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SMOOTH YAMABE INVARIANT AND SURGERY

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Abstract. We prove a surgery formula for the smooth Yamabe invariant $\sigma(M)$ of a compact manifold M. Assume that N is obtained from M by surgery of codimension at least 3. We prove the existence of a positive number Λ_n, depending only on the dimension n of M, such that

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_n\}.$$

Contents

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1. Main Result

The smooth Yamabe invariant, also called Schoen's σ-invariant, of a compact manifold M is defined as

$$\sigma(M) := \sup \inf \int_M \mathrm{Scal}^g \, dv^g,$$

where the supremum runs over all conformal classes $[g_0]$ on M and the infimum runs over all metrics g of volume 1 in $[g_0]$. The integral $\mathcal{E}(g) := \int_M \mathrm{Scal}^g \, dv^g$ is the integral of the scalar curvature of g integrated with respect the volume element of g and is known as the Einstein-Hilbert-functional.

Let $n = \dim M$. We assume that N is obtained from M by surgery of codimension $k \geq 3$. That is for a given embedding $S^k \hookrightarrow M$, with trivial normal bundle, $0 \leq k \leq n-3$, we remove a tubular neighborhood of this embedding. The resulting manifold has boundary $S^k \times S^{n-k-1}$. This boundary is glued together with the boundary of $B^{k+1} \times S^{n-k-1}$, and we thus obtain the closed smooth manifold

$$N := (M \setminus U_\epsilon(S^k)) \cup_{S^k \times S^{n-k-1}} (B^{k+1} \times S^{n-k-1}).$$

Our main result is the existence of a positive constant Λ_n depending only on n such that

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_n\}.$$

This formula unifies and generalizes previous results by Gromov-Lawson, Schoen-Yau, Kobayashi, Petean-Yun and allows many conclusions by using bordism theory.

In Section 2 we give a detailed description of the background of our result. The construction of a generalization of surgery is recalled in Section 3. Then, in Section 4 the constant Λ_n is described and it is proven to be positive. After the proof of some preliminary results on limit spaces in Section 5, we derive in Section 6 a key estimate of this article, namely an estimate for the L^2-norm of solutions of a perturbed Yamabe equation on a special kind of sphere bundle, called WS-bundle. The last section contains the proof of the main theorem, Theorem 2.3.

2. Background

We denote by $B^n(r)$ the open ball of radius r around 0 in $\mathbb{R}^n$ and we set $B^n := B^n(1)$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. By ξ^n we denote the standard flat metric on $\mathbb{R}^n$ and by σ^{n-1} the standard metric of constant sectional curvature 1 on S^{n-1}. We denote the Riemannian manifold (S^{n-1}, σ^{n-1}) by $\mathbb{S}^{n-1}$.

Let (M, g) be a Riemannian manifold of dimension n. The Yamabe operator (or Conformal Laplacian) acting on smooth functions on M is defined by

$$L^g u = a\Delta^g u + \mathrm{Scal}^g u,$$

where $a = \frac{4(n-1)}{n-2}$. Let $p = \frac{2n}{n-2}$. Define the functional J^g acting on non-zero compactly supported smooth functions on M by

$$J^g(u) := \frac{\int_M u L^g u \, dv^g}{\left(\int_M u^p \, dv^g\right)^{\frac{2}{p}}}. \tag{1}$$

If g and $\tilde{g} = f^{\frac{4}{n-2}} g = f^{p-2} g$ are conformal metrics on M then the corresponding Yamabe operators are related by

$$L^{\tilde{g}} u = f^{-\frac{n+2}{n-2}} L^g(fu) = f^{1-p} L^g(fu). \tag{2}$$

It follows that

$$J^{\tilde{g}}(u) = J^g(fu). \tag{3}$$

For a compact Riemannian manifold (M, g) the conformal Yamabe invariant is defined by

$$\mu(M, g) := \inf J^g(u) \in \mathbb{R},$$

where the infimum is taken over all non-zero smooth functions u on M. The same value of $\mu(M, g)$ is obtained by taking the infimum over positive smooth functions. From (3) it follows that the invariant μ depends only on the conformal class $[g]$ of g, and the notation $\mu(M, [g]) = \mu(M, g)$ is also used. For the standard sphere we have

$$\mu(\mathbb{S}^n) = n(n-1){\omega_n}^{2/n}, \tag{4}$$

where ω_n denotes the volume of $\mathbb{S}^n$. This value is a universal upper bound for μ.

Theorem 2.1 ([7, Lemma 3]). *The inequality*

$$\mu(M, g) \leq \mu(\mathbb{S}^n)$$

holds for any compact Riemannian manifold (M, g).

For $u > 0$ the J^g-functional is related to the Einstein-Hilbert-functional via

$$J^g(u) = \frac{\mathcal{E}(u^{4/(n-2)}g)}{\mathrm{Vol}(M, u^{4/(n-2)}g)^{\frac{n-2}{n}}}, \qquad \forall u \in C^\infty(M, \mathbb{R}^+),$$

and it follows that $\mu(M, g)$ has the alternative characterization

$$\mu(M, g) = \inf_{\tilde{g} \in [g]} \frac{\mathcal{E}(\tilde{g})}{\mathrm{Vol}(M, \tilde{g})^{\frac{n-2}{n}}}.$$

Critical points of the functional J^g are given by solutions of the Yamabe equation

$$L^g u = \mu |u|^{p-2} u$$

for some $\mu \in \mathbb{R}$.

If the inequality in Theorem 2.1 is satisfied strictly, i.e. if $\mu(M, g) < \mu(\mathbb{S}^n)$, then the infimum in the definition of $\mu(M, g)$ is attained.

Theorem 2.2 ([36, 7]). *Let M be connected. If $\mu(M, g) < \mu(\mathbb{S}^n)$ then there exists a smooth positive function u with $J^g(u) = \mu$ and $\|u\|_{L^p} = 1$. This implies that u solves* (5) *with $\mu = \mu(M, g)$. The minimizer u is unique if $\mu \leq 0$.*

The inequality $\mu(M, g) < \mu(\mathbb{S}^n)$ was shown by Aubin [7] and Schoen [31] for all compact manifolds not conformal to the standard sphere. We thus have a solution of

$$L^g u = \mu u^{p-1}, \qquad u > 0. \tag{5}$$

To explain the geometric meaning of these results we recall a few facts about the Yamabe problem, see for example [27] for a clear and detailed overview of this material. For a given compact Riemannian manifold (M, g) the Yamabe problem consists of finding a metric of constant scalar curvature in the conformal class of g. The above results yield a minimizer u for J^g. Equation (5) is equivalent to the fact that the scalar curvature of the metric $u^{4/(n-2)}g$ is everywhere equal to μ. Thus, the above Theorem, together with $\mu(M, g) < \mu(\mathbb{S}^n)$, resolves the Yamabe problem.

A conformal class $[g]$ on M contains a metric of positive scalar curvature if and only if $\mu(M,[g]) > 0$. If $M = M_1 \amalg M_2$ is a disjoint union of M_1 and M_2 and if g_i is the restriction of g to M_i, then

$$\mu(M,[g]) = \min\{\mu(M_1,[g_1]), \mu(M_2,[g_2])\}$$

if $\mu(M_1,[g_1]) \geq 0$ or $\mu(M_2,[g_2]) \geq 0$, and otherwise

$$\mu(M,[g]) = -\left(|\mu(M_1,[g_1])|^{n/2} + |\mu(M_2,[g_2])|^{n/2}\right)^{2/n}.$$

One now defines the smooth Yamabe invariant as

$$\sigma(M) := \sup \mu(M,[g]) \leq n(n-1)\omega_n^{2/n},$$

where the supremum is taken over all conformal classes $[g]$ on M.

The introduction of this invariant was originally motivated by Yamabe's attempt to find Einstein metrics on a given compact manifold, see [32] and [24]. Yamabe's idea in the early 1960's was to search for a conformal class $[g_{\mathrm{sup}}]$ that attains the supremum. The minimizer g_0 of $\mathcal{E}$ among all unit volume metrics in $[g_{\mathrm{sup}}]$ exists according to Theorem 2.2, and Yamabe hoped that the g_0 obtained with this minimax procedure would be a stationary point of $\mathcal{E}$ among all unit volume metrics (without fixed conformal class), which is equivalent to g_0 being an Einstein metric.

Yamabe's approach was very ambitious. If M is a simply connected compact 3-manifold, then an Einstein metric on M is necessarily a round metric on S^3, hence the 3-dimensional Poincaré conjecture would follow. It turned out, that his approach actually yields an Einstein metric in some special cases. For example, LeBrun [25] showed that if a compact 4-dimensional M carries a Kähler-Einstein metric with nonpositive scalar curvature, then the supremum is attained by the conformal class of this metric. Moreover, in any maximizing conformal class the minimizer is a Kähler-Einstein metric.

Compact quotients $M = \Gamma\backslash\mathbb{H}^3$ of 3-dimensional hyperbolic space $\mathbb{H}^3$ yield other examples on which Yamabe's approach yields an Einstein metric. On such quotients the supremum is attained by the hyperbolic metric on M. The proof of this statement uses Perelman's proof of the Geometrization conjecture, see [6]. In particular, $\sigma(\Gamma\backslash\mathbb{H}^3) = -6(v_\Gamma)^{2/3}$ where v_Γ is the volume of $\Gamma\backslash\mathbb{H}^3$ with respect to the hyperbolic metric.

On a general manifold, Yamabe's approach failed for various reasons. In dimension 3 and 4 obstructions against the existence of Einstein metrics are known today, see for example [23, 26]. In many cases the supremum is not attained.

R. Schoen and O. Kobayashi started to study the σ-invariant systematically in the late 1980's, [32, 33, 20, 21]. In particular, they determined $\sigma(S^{n-1} \times S^1)$ to be $\sigma(S^n) = n(n-1)\omega_n^{2/n}$. On $S^{n-1} \times S^1$ the supremum in the definition of σ is not attained. In order to commemorate Schoen's important contributions in these articles, the σ-invariant is also often called Schoen's σ-constant.

The smooth Yamabe invariant determines the existence of positive scalar curvature metrics. Namely, it follows from above that the smooth Yamabe invariant $\sigma(M)$ is positive if and only if the manifold M admits a metric of positive scalar curvature. Thus the value of $\sigma(M)$ can be interpreted as a quantitative refinement of the property of admitting a positive scalar curvature metric.

In general calculating σ is very difficult. LeBrun [23, Section 5], [25] showed that the σ-invariant of a complex algebraic surfaces is negative (resp. zero) if and only if it is of general type (resp. of Kodaira dimension 0 or 1), and the value of $\sigma(M)$ can be calculated explicitly in these cases. As already explained above, the σ-invariant can also be calculated for hyperbolic 3-manifolds, they are realized by the hyperbolic metrics.

There are many manifolds admitting a Ricci-flat metric, but no metric of positive scalar curvature, for example tori, K3-surfaces and compact connected 8-dimensional manifolds admitting metrics with holonomy Spin(7). These conditions imply $\sigma(M) = 0$, and the supremum is attained.

Conversely, Bourguignon showed that if $\sigma(M) = 0$ and if the supremum is attained by a conformal class $[g_{\mathrm{sup}}]$, then $\mathcal{E} : [g_{\mathrm{sup}}] \to \mathbb{R}$ attains its minimum in a Ricci-flat metric $g_0 \in [g_{\mathrm{sup}}]$, thus Cheeger's splitting principle implies restrictions on M. In particular, a compact quotient $\Gamma\backslash N$ of a non-abelian nilpotent Lie group N does not admit metrics of non-negative scalar curvature, but it admits a sequence of metrics g_i with $\mu(\Gamma\backslash N, g_i) \to 0$. Thus $\Gamma\backslash N$ is an example of a manifold for which $\sigma(\Gamma\backslash N) = 0$, but where the supremum is not attained.

All the examples mentioned up to here have $\sigma(M) \leq 0$. Positive smooth Yamabe invariants are even harder to determine. The calculation of non-positive $\sigma(M)$ often relies on the formula

$$|\min\{\sigma(M), 0\}|^{n/2} = \inf_g \int_M |\mathrm{Scal}^g|^{n/2} dv^g$$

where the infimum runs over all metrics on M. This formula does not distinguis between different positive values of $\sigma(M)$, and thus it is useless in the positive case.

It is conjectured [33, Page 10, lines 6–11] that all finite quotients of round spheres satisfy $\sigma(S^n/\Gamma) = (\#\Gamma)^{-2/n} Y(\mathbb{S}^n)$, but this conjecture is only verified for $\mathbb{R}P^3$ [10], namely $\sigma(\mathbb{R}P^3) = 6(\omega_3/2)^{2/3}$. The σ-invariant is also known for connected sums of $\mathbb{R}P^3$:s with $S^2 \times S^1$:s [3], for $\mathbb{C}P^2$ [16] and for connected sums of $\mathbb{C}P^2$ with several copies of $S^3 \times S^1$. With similar methods, it can also be determined for some related manifolds, but even $\sigma(S^2 \times S^2)$ is not known. To the knowledge of the authors there are no manifolds M of dimension $n \geq 5$ for which $\sigma(M) \in (0, \sigma(S^n))$ has been shown, but due to Schoen's conjecture finite quotients of spheres would be examples of such manifolds.

As explicit calculation is difficult, it is natural to use surgery theory to get estimates for more complicated examples. Several articles study the behavior of the smooth Yamabe invariant under surgery. In [15] and [34] it is proven that the existence of a positive scalar curvature metric is preserved under surgeries of codimension at least 3. In terms of the σ-invariant this means that if N is obtained from a compact manifold M by surgery of codimension at least 3 and $\sigma(M) > 0$, then $\sigma(N) > 0$.

Later Kobayashi proved in [21] that if N is obtained from M by 0-dimensional surgery, then $\sigma(N) \geq \sigma(M)$. A first consequence is an alternative deduction of $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$ using the fact that $S^{n-1} \times S^1$ is obtained from S^n by 0-dimensional surgery. More generally one sees that $\sigma(S^{n-1} \times S^1 \# \cdots \# S^{n-1} \times S^1) = \sigma(S^n)$ as this connected sum is obtained from S^n by 0-dimensional surgeries as well.

Note that it follows from what we said above that the smooth Yamabe invariant of disjoint unions $M = M_1 \amalg M_2$ satisfies

$$\sigma(M) = \min\{\sigma(M_1), \sigma(M_2)\}$$

if $\sigma(M_1) \geq 0$ or $\sigma(M_2) \geq 0$, and otherwise

$$\sigma(M) = -\left(|\sigma(M_1)|^{n/2} + |\sigma(M_2)|^{n/2}\right)^{2/n}.$$

Kobayashi's result then implies $\sigma(M_1 \# M_2) \geq \sigma(M_1 \amalg M_2)$, and thus yields a lower bound for $\sigma(M_1 \# M_2)$ in terms of $\sigma(M_1)$ and $\sigma(M_2)$.

A similar monotonicity formula for the σ-invariant was proved by Petean and Yun in [29]. They prove that $\sigma(N) \geq \min\{\sigma(M), 0\}$ if N is obtained from M by surgery of codimension at least 3. See also [**?**, Proposition 4.1], [1] for other approaches to this result. Clearly, this surgery result is particularly interesting in the case $\sigma(M) \leq 0$, and it has several fruitful applications. In particular, any simply connected compact manifold of dimension at least 5 has $\sigma(M) \geq 0$, [28]. This result was generalized to manifolds with certain types of fundamental group in [9].

In the present article we show a surgery formula that is stronger than the Gromov-Lawson/Schoen-Yau surgery formula, the Kobayashi surgery formula and the Petean-Yun surgery formula described above. Suppose that M_1 and M_2 are compact manifolds of dimension n and that W is a compact manifold of dimension k. Let embeddings $W \hookrightarrow M_1$ and $W \hookrightarrow M_2$ be given. We assume further that the normal bundles of these embeddings are trivial. Removing tubular neighborhoods of the images of W in M_1 and M_2, and gluing together these manifolds along their common boundary, we get a new compact manifold N, the connected sum of M_1 and M_2 along W. Strictly speaking N also depends on the choice of trivialization of the normal bundle. See section 3 for more details.

Surgery is a special case of this construction: if $M_2 = S^n$, $W = S^k$ and if $S^k \hookrightarrow S^n$ is the standard embedding, then N is obtained from M_1 via k-dimensional surgery along $S^k \hookrightarrow M_1$.

Theorem 2.3. *Let M_1 and M_2 be compact manifolds of dimension n. If N is obtained as a connected sum of M_1 and M_2 along a k-dimensional submanifold where $k \leq n-3$, then*

$$\sigma(N) \geq \min\{\sigma(M_1 \amalg M_2), \Lambda_{n,k}\}$$

where $\Lambda_{n,k}$ is positive, and only depends on n and k. Furthermore $\Lambda_{n,0} = \sigma(S^n)$.

From Theorem 2.1 we know that $\sigma(M) \leq \sigma(S^n)$ and thus $\sigma(M \amalg S^n) = \sigma(M)$ for all compact M. Hence, we obtain for the special case of surgery the following corollary.

Corollary 2.4. *Let M be a compact manifold of dimension n. Assume that N is obtained from M via surgery along a k-dimensional sphere W, $k \leq n-3$. We then have*

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_{n,k}\}$$

This surgery result can be combined with standard techniques of bordism theory. Such applications will be the subject of a sequel to this article, and we will only give some typical conclusions as examples.

The first corollary uses the fact that spin bordism groups and oriented bordism groups are finitely generated together with techniques developed for the proof of the h-cobordism theorem.

Corollary 2.5. *For any $n \geq 5$ there is a constant $C_n > 0$, depending only on n, such that*

$$\sigma(M) \in \{0\} \cup [C_n, \sigma(S^n)]$$

for any simply-connected compact manifold M of dimension n.

Setting $\bar{\sigma}(M) := \min\{\sigma(M), \Lambda_{n,1}, \ldots, \Lambda_{n,n-3}\}$ one sees that $\bar{\sigma}(M)$ is a bordism invariant, where the precise meaning of the expression "bordism invariant depends on some topological properties of the manifold M. For example $\bar{\sigma}(M)$ is a spin-bordism invariant of simply connected spin manifolds of dimension ≥ 5. It is an oriented bordism invariant of simply connected oriented non-spin manifolds of dimension ≥ 5. Non-simply connected manifolds can be dealt with by considering bordisms with maps to $B\pi_1(M)$.

