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Guido Kings

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Introduction

Cohomology classes defined by polylogarithms have been one of the main tools to study special values of L-functions. Most notably, they play a decisive role in the study of the Tamagawa number conjecture for abelian number fields ([Be2], [Del], [HuW] [Hu-Ki]), CM elliptic curves ([Den], [Ki2]) and modular forms ([Be1], [Ka]).

Polylogarithms have been defined for relative curves by Beilinson and Levin (unpublished) and for abelian schemes by Wildeshaus [Wi] in the context of mixed Shimura varieties. In general, the nature of these extension classes is not well understood.

The aim of this note is to show that there is a close connection between the polylogarithm extension on curves and on abelian schemes. It turns out that the polylog on an abelian scheme is roughly the push-forward of the polylog on a sub-curve. If we apply this to the embedding of a curve into its Jacobian, we can give a more precise statement: the polylog on the Jacobian is the cup product of the polylog on the curve with the fundamental class of the curve (see theorem 3.2.1). With this result it is possible to understand the nature of the polylog extension on abelian schemes in a better way.

The polylog extension on curves has the advantage of being a one extension of lisse sheaves. Thus, itself can be represented by a lisse sheaf. The polylog extension on the abelian scheme on the contrary is a $2d-1$ extension, where d is the relative dimension of the abelian scheme.

The contents of this note is as follows: To simplify the exposition we only treat the étale realization. First we define the polylog extension on curves and abelian schemes in a unified way for integral coefficients. To our knowledge this and the construction on curves is not published but goes back to an earlier version of [Be-Le1]. The case of abelian schemes is treated in [Wi] (for $\mathbb{Q}_l$-sheaves), which we mildly generalize to $\mathbb{Z}/l^r\mathbb{Z}$- and $\mathbb{Z}_l$-sheaves. All the main ideas are of course already in [Be-Le1].

The second part gives three important properties of the polylog extension, namely compatibility with base change, norm compatibility and the splitting principle.

In the last part we show that the push-forward of the polylog on a subcurve of an abelian scheme gives the polylogarithmic extension on the abelian scheme and prove our main theorema about the polylog on the Jacobian.

1 Definition of the polylogarithm extension

The first part of this paper recalls the definition of the polylogarithmic extension for curves and abelian schemes.

The case of elliptic curves was treated by Beilinson and Levin [Be-Le1] in analogy with the cyclotomic case considered by Beilinson and Deligne. An earlier version of [Be-Le1] contained also the case of general curves. Polylogarithmic extensions for abelian schemes and more generally certain semi-abelian schemes were first considered by Wildeshaus in [Wi] in the context of mixed Shimura varieties.

1.1 The logarithm sheaf

In this section we recall the definition of the logarithm sheaf for curves and abelian schemes.

Let S be a connected scheme, and l be a prime number invertible on S. We fix a base ring Λ which is either $\mathbb{Z}/l^r\mathbb{Z}$ or $\mathbb{Z}_l$. The cohomology in this paper is always continuous cohomology in the sense of Jannsen [Ja].

Definition 1.1.1. *A* curve *is a smooth proper morphism $\pi : C \to S$ together with a section $e : S \to C$, such that the geometric fibers $C_{\overline{s}}$ of π are connected curves of genus ≥ 1.*

In addition to the curves we consider also abelian schemes $\pi : A \to S$ with unit section $e : S \to A$. For brevity we use the following notation: $\pi : X \to S$ will denote either a curve in the sense of 1.1.1 or an abelian scheme over S. The section will be denoted by $e : S \to X$. The relative dimension of X/S is d.

Let us describe a Λ-version of the theory in [Wi] I, chapter 3. In the case of an elliptic curve this coincides with [Be-Le1]. Let $\overline{s}$ be a geometric point of S and denote by $\overline{x} := e(\overline{s})$ the corresponding geometric point of X. Denote the fiber over $\overline{s}$ by $X_{\overline{s}}$ and consider the split exact sequence of fundamental groups

$$1 \to \pi_1'(X_{\overline{s}}, \overline{x}) \to \pi_1'(X, \overline{x}) \xrightarrow{\pi_*} \pi_1(S, \overline{s}) \to 1 \tag{1}$$

(cf. [SGA1] XIII 4.3), where $\pi_1'(X_{\overline{s}}, \overline{x})$ is the largest pro-l-quotient of $\pi_1(X_{\overline{s}}, \overline{x})$ and if $\ker(\pi_*)/N$ denotes the largest pro-l-quotient of $\ker(\pi_*)$, then $\pi_1'(X, \overline{x}) :=$

