Universität Regensburg
Mathematik

Surgery and the spinorial τ-invariant

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Preprint Nr. 27/2007

SURGERY AND THE SPINORIAL τ-INVARIANT

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ABSTRACT. We associate to a compact spin manifold M a real-valued invariant $\tau(M)$ by taking the supremum over all conformal classes of the infimum inside each conformal class of the first positive Dirac eigenvalue, normalized to volume 1. This invariant is a spinorial analogue of Schoen's σ-constant, also known as the smooth Yamabe number.

We prove that if N is obtained from M by surgery of codimension at least 2, then $\tau(N) \geq \min\{\tau(M), \Lambda_n\}$ with $\Lambda_n > 0$. Various topological conclusions can be drawn, in particular that τ is a spin-bordism invariant below Λ_n. Below Λ_n, the values of τ cannot accumulate from above when varied over all manifolds of a fixed dimension.

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Date: November 7, 2007.
2000 *Mathematics Subject Classification.* 53C27 (Primary) 55N22, 57R65 (Secondary).
Key words and phrases. Dirac operator, eigenvalue, surgery.
We thank Victor Nistor, Penn State University, for some helpful discussions about Sobolev inequalities on non-compact manifolds with ends that are asymptotic to products of hyperbolic spaces and spheres. Also Mattias Dahl thanks the Institut Élie Cartan, Nancy, for its kind hospitality and support during visits when this work was initiated.

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1. Introduction

1.1. **Spin manifolds, Dirac operators and choice of Clifford representation.** Let M be a compact n-dimensional spin manifold without boundary. We will always consider spin manifolds as equipped with an orientation and a spin structure. The existence of these structures is equivalent to the vanishing of the first and second Stiefel-Whitney class. In some of the literature, the word "spin" only means that such structures exist. However, we use the word "spin" in the sense that M actually comes with a choice of orientation and spin structure.

As explained in [19, 10, 14] one associates to the spin structure, to a Riemannian metric g on M and to a complex irreducible representation ρ of the Clifford algebra over $\mathbb{R}^n$ a complex vector bundle, the spinor bundle $\Sigma^g_\rho(M)$. The Dirac operator D^g_ρ is a self-adjoint elliptic first order differential operator acting on smooth sections of the spinor bundle $\Sigma^g(M)$. It has a spectrum consisting only of real eigenvalues of finite multiplicity. The Dirac operator depends on the choice of spin structure, on the metric g and a priori on the representation ρ. In even dimensions n, the representation ρ is unique (up to equivalence). In odd dimensions there are two choices $\rho\{+\}$ and $\rho\{-\}$. Exchanging the representation results in reversing the spectrum, i.e. if λ is an eigenvalue of $D^g_{\rho\{+\}}$ then $-\lambda$ is an eigenvalue of $D^g_{\rho\{-\}}$ with the same multiplicity, and vice versa. This has no effect if $n \equiv 1 \mod 4$ since the real/quaternionic structure on $\Sigma^g(M)$ anti-commutes with the Dirac operator and the spectrum therefore is symmetric, see [10, Section 1.7]. However, in dimension $n \equiv 3 \mod 4$ the choice of ρ matters. In this case we choose the representation such that Clifford multiplication of $e_1 \cdot e_2 \cdots e_n$ acts as the identity, where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$. We thus can and will suppress ρ in the notation.

1.2. **The τ-invariant.** We denote by $\lambda^+_1(D^{\tilde{g}})$ the first non-negative eigenvalue of $D^{\tilde{g}}$. For a metric g on M we define

$$\lambda^+_{\min}(M, g) := \inf \lambda^+_1(D^{\tilde{g}})\mathrm{Vol}(M, \tilde{g})^{\frac{1}{n}}$$

where the infimum is taken over all metrics $\tilde{g}$ conformal to g. Further we define

$$\tau^+(M) := \sup \lambda^+_{\min}(M, g)$$

where the supremum is taken over all metrics g on M. This yields an invariant of the spin manifold M, observe that we do not require M to be connected.

We begin by noting some simple properties of the invariant τ^+. Let (S^n, σ^n) denote the sphere with its standard metric. We have

$$\lambda^+_{\min}(S^n, \sigma^n) = \frac{n}{2}\omega_n^{1/n}$$

where ω_n is the volume of (S^n, σ^n). Moreover it is shown in [2, 6] that

$$\lambda^+_{\min}(M, g) \le \lambda^+_{\min}(S^n, \sigma^n)$$

for any compact Riemannian spin manifold. Together with Inequality (1) below we get

$$\tau^+(S^n) = \lambda^+_{\min}(S^n, \sigma^n) = \frac{n}{2}\omega_n^{1/n}$$

so for all compact spin manifolds M we have

$$\tau^+(M) \le \tau^+(S^n).$$

If the kernel of D^g is non-trivial, then obviously $\lambda^+_{\min}(M,g) = 0$. Conversely, it was shown in [2] that if D^g is invertible, then $\lambda^+_{\min}(M,g) > 0$. It follows that $\tau^+(M) > 0$ is equivalent to the existence of a metric g on M for which the Dirac operator D^g is invertible. It is a further fact that $\tau^+(M) = 0$ precisely when $\alpha(M) \neq 0$, where $\alpha(M)$ is the alpha-invariant which equals the index of the Dirac operator for any metric on M, see [4].

If (M_1, g_1) and (M_2, g_2) are compact Riemannian spin manifolds we denote by $M_1 \amalg M_2$ the disjoint union of M_1 and M_2 equipped with the natural metric $g_1 \amalg g_2$. It is not difficult to see that

$$\lambda^+_{\min}(M_1 \amalg M_2, g_1 \amalg g_2) = \min\{\lambda^+_{\min}(M_1, g_1), \lambda^+_{\min}(M_2, g_2)\}.$$

This implies

$$\tau^+(M_1 \amalg M_2) = \min\{\tau^+(M_1), \tau^+(M_2)\}.$$

We denote by $-M$ the manifold M equipped with the opposite orientation. The Dirac operator changes sign when the orientation of the manifold is reversed. In dimensions $\not\equiv 3 \mod 4$ this does not change the first positive eigenvalue of D since the spectrum is symmetric, so in those dimensions we have $\lambda^+_{\min}(-M,g) = \lambda^+_{\min}(M,g)$ and $\tau^+(-M) = \tau^+(M)$. For dimensions $\equiv 3 \mod 4$ we define $\lambda^-_{\min}(M,g)$ and $\tau^-(M)$ similar to $\lambda^+_{\min}(M,g)$ and $\tau^+(M)$ by replacing λ_1^+ by the absolute value of the first non-positive eigenvalue. We then have $\lambda^+_{\min}(-M,g) = \lambda^-_{\min}(M,g)$ and $\tau^+(-M) = \tau^-(M)$.

1.3. **The σ-constant.** The τ-invariant is a spinorial analogue of the σ-constant [17, 21] which is defined for a compact manifold M by

$$\sigma(M) := \sup \inf \frac{\int \mathrm{Scal}^{\tilde g}\, dv^{\tilde g}}{\mathrm{Vol}(M,\tilde g)^{\frac{n-2}{n}}}$$

where the infimum runs over all metrics $\tilde g$ in a conformal class and the supremum runs over all conformal classes ($\sigma(M)$ is also known as the Yamabe invariant of M.) When $\sigma(M)$ is positive it can be computed in a way analogous to $\tau^+(M)$ using the smallest eigenvalue of the conformal Laplacian $L^g = 4\frac{n-1}{n-2}\Delta^g + \mathrm{Scal}^g$ instead of $\lambda_1^+(D^g)$. Hijazi's inequality [12, 13] gives a comparison of the two invariants,

$$\tau^{\pm}(M)^2 \ge \frac{n}{4(n-1)}\sigma(M). \tag{1}$$

For $M = S^n$ equality is attained in this inequality. Upper bounds for $\tau^{\pm}(M)$ may help to determine the σ-constant.

We are currently working out an analogous surgery formula for the σ-invariant under surgeries of codimension at least 3. If the dimension of the surgery is smaller than $[n/2]-1$, then the techniques of the present article can be carried over to the σ-invariant, but for higher-dimensional surgeries other techniques will be used. See [5] for an announcement of these results.

1.4. **Geometric constants.** We are going to prove a surgery formula for the invariant τ^+. This formula involves some geometric constants $\Lambda_{n,k}$ which we now define.

For a complete spin manifold (V, g) we set

$$\widetilde{\lambda^+_{\min}}(V,g) := \inf \lambda \in [0,\infty]$$

where the infimum is taken over all $\lambda \in (0,\infty)$ for which there is a non-zero spinor field $\varphi \in L^\infty \cap L^2 \cap C^1_{\mathrm{loc}}$ such that $\|\varphi\|_{L^{\frac{2n}{n-1}}} \leq 1$ and

$$D^g \varphi = \lambda |\varphi|^{\frac{2}{n-1}} \varphi. \tag{2}$$

If there are no such solutions of (2) on V then $\widetilde{\lambda^+_{\min}}(V,g) = \infty$.

For k a positive integer we let ξ^k be the Euclidean metric on $\mathbb{R}^k$. For $c \in \mathbb{R}$ we let $\eta_c^{k+1} = e^{2ct}\xi^k + dt^2$ be the hyperbolic metric of sectional curvature $-c^2$ on $\mathbb{R}^{k+1}$. As above σ^{n-k-1} denotes the metric of curvature 1 on S^{n-k-1}. We define our geometric invariants as

$$\Lambda_{n,k} := \inf_{c \in [-1,1]} \widetilde{\lambda^+_{\min}}(\mathbb{R}^{k+1} \times S^{n-k-1}, \eta_c^{k+1} + \sigma^{n-k-1}), \qquad \Lambda_n := \min_{0 \leq k \leq n-2} \Lambda_{n,k}.$$

Note that the infimum could as well be taken over $c \in [0,1]$. It is easy to show that $\Lambda_{n,0} = \lambda^+_{\min}(S^n, \sigma^n)$. For $k > 0$ we are not able to compute these constants, but at least we can show that they are positive.

Theorem 1.1. *For $0 \leq k \leq n-2$ we have $\Lambda_{n,k} > 0$.*

1.5. **Joining manifolds.** We are going to study the behaviour of τ^+ when two compact Riemannian spin manifolds are joined along a common submanifold. Let (M_1, g_1) and (M_2, g_2) be compact spin manifolds of dimension n and let N be obtained by joining M_1 and M_2 along a submanifold of dimension k as described in Section 2.2. The manifold N is spin and from the construction follows a natural choice of spin structure on N. The following results make it possible to compare $\tau^+(M_1 \amalg M_2)$ and $\tau^+(N)$.

Theorem 1.2. *Assume that $k = \dim W$ satisfies $0 \leq k \leq n-2$. Let $w_i : W \times B^{n-k} \to M_i^n$ be embeddings and let N be obtained by joining M_1 and M_2 along W. Assume further that both D^{g_1} and D^{g_2} have trivial kernel. Then there is a sequence of metrics g_θ, $\theta \to 0$, such that*

$$\min\{\lambda^+_{\min}(M_1 \amalg M_2, g_1 \amalg g_2), \Lambda_{n,k}\} \leq \lim_{\theta \to 0} \lambda^+_{\min}(N, g_\theta) \leq \lambda^+_{\min}(M_1 \amalg M_2, g_1 \amalg g_2).$$

Taking the supremum over all metrics on $M_1 \amalg M_2$ the first inequality gives us the following corollary.

Corollary 1.3. *We have*

$$\tau^+(N) \geq \min\{\tau^+(M_1 \amalg M_2), \Lambda_{n,k}\} \geq \min\{\tau^+(M_1), \tau^+(M_2), \Lambda_n\}.$$

Note that these estimates on τ^+ would be trivial without Theorem 1.1.

1.6. **Surgery and bordism.** Performing surgery on a spin manifold is a special case of joining manifolds, this is discussed in more detail in Section 2.2 below. As a special case of Corollary 1.3 we get an inequality relating the τ-invariant before and after surgery. For a compact spin manifold M of dimension n we define

$$\overline{\tau}^+(M) := \min\{\tau^+(M), \Lambda_n\}.$$

It will also be convenient to introduce

$$\overline{\tau}(M) := \min\{\tau^+(M), \tau^-(M), \Lambda_n\}.$$

As already explained, in the case $n \not\equiv 3 \mod 4$, one has $\overline{\tau}(M) = \overline{\tau}^+(M)$. As before, all results for $\overline{\tau}^+(M)$ also hold for $\overline{\tau}^-(M) := \min\{\tau^-(M), \Lambda_n\}$.

Corollary 1.4. *Assume that M is a spin manifold of dimension n and that N is obtained from M by a surgery of codimension $n-k \geq 2$, then*

$$\tau^+(N) \geq \min\{\tau^+(M), \Lambda_{n,k}\} \geq \min\{\tau^+(M), \Lambda_n\}.$$

The Corollary implies, in particular,

$$\overline{\tau}^+(N) \geq \overline{\tau}^+(M), \quad \overline{\tau}(N) \geq \overline{\tau}(M).$$

Two compact spin manifolds M and N are spin bordant if there is a spin diffeomorphism from their disjoint union to the boundary of a spin manifold of one dimension higher, and this diffeomorphism respects the orientation of N and reverses that of M. This happens if and only if N can be obtained from M by a sequence of surgeries. To apply Corollary 1.4 we need to know when this sequence of surgeries can be chosen to include only surgeries of codimension at least two. The theory of handle decompositions of bordisms tells us that this can be done when N is connected, see [16, VII Theorem 3] for dimension 3 and [18, VIII Theorem 3.1] for higher dimensions.