The constants $\Lambda_{n,k}$ will be characterized in section 4. In the case $k = 0$ we prove that $\Lambda_{n,0} = \mu(\mathbb{S}^n)$ in Subsection 4.4. However an explicit calculation for $k > 0$ seems very difficult. The main problem consists in calculating the conformal Yamabe invariant of certain Riemannian products, which is in general a hard problem. See [2] for recent progress on this problem.

An analogous surgery formula holds if we replace the Conformal Laplacian by the Dirac operator, see [4] for details and applications.

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3. The connected sum along a submanifold

In this section we are going to describe how two manifolds are joined along a common submanifold with trivialized normal bundle. Strictly speaking this is a differential topological construction, but since we work with Riemannian manifolds we will make the construction adapted to the Riemannian metrics and use distance neighborhoods defined by the metrics etc.

Let (M_1, g_1) and (M_2, g_2) be complete Riemannian manifolds of dimension n. Let W be a compact manifold of dimension k, where $0 \leq k \leq n$. Let $\bar{w}_i : W \times \mathbb{R}^{n-k} \to TM_i$, $i = 1, 2$, be smooth embeddings. We assume that $\bar{w}_i$ restricted to $W \times \{0\}$ maps to the zero section of TM_i (which we identify with M_i) and thus gives an embedding $W \to M_i$. The image of this embedding is denoted by W_i'. Further we assume that $\bar{w}_i$ restrict to linear isomorphisms $\{p\} \times \mathbb{R}^{n-k} \to N_{\bar{w}_i(p,0)} W_i'$ for all $p \in W_i$, where NW_i' denotes the normal bundle of W_i' defined using g_i.

We set $w_i := \exp^{g_i} \circ \bar{w}_i$. This gives embeddings $w_i : W \times B^{n-k}(R_{\max}) \to M_i$ for some $R_{\max} > 0$ and $i = 1, 2$. We have $W_i' = w_i(W \times \{0\})$ and we define the disjoint union

$$(M, g) := (M_1 \amalg M_2, g_1 \amalg g_2),$$

and

$$W' := W_1' \amalg W_2'.$$

Let r_i be the function on M_i giving the distance to W_i'. Then $r_1 \circ w_1(p, x) = r_2 \circ w_2(p, x) = |x|$ for $p \in W$, $x \in B^{n-k}(R_{\max})$. Let r be the function on M defined by $r(x) := r_i(x)$ for $x \in M_i$, $i = 1, 2$. For $0 < \epsilon$ we set $U_i(\epsilon) := \{x \in M_i \, : \, r_i(x) < \epsilon\}$ and $U(\epsilon) := U_1(\epsilon) \cup U_2(\epsilon)$. For $0 < \epsilon < \theta$ we define

$$N_\epsilon := (M_1 \setminus U_1(\epsilon)) \cup (M_2 \setminus U_2(\epsilon))/\!\sim,$$

and

$$U_\epsilon^N(\theta) := (U(\theta) \setminus U(\epsilon))/\!\sim$$

where $\sim$ indicates that we identify $x \in \partial U_1(\epsilon)$ with $w_2 \circ w_1^{-1}(x) \in \partial U_2(\epsilon)$. Hence

$$N_\epsilon = (M \setminus U(\theta)) \cup U_\epsilon^N(\theta).$$

We say that N_ϵ is obtained from M_1, M_2 (and $\bar{w}_1$, $\bar{w}_2$) by a connected sum along W with parameter ϵ.

The diffeomorphism type of N_ϵ is independent of ϵ, hence we will usually write $N = N_\epsilon$. However, in situations when dropping the index causes ambiguities we will keep the notation N_ϵ. For example the function $r : M \to [0, \infty)$ gives a continuous function $r_\epsilon : N_\epsilon \to [\epsilon, \infty)$ whose domain depends on ϵ. It is also going to be important to keep track of the subscript ϵ on $U_\epsilon^N(\theta)$ since crucial estimates on solutions of the Yamabe equation will be carried out on this set.

The surgery operation on a manifold is a special case of taking connected sum along a submanifold. Indeed, let M be a compact manifold of dimension n and let $M_1 = M$, $M_2 = S^n$, $W = S^k$. Let $w_1 : S^k \times B^{n-k} \to M$ be an embedding defining a surgery and let $w_2 : S^k \times B^{n-k} \to S^n$ be the standard embedding. Since $S^n \setminus w_2(S^k \times B^{n-k})$ is diffeomorphic to $B^{k+1} \times S^{n-k-1}$ we have in this situation that N is obtained from M using surgery on w_1, see [22, Section VI, 9].

4. The constants $\Lambda_{n,k}$

In Section 2 we defined the conformal Yamabe invariant only for compact manifolds. There are several ways to generalize the conformal Yamabe invariant to non-compact manifolds. In this section we define two such generalizations $\mu^{(1)}$ and μ^K, and also introduce a related quantity called $\mu^{(2)}$. These invariants will be needed to define the numbers $\Lambda_{n,k}$ and to prove their positivity and to prove their positivity on our model spaces $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$.

The definition of $\mu^{(2)}$ comes from a technical difficulty in the proof of Theorem 7.1 and is only relevant in the case $k = n - 3 \geq 3$, see Remark 4.4.

4.1. The manifolds $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$.

For $c \in \mathbb{R}$ we define the metric $\eta_c^{k+1} := e^{2ct}\xi^k + dt^2$ on $\mathbb{R}^k \times \mathbb{R}$ and we write

$$\mathbb{H}_c^{k+1} := (\mathbb{R}^k \times \mathbb{R}, \eta_c^{k+1}).$$

We denote by

$$G_c := \eta_c^{k+1} + \sigma^{n-k-1}$$

the product metric on $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$. The scalar curvature of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ is $\mathrm{Scal}^{G_c} = -k(k+1)c^2 + (n-k-1)(n-k-2)$.

Proposition 4.1. *$\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}$ is conformal to $\mathbb{S}^n \setminus \mathbb{S}^k$.*

Proof. Let $\mathbb{S}^k$ be embedded in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by setting the last $n-k$ coordinates to zero and let $s := d(\cdot, \mathbb{S}^k)$ be the distance to $\mathbb{S}^k$. Here the distance is meant as the intrinsic distance in $\mathbb{S}^n$. Then the function $\sin s$ is smooth and positive on $S^n \setminus S^k$. The points of maximal distance $\pi/2$ to $\mathbb{S}^k$ lie on an $(n-k-1)$-sphere, denoted by $(\mathbb{S}^k)^\perp$. On $\mathbb{S}^n \setminus (\mathbb{S}^k \cup (\mathbb{S}^k)^\perp)$ the round metric is

$$\sigma^n = (\cos s)^2 \sigma^k + ds^2 + (\sin s)^2 \sigma^{n-k-1}.$$

Substitute $s \in (0, \pi/2)$ by $t \in (0, \infty)$ such that $\sinh t = \cot s$. Then $\cosh t = (\sin s)^{-1}$ and $\cosh t \, dt = -(\sin s)^{-2} \, ds$, so σ^n is conformal to

$$(\sin s)^{-2} \sigma^n = (\sinh t)^2 \sigma^k + dt^2 + \sigma^{n-k-1}.$$

Here we see that the first two terms give a metric

$$(\sinh t)^2 \sigma^k + dt^2$$

on $S^k \times (0, \infty)$. This is just the standard metric on $\mathbb{H}_1^{k+1} \setminus \{p_0\}$ where $t = d(\cdot, p_0)$, written in polar normal coordinates. In the case $k \geq 1$ it is evident that the conformal diffeomorphism $\mathbb{S}^n \setminus (\mathbb{S}^k \cup (\mathbb{S}^k)^\perp) \to (\mathbb{H}_1^{k+1} \setminus \{p_0\}) \times \mathbb{S}^{n-k-1}$ extends to a conformal diffeomorphism $\mathbb{S}^n \setminus \mathbb{S}^k \to \mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}$.

In the case $k = 0$ we equip s and t with a sign, that is we let $s > 0$ and $t > 0$ on one of the components of $\mathbb{S}^n \setminus (\mathbb{S}^0 \cup (\mathbb{S}^0)^\perp)$, and $s < 0$ and $t < 0$ on the other component. The functions s and t are then smooth on $\mathbb{S}^n \setminus \mathbb{S}^0$ and take values $s \in (-\pi/2, \pi/2)$ and $t \in \mathbb{R}$. Then the argument is the same as above. ☐

4.2. **Definition of $\Lambda_{n,k}$.** Let (N, h) be a Riemannian manifold of dimension n. For $i = 1, 2$ we let $\Omega^{(i)}(N, h)$ be the set of non-negative C^2 functions u which solve the Yamabe equation

$$L^h u = \mu u^{p-1} \tag{6}$$

for some $\mu = \mu(u) \in \mathbb{R}$ and satisfy

- $u \not\equiv 0$,
- $\|u\|_{L^p(N)} \leq 1$,
- $u \in L^\infty(N)$,

together with

- $u \in L^2(N)$, for $i = 1$,

or

- $\mu(u) \|u\|_{L^\infty(N)}^{p-2} \geq \frac{(n-k-2)^2(n-1)}{8(n-2)}$, for $i = 2$.

For $i = 1, 2$ we set

$$\mu^{(i)}(N, h) := \inf_{u \in \Omega^{(i)}(N,h)} \mu(u).$$

In particular, if $\Omega^{(i)}(N, h)$ is empty then $\mu^{(i)}(N, h) = \infty$.

Definition 4.2. For integers $n \geq 3$ and $0 \leq k \leq n - 3$ let

$$\Lambda_{n,k}^{(i)} := \inf_{c \in [-1,1]} \mu^{(i)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

and

$$\Lambda_{n,k} := \min \left\{ \Lambda_{n,k}^{(1)}, \Lambda_{n,k}^{(2)} \right\}.$$

Note that the infimum could just as well be taken over $c \in [0, 1]$ since $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ and $\mathbb{H}_{-c}^{k+1} \times \mathbb{S}^{n-k-1}$ are isometric. We are going to prove that these constants are positive.

Theorem 4.3. *For all $n \geq 3$ and $0 \leq k \leq n - 3$, we have $\Lambda_{n,k} > 0$.*

To prove Theorem 4.3 we have to prove that $\Lambda_{n,k}^{(1)} > 0$ and that $\Lambda_{n,k}^{(2)} > 0$. This is the object of the following two subsections. In the final subsection we prove that $\Lambda_{n,0} = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$.

Remark 4.4. Suppose that either $k \le n-4$ or $k = n-3 \le 2$. With similar methods as in Section 6 one can show that under these dimension restrictions any L^p solution of (6) on the model spaces is also L^2. This implies that $\Lambda^{(2)}_{n,k} \ge \Lambda^{(1)}_{n,k}$ in these dimensions, and hence

$$\Lambda_{n,k} = \Lambda^{(1)}_{n,k}.$$

In the case $k = n-3 \ge 4$ there are L^p-solutions of (6) on $\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}$ which are not L^2.

4.3. **Proof of $\Lambda^{(1)}_{n,k} > 0$.** The proof proceeds in several steps. We first introduce a conformal Yamabe invariant for non-compact manifolds and show that it gives a lower bound for $\mu^{(1)}$. We will conclude by studying this conformal invariant.

Let (N,h) be a Riemannian manifold which is not necessarily compact or complete. We define the conformal Yamabe invariant μ^{K} of (N,h) following Kim [19] as

$$\mu^{\mathrm{K}}(N,h) := \inf J^h(u)$$

where J^h is defined in (1) and the infimum runs over all non-zero compactly supported smooth functions u on N. If h and $\tilde{h}$ are conformal metrics on N it follows from (3) that $\mu^{\mathrm{K}}(N,h) = \mu^{\mathrm{K}}(N,\tilde{h})$.

Lemma 4.5. *Let $0 \le k \le n-3$. Then*

$$\mu^{(1)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}) \ge \mu^{\mathrm{K}}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$$

for all $c \in \mathbb{R}$.

Proof. Suppose that $u \in \Omega^{(1)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$ is a solution of (6) on $\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}$ with $\mu = \mu(u)$ close to $\mu^{(1)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$. Let χ_α be a cut-off function on $\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}$ depending only on the distance r to a fixed point, such that $\chi_\alpha(r) = 1$ for $r \le \alpha$, $\chi_\alpha(r) = 0$ for $r \ge \alpha + 2$, and $|d\chi_\alpha| \le 1$. We are going to see that

$$\lim_{\alpha\to\infty} J^{G_c}(\chi_\alpha u) = \mu \|u\|^{p-2}_{L^p(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})} \le \mu. \tag{7}$$

Integrating by parts and using Equation (6) we get

$$\begin{aligned}\int_{\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}} (\chi_\alpha u) L^{G_c}(\chi_\alpha u)\, dv^{G_c} &= \int_{\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}} \chi_\alpha^2 u L^{G_c} u \, dv^{G_c} \\ &\quad + a \int_{\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}} |d\chi_\alpha|^2 u^2 \, dv^{G_c} \\ &= \mu \int_{\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}} \chi_\alpha^2 u^p \, dv^{G_c} \\ &\quad + a \int_{\mathrm{Supp}(d\chi_\alpha)} |d\chi_\alpha|^2 u^2 \, dv^{G_c}.\end{aligned}$$

Since $u \in L^2(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$ and $|d\chi_\alpha| \le 1$ the last integral goes to zero as $\alpha \to \infty$ and we conclude that

$$\lim_{\alpha\to\infty} \int_{\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}} (\chi_\alpha u) L^{G_c}(\chi_\alpha u)\, dv^{G_c} = \mu \|u\|^p_{L^p(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})}.$$

Going back to the definition of J^{G_c} we easily get (7) and Lemma 4.5 follows. $\square$

We define

$$\Lambda^{\mathrm{K}}_{n,k} := \inf_{c\in[-1,1]} \mu^{\mathrm{K}}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}).$$

Then Lemma 4.5 tells us that $\Lambda^{(1)}_{n,k} \geq \Lambda^{\mathrm{K}}_{n,k}$, so we are done if we prove that $\Lambda^{\mathrm{K}}_{n,k} > 0$. To do this we need two lemmas.

Lemma 4.6. *Let* $0 \leq k \leq n-3$. *Then*

$$\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}) = \mu(\mathbb{S}^n).$$

Proof. The inequality $\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}) \leq \mu(\mathbb{S}^n)$ is completely analogous to [7, Lemma 3]. As we do not need this inequality later, we skip the proof. The opposite inequality $\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}) \geq \mu(\mathbb{S}^n)$ can either be derived from results in [19] or proven directly with the following simple cut-off argument.

Proposition 4.1 together with the conformal invariance of μ^{K} tells us that

$$\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}) = \mu^{\mathrm{K}}(\mathbb{S}^n \setminus \mathbb{S}^k).$$

Now, if u is compactly supported in $\mathbb{S}^n \setminus \mathbb{S}^k$, then u can be seen as a smooth function on $\mathbb{S}^n$. Let ϵ be a small positive number and choose u such that

$$\frac{\int_{\mathbb{S}^n\setminus\mathbb{S}^k} uL^{\sigma^n} u \, dv^{\sigma^n}}{\|u\|^2_{L^p(\mathbb{S}^n\setminus\mathbb{S}^k)}} \leq \mu^{\mathrm{K}}(\mathbb{S}^n \setminus \mathbb{S}^k) + \epsilon.$$

Then, by definition of $\mu(\mathbb{S}^n)$, we have

$$\begin{aligned}
\mu(\mathbb{S}^n) &\leq \frac{\int_{\mathbb{S}^n} uL^{\sigma^n} u \, dv^{\sigma^n}}{\|u\|^2_{L^p(\mathbb{S}^n)}} \\
&= \frac{\int_{\mathbb{S}^n\setminus\mathbb{S}^k} uL^{\sigma^n} u \, dv^{\sigma^n}}{\|u\|^2_{L^p(\mathbb{S}^n\setminus\mathbb{S}^k)}} \\
&\leq \mu^{\mathrm{K}}(\mathbb{S}^n \setminus \mathbb{S}^k) + \epsilon \\
&= \mu^{\mathrm{K}}(\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}) + \epsilon.
\end{aligned}$$

Lemma 4.6 follows since we can take ϵ to be arbitrarily small. □

Lemma 4.7. *Let* $0 \leq k \leq n-2$ *and* $0 < c_0 \leq c_1$. *Then*

$$\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_{c_0} \times \mathbb{S}^{n-k-1}) \geq \left(\frac{c_0}{c_1}\right)^{\frac{2(n-k-1)}{n}} \mu^{\mathrm{K}}(\mathbb{H}^{k+1}_{c_1} \times \mathbb{S}^{n-k-1}).$$

Proof. Let $c > 0$. Setting $s = ct + \ln c$ we see that

$$G_c = e^{2ct}\xi^k + dt^2 + \sigma^{n-k-1} = \frac{1}{c^2}\left(e^{2s}\xi^k + ds^2\right) + \sigma^{n-k-1}.$$

Hence G_c is conformal to the metric

$$\tilde{G}_c := e^{2s}\xi^k + ds^2 + c^2\sigma^{n-k-1}$$

and by the conformal invariance of μ^{K} we get that

$$\mu^{\mathrm{K}}(\mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1}) = \mu^{\mathrm{K}}(\mathbb{R}^k \times \mathbb{R} \times S^{n-k-1}, \tilde{G}_{c_i})$$

for $i = 0, 1$. In these coordinates we easily compute that $\mathrm{Scal}^{\tilde{G}_{c_0}} \geq \mathrm{Scal}^{\tilde{G}_{c_1}}$, $|du|^2_{\tilde{G}'_{c_0}} \geq |du|^2_{\tilde{G}_{c_1}}$, and $dv^{\tilde{G}_{c_0}} = \left(\frac{c_0}{c_1}\right)^{n-k-1} dv^{\tilde{G}_{c_1}}$. We conclude that

$$J^{\tilde{G}_{c_0}}(u) \geq \left(\frac{c_0}{c_1}\right)^{\frac{2(n-k-1)}{n}} J^{\tilde{G}_{c_1}}(u)$$

for all functions u on $\mathbb{R}^k \times \mathbb{R} \times S^{n-k-1}$ and Lemma 4.7 follows. □

If we set $c_1 = 1$ and use Lemma 4.6 together with (4) we get the following result.