$\pi_1(X,\overline{x})/N$. The splitting is given by e_*. Now $\pi_1'(X_{\overline{s}},\overline{x})$ is a pro-finite group and we fix a fundamental system of open neighborhoods Γ_j of the identity with $j \in J$, such that

$$\pi_1'(X_{\overline{s}},\overline{x}) = \varprojlim_j \pi_1'(X_{\overline{s}},\overline{x})/\Gamma_j.$$

Define

$$H_j := \pi_1'(X_{\overline{s}},\overline{x})/\Gamma_j$$

and let us agree that in the case where X is an abelian scheme we choose the Γ_j in such a way that $H_j = \ker[l^j]$ is the kernel of the $[l^j]$-multiplication. Let us also fix a projective system X_j of étale H_j-torsors (i.e. Galois coverings of X with group H_j) such that

$$\begin{array}{ccc} H_j & \xrightarrow{h_j} & X_j \\ \downarrow & & \downarrow{\scriptstyle p_j} \\ S & \xrightarrow{e} & X, \end{array}$$

is Cartesian. In the case of an abelian scheme we take $X_j = A$ and $p_j = [l^j]$. For $j' \to j$ we have the trace map

$$p_{j'*}\Lambda \to p_{j*}\Lambda$$

and we define:

Definition 1.1.2. *The* logarithm sheaf *is the étale sheaf*

$$\mathcal{L}\mathrm{og}_{X,\Lambda} := \varprojlim_j (p_{j*}\Lambda)$$

where the transition maps are the above trace maps.

The stalk of $\mathcal{L}\mathrm{og}_{X,\Lambda}$ at $\overline{x}$ is just the Iwasawa algebra of the profinite group $\pi_1'(X_{\overline{s}},\overline{x})$

$$\mathcal{L}\mathrm{og}_{X,\Lambda,\overline{x}} = \Lambda[[\pi_1'(X_{\overline{s}},\overline{x})]]$$

with the canonical action of the semi-direct product $\pi_1'(X,\overline{x})$ given by multiplication on $\pi_1'(X_{\overline{s}},\overline{x})$ and by conjugation on the quotient $\pi_1(S,\overline{s})$.

Let

$$\mathcal{R}_{X,\Lambda} := e^* \mathcal{L}\mathrm{og}_{X,\Lambda}$$

be the pull-back of $\mathcal{L}\mathrm{og}_{X,\Lambda}$ along the unit section e.

Note that that there is a canonical map $\mathbf{1} : \Lambda \to \mathcal{R}_{X,\Lambda}$ and that $\mathcal{R}_{X,\Lambda}$ has a ring structure given by group multiplication as usual. We denote by

$$\mathcal{I}_{X,\Lambda} := \ker(\mathcal{R}_{X,\Lambda} \to \Lambda)$$

the augmentation ideal. The logarithm sheaf has a canonical action of $\pi^*\mathcal{R}_{X,\Lambda}$ (again induced by group multiplication)

$$\pi^*\mathcal{R}_{X,\Lambda} \otimes_\Lambda \mathcal{L}\mathrm{og}_{X,\Lambda} \to \mathcal{L}\mathrm{og}_{X,\Lambda},$$

which defines on $\mathcal{L}\mathrm{og}_{X,\Lambda}$ the structure of an $\pi^*\mathcal{R}_{X,\Lambda}$-torsor.

It is very useful to consider the abelianized version of the logarithm sheaf.

Definition 1.1.3. *Let $\pi_1'(X_{\overline{s}}, \overline{x})^{\mathrm{ab}}$ be the maximal abelian quotient of $\pi_1'(X_{\overline{s}}, \overline{x})$ and define the* abelian logarithm sheaf *to be the lisse sheaf defined by the $\pi_1'(X, \overline{x})$-representation*

$$\mathcal{L}\mathrm{og}^{\mathrm{ab}}_{X,\Lambda,\overline{x}} := \Lambda[[\pi_1'(X_{\overline{s}}, \overline{x})^{\mathrm{ab}}]].$$

Note that in the our case $\pi_1'(X_{\overline{s}}, \overline{x})^{\mathrm{ab}} \cong \mathbb{Z}_l^r$ is a free modul over $\mathbb{Z}_l$ of rank twice the genus of the curve or twice the dimension of the ablian scheme. In particular,

$$\Lambda[[\pi_1'(X_{\overline{s}}, \overline{x})^{\mathrm{ab}}]] \cong \Lambda[[x_1, \ldots, x_r]]$$

is isomorphic to a power series ring in r variables.

The whole theory of the polylogarithm sheaves relies on the fact that the higher direct images of $\mathcal{L}\mathrm{og}_{X,\Lambda}$ can be computed for curves and abelian schemes.

