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Abstract

The Cauchy problem is considered for the scalar wave equation in the Schwarzschild geometry. Using an integral spectral representation we derive the exact decay rate for solutions of the Cauchy problem with spherical symmetric initial data, which is smooth and compactly supported outside the event horizon.

1 Introduction

In this paper we study the decay rate for solutions of the scalar wave equation with spherical symmetric initial data in the Schwarzschild geometry, which is smooth and compactly supported outside the event horizon. We prove that these solutions decay at the rate t^{-3} and t^{-4} for momentarily stationary initial data, respectively, as it was earlier predicted by Price [16], though not rigorously proved. In [15] we have already shown pointwise decay for solutions of the same kind of initial data not necessarily spherical symmetric. To this end, we have derived an integral spectral representation for the solutions applying Hilbert space methods in terms of special solutions of the Schrödinger equation the so-called Jost solutions.

In order to set up some notation, recall that in Schwarzschild coordinates $(t, r, \vartheta, \varphi)$, the Schwarzschild metric takes the form

$$\begin{aligned} ds^2 &= g_{ij}\, dx^i dx^j \\ &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \end{aligned} \tag{1.1}$$

with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$. The metric has two singularities at $r = 0$ and $r = 2M$. The latter is called the *event horizon* and can be resolved by a coordinate transformation. We consider the scalar wave equation in the region $r > 2M$ outside the event horizon, which is given by

$$\Box\phi := g^{ij}\nabla_i\nabla_j\phi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\, g^{ij}\frac{\partial}{\partial x^j}\right)\phi = 0 \tag{1.2}$$

where g denotes the determinant of the metric g_{ij}. We now state our main result.

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Theorem 1. Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry

$$\Box\phi = 0 \;, \quad (\phi_0, i\partial_t\phi_0)(0, r, x) = \Phi_0(r, x)$$

for smooth spherical symmetric initial data $\Phi_0 \in C_0^\infty((2M, \infty) \times S^2)^2$ which is compactly supported outside the event horizon. Let $\Phi(t) = (\phi(t), i\partial_t\phi(t)) \in C^\infty(\mathbb{R} \times (2M, \infty) \times S^2)^2$ be the unique global solution which is compactly supported for all times t. Then for fixed r there is a constant $c = c(r, \Phi_0)$ such that for large t

$$|\phi(t)| \leq \frac{c}{t^3} \;.$$

Moreover, if we have initially momentarily static initial data, i.e. $\partial_t\phi_0 \equiv 0$, the solution $\phi(t)$ satisfies

$$|\phi(t)| \leq \frac{c}{t^4} \;.$$

There has been significant work in the study of linear hyperbolic equations in black hole spacetimes. The first major contribution in this topic was made in 1957, when Regge and Wheeler studied the linearized equations for perturbations of the Schwarzschild metric [17]. This work was continued in [22, 26], while more recently the decay of the perturbation and all of its derivatives was shown in [12] using a theorem by Wilcox. By heuristic arguments, in 1972 Price [16] got evidence for polynomial decay of solutions of the scalar wave equation in Schwarzschild, where the power depends explicitely on the angular mode. In 1973, Teukolsky [19] could derive by means of the Newman Penrose formalism one single master equation that describes in the Kerr background the evolution of a test scalar field ($s = 0$), a test neutrino field ($s = \pm 1/2$), a test electromagnetic field ($s = \pm 1$) and linearized gravitational waves ($s = \pm 2$). Here, the parameter s is also called the spin weight of the field. Note that it is a quite complicated task in the case $s \neq 0$ to recover all the components of the corresponding field from a solution of this equation. For further details see [3, 24]. In two subsequent papers [20, 21], Teukolsky and Press discussed the physical consequences of these perturbations. Note that the rigorous analysis of the equation remains a quite subtle point, though any linearized perturbation is given by this equation. For instance, in the case $s \neq 0$ *complex* coefficients are involved, which makes the analysis very complicated. Hence, until now there are just a few rigorous results in this case. In [7] local decay was proven for the Dirac equation ($s = \frac{1}{2}$) in the Kerr geometry (in the massless and massive case). Moreover, a precise decay rate has been specified in the massive case [8]. More recently, there has been a linear stability result for the Schwarzschild geometry under electromagnetic and gravitational perturbations [9]. This result relies on the mode analysis, which has been carried out in [23]. More work has been done on the case $s = 0$, where the Teukolsky equation reduces to the scalar wave equation. In the Schwarzschild case, Kay and Wald [14] proved a time independent L^∞-bound for solutions of the Klein-Gordon equation. In [5], a mathematical proof is given for the decay rate of solutions with spherical symmetric initial data, which has been predicted by Price [16], which is not sharp, however. For general initial data, the same authors derived another decay result [6]. Pointwise decay in the Kerr geometry was proven rigorously [10, 11]. Furthermore, Morawetz and Strichartz-type estimates for a massless scalar field without charge in a Reissner Nordstrøm background with naked singularity are developed in [18]. And in [2] a Morawetz-type inequality was proven for the semi-linear wave equation in Schwarzschild, which is also supposed to yield decay rates.