Corollary 4.8. *For $c_0 > 0$ we have*

$$\inf_{c \in [c_0, 1]} \mu^{\mathrm{K}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq n(n-1){\omega_n}^{2/n} {c_0}^{\frac{4}{n}}.$$

Finally, we are ready to prove that $\Lambda^{\mathrm{K}}_{n,k}$ is positive.

Theorem 4.9. *Let $0 \leq k \leq n-3$. Then $\Lambda^{\mathrm{K}}_{n,k} > 0$.*

Proof. Choose $c_0 > 0$ small enough so that $\mathrm{Scal}^{G_{c_0}} > 0$. We then have $\mathrm{Scal}^{G_c} \geq \mathrm{Scal}^{G_{c_0}}$ for all $c \in [0, c_0]$. Hence

$$\mu^{\mathrm{K}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq \inf \frac{\int_{\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}} \left(a|du|^2_{G_c} + \mathrm{Scal}^{G_{c_0}} u^2\right) dv^{G_c}}{\|u\|^2_{L^p(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})}}.$$

By Hebey [17, Theorem 4.6, page 64], there exists a constant $A > 0$ such that for all $c \in [0, c_0]$ and all smooth non-zero functions u compactly supported in $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ we have

$$\|u\|^2_{L^p(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})} \leq A \int_{\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}} \left(|du|^2_{G_c} + u^2\right) dv^{G_c}.$$

This implies that

$$\mu^{\mathrm{K}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq \frac{1}{A} \min\left\{a, \mathrm{Scal}^{G_{c_0}}\right\} > 0$$

for all $c \in [0, c_0]$, and together with Lemma 4.7 we obtain that

$$\inf_{c \in [0,1]} \mu^{\mathrm{K}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) > 0.$$

Since $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ and $\mathbb{H}_{-c}^{k+1} \times \mathbb{S}^{n-k-1}$ are isometric we have

$$\Lambda^{\mathrm{K}}_{n,k} = \inf_{c \in [-1,1]} \mu^{\mathrm{K}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) > 0.$$

This ends the proof of Theorem 4.9. □

As an immediate consequence we obtain that $\Lambda^{(1)}_{n,k}$ is positive.

Corollary 4.10. *Let $0 \leq k \leq n-3$. Then $\Lambda^{(1)}_{n,k} > 0$.*

4.4. **Proof of $\Lambda^{(2)}_{n,k} > 0$.**

Theorem 4.11. *Let $0 \le k \le n-3$. Then $\Lambda^{(2)}_{n,k} > 0$.*

Proof. We prove this by contradiction. Assume that there exists a sequence (c_i) of $c_i \in [-1,1]$ for which $\mu_i := \mu^{(2)}(\mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1})$ tends to a limit $l \le 0$ as $i \to \infty$. After removing the indices i for which μ_i is infinite we get for every i a solution $u_i \in \Omega^2(\mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1})$ of the equation

$$L^{G_{c_i}} u_i = \mu_i u_i^{p-1}.$$

By definition of $\Omega^{(2)}(\mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1})$ we have

$$\frac{(n-k-2)^2(n-1)}{8(n-2)} \le \mu_i \|u_i\|^{p-2}_{L^\infty}, \tag{8}$$

which implies that $\mu_i > 0$. We conclude that $l := \lim_i \mu_i = 0$. We cannot assume that $\|u_i\|_{L^\infty}$ is attained but we can choose points $x_i \in \mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1}$ such that $u_i(x_i) \ge \frac{1}{2}\|u_i\|_{L^\infty}$. Moreover, we can compose the functions u_i with isometries so that all the x_i are the same point x. From (8) we get

$$\frac{1}{2}\left(\frac{(n-k-2)^2(n-1)}{8(n-2)\mu_i}\right)^{\frac{1}{p-2}} \le u_i(x).$$

We define $m_i := u_i(x)$. Since $\lim_{i\to\infty} \mu_i = 0$ we have $\lim_{i\to\infty} m_i = \infty$. Restricting to a subsequence we can assume that $c := \lim_i c_i \in [-1,1]$ exists. Define $\tilde{g}_i := m_i^{\frac{4}{n-2}} G_{c_i}$. We apply Lemma 5.1 with $\alpha = 1/i$, $(V,\gamma_\alpha) = \mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1}$, $(V,\gamma_0) = \mathbb{H}^{k+1}_{c} \times \mathbb{S}^{n-k-1}$, $q_\alpha = x_i = x$, and $b_\alpha = m_i^{\frac{2}{n-2}}$. For $r > 0$ we obtain a diffeomorphism

$$\Theta_i : B^n(r) \to B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r)$$

such that $\Theta_i^*(\tilde{g}_i)$ tends to the flat metric ξ^n on $B^n(r)$. We let $\tilde{u}_i := m_i^{-1} u_i$. By (2) we then have

$$L^{\tilde{g}_i} \tilde{u}_i = \mu_i \tilde{u}_i^{p-1}$$

on $B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)$ and

$$\begin{aligned} \int_{B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)} \tilde{u}_i^p \, dv^{\tilde{g}_i} &= \int_{B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)} u_i^p \, dv^{G_{c_i}} \\ &\le \int_N u_i^p dv^{G_{c_i}} \\ &\le 1. \end{aligned}$$

Here we used $dv^{\tilde{g}_i} = m_i^p \, dv^{G_{c_i}}$. The last inequality comes from the fact that any function in $\Omega^{(2)}(\mathbb{H}^{k+1}_{c_i} \times \mathbb{S}^{n-k-1})$ has L^p-norm smaller than 1. Since

$$\Theta_i : (B^n(r), \Theta_i^*(\tilde{g}_i)) \to (B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r), \tilde{g}_i)$$

is an isometry we can consider $\tilde{u}_i$ as a solution of

$$L^{\Theta_i^*(\tilde{g}_i)} \tilde{u}_i = \mu_i \tilde{u}_i^{p-1}$$

on $B^n(r)$ with $\int_{B^n(r)} \tilde{u}_i^p \, dv^{\Theta_i^*(\tilde{g}_i)} \le 1$. Since $\|\tilde{u}_i\|_{L^\infty(B^n(r))} = |\tilde{u}_i(0)| = 1$ we can apply Lemma 5.2 with $V = \mathbb{R}^n$, $\alpha = 1/i$, $g_\alpha = \Theta_i^*(\tilde{g}_i)$, and $u_\alpha = \tilde{u}_i$ (we can apply this lemma since each compact set of $\mathbb{R}^n$ is contained in some ball $B^n(r)$). This shows that there exists a non-negative function $u \not\equiv 0$ (since $u(0) = 1$) of class C^2 on $(\mathbb{R}^n, \xi^n)$ satisfying

$$L^{\xi^n} u = a\Delta^{\xi^n} u = \bar{\mu} u^{p-1}$$

where $\bar{\mu} = 0$. By (12) we further have

$$\int_{B^n(r)} u^p \, dv^{\xi^n} = \lim_{i \to 0} \int_{B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r)} u_i^p \, dv^{G_{c_i}} \leq 1$$

for any $r > 0$. In particular,

$$\int_{\mathbb{R}^n} u^p \, dv^{\xi^n} \leq 1.$$

Lemma 5.3 below then implies the contradiction $0 = \bar{\mu} \geq \mu(\mathbb{S}^n)$. This proves that $\Lambda^{(2)}_{n,k}$ is positive. □

4.5. **The constant** $\Lambda_{n,0}$**.** Now we show that $\Lambda_{n,0} = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$. The corresponding model spaces $\mathbb{H}^1_c \times \mathbb{S}^{n-1}$ carry the standard product metric $dt^2 + \sigma^{n-1}$ of $\mathbb{R} \times \mathbb{S}^{n-1}$, independently of $c \in [-1, 1]$. Thus $\Lambda^{(i)}_{n,0} = \mu^{(i)}(\mathbb{R} \times \mathbb{S}^{n-1})$. Proposition 4.1 yields a conformal diffeomorphism from the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ to $\mathbb{S}^n \setminus \mathbb{S}^0$, the n-sphere with North and South pole removed.

Lemma 4.12.

$$\Lambda^{(i)}_{n,0} \leq \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$$

for $i = 1, 2$.

Proof. We use the notation of Proposition 4.1 with $k = 0$. Then the standard metric on S^n is

$$\sigma^n = (\sin s)^2 (dt^2 + \sigma^{n-1}) = (\cosh t)^{-2} (dt^2 + \sigma^{n-1}).$$

It follows that $(\omega_n)^{-2/n} (\cosh t)^{-2} (dt^2 + \sigma^{n-1})$ is a (non-complete) metric of volume 1 and scalar curvature $n(n-1)\omega^{2/n} = \mu(\mathbb{S}^n)$ on $\mathbb{H}^1_c \times \mathbb{S}^{n-1} = \mathbb{R} \times \mathbb{S}^{n-1}$. This is equivalent to saying that

$$u(t) := \omega_n^{-\frac{n-2}{2n}} (\cosh t)^{-\frac{n-2}{2}}$$

is a solution of (6) with $\mu = \mu(\mathbb{S}^n)$ and $\|u\|_{L^p} = 1$ on $\mathbb{H}^1_c \times \mathbb{S}^{n-1}$. Obviously $u \in L^2$, and $\|u\|_{L^\infty} = \omega_n^{-\frac{n-2}{2n}} < \infty$. Thus $u \in \Omega^{(1)}(\mathbb{H}^1_c \times \mathbb{S}^{n-1})$. This implies $\Lambda^{(1)}_{n,0} \leq n(n-1)\omega_n^{2/n}$.

Further, we have

$$\mu(\mathbb{S}^n) \|u\|_{L^\infty}^{p-2} = n(n-1) > \frac{(n-0-2)^2(n-1)}{8(n-2)}$$

and thus $u \in \Omega^{(2)}(\mathbb{H}^1_c \times \mathbb{S}^{n-1})$ which implies $\Lambda^{(2)}_{n,0} \leq n(n-1)\omega_n^{2/n}$. □

Lemma 4.13. *Let* $u \in C^2(\mathbb{R} \times \mathbb{S}^{n-1})$ *be a solution of* (6) *on* $\mathbb{R} \times \mathbb{S}^{n-1}$ *with* $\|u\|_{L^p} \leq 1$, $u \not\equiv 0$. *Then* $\mu \geq \mu(\mathbb{S}^n)$.

Proof. As above $\sigma^n = (\sin s)^2 (dt^2 + \sigma^{n-1})$. If u solves (6) for $h = dt^2 + \sigma^{n-1}$ then $\tilde{u} := (\sin s)^{-\frac{n-2}{2}} u$ solves

$$L^{\sigma^n} \tilde{u} = \mu \tilde{u}^{p-1}.$$

Further $\tilde{u}^p\, dv^{\sigma^n} = u^p\, dv^h$, hence $\nu := \|\tilde{u}\|_{L^p(S^n\setminus S^0,\sigma^n)} \leq 1$. For $\alpha > 0$ we choose a smooth cut-off function $\chi_\alpha : S^n \to [0,1]$ that is 1 on $S^n \setminus U_\alpha(S^0)$, with support disjoint from S^0, and with $|d\chi|_{\sigma^n} \leq 2/\alpha$. Then using (65) we see that

$$\int_{\mathbb{S}^n} (\chi_\alpha \tilde{u}) L^{\sigma^n} (\chi_\alpha \tilde{u})\, dv^{\sigma^n} = \mu \int_{\mathbb{S}^n} u^p \chi_\alpha^2\, dv^{\sigma^n} + a \int_{\mathbb{S}^n} |d\chi|^2_{\sigma^n} \tilde{u}^2\, dv^{\sigma^n}.$$

The first summand tends to $\mu\nu^p$ as $\alpha \to 0$. By Hölder's inequality the second summand is bounded by

$$\frac{4a}{\alpha^2} \|\tilde{u}\|^2_{L^p(U_\alpha(S^0)\setminus S^0,\sigma^n)} \mathrm{Vol}(U_\alpha(S^0) \setminus S^0, \sigma^n)^{2/n} \leq C \|\tilde{u}\|^2_{L^p(U_\alpha(S^0)\setminus S^0,\sigma^n)} \to 0$$

as $\alpha \to 0$. Together with $\lim_{\alpha\to 0} \|\chi_\alpha \tilde{u}\|_{L^p(S^n\setminus S^0,\sigma^n)} = \nu$ we obtain

$$\mu(\mathbb{S}^n) \leq J^{\sigma^n}(\chi_\alpha \tilde{u}) \to \mu\nu^{p-2} \leq \mu$$

as $\alpha \to 0$. □

This lemma obviously implies $\Lambda^{(i)}_{n,0} \geq \mu(\mathbb{S}^n)$ for $i = 1, 2$, and thus we have

$$\Lambda_{n,0} = \Lambda^{(1)}_{n,0} = \Lambda^{(2)}_{n,0} = \mu(\mathbb{S}^n).$$

5. Limit spaces and limit solutions

In the proofs of the main theorems we will construct limit solutions of the Yamabe equation on certain limit spaces. For this we need the following two lemmas.

Lemma 5.1. *Let V be an n-dimensional manifold. Let (q_α) be a sequence of points in V which converges to a point q as $\alpha \to 0$. Let (γ_α) be a sequence of metrics defined on a neighborhood O of q which converges to a metric γ_0 in the $C^2(O)$-topology. Finally, let (b_α) be a sequence of positive real numbers such that $\lim_{\alpha\to 0} b_\alpha = \infty$. Then for $r > 0$ there exists for α small enough a diffeomorphism*

$$\Theta_\alpha : B^n(r) \to B^{\gamma_\alpha}(q_\alpha, b_\alpha^{-1} r)$$

with $\Theta_\alpha(0) = q_\alpha$ such that the metric $\Theta_\alpha^*(b_\alpha^2 \gamma_\alpha)$ tends to the flat metric ξ^n in $C^2(B^n(r))$.

Proof. Denote by $\exp^{\gamma_\alpha}_{q_\alpha} : U_\alpha \to O_\alpha$ the exponential map at the point q_α defined with respect to the metric γ_α. Here O_α is a neighborhood of q_α in V and U_α is a neighborhood of the origin in $\mathbb{R}^n$. We set

$$\Theta_\alpha : B^n(r) \ni x \mapsto \exp^{\gamma_\alpha}_{q_\alpha}(b_\alpha^{-1} x) \in B^{\gamma_\alpha}(q_\alpha, b_\alpha^{-1} r).$$

It is easily checked that Θ_α is the desired diffeomorphism. □

Lemma 5.2. *Let V be an n-dimensional manifold. Let (g_α) be a sequence of metrics which converges to a metric g in C^2 on all compact sets $K \subset V$ as $\alpha \to 0$. Assume that (U_α) is an increasing sequence of subdomains of V such that $\bigcup_\alpha U_\alpha = V$. Let $u_\alpha \in C^2(U_\alpha)$ be a sequence of positive functions such that $\|u_\alpha\|_{L^\infty(U_\alpha)}$ is bounded independently of α. We assume*

$$L^{g_\alpha} u_\alpha = \mu_\alpha u_\alpha^{p-1} \tag{9}$$

where the μ_α are numbers tending to $\bar{\mu}$. Then there exists a non-negative function $u \in C^2(V)$, satisfying

$$L^g u = \bar{\mu} u^{p-1} \tag{10}$$

on V and a subsequence of u_α which tends to u in C^1 on each open set $\Omega \subset V$ with compact closure. In particular

$$\|u\|_{L^\infty(K)} = \lim_{\alpha\to 0} \|u_\alpha\|_{L^\infty(K)}, \tag{11}$$

and

$$\int_K u^r \, dv^g = \lim_{\alpha\to 0} \int_K u_\alpha^r \, dv^{g_\alpha} \tag{12}$$

for any compact set K and any $r \geq 1$.

Proof. Let K be a compact subset of V and let Ω be an open set with smooth boundary and compact closure in V such that $K \subset \Omega$. From equation (9) and the boundedness of $\|u_\alpha\|_\infty$ we see with standard results on ellitpic regularity (see e.g. [12]) that (u_α) is bounded in the Sobolev space $H^{2,2n}(\Omega, g)$, i.e. all derivatives of $u_\alpha|_\Omega$ up to second order are bounded in $L^{2n}(\Omega)$). As this Sobolev space embeds compactly into $C^1(\Omega)$, a subsequence of (u_α) converges in $C^1(\Omega)$ to a function $u^\Omega \in C^1(\Omega)$, $u^\Omega \geq 0$, depending on Ω. Let $\varphi \in C^\infty(\Omega)$ be compactly supported in Ω. Multiplying Equation (9) by φ and integrating over Ω, we obtain that u^Ω satisfies Equation (10) weakly on Ω. By standard regularity results $u^\Omega \in C^2(\Omega)$ and satisfies Equation (10).

Now we choose an increasing sequence of compact sets K_m such that $\bigcup_m K_m = V$. Using the above arguments and taking successive subsequences it follows that (u_α) converges to functions $u_m \in C^2(K_m)$ which solve Equation (10) and satisfy $u_m \geq 0$ and $u_m|_{K_{m-1}} = u_{m-1}$. We define u on V by $u = u_m$ on K_m. By taking a diagonal subsequence of (u_α) we get that (u_α) tends to u in C^1 on any compact set $K \subset V$. This ends the proof of Lemma 5.2. □

Lemma 5.2 will be applied several times in the article, in most applications the limit space $\mathbb{R}^n$ will be obtained. In this situation, the following lemma will be helpful.

Lemma 5.3. *Let ξ^n be the standard flat metric on $\mathbb{R}^n$ and assume that $u \in C^2(\mathbb{R}^n)$, $u \geq 0$, $u \not\equiv 0$ satisfies*

$$L^{\xi^n} u = \mu u^{p-1} \tag{13}$$

for some $\mu \in \mathbb{R}$. Assume in addition that $u \in L^p(\mathbb{R}^n)$ and that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq 1.$$

Then $\mu \geq \mu(\mathbb{S}^n)$.