Theorem 1.1.4 ([Be-Le1],[Be-Le2],[Wi]). *Let d be the relative dimension of $X = C, A$ over S. Then the étale sheaf*

$$R^i\pi_* \mathcal{L}\mathrm{og}_{X,\Lambda}$$

is zero for $i \neq 2d$ and

$$R^{2d}\pi_* \mathcal{L}\mathrm{og}_{X,\Lambda} \cong \Lambda(-d).$$

Proof. We use the spectral sequence [Ja] 3.10. We have to compute the transition maps in the inverse system $(R^i\pi_* p_{j_*}\Lambda)_j$, which by Poincaré duality can by written as an inductive system

$$R^{2d-i}\pi_! p_{j_*}\Lambda(d).$$

Here the transition maps are induced by the pull-back maps $j' \to j$

$$p_{j*}\Lambda(d) \to p_{j'*}\Lambda(d).$$

Let us first show that the inductive system $(R^{2d-i}\pi_! p_{j*}\Lambda(d))_j$ is zero for $i < 2d$. By base change we may assume that S is the spectrum of an algebraically closed field, so that we have to compute the transition maps for

$$H^{2d-i}_{\text{et}}(X_j, \Lambda(d)).$$

We have

$$H^1_{\text{et}}(X_j, \Lambda(d)) = \text{Hom}(\pi_1(X_j), \Lambda(d)) = \text{Hom}(\Gamma_j, \Lambda(d)).$$

For every homomorphism in $\text{Hom}(\Gamma_j, \Lambda(d))$ there is an j', such that its restriction to $\Gamma_{j'}$ is trivial. Thus the maps

$$H^1_{\text{et}}(X_j, \Lambda(d)) \to H^1_{\text{et}}(X_{j'}, \Lambda(d))$$

are zero. For an abelian scheme we have

$$H^{2d-i}_{\text{et}}(X_j, \Lambda(d)) \cong \Lambda^{2d-i} H^1_{\text{et}}(X_j, \Lambda(d))$$

hence we get the same result for

$$H^{2d-i}_{\text{et}}(X_j, \Lambda(d)) = 0$$

for $i < 2d$. Now $H^0_{\text{et}}(X_j, \Lambda(d)) = \Lambda(d)$ is constant, so that the natural map $R^{2d}\pi_! p_{j*}\Lambda(d) \to \Lambda$ is an isomorphism. Thus the result is proven for abelian schemes. If X is a curve it remains to consider

$$H^2_{\text{et}}(X_j, \Lambda(1))$$

We have $H^2_{\text{et}}(X_j, \Lambda(1)) = \Lambda$ and the transition maps are given by multiplication with the degree of $X_{j'} \to X_j$. Thus for j' large enough the transition maps are zero. $\square$

1.2 The polylogarithm extension

Let X be a curve or an abelian scheme and $U := X \setminus e(S)$. Denote by $\pi_U : U \to S$ the restriction of π to U. We let

$$j : U \to X$$

be the open immersion of U into X. The restriction of $\mathcal{L}\text{og}_{X,\Lambda}$ to U is denoted by $\mathcal{L}\text{og}_{U,\Lambda}$.

Proposition 1.2.1. *Let d be the relative dimension of X/S and recall that $\mathcal{I}_{X,\Lambda}$ is the kernel of the augmentation map of $\mathcal{R}_{X,\Lambda}$. Then the étale sheaf*

$$R^i\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d)$$

is zero for $i \neq 2d-1$ and for $i = 2d-1$ there is an isomorphism of sheaves

$$R^{2d-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d) \cong \mathcal{I}_{X,\Lambda}.$$

Proof. This follows immediately from the localization sequence

$$R^{i-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d) \to R^i e^!\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d) \to R^i\pi_*\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d)$$

and the purity isomorphism $e^!\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d) \cong e^*\,\mathcal{L}\mathrm{og}_{X,\Lambda}[-2d] = \mathcal{R}_{X,\Lambda}[-2d]$. By theorem 1.1.4 the sheaf $R^i\pi_*\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d)$ is zero for $i \neq 2d$, so that we get an exact sequence

$$R^{2d-1}\pi_*\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d) \to R^{2d-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d) \to \mathcal{R}_{X,\Lambda} \to R^{2d}\pi_*\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d) \to 0.$$

The identification $R^{2d}\pi_*\,\mathcal{L}\mathrm{og}_{X,\Lambda}(d) \cong \Lambda$ gives an identification of the last map with the augmentation map $\mathcal{R}_{X,\Lambda} \to \Lambda$ and $R^{2d-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d)$ becomes isomorphic to the augmentation ideal. □

Consider the extension

$$\mathrm{Ext}^i_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}_{U,\Lambda}(d)).$$

Corollary 1.2.2. *There is an isomorphism*

$$\mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}_{U,\Lambda}(d)) \cong \mathrm{Hom}_S(\mathcal{I}_{X,\Lambda}, \mathcal{I}_{X,\Lambda})$$

given by the edge morphism in the Leray spectral sequence for $R\pi_{U*}$.