Corollary 1.5. *Let M and N be spin bordant manifolds of dimension at least 3 and assume that N is connected. Then $\overline{\tau}(N) \geq \overline{\tau}(M)$. In particular, if M is also connected we have $\overline{\tau}(N) = \overline{\tau}(M)$.*

The corollary can also be shown in dimension 2 with similar arguments [7, Theorem 1.3].

The spin bordism group Ω_n^{spin} is the set of equivalence classes of spin bordant manifolds of dimension n with disjoint union as addition. Since every element in Ω_n^{spin} can be represented by a connected manifold we obtain a well-defined map $\overline{\tau} : \Omega_n^{\text{spin}} \to [0, \Lambda_n]$ which sends the equivalence class $[M]$ of a connected spin manifold M to $\overline{\tau}(M)$.

Corollary 1.6. *There is a positive constant ε_n such that*

$$\tau^+(M) \in \{0\} \cup [\varepsilon_n, \lambda^+_{\min}(S^n, \sigma^n)].$$

for all spin manifolds M of dimension n.

Proof. The spin bordism group Ω_n^{spin} is finitely generated [22, page 336]. This implies that the kernel of the map $\alpha : \Omega_n^{\text{spin}} \to KO_n$ is also finitely generated. Let $[N_1], \ldots, [N_r]$ be generators of this kernel, we assume that all N_i are connected. Since $\tau(M) = 0$ if and only if $\alpha(M) \neq 0$ we obtain the corollary for

$$\varepsilon_n := \min\{\Lambda_n, \overline{\tau}(N_1), \ldots, \overline{\tau}(N_r)\}.$$

□

The α-map is injective when $n < 8$, and then $\varepsilon_n = \Lambda_n$. Unfortunately, we do not know whether there are $n \in \mathbb{N}$ with $\varepsilon_n < \Lambda_n$. In other words, we do not know if there are n-dimensional manifolds M with $0 < \tau^+(M) < \Lambda_n$. If such manifolds exist, the following conclusions might be interesting.

At first, if M is a spin manifold with $\tau^+(M) < \Lambda_n$, then it follows from the Corollary 1.5 that the σ-invariant of any manifold N spin bordant to M satisfies

$$\sigma(N) \leq \frac{4(n-1)}{n}\tau^+(M)^2.$$

For the next application we define

$$S(t) := \{[M] \in \Omega_n^{\mathrm{spin}} \,|\, \overline{\tau}(M) \geq t\} \qquad S^+(t) := \{[M] \in \Omega_n^{\mathrm{spin}} \,|\, \overline{\tau}^+(M) \geq t\}$$

and

$$T(t) := \{[M] \in \Omega_n^{\mathrm{spin}} \,|\, \overline{\tau}(M) > t\} \qquad T^+(t) := \{[M] \in \Omega_n^{\mathrm{spin}} \,|\, \overline{\tau}^+(M) > t\}.$$

Obviously $S(t) = S^+(t)$ and $T(t) = T^+(t)$ in dimensions $n \not\equiv 3 \mod 4$.

Corollary 1.7. *$S(t)$ is a subgroup of Ω_n^{spin} for $t \in [0, \Lambda_n]$ and $T(t)$ is a subgroup of Ω_n^{spin} for $t \in [0, \Lambda_n)$. If $n \equiv 3 \mod 4$, then $S^+(t)$ and $T^+(t)$ are subsemigroups.*

Corollary 1.8. *The values of $\overline{\tau}$ cannot accumulate from above.*

Proof. Assume that $\overline{\tau}(M_i) = t_i$, $i \in \mathbb{N}$ is a sequence of values of $\overline{\tau}$. We want to show that the infimum $\inf t_i$ is attained by a t_i. Assume that the infimum is not attained. After passing to a subsequence we can assume that the sequence t_i is decreasing. Then $S(t_i) \subset S(t_{i+1})$, hence $\bigcup S(t_i) = T(t_\infty)$ is a subgroup of the finitely generated group Ω_n^{spin}. It is thus finitely generated itself. Hence there exists $r \in \mathbb{N}$ such that $S(t_r)$ contains the finite set of generators, and thus $S(t_r) = T(t_\infty)$. Hence $[M_i] \in S(t_r)$ for all i, which implies $t_i \geq t_r$, i.e. we obtain the contradiction $t_r = \inf t_i$. □

We do not know whether the same statement holds for $\overline{\tau}^+$ in dimensions $n \equiv 3 \mod 4$.

1.7. **Variants of the results.** We already remarked earlier that if the alpha-genus $\alpha(M)$ of a spin manifold M does not vanish, then the index theorem tells us that for any metric g on M, the kernel of D^g is non-trivial, and hence $\tau^+(M) = 0$. More exactly the index theorem implies for connected spin manifold M, that the kernel of the Dirac operator has at least dimension

$$a(M) := \begin{cases} |\widehat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let us modify the definition of τ^+ and use the k-th non-negative eigenvalue of the Dirac operator instead of the first one. The quantity thus obtained, denoted by $\tau_k^+(M)$ is zero if $k \leq a(M)$. It follows from [2] and [4] that $\tau_{a(M)+1}^+(M) > 0$. We expect that our methods generalize to this situation and yield similar surgery formulas for τ_k^+.

2. Preliminaries

2.1. **Notation for balls and neighbourhoods.** We write $B^n(r)$ for the open standard ball of radius r around 0 in $\mathbb{R}^n$, and set $B^n := B^n(1)$. For a Riemannian manifold (M, g) we let $B^g(p, r)$ denote the open ball of radius r around $p \in M$. If the Riemannian metric is clear from the context we will write $B(p, r)$. For a

Riemannian manifold (M, g) with subset S we let $B^g(S, r) := \bigcup_{x\in S} B^g(p, r)$, the r-neighbourhood of S. Again, if the Riemannian metric is clear from the context we abbreviate to $B(S, r)$.

2.2. **Joining manifolds along submanifolds.** Let (M_1, g_1) and (M_2, g_2) be compact Riemannian spin manifolds of dimension n. Let W be a compact k-dimensional spin manifold where $0 \le k \le n$, and assume that $w_i : W \times B^{n-k} \to M_i$, $i = 1, 2$ are embeddings such that w_1 is orientation reserving and w_2 is orientation preversing and they both preserve spin structures. Let N be obtained by joining M_1 and M_2 along W. The manifold N is naturally equipped with a spin structure.

We define

$$W' := w_1(W \times \{0\}) \amalg w_2(W \times \{0\}).$$

When we restrict g_1 to $w_1(W \times \{0\})$ and pull it back to W, we obtain a metric on W, called h_1. Similarly one obtains the metric h_2 on W by restricting and pulling-back g_2 via w_2. Let r_i denote the distance function in (M_i, g_i) to $w_i(W \times \{0\})$ and let

$$r(x) := \begin{cases} r_1(x), & \text{if } x \in M_1; \\ r_2(x), & \text{if } x \in M_2. \end{cases}$$

For $0 < \varepsilon$ we define $U_i(\varepsilon) := \{x \in M_i : r_i(x) < \varepsilon\}$ and $U(\varepsilon) := U_1(\varepsilon) \cup U_2(\varepsilon)$. For $0 < \varepsilon < \theta$ we define

$$N_\varepsilon := (M_1 \setminus U_1(\varepsilon)) \cup (M_2 \setminus U_2(\varepsilon))/ \sim,$$

and

$$U^N_\varepsilon(\theta) := (U(\theta) \setminus U(\varepsilon))/ \sim$$

where $\sim$ indicates that we identify $x \in \partial U_1(\varepsilon)$ with $w_2 \circ w_1^{-1}(x) \in \partial U_2(\varepsilon)$. Hence $N_\varepsilon = (M_1 \amalg M_2 \setminus U(\theta)) \cup U^N_\varepsilon(\theta)$. We say that N_ε *is obtained from* $M_1 \amalg M_2$ *by a connected sum along* W *with parameter* ε*.*

The operation of doing surgery on a manifold is a special case of this construction. Indeed, let M be a compact spin manifold and set $M_1 = M$, $M_2 = S^n$, $W = S^k$, $w_1 : S^k \times B^{n-k} \to M$ an embedding defining a surgery of dimension k, $w_2 : S^k \times B^{n-k} \to S^n$ the standard embedding and let N_ε be obtained from M by a surgery along W. Since $S^n \setminus w_2(S_k \times \{0\})$ is diffeomorphic to $B^{k+1} \times S^{n-k-1}$ we have that N_ε is obtained from M by a surgery of dimension k, see [18, Section VI.9].

The diffeomorphism type of N_ε is independent of ε, hence we will usually write $N = N_\varepsilon$. However, in some situations where dropping the index ε might cause ambiguites we will write N_ε. For example the function $r : M_1 \amalg M_2 \to [0, \infty)$ also defines a continuous function $r : N_\varepsilon \to [\varepsilon, \infty)$ whose definition depends on ε. We will also keep the ε-subscript for $U^N_\varepsilon(\theta)$ as important estimates for spinors will be carried out on $U^N_\varepsilon(\theta)$, and these estimates are not invariant if one applies a diffeomorphism of M to $U^N_\varepsilon(\theta)$ without applying it to the spinor. As the embeddings w_1 and w_2 preserve the spin structure, the manifold N carries a spin structure such that its restriction to $(M_1 \setminus w_1(W \times B^{n-k})) \amalg (M_2 \setminus w_2(W \times B^{n-k}))$ coincides with the restriction of the given spin structure on $M_1 \amalg M_2$. If W is not connected, then this choice is not unique. The statements of our theorem hold for any such spin structure on N.

2.3. **Comparing spinors for different metrics.** Let M be a spin manifold of dimension n and let g, g' be Riemannian metrics on M. The goal of this paragraph is to identify the spinor bundles of (M,g) and (M,g') following Bourguignon and Gauduchon [9].

There exists a unique endomorphism $b^g_{g'}$ of TM which is positive, symmetric with respect to g, and satisfies $g(X,Y) = g'(b^g_{g'}X, b^g_{g'}Y)$ for all $X,Y \in TM$. This endomorphism maps g-orthonormal frames at a point to g'-orthonormal frames at the same point and we get a map $b^g_{g'} : \mathrm{SO}(M,g) \to \mathrm{SO}(M,g')$ of $\mathrm{SO}(n)$-principal bundles. If we assume that $\mathrm{Spin}(M,g)$ and $\mathrm{Spin}(M,g')$ are equivalent spin structures on M the map $b^g_{g'}$ lifts to a map $\beta^g_{g'}$ of $\mathrm{Spin}(n)$-principal bundles,

$$\begin{array}{ccc} \mathrm{Spin}(M,g) & \xrightarrow{\beta^g_{g'}} & \mathrm{Spin}(M,g') \\ \downarrow & & \downarrow \\ \mathrm{SO}(M,g) & \xrightarrow{b^g_{g'}} & \mathrm{SO}(M,g') \end{array}.$$

From this we get a map between the spinor bundles $\Sigma^g M$ and $\Sigma^{g'} M$ denoted by the same symbol and defined by

$$\Sigma^g M = \mathrm{Spin}(M,g) \times_\sigma \Sigma_n \to \mathrm{Spin}(M,g') \times_\sigma \Sigma_n = \Sigma^{g'} M$$
$$\psi = [s,\varphi] \mapsto [\beta^g_{g'} s, \varphi] = \beta^g_{g'}\psi$$

where (σ, Σ_n) is the complex spinor representation, and where $[s,\varphi]$ denotes an element of $\mathrm{Spin}(M,g) \times_\sigma \Sigma_n$. Note that the map $\beta^g_{g'}$ is fiberwise an isometry.

We define the Dirac operator $D^{g'}$ acting on sections of the spinor bundle for g by

$${}^g D^{g'} := (\beta^g_{g'})^{-1} \circ D^{g'} \circ \beta^g_{g'}$$

In [9, Theorem 20] the operator ${}^g D^{g'}$ is computed in terms of D^g and some extra terms which are small if g and g' are close. Formulated in a way convenient for us the relationship is

$${}^g D^{g'} \psi = D^g \psi + A^g_{g'}(\nabla^g \psi) + B^g_{g'}(\psi) \tag{3}$$

where $A^g_{g'} \in \mathrm{hom}(T^*M \otimes \Sigma^g M, \Sigma^g M)$ satisfies

$$|A^g_{g'}| \leq C|g - g'|_g \tag{4}$$

and $B^g_{g'} \in \mathrm{hom}(\Sigma^g M, \Sigma^g M)$ satisfies

$$|B^g_{g'}| \leq C(|g - g'|_g + |\nabla^g (g - g')|_g) \tag{5}$$

for some constant C.

In the special case that g' and g are conformal with $g' = F^2 g$ for a positive smooth function F the formula simplifies considerably, and one obtains

$${}^g D^{g'} (F^{-\frac{n-1}{2}} \psi) = F^{-\frac{n+1}{2}} D^g \psi \tag{6}$$

see for instance [15, 8].

2.4. **Regularity results.** By standard elliptic theory we have the following lemma (see for example [3] where the corresponding results of [11] are adjusted to the Dirac operator).

Lemma 2.1. *Let (V, g) be a Riemannian spin manifold and $\Omega \subset V$ an open set with compact closure in V. Let also $r \in (1, \infty)$. Then there is a constant C so that*

$$\int_\Omega |\nabla^g \varphi|^r \, dv^g \leq C \left(\int_\Omega |D^g \varphi|^r \, dv^g + \int_\Omega |\varphi|^r \, dv^g \right) \tag{7}$$

for all $\varphi \in \Gamma(\Sigma^g \Omega)$ which are of class C^1 and compactly supported in Ω.