Proof. The map $\varphi : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$, $\varphi(t, x) = e^t x$, is a conformal diffeomorphism with

$$dt^2 + \sigma^{n-1} = e^{-2t} \varphi^* \xi^n.$$

Thus if u is a solution of (13), then $\hat{u} := e^{(n-2)t/2} u \circ \varphi$ is a solution of $L^{dt^2+\sigma^{n-1}} \hat{u} = \mu \hat{u}^{p-1}$ and $\|\hat{u}\|_{L^p(\mathbb{R}\times\mathbb{S}^{n-1})} = \|u\|_{L^p(\mathbb{R}^n)} \leq 1$. The lemma now follows from Lemma 4.13. □

6. L^2-estimates on WS-bundles

Manifolds with a certain structure of a double bundle will appear in the proofs of our main results. We have chosen to call manifolds of this type WS-bundles. In this section we assume that a solution of the Yamabe equation (or a perturbed Yamabe equation) is given on a WS-bundle. We derive L^2-estimates for such solutions.

6.1. **Definition and statement of the result.** Let $n \geq 1$ and $0 \leq k \leq n-3$ be integers. Let W be a closed manifold of dimension k and let I be an interval. By a *WS-bundle* we will mean the product $P := I \times W \times S^{n-k-1}$ equipped with a metric of the form

$$g_{\mathrm{WS}} = dt^2 + e^{2\varphi(t)} h_t + \sigma^{n-k-1} \tag{14}$$

where h_t is a smooth family of metrics on W depending on $t \in I$ and φ is a function on I. Let $\pi : P \to I$ be the projection onto the first factor and let $F_t := \pi^{-1}(t) = \{t\} \times W \times S^{n-k-1}$. The metric induced on F_t is $g_t := e^{2\varphi(t)} h_t + \sigma^{n-k-1}$. Let H_t be the mean curvature of F_t in P, that is $H_t \partial_t$ is the mean curvature vector of F_t. The mean curvature is given by the following formula

$$H_t = -\frac{k}{n-1} \varphi'(t) + e(h_t) \tag{15}$$

with $e(h_t) := \frac{1}{2} \mathrm{tr}_{h_t}(\partial_t h_t)$. Obviously, $e(h_t) = 0$ if $t \mapsto h_t$ is constant. The derivative of the volume element dv^{g_t} of F_t is then

$$\partial_t dv^{g_t} = -(n-1) H_t dv^{g_t}.$$

It is straightforward to check that the scalar curvatures of g_{WS} and h_t are related by (see Appendix B for details)

$$\begin{aligned} \mathrm{Scal}^{g_{\mathrm{WS}}} &= e^{-2\varphi(t)} \mathrm{Scal}^{h_t} + (n-k-1)(n-k-2) \\ &\quad - k(k+1)\varphi'(t)^2 - 2k\varphi''(t) - (k+1)\varphi'(t) \mathrm{tr}(h_t^{-1} \partial_t h_t) \\ &\quad + \frac{3}{4} \mathrm{tr}((h_t^{-1} \partial_t h_t)^2) - \frac{1}{4} \mathrm{tr}(h_t^{-1} \partial_t h_t)^2 - \mathrm{tr}_{h_t}(\partial_t^2 h_t). \end{aligned} \tag{16}$$

Definition 6.1. We say that condition (A_t) holds if the following assumptions are true:

$$\begin{array}{ll} (1) & t \mapsto h_t \text{ is constant,} \\ (2) & e^{-2\varphi(t)} \inf_{x \in W} \mathrm{Scal}^{h_t}(x) \geq -\frac{n-k-2}{32} a, \\ (3) & |\varphi'(t)| \leq 1, \\ (4) & 0 \leq -2k\varphi''(t) \leq \frac{1}{2}(n-1)(n-k-2)^2. \end{array} \tag{A_t}$$

Similarly we say that (B_t) holds if the following assumptions are true:

$$\begin{array}{ll} (1) & t \mapsto \varphi(t) \text{ is constant,} \\ (2) & \inf_{x \in F_t} \mathrm{Scal}^{g_{\mathrm{WS}}}(x) \geq \frac{1}{2} \mathrm{Scal}^{\sigma^{n-k-1}} = \frac{1}{2}(n-k-1)(n-k-2), \\ (3) & \frac{(n-1)^2}{2} e(h_t)^2 - \frac{n-1}{2} \partial_t e(h_t) \geq -\frac{3}{64}(n-k-2). \end{array} \tag{B_t}$$

Let P be WS-bundle equipped with a metric G which is close to g_{WS} in a sense to be made precise later. Let $\alpha, \beta \in \mathbb{R}$ be such that $[\alpha, \beta] \subset I$. Our goal in the following is to derive an estimate for the distribution of L^2-norm of a positive solution to the Yamabe equation

$$L^G u = \mu u^{p-1}.$$

If we write this equation in terms of the metric g_{WS} we get a perturbed version of the Yamabe equation for g_{WS}. We assume that we have a smooth positive solution u of the equation

$$L^{g_{\mathrm{WS}}} u = a\Delta^{g_{\mathrm{WS}}} u + \mathrm{Scal}^{g_{\mathrm{WS}}} u = \mu u^{p-1} + d^* A(du) + Xu + \epsilon \partial_t u - su \tag{17}$$

where $s, \epsilon \in C^\infty(P)$, $A \in \mathrm{End}(T^*P)$, and $X \in \Gamma(TP)$ are perturbation terms coming from the difference between G and g_{WS}. We assume that the endomorphism A is symmetric and that X and A are vertical, that is $dt(X) = 0$ and $A(dt) = 0$.

Theorem 6.2. *Assume that P carries a metric g_{WS} of the form (14). Let $\alpha, \beta \in \mathbb{R}$ be such that $[\alpha, \beta] \subset I$. Assume further that for each $t \in I$ either condition (A_t) or condition (B_t) is true. We also assume that u is a positive solution of (17) satisfying*

$$\mu \|u\|^{p-2}_{L^\infty(P)} \leq \frac{(n-k-2)^2(n-1)}{8(n-2)}. \tag{18}$$

Then there exists $c_0 > 0$ independent of α, β, and φ, such that if

$$\|A\|_{L^\infty(P)}, \|X\|_{L^\infty(P)}, \|s\|_{L^\infty(P)}, \|\epsilon\|_{L^\infty(P)}, \|e(h_t)\|_{L^\infty(P)} \leq c_0$$

then

$$\int_{\pi^{-1}((\alpha+\gamma,\beta-\gamma))} u^2\, dv^{g_{\mathrm{WS}}} \leq \frac{4\|u\|^2_{L^\infty}}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta)\right),$$

where $\gamma := \frac{\sqrt{32}}{n-k-2}$.

Note that this theorem only gives information when $\beta - \alpha > 2\gamma$.

6.2. **Proof of Theorem 6.2.** Before proving the theorem we prove the following lemma.

Lemma 6.3. *Suppose T is a positive number. Let $w : [-T-\gamma, T+\gamma] \to \mathbb{R}$ be a smooth positive function satisfying*

$$w''(t) \geq \frac{w(t)}{\gamma^2}. \tag{19}$$

Then

$$\int_{-T}^{T} w(t)^m\, dt \leq \frac{\gamma}{m}\left(w(T+\gamma)^m + w(-T-\gamma)^m\right) \tag{20}$$

for all $m \geq 1$.

Proof. Since $w'' \geq w/\gamma^2 > 0$ there exists a $t_0 \in [-T-\gamma, T+\gamma]$ such that $w'(t) > 0$ if $t \in (t_0, T+\gamma)$, and $w'(t) < 0$ if $t \in (-T-\gamma, t_0)$. We first study the case when $t_0 \in (-T, T)$. We define $W(t) := w(t) + \gamma w'(t)$. As w and w' are increasing we get

$$\begin{aligned} W(T) &= w(T) + \int_T^{T+\gamma} w'(T)\, dt \\ &\leq w(T) + \int_T^{T+\gamma} w'(t)\, dt \\ &= w(T+\gamma). \end{aligned} \tag{21}$$

From (19) we see that $W'(t) \geq W(t)/\gamma$, or $\partial_t \ln W(t) \geq 1/\gamma$. Integrating this relation between $t \in (t_0, T)$ and T we get

$$W(t) \leq e^{-\frac{T-t}{\gamma}} W(T).$$

Using that $w \leq W$ on (t_0, T) together with (21) we obtain

$$w(t) \leq W(t) \leq e^{-\frac{T-t}{\gamma}} w(T+\gamma),$$

and hence

$$w(t)^m \leq e^{-m\frac{T-t}{\gamma}} w(T+\gamma)^m$$

for all $t \in [t_0, T]$ and $m \geq 1$. Integrating this relation over $t \in [t_0, T]$ we get

$$\int_{t_0}^{T} w(t)^m \, dt \leq \frac{\gamma(1 - e^{-m\frac{T-t_0}{\gamma}})}{m} w(T+\gamma)^m \leq \frac{\gamma}{m} w(T+\gamma)^m. \tag{22}$$

Similarly we conclude that

$$\int_{-T}^{t_0} w(t)^m \, dt \leq \frac{\gamma}{m} w(-T-\gamma)^m. \tag{23}$$

This proves relation (20) in this case. In the case that $t_0 \leq -T$ relation (22) remains valid. Using

$$\int_{-T}^{T} w(t)^m \, dt \leq \int_{t_0}^{T} w(t)^m \, dt$$

and

$$w(T+\gamma)^m \leq w(T+\gamma)^m + w(-T-\gamma)^m,$$

we obtain relation (20). We proceed in a similar way using (23) in case $t_0 \geq T$. This ends the proof of Lemma 6.3. □

Proof of Theorem 6.2. The Laplacian $\Delta^{g_{\mathrm{WS}}}$ on P is related to the Laplacian Δ^{g_t} on F_t through the formula

$$\Delta^{g_{\mathrm{WS}}} = \Delta^{g_t} - \partial_t^2 + (n-1)H_t\partial_t,$$

so

$$\begin{aligned}\int_{F_t} u\Delta^{g_{\mathrm{WS}}} u \, dv^{g_t} &= \int_{F_t} \left(u\Delta^{g_t} u - u(\partial_t^2 u) + (n-1)H_t u(\partial_t u)\right) dv^{g_t} \\ &= \int_{F_t} \left(|d_{\mathrm{vert}} u|^2 - u(\partial_t^2 u) + (n-1)H_t u(\partial_t u)\right) dv^{g_t}.\end{aligned}$$

Together with (17) we get

$$\begin{aligned}a\int_{F_t} u\partial_t^2 u \, dv^{g_t} = \int_{F_t} \Big(&a|d_{\mathrm{vert}} u|^2 + a(n-1)H_t u\partial_t u \\ &- \langle d_{\mathrm{vert}} u, A(d_{\mathrm{vert}} u)\rangle - uXu - \epsilon u\partial_t u \\ &+ (\mathrm{Scal}^{g_{\mathrm{WS}}} + s)u^2 - \mu u^p\Big) dv^{g_t}.\end{aligned}$$

In the following we denote by $\delta(c_0)$ a positive constant which goes to 0 if c_0 tends to 0 and whose convergence depends only on n, μ, and h. We set $S_t := \inf_{F_t} \mathrm{Scal}^{g_{\mathrm{WS}}}$. If we use the inequality $2\int |ab| \leq \int (a^2 + b^2)$ to simplify the terms involving X and ϵ we obtain

$$\begin{aligned}a\int_{F_t} u\partial_t^2 u \, dv^{g_t} \geq \int_{F_t} \Big(&(a - \delta(c_0))|d_{\mathrm{vert}} u|^2 + a(n-1)H_t u\partial_t u \\ &- \delta(c_0)(\partial_t u)^2 + (S_t - \delta(c_0))u^2 - \mu u^p\Big) dv^{g_t}.\end{aligned}$$

If c_0 is small enough so that $a - \delta(c_0) > 0$ we conclude that

$$\begin{aligned}a\int_{F_t} \Big(u\partial_t^2 u - (n-1)H_t u(\partial_t u)\Big) dv^{g_t} &\geq (S_t - \delta(c_0))w(t)^2 \\ &\quad - \int_{F_t} \Big(\delta(c_0)(\partial_t u)^2 + \mu u^p\Big) dv^{g_t}.\end{aligned} \tag{24}$$

We define

$$w(t) := \|u\|_{L^2(F_t)} = \left(\int_{F_t} u^2\, dv^{g_t}\right)^{1/2}.$$

Differentiating this we get

$$\begin{aligned} 2w'(t)w(t) &= \partial_t \int_{F_t} u^2\, dv^{g_t} \\ &= \int_{F_t} \Big(2u(\partial_t u) - (n-1)H_t u^2\Big)\, dv^{g_t}. \end{aligned} \tag{25}$$

We now assume that (A_t) holds. Then (15) tells us that

$$H_t = -\frac{k}{n-1}\varphi'(t),$$

so (25) becomes

$$w'(t)w(t) = \int_{F_t} u(\partial_t u)\, dv^{g_t} + \frac{k}{2}\varphi'(t)w(t)^2. \tag{26}$$

We differentiate this and obtain

$$\begin{aligned} w'(t)^2 + w''(t)w(t) = &\int_{F_t} (\partial_t u)^2\, dv^{g_t} \\ &+ \int_{F_t} \Big(u\partial_t^2 u - (n-1)H_t u \partial_t u\Big)\, dv^{g_t} \\ &+ \frac{k}{2}\varphi''(t)w(t)^2 + k\varphi'(t)w'(t)w(t). \end{aligned}$$

From (24) we get

$$\begin{aligned} w'(t)^2 + w''(t)w(t) \geq &\left(1 - \frac{\delta(c_0)}{a}\right)\int_{F_t} (\partial_t u)^2\, dv^{g_t} \\ &+ \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t)\right) w(t)^2 \\ &- \frac{1}{a}\int_{F_t} \mu u^p\, dv^{g_t} + k\varphi'(t)w'(t)w(t). \end{aligned} \tag{27}$$

We now use (26) to get

$$\begin{aligned} w(t)^2 \int_{F_t} (\partial_t u)^2\, dv^{g_t} &\geq \left(\int_{F_t} u(\partial_t u)\, dv^{g_t}\right)^2 \\ &= \left(w'(t)w(t) - \frac{k}{2}\varphi'(t)w(t)^2\right)^2, \end{aligned}$$

or

$$\int_{F_t} (\partial_t u)^2\, dv^{g_t} \geq \left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2. \tag{28}$$

From assumption (18) it follows that

$$\frac{\mu}{a}\int_{F_t} u^p\, dv^{g_t} \leq \frac{(n-k-2)^2}{32} w(t)^2. \tag{29}$$

Inserting (28) and (29) into (27) we obtain

$$\begin{aligned} w'(t)^2 + w''(t)w(t) \geq & \left(1 - \frac{\delta(c_0)}{a}\right)\left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2 \\ & + \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t)\right) w(t)^2 \\ & - \frac{(n-k-2)^2}{32} w(t)^2 + k\varphi'(t)w'(t)w(t), \end{aligned}$$

or after some rearranging,

$$\begin{aligned} w''(t)w(t) \geq & -\frac{\delta(c_0)}{a}\left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2 \\ & + \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t) + \frac{k^2}{4}\varphi'(t)^2 - \frac{(n-k-2)^2}{32}\right) w(t)^2. \end{aligned} \tag{30}$$

Next we estimate the coefficient of $w(t)^2$ in the last line of (30). We denote this coefficient by D. Using (16) and assumption (A_t), (1), which tells us that $f(h_t) = 0$ we get

$$\begin{aligned} D &= \frac{1}{a}\left(e^{-2\varphi(t)}\mathrm{Scal}^{h_t} - k(k+1)\varphi'(t)^2 - 2k\varphi''(t) + (n-k-1)(n-k-2)\right) \\ &\quad - \frac{\delta(c_0)}{a} + \frac{k}{2}\varphi''(t) + \frac{k^2}{4}\varphi'(t)^2 - \frac{(n-k-2)^2}{32} \\ &= \frac{1}{a}e^{-2\varphi(t)}\mathrm{Scal}^{h_t} + \frac{1}{a}\left((n-k-1)(n-k-2) - \delta(c_0)\right) + \frac{k}{2(n-1)}\varphi''(t) \\ &\quad - \frac{k}{4(n-1)}(n-k-2)\varphi'(t)^2 - \frac{(n-k-2)^2}{32}. \end{aligned}$$

From assumptions (A_t), (2) and (3), we obtain

$$\begin{aligned} D &\geq -\frac{n-k-2}{32} + \frac{1}{a}\left((n-k-1)(n-k-2) - \delta(c_0)\right) + \frac{k}{2(n-1)}\varphi''(t) \\ &\quad - \frac{k}{4(n-1)}(n-k-2) - \frac{(n-k-2)^2}{32} \\ &= \frac{1}{4(n-1)}\left((n-1)(n-k-2)^2 + 2k\varphi''(t)\right) \\ &\quad - \frac{n-k-2}{32} - \frac{(n-k-2)^2}{32} - \frac{\delta(c_0)}{a}. \end{aligned}$$

Using (A_t), (4), and $n-k-2 \geq 1$ we further obtain

$$\begin{aligned} D &\geq \frac{1}{4(n-1)}\left(\frac{1}{2}(n-1)(n-k-2)^2\right) \\ &\quad - \frac{(n-k-2)^2}{32} - \frac{(n-k-2)^2}{32} - \frac{\delta(c_0)}{a} \\ &= \frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}. \end{aligned}$$

Inserting this in (30) we get

$$\begin{aligned} w''(t)w(t) \geq & -\frac{\delta(c_0)}{a}\left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2 \\ & + \left(\frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\right) w(t)^2 \\ \geq & -\frac{2\delta(c_0)}{a} w'(t)^2 \\ & + \left(-\frac{2\delta(c_0)}{a}\frac{k^2}{4}\varphi'(t)^2 + \frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\right) w(t)^2, \end{aligned}$$

where we also used $(a-b)^2 \leq 2a^2 + 2b^2$. Again using (A_t), (3), we conclude

$$\begin{aligned} w''(t)w(t) \geq & -\frac{2\delta(c_0)}{a} w'(t)^2 \\ & + \left(\frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\left(1 + \frac{k^2}{2}\right)\right) w(t)^2. \end{aligned} \tag{31}$$