Proof. In the Leray spectral sequence for $R\pi_{U*}$ all higher direct images except $R^{2d-1}\pi_{U*}\mathcal{L}_j|_U(d)$ are zero. Thus all higher Ext groups vanish except

$$\mathrm{Hom}_S(\mathcal{I}_{X,\Lambda}, R^{2d-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d)).$$

The above theorem identifies $R^{2d-1}\pi_{U*}\,\mathcal{L}\mathrm{og}_{U,\Lambda}(d)$ with $\mathcal{I}_{X,\Lambda}$ and the result follows. □

Definition 1.2.3. *The* large polylogarithm extension *on X is the extension class $\mathcal{P}\mathrm{ol}_{X,\Lambda}$ in*

$$\mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}_{U,\Lambda}(d)) \cong \mathrm{Hom}_S(\mathcal{I}_{X,\Lambda}, \mathcal{I}_{X,\Lambda})$$

corresponding to the identity in $\mathrm{Hom}_S(\mathcal{I}_{X,\Lambda}, \mathcal{I}_{X,\Lambda})$.

Remark: a) In the case of an elliptic curve the polylogarithm is the one considered by Beilinson and Levin [Be-Le1]. In the case of an abelian scheme our definition gives a $\mathbb{Z}/l^r\mathbb{Z}$-version of the construction in Wildeshaus [Wi].
b) This polylog extension should be more precisely called the étale realization of the polylog. The Hodge-realization can be defined in a similar way.
c) In the case of an abelian scheme, this class is in the image of the regulator coming from K-theory (see [Ki1]).

It is useful to make also the following definition:

Definition 1.2.4. *The* abelian polylogarithm extension *on X is the extension class $\mathcal{P}\mathrm{ol}^{\mathrm{ab}}_{X,\Lambda}$ in*

$$\mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}^{\mathrm{ab}}_{U,\Lambda}(d)),$$

which is the image of $\mathcal{P}\mathrm{ol}_{X,\Lambda}$ under the canonical map

$$\mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}_{U,\Lambda}(d)) \to \mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{L}\mathrm{og}^{\mathrm{ab}}_{U,\Lambda}(d)).$$

2 Properties of the polylog

We consider compatibilities of the polylog, namely its behavior under base change and finite étale morphisms. Finally, we explain the splitting principle in our situation.

2.1 Compatibility with base change

Let $S' \to S$ be a scheme over S and X' be the fiber product $X \times_S S'$, where X is C or A as usual. We get a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S. \end{array}$$

and let $U' := X' \smallsetminus e'(S')$ and $U := X \smallsetminus e(S)$. Here e and e' are the unit sections of X and X' respectively. The obvious map of fundamental groups $\pi_1'(X_{\overline{s}}, \overline{x}) \to \pi_1'(X'_{\overline{s}'}, \overline{x}')$ induces

$$\mathcal{L}\mathrm{og}_{X'} \to f^* \mathcal{L}\mathrm{og}_X$$

which is called the canonical map.

Lemma 2.1.1. *The canonical map*

$$\mathcal{L}\mathrm{og}_{X'} \to f^* \mathcal{L}\mathrm{og}_X$$

is an isomorphism.

Proof. It suffices to remark that pull-back by f induces an isomorphism of the relative fundamental groups

$$\pi_1'(X_{\overline{s}}, \overline{x}) \cong \pi_1'(X'_{\overline{s}'}, \overline{x}')$$

from equation (1) in section 1.1. Here we use of course the base points $f(\overline{x}') = \overline{x}$ and $f(\overline{s}') = \overline{s}$. $\square$

In particular the canonical map induces an isomorphism

$$\pi'^* \mathcal{I}_{X'} \cong f^* \pi^* \mathcal{I}_X$$

so that we have

$$\mathrm{Ext}_U^{2d-1}(f^* \pi_U^* \mathcal{I}_X, f^* \mathcal{L}\mathrm{og}_U(d)) \cong \mathrm{Ext}_{U'}^{2d-1}({\pi'_{U'}}^* \mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{U'}(d)).$$