In case of a compact Riemannian manifold with invertible Dirac operator we have the following special case.

Lemma 2.2. *Let (V, g) be a compact Riemannian spin manifold with invertible Dirac operator. Then there exists a constant C such that*

$$\int_V |\nabla^g \varphi|^{\frac{2n}{n+1}} \, dv^g \leq C \int_V |D^g \varphi|^{\frac{2n}{n+1}} \, dv^g. \tag{8}$$

for all $\varphi \in \Gamma(\Sigma^g V)$ of class $C^1(V)$.

2.5. **The associated variational problem.** Let (M, g) be a compact spin manifold of dimension n with $\ker D^g = \{0\}$. For $\psi \in \Gamma(\Sigma M)$ we define

$$J(\psi) := \frac{\left(\int_M |D\psi|^{\frac{2n}{n+1}} \, dv^g \right)^{\frac{n+1}{n}}}{\int_M \langle D\psi, \psi \rangle \, dv^g}.$$

whenever the denominator is non-zero. Using techniques from [20] it was proved in [2] that

$$\lambda^+_{\min}(M, g) = \inf_\psi J(\psi) \tag{9}$$

where the infimum is taken over the set of smooth spinor fields satisfying

$$\int_M \langle D\psi, \psi \rangle \, dv^g > 0.$$

If g and $\tilde{g} = F^2 g$ are conformal metrics on M and if J and $\tilde{J}$ are the associated functionals, then by Relation (6) one computes that for all smooth $\psi \in \Gamma(\Sigma^g M))$

$$\tilde{J}(F^{-\frac{n-1}{2}} \psi) = J(\psi). \tag{10}$$

The following result gives a universal upper bound on $\lambda^+_{\min}(M, g)$.

Proposition 2.3. *Let (M, g) be a compact spin manifolds of dimension $n \geq 2$. Then*

$$\lambda^+_{\min}(M, g) \leq \lambda^+_{\min}(S^n, \sigma^n) = \frac{n}{2} \, \omega_n^{1/n} \tag{11}$$

where ω_n is the volume of (S^n, σ^n).

The proposition was proven for $n \geq 3$ in [2] using geometric methods. In the case $n = 2$ the article [2] only provides a proof if $\ker D = \{0\}$. Another method that yields the proposition in full generality is to construct for any $p \in M$ and $\varepsilon > 0$ a suitable test spinor field ψ_ε supported in $B^g(p, \varepsilon)$ satisfying $J(\psi_\varepsilon) \leq \lambda^+_{\min}(S^n, \sigma^n) + o(\varepsilon)$, see [6] for details.

If inequality (11) holds strictly then one can show that the infimum in equation (9) is attained by a spinor field φ. The following theorem will be a central ingredient in the proof of Theorem 1.2.

Theorem 2.4 ([1, 3])**.** *Let (M,g) be a compact spin manifold of dimension n for which inequality (11) holds strictly. Then there exists a spinor field $\varphi \in C^{2,\alpha}(\Sigma M) \cap C^\infty(\Sigma M \setminus \varphi^{-1}(0))$ where $\alpha \in (0,1) \cap (0, 2/(n-1)]$ such that $\|\varphi\|_{L^{\frac{2n}{n-1}}} = 1$ and*

$$D\varphi = \lambda^+_{\min}(M,g)|\varphi|^{\frac{2}{n-1}}\varphi.$$

Furthermore the infimum in the definition of $\lambda^+_{\min}(M,g)$ is attained by the generalized conformal metric $\tilde{g} = |\varphi|^{4/(n-1)}g$, see [1] for details.

3. Preparations for proofs

3.1. **Removal of singularities.** The following theorem gives conditions for when singularities of solutions to Dirac equations can be removed.

Theorem 3.1. *Let (V,g) be a (not necessarily complete) Riemannian spin manifold and let S be a compact submanifold of V of codimension $m \geq 2$. Assume that $\varphi \in L^p(\Sigma(V \setminus S))$, $p \geq m/(m-1)$, satisfies the equation*

$$D\varphi = \rho$$

weakly on $V \setminus S$ where $\rho \in L^1(\Sigma(V \setminus S)) = L^1(\Sigma V)$. Then this equation holds weakly on V. In particular the singular support of the distribution $D\varphi$ is empty.

Proof. Let ψ be a smooth compactly supported spinor. We have to show that

$$\int_V \langle \varphi, D\psi \rangle \, dv = \int_V \langle \rho, \psi \rangle \, dv.$$

Recall that for $\varepsilon > 0$ we denote the set of points in V of distance less than ε to S by $B(S,\varepsilon)$. We choose a smooth cut-off function $\chi_\varepsilon : V \to [0,1]$ with support in $B(S, 2\varepsilon)$, $\chi_\varepsilon = 1$ on $B(S,\varepsilon)$, and $|\mathrm{grad}\chi_\varepsilon| \leq 2/\varepsilon$. We then have

$$\begin{aligned}
\int_V \langle \varphi, D\psi \rangle \, dv - \int_V \langle \rho, \psi \rangle \, dv &= \int_V \langle \varphi, D((1-\chi_\varepsilon)\psi + \chi_\varepsilon\psi) \rangle \, dv - \int_V \langle \rho, \psi \rangle \, dv \\
&= \int_V \langle D\varphi, (1-\chi_\varepsilon)\psi \rangle \, dv + \int_V \langle \varphi, \chi_\varepsilon D\psi \rangle \, dv \\
&\quad + \int_V \langle \varphi, \mathrm{grad}\chi_\varepsilon \cdot \psi \rangle \, dv - \int_V \langle \rho, \psi \rangle \, dv \\
&= -\int_V \langle \rho, \chi_\varepsilon \psi \rangle \, dv + \int_V \langle \varphi, \chi_\varepsilon D\psi \rangle \, dv \\
&\quad + \int_V \langle \varphi, \mathrm{grad}\chi_\varepsilon \cdot \psi \rangle \, dv,
\end{aligned}$$

where $D\varphi = \rho$ is used in the last equality. Let q be related to p via $1/q + 1/p = 1$. It then follows that

$$\begin{aligned}
\left| \int_V \langle \varphi, D\psi \rangle \, dv - \int_V \langle \rho, \psi \rangle \, dv \right| &\leq \left(\sup_{B(S,2\varepsilon)} |\psi| \right) \int_{B(S,2\varepsilon)} |\rho| \, dv \\
&\quad + \left(\frac{2}{\varepsilon} \sup_{B(S,2\varepsilon)} |\psi| + \sup_{B(S,2\varepsilon)} |D\psi| \right) \int_{B(S,2\varepsilon)} |\varphi| \, dv \\
&\leq o(1) + \frac{C}{\varepsilon} \|\varphi\|_{L^p(V_{2\varepsilon}(S))} \mathrm{Vol}(B(S,2\varepsilon))^{1/q} \\
&\leq o(1) + o(\varepsilon^{(m/q)-1}),
\end{aligned}$$

where $o(1)$ denotes a term tending to 0 as $\varepsilon \to 0$. Since $m/q \geq 1$ is equivalent to $p \geq m/(m-1)$ the statement follows. □

Applying this result to the non-linear equation in Theorem 2.4 we get the following corollary.

Corollary 3.2. *Let V and S be as above. Then any L^p-solution, $p = 2n/(n-1)$, of*

$$D\varphi = \lambda|\varphi|^{p-2}\varphi \tag{12}$$

on $V \setminus S$ is also a weak L^p-solution of (12) on V.

3.2. **Limit spaces and limit solutions.** In the proofs of the main theorems we will construct limit solutions of a partial differential equation on certain limit spaces. For this we need the following two lemmas.

Lemma 3.3. *Let V be an n-dimensional manifold. Let (p_α) be a sequence of points in V which converges to a point p as $\alpha \to 0$. Let (γ_α) be a sequence of metrics defined on a neighbourhood O of p which converges to a metric γ_0 in the $C^2(O)$-topology. Finally, let (b_α) be a sequence of positive real numbers such that $\lim_{\alpha\to 0} b_\alpha = +\infty$. Then for $r > 0$ there exists for α small enough a diffeomorphism*

$$\Theta_\alpha : B^n(r) \to B^{\gamma_\alpha}(p_\alpha, b_\alpha^{-1} r)$$

with $\Theta_\alpha(0) = p_\alpha$ such that the metric $\Theta_\alpha^*(b_\alpha^2\gamma_\alpha)$ tends to the Euclidean metric ξ^n in $C^1(B^n(r))$.

Proof. Denote by $\exp_\alpha : U_\alpha \to O_\alpha$ the exponential map at the point p_α defined with respect to the metric γ_α. Here O_α is a neighbourhood of p_α in V and U_α is a neighbourhood of the origin in $\mathbb{R}^n$. We set

$$\Theta_\alpha : B^n(r) \ni x \mapsto \exp_\alpha(b_\alpha^{-1}x) \in B^{\gamma_\alpha}(p_\alpha, b_\alpha^{-1} r)$$

It is easily checked that Θ_α is the desired diffeomorphism. □

Lemma 3.4. *Let V an n-dimensional spin manifold. Let (g_α) be a sequence of metrics which converges to a metric g in C^1 on all compact sets $K \subset V$ as $\alpha \to 0$. Assume that (U_α) is an increasing sequence of subdomains of V such that $\cup_\alpha U_\alpha = V$. Let $\psi_\alpha \in \Gamma(\Sigma^{g_\alpha} U_\alpha)$ be a sequence of spinors of class $C^1(U_\alpha)$ such that $\|\psi_\alpha\|_{L^\infty(U_\alpha)} \leq C$ where C does not depend on α, and*

$$D^{g_\alpha}\psi_\alpha = \lambda_\alpha|\psi_\alpha|^{\frac{2}{n-1}}\psi_\alpha \tag{13}$$

where the λ_α are positive numbers which tend to $\bar\lambda \geq 0$. Then there exists a spinor $\psi \in \Gamma(\Sigma^g V)$ of class $C^1(V)$ such that

$$D^g\psi = \bar\lambda|\psi|^{\frac{2}{n-1}}\psi \tag{14}$$

on V and a subsequence of $(\beta_g^{g_\alpha}\psi_\alpha)$ tends to ψ in $C^0(K)$ for any compact set $K \subset V$. In particular

$$\|\psi\|_{L^\infty(K)} = \lim_{\alpha\to 0} \|\psi_\alpha\|_{L^\infty(K)}, \tag{15}$$

and

$$\int_K |\psi|^r \, dv^g = \lim_{\alpha\to 0} \int_K |\psi_\alpha|^r \, dv^{g_\alpha} \tag{16}$$

for any compact set K and any $r \geq 1$.

Proof. Let K be a compact subset of V and let Ω be an open set in V with compact closure such that $K \subset \Omega$. Let $\chi \in C^\infty(V)$ with $0 \le \chi \le 1$ be compactly supported in Ω and satisfy $\chi = 1$ on a neighbourhood $\widetilde{\Omega}$ of K. We set $\varphi_\alpha = (\beta^g_{g_\alpha})^{-1}\psi_\alpha$. Using Equations (13) and (3) we get

$$D^g(\chi\varphi_\alpha) = \mathrm{grad}^g\chi \cdot \varphi_\alpha + \chi\lambda_\alpha|\varphi_\alpha|^{\frac{2}{n-1}}\varphi_\alpha - \chi A^g_{g_\alpha}(\nabla^g\varphi_\alpha) - \chi B^g_{g_\alpha}(\varphi_\alpha). \qquad (17)$$

and hence using the fact that $|a+b+c|^r \le 3^r(|a|^r + |b|^r + |c|^r)$ for $a, b, c \in \mathbb{R}$ we get for $r > 1$ that

$$|D^g(\chi\varphi_\alpha)|^r \le 3^r\Big(|\mathrm{grad}^g\chi \cdot \varphi_\alpha + \chi\lambda_\alpha|\varphi_\alpha|^{\frac{2}{n-1}}\varphi_\alpha|^r + |\chi A^g_{g_\alpha}(\nabla^g\varphi_\alpha)|^r + |\chi B^g_{g_\alpha}(\varphi_\alpha)|^r\Big).$$

Since $\|\varphi_\alpha\|_{L^\infty(V)} = \|\psi_\alpha\|_{L^\infty(V)} \le C$ we have

$$|\mathrm{grad}^g\chi \cdot \varphi_\alpha + \chi\lambda_\alpha|\varphi_\alpha|^{\frac{2}{n-1}}\varphi_\alpha|^r \le C.$$

By Relations (4) and (5) and since $\lim_{\alpha\to 0}\|g_\alpha - g\|_{C^1(\Omega)} = 0$ we get

$$\begin{aligned} |\chi A^g_{g_\alpha}(\nabla^g\varphi_\alpha)|^r + |\chi B^g_{g_\alpha}(\varphi_\alpha)|^r &\le o(1)\left(|\nabla^g(\chi\varphi_\alpha)|^r + |\mathrm{grad}^g\chi \cdot \varphi_\alpha|^r + |\chi\varphi_\alpha|^r\right) \\ &\le o(1)\left(|\nabla^g(\chi\varphi_\alpha)|^r + C\right) \end{aligned}$$

where $o(1)$ tends to 0 with α. It follows that

$$|D^g(\chi\varphi_\alpha)|^r \le C + o(1)|\nabla^g(\chi\varphi_\alpha)|^r.$$

Setting $\varphi = \chi\varphi_\alpha$ in Inequality (7) and again using that $\|\varphi_\alpha\|_{L^\infty(\Omega)}$ is uniformly bounded we get that

$$\int_\Omega |\nabla^g(\chi\varphi_\alpha)|^r\, dv^g \le C + o(1)\int_\Omega |\nabla^g(\chi\varphi_\alpha)|^r\, dv^g.$$

In particular $(\chi\varphi_\alpha)$ is bounded in $H_0^{1,r}(\Omega)$. Let $a \in (0,1)$. By the Sobolev Embedding Theorem this implies that a subsequence of $(\chi\varphi_\alpha)$ converges in $C^{0,a}(\Omega)$ to $\psi_K \in \Gamma(\Sigma^{g_\alpha}\Omega)$ of class $C^{0,a}$. We take the inner product of (17) with a smooth spinor $\widetilde{\varphi}$ which is compactly supported in $\widetilde{\Omega}$ and integrate over Ω. Since $\chi = 1$ on the support of $\widetilde{\varphi}$ the result is

$$\begin{aligned} \int_\Omega \langle\varphi_\alpha, D^g\widetilde{\varphi}\rangle\, dv^g = &\int_\Omega \lambda_\alpha|\varphi_\alpha|^{\frac{2}{n-1}}\langle\varphi_\alpha, \widetilde{\varphi}\rangle\, dv^g \\ &- \int_\Omega \langle A^g_{g_\alpha}(\nabla^g\varphi_\alpha), \widetilde{\varphi}\rangle\, dv^g - \int_\Omega \langle B^g_{g_\alpha}(\varphi_\alpha), \widetilde{\varphi}\rangle\, dv^g. \end{aligned}$$

Passing to the limit in α and again using (4) and (5) we get

$$\int_\Omega \langle\psi_K, D^g\widetilde{\varphi}\rangle\, dv^g = \int_\Omega \bar{\lambda}|\psi_K|^{\frac{2}{n-1}}\langle\psi_K, \widetilde{\varphi}\rangle\, dv^g.$$

Hence, ψ_K satisfies Equation (14) weakly on K. By standard regularity theorems we conclude that $\psi_K \in C^1(K)$.