In this paper we first recapitulate the framework and some notations of the foregoing paper [15]. Afterwards, we give an explicit expansion of the Jost solutions $\grave{\phi}$ of the Schrödinger equation which also were derived in [15]. At the end we show how to derive the exact decay rate out of this expansion.

2 Preliminaries

We usually replace the Schwarzschild radius r by the Regge-Wheeler coordinate $u \in \mathbb{R}$ given by

$$u(r) := r + 2M \log \left(\frac{r}{2M} - 1 \right) . \tag{2.1}$$

After having separated the angular modes l, m using spherical harmonics, it is convenient to write the Cauchy problem in Hamiltonian formalism

$$i\partial_t \Psi = H\Psi , \quad \Psi\big|_{t=0} = \Psi_0 \tag{2.2}$$

where $\Psi = (\psi, i\partial_t\psi)^T$ is a two component vector representing the wave function and its first time derivative and H is the Hamiltonian

$$\begin{pmatrix} 0 & 1 \\ -\partial_u^2 + V_l(u) & 0 \end{pmatrix} , \tag{2.3}$$

with the potential

$$V_l(u) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) . \tag{2.4}$$

Constructing the resolvent of the operator H and using Stone's formula we have derived an integral spectral representation for the solutions of the Cauchy problem of the following form

$$\Psi(t,u) = e^{-itH}\Psi_0(u) =$$
$$-\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\left(\int_{\text{supp }\Psi_0} \operatorname{Im}\left(\frac{\acute{\phi}_{\omega l}(u)\grave{\phi}_{\omega l}(v)}{w(\acute{\phi}_{\omega l}, \grave{\phi}_{\omega l})}\right)\begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix}\Psi_0(v)dv\right) d\omega , \tag{2.5}$$

where the integrand is in L^1 with respect to ω.

At this point, the functions $\acute{\phi}, \grave{\phi}$ play an important role. These functions are a fundamental system of the Schrödinger equation

$$\left(-\partial_u^2 + V_\omega(u)\right)\phi(u) = 0 \tag{2.6}$$

with the potential

$$V_\omega(u) = -\omega^2 + V_l(u) = -\omega^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) \tag{2.7}$$

with boundary conditions

$$\lim_{u\to-\infty} e^{-i\omega u}\acute{\phi}_\omega(u) = 1 , \qquad \lim_{u\to-\infty}\left(e^{-i\omega u}\acute{\phi}_\omega(u)\right)' = 0 \tag{2.8}$$

$$\lim_{u\to+\infty} e^{i\omega u}\grave{\phi}_\omega(u) = 1\,, \qquad \lim_{u\to+\infty}\left(e^{i\omega u}\grave{\phi}_\omega(u)\right)' = 0\,. \tag{2.9}$$

in the case $\operatorname{Im}(\omega) < 0$. We derived these solutions using the corresponding integral equation, the so-called Jost equation, which is given by

$$\phi_\omega(u) = e^{i\omega u} + \int_{-\infty}^{u} \frac{1}{\omega}\sin(\omega(u-v))V_l(v)\phi_\omega(v)\,dv\,, \tag{2.10}$$

