes_{\mathsf{MU}_*} \mathsf{F}_*^{\mathsf{Top}} \cong \mathsf{F}_{**}$$

$$\pi_Y^* \pi_{Y,*}\mathcal{O}_Y \cong \mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}$$

$$\mathcal{A}(Y) \otimes_{\mathcal{O}_Y} f_*\mathcal{O}_X \cong \mathsf{F}_{**}$$

$$\mathcal{O}_Y \cong \mathsf{E}_*^{\mathsf{Top}}$$

we obtain from Proposition 2.3

$$\mathsf{Ext}^n_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{E}, \mathsf{F}_{**}) \cong \begin{cases} 0 & n \geq 1, \\ \mathsf{Hom}_{\mathsf{E}_*^{\mathsf{Top}}}(\mathsf{E}_*^{\mathsf{Top}}\mathsf{E}^{\mathsf{Top}}, \mathsf{F}_{**}) & n = 0. \end{cases}$$

Hence (i) follows from Theorem 9.5 and (ii) by unwinding the definitions. $\square$

10 A Chern character

In what follows we define a ring map from KGL to periodized rational motivic cohomology which induces the Chern character (or regulator map) from K-theory to (higher) Chow groups in the case when the base is a smooth scheme over a field.

Let $\mathsf{M}\mathbf{Z}$ denote the integral motivic Eilenberg-MacLane ring spectrum introduced by Voevodsky [29, §6.1], cf. [8, Example 3.4]. Next we recall the canonical orientation on $\mathsf{M}\mathbf{Z}$, in particular the construction of a map $\mathbb{P}^\infty_+ \to K(\mathbf{Z}(1),2) = L((\mathbb{P}^1,\infty))$.

Recall the space $L(X)$ assigns to any U the group of proper relative cycles on $U \times_S X$ over U of relative dimension 0 which have universally integral coefficients. The line bundle $\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(n)$ carries the section $l_n \equiv T_n x_0^n + T_{n-1}x_0^{n-1}x_1 + \cdots + T_0 x_1^n$, $[T_0 : \cdots : T_n]$ homogeneous coordinates on $\mathbb{P}^n$, $[x_0 : x_1]$ coordinates on $\mathbb{P}^1$. Its zero locus is a relative divisor of degree n on $\mathbb{P}^1$ which induces a map $\mathbb{P}^n \to L(\mathbb{P}^1)$. These maps arrange to maps $\mathbb{P}^n \to L((\mathbb{P}^1,\infty))$ compatible with the inclusions $\mathbb{P}^n \to \mathbb{P}^{n+1}$ inducing a map $\varphi\colon \mathbb{P}^\infty \to K(\mathbf{Z}(1),2)$. Moreover the maps $\mathbb{P}^n \to L(\mathbb{P}^1)$ are additive for the addition $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m}$ induced by multiplication of the sections l_n. Hence φ is a map of commutative monoids and it restricts to the canonical map $\mathbb{P}^1 \to K(\mathbf{Z}(1),2)$. This establishes an orientation on $\mathsf{M}\mathbf{Z}$ with additive formal group law.

Let $\mathsf{M}\mathbf{Q}$ be the rationalization of $\mathsf{M}\mathbf{Z}$. In order to apply the spectral sequence of Proposition 9.4 to $\mathsf{M}\mathbf{Q}$ we equip it with an MGL-module structure. Note that both $\mathsf{M}\mathbf{Z}$ and $\mathsf{M}\mathbf{Q}$ have canonical E_∞-structures. Thus $\mathsf{M}\mathbf{Q} \wedge \mathsf{MGL}$ is also E_∞. As an $\mathsf{M}\mathbf{Q}$-module it has the form $\mathsf{M}\mathbf{Q}[b_1, b_2, \ldots]$. For any generator b_i we let $\iota_i\colon \Sigma^{2i,i}\mathsf{M}\mathbf{Q} \to \mathsf{M}\mathbf{Q} \wedge \mathsf{MGL}$ be the corresponding map. Taking its adjoint provides a map φ from the free $\mathsf{M}\mathbf{Q}$-E_∞-algebra on $\bigvee_{i>0} S^{2i,i}$ to $\mathsf{M}\mathbf{Q} \wedge \mathsf{MGL}$. Since everything is rational the contraction of these cells in E_∞-algebras is isomorphic to $\mathsf{M}\mathbf{Q}$. Hence we get a map $\mathsf{MGL} \to \mathsf{M}\mathbf{Q}$ in E_∞-algebras. This provides us in particular with an MGL-module structure on $\mathsf{M}\mathbf{Q}$.