Now we choose an increasing sequence of compact sets K_m such that $\cup_m K_m = V$. Using the above arguments and taking successive subsequences it follows that (φ_α) converge to spinor fields ψ_m on K_m with $\psi_m|_{K_{m-1}} = \psi_{m-1}$. We define ψ on V by $\psi := \psi_m$ on K_m. By taking a diagonal subsequence of (φ_α) we get that (φ_α) tends to ψ in C^0 on any compact set $K \subset V$.

The relations (15) and (16) follow immediately since $\beta^g_{g_\alpha}$ is an isometry, since $\varphi_\alpha = (\beta^g_{g_\alpha})^{-1}\psi_\alpha$ and since (g_α) (resp. (φ_α)) tends to g (resp. ψ) in C^0 on K. This ends the proof of Lemma 3.4. □

3.3. **Dirac spectral bounds on products with spheres.** In the following lemma we assume (in the case $m = 1$) that S^1 carries the spin structure which is obtained by restricting the unique spin structure on the 2-ball to its boundary. The proof is a simple application of the formula for the squared Dirac operator on a product manifold and the lower bound of its spectrum on the standard sphere.

Lemma 3.5. *Let (V, g) be a complete Riemannian spin manifold. Then any L^2-spinor ψ on $(V \times S^m, g + \sigma^m)$ satisfies*

$$\int_{V\times S^m} |D\psi|^2\, dv^{g+\sigma^m} \geq \frac{m^2}{4} \int_{V\times S^m} |\psi|^2\, dv^{g+\sigma^m}.$$

3.4. **Approximation by local product metrics.** The goal of this paragraph is to prove that we can assume that in a neighbourhood of $w_i(W \times \{0\})$ in M_i we have

$$g_i = h_i + dr_i^2 + r_i^2\sigma^{n-k-1}.$$

We are going to prove

Lemma 3.6. *Let (V, g) be a compact Riemannian manifold of dimension n and let S be a closed submanifold of dimension k where $0 \leq k \leq n - 2$ with a trivialization of its normal bundle. Assume that D^g is invertible. Then there exists a sequence (g_α) of metrics on V such that*

$$\lim_{\alpha\to 0} \lambda^+_{\min}(V, g_\alpha) = \lambda^+_{\min}(V, g)$$

and

$$g_\alpha = h + dr^2 + r^2\sigma^{n-k-1}$$

in a neighbourhood $B^g(V, \alpha)$ of S. Here h is the restriction of the metric g to S and $r(x) = d^g(S, x)$.

Proof. Define the metric G on a neighbourhood of S as $G := h+dr^2+r^2\sigma^{n-k-1}$. Let $B^g(V, \alpha)$ be the set of points $x \in V$ such that $r(x) < \alpha$ and let $\chi_\alpha \in C^\infty(M)$, $0 \leq \chi \leq 1$, be a cut-off function such that $\chi = 1$ on $B^g(V, \alpha)$, $\chi = 0$ on $M \setminus B^g(V, 2\alpha)$, and $|d\chi_\alpha| \leq 2/\alpha$. We define

$$g_\alpha := \chi_\alpha G + (1 - \chi_\alpha)g.$$

For convenience we introduce the notation $\lambda_\alpha := \lambda^+_{\min}(V, g_\alpha)$ and $\lambda := \lambda^+_{\min}(V, g)$. After possibly taking a subsequence we assume that $\lim_{\alpha\to 0} \lambda_\alpha$ exists and we denote the limit by $\bar\lambda$.

We begin by proving that

$$\bar\lambda \leq \lambda \tag{18}$$

which is the simpler part of the proof. Let J and J_α be the functionals associated to g and g_α and let $\delta > 0$ be a small number. We set $\chi'_\alpha := 1 - \chi_{2\alpha}$ so that $\chi'_\alpha = 1$ on $V \setminus B^g(V, 4\alpha)$, $\chi'_\alpha = 0$ on $B^g(V, 2\alpha)$, and $|d\chi'_\alpha| \leq 1/\alpha$. We see that $g = g_\alpha$ on the support of η'_α. Let ψ be a smooth spinor such that $J(\psi) \leq \lambda+\delta$. We then have

$$\int_V \langle D^g(\chi'_\alpha\psi), \chi'_\alpha\psi\rangle\, dv^g = \int_V \chi'^2_\alpha\langle D^g\psi, \psi\rangle\, dv^g + \int_V \langle \mathrm{grad}^g\chi'_\alpha \cdot \psi, \chi'_\alpha\psi\rangle\, dv^g.$$

Since the last term here is purely imaginary we obtain

$$\lim_{\alpha\to 0}\int_V \langle D^g(\chi'_\alpha\psi),\chi'_\alpha\psi\rangle\, dv^g = \lim_{\alpha\to 0}\mathrm{Re}\int_V \chi'^2_\alpha\langle D^g\psi,\psi\rangle\, dv^g = \int_V\langle D^g\psi,\psi\rangle\, dv^g. \quad (19)$$

We compute

$$\begin{aligned}\int_V |D^g(\chi'_\alpha\psi)|^{\frac{2n}{n+1}}\, dv^g = &\int_{V\setminus B^g(V,4\alpha)} |D^g\psi|^{\frac{2n}{n+1}}\, dv^g\\ &+\int_{B^g(V,4\alpha)\setminus B^g(V,2\alpha)} |\mathrm{grad}^g\chi'_\alpha\cdot\psi+\chi'_\alpha D^g\psi|^{\frac{2n}{n+1}}\, dv^g.\end{aligned} \quad (20)$$

Using the fact that $|a+b|^{\frac{2n}{n+1}} \leq 2^{\frac{2n}{n+1}}(|a|^{\frac{2n}{n+1}}+|b|^{\frac{2n}{n+1}})$ for $a,b\in\mathbb{R}$ we have

$$\begin{aligned}&|\mathrm{grad}^g\chi'_\alpha\cdot\psi+\chi'_\alpha D^g\psi|^{\frac{2n}{n+1}}\\ &\quad\leq 2^{\frac{2n}{n+1}}\left(|\mathrm{grad}^g\chi'_\alpha|^{\frac{2n}{n+1}}|\psi|^{\frac{2n}{n+1}}+|\chi'_\alpha|^{\frac{2n}{n+1}}|D^g\psi|^{\frac{2n}{n+1}}\right)\\ &\quad\leq 2^{\frac{2n}{n+1}}\left(C_1\alpha^{-\frac{2n}{n+1}}+C_2\right)\end{aligned}$$

where C_1 and C_2 are bounds on $|\psi|$ and $|D\psi|$. Since $\mathrm{Vol}(B^g(V,4\alpha)\setminus B^g(V,2\alpha)) \leq C\alpha^{n-k}\leq C\alpha^2$ it follows that

$$\lim_{\alpha\to 0}\int_{B^g(V,4\alpha)\setminus B^g(V,2\alpha)} |\mathrm{grad}^g\chi'_\alpha\cdot\psi+\chi'_\alpha D^g\psi|^{\frac{2n}{n+1}}\, dv^g = 0.$$

It is clear that $\lim_{\alpha\to 0}\int_{V\setminus B^g(V,4\alpha)}|D^g\psi|^{\frac{2n}{n+1}}\, dv^g = \int_V |D^g\psi|^{\frac{2n}{n+1}}\, dv^g$ so Equation (20) tells us that

$$\lim_{\alpha\to 0}\int_V |D^g(\chi'_\alpha\psi)|^{\frac{2n}{n+1}}\, dv^g = \int_V |D^g\psi|^{\frac{2n}{n+1}}\, dv^g.$$

Together with Equation (19) this proves that $\lim_{\alpha\to 0}J(\chi'_\alpha\psi) = J(\psi)\leq\lambda+\delta$. Since $g_\alpha = g$ on the support of $\chi'_\alpha\psi$, we have $J_\alpha(\chi'_\alpha\psi) = J(\chi'_\alpha\psi)$. Relation (18) now follows since $\lambda_\alpha\leq J_\alpha(\chi'_\alpha\psi)$ and δ is arbitrary.

The second and harder part of the proof is to show that

$$\bar\lambda\geq\lambda. \quad (21)$$

Recall that due to Proposition 2.3 we know $\lambda_\alpha\leq\lambda^+_{\min}(S^n,\sigma^n)$, $\bar\lambda\leq\lambda^+_{\min}(S^n,\sigma^n)$, and $\lambda\leq\lambda^+_{\min}(S^n,\sigma^n)$. Inequality (21) is obvious for $\bar\lambda = \lambda^+_{\min}(S^n,\sigma^n)$. Hence we will assume $\lambda_\alpha<\lambda^+_{\min}(S^n,\sigma^n)$ for a sequence $\alpha\to 0$. As the Dirac operator is invertible we know that (8) holds. By Theorem 2.4 there exists for all α spinor fields $\psi_\alpha\in\Gamma(\Sigma^{g_\alpha}v)$ of class C^1 such that

$$D^{g_\alpha}\psi_\alpha = \lambda_\alpha|\psi_\alpha|^{\frac{2}{n-1}}\psi_\alpha \quad (22)$$

and

$$\int_V |\psi_\alpha|^{\frac{2n}{n-1}}\, dv^{g_\alpha} = 1. \quad (23)$$

We set $\varphi_\alpha = (\beta^g_{g_\alpha})^{-1}\psi_\alpha$. Since $g_\alpha\to g$ it is easily seen that the sequence (φ_α) is bounded in $L^{\frac{2n}{n-1}}(V,g)$. By (3) and (22) we have

$$D^g\varphi_\alpha = \lambda_\alpha|\varphi_\alpha|^{\frac{2}{n-1}}\varphi_\alpha - A^g_{g_\alpha}(\nabla^g\varphi_\alpha) - B^g_{g_\alpha}(\varphi_\alpha), \quad (24)$$

together with $|a+b+c|^{\frac{2n}{n+1}} \leq 3^{\frac{2n}{n+1}}(|a|^{\frac{2n}{n+1}} + |b|^{\frac{2n}{n+1}} + |c|^{\frac{2n}{n+1}})$ for $a,b,c \in \mathbb{R}$ this implies

$$|D^g \varphi_\alpha|^{\frac{2n}{n+1}} \leq C\Big(\lambda_\alpha^{\frac{2n}{n+1}} |\varphi_\alpha|^{\frac{2n}{n-1}} + |A^g_{g_\alpha}(\nabla^g \varphi_\alpha)|^{\frac{2n}{n+1}} + |B^g_{g_\alpha}(\varphi_\alpha)|^{\frac{2n}{n+1}}\Big). \tag{25}$$

We also have

$$|A^g_{g_\alpha}(\nabla^g \varphi_\alpha)| \leq \|g - g_\alpha\|_{C^0(V)} |\nabla^g \varphi_\alpha| \leq C\alpha |\nabla^g \varphi_\alpha| \tag{26}$$

and

$$|B^g_{g_\alpha}(\varphi_\alpha)| \leq \|g - g_\alpha\|_{C^1(V)} |\varphi_\alpha| \leq C|\varphi_\alpha|. \tag{27}$$

Indeed, since g and g_α coincide on S, there exists a constant C so that $\|g - g_\alpha\|_{B^g(V,\alpha)} \leq C\alpha$. Together with the fact that $|d\chi_\alpha| \leq 2/\alpha$ and using the definition of g_α, this immediately implies that $\|g - g_\alpha\|_{C^1(V)} \leq C$. Using Relation (8) and integrating (25) we find that

$$\begin{aligned}
&\int_V |\nabla^g \varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g \\
&\quad \leq C\left(\lambda_\alpha^{\frac{2n}{n+1}} \int_V |\varphi_\alpha|^{\frac{2n}{n-1}} \, dv^g + \alpha^{\frac{2n}{n+1}} \int_V |\nabla^g \varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g + \int_V |B^g_{g_\alpha}(\varphi_\alpha)|^{\frac{2n}{n+1}} \, dv^g\right).
\end{aligned}$$

As g and g_α coincide on $V \setminus B^g(V, 2\alpha)$ we conclude that $B^g_{g_\alpha}(\varphi_\alpha) = 0$ on this set. Together with (27) we have

$$\begin{aligned}
\int_V |B^g_{g_\alpha}(\varphi_\alpha)|^{\frac{2n}{n+1}} \, dv^g &\leq C \int_{B^g(V,2\alpha)} |\varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g \\
&\leq C \operatorname{Vol}(B^g(V,2\alpha))^{\frac{2}{n+1}} \left(\int_{B^g(V,2\alpha)} |\varphi_\alpha|^{\frac{2n}{n-1}} \, dv^g\right)^{\frac{n-1}{n+1}} \\
&= o(1)
\end{aligned}$$

where $o(1)$ tends to 0 with α. Hence

$$\int_V |\nabla^g \varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g \leq C\left(\lambda_\alpha^{\frac{2n}{n+1}} \int_V |\varphi_\alpha|^{\frac{2n}{n-1}} \, dv^g + \alpha \int_V |\nabla^g \varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g + o(1)\right) \tag{28}$$

This implies in particular that (φ_α) is bounded in $H_1^{\frac{2n}{n+1}}(V)$ and hence after passing to a subsequence (φ_α) converges weakly to a limit φ in $H_1^{\frac{2n}{n+1}}(V)$.