in the case of boundary conditions at $-\infty$ (an analog equation is considered for the boundary conditions at ∞). Now, we have constructed the solution $\acute{\phi}$ with the series ansatz

$$\acute{\phi}_\omega = \sum_{k=0}^{\infty}\phi_\omega^{(k)}\,, \tag{2.11}$$

together with the iteration scheme

$$\left.\begin{array}{rcl} \phi_\omega^{(0)}(u) & = & e^{i\omega u} \\ & \vdots & \\ \phi_\omega^{(k+1)}(u) & = & \displaystyle\int_{-\infty}^{u}\frac{1}{\omega}\sin(\omega(u-v))V_l(v)\phi_\omega^{(k)}(v)\,dv \end{array}\right\}\,. \tag{2.12}$$

Using this we have proven that the solutions $\acute{\phi}_\omega(u)$ are analytic with respect to ω for fixed u in the region $\operatorname{Im}(\omega) < 0$. Moreover $\acute{\phi}_\omega$ can be analytically extended to the region $\operatorname{Im}(\omega) \le \frac{1}{4M}$, whereas the solution $\omega^l\grave{\phi}_\omega$ can be only continuously extended to the real axis. Thus, in order to obtain the exact decay rates it is important to analyze the behavior of $\grave{\phi}_\omega$ with respect to ω at the real axis in more details.

3 Expansion of the Jost solutions $\grave{\phi}_\omega$

Since the ω-dependence of the Jost solutions $\grave{\phi}_\omega$ plays an essential role in the analysis of the integral representation, we show in this section a method to expand these solutions at the critical point $\omega = 0$. We start with an explicit calculation:

Lemma 3.1. *For all $u > 0$, $\omega \in \mathbb{R}\setminus\{0\}$, $\varepsilon > 0$, $q \in \mathbb{N}_0$ and $p \in \mathbb{N}$,*

$$\begin{aligned} \int_u^\infty e^{-2i\omega x-\varepsilon x}\frac{\log^q(x)}{x^p}\,dx = \sum_{m=0}^{q}\binom{q}{m}\log^{q-m}(u)\Bigg\{\,(2i\omega+\varepsilon)^{p-1} \\ \times\left[\frac{(-1)^{p-1}}{(p-1)!}\frac{(-1)^{m+1}}{m+1}\log^{m+1}[(2i\omega+\varepsilon)u] + \sum_{k=0}^{m}c_k(m)\log^k[(2i\omega+\varepsilon)u]\right] \\ -u^{-p+1}\sum_{k=0,k\neq p-1}^{\infty}\frac{(-1)^k(-1)^m\,m!}{(k-p+1)^{m+1}k!}[(2i\omega+\varepsilon)u]^k\Bigg\}\,, \end{aligned} \tag{3.1}$$

where the coefficients c_k involve the coefficients $a_0,\ldots,a_q$ of the series expansion of the Γ-function at $1-p$.

Proof. In order to prove this, we write the integral as λ-derivatives,

$$\int_u^\infty e^{-2i\omega x-\varepsilon x}\frac{\log^q(x)}{x^p}\,dx = \frac{d^q}{d\lambda^q}F_p(\lambda)\bigg|_{\lambda=0}\ , \tag{3.2}$$

with the generating functional,

$$F_p(\lambda) = \int_u^\infty e^{(-2i\omega-\varepsilon)x}\frac{1}{x^{p-\lambda}}\,dx = u^{-p+\lambda+1}\int_1^\infty e^{(-2i\omega-\varepsilon)uv}\frac{1}{v^{p-\lambda}}\,dv\ ,$$

where in the last step we introduced the new integration variable $v=\frac{x}{u}$. In the following we will write $z=(2i\omega+\varepsilon)u$ for reasons of convenience. The integral on the right hand side is also known as the Exponential Integral $E_{p-\lambda}(z)$ with the series expansion

$$E_{p-\lambda}(z) = \Gamma(1-p+\lambda)\,z^{p-\lambda-1} - \sum_{k=0}^\infty \frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\ ,$$