Corollary 2.1.2. *Via this identification of* Ext*-groups*

$$f^* \mathcal{P}\mathrm{ol}_X = \mathcal{P}\mathrm{ol}_{X'} .$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_U^{2d-1}(\pi_U^* \mathcal{I}_X, \mathcal{L}\mathrm{og}_U(d)) & \xrightarrow{f^*} & \mathrm{Ext}_{U'}^{2d-1}({\pi'_{U'}}^* \mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{U'}(d)) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_S(\mathcal{I}_X, \mathcal{I}_X) & \xrightarrow{g^*} & \mathrm{Hom}_{S'}(\mathcal{I}_{X'}, \mathcal{I}_{X'})
\end{array}$$

and the identity is mapped to the identity under g^*. $\square$

2.2 Norm compatibility for finite étale morphisms

Let $f : X' \to X$ be a finite étale pointed morphism over S, i.e. $f \circ e' = e$, and denote by π' and π the structure maps of X' and X. Let U and U' be as above. Define Z' and $\widetilde{U}$ by the Cartesian diagram:

$$\begin{array}{ccccc}
Z' & \xrightarrow{\varepsilon} & X' & \xleftarrow{\widetilde{j}} & \widetilde{U} \\
\downarrow f & & \downarrow f & & \downarrow f \\
S & \xrightarrow{e} & X & \longleftarrow & U.
\end{array}$$

Observe that $\widetilde{U} \subset U'$ and denote by $U' \xrightarrow{j'} X'$ the open immersion. Restriction to $\widetilde{U}$ gives a map

$$\text{(2)} \qquad \operatorname{Ext}^{2d-1}_{U'}({\pi'_{U'}}^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{U'}(d)) \to \operatorname{Ext}^{2d-1}_{\widetilde{U}}({\pi'_{\widetilde{U}}}^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{\widetilde{U}}(d)).$$

On the other hand we have an adjunction map (${\pi'_{\widetilde{U}}}^* = f^* \circ {\pi_U}^*$)

$$\text{(3)} \qquad \operatorname{Ext}^{2d-1}_{\widetilde{U}}(f^*{\pi_U}^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{\widetilde{U}}(d)) \to \operatorname{Ext}^{2d-1}_{U}({\pi_U}^*\mathcal{I}_{X'}, f_*\,\mathcal{L}\mathrm{og}_{\widetilde{U}}(d)).$$

If we apply f_* to the canonical map $\mathcal{L}\mathrm{og}_{X'} \to f^*\,\mathcal{L}\mathrm{og}_X$ we get a map

$$f_*\,\mathcal{L}\mathrm{og}_{\widetilde{U}} \to f_*f^*\,\mathcal{L}\mathrm{og}_U \xrightarrow{\mathrm{Tr}} \mathcal{L}\mathrm{og}_U$$

and hence a map

$$\text{(4)} \qquad \operatorname{Ext}^{2d-1}_{U}({\pi_U}^*\mathcal{I}_{X'}, f_*\,\mathcal{L}\mathrm{og}_{\widetilde{U}}(d)) \to \operatorname{Ext}^{2d-1}_{U}(\pi_U^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_U(d)).$$

Denote by N_f the resulting composition of (2), (3) and (4)

$$\mathrm{N}_f : \operatorname{Ext}^{2d-1}_{U'}({\pi'_{U'}}^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_{U'}(d)) \to \operatorname{Ext}^{2d-1}_{U}(\pi_U^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_U(d)).$$

Consider also the map $\mathcal{I}_{X'} \to \mathcal{I}_X$ induced by the canonical map $\mathcal{L}\mathrm{og}_{X'} \to f^*\,\mathcal{L}\mathrm{og}_X$ and let

$$\alpha^* : \operatorname{Ext}^{2d-1}_{U}(\pi_U^*\mathcal{I}_X, \mathcal{L}\mathrm{og}_U(d)) \to \operatorname{Ext}^{2d-1}_{U}(\pi_U^*\mathcal{I}_{X'}, \mathcal{L}\mathrm{og}_U(d))$$

be the associated pull-back.

Proposition 2.2.1. (Norm compatibility) *With the maps* N_f *and* α^* *defined above*

$$\mathrm{N}_f(\mathcal{P}\mathrm{ol}_{X'}) = \alpha^*(\mathcal{P}\mathrm{ol}_X).$$