The next step is to prove that $\bar{\lambda} = \lim_{\alpha\to 0} \lambda_\alpha$ is not zero. To get a contradiction let us assume that $\bar{\lambda} = 0$. We then obtain from (28) that

$$\int_V |\nabla^g \varphi|^{\frac{2n}{n+1}} \, dv^g \leq \lim_{\alpha\to 0} \int_V |\nabla^g \varphi_\alpha|^{\frac{2n}{n+1}} \, dv^g = 0.$$

So φ is parallel and since D^g is invertible we conclude $\varphi = 0$, in other words (φ_α) converges weakly to zero in $H_1^{\frac{2n}{n+1}}(V)$. As this space embeds compactly into $L^{\frac{2n}{n+1}}(V)$ we have

$$\lim_{\alpha\to 0} \|\varphi_\alpha\|_{L^{\frac{2n}{n+1}}(V)} = \|\varphi\|_{L^{\frac{2n}{n+1}}(V)} = 0$$

and hence (φ_α) converges strongly to zero in $H_1^{\frac{2n}{n+1}}(V)$. As this space embeds continuously into $L^{\frac{2n}{n-1}}(V)$ we conclude that the sequence converges strongly to zero in $L^{\frac{2n}{n-1}}(V)$. This is impossible since by Relation (23) we easily get that

$$\lim_{\alpha\to 0} \|\varphi_\alpha\|_{L^{\frac{2n}{n-1}}(V)} = 1.$$

From this contradiction we conclude

$$\bar{\lambda} > 0. \tag{29}$$

From (24) we have

$$\|D^g\varphi_\alpha\|_{L^{\frac{2n}{n+1}}(V)} \leq \lambda_\alpha \|\varphi_\alpha\|^{\frac{n-1}{n+1}}_{L^{\frac{2n}{n-1}}(V)} + \|A^g_{g_\alpha}(\nabla^g\varphi_\alpha)\|_{L^{\frac{2n}{n+1}}(V)} + \|B^g_{g_\alpha}(\varphi_\alpha)\|_{L^{\frac{2n}{n+1}}(V)}.$$

We already proved above that

$$\lim_{\alpha\to 0} \|B^g_{g_\alpha}(\varphi_\alpha)\|_{L^{\frac{2n}{n+1}}(V)} = 0.$$

Using Relation (26) we get similarily

$$\lim_{\alpha\to 0} \|A^g_{g_\alpha}(\nabla^g\varphi_\alpha)\|_{L^{\frac{2n}{n+1}}(V)} = 0.$$

Moreover since $dv^{g_\alpha} = (1+o(1))\, dv^g$ it follows from (23) that

$$\lambda_\alpha \|\varphi_\alpha\|^{\frac{n+1}{n-1}}_{L^{\frac{2n}{n-1}}(V)} = \lambda_\alpha(1+o(1)).$$

We obtain that

$$\|D^g\varphi_\alpha\|_{L^{\frac{2n}{n+1}}(V)} \leq \lambda_\alpha + o(1). \tag{30}$$

Starting from Equation (24) we can prove in a similar way that

$$\int_V \langle D^g\varphi_\alpha, \varphi_\alpha\rangle\, dv^g \geq \lambda_\alpha + o(1). \tag{31}$$

From (29), (30), and (31) it follows that $\lambda \leq \lim_{\alpha\to 0} J(\varphi_\alpha) = \bar{\lambda}$. This ends the proof of (21). Together with (18) this proves Lemma 3.6. □

4. Proofs

4.1. **Proof of Theorem 1.2.** This section is devoted to the proof of Theorem 1.2. Our goal is to construct a sequence of metrics (g_θ) which satisfies the conclusion of Theorem (1.2) when θ is small.

Applying Lemma 3.6 with $V = M = M_1 \amalg M_2$ and $S = W' = w_1(W \times \{0\}) \amalg w_2(W \times \{0\})$ it follows that we may assume that

$$g = h + dr^2 + r^2\sigma^{n-k-1} \tag{32}$$

in a neighbourhood $U(R_{\max})$ of W', $R_{\max} > 0$. We fix numbers $R_0, R_1 \in \mathbb{R}$ with $R_{\max} > R_1 > R_0 > 0$ and we choose a function $F : M \setminus S \to \mathbb{R}^+$ such that

$$F(x) = \begin{cases} 1, & \text{if } x \in M_i \setminus U_i(R_1); \\ r_i^{-1} & \text{if } x \in U_i(R_0) \setminus S. \end{cases}$$

We further choose $\theta \in (0, R_0)$, later we will let $\theta \to 0$. It is not difficult to see that there is a smooth function $f : U(R_{\max}) \to \mathbb{R}$ (depending only on r), real numbers $\delta_1 = \delta_1(\theta)$ and $\delta_2 = \delta_2(\theta)$ with $\theta > \delta_2 > \delta_1 > 0$ and a real number $A_\theta \in [-\ln\theta, -\ln\delta_1)$ such that

$$f(x) = \begin{cases} -\ln r & \text{if } x \in U(R_{\max}) \setminus U(\theta); \\ \ln A_\theta & \text{if } x \in U(\delta_2), \end{cases}$$

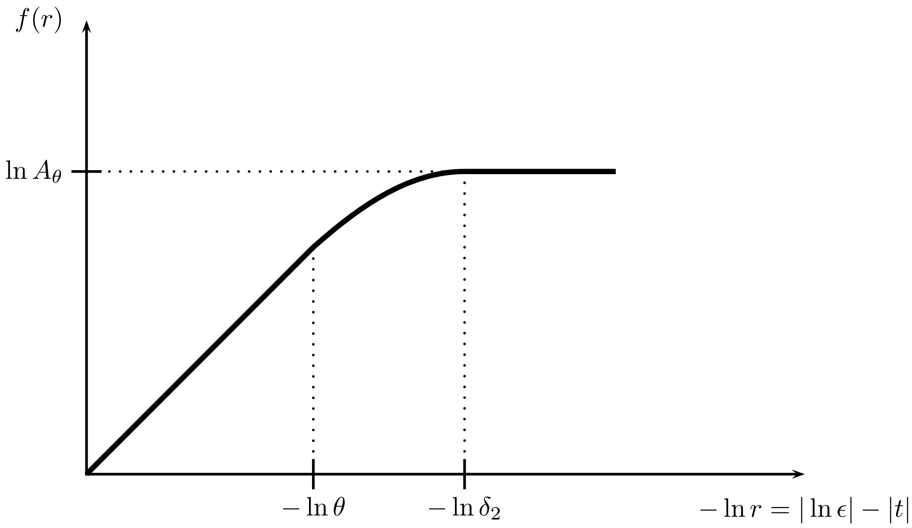

FIGURE 1. The function $-\ln r \mapsto f(r)$

and such that

$$\left|r\frac{df}{dr}\right| = \left|\frac{df}{d(\ln r)}\right| \leq 1,$$

and

$$\left\|r\frac{d}{dr}\left(r\frac{df}{dr}\right)\right\|_{L^\infty} = \left\|\frac{d^2 f}{d^2(\ln r)}\right\|_{L^\infty} \to 0$$

as $\theta \to 0$. It follows that $\lim_{\theta\to 0} A_\theta = \infty$.

After these choices we set $\varepsilon := e^{-A_\theta}\delta_1$. We assume that N is obtained from M by a connected sum along W with parameter ε, as explained in section 2.2. In particular, recall that $U^N_\varepsilon(s) = U(s) \setminus U(\varepsilon)/\sim$ for all $s \geq \varepsilon$. On the set $U^N_\varepsilon(R_{\max}) = U(R_{\max}) \setminus U(\varepsilon)/\sim$ we define the variable t by

$$t := -\ln r_1 + \ln\varepsilon \leq 0$$

on $U_1(R_{\max})$ and

$$t := \ln r_2 - \ln\varepsilon \geq 0$$

on $U_2(R_{\max})$. This implies

$$r_i = e^{|t|+\ln\varepsilon} = \varepsilon e^{|t|}.$$

The choices imply that $t : U^N_\varepsilon(R_{\max}) \to \mathbb{R}$ is a smooth function with $t \leq 0$ on $U^N_\varepsilon(R_{\max}) \cap M_1$, $t \geq 0$ on $U^N_\varepsilon(R_{\max}) \cap M_2$, and $t = 0$ is the common boundary $\partial U_1(\varepsilon)$ identified in N with $\partial U_2(\varepsilon)$. Then equation (32) tells us that

$$r^{-2}g = \varepsilon^{-2}e^{-2|t|}h_i + dt^2 + \sigma^{n-k-1}.$$

Expressed in the new variable t we have

$$F(x) = \varepsilon^{-1}e^{-|t|}$$

Hierarchy of Variables
$R_{\max} > R_1 > R_0 > \theta > \delta_2 > \delta_1 > \varepsilon > 0$
We choose in the order $R_{\max}, R_1, R_0, \theta, \delta_2, \delta_1, A_\theta$ We can assume for example that $\varepsilon = e^{-A_\theta}\delta_1$. This implies $\lvert t\rvert = A_\theta \Leftrightarrow r_i = \delta_1$.

Figure 2. Hierarchy of variables

if $x \in U(R_0) \setminus U^N(\theta)$ or in other words if $|t| + \ln\varepsilon \le \ln R_0$, and

$$f(t) = \begin{cases} -|t| - \ln\varepsilon & \text{if } |t| + \ln\varepsilon \in (\theta, R_{\max}), \\ \ln A_\theta & \text{if } |t| + \ln\varepsilon \le \ln\delta_2, \end{cases}$$

and $|df/dt| \le 1$, $\|d^2f/dt^2\|_{L^\infty} \to 0$. After choosing a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[1, \infty)$, we define

$$g_\theta(x) := \begin{cases} F^2 g_i & \text{if } x \in M_i \setminus U_i(\theta); \\ e^{2f(t)}h_i + dt^2 + \sigma^{n-k-1} & \text{if } x \in U_i(\theta) \setminus U_i(\delta_1); \\ A_\theta^2\chi(A_\theta^{-1}t)h_2 + A_\theta^2(1-\chi(A_\theta^{-1}t))h_1 + dt^2 + \sigma^{n-k-1} & \text{if } x \in U_i(\delta_1) \setminus U_i(\varepsilon). \end{cases}$$

(Recall that h_i were define to be the pullback via w_i of the metric g_i on M_i, composed with restriction to $W = W \times \{0\}$.)

On $U(R_0)$ we write g_θ as

$$g_\theta = \alpha_t^2\tilde{h}_t + dt^2 + \sigma^{n-k-1}$$

where the metric $\tilde{h}_t$ is defined for $t \in \mathbb{R}$ by

$$\tilde{h}_t := \chi(A_\theta^{-1}t)h_2 + (1-\chi(A_\theta^{-1}t))h_1 \tag{33}$$

and where

$$\alpha_t := e^{f(t)}. \tag{34}$$

The rest of the proof is devoted to show that a subsequence of (g_θ) is the desired sequence of metrics. In the following we keep the notation (g_θ) for any subsequence of (g_θ). We set $\lambda := \lambda^+_{\min}(M_1 \amalg M_2, g)$, $\lambda_\theta := \lambda^+_{\min}(N, g_\theta)$, and $\bar{\lambda} := \lim_{\theta\to 0}\lambda_\theta$ (after passing to a subsequence we can assume that this limit exists). Let J and J_θ be the functionals associated respectively to λ and λ_θ.

The easier part of the argument is to show that

$$\bar{\lambda} \le \lambda. \tag{35}$$

For this let $\alpha > 0$ be a small number. We choose a smooth cut-off function $\chi_\alpha : M_1 \amalg M_2 \to [0,1]$ such that $\chi_\alpha = 1$ on $M_1 \amalg M_2 \setminus U(2\alpha)$, $|d\chi_\alpha| \le \frac{2}{\alpha}$, and $\chi_\alpha = 0$ on $U(\alpha)$. Let ψ be a smooth non-zero spinor such that $J(\psi) \le \lambda + \delta$ where δ is a small positive number. On the support of χ_α the metrics g and g_α are conformal since $g_\theta = F^2 g$ and hence by Formula (10) we have

$$\lambda_\theta \le J_\theta\left(\chi_\alpha\beta^g_{g_\theta}(F^{-\frac{n-1}{2}}\psi)\right) = J(\chi_\alpha\psi)$$

for $\theta < \alpha$. Proceeding exactly as in the first part of the proof of Lemma 3.6 we show that $\lim_{\alpha\to 0} J(\chi_\alpha\psi) = J(\psi) \le \lambda + \delta$. From this follows Relation (35).