for small $\lambda\neq 0$ [as a reference cf. [25]]. Using the series expansion of the Γ-function at $1-p\in\mathbb{Z}\setminus\mathbb{N}$, where the Γ-function has a pole of first-order, we obtain

$$\begin{aligned}
F_p(\lambda) = u^{-p+\lambda+1}&\left[\left(\frac{(-1)^{p-1}}{(p-1)!\,\lambda}+\sum_{n=0}^\infty a_n\lambda^n\right)z^{p-\lambda-1}\right.\\
&\qquad\qquad\left.-\sum_{k=0}^\infty\frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\right]\\
= u^{-p+\lambda+1}&\left[z^{p-1}\left(\frac{(-1)^{p-1}}{(p-1)!}\left(\frac{z^{-\lambda}-1}{\lambda}\right)+z^{-\lambda}\sum_{n=0}^\infty a_n\lambda^n\right)\right.\\
&\qquad\left.-\sum_{k=0,k\neq p-1}^\infty\frac{(-1)^k}{(k-p+\lambda+1)k!}\,z^k\right]\ .
\end{aligned}\tag{3.3}$$

Using $z^{-\lambda}=e^{-\lambda\log z}$, we immediately get the formulas

$$\begin{aligned}
\frac{d^n}{d\lambda^n}\left(\frac{z^{-\lambda}-1}{\lambda}\right)\bigg|_{\lambda=0} &= \frac{(-1)^{n+1}\log^{n+1}(z)}{n+1}\\
\frac{d^m}{d\lambda^m}\left(u^\lambda\right)\bigg|_{\lambda=0} &= \log^m(u)\\
\frac{d^m}{d\lambda^m}\left(z^{-\lambda}\right)\bigg|_{\lambda=0} &= (-1)^m\log^m(z)
\end{aligned}$$

one directly verifies the claim setting (3.3) in (3.2). ∎

Directly in the same way, one proves an analogue lemma for the case $p\in\mathbb{Z}\setminus\mathbb{N}$:

Lemma 3.2. *For all $u>0$, $\omega\in\mathbb{R}\setminus\{0\}$, $\varepsilon>0$, $q\in\mathbb{N}_0$ and $p\in\mathbb{Z}\setminus\mathbb{N}$,*

$$\int_u^\infty e^{-2i\omega x-\varepsilon x}\frac{\log^q(x)}{x^p}\,dx =$$

$$\sum_{m=0}^{q} \binom{q}{m} \log^{q-m}(u) \Bigg\{ (2i\omega+\varepsilon)^{p-1} \sum_{k=0}^{m} c_k(m) \log^k \left[(2i\omega+\varepsilon)u\right]$$
$$-u^{-p+1} \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^m\, m!}{(k-p+1)^{m+1} k!} \left[(2i\omega+\varepsilon)u\right]^k \Bigg\} \quad (3.4)$$

where the coefficients c_k involve the coefficients $a_0, ..., a_q$ of the series expansion of the Γ-function at $1-p$.

Compared to Lemma 3.1, here the logarithmic term is of lower order due to the fact that the Gamma-function has no singularity for positive integers.

In order to apply this lemma to our integral representation, we have to derive an asymptotic expansion for the potential $V_l(u)$ at $+\infty$. Therefore, we have the following

Lemma 3.3. *For the potential $V_l(u) = \left(1 - \frac{2M}{r(u)}\right)\left(\frac{2M}{r(u)^3} + \frac{l(l+1)}{r(u)^2}\right)$ we have the asymptotic expansion*

$$V_l(u) = \sum_{p=2}^{k} \sum_{q=0}^{p-2} c_{pq} \frac{\log^q(u)}{u^p} + c_{k+1,k-1} \frac{\log^{k-1}(u)}{u^{k+1}} + \mathcal{O}\left(\frac{\log^{k-2}(u)}{u^{k+1}}\right) , \quad (3.5)$$

as $u \to \infty$, with $k \geq 2$ and real coefficients c_{pq}, where e.g. the first coefficients are given by

$$c_{20} = l(l+1) , \quad c_{31} = 4l(l+1)M ,$$
$$c_{30} = 2M - 2Ml(l+1)(1+2\log(2)) - 4Ml(l+1)\log(M)$$
$$c_{42} = 12l(l+1)M^2 ,$$
$$c_{41} = -4M^2(-3 + l(l+1)(5 + 8\log(8)) + 6l(l+1)\log(M)) , \ \ldots \ .$$

Furthermore, in the case $l = 0$ the coefficients $c_{n,n-2}$ vanish.