Fix a small positive number $\hat{\delta}$. Choose c_0 small so that $\delta(c_0)$ is also small. Then (31) tells us that

$$w''(t)w(t) \geq \frac{(n-k-2)^2}{32} w(t)^2 - \hat{\delta} w'(t)^2. \tag{32}$$

Define $v(t) := w(t)^{1+\hat{\delta}}$. This function satisfies

$$\begin{aligned} v''(t) &= (1+\hat{\delta}) w''(t) w(t)^{\hat{\delta}} + \hat{\delta}(1+\hat{\delta}) w'(t)^2 w(t)^{\hat{\delta}-1} \\ &\geq (1+\hat{\delta}) \frac{(n-k-2)^2}{32} w(t)^{1+\hat{\delta}} \\ &\geq \frac{(n-k-2)^2}{32} v(t). \end{aligned}$$

Next we assume that (B_t) holds. Then (15) becomes

$$H_t = e(h_t),$$

and from (25) we get

$$w'(t)w(t) = \int_{F_t} \left(u(\partial_t u) - \frac{n-1}{2} e(h_t) u^2\right) dv^{g_t}. \tag{33}$$

Differentiating this we get

$$\begin{aligned} w'(t)^2 + w''(t)w(t) = & \int_{F_t} \Big((\partial_t u)^2 - (n-1) e(h_t) u \partial_t u \\ & \quad + \left(\frac{(n-1)^2}{2} e(h_t)^2 - \frac{n-1}{2} \partial_t e(h_t)\right) u^2\Big) dv^{g_t} \\ & + \int_{F_t} \left(u \partial_t^2 u - (n-1) H_t u \partial_t u\right) dv^{g_t}. \end{aligned}$$

Next we use (24) followed by assumptions (B_t), (2) and (3), to obtain

$$
\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq \int_{F_t} \Big(&(\partial_t u)^2 - (n-1)e(h_t)u\partial_t u \\
&+ \left(\frac{(n-1)^2}{2} e(h_t)^2 - \frac{n-1}{2}\partial_t e(h_t) \right) u^2 \\
&- \frac{\delta(c_0)}{a}(\partial_t u)^2 - \frac{\mu}{a}u^p \Big)\, dv^{g_t} \\
&+ \frac{1}{a}(S_t - \delta(c_0))w(t)^2 \\
\geq \int_{F_t} &\left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 - (n-1)e(h_t)u\partial_t u - \frac{\mu}{a}u^p \right) dv^{g_t} \\
&+ \left(\frac{1}{2a}(n-k-1)(n-k-2) - \frac{3}{64}(n-k-2) - \frac{\delta(c_0)}{a} \right) w(t)^2.
\end{aligned}
$$

From (29) we further get

$$
\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq \int_{F_t} &\left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 - (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
&+ \Big(\frac{1}{2a}(n-k-1)(n-k-2) - \frac{3}{64}(n-k-2) \\
&\quad - \frac{1}{32}(n-k-2)^2 - \frac{\delta(c_0)}{a} \Big) w(t)^2 \\
\geq \int_{F_t} &\left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 - (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
&+ \left(\frac{1}{32}(n-k-2)(n-k-3/2) - \frac{\delta(c_0)}{a} \right) w(t)^2 \\
\geq \int_{F_t} &\left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 - (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
&+ \left(\frac{1}{32}(n-k-2)^2 + \frac{1}{64} - \frac{\delta(c_0)}{a} \right) w(t)^2.
\end{aligned}
\tag{34}
$$

We set $E_t := \sup_{F_t} |e(h_t)|$ and use (33) to compute

$$\begin{aligned}
w(t)^2 \int_{F_t} (\partial_t u)^2 \, dv^{g_t} &\geq \left(\int_{F_t} u(\partial_t u) \, dv^{g_t} \right)^2 \\
&= \left(w'(t)w(t) + \frac{n-1}{2} \int_{F_t} e(h_t) u^2 \, dv^{g_t} \right)^2 \\
&= (w'(t)w(t))^2 + \left(\frac{n-1}{2} \int_{F_t} e(h_t) u^2 \, dv^{g_t} \right)^2 \\
&\quad + (n-1) w'(t)w(t) \int_{F_t} e(h_t) u^2 \, dv^{g_t} \\
&\geq w'(t)^2 w(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^4 \\
&\quad - (n-1)|w'(t)| w(t) \int_{F_t} |e(h_t)| u^2 \, dv^{g_t} \\
&\geq w'(t)^2 w(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^4 \\
&\quad - (n-1) E_t |w'(t)| w(t)^3.
\end{aligned}$$

Next we divide by $w(t)^2$ and obtain

$$\begin{aligned}
\int_{F_t} (\partial_t u)^2 \, dv^{g_t} &\geq w'(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^2 - (n-1) E_t |w'(t)| w(t) \\
&\geq w'(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^2 - \frac{n-1}{2} E_t \left(w'(t)^2 + w(t)^2 \right) \\
&= \left(1 - \frac{n-1}{2} E_t \right) w'(t)^2 - \left(\frac{n-1}{2} E_t + \left(\frac{n-1}{2} \right)^2 E_t^2 \right) w(t)^2.
\end{aligned} \tag{35}$$

Also

$$\begin{aligned}
\int_{F_t} e(h_t) u \partial_t u \, dv^{g_t} &\leq \int_{F_t} |e(h_t) u \partial_t u| \, dv^{g_t} \\
&\leq E_t \int_{F_t} |u \partial_t u| \, dv^{g_t} \\
&\leq \frac{1}{2} E_t \int_{F_t} \left(u^2 + (\partial_t u)^2 \right) \, dv^{g_t},
\end{aligned}$$

so

$$\int_{F_t} \left(-(n-1) e(h_t) u \partial_t u \right) \, dv^{g_t} \geq -\frac{n-1}{2} E_t \int_{F_t} \left(u^2 + (\partial_t u)^2 \right) \, dv^{g_t}. \tag{36}$$

Fix a small number $\hat{\delta} > 0$. We insert (35) and (36) in (34) and choose c_0 small enough so that $\delta(c_0)$ and E_t are small. Then we get that $w(t)$ satisfies the same inequality (32) as we obtained under the assumption (A_t). We have showed that in both cases (A_t) and (B_t) the function $v(t) := w(t)^{1+\hat{\delta}}$ satisfies

$$v''(t) \geq v(t)/\gamma^2$$

since $\frac{32}{(n-k-2)^2} = \gamma^2$.

Now we apply Lemma 6.3 to the function $\tilde{v}(t) := v(t + \frac{\beta+\alpha}{2})$ with $T = \frac{\beta-\alpha}{2} - \gamma$ and $m = \frac{2}{1+\hat{\delta}}$. From this we obtain

$$\frac{\gamma}{m}\left(\tilde{v}(T+\gamma)^m + \tilde{v}(-T-\gamma)^m\right) \geq \int_{-T}^{T} \tilde{v}^m \, dt. \tag{37}$$

We further have

$$\int_{-T}^{T} \tilde{v}^m \, dt = \int_{-\frac{\beta-\alpha}{2}+\gamma}^{\frac{\beta-\alpha}{2}-\gamma} (w^{(1+\hat{\delta})})^m \left(t + \frac{\beta+\alpha}{2}\right) dt$$

We set $s = t + \frac{\beta+\alpha}{2}$ and we obtain

$$\int_{-T}^{T} \tilde{v}^m \, dt = \int_{\alpha+\gamma}^{\beta-\gamma} w^2 \, ds.$$

From the definition of w we obtain

$$\int_{-T}^{T} \tilde{v}^m \, dt = \int_{\pi^{-1}((\alpha+\gamma,\beta-\gamma))} u^2 \, dv^{g_{\rm WS}}.$$

In addition, we have

$$\begin{aligned}(\tilde{v}(T+b)^m + \tilde{v}(-T-b)^m) &= \int_{F_\alpha} u^2 \, dv^{g_\alpha} + \int_{F_\beta} u^2 \, dv^{g_\beta} \\ &\leq \|u\|^2_{L^\infty(P)} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta)\right).\end{aligned}$$

Choosing $\hat{\delta}$ small we may assume $m \geq \sqrt{2}$. This together with (37) and $\gamma = \frac{\sqrt{32}}{n-k-2}$ gives us

$$\int_{\pi^{-1}((\alpha+\gamma,\beta-\gamma))} u^2 \, dv^{g_{\rm WS}} \leq \frac{4\|u\|^2_{L^\infty}}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta)\right).$$

This proves Theorem 6.2. □

7. Proof of Theorem 2.3

7.1. **Stronger version of the Theorem 2.3.** In this section we prove the following Theorem 7.1. By taking the supremum over all conformal classes Theorem 7.1 implies Theorem 2.3.

Theorem 7.1. *Suppose that (M_1, g_1) and (M_2, g_2) are compact Riemannian manifolds of dimension n. Let N be obtained from M_1, M_2, by a connected sum along W as described in Section 3. Then there is a sequence of metrics g_θ on N satisfying*

$$\min\left\{\mu(M_1 \amalg M_2, g_1 \amalg g_2), \Lambda_{n,k}\right\} \leq \lim_{\theta\to 0} \mu(N, g_\theta) \leq \mu(M_1 \amalg M_2, g_1 \amalg g_2).$$

In the following we define suitable metrics g_θ, and then we show that they satisfy these inequalities.

Hierarchy of parameters
$R_{\max} > R_0 > \theta > \delta_0 > \epsilon > 0$
We choose parameters in the order $R_{\max}, R_0, \theta, \delta_0, A_\theta$. We then set $\epsilon := e^{-A_\theta}\delta_0$. This implies $|t| = A_\theta \Leftrightarrow r_i = \delta_0$.

Figure 1. Hierarchy of parameters

7.2. **Definition of the metrics g_θ.** We continue to use the notation of Section 3. In the following, C denotes a constant that might change its value between lines. Recall that $(M, g) = (M_1 \amalg M_2, g_1 \amalg g_2)$. For $i = 1, 2$ we define the metric h_i as the restriction of g_i to $W_i' = w_i(W \times \{0\})$, and we set $h := h_1 \amalg h_2$ on $W' = W_1' \amalg W_2'$. As already explained, the normal exponential map of $W' \subset M$ defines a diffeomorphism

$$w_i : W \times B^{n-k}(R_{\max}) \to U_i(R_{\max}), \quad i = 1, 2,$$

which decomposes $U(R_{\max}) = U_1(R_{\max}) \amalg U_2(R_{\max})$ as a product $W' \times B^{n-k}(R_{\max})$. In general the Riemannian metric g does not have a corresponding product structure, and we introduce an error term T measuring the difference to the product metric. If r denotes the distance function to W', then the metric g can be written on $U(R_{\max}) \setminus W' \cong W' \times (0, R_{\max}) \times S^{n-k-1}$ as

$$g = h + \xi^{n-k} + T = h + dr^2 + r^2 \sigma^{n-k-1} + T. \tag{38}$$

where T is a symmetric $(2,0)$-tensor vanishing on W' (in the sense of sections of $(T^*M \otimes T^*M)|_{W'}$). We also define the product metric

$$g' := h + \xi^{n-k} = h + dr^2 + r^2 \sigma^{n-k-1}, \tag{39}$$

on $U(R_{\max}) \setminus W'$. Thus $g = g' + T$. Since T vanishes on W' we have

$$|T(X, Y)| \le Cr|X|_{g'}|Y|_{g'} \tag{40}$$

for any $X, Y \in T_xM$ where $x \in U(R_{\max})$. Since T is smooth we have

$$|(\nabla_U T)(X, Y)| \le C|X|_{g'}|Y|_{g'}|U|_{g'},$$

and

$$|(\nabla^2_{U,V})T(X, Y)| \le C|X|_{g'}|Y|_{g'}|U|_{g'}|V|_{g'},$$

for $X, Y, U, V \in T_xM$. We define $T_i := T|_{M_i}$ for $i = 1, 2$.

For a fixed $R_0 \in (0, R_{\max})$ we choose a smooth positive function $F : M \setminus W' \to \mathbb{R}$ such that

$$F(x) = \begin{cases} 1, & \text{if } x \in M_i \setminus U_i(R_{\max}); \\ r_i(x)^{-1}, & \text{if } x \in U_i(R_0) \setminus W'. \end{cases}$$

Next we choose small numbers $\theta, \delta_0 \in (0, R_0)$ with $\theta > \delta_0 > 0$. Here "small" means that we first choose a sequence $\theta = \theta_j$ of small positive numbers tending to zero, such that all following arguments hold for all θ. Then we choose for any given θ a number $\delta_0 = \delta_0(\theta) \in (0, \theta)$ such that all arguments which need δ_0 to be small will hold, see Figure 1. For any $\theta > 0$ and sufficiently small δ_0 there is $A_\theta \in [\theta^{-1}, (\delta_0)^{-1})$ and a smooth function $f : U(R_{\max}) \to \mathbb{R}$ depending only on the coordinate r such that

$$f(x) = \begin{cases} -\ln r(x), & \text{if } x \in U(R_{\max}) \setminus U(\theta); \\ \ln A_\theta, & \text{if } x \in U(\delta_0), \end{cases}$$

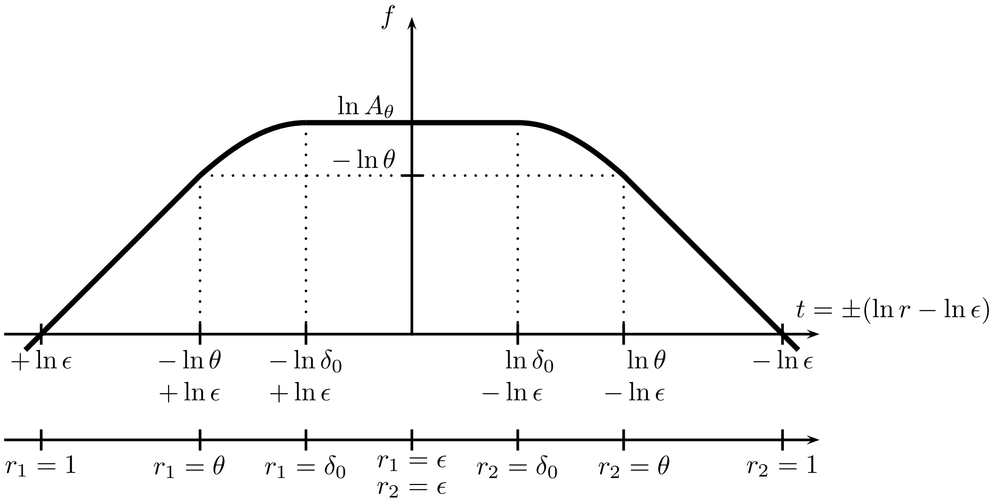

FIGURE 2. The function f

and such that

$$\left|r\frac{df}{dr}\right| = \left|\frac{df}{d(\ln r)}\right| \leq 1, \quad \text{and} \quad \left\|r\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\|_{L^\infty} = \left\|\frac{d^2 f}{d^2(\ln r)}\right\|_{L^\infty} \to 0 \qquad (41)$$

as $\theta \to 0$. See Figure 2.

We set $\epsilon = e^{-A_\theta}\delta_0$. We can and will assume that $\epsilon < 1$.

Let N be obtained from M by a connected sum along W with parameter ϵ, as described in Section 3. In particular, $U^N_\epsilon(s) = U(s) \setminus U(\epsilon)/\sim$ for all $s \geq \epsilon$. On the set $U^N_\epsilon(R_{\max}) = U(R_{\max}) \setminus U(\epsilon)/\sim$ we define the variable t by

$$t := \begin{cases} -\ln r_1 + \ln \epsilon, & \text{on } U_1(R_{\max}) \setminus U_1(\epsilon); \\ \ln r_2 - \ln \epsilon, & \text{on } U_2(R_{\max}) \setminus U_2(\epsilon). \end{cases}$$

Note that $t \leq 0$ on $U_1(R_{\max}) \setminus U_1(\epsilon)$ and $t \geq 0$ on $U_2(R_{\max}) \setminus U_2(\epsilon)$, with $t = 0$ precisely on the common boundary $\partial U_1(\epsilon)$ identified with $\partial U_2(\epsilon)$ in N. It follows that

$$r_i = e^{|t| + \ln \epsilon} = \epsilon e^{|t|}.$$

We can assume that $t : U^N_\epsilon(R_{\max}) \to \mathbb{R}$ is smooth. Expressed in the variable t we have

$$F(x) = \epsilon^{-1} e^{-|t|}$$

for $x \in U(R_0) \setminus U^N(\theta)$, or in other words if $|t| + \ln \epsilon \leq \ln R_0$. Then Equation (38) tells us that

$$F^2 g = \epsilon^{-2} e^{-2|t|}(h + T) + dt^2 + \sigma^{n-k-1}$$

on $U(R_0) \setminus U^N(\theta)$. If we view f as a function of t, then

$$f(t) = \begin{cases} -|t| - \ln \epsilon, & \text{if } \ln \theta - \ln \epsilon \leq |t| \leq \ln R_{\max} - \ln \epsilon; \\ \ln A_\theta, & \text{if } |t| \leq \ln \delta_0 - \ln \epsilon; \end{cases}$$

and $|df/dt| \leq 1$, $\|d^2 f/dt^2\|_{L^\infty} \to 0$. We choose a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi = 0$ on $(-\infty, -1]$, $|d\chi| \leq 1$, and $\chi = 1$ on $[1, \infty)$. With these choices we define

$$g_\theta := \begin{cases} F^2 g_i, & \text{on } M_i \setminus U_i(\theta); \\ e^{2f(t)}(h_i + T_i) + dt^2 + \sigma^{n-k-1}, & \text{on } U_i(\theta) \setminus U_i(\delta_0); \\ A_\theta^2 \chi(t/A_\theta)(h_2 + T_2) + A_\theta^2(1 - \chi(t/A_\theta))(h_1 + T_1) & \\ \quad + dt^2 + \sigma^{n-k-1}, & \text{on } U_\epsilon^N(\delta_0). \end{cases}$$

On $U^N(R_0)$ we write g_θ as

$$g_\theta = e^{2f(t)} \tilde{h}_t + dt^2 + \sigma^{n-k-1} + \widetilde{T}_t,$$

where the metric $\tilde{h}_t$ is defined for $t \in \mathbb{R}$ by

$$\tilde{h}_t := \chi(t/A_\theta) h_2 + (1 - \chi(t/A_\theta)) h_1,$$

and where the error term $\widetilde{T}_t$ is equal to

$$\widetilde{T}_t := e^{2f(t)} \left(\chi(t/A_\theta) T_2 + (1 - \chi(t/A_\theta)) T_1 \right).$$

On $U^N(R_0)$ we also define the metric without error term

$$g'_\theta := g_\theta - \widetilde{T}_t = e^{2f(t)} \tilde{h}_t + dt^2 + \sigma^{n-k-1}. \tag{42}$$

An upper bound for the error term $\widetilde{T}_t$ will be needed in the following. We claim that

$$|X|_{g'} \leq C e^{-f(t)} |X|_{g'_\theta} \tag{43}$$

for $X \in T_x N$, where g' is the metric defined by (39). To prove the claim, we decompose X in a radial part, a part parallel to W', and a part parallel to S^{n-k-1}. This decomposition is orthogonal with respect to both g' and g'_θ. For $X = \frac{\partial}{\partial t} = \pm \epsilon e^{|t|} \frac{\partial}{\partial r}$ we have that $1 = |X|_{g'_\theta}$ and $|X|_{g'} = \epsilon e^{|t|} \leq e^{-f(t)}$ since $f(t) \leq -|t| - \ln(\epsilon)$. The argument is similar if X is parallel to S^{n-k-1}. If X is tangent to W', then $|X|_g = |X|_h \leq C |X|_{\tilde{h}_t} \leq C e^{-f(t)} |X|_{g'_\theta}$, and the claim follows.