Now we consider the more difficult part of the proof, that

$$\bar{\lambda} \ge \min\{\lambda, \Lambda_{n,k}\}. \tag{36}$$

By Proposition 2.3 we can assume that for all θ, $\lambda_\theta < \lambda^+_{\min}(S^n, \sigma^n)$. Otherwise Relation (36) is trivial. From Theorem 2.4 we know that there exists a spinor field $\psi_\theta \in \Gamma(\Sigma^{g_\alpha} N)$ of class C^2 such that

$$\int_N |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} = 1$$

and

$$D^{g_\theta}\psi_\theta = \lambda_\theta |\psi_\theta|^{\frac{2}{n-1}} \psi_\theta. \tag{37}$$

We let x_θ in N be such that $|\psi_\theta(x_\theta)| = m_\theta$ where $m_\theta := \|\psi_\theta\|_{L^\infty(N)}$.

The proof continues divided in cases.

Case I. *The sequence (m_θ) is not bounded.*

After taking a subsequence, we can assume that $\lim_{\theta\to 0} m_\theta = +\infty$. We consider two subcases.

Subcase I.1. *There exists $a > 0$ such that $x_\theta \in N \setminus U^N(a)$ for an infinite number of θ.*

We recall that $N \setminus U^N(a) = N_\varepsilon \setminus U^N_\varepsilon(a) = M_1 \amalg M_2 \setminus U(a)$. By taking a subsequence we can assume that there exists $\bar{x} \in M_1 \amalg M_2 \setminus U(a)$ such that $\lim x_\theta = \bar{x}$. We let $g'_\theta := m_\theta^{\frac{4}{n-1}} g_\theta$. In a neighbourhood U of $\bar{x}$ the metric $g_\theta = F^2 g$ does not depend on θ. We apply Lemma 3.3 with $O = U$, $\alpha = \theta$, $p_\alpha = x_\theta$, $p = \bar{x}$, $\gamma_\alpha = g_\theta = F^2 g$, and $b_\alpha = m_\theta^{\frac{2}{n-1}}$. Let $r > 0$. For θ small enough Lemma 3.3 gives us a diffeomorphism

$$\Theta_\theta : B^n(r) \to B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r)$$

such that the sequence of metrics $(\Theta^*_\theta(g'_\theta))$ tends to the Euclidean metric ξ^n in $C^1(B^n(r))$. We let $\psi'_\theta := m_\theta^{-1}\psi_\theta$. By (6) we then have

$$D^{g'_\theta}\psi'_\theta = \lambda_\theta |\psi'_\theta|^{\frac{2}{n-1}} \psi'_\theta$$

on $B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r)$ and

$$\begin{aligned}\int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r)} |\psi'_\theta|^{\frac{2n}{n-1}} \, dv^{g'_\theta} &= \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} \\ &\le \int_N |\psi_\theta|^{\frac{2n}{n-1}} dv^{g_\theta} \\ &= 1.\end{aligned}$$

Here we used the fact that $dv^{g'_\theta} = m_\theta^{\frac{2n}{n-1}} \, dv^{g_\theta}$. Since

$$\Theta_\theta : (B^n(r), \Theta^*_\theta(g'_\theta)) \to (B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r), g'_\theta)$$

is an isometry we can consider ψ'_θ as a solution of

$$D^{\Theta^*_\theta(g'_\theta)}\psi'_\theta = \lambda_\theta |\psi'_\theta|^{\frac{2}{n-1}} \psi'_\theta$$

on $B^n(r)$ with $\int_{B^n(r)} |\psi'_\theta|^{\frac{2n}{n-1}} \, dv^{\Theta^*_\theta(g'_\theta)} \le 1$. Since $\|\psi_\theta\|_{L^\infty(B^n(r))} = |\psi'_\theta(0)| = 1$ we can apply Lemma 3.4 with $V = \mathbb{R}^n$, $\alpha = \theta$, $g_\alpha = \Theta^*_\theta(g'_\theta)$, and $\psi_\alpha = \psi'_\theta$ (we can apply this lemma since each compact set of $\mathbb{R}^n$ is contained in some ball $B^n(r)$). This shows that there exists a spinor ψ of class C^1 on $(\mathbb{R}^n, \xi^n)$ which satisfies

$$D^{\xi^n}\psi = \bar{\lambda}|\psi|^{\frac{2}{n-1}}\psi.$$

Furthermore by (16) we have

$$\int_{B^n(r)} |\psi|^{\frac{2n}{n-1}}\, dv^{\xi^n} = \lim_{\theta\to 0} \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-1}} r)} |\psi_\theta|^{\frac{2n}{n-1}}\, dv^{g_\theta} \le 1$$

for any $r > 0$. We conclude that $\int_{\mathbb{R}^n} |\psi|^{\frac{2n}{n-1}}\, dv^{\xi^n} \le 1$. Since $|\psi(0)| = 1$ we also see that ψ is not identically zero. As $(\mathbb{R}^n, \xi^n)$ and $(S^n \setminus \{\mathrm{pt}\}, \sigma^n)$ are conformal we can write $\sigma^n = \Phi^2 \xi^n$ for a positive function Φ. We define $\varphi := \Phi^{-\frac{n-1}{2}} \beta_{\sigma^n}^{\xi^n} \psi$. By Relation (6) it follows that $\varphi \in L^{\frac{2n}{n-1}}(S^n)$ is a solution of

$$D^{\sigma^n} \varphi = \bar\lambda |\varphi|^{\frac{2}{n-1}} \varphi \tag{38}$$

on $S^n \setminus \{\mathrm{pt}\}$ of class C^1. By Corollary 3.2 we know that φ can be extended to a weak solution of (38) on all S^n and by standard regularity theorems it follows that $\varphi \in C^1(S^n)$. Let J^{σ^n} be the functional associated to (S^n, σ^n). By Equation (38) we have

$$\lambda^+_{\min}(S^n, \sigma^n) \le J^{\sigma^n}(\varphi) = \bar\lambda$$

where the inequality comes from Proposition 2.3. We have proved Relation (36) in this subcase.

Subcase I.2. *For all $a > 0$ it holds that $x_\theta \notin M_1 \amalg M_2 \setminus U(a)$ for θ sufficiently small.*

This means that x_θ belongs to $U^N(a)$ if θ is sufficiently small. This subset is diffeomorphic to $W \times I \times S^{n-k-1}$ where I is an interval. Hence x_θ can be written as

$$x_\theta = (y_\theta, t_\theta, z_\theta)$$

where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln\varepsilon, -\ln\varepsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By taking a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge respectively to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. We apply Lemma 3.3 with $V = W$, $\alpha = \theta$, $p_\alpha = y_\theta$, $p = y$, $\gamma_\alpha = \tilde h_{t_\theta}$, $\gamma_0 = \tilde h_T$ (we define $\tilde h_{-\infty} := h_1$ and $\tilde h_{+\infty} := h_2$), and $b_\alpha = m_\theta^{\frac{2}{n-1}} \alpha_{t_\theta}$. The lemma provides diffeomorphisms

$$\Theta_\theta^y : B^k(r) \to B^{\tilde h_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-1}} \alpha_{t_\theta}^{-1} r)$$

for $r > 0$ such that $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-1}} \alpha_{t_\theta}^2 \tilde h_{t_\theta})$ tends to the Euclidean metric ξ^k on $B^k(r)$ as $\theta \to 0$. Next we apply Lemma 3.3 with $V = S^{n-k-1}$, $\alpha = \theta$, $p_\alpha = z_\theta$, $\gamma_\alpha = \gamma_0 = \sigma^{n-k-1}$, and $b_\alpha = m_\theta^{\frac{2}{n-1}}$. For $r' > 0$ we get the existence of diffeomorphisms

$$\Theta_\theta^z : B^{n-k-1}(r') \to B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-1}} r')$$

such that $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-1}} \sigma^{n-k-1})$ converges to ξ^{n-k-1} on $B^{n-k-1}(r')$ as $\theta \to 0$. For $r, r', r'' > 0$ we define

$$\begin{aligned} U_\theta(r, r', r'') := {} & B^{\tilde h_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-1}} \alpha_{t_\theta}^{-1} r) \times [t_\theta - m_\theta^{-\frac{2}{n-1}} r'', t_\theta + m_\theta^{-\frac{2}{n-1}} r''] \\ & \times B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-1}} r') \end{aligned}$$

and

$$\begin{aligned} \Theta_\theta : B^k(r) \times [-r'', r''] \times B^{n-k-1}(r') &\to U_\theta(r, r', r'') \\ (y, s, z) &\mapsto (\Theta_\theta^y(y), t(s), \Theta_\theta^z(z)), \end{aligned}$$

where $t(s) := t_\theta + m_\theta^{\frac{2}{n-1}} s$. By construction Θ_θ is a diffeomorphism. As is readily seen

$$\Theta_\theta^*(m_\theta^{\frac{4}{n-1}} g_\theta) = (\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-1}} \alpha_t^2 \tilde{h}_t) + ds^2 + (\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-1}} \sigma^{n-k-1}). \tag{39}$$

By construction of α_t one can verify that

$$\lim_{\theta \to 0} \left\| \frac{\alpha_{t_\theta}}{\alpha_t} - 1 \right\|_{C^1([t_\theta - m_\theta^{-\frac{2}{n-1}} r'', t_\theta + m_\theta^{-\frac{2}{n-1}} r''])} = 0$$

for all $R > 0$ since $\frac{df}{dt}$ and $\frac{d^2 f}{dt^2}$ are uniformly bounded. Moreover it is clear that

$$\lim_{\theta \to 0} \left| \tilde{h}_t - \tilde{h}_{t_\theta} \right|_{C^1(B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-1}} \alpha_{t_\theta}^{-1} R))} = 0$$

uniformly in $t \in [t_\theta - m_\theta^{-\frac{2}{n-1}} r'', t_\theta + m_\theta^{-\frac{2}{n-1}} r'']$. As a consequence

$$\lim_{\theta \to 0} \left| (\Theta_\theta^y)^* \left(m_\theta^{\frac{4}{n-1}} \left(\alpha_t^2 \tilde{h}_t - \alpha_{t_\theta}^2 \tilde{h}_{t_\theta} \right) \right) \right|_{C^1(B^k(r))} = 0$$

uniformly in t. This implies that the sequence $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-1}} \alpha_t^2 \tilde{h}_t)$ tends to the Euclidean metric ξ^k in $C^1(B^k(r))$ uniformly in t as $\theta \to 0$. From (39) we know that the sequence $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-1}} \sigma^{n-k-1})$ tends to the Euclidean metric ξ^{n-k-1} on $B^{n-k-1}(r')$ as $\theta \to 0$. Returning to (39) we obtain that the sequence $\Theta_\theta^*(m_\theta^{\frac{4}{n-1}} g_\theta)$ tends to $\xi^n = \xi^k + ds^2 + \xi^{n-k-1}$ on $B^k(r) \times [-r'', r''] \times B^{n-k-1}(r')$. As in Subcase I.1 we apply Lemma 3.4 to get a spinor ψ of class C^1 on $\mathbb{R}^n$ which satisfies

$$D^{\xi^n} \psi = \bar{\lambda} |\psi|^{\frac{2}{n-1}} \psi$$

with $\int_{B^n(r)} |\psi|^{\frac{2n}{n-1}} dx \leq 1$ for all $r \in \mathbb{R}^+$. Lemma 3.4 tells us that $|\psi(0)| = 1$ so ψ does not vanish identically. Just as in Subcase I.1 we deduce that

$$\lambda \leq \lambda_{\min}^+(S^n, \sigma^n) \leq \bar{\lambda}.$$

This ends the proof of Theorem 1.2 in Case I.

Case II. *There exists a constant C_1 such that $m_\theta \leq C_1$ for all θ.*

As in Case I we consider two subcases.

Subcase II.1. *Assume that*

$$\liminf_{\theta \to 0} \int_{N \setminus U^N(a)} |\psi_\theta|^{\frac{2n}{n-1}} dv^{g_\theta} > 0 \tag{40}$$

for some number $a > 0$.