Proof. First we have to find an expression for r in terms of the Regge-Wheeler coordinate u. Remember that $u = r + 2M\log(\frac{r}{2M} - 1)$, which is equivalent to

$$e^{\frac{u}{2M}-1} = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}-1} .$$

In order to resolve this equation with respect to r, we use the principal branch of the Lambert W function denoted by $W(z)$. This is just the inverse function of $f(x) = xe^x$ on the positive real axis. [As a reference cf. [4].] Hence, we obtain

$$r = 2M + 2M\, W(e^{\frac{u}{2M}-1}) . \quad (3.6)$$

Moreover, for W we have the asymptotic expansion

$$W(z) = \log z - \log(\log z) + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} c_{km} \left(\log(\log z)\right)^{m+1} (\log z)^{-k-m-1} , \quad (3.7)$$

as $z \to \infty$. Here, the coefficients c_{km} are given by $c_{km} = \frac{1}{m!}(-1)^k \begin{bmatrix} k+m \\ k+1 \end{bmatrix}$, where $\begin{bmatrix} k+m \\ k+1 \end{bmatrix}$ is a Stirling cycle number. In particular, applying this expansion to (3.6), we get the series representation

$$
\begin{aligned}
r(u) = 2M + 2M\Big[&\frac{u}{2M} - 1 - \log\Big(\frac{u}{2M} - 1\Big) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km}\,\Big(\log\Big(\frac{u}{2M} - 1\Big)\Big)^{m+1}\Big(\frac{u}{2M} - 1\Big)^{-k-m-1}\Big]\,.
\end{aligned}
$$

This allows us to expand the powers $\frac{1}{r^2}$, $\frac{1}{r^3}$ and $\frac{1}{r^4}$ to any order in $u/2M - 1$ using the method of the geometric series. Together with the expansion

$$
\log\Big(\frac{u}{2M} - 1\Big) = \log\Big[\frac{u}{2M}\Big(1 - \frac{2M}{u}\Big)\Big] = \log u - \log(2M) - \sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{2M}{u}\Big)^n,
$$

which holds for $u > 2M$, the result follows. □

These two lemmas let us expand the solution $\grave{\phi}_\omega(u)$ in the following way.

Lemma 3.4. *For $l = 0$, $\omega \in \mathbb{R}\setminus\{0\}$ and fixed $u > 0$, the fundamental solution $\grave{\phi}_\omega(u)$ can be represented as*

$$
\grave{\phi}_\omega(u) = e^{-i\omega u} + g_0(\omega, u) + 2i\omega\log(2i\omega)g_1(\omega, u) + 2i\omega g_2(\omega, u)\,, \tag{3.8}
$$

where the functions g_0, g_1 and g_2 are $C^1(\mathbb{R})$ with respect to ω.

In order to prove this, we need the following lemma

Lemma 3.5. *For all $u \in \mathbb{C}$ and $n \in \mathbb{N}_0$,*

$$
\Big|\partial_u^n\Big(\frac{1}{u}\sin u\Big)\Big| \le \frac{2^{n+1}}{1 + |u|}e^{|\operatorname{Im} u|}\,. \tag{3.9}
$$

Moreover, if $\omega \neq 0$ and $v \ge u > 0$,

$$
\Big|\partial_\omega^n\big[\frac{1}{\omega}\sin(\omega(u - v))\big]\Big| \le \frac{C(n)\,v^{n+1}}{1 + |\omega v|}e^{v|\operatorname{Im}\omega| + u\operatorname{Im}\omega}\,, \tag{3.10}
$$

for some constant $C(n)$, which is just depending on n.