The Relations (43) and (40) imply

$$\begin{aligned} |\widetilde{T}_t(X,Y)| &\leq C e^{2f(t)} |T(X,Y)| \\ &\leq C e^{2f(t)} r |X|_{g'} |Y|_{g'} \\ &\leq C r |X|_{g'_\theta} |X|_{g'_\theta} \end{aligned}$$

for all X, Y. In other words this means

$$|\widetilde{T}_t|_{g'_\theta} \leq C r = C \epsilon e^{|t|} \leq C e^{-f(t)}. \tag{44}$$

One can calculate that

$$|\nabla \widetilde{T}_t|_{g'_\theta} \leq C e^{-f(t)}, \tag{45}$$

and

$$|\nabla^2 \widetilde{T}_t|_{g'_\theta} \leq C e^{-f(t)}. \tag{46}$$

Here ∇ denotes the Levi-Civita-connection with respect to g'_θ. In particular we see with Corollary A.2

$$|\mathrm{Scal}^{g_\theta} - \mathrm{Scal}^{g'_\theta}| \leq C e^{-f(t)}. \tag{47}$$

7.3. **Geometric description of the new metrics.** In this subsection we collect some facts about the geometry of F^2g and g'_θ. Most of the results are not needed for the proof of our result, but are useful to understand the underlying geometric concept of the argument. We will thus skip most of the proofs in this subsection.

The first proposition explains the special role of $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$.

Proposition 7.2. *Let x_i be a sequence of points in $M \setminus W$, converging to W. Then the Riemann tensor of F^2g in x_i converges to the Riemann tensor of $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$. The covariant derivative of the Riemann tensor of F^2g converges to zero. For any fixed $R > 0$ these convergences are uniform on balls (with respect to the metric F^2g) of radius R.*

It follows that for any fixed $R > 0$ the balls $(B^{F^2g}(x_i, R), x_i, F^2g)$ converge to a ball of radius R in $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$ in the $C^{2,\alpha}$-topology of Riemannian manifolds with base point. This topology has its origins in Cheeger's finiteness theorem [11] and in the work of Gromov [13], [14]. The article by Petersen [30, Pages 167–202] is a good overview of the subject.

In the limit $r \to 0$ (or equivalently $t \to \infty$) the W-component of the metrics F^2g grows exponentially. The motivation for introducing the function f into the definition of g_θ is to slow down this exponential growth: the diameter of the W-component with respect to g_θ is then bounded by $A_\theta \mathrm{diam}(W, g)$, where $\mathrm{diam}(W, g)$ is the diameter of W with respect to g. This slowing down has to be done carefully in order to get nice limit spaces. The properties claimed for f imply the following result.

Proposition 7.3. *Let θ_i be a sequence of positive numbers tending to zero, and let $x_i \in U_\epsilon^N(R_{max})$ be a sequence of points such that the limit $c := \lim(\frac{\partial}{\partial t} f)(t(x_i))$ exists. Then the Riemann tensor of g_{θ_i} in x_i converges to the Riemann tensor of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$. The covariant derivative of the Riemann tensor of F^2g converges to zero. For any fixed $R > 0$ these convergences are uniform on balls (with respect to the metric F^2g) of radius R.*

From this proposition it follows that the balls $(B^{F^2g}(x_i, R), x_i, F^2g)$ converge to a ball of radius R in $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ in the $C^{2,\alpha}$-topology of Riemannian manifolds with base point. Thus, we get an explanation why the spaces $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ appear as limit spaces.

The sectional curvature of $\mathbb{H}_c^{k+1}$ is $-c^2$. Hence the sectional curvatures of the product $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ are in the interval $[-c^2, 1]$. Using this fact we can prove the following Proposition.

Proposition 7.4. *The scalar curvatures of g_θ and g'_θ are bounded by a constant independent of θ.*

Proof. The metric g'_θ is the metric of a WS-bundle. Hence (16) is valid. We calculate $\partial_t \tilde{h}_t = (1/A_\theta)\chi'(t/A_\theta)(h_2 - h_1)$ and $\partial_t^2 \tilde{h}_t = (1/A_\theta)^2 \chi''(t/A_\theta)(h_2 - h_1)$. This implies $|\mathrm{tr}_{\tilde{h}_t} \partial_t \tilde{h}_t| \leq C/A_\theta$, $|\mathrm{tr}(\tilde{h}_t^{-1} \partial_t \tilde{h}_t)^2| \leq C/A_\theta^2$, and $|\mathrm{tr}_{\tilde{h}_t} \partial_t^2 \tilde{h}_t| \leq C/A_\theta^2$. From (16) it follows that $\mathrm{Scal}^{g'_\theta}$ is bounded. Equation (47) then implies that Scal^{g_θ} is bounded. □

The geometry close to the gluing of $M_1 \setminus U_1(\epsilon)$ with $M_2 \setminus U_2(\epsilon)$ is described by the following simple proposition.

Proposition 7.5. *Let H be the metric on $W \times (-1,1)$ given by $(\chi(t)h_2 + (1-\chi(t))h_1) + dt^2$. Then $(U^N(\delta_0), g'_\theta)$ is isometric to $(W \times (-1,1) \times S^{n-k-1}, A_\theta^2 H + \sigma^{n-k-1})$.*

7.4. **Proof of Theorem 7.1.** The metrics g_θ are defined for small $\theta > 0$ as described above. In order to prove Theorem 7.1 it is sufficient to prove

$$\min\{\mu(M,g), \Lambda_{n,k}\} \le \lim_{i\to\infty} \mu(N, g_{\theta_i}) \le \mu(M,g)$$

for any sequence $\theta_i \to 0$ as $i \to \infty$. Recall that $(M,g) = (M_1 \amalg M_2, g_1 \amalg g_2)$.

The upper bound on $\lim_{i\to\infty} \mu(N, g_{\theta_i})$ is easy to prove. The proof of the lower bound is inspired by the compactness-concentration principle in analysis, see for example [35, I.4] for a good overview (but be aware of some misleading typos).

For each metric g_θ we have a solution of the Yamabe equation (5). We take a sequence of θ tending to 0. Following the compactness-concentration principle, this sequence of solutions can concentrate in points or converge to a non-trivial solution or do both at the same time. The concentration in points can be used to construct a non-trivial solution on a sphere by blowing up the metrics.

In our situation we may have concentration in a fixed point (subcase I.1) or in a wandering point (subcase I.2), and we may have convergence to a non-trivial solution on the original manifold (subcase II.1.2) or in the attached part (subcases II.1.1 and II.2). In each of these cases we obtain a different lower bound for $\lim_{i\to\infty} \mu(N, g_{\theta_i})$: In the subcases I.1 and I.2 the lower bound is $\mu(\mathbb{S}^n)$, in subcase II.1.2 it is $\mu(M,g)$, and in the subcases II.1.1 and II.2 we obtain $\Lambda_{n,k}^{(1)}$ and $\Lambda_{n,k}^{(2)}$ as lower bounds. Together these cases give the lower bound of Theorem 7.1.

The cases here are not exclusive. For example it is possible that the solutions may both concentrate in a point and converge to a non-trivial solution on the original manifold.

In our arguments we will often pass to subsequences. To avoid complicated notation we write $\theta \to 0$ for a sequence $(\theta_i)_{i\in\mathbb{N}}$ converging to zero, and we will pass successively to subsequences without changing notation. Similarly $\lim_{\theta\to 0} h(\theta)$ should be read as $\lim_{i\to\infty} h(\theta_i)$.

We set $\mu := \mu(M,g)$ and $\mu_\theta := \mu(N, g_\theta)$. From Theorem 2.1 we have

$$\mu, \mu_\theta \le \mu(\mathbb{S}^n). \tag{48}$$

After passing to a subsequence, the limit

$$\bar{\mu} := \lim_{\theta\to 0} \mu_\theta \in [-\infty, \mu(\mathbb{S}^n)]$$

exists. Let $J := J^g$ and $J_\theta := J^{g_\theta}$ be defined as in (1).

We start with the easier part of the argument, namely with

$$\bar{\mu} \le \mu. \tag{49}$$

For this let $\alpha > 0$ be a small number. We choose a smooth cut-off function χ_α on M such that $\chi_\alpha = 1$ on $M \setminus U(2\alpha)$, $|d\chi_\alpha| \le 2/\alpha$, and $\chi_\alpha = 0$ on $U(\alpha)$. Let u be a smooth non-zero function such that $J(u) \le \mu + \delta$ where δ is a small positive number. On the support of χ_α the metrics g and g_θ are conformal since $g_\theta = F^2 g$ and hence by (3) we have

$$\mu_\theta \le J_\theta\left(\chi_\alpha F^{-\frac{n-2}{2}} u\right) = J(\chi_\alpha u)$$

for $\theta < \alpha$. It is straightforward to compute that $\lim_{\alpha\to 0} J(\chi_\alpha u) = J(u) \leq \mu + \delta$. From this Relation (49) follows.

Now we turn to the more difficult part of the proof, namely the inequality

$$\bar{\mu} \geq \min\{\mu, \Lambda_{n,k}\}. \tag{50}$$

In the case $\bar{\mu} = \mu(\mathbb{S}^n)$ this inequality follows trivially from (48). Hence we assume $\bar{\mu} < \mu(\mathbb{S}^n)$ in the following, which implies $\mu_\theta < \mu(\mathbb{S}^n)$ if θ is sufficiently small. From Theorem 2.2 we know that there exist positive functions $u_\theta \in C^2(M)$ such that

$$L^{g_\theta} u_\theta = \mu_\theta u_\theta^{p-1}, \tag{51}$$

and

$$\int_N u_\theta^p \, dv^{g_\theta} = 1.$$

We begin by proving a lemma which yields a bound of the L^2-norm of u_θ in terms of the L^∞-norm. This lemma is non-trivial since $\mathrm{Vol}(N, g_\theta) \to \infty$ as $\theta \to 0$.

Lemma 7.6. *Assume that there exists $b > 0$ such that*

$$\mu_\theta \sup_{U^N(b)} u_\theta^{p-2} \leq \frac{(n-k-2)^2(n-1)}{8(n-2)}$$

for θ small enough. Then there exist constants $c_1, c_2 > 0$ independent of θ such that

$$\int_N u_\theta^2 \, dv^{g_\theta} \leq c_1 \|u_\theta\|^2_{L^\infty(N)} + c_2$$

for all sufficiently small θ. In particular, if $\|u_\theta\|_{L^\infty(N)}$ is bounded, so is $\|u_\theta\|_{L^2(N)}$.

Proof. Let $\tilde{r} \in (0, b)$ be fixed and set $P = U(\tilde{r})$. Then P is a WS-bundle where, with the notation of Section 6, $I = (\alpha, \beta)$ with $\alpha = -\ln(\tilde{r}) + \ln(\epsilon)$ and $\beta = \ln(\tilde{r}) - \ln(\epsilon)$. On P we have two natural metrics: g_θ and $g_{\mathrm{WS}} = g'_\theta = g_\theta - \widetilde{T}_t$. The metric g_{WS} has exactly the form (14) with $\varphi = f$ and $h_t = \tilde{h}_t$. Let θ be small enough and let $t \in (-\ln(\tilde{r}) + \ln(\epsilon), -\ln(\delta_0) + \ln(\epsilon)) \cup (\ln(\delta_0) - \ln(\epsilon), \ln(\tilde{r}) - \ln(\epsilon))$. Then assumption (A_t) of Theorem 6.2 is true. Now, again if θ is small enough, we have for all $t \in (-\ln(\delta_0) + \ln(\epsilon), \ln(\delta_0) - \ln(\epsilon))$ the relation $\mathrm{Scal}^{g_{\mathrm{WS}}} = \mathrm{Scal}^{\sigma^{n-k-1}} + O(1/A_\theta)$. The error term $e(\tilde{h}_t)$ from (B_t) in this case satisfies

$$|e(\tilde{h}_t)| \leq \left|\mathrm{tr}_{\tilde{h}_t} \partial_t \tilde{h}_t\right| = \left|\mathrm{tr}_{\tilde{h}_t}\left(\chi'(t/A_\theta)\frac{h_2 - h_1}{A_\theta}\right)\right| \leq \frac{C}{A_\theta},$$

and

$$|\partial_t e(\tilde{h}_t)| = \left|\mathrm{tr}\left(\tilde{h}_t^{-1}(\partial_t \tilde{h}_t)\tilde{h}_t^{-1}(\partial_t \tilde{h}_t)\right)\right| + \left|\mathrm{tr}_{\tilde{h}_t}\partial_t^2 \tilde{h}_t\right| \leq \frac{C}{A_\theta^2}.$$

Because of $1/A_\theta \leq \theta$ condition (B_t) is true. Equation (51) is written in the metric g_θ. Using the expression of the Laplacian in local coordinates,

$$\Delta^{g_\theta} u = -\sum_{i,j} (\det g_\theta)^{-1/2} \partial_i \left(g_\theta^{ij} (\det g_\theta)^{1/2} \partial_j u\right),$$

one can check that if we write Equation (51) in the metric g_{WS} we obtain an equation of the form (17) with $\mu = \mu_\theta$. Together with (44), (45) and (47), one verifies that the error terms satisfy

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |\epsilon(x)|_{g_{\mathrm{WS}}} \leq C e^{-f(t)},$$

where $|\cdot|_{g_{\mathrm{WS}}}$ denotes the pointwise norm at a point in $U^N(R_0)$, and where C is a constant independent of θ. In particular for any $c_0 > 0$, we obtain on $U^N(\theta)$ for small θ

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |e(\tilde{h}_t)(x)|_{g_{\mathrm{WS}}}, |\epsilon(x)|_{g_{\mathrm{WS}}} \le c_0.$$

This estimate allows us to apply Theorem 6.2. By the assumptions of Lemma 7.6, if $\tilde{r} \in (0, b)$ is small enough, Assumption (18) of Theorem 6.2 is true. Thus, all hypotheses of Theorem 6.2 hold for $-\alpha := \beta := \ln \tilde{r} - \ln \epsilon$, and hence

$$\int_{P'} u_\theta^2 \, dv^{g_{\mathrm{WS}}} \le \frac{4\|u_\theta\|_{L^\infty}^2}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta) \right).$$

where $P' := U^N(\tilde{r}e^{-\gamma})$. Now observe that

$$C := \frac{4}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta) \right)$$

does not depend on θ (since F_α and F_β correspond to the hypersurface $r = \tilde{r}$). This implies that

$$\int_{P'} u_\theta^2 \, dv^{g_{\mathrm{WS}}} \le C \|u_\theta\|_{L^\infty(N)}^2$$

where $C > 0$ is independent of θ. Since if $\tilde{r}$ is small enough, we clearly have

$$dv^{g_\theta} \le 2 dv^{g_{\mathrm{WS}}},$$

we obtain that

$$\int_{P'} u_\theta^2 \, dv^{g_\theta} \le c_1 \|u_\theta\|_{L^\infty(N)}^2$$

where $c_1 := 2C > 0$ is independent of θ. Now observe that $\mathrm{Vol}^{g_\theta}(N \setminus P')$ is bounded by a constant independent of θ. Using the Hölder inequality we obtain

$$\begin{aligned} \int_N u_\theta^2 \, dv^{g_\theta} &= \int_{P'} u_\theta^2 \, dv^{g_\theta} + \int_{N \setminus P'} u_\theta^2 \, dv^{g_\theta} \\ &\le c_1 \|u_\theta\|_{L^\infty(N)}^2 + \mathrm{Vol}^{g_\theta}(N \setminus P')^{\frac{2}{n}} \left(\int_{N \setminus P'} u_\theta^p \, dv \right)^{\frac{n-2}{n}}. \end{aligned}$$

Since $\|u_\theta\|_{L^p(N)} = 1$, this proves Lemma 7.6 with c_1 as defined above and with $c_2 := \mathrm{Vol}^{g_\theta}(N \setminus P')^{\frac{2}{n}}$. For small θ, the metric $g_\theta|_{N \setminus P'}$ is independent of θ, and thus c_2 does not depend on θ. □

Corollary 7.7.

$$\liminf_{\theta \to 0} \|u_\theta\|_{L^\infty(N)} > 0.$$

Proof. We set $m_\theta := \|u_\theta\|_{L^\infty(N)}$ and we choose x_θ in N such that $u_\theta(x_\theta) = m_\theta$. In order to prove the corollary by contradiction we assume $\lim_{\theta \to 0} m_\theta = 0$. Then since $\mu_\theta \le \mu(\mathbb{S}^n)$ the assumption of Lemma 7.6 is satisfied for all $b > 0$ for which $U^N(b)$ is defined. We get the contradiction

$$1 = \int_N u_\theta^p dv^{g_\theta} \le m_\theta^{p-2} \int_N u_\theta^2 dv^g \le m_\theta^{p-2}(c_1 m_\theta^2 + c_2) \to 0$$

as $\theta \to 0$. □

Corollary 7.8.

$$\bar{\mu} = \lim_{\theta \to 0} \mu_\theta > -\infty.$$

Proof. Choose x_θ as above. We then have $\Delta^{g_\theta} u_\theta(x_\theta) \geq 0$, which together with (51) gives us

$$\mathrm{Scal}^{g_\theta}(x_\theta)\|u_\theta\|_{L^\infty(N)} \leq \mu_\theta \|u_\theta\|^{p-1}_{L^\infty(N)}.$$

Proposition 7.4 and the previous corollary then imply that μ_θ is bounded from below. □

In addition, by Theorem 2.1, μ_θ is bounded from above by $\mu(\mathbb{S}^n)$. It follows that $\bar{\mu} \in \mathbb{R}$. The rest of the proof proceeds in several cases.