Proof. First of all we remark that

$$R^i\pi_{\widetilde{U},*}\,\mathcal{L}\mathrm{og}_{\widetilde{U}}$$

is zero for $i \neq 2d-1$ and isomorphic to

$$\ker(f_*\varepsilon^*\,\mathcal{L}\mathrm{og}_{X'} \to \Lambda)$$

for $\varepsilon : Z' \hookrightarrow X'$ and $i = 2d-1$. This follows as in corollary 1.2.2 from the localization sequence

$$\varepsilon_*\varepsilon^!\,\mathcal{L}\mathrm{og}_{X'}(d) \to \mathcal{L}\mathrm{og}_{X'}(d) \to R\widetilde{j}_*\,\mathcal{L}\mathrm{og}_X(d)$$

and the purity isomorphism $\varepsilon^! \mathcal{L}\mathrm{og}_{X'}(d) \cong \varepsilon^* \mathcal{L}\mathrm{og}_{X'}[-2d]$. Thus the map N_f on Ext-groups can be identified with the following composition of maps on Hom-groups:

$$\mathrm{Hom}_S(\mathcal{I}_{X'}, \mathcal{I}_{X'}) \to \mathrm{Hom}_S(\mathcal{I}_{X'}, \ker(f_*\varepsilon^* \mathcal{L}\mathrm{og}_{X'} \to \Lambda)) \to \mathrm{Hom}_S(\mathcal{I}_{X'}, \mathcal{I}_X),$$

where the first map is induced by the natural inclusion of $\mathcal{I}_{X'}$ into the kernel of $f_*\varepsilon^* \mathcal{L}\mathrm{og}_{X'} \to \Lambda$ and the second is induced by the trace map. We have to show that the identity in $\mathrm{Hom}_S(\mathcal{I}_{X'}, \mathcal{I}_{X'})$ maps to the canonical map in $\mathrm{Hom}_S(\mathcal{I}_{X'}, \mathcal{I}_X)$ or that

$$e'^* \mathcal{L}\mathrm{og}_{X'} \to f_*\varepsilon^* \mathcal{L}\mathrm{og}_{X'} \to e^* \mathcal{L}\mathrm{og}_X$$

is the canonical map. As the first map is f_* applied to $e'_*e'^* \mathcal{L}\mathrm{og}_{X'} \to \varepsilon_*\varepsilon^* \mathcal{L}\mathrm{og}_{X'}$ this is clear. □

2.3 The splitting principle

Let $\phi : X \to X'$ be a finite Galois covering over S (we assume that ϕ is a pointed map) with covering group G, where X, X' are either curves or abelian schemes. Then we have a map

$$\phi_* : \mathcal{L}\mathrm{og}_{X,\Lambda} \to \phi^* \mathcal{L}\mathrm{og}_{X',\Lambda}$$

induced by the canonical map $\phi_* : \pi_1'(X_{\overline{s}}, \overline{x}) \to \pi_1'(X'_{\overline{s}}, \overline{x})$ of relative fundamental groups.

Lemma 2.3.1 (Splitting principle). *Assume in the above situation that the order of G is prime to l, then*

$$\phi_* : \mathcal{L}\mathrm{og}_{X,\Lambda} \to \phi^* \mathcal{L}\mathrm{og}_{X',\Lambda}$$

is an isomorphism. In particular, for any $x \in X$, which is in the G-orbit of e, one has a canonical isomorphism

$$\mathcal{R}_{X,\Lambda} \cong x^* \mathcal{L}\mathrm{og}_{X,\Lambda}.$$

Proof. Obvious, as in this case $\phi_* : \pi_1'(X_{\overline{s}}, \overline{x}) \to \pi_1'(X'_{\overline{s}}, \overline{x})$ is an isomorphism. □

This lemma is very useful in the case of the abelian polylog. One gets

$$x^* \mathcal{P}\mathrm{ol}^{\mathrm{ab}}_{X,\Lambda} \in \mathrm{Ext}^{2d-1}_U(\pi_U^*\mathcal{I}_{X,\Lambda}, \mathcal{R}^{\mathrm{ab}}_{X,\Lambda}(d))$$

and $\mathcal{R}^{\mathrm{ab}}_{X,\Lambda}$ is isomorphic to a power series ring over Λ.

3 The connection between the abelian polylogarithm on curves and on abelian schemes

In this section we show that the polylogarithm on an abelian scheme is induced from the abelian polylogarithm on a sub-curve.

3.1 The polylog on an abelian scheme as push-forward

We consider curves with the following:

Conditions 3.1.1. *a) $i : C \hookrightarrow A$ is a closed embedding and C is is a curve as in definition 1.1.1.*

b) The map $\varrho : R^1\pi_{C*}\Lambda \to R^1\pi_{A*}\Lambda$ induced by i is surjective

c) The section e of C is mapped under i to the unit section e of A

The structure maps of C and A will be denoted by π_C and π_A.
Remark: The typical case to consider here is $A = \mathrm{P}^0(C/S)$ and $i : C \to A$ the embedding $t \mapsto \mathcal{O}((e) - (t))$.