Let $\mathsf{PM}\mathbf{Q}$ be the periodized rational Eilenberg-MacLane spectrum considered as an MGL-module, and $\mathsf{L}\mathbf{Q}$ the Landweber spectrum corresponding to the additive formal group law over $\mathbf{Q}$. By Remark 9.8 $\mathsf{L}\mathbf{Q}$ is a ring spectrum. We let $\mathsf{PL}\mathbf{Q}$ be the periodic version. Both $\mathsf{L}\mathbf{Q}$ and $\mathsf{PL}\mathbf{Q}$ have canonical structures of MGL-modules. Finally, let $\mathsf{PH}\mathbf{Q}$ be the periodized rational topological Eilenberg-MacLane spectrum.

Recall the map $\psi\colon \mathsf{KU}_* \to \mathsf{PH}\mathbf{Q}_*$ sending the Bott element to the canonical element in degree 2. The exponential map establishes an isomorphism from the additive formal group law over $\mathsf{PH}\mathbf{Q}_*$ to the pushforward of the multiplicative formal group law over

KU_* with respect to ψ. By Theorem 9.7 and Remark 9.8(iii) there is an induced map of ring spectra $C\colon \mathsf{KGL} \to \mathsf{PL}\mathbf{Q}$.

Theorem 10.1: *The rationalization*

$$C_{\mathbf{Q}}\colon \mathsf{KGL}_{\mathbf{Q}} \longrightarrow \mathsf{PL}\mathbf{Q}$$

of the map C is an isomorphism.

Proof. This follows directly from the fact that the rationalization of ψ is an isomorphism. □

Theorem 9.5 shows there is a short exact sequence

$$0 \longrightarrow \mathsf{Ext}^{1,(p-1,q)}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{L}\mathbf{Q}, \mathsf{M}\mathbf{Q}_{**}) \longrightarrow \mathsf{M}\mathbf{Q}^{p,q}\mathsf{L}\mathbf{Q}$$

$$\xrightarrow{\ \pi\ } \mathsf{Hom}^{p,q}_{\mathsf{MGL}_{**}}(\mathsf{MGL}_{**}\mathsf{L}\mathbf{Q}, \mathsf{M}\mathbf{Q}_{**}) \longrightarrow 0.$$

Now since $\mathsf{M}\mathbf{Q}$ carries the additive formal group law there is a natural transformation of homology theories

$$\mathsf{L}\mathbf{Q}_{**}(-) \longrightarrow \mathsf{M}\mathbf{Q}_{**}(-).$$

The methods of Theorem 9.7 apply likewise to $\mathsf{E} = \mathsf{L}\mathbf{Q}$, $\mathsf{F} = \mathsf{M}\mathbf{Q}$ and it follows that the above transformation again lifts uniquely to a map of ring spectra

$$\iota\colon \mathsf{L}\mathbf{Q} \longrightarrow \mathsf{M}\mathbf{Q}$$

which can be prolonged to a map $\mathsf{PL}\mathbf{Q} \to \mathsf{PM}\mathbf{Q}$ (denoted by the same symbol).

The composition

$$\iota \circ C : \mathsf{KGL} \longrightarrow \mathsf{M}\mathbf{Q}$$

is called the Chern character. By construction it is functorial in the base scheme with respect to the natural map $\mathbf{L}f^*\mathsf{M}\mathbf{Q}_S \to \mathsf{M}\mathbf{Q}_{S'}$ for $f\colon S' \to S$. It is easily seen that over fields the map C coincides with the usual Chern character from K-theory to higher Chow groups with respect to the identification of higher Chow groups and motivic cohomology in [30].

For smooth quasi-projective schemes over fields this is known to be an isomorphism after rationalization [4] (a map $\mathsf{E} \to \mathsf{F}$ between periodic spectra is an isomorphism if it induces isomorphisms $\mathsf{E}^{-i,0}(X) \to \mathsf{F}^{-i,0}(X)$ for all smooth schemes X over S and $i \geq 0$). By Mayer-Vietoris this holds in general for smooth schemes over fields.

Corollary 10.2: *For smooth schemes over fields the map*

$$\iota\colon \mathsf{LQ} \longrightarrow \mathsf{MQ}$$

is an isomorphism.

Corollary 10.3: *For smooth schemes over fields*

$$\mathsf{MQ}_{**}(-)$$

is the universal oriented homology theory with rational coefficients and additive formal group law.

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