Case I. $\limsup_{\theta\to 0} \|u_\theta\|_{L^\infty(N)} = \infty$.

As before we set $m_\theta := \|u_\theta\|_{L^\infty(N)}$ and we choose $x_\theta \in N$ with $u_\theta(x_\theta) = m_\theta$. After taking a subsequence we can assume that $\lim_{\theta\to 0} m_\theta = \infty$. We consider two subcases.

Subcase I.1. There exists $b > 0$ such that $x_\theta \in N \setminus U^N(b)$ for an infinite number of θ.

We recall that $N\setminus U^N(b) = N_\epsilon \setminus U^N_\epsilon(b) = M_1 \amalg M_2 \setminus U(b)$. By taking a subsequence we can assume that there exists $\bar{x} \in M_1 \amalg M_2 \setminus U(b)$ such that $\lim_{\theta\to 0} x_\theta = \bar{x}$. We let $\tilde{g}_\theta := m_\theta^{\frac{4}{n-2}} g_\theta$. In a neighborhood U of $\bar{x}$ the metric $g_\theta = F^2 g$ does not depend on θ. We apply Lemma 5.1 with $O = U$, $\alpha = \theta$, $q_\alpha = x_\theta$, $q = \bar{x}$, $\gamma_\alpha = g_\theta = F^2 g$, and $b_\alpha = m_\theta^{\frac{2}{n-2}}$. Let $r > 0$. For θ small enough Lemma 5.1 gives us a diffeomorphism

$$\Theta_\theta : B^n(r) \to B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)$$

such that the sequence of metrics $(\Theta_\theta^*(\tilde{g}_\theta))$ tends to the flat metric ξ^n in $C^2(B^n(r))$. We let $\tilde{u}_\theta := m_\theta^{-1} u_\theta$. By (2) we then have

$$L^{\tilde{g}_\theta} \tilde{u}_\theta = \mu_\theta \tilde{u}_\theta^{p-1}$$

on $B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)$ and

$$\begin{aligned}
\int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} \tilde{u}_\theta^p \, dv^{\tilde{g}_\theta} &= \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} u_\theta^p \, dv^{g_\theta} \\
&\leq \int_N u_\theta^p dv^{g_\theta} \\
&= 1.
\end{aligned}$$

Here we used the fact that $dv^{\tilde{g}_\theta} = m_\theta^p \, dv^{g_\theta}$. Since

$$\Theta_\theta : (B^n(r), \Theta_\theta^*(\tilde{g}_\theta)) \to (B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r), \tilde{g}_\theta)$$

is an isometry we can consider $\tilde{u}_\theta$ as a solution of

$$L^{\Theta_\theta^*(\tilde{g}_\theta)} \tilde{u}_\theta = \mu_\theta \tilde{u}_\theta^{p-1}$$

on $B^n(r)$ with $\int_{B^n(r)} \tilde{u}_\theta^p \, dv^{\Theta_\theta^*(\tilde{g}_\theta)} \leq 1$. Since $\|\tilde{u}_\theta\|_{L^\infty(B^n(r))} = |\tilde{u}_\theta(0)| = 1$ we can apply Lemma 5.2 with $V = \mathbb{R}^n$, $\alpha = \theta$, $g_\alpha = \Theta_\theta^*(\tilde{g}_\theta)$, and $u_\alpha = \tilde{u}_\theta$ (we can apply this lemma since each compact set of $\mathbb{R}^n$ is contained in some ball $B^n(r)$). This shows that there exists a non-negative function $u \not\equiv 0$ (since $u(0) = 1$) of class C^2 on $(\mathbb{R}^n, \xi^n)$ which satisfies

$$L^{\xi^n} u = a\Delta^{\xi^n} u = \bar{\mu} u^{p-1}.$$

By (12) we further have

$$\int_{B^n(r)} u^p \, dv^{\xi^n} = \lim_{\theta \to 0} \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} u_\theta^p \, dv^{g_\theta} \leq 1$$

for any $r > 0$. In particular,

$$\int_{\mathbb{R}^n} u^p \, dv^{\xi^n} \leq 1.$$

From Lemma 5.3, we get that $\bar{\mu} \geq \mu(\mathbb{S}^n) \geq \min\{\mu, \Lambda_{n,k}\}$. We have proved (50) in this subcase.

Subcase I.2. For all $b > 0$ it holds that $x_\theta \in U^N(b)$ for θ sufficiently small.

The subset $U^N(b)$ is diffeomorphic to $W \times I \times S^{n-k-1}$ where I is an interval. We identify

$$x_\theta = (y_\theta, t_\theta, z_\theta)$$

where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln \epsilon, -\ln \epsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By taking a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge respectively to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. First we apply Lemma 5.1 with $V = W$, $\alpha = \theta$, $q_\alpha = y_\theta$, $q = y$, $\gamma_\alpha = \tilde{h}_{t_\theta}$, $\gamma_0 = \tilde{h}_T$ (we define $\tilde{h}_{-\infty} = h_1$ and $\tilde{h}_{+\infty} = h_2$), and $b_\alpha = m_\theta^{\frac{2}{n-2}} e^{f(t_\theta)}$. The lemma provides diffeomorphisms

$$\Theta_\theta^y : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r)$$

for $r > 0$ such that $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t_\theta)} \tilde{h}_{t_\theta})$ tends to the flat metric ξ^k on $B^k(r)$ as $\theta \to 0$. Second we apply Lemma 5.1 with $V = S^{n-k-1}$, $\alpha = \theta$, $q_\alpha = z_\theta$, $\gamma_\alpha = \gamma_0 = \sigma^{n-k-1}$, and $b_\alpha = m_\theta^{\frac{2}{n-2}}$. For $r' > 0$ we get diffeomorphisms

$$\Theta_\theta^z : B^{n-k-1}(r') \to B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-2}} r')$$

such that $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}} \sigma^{n-k-1})$ converges to ξ^{n-k-1} on $B^{n-k-1}(r')$ as $\theta \to 0$. For $r, r', r'' > 0$ we define

$$\begin{aligned} U_\theta(r, r', r'') := {} & B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r) \times [t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r''] \\ & \times B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-2}} r'), \end{aligned}$$

and

$$\Theta_\theta : B^k(r) \times [-r'', r''] \times B^{n-k-1}(r') \to U_\theta(r, r', r'')$$

by

$$\Theta_\theta(y, s, z) := (\Theta_\theta^y(y), t(s), \Theta_\theta^z(z)),$$

where $t(s) := t_\theta + m_\theta^{\frac{2}{n-2}} s$. By construction Θ_θ is a diffeomorphism, and we see that

$$\begin{aligned} \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g_\theta) = {} & (\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t)} \tilde{h}_t) + ds^2 \\ & + (\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}} \sigma^{n-k-1}) + \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} \widetilde{T}_t). \end{aligned} \tag{52}$$

Next we study the first term on the right hand side of (52). Note that it is here evaluated at t, while we have information above when evaluated at t_θ. By construction of $f(t)$ one can verify that

$$\lim_{\theta \to 0} \left\| \frac{e^{f(t_\theta)}}{e^{f(t)}} - 1 \right\|_{C^2([t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r''])} = 0$$

since $\frac{df}{dt}$ and $\frac{d^2f}{dt^2}$ are uniformly bounded. Moreover it is clear that

$$\lim_{\theta\to 0}\left\|\tilde{h}_t - \tilde{h}_{t_\theta}\right\|_{C^2(B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r))} = 0$$

uniformly in $t \in [t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r'']$. As a consequence

$$\lim_{\theta\to 0}\left\|(\Theta_\theta^y)^*\left(m_\theta^{\frac{4}{n-2}}\left(e^{2f(t)}\tilde{h}_t - e^{2f(t_\theta)}\tilde{h}_{t_\theta}\right)\right)\right\|_{C^2(B^k(r))} = 0$$

uniformly in t. This implies that the sequence $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t)}\tilde{h}_t)$ tends to the flat metric ξ^k in $C^2(B^k(r))$ uniformly in t as $\theta \to 0$. We also know that the sequence $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}}\sigma^{n-k-1})$ tends to ξ^{n-k-1} in $C^2(B^{n-k-1}(r'))$ as $\theta \to 0$. Recall from (42) that $g'_\theta = g_\theta - \widetilde{T}_t$, we have proved that $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)$ tends to the flat metric in $C^2(B^k(r) \times [-r'', r''] \times B^{n-k-1}(r'))$. Finally we are going to show that the last term of (52) tends to zero in C^2. It follows from (44) that

$$\lim_{\theta\to 0}\left\|\Theta_\theta^*(m_\theta^{\frac{4}{n-2}}\widetilde{T}_t)\right\| = 0. \tag{53}$$

Indeed, (44) tells us that

$$\begin{aligned}\left|\Theta_\theta^*(m_\theta^{\frac{4}{n-2}}\widetilde{T}_t)(X,Y)\right| &= m_\theta^{\frac{4}{n-2}}\left|\widetilde{T}_t(\Theta_{\theta*}(X), \Theta_{\theta*}(Y))\right| \\ &\le Crm_\theta^{\frac{4}{n-2}}|\Theta_{\theta*}(X)|_{g'_\theta}|\Theta_{\theta*}(Y)|_{g'_\theta} \\ &\le Cr|X|_{\Theta_\theta^*(m_\theta^{\frac{4}{n-2}}g'_\theta)}|X|_{\Theta_\theta^*(m_\theta^{\frac{4}{n-2}}g'_\theta)},\end{aligned}$$

and since $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)$ tends to the flat metric we get (53). Doing the same with $\nabla\widetilde{T}_t$ and $\nabla^2\widetilde{T}_t$ using (45) and (46), we obtain that

$$\lim_{\theta\to 0}\Theta_\theta^*(m_\theta^{\frac{4}{n-2}}\widetilde{T}_t) = 0 \tag{54}$$

in $C^2(B^k(r)\times[-r'', r'']\times B^{n-k-1}(r'))$. Returning to (52) we see that the sequence $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g_\theta)$ tends to $\xi^n = \xi^k + ds^2 + \xi^{n-k-1}$ on $B^k(r)\times[-r'', r'']\times B^{n-k-1}(r')$. We proceed as in Subcase I.1 to show that $\bar{\mu} \ge \mu(\mathbb{S}^n) \ge \min\{\mu, \Lambda_{n,k}\}$, which proves Relation (50) in this subcase. This ends the proof of Theorem 7.1 in Case I.

Case II. There exists a constant C_1 such that $\|u_\theta\|_{L^\infty(N)} \le C_1$ for all θ.

As in Case I we consider two subcases.

Subcase II.1. There exists $b > 0$ such that

$$\liminf_{\theta\to 0}\left(\mu_\theta \sup_{U^N(b)} u_\theta^{p-2}\right) < \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

By restricting to a subsequence we can assume that

$$\mu_\theta \sup_{U^N(b)} u_\theta^{p-2} < \frac{(n-k-2)^2(n-1)}{8(n-2)}$$

for all θ. Lemma 7.6 tells us that there is a constant $A_0 > 0$ such that

$$\|u_\theta\|_{L^2(N,g_\theta)} \le A_0. \tag{55}$$

We split the treatment of Subcase II.1. into two subsubcases.

Subsubcase II.1.1. $\limsup_{b\to 0}\limsup_{\theta\to 0}\sup_{U^N(b)} u_\theta > 0$.

We set $D_0 := \frac{1}{2}\limsup_{b\to 0}\limsup_{\theta\to 0}\sup_{U^N(b)} u_\theta > 0$. Then there are sequences (b_i) and (θ_i) of positive numbers converging to 0 such that

$$\sup_{U^N(b_i)} u_{\theta_i} \geq D_0,$$

for all i. For brevity of notation we write θ for θ_i and b_θ for b_i. Let $x'_\theta \in \overline{U^N(b_\theta)}$ be such that

$$u_\theta(x'_\theta) \geq D_0. \tag{56}$$

As in Subcase I.2 above we write $x'_\theta = (y_\theta, t_\theta, z_\theta)$ where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln\epsilon, -\ln\epsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By restricting to a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge respectively to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. We apply Lemma 5.1 with $V = W$, $\alpha = \theta$, $q_\alpha = y_\theta$, $q = y$, $\gamma_\alpha = \tilde{h}_{t_\theta}$, $\gamma_0 = \tilde{h}_T$, and $b_\alpha = e^{f(t_\theta)}$ and conclude that there is a diffeomorphism

$$\Theta^y_\theta : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, e^{-f(t_\theta)}r)$$

for $r > 0$ such that $(\Theta^y_\theta)^*(e^{2f(t_\theta)}\tilde{h}_{t_\theta})$ converges to the flat metric ξ^k on $B^k(r)$. For $r, r' > 0$ we define

$$U_\theta(r, r') := B^{\tilde{h}_{t_\theta}}(y_\theta, e^{-f(t_\theta)}r) \times [t_\theta - r', t_\theta + r'] \times S^{n-k-1},$$

and

$$\Theta_\theta : B^k(r) \times [-r', r'] \times S^{n-k-1} \to U_\theta(r, r')$$

by

$$\Theta_\theta(y, s, z) := (\Theta^y_\theta(y), t(s), z),$$

where $t(s) := t_\theta + s$. By construction, Θ_θ is a diffeomorphism, and we see that

$$\Theta^*_\theta(g_\theta) = \frac{e^{2f(t)}}{e^{2f(t_\theta)}}(\Theta^y_\theta)^*(e^{2f(t_\theta)}\tilde{h}_t) + ds^2 + \sigma^{n-k-1} + \Theta^*_\theta(\widetilde{T}_t) \tag{57}$$

We will now find the limit of $\Theta^*_\theta(g_\theta)$ in the C^2 topology. We define $c := \lim_{\theta\to 0} f'(t_\theta)$.

Lemma 7.9. *For fixed $r, r' > 0$ the sequence of metrics $\Theta^*_\theta(g_\theta)$ tends to $G_c = \eta^{k+1}_c + \sigma^{n-k-1} = e^{2cs}\xi^k + ds^2 + \sigma^{n-k-1}$ in $C^2(B^k(r) \times [-r', r'] \times S^{n-k-1})$.*

As this lemma coincides with [4, Lemma 4.1] we only sketch the proof.

Proof. The intermediate value theorem tells us that

$$|f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)| \leq \frac{r'^2}{2}\max_{s\in[t_\theta - r', t_\theta + r']}|f''(s)|$$

for all $t \in [t_\theta - r', t_\theta + r']$. Because of (41) we also have $\|f''\|_{L^\infty} \to 0$ for $\theta \to 0$, and hence

$$\lim_{\theta\to 0}\|f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)\|_{C^0([t_\theta - r', t_\theta + r'])} = 0$$

for r' fixed. Further we have

$$\begin{aligned}\left|\frac{d}{dt}\Big(f(t)-f(t_\theta)-f'(t_\theta)(t-t_\theta)\Big)\right| &= |f'(t)-f'(t_\theta)| \\ &= \left|\int_{t_\theta}^{t} f''(s)\,ds\right| \\ &\leq r' \max_{s\in[t_\theta-r',t_\theta+r']} |f''(s)| \\ &\to 0\end{aligned}$$

as $\theta\to 0$, and and finally

$$\left|\frac{d^2}{dt^2}\left(f(t)-f(t_\theta)-f'(t_\theta)(t-t_\theta)\right)\right| = |f''(t)|\to 0$$

as $\theta\to 0$. Together with $c=\lim_{\theta\to 0} f'(t_\theta)$ we have shown that

$$\lim_{\theta\to 0}\|f(t)-f(t_\theta)-c(t-t_\theta)\|_{C^2([t_\theta-r',t_\theta+r'])}=0.$$

Hence

$$\lim_{\theta\to 0}\left\|e^{f(t)-f(t_\theta)}-e^{c(t-t_\theta)}\right\|_{C^2([t_\theta-r',t_\theta+r'])}=0.$$

We now write $e^{2f(t)}\tilde{h}_t = e^{2f(t)}(\tilde{h}_t-\tilde{h}_{t_\theta})+\frac{e^{2f(t)}}{e^{2f(t_\theta)}}e^{2f(t_\theta)}\tilde{h}_{t_\theta}$. Using the fact that

$$\lim_{\theta\to 0}\left\|\tilde{h}_t-\tilde{h}_{t_\theta}\right\|_{C^2(B^{\tilde{h}_{t_\theta}}(y_\theta,e^{-f(t_\theta)}r))}=0$$

uniformly for $t\in[t_\theta-r',t_\theta-r']$ we get that the sequence $\frac{e^{2f(t)}}{e^{2f(t_\theta)}}(\Theta_\theta^y)^*(e^{2f(t_\theta)}\tilde{h}_t)$ tends to $e^{2cs}\xi^k$ in $C^2(B^k(r))$ where again $s=t-t_\theta\in[-r',r']$. Finally, proceeding exactly as we did to get Relation (54), we have that

$$\lim_{\theta\to 0}\Theta_\theta^*(\widetilde{T}_t)=0$$

in $C^2(B^k(r)\times[-r',r']\times S^{n-k-1})$. Going back to (57) this proves Lemma 7.9. □

We continue with the proof of Subsubcase II.1.1. As in Subcases I.1 and I.2 we apply Lemma 5.2 with $(V,g)=(\mathbb{R}^{k+1}\times S^{n-k-1},G_c)$, $\alpha=\theta$, and $g_\alpha=\Theta_\theta^*(g_\theta)$ (we can apply this lemma since any compact subset of $\mathbb{R}^{k+1}\times S^{n-k-1}$ is contained in some $B^k(r)\times[-r',r']\times S^{n-k-1}$). We obtain a C^2 function $u\geq 0$ which is a solution of

$$L^{G_c}u=\bar{\mu}u^{p-1}$$

on $\mathbb{R}^{k+1}\times S^{n-k-1}$. From (12) it follows that

$$\int_{\mathbb{R}^{k+1}\times S^{n-k-1}} u^p\,dv^{G_c}\leq 1.$$

From (11) it follows that $u\in L^\infty(\mathbb{R}^{k+1}\times S^{n-k-1})$. With (56), we see that $u(0)\geq D_0$ and thus, $u\not\equiv 0$. By (55), we also get that $u\in L^2(\mathbb{R}^{k+1}\times S^{n-k-1})$. By the definition of $\Lambda_{n,k}^{(1)}$ we have that $\bar{\mu}\geq\Lambda_{n,k}^{(1)}\geq\Lambda_{n,k}$. This ends the proof of Theorem 7.1 in this subsubcase.