Lemma 3.1.2. *Let $i : C \hookrightarrow A$ satisfy the conditions 3.1.1, then the canonical map*

$$\varrho : \mathcal{L}\mathrm{og}_C \to i^* \mathcal{L}\mathrm{og}_A$$

is surjective (and factors through $\mathcal{L}\mathrm{og}_C^{\mathrm{ab}}$). In particular, the canonical maps $\varrho : \mathcal{R}_C \to \mathcal{R}_A$ and $\varrho : \mathcal{I}_C \to \mathcal{I}_A$ are surjective.

Proof. The condition 3.1.1 b) implies that the pull-back of the coverings $A_j \to A$ to C are quotients of the maximal pro-l-covering of C. Applying this to the definition of $\mathcal{L}\mathrm{og}_C$ and $\mathcal{L}\mathrm{og}_A$ gives the desired result. □

Consider the adjunction map

$$i_* i^! \mathcal{L}\mathrm{og}_A \to \mathcal{L}\mathrm{og}_A$$

and observe that by purity $i^! = i^*[-2d + 2](-d + 1)$ as C is smooth, so that the canonical map ϱ gives

$$\lambda : i_* \mathcal{L}\mathrm{og}_C \to \mathcal{L}\mathrm{og}_A[2d - 2](d - 1).$$

This gives us two diagrams

$$\begin{array}{ccc} \mathrm{Ext}^1_{C \smallsetminus e(S)}(\pi_C^* \mathcal{I}_C, \mathcal{L}\mathrm{og}_C(1)) & \xrightarrow{\alpha} & \mathrm{Ext}^{2d-1}_{A \smallsetminus e(S)}(\pi_A^* \mathcal{I}_C, \mathcal{L}\mathrm{og}_A(d)) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_S(\mathcal{I}_C, \mathcal{I}_C) & \xrightarrow{\varrho_*} & \mathrm{Hom}_S(\mathcal{I}_C, \mathcal{I}_A), \end{array}$$

where α is given by the adjunction map composed with λ and

$$\begin{array}{ccc} \operatorname{Ext}^{2d-1}_{A\smallsetminus e(S)}(\pi_A^*\mathcal{I}_A, \mathcal{L}\mathrm{og}_A(d)) & \xrightarrow{\beta} & \operatorname{Ext}^{2d-1}_{A\smallsetminus e(S)}(\pi_A^*\mathcal{I}_C, \mathcal{L}\mathrm{og}_A(d)) \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Hom}_S(\mathcal{I}_A, \mathcal{I}_A) & \xrightarrow{\varrho^*} & \operatorname{Hom}_S(\mathcal{I}_C, \mathcal{I}_A), \end{array}$$

where β is the pull-back by ϱ. Note that the isomorphisms for the right vertical arrows follow from a trivial extension of corollary 1.2.2.

Proposition 3.1.3. *With the above notation,*

$$\alpha(\mathcal{P}\mathrm{ol}_C) = \beta(\mathcal{P}\mathrm{ol}_A).$$

Proof. Results immediately from the above diagrams. □

3.2 The polylog on the Jacobian as cup-product

The push forward of the polylog on a curve to its Jacobian can also be written as a cup-product. This sheds some light on the nature of the extensions involved. Note that the polylog on curves is a one extension of lisse sheaves, hence itself represented by a lisse sheaf. It turns out, that the polylog on the Jacobian is the Yoneda product of this extension with the fundamental class of the curve.

Let $J = \mathrm{P}^0(C/S)$ be the Jacobian of C and $i : C \to J$ be an embedding satisfying the conditions in 3.1.1. Let g be the genus of C.

Consider in

$$\operatorname{Ext}^{2g-2}_{J\smallsetminus e(S)}(i_*i^*\Lambda, \Lambda(g-1)) \cong \operatorname{Hom}_{C\smallsetminus e(S)}(\Lambda, \Lambda)$$

the class $cl(C)$ corresponding to the identity. This is the fundamental class of $C \smallsetminus e(S)$ in $J \smallsetminus e(S)$. If we tensor this class with $\mathcal{L}\mathrm{og}_J(1)$ we get

$$cl(C) \otimes \mathcal{L}\mathrm{og}_J(1) \in \operatorname{Ext}^{2g-2}_{J\smallsetminus e(S)}(i_*i^*\,\mathcal{L}\mathrm{og}_J(1), \mathcal{L}\mathrm{og}_J(g)).$$

On the other hand we have

$$i_*\,\mathcal{P}\mathrm{ol}_C \in \operatorname{Ext}^1_{J\smallsetminus e(S)}(i_*\pi_C^*\mathcal{I}_C, i_*\,\mathcal{L}\mathrm{og}_C(1))$$

and with the canonical map $\varrho : i_*\,\mathcal{L}\mathrm{og}_C(1) \to i_*i^*\,\mathcal{L}\mathrm{og}_J(1)$ we get

$$\varrho_* i_*\,\mathcal{P}\mathrm{ol}_C \in \operatorname{Ext}^1_{J\smallsetminus e(S)}(i_*\pi_C^*\mathcal{I}_C, i_*i^*\,\mathcal{L}\mathrm{og}_J(1)).$$