Let K a compact subset such that $K \subset M_1 \amalg M_2 \setminus W'$. We choose a small number b such that $K \subset M_1 \amalg M_2 \setminus U(2b) = N \setminus U^N(2b)$. Let $\chi \in C^\infty(M_1 \amalg M_2)$, $0 \leq \chi \leq 1$, be a cut-off function equal to 1 on $M_1 \amalg M_2 \setminus U(2b)$ and equal to 0 on $U(b)$. We set $\psi'_\theta := F^{\frac{n-1}{2}} (\beta_{g_\theta}^g)^{-1} \psi_\theta$. Since $g_\theta = F^2 g$ on the support of χ we have

$$D^g \psi'_\theta = \lambda_\theta |\psi'_\theta|^{\frac{2}{n-1}} \psi'_\theta$$

on this set. For some $r > 0$ we have

$$\begin{aligned}\int_{M_1 \amalg M_2} & |D^g(\chi\psi'_\theta)|^r \, dv^g \\ &= \int_{M_1 \amalg M_2} \left| \mathrm{grad}^g \chi \cdot \psi'_\theta + \chi \lambda_\theta |\psi'_\theta|^{\frac{2}{n-1}} \psi'_\theta \right|^r \, dv^g \\ &\leq 2^r \left(\int_V |\mathrm{grad}^g \chi|^r |\psi'_\theta|^r \, dv^g + \lambda_\theta^r \int_{M_1 \amalg M_2} \chi^r |\psi'_\theta|^{\frac{(n+1)r}{n-1}} \, dv^g \right) \\ &\leq C.\end{aligned}$$

since $m_\theta \leq C_1$. Together with Relation (8) we get that the sequence $(\chi\psi'_\theta)$ is bounded in $H_1^r(M_1 \amalg M_2)$ for all $r > 0$. Proceeding as in the proof of Lemma 3.4 we get a C^1 spinor ψ_0 defined on K such that a subsequence of (ψ'_θ) converges to ψ_0 in $C^0(K)$ and which satisfies

$$D^g \psi_0 = \bar{\lambda} |\psi_0|^{\frac{2}{n-1}} \psi_0. \tag{41}$$

Furthermore the convergence in C^0 implies that

$$\int_K |\psi_0|^{\frac{2n}{n-1}} dv^g \leq \liminf_{\theta \to 0} \int_K |\psi'_\theta|^{\frac{2n}{n-1}} dv^g = \liminf_{\theta \to 0} \int_K |\psi_\theta|^{\frac{2n}{n-1}} dv^{g_\theta} \leq 1.$$

Repeating the same for a sequence of compact sets K_m which exhausts $M_1 \amalg M_2 \setminus W'$ and taking a diagonal subsequence we can extend ψ_0 to $M_1 \amalg M_2 \setminus W'$. Since $\psi_0 \in L^{\frac{2n}{n-1}}(M_1 \amalg M_2 \setminus W') = L^{\frac{2n}{n-1}}(M_1 \amalg M_2)$ we can use Theorem 3.2 to extend ψ_0 to a weak solution of Equation (41). Note here that since D^g is invertible we have $\bar{\lambda} > 0$. By standard regularity theorems we conclude that $\psi_0 \in C^1(M_1 \amalg M_2)$. By assumption (40) we have

$$\begin{aligned}\int_{M_1 \amalg M_2 \setminus U(a)} |\psi_0|^{\frac{2n}{n-1}} \, dv^g &= \lim_{\theta \to 0} \int_{M_1 \amalg M_2 \setminus U(a)} |\psi'_\theta|^{\frac{2n}{n-1}} \, dv^g \\ &= \lim_{\theta \to 0} \int_{M_1 \amalg M_2 \setminus U(a)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} \\ &> 0.\end{aligned}$$

We conclude that $\psi_0 \neq 0$. If J denotes the functional associated to g then Equation (41) leads to

$$\lambda \leq J(\psi_0) = \bar{\lambda} \left(\int_{M_1 \amalg M_2} |\psi_0|^{\frac{2n}{n-1}} \, dv^g \right)^{\frac{n+1}{n} - 1} \leq \bar{\lambda},$$

which proves Theorem 1.2 in this case.

Subcase II.2. *We have*

$$\liminf_{\theta \to 0} \int_{N \setminus U^N(a)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} = 0 \tag{42}$$

for all $a > 0$.

This case is the most difficult one and we proceed in several steps. The assumption here is that we have a sequence (θ_i) which tends to zero as $i \to \infty$ with the property that the integral above tends to zero for all $a > 0$. We will abuse notation and write $\lim_{\theta \to 0}$ for what should be a limit as $i \to \infty$ or a limit of a subsequence of it.

For positive a and θ let

$$\gamma_\theta(a) := \frac{\int_{N\setminus U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}}{\int_{U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}}$$

The first step is to establish an estimate we will need later.

Step 1. *For $a > 0$ we have*

$$1 \le C_0 \left(\gamma_\theta(a) + \|\psi_\theta\|^{\frac{4}{n-1}}_{L^\infty(U^N(2a))} \right) \tag{43}$$

where $C_0 > 0$ is a constant which does not depend on a.

Let $\chi \in C^\infty(N)$, $0 \le \chi \le 1$ be a cut-off function with $\chi = 1$ on $U^N(a)$ and $\chi = 0$ on $N \setminus U^N(2a) = M_1 \amalg M_2 \setminus U(2a)$. Since the definitions of $U^N(a)$ and $U(a)$ use the distance to W' for the metric g we can and do assume that $|d\chi|_g \le \frac{2}{a}$. In the metric g_θ this gives

$$|d\chi|_{g_\theta} = F^{-1}|d\chi|_g = r|d\chi|_g \le 2a\frac{2}{a} = 4.$$

From Proposition 3.5 and Equation (37) it follows that

$$\begin{aligned}
\frac{(n-1-k)^2}{4} &\le \frac{\int_N |D^{g_\theta}(\chi\psi_\theta)|^2\, dv^{g_\theta}}{\int_N |\chi\psi_\theta|^2\, dv^{g_\theta}} \\
&= \frac{\int_N |d\chi|^2_{g_\theta} |\psi_\theta|^2\, dv^{g_\theta} + \lambda_\theta^2 \int_N \chi^2 |\psi_\theta|^{\frac{2(n+1)}{n-1}}\, dv^{g_\theta}}{\int_N |\chi\psi_\theta|^2\, dv^{g_\theta}} \\
&\le \frac{16\int_{U^N(2a)\setminus U^N(a)} |\psi_\theta|^2\, dv^{g_\theta} + \lambda_\theta^2 \|\psi_\theta\|^{\frac{4}{n-1}}_{L^\infty(U^N(2a))} \int_N |\chi\psi_\theta|^2\, dv^{g_\theta}}{\int_N |\chi\psi_\theta|^2\, dv^{g_\theta}} \\
&\le \frac{16\int_{U^N(2a)\setminus U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}}{\int_{U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}} + \lambda_\theta^2 \|\psi_\theta\|^{\frac{4}{n-1}}_{L^\infty(U^N(2a))} \\
&\le 16\gamma_\theta(a) + \lambda_\theta^2 \|\psi_\theta\|^{\frac{4}{n-1}}_{L^\infty(U^N(2a))}.
\end{aligned}$$

Using that $\lambda_\theta \le \lambda^+_{\min}(S^n, \sigma^n)$ by Proposition 2.3 we obtain Relation (43) with

$$C_0 := \frac{4}{(n-1-k)^2} \max\left\{16, \lambda^+_{\min}(S^n)^2\right\}.$$

This ends the proof of Step 1.

Step 2. *There exist a sequence of positive numbers (a_θ) which tends to 0 with θ and constants $0 < m < M$ such that*

$$m \le \|\psi_\theta\|_{L^\infty(U^N(2a_\theta))} \le M \tag{44}$$

for all θ.

By (42) we have for all a

$$\lim_{\theta\to 0} \int_{N\setminus U^N(a)} |\psi_\theta|^{\frac{2n}{n-1}}\, dv^{g_\theta} = 0$$

for all $a > 0$. Since $\mathrm{Vol}\big(N \setminus U^N(a), g_\theta\big)$ does not depend on θ if $\theta < a$ it follows that

$$\lim_{\theta \to 0} \left(\int_{N \setminus U^N(a)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} \right)^{\frac{n-1}{n}} \mathrm{Vol}(N \setminus U^N(a), g_\theta)^{\frac{1}{n}} = 0$$

for any a. Hence we can take a sequence (a_θ) which tends sufficiently slowly to 0 so that

$$\lim_{\theta \to 0} \left(\int_{N \setminus U^N(a_\theta)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} \right)^{\frac{n-1}{n}} \mathrm{Vol}(N \setminus U^N(a_\theta), g_\theta)^{\frac{1}{n}} = 0. \tag{45}$$

Using the Hölder inequality we get

$$\begin{aligned} \gamma_\theta(a_\theta) &= \frac{\int_{N \setminus U^N(a_\theta)} |\psi_\theta|^2 \, dv^{g_\theta}}{\int_{U^N(a_\theta)} |\psi_\theta|^2 \, dv^{g_\theta}} \\ &\le \frac{\big(\int_{N \setminus U^N(a_\theta)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta}\big)^{\frac{n-1}{n}} \mathrm{Vol}(N \setminus U^N(a_\theta), g_\theta)^{\frac{1}{n}}}{\|\psi_\theta\|_{L^\infty(U^N(a_\theta))}^{-\frac{2}{n-1}} \int_{U^N(a_\theta)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta}}. \end{aligned}$$

The numerator of this expression tends to 0 by Relation (45). Further by (45) we have

$$\begin{aligned} \lim_{\theta \to 0} \int_{U^N(a_\theta)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} &= \lim_{\theta \to 0} \int_N |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} - \int_{N \setminus U^N(a_\theta)} |\psi_\theta|^{\frac{2n}{n-1}} \, dv^{g_\theta} \\ &= 1. \end{aligned}$$

Together with the fact that $\|\psi_\theta\|_{L^\infty(U^N(a_\theta))} \le m_\theta \le C_1$ we obtain that

$$\lim_{\theta \to 0} \gamma_\theta(a_\theta) = 0.$$

From Relation (43) applied with $a = a_\theta$ we know that $\|\psi_\theta\|_{L^\infty(U^N(2a_\theta))}$ is bounded from below. Moreover we have by the assumption of Case II that $\|\psi_\theta\|_{L^\infty(U^N(2a_\theta))} \le m_\theta \le C_1$. This finishes the proof of Step 2.

Step 3. *We have*

$$\bar{\lambda} \ge \Lambda_{n,k}.$$

Let x_θ be a point in the closure of $U^N(2a_\theta)$ such that $|\psi_\theta(x_\theta)| = \|\psi_\theta\|_{L^\infty(U^N(2a_\theta))}$. As in Subcase I.2 we write $x_\theta = (y_\theta, t_\theta, z_\theta)$ where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln \varepsilon, -\ln \varepsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By restricting to a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge respectively to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. We apply Lemma 3.3 with $V = W$, $\alpha = \theta$, $p_\alpha = y_\theta$, $p = y$, $\gamma_\alpha = \tilde{h}_{t_\theta}$, $\gamma_0 = \tilde{h}_T$, and $b_\alpha = \alpha_{t_\theta}$ (recall that $\tilde{h}_t$ and α_t were defined in (33) and (34)) and conclude that there is a diffeomorphism

$$\Theta_\theta^y : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, \alpha_{t_\theta}^{-1} r)$$

for $r > 0$ such that $(\Theta_\theta^y)^*(\alpha_{t_\theta}^2 \tilde{h}_{t_\theta})$ converges to the Euclidean metric ξ^k on $B^k(r)$. For $r, r' > 0$ we define

$$U_\theta(r, r') := B^{\tilde{h}_{t_\theta}}(y_\theta, \alpha_{t_\theta}^{-1} r) \times [t_\theta - r', t_\theta + r'] \times S^{n-k-1}$$

and

$$\begin{aligned} \Theta_\theta : B^k(r) \times [-r', r'] \times S^{n-k-1} &\to U_\theta(r, r') \\ (y, s, z) &\mapsto (\Theta_\theta^y(y), t(s), z), \end{aligned}$$

where $t(s) := t_\theta + s$. By construction Θ_θ is a diffeomorphism. Since $g_\theta = \alpha_t^2 \tilde{h}_t + dt^2 + \sigma^{n-k-1}$ we see that

$$\Theta_\theta^*(g_\theta) = \frac{\alpha_t^2}{\alpha_{t_\theta}^2} (\Theta_\theta^y)^* (\alpha_{t_\theta}^2 \tilde{h}_t) + ds^2 + \sigma^{n-k-1}. \tag{46}$$

We will now find the limit of $\Theta_\theta^*(g_\theta)$ in the C^1 topology. We define $c := \lim_{\theta\to 0} f'(t_\theta)$.