Subsubcase II.1.2. $\lim_{b\to 0}\limsup_{\theta\to 0}\sup_{U^N(b)}u_\theta=0$.

The proof in this subsubcase proceeds in several steps.

Step 1. *We prove* $\lim_{b\to 0}\limsup_{\theta\to 0}\int_{U^N(b)}u_\theta^p\,dv^{g_\theta}=0$.

Let $b > 0$. Using (55) we have

$$\int_{U^N(b)} u_\theta^p \, dv^{g_\theta} \le A_0 \sup_{U^N(b)} u_\theta^{p-2}$$

where the constant A_0 is independent of b and θ. Step 1 follows.

Step 2. *We show* $\liminf_{b\to 0} \liminf_{\theta\to 0} \int_{U^N(2b)\setminus U^N(b)} u_\theta^2 \, dv^{g_\theta} = 0$.

Let

$$d_0 := \liminf_{b\to 0} \liminf_{\theta\to 0} \int_{U^N(2b)\setminus U^N(b)} u_\theta^2 \, dv^{g_\theta}.$$

We prove this step by contradiction and assume that $d_0 > 0$. Then there exists $\delta > 0$ such that for all $b \in (0, \delta]$,

$$\liminf_{\theta\to 0} \int_{U^N(2b)\setminus U^N(b)} u_\theta^2 \, dv^{g_\theta} \ge \frac{d_0}{2}.$$

For $m \in \mathbb{N}$ we set $V_m := U(2^{-m}\delta) \setminus U(2^{-(m+1)}\delta)$. In particular we have

$$\liminf_{\theta\to 0} \int_{V_m} u_\theta^2 \, dv^{g_\theta} \ge \frac{d_0}{2}$$

for all m. Let $N_0 \in \mathbb{N}$. For $m \ne m'$ the sets V_m and $V_{m'}$ are disjoint. Hence we can write

$$\int_N u_\theta^2 \, dv^{g_\theta} \ge \int_{\bigcup_{m=0}^{N_0} V_m} u_\theta^2 \, dv^{g_\theta} \ge \sum_{m=0}^{N_0} \int_{V_m} u_\theta^2 \, dv^{g_\theta}$$

for θ small enough. This leads to

$$\begin{aligned}\liminf_{\theta\to 0} \int_N u_\theta^2 \, dv^{g_\theta} &\ge \liminf_{\theta\to 0} \sum_{m=0}^{N_0} \int_{V_m} u_\theta^2 \, dv^{g_\theta} \\ &\ge \sum_{m=0}^{N_0} \liminf_{\theta\to 0} \int_{V_m} u_\theta^2 \, dv^{g_\theta} \\ &\ge (N_0+1)\frac{d_0}{2}.\end{aligned}$$

Since N_0 is arbitrary, this contradicts that (u_θ) is bounded in $L^2(N)$ and proves Step 2.

Step 3. *Conclusion.*

Let $d_0 > 0$. By Steps 1 and 2 we can find $b > 0$ such that after passing to a subsequence, we have for all θ close to 0

$$\int_{N\setminus U^N(2b)} u_\theta^p \, dv^{g_\theta} \ge 1 - d_0 \tag{58}$$

and

$$\int_{U^N(2b)\setminus U^N(b)} u_\theta^2 \, dv^{g_\theta} \le d_0. \tag{59}$$

Let $\chi \in C^\infty(M)$, $0 \le \chi \le 1$, be a cut-off function equal to 0 on $U^N(b)$ and equal to 1 on $N \setminus U^N(2b)$. Since the set $U^N(2b) \setminus U^N(b)$ corresponds to $t \in [t_0 - \ln(2), t_0] \cup [t_1, t_1 + \ln 2]$ with $t_0 = -\ln(b) + \ln(\epsilon)$ and $t_1 = \ln(b) - \ln(\epsilon)$ we can assume that

$$|d\chi|_{g_\theta} \le 2\ln 2. \tag{60}$$

We will use the function χu_θ to estimate μ. This function is supported in $N \setminus U^N(b)$. If θ is smaller than b, then $(N \setminus U^N(b), g_\theta)$ is isometric to $(M \setminus U^M(b), F^2 g)$. In other words $(N \setminus U^N(b), g_\theta)$ is conformally equivalent to $(M \setminus U^M(b), g)$. Relation (3) implies that

$$\mu \le J_\theta(\chi u_\theta) = \frac{\int_N (a|d(\chi u_\theta)|^2_{g_\theta} + \mathrm{Scal}^{g_\theta}(\chi u_\theta)^2)\, dv^{g_\theta}}{\left(\int_N (\chi u_\theta)^p \, dv^{g_\theta}\right)^{\frac{n-2}{n}}}. \tag{61}$$

We multiply Equation (51) by $\chi^2 u_\theta$ and integrate over N. From (65) we see

$$\int_N |d(\chi u_\theta)|^2_{g_\theta}\, dv^{g_\theta} = \int_N \chi^2 u_\theta \Delta^{g_\theta} u_\theta \, dv^{g_\theta} + \int_N |d\chi|^2_{g_\theta} u_\theta^2 \, dv^{g_\theta},$$

and we obtain

$$\begin{aligned}\int_N \left(a|d(\chi u_\theta)|^2_{g_\theta} + \mathrm{Scal}^{g_\theta}(\chi u_\theta)^2\right) dv^{g_\theta} &= \mu_\theta \int_N u_\theta^p \chi^2 \, dv^{g_\theta} + a\int_N |d\chi|^2_{g_\theta} u_\theta^2 \, dv^{g_\theta} \\ &\le \mu_\theta \int_N u_\theta^p \, dv^{g_\theta} + |\mu_\theta| \int_{U^N(2b)} u_\theta^p \, dv^{g_\theta} \\ &\quad + a\int_N |d\chi|^2_{g_\theta} u_\theta^2 \, dv^{g_\theta}.\end{aligned}$$

Using (59) and (60), we have

$$\int_N |d\chi|^2_{g_\theta} u_\theta^2 \, dv^{g_\theta} = \int_{U^N(2b)\setminus U^N(b)} |d\chi|^2_{g_\theta} u_\theta^2 \, dv^{g_\theta} \le 4(\ln 2)^2 d_0.$$

Relation (58) implies $\int_{U^N(2b)} u_\theta^p \, dv^{g_\theta} \le d_0$. Together with $\int_N u_\theta^p \, dv^{g_\theta} = 1$

$$\int_N (a|d(\chi u_\theta)|^2_{g_\theta} + \mathrm{Scal}^{g_\theta}(\chi u_\theta)^2)\, dv^{g_\theta} \le \mu_\theta + |\mu_\theta| d_0 + 4(\ln 2)^2 a d_0. \tag{62}$$

In addition, by Relation (58),

$$\int_N (\chi u_\theta)^p \, dv^{g_\theta} \ge 1 - d_0. \tag{63}$$

Plugging (62) and (63) in (61) we get

$$\mu \le \frac{\mu_\theta + |\mu_\theta| d_0 + 4(\ln 2)^2 a d_0}{(1-d_0)^{\frac{n-2}{n}}}$$

for small θ. By taking the limit $\theta \to 0$ we can replace μ_θ by $\bar{\mu}$ in this inequality. Since d_0 can be chosen arbitrarily small we finally obtain $\mu \le \bar{\mu}$. This proves Theorem 7.1 in Subcase II.1.

Subcase II.2. For all $b > 0$, we have

$$\liminf_{\theta \to 0} \left(\mu_\theta \sup_{U^N(b)} u_\theta^{p-2} \right) \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

Hence, we can construct a subsequence of θ and a sequence (b_θ) of positive numbers converging to 0 with

$$\liminf_{\theta \to 0} \left(\mu_\theta \sup_{U^N(b_\theta)} u_\theta^{p-2} \right) \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

Choose a point $x''_\theta \in \overline{U^N(b_\theta)}$ such that $u_\theta(x''_\theta) = \sup_{U^N(b_\theta)} u_\theta$. Since $\mu_\theta \le \mu(\mathbb{S}^n)$, we have

$$u_\theta(x''_\theta) \ge D_1 := \left(\frac{(n-k-2)^2(n-1)}{8\mu(\mathbb{S}^n)(n-2)} \right)^{\frac{1}{p-2}}.$$

With similar arguments as in Subcase II.1.1 (just replace x'_θ by x''_θ and D_0 by D_1), we get the existence of a C^2 function $u \ge 0$ which is a solution of

$$L^{G_c} u = \bar{\mu} u^{p-1}$$

on $\mathbb{H}_c^{k+1} \times S^{n-k-1}$. As in Subsubcase II.1.1, $u \not\equiv 0$, $u \in L^\infty(\mathbb{H}_c^{k+1} \times S^{n-k-1})$, and

$$\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} u^p \, dv^{G_c} \le 1.$$

Moreover, the assumption of Subcase II.2 implies that

$$\bar{\mu} u^{p-2}(0) = \lim_{\theta \to 0} \mu_\theta u_\theta^{p-2}(x''_\theta) \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

By the definition of $\Lambda_{n,k}^{(2)}$ we have that $\bar{\mu} \ge \Lambda_{n,k}^{(2)} \ge \Lambda_{n,k}$.

Appendix A. Scalar curvature

In this section U denotes an open subset of a manifold and $q \in U$ a fixed point.

Proposition A.1. *Let g be a Riemannian metric on U and T a symmetric 2-tensor such that $\tilde{g} := g + T$ is positive definite. Then the scalar curvature $\mathrm{Scal}^{g+T}(q)$ of $g + T$ in $q \in U$ is a smooth function of the Riemann tensor of g in q, of $T|_q$, $\nabla T|_q$ and $\nabla^2 T|_q$. Here ∇ is the Levi-Civita connection of g. Moreover the operator $T \mapsto \mathrm{Scal}^{g+T}(q)$ is a quasilinear partial differential operator of second order.*

Proof. Here we denote the Riemann curvature of $\tilde{g}$ by $\tilde{R}$, etc. We use to the notation from [18] which also coincide with those in [5].

$$R_{ijkl} = g(R(\partial_k, \partial_l)\partial_j, \partial_i), \qquad \tilde{R}_{ijkl} = \tilde{g}(\tilde{R}(\partial_k, \partial_l)\partial_j, \partial_i)$$

We work in normal coordinates for the metric g centered in q, indices of partial derivatives in coordinates are added and separated with , and covariant ones with respect to g separated with ;. In particular $T = T_{ij} dx^i \, dx^j$,

$$T_{kl;i} = (\nabla_i T)(\partial_k, \partial_l) = \partial_i T_{kl} - T_{ml}\Gamma_{ik}^m - T_{km}\Gamma_{il}^m.$$

In particular in q we have $\tilde{g}_{kl,i} = T_{kl;i}$.

As explained in [5, Formula (13)] we have in q

$$\nabla_\alpha \Gamma_{ij}^k = \partial_\alpha \Gamma_{ij}^k = -\frac{1}{3}\left(R_{ik\alpha j} + R_{i\alpha kj}\right).$$

Hence in q,

$$T_{kl;rs} = (\nabla_{rs}^2 T)(\partial_k, \partial_l) = \partial_r \partial_s T_{kl} + \frac{1}{3} T_{ml}(R_{smrk} + R_{srmk}) + \frac{1}{3} T_{mk}(R_{smrl} + R_{srml}).$$

In order to calculate the scalar curvature $\mathrm{Scal}^{\tilde{g}}(q)$ of $\tilde{g}$ in q we use the curvature formula as in [18] and contract twice. We obtain

$$\mathrm{Scal}^{\tilde{g}}(q) = \tilde{g}^{ik}\tilde{g}^{jm}(\tilde{g}_{km,ij} - \tilde{g}_{ki,mj}) + P(\tilde{g}^{rm}, \tilde{g}_{ij,k}) \tag{64}$$

where P is an invariant polynomial expression in $\tilde{g}^{-1}$ and $\partial\tilde{g}$ that is cubic in $\tilde{g}^{-1} = \tilde{g}^{rm}$ and quadratic in $\tilde{g}_{ij,k}$. (Note that formula (64) holds for an arbitrary metric in arbitrary coordinates.) The polynomial P vanishes for $T = 0$ in normal coordinates for g. □

Corollary A.2. *Let $\mathcal{R} \subset T_q^*M \otimes T_q^*M \otimes T_q^*M \otimes T_qM$ be a bounded set of curvature tensors. Then there is an $\epsilon > 0$ and $C \in \mathbb{R}$ such that for all metrics g on U with $R^g|_q \in \mathcal{R}$ we have: if*

$$\max_{i \in \{0,1,2\}} \left|(\nabla^i T)_q\right| < \epsilon,$$

then

$$|\mathrm{Scal}^{g+T}(q) - \mathrm{Scal}^g(q)| \leq C \left(\left|\nabla^2 T|_q\right| + |\nabla T|_q|^2 + |T|_q| \right).$$

Appendix B. Details for equation (16)

We compute the scalar curvature of the metric $dt^2 + e^{2\varphi(t)} h_t$ on $I \times W$. This is a generalized cylinder metric as studied in [8]. In the following computations we use the notation from [8], so $g_t = e^{2\varphi(t)} h_t$ and we have

$$\dot{g}_t = 2\varphi'(t) e^{2\varphi(t)} h_t + e^{2\varphi(t)} \partial_t h_t,$$

and

$$\ddot{g}_t = (2\varphi''(t) + 4\varphi'(t)^2) e^{2\varphi(t)} h_t + 4\varphi'(t) e^{2\varphi(t)} \partial_t h_t + e^{2\varphi(t)} \partial_t^2 h_t.$$

This implies that the shape operator S of the hypersurfaces of constant t is given by

$$S = -\varphi' - \frac{1}{2} h_t^{-1} \partial_t h_t,$$

so

$$\mathrm{tr}(S^2) = k\varphi'(t)^2 + \varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \frac{1}{4}\mathrm{tr}((h_t^{-1}\partial_t h_t)^2),$$

and

$$\mathrm{tr}(S)^2 = k^2\varphi'(t)^2 + k\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \frac{1}{4}\mathrm{tr}(h_t^{-1}\partial_t h_t)^2.$$

Further

$$\begin{aligned}\mathrm{tr}_{g_t} \ddot{g}_t &= (2\varphi''(t) + 4\varphi'(t)^2)k + 4\varphi'(t)\mathrm{tr}_{h_t}(\partial_t h_t) + \mathrm{tr}_{h_t}(\partial_t^2 h_t) \\ &= (2\varphi''(t) + 4\varphi'(t)^2)k + 4\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \mathrm{tr}_{h_t}(\partial_t^2 h_t).\end{aligned}$$

From [8, Proposition 4.1, (21)] we have

$$\begin{aligned}\mathrm{Scal}^{e^{2\varphi(t)}h_t + dt^2} &= \mathrm{Scal}^{e^{2\varphi(t)}h_t} + 3\mathrm{tr}(S^2) - \mathrm{tr}(S)^2 - \mathrm{tr}_{g_t}\ddot{g}_t \\ &= e^{-2\varphi(t)}\mathrm{Scal}^{h_t} - k(k+1)\varphi'(t)^2 - (k+1)\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) \\ &\quad - 2k\varphi''(t) + \frac{3}{4}\mathrm{tr}((h_t^{-1}\partial_t h_t)^2) - \frac{1}{4}\mathrm{tr}(h_t^{-1}\partial_t h_t)^2 - \mathrm{tr}_{h_t}(\partial_t^2 h_t).\end{aligned}$$

When we add the scalar curvature of σ^{n-k-1} we get Formula (16) for the scalar curvature of $g_{\mathrm{WS}} = dt^2 + e^{2\varphi(t)} h_t + \sigma^{n-k-1}$.

Appendix C. A Cut-off formula

Here we state a formula used several times in the article. The functions u and χ are smooth functions on a Riemannian manifold (N, h). We assume that χ is compactly supported. Then

$$\begin{aligned}\int_N |d(\chi u)|^2 \, dv^h &= \int_N u \left(\langle d\chi, d(\chi u)\rangle + \langle \chi du, d(\chi u)\rangle \right) dv^h \\ &= \int_N \left(u^2 |d\chi|^2 + u\chi \langle d\chi, du\rangle + \chi u d^*(\chi du) \right) dv^h \\ &= \int_N \left(u^2 |d\chi|^2 + \chi^2 u \Delta u \right) dv^h.\end{aligned} \tag{65}$$

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