Denote by $\gamma(\mathcal{P}\mathrm{ol}_C)$ the image of this class under the pull-back by the adjunction map $\pi_J^*\mathcal{I}_C \to i_*\pi_C^*\mathcal{I}_C$ in

$$\mathrm{Ext}^1_{J\smallsetminus e(S)}(\pi_J^*\mathcal{I}_C, i_*i^*\,\mathcal{L}\mathrm{og}_J(1)).$$

The Yoneda product gives a class

$$\gamma(\mathcal{P}\mathrm{ol}_C) \cup (cl(C) \otimes \mathcal{L}\mathrm{og}_J(1)) \in \mathrm{Ext}^{2g-1}_{J\smallsetminus e(S)}(\pi_J^*\mathcal{I}_C, \mathcal{L}\mathrm{og}_J(g)).$$

The canonical map $\varrho : \mathcal{I}_C \to \mathcal{I}_J$ induces a map $\pi_J^*\mathcal{I}_C \to \pi_J^*\mathcal{I}_J$, which gives

$$\mathrm{Ext}^{2g-1}_{J\smallsetminus e(S)}(\pi_J^*\mathcal{I}_J, \mathcal{L}\mathrm{og}_J(g)) \xrightarrow{\beta} \mathrm{Ext}^{2g-1}_{J\smallsetminus e(S)}(\pi_J^*\mathcal{I}_C, \mathcal{L}\mathrm{og}_J(g)).$$

Theorem 3.2.1. *In* $\mathrm{Ext}^{2g-1}_{J\smallsetminus e(S)}(\pi_J^*\mathcal{I}_C, \mathcal{L}\mathrm{og}_J(g))$ *holds the equality*

$$\beta(\mathcal{P}\mathrm{ol}_J) = \gamma(\mathcal{P}\mathrm{ol}_C) \cup (cl(C) \otimes \mathcal{L}\mathrm{og}_J(1)).$$

Proof. Let us write the extension classes as morphisms in the derived category. We have

$$\begin{aligned} \gamma(\mathcal{P}\mathrm{ol}_C) :& \pi_J^*\mathcal{I}_C \longrightarrow i_*i^*\,\mathcal{L}\mathrm{og}_J(1)[1] \\ \mathrm{cl}(C) \otimes \mathcal{L}\mathrm{og}_J(1)[1] :& i_*i^*\,\mathcal{L}\mathrm{og}_J(1)[1] \longrightarrow \mathcal{L}\mathrm{og}_J(g)[2g-1] \end{aligned}$$

so that

$$\gamma(\mathcal{P}\mathrm{ol}_C) \cup (cl(C) \otimes \mathcal{L}\mathrm{og}_J(1)[1]) : i_*\pi_C^*\mathcal{I}_C \longrightarrow \mathcal{L}\mathrm{og}_J(g)[2g-1]$$

is the composition. This morphism is the adjoint of

$$\pi_C^*\mathcal{I}_C \xrightarrow{\mathcal{P}\mathrm{ol}_C} \mathcal{L}\mathrm{og}_C(1)[1] \to i^*\,\mathcal{L}\mathrm{og}_J(1)[1]$$

because the adjoint of $cl(C) \otimes \mathcal{L}\mathrm{og}_J(1)[1]$ is the identity. We have a commutative diagram

$$\begin{array}{ccc} \pi_J^*\mathcal{I}_C & \longrightarrow & i_*i^*\,\mathcal{L}\mathrm{og}_J(1)[1] \\ \downarrow & & \downarrow \\ i_*\pi_C^*\mathcal{I}_C & \longrightarrow & \mathcal{L}\mathrm{og}_J(g)[2g-1] \end{array}$$

where the left and right vertical arrows are adjunction maps and the horizontal arrows are adjoint to $\mathcal{P}\mathrm{ol}_C \circ \varrho$. The composition of these maps is the morphism in the proposition and

$$\pi_J^*\mathcal{I}_C \to i_*i^*\,\mathcal{L}\mathrm{og}_J(1)[1] \to \mathcal{L}\mathrm{og}_J(2g-1)[2g-1]$$

is just $\alpha(\mathcal{P}\mathrm{ol}_C)$ with the notation of proposition 3.1.3. Our result follows from this proposition. □

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Guido Kings
NWF-I Mathematik
Universität Regensburg
93040 Regensburg
guido.kings@mathematik.uni-regensburg.de