Lemma 4.1. *The sequence of metrics $\Theta_\theta^*(g_\theta)$ tends to $\eta_c^{k+1} + \sigma^{n-k-1} = e^{2cs}\xi^k + ds^2 + \sigma^{n-k-1}$ in C^1 on $B^k(r) \times [-r', r'] \times S^{n-k-1}$ for fixed $r, r' > 0$.*

Proof. Recall that $\alpha_t = e^{f(t)}$. The intermediate value theorem says that

$$\Big| f(t) - f(t_\theta) - (\frac{d}{dt} f)(t_\theta)(t - t_\theta) \Big| \le \frac{r'^2}{2} \max_{\xi \in [t_\theta - r', t_\theta + r']} \Big| \frac{d^2}{dt^2} f(\xi) \Big|$$

for all $t \in [t_\theta - r', t_\theta + r']$. On the other hand we claimed $f''(t) \to 0$ for $t \to \infty$, hence

$$\Big\| f(t) - f(t_\theta) - (\frac{d}{dt} f)(t_\theta)(t - t_\theta) \Big\|_{C^0([t_\theta - r', t_\theta + r'])} \to 0$$

for $\theta \to 0$ (and r' fixed). Furthermore

$$\begin{aligned} \Big| \frac{d}{dt} \Big(f(t) - f(t_\theta) - (\frac{d}{dt} f)(t_\theta)(t - t_\theta) \Big) \Big| &= \Big| \frac{d}{dt} f(t) - \frac{d}{dt} f(t_\theta) \Big| \\ &= \Big| \int_{t_\theta}^t \frac{d^2}{dt^2} f(s)\, ds \Big| \\ &\le r' \max_{\xi \in [t_\theta - r', t_\theta + r']} \Big| \frac{d^2}{dt^2} f(\xi) \Big| \\ &\to 0 \end{aligned}$$

as $\theta \to 0$. Together with $c = \lim_{\theta\to 0} f'(t_\theta)$ we have

$$\Big\| f(t) - f(t_\theta) - c(t - t_\theta) \Big\|_{C^1([t_\theta - r', t_\theta + r'])} \to 0$$

Exponentiation of functions $[t_\theta - r', t_\theta + r'] \to \mathbb{R}$ is a continuous map

$$C^1([t_\theta - r', t_\theta + r']) \to C^1([t_\theta - r', t_\theta + r']), \quad \tilde{f} \mapsto \exp \circ \tilde{f}.$$

Hence

$$\Big\| \frac{\alpha_t}{\alpha_{t_\theta}} - e^{c(t - t_\theta)} \Big\|_{C^1([t_\theta - r', t_\theta + r'])} = \Big\| e^{f(t) - f(t_\theta)} - e^{c(t - t_\theta)} \Big\|_{C^1([t_\theta - r', t_\theta + r'])} \to 0$$

We now write $\alpha_t^2 \tilde{h}_t = \alpha_t^2 (\tilde{h}_t - \tilde{h}_{t_\theta}) + \frac{\alpha_t^2}{\alpha_{t_\theta}^2} \alpha_{t_\theta}^2 \tilde{h}_{t_\theta}$. Using the fact that

$$\lim_{\theta\to 0} \Big\| \tilde{h}_t - \tilde{h}_{t_\theta} \Big\|_{C^1(B_{\tilde{h}_{t_\theta}}(y_\theta, \alpha_{t_\theta}^{-1} R))} = 0$$

uniformly for $t \in [t_\theta - r', t_\theta - r']$ we get that the sequence $\frac{\alpha_t^2}{\alpha_{t_\theta}^2} (\Theta_\theta^y)^* (\alpha_{t_\theta}^2 \tilde{h}_t)$ tends to $e^{2cs}\xi^k$ in C^1 on $B^k(r)$). Going back to Relation (46), this proves Lemma 4.1. □

We continue with the proof of Step 3. As in subcases I.1 and I.2 we apply Lemma 3.4 with $(V, g) = (\mathbb{R}^{k+1} \times S^{n-k-1}, \eta_c^{k+1} + \sigma^{n-k-1})$, $\alpha = \theta$, and $g_\alpha = \Theta_\theta^*(g_\theta)$ (we can apply this lemma since any compact subset of $\mathbb{R}^{k+1} \times S^{n-k-1}$ is contained in some $B^k(r) \times [-r', r'] \times S^{n-k-1}$). We obtain a C^1 spinor ψ which is a solution of

$$D^{\eta_c^{k+1}+\sigma^{n-k-1}}\psi = \bar{\lambda}|\psi|^{\frac{2}{n-1}}\psi$$

on $(\mathbb{R}^{k+1} \times S^{n-k-1}, \eta_c^{k+1} + \sigma^{n-k-1})$. From (16) it follows that

$$\int_{\mathbb{R}^{k+1}\times S^{n-k-1}} |\psi|^{\frac{2n}{n-1}}\, dv^{\eta_c^{k+1}+\sigma^{n-k-1}} \leq 1.$$

From (15) it follows that $\psi \in L^\infty(\mathbb{R}^{k+1} \times S^{n-k-1})$ and from (15) and (44) it follows that ψ is non-zero. We want to show that $\psi \in L^2(\mathbb{R}^{k+1} \times S^{n-k-1})$. From (16) we get that

$$\begin{aligned} \int_{B^k(r)\times[-r',r']\times S^{n-k-1}} |\psi|^2\, dv^{\eta_c^{k+1}+\sigma^{n-k-1}} &= \lim_{\theta\to 0}\int_{U_\theta(r,r')} |\psi_\theta|^2 dv^{g_\theta} \\ &\leq \lim_{\theta\to 0}\int_{U^N(a)} |\psi_\theta|^2 dv^{g_\theta} \end{aligned} \tag{47}$$

for some fixed number $a > 0$ independent of r, r' and θ. Let χ be defined as in Step 1. Using the Hölder inequality, Proposition 3.5, and Equation (37) we see that

$$\begin{aligned} &\frac{(n-1-k)^2}{4} \\ &\leq \frac{\int_N |D^{g_\theta}(\chi\psi_\theta)|^2\, dv^{g_\theta}}{\int_N |\chi\psi_\theta|^2\, dv^{g_\theta}} \\ &= \frac{\int_N |d\chi|_{g_\theta}^2 |\psi_\theta|^2\, dv^{g_\theta} + \lambda_\theta^2 \int_N \chi^2 |\psi_\theta|^{\frac{2(n+1)}{n-1}}\, dv^{g_\theta}}{\int_N |\chi\psi_\theta|^2\, dv^{g_\theta}} \\ &\leq \frac{16\int_{U^N(2a)\setminus U^N(a)} |\psi_\theta|^2\, dv^{g_\theta} + \lambda_\theta^2 \|\psi_\theta\|_{L^\infty(U^N(2a))}^{\frac{2}{n-1}} \int_{U^N(2a)} |\psi_\theta|^{\frac{2n}{n-1}}\, dv^{g_\theta}}{\int_{U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}}. \end{aligned}$$

We have

$$\lambda_\theta^2 \|\psi_\theta\|_{L^\infty(U^N(2a))}^{\frac{2}{n-1}} \int_{U^N(2a)} |\psi_\theta|^{\frac{2n}{n-1}}\, dv^{g_\theta} \leq \lambda_{\min}^+(S^n, \sigma^n)^2 C_1^{\frac{2}{n-1}}$$

and

$$\begin{aligned} &\int_{U^N(2a)\setminus U^N(a)} |\psi_\theta|^2\, dv^{g_\theta} \\ &\quad\leq \left(\int_{U^N(2a)\setminus U^N(a)} |\psi_\theta|^{\frac{2n}{n-1}}\, dv^{g_\theta}\right)^{\frac{n-1}{n}} \mathrm{Vol}\left(U^N(2a)\setminus U^N(a), g_\theta\right)^{\frac{1}{n}} \\ &\quad\leq \mathrm{Vol}\left(U^N(2a)\setminus U^N(a), g_\theta\right)^{\frac{1}{n}}. \end{aligned}$$

Since g_θ does not depend on θ on $U^N(2a) \setminus U^N(a)$ for $\theta < a$, we get the existence of a constant C such that

$$\frac{(n-1-k)^2}{4} \leq \frac{C}{\int_{U^N(a)} |\psi_\theta|^2\, dv^{g_\theta}}.$$

Together with (47) we obtain that

$$\int_{B^k(r)\times[-r',r']\times S^{n-k-1}} |\psi|^2\, dv^{\eta_c^{k+1}+\sigma^{n-k-1}} \leq C$$

where C is independent of r and r'. This proves that $\psi \in L^2(\mathbb{R}^{k+1} \times S^{n-k-1})$. Since the spinor ψ is non-zero and

$$\psi \in L^\infty(\mathbb{R}^{k+1} \times S^{n-k-1}) \cap C^1_{\mathrm{loc}}(\mathbb{R}^{k+1} \times S^{n-k-1}) \cap L^2(\mathbb{R}^{k+1} \times S^{n-k-1})$$

with

$$\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |\psi|^{\frac{2n}{n-1}} \, dv^{\eta_c^{k+1}+\sigma^{n-k-1}} \leq 1$$

we get that $\bar{\lambda} \geq \Lambda_{n,k}$ from the definition of $\Lambda_{n,k}$. This ends the proof of this subcase and the proof of Theorem 1.2.

4.2. **Proof of Theorem 1.1.** We prove Theorem 1.1 by contradiction. Assume that there is a sequence (c_i), $i \in \mathbb{N}$, of $c_i \in [-1,1]$ for which

$$\widetilde{\lambda^+_{\min}}(\mathbb{R}^{k+1} \times S^{n-k-1}, \eta^{k+1}_{c_i} + \sigma^{n-k-1}) \to 0.$$

After removing the indices i for which $\widetilde{\lambda^+_{\min}}$ is infinite we obtain for all i a solution of

$$D^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}} \psi_i = \lambda_i |\psi_i|^{\frac{2}{n-1}} \psi_i \tag{48}$$

where $\lambda_i \to 0$. Moreover the spinors ψ_i are in $L^\infty \cap L^2 \cap C^1_{\mathrm{loc}}$ and

$$\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |\psi_i|^{\frac{2n}{n-1}} \, dv^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}} \leq 1.$$

Let $m_i := \|\psi_i\|_{L^\infty}$. We cannot assume that m_i is attained, but since $(\mathbb{R}^{k+1} \times S^{n-k-1}, \eta^{k+1}_{c_i} + \sigma^{n-k-1})$ is a symmetric space we can assume that $|\psi_i(P)| > \frac{m_i}{2}$ for some fixed point $P \in \mathbb{R}^{k+1} \times S^{n-k-1}$. First we prove that

$$\lim_i m_i = \infty. \tag{49}$$

By Lemma 3.5 and Equation (48) we have

$$\begin{aligned} \frac{(n-k-1)^2}{4} &\leq \frac{\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |D^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}} \psi_i|^2 \, dv^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}}}{\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |\psi_i|^2 \, dv^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}}} \\ &\leq \frac{\lambda_i^2 \int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |\psi_i|^{\frac{2(n+1)}{n-1}} \, dv^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}}}{\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} |\psi_i|^2 \, dv^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}}} \\ &\leq \lambda_i^2 m_i^{\frac{4}{n-1}}. \end{aligned}$$

Since $\lim_i \lambda_i = 0$ this proves (49). Restricting to subsequence we can assume that $\lim_i c_i$ exists and we denote this limit by $c \in [-1,1]$. We apply Lemma 3.3 with $\alpha = \frac{1}{i}$, $(V, \gamma_\alpha) = (\mathbb{R}^{k+1} \times S^{n-k-1}, \eta^{k+1}_{c_i} + \sigma^{n-k-1})$, $(V, \gamma_0) = (\mathbb{R}^{k+1} \times S^{n-k-1}, \eta^{k+1}_c + \sigma^{n-k-1})$, $p_\alpha = p = P$, and $b_\alpha = m_i^{\frac{2}{n-1}}$. For $r > 0$ we obtain a diffeomorphism

$$\Theta_i : B^n(r) \to B^{\eta^{k+1}_{c_i}+\sigma^{n-k-1}}(P, m_i^{\frac{2}{n-1}} r)$$

such that $\Theta_i^*(m_i^{\frac{4}{n-1}}(\eta^{k+1}_{c_i} + \sigma^{n-k-1}))$ tends to the Euclidean metric ξ^n on $B^n(r)$. Proceeding exactly as in Subcase I.1 of Theorem 1.2 we construct a non-zero spinor ψ belonging to $L^{\frac{2n}{n-1}}(\mathbb{R}^n)$ such that

$$D^{\xi^n} \psi = \lim_i \lambda_i |\psi|^{\frac{2}{n-1}} \psi = 0.$$

Again as in in Subcase I.1 of Theorem 1.2 we get $0 \geq \lambda^+_{\min}(S^n, \sigma^n)$, which is false. This proves Theorem 1.1.

References

[1] B. Ammann, *The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions*, Preprint, 2003.

[2] ———, *A spin-conformal lower bound of the first positive Dirac eigenvalue*, Diff. Geom. Appl. **18** (2003), 21–32.

[3] ———, *A variational problem in conformal spin geometry*, Habilitationsschrift, Universität Hamburg, 2003.

[4] B. Ammann, M. Dahl, and E. Humbert, *Surgery and harmonic spinors*, Preprint, 2006.

[5] ———, *A surgery formula for the smooth Yamabe invariant*, Announcement at an Oberwolfach conference, Oberwolfach Report no. 41/2007, 2007.

[6] B. Ammann, J. F. Grosjean, E. Humbert, and B. Morel, *A spinorial analogue of Aubin's inequality*, to appear in Math. Z.

[7] B. Ammann and E. Humbert, *The spinorial τ-invariant and 0-dimensional surgery*, ArXiv math.DG/0607716, to appear in J. Reine Angew. M., 2006.

[8] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Teubner Verlag, 1981.

[9] J.-P. Bourguignon and P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*, Comm. Math. Phys. **144** (1992), 581–599.

[10] T. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics 25, AMS, Providence, Rhode Island, 2000.

[11] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der mathematischen Wissenschaften, no. 224, Springer-Verlag, 1977.

[12] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), 151–162.

[13] ———, *Première valeur propre de l'opérateur de Dirac et nombre de Yamabe*, C. R. Acad. Sci. Paris **t. 313, Série I** (1991), 865–868.

[14] ———, *Spectral properties of the Dirac operator and geometrical structures.*, Ocampo, Hernan (ed.) et al., Geometric methods for quantum field theory. Proceedings of the summer school, Villa de Leyva, Colombia, July 12-30, 1999. Singapore: World Scientific. 116–169 , 2001.

[15] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55.

[16] R. C. Kirby, *The topology of 4-manifolds*, Lecture Notes in Mathematics, vol. 1374, Springer-Verlag, Berlin, 1989.

[17] O. Kobayashi, *Scalar curvature of a metric with unit volume*, Math. Ann. **279** (1987), no. 2, 253–265.

[18] A. A. Kosinski, *Differential manifolds*, Pure and Applied Mathematics, vol. 138, Academic Press Inc., Boston, MA, 1993.

[19] H.-B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.

[20] J. Lott, *Eigenvalue bounds for the Dirac operator*, Pacific J. of Math. **125** (1986), 117–126.

[21] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related Topics.*, Topics in calculus of variations, Lect. 2nd Sess., Montecatini/Italy 1987, Lect. Notes Math. 1365, 120-154 , 1989 (English).

[22] R. E. Stong, *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J., 1968.

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