Proof. In the case $|u| \ge 1$, Eulers formula for the sin-function inductively yields

$$
(1 + |u|)\Big|\partial_u^n\Big(\frac{1}{u}\sin u\Big)\Big| \le 2^{n+1}e^{|\operatorname{Im} u|}\,.
$$

For $|u| < 1$ we rewrite $(1/u)\sin u$ as an integral, in order to obtain the estimate

$$
(1 + |u|)\Big|\partial_u^n\Big(\frac{1}{u}\sin u\Big)\Big| = (1 + |u|)\Big|\frac{1}{2}\int_{-1}^{1}(i\tau)^n e^{iu\tau}\,d\tau\Big| \le 2e^{|\operatorname{Im} u|}\,,
$$

which shows the first claim. As a consequence, we get for $\omega \neq 0$ and all $n \in \mathbb{N}$ the estimate

$$
\Big|\partial_\omega^n\Big(\frac{1}{\omega}\sin(\omega u)\Big)\Big| \quad = \quad \Big|u^{n+1}\partial_{\omega u}^n\Big(\frac{1}{\omega u}\sin(\omega u)\Big)\Big|
$$

$$\leq \quad \frac{2^{n+1}|u|^{n+1}}{1+|\omega u|} e^{|\,\mathrm{Im}\,(\omega u)|} \,. \tag{3.11}$$

In order to show (3.10) we use the identity

$$\frac{1}{\omega}\sin(\omega(u-v)) = \frac{1}{\omega}\left(\sin(\omega u)e^{i\omega v} - \sin(\omega v)e^{i\omega u}\right) \tag{3.12}$$

and apply (3.9), $n=0$,

$$\begin{aligned}
\left|\frac{1}{\omega}\sin(\omega(u-v))\right| \;&\leq\; \frac{1}{|\omega|}\left(\left|\sin(\omega u)e^{i\omega v}\right| + \left|\sin(\omega v)e^{i\omega u}\right|\right) \\
&\leq \frac{2|u|}{1+|\omega u|}\, e^{|u\,\mathrm{Im}\,\omega|}e^{-v\,\mathrm{Im}\,\omega} + \frac{2|v|}{1+|\omega v|}\, e^{|v\,\mathrm{Im}\,\omega|}e^{-u\,\mathrm{Im}\,\omega} \,.
\end{aligned} \tag{3.13}$$

Due to the assumption $v \geq u \geq 0$, we know that $|v| \geq |u|$ and thus

$$\frac{2|u|}{1+|\omega u|} \leq \frac{2|v|}{1+|\omega v|}\,, \qquad u|\,\mathrm{Im}\,\omega| + v\,\mathrm{Im}\,\omega \geq v|\,\mathrm{Im}\,\omega| + u\,\mathrm{Im}\,\omega \,.$$

Using these inequalities in (3.13) the claim follows for $n=0$.

Once again using the identity 3.12 we get

$$\begin{aligned}
\left|\partial_\omega\left(\frac{1}{\omega}\sin(\omega(u-v))\right)\right| &\leq \left|\frac{1}{\omega}\left(\sin(\omega u)(-iv)e^{-i\omega v} - \sin(\omega v)(-iu)e^{-i\omega u}\right)\right| \\
&\quad + \left|\partial_\omega\left(\frac{1}{\omega}\sin(\omega u)\right)e^{-i\omega v} - \partial_\omega\left(\frac{1}{\omega}\sin(\omega v)\right)e^{-i\omega u}\right| .
\end{aligned}$$

Using the estimates (3.11) and (3.9) for $n=0$ together with the assumption $v \geq u > 0$, we see as before that the first term is bounded by

$$\left|\frac{1}{\omega}\left(\sin(\omega u)(-iv)e^{-i\omega v} - \sin(\omega v)(-iu)e^{-i\omega u}\right)\right| \leq \frac{4v^2}{1+|\omega v|}e^{v|\,\mathrm{Im}\,\omega| + u\,\mathrm{Im}\,\omega} \,.$$

For the second term we use (3.11)

$$\begin{aligned}
&\left|\partial_\omega\left(\frac{1}{\omega}\sin(\omega u)\right)e^{-i\omega v} - \partial_\omega\left(\frac{1}{\omega}\sin(\omega v)\right)e^{-i\omega u}\right| \\
&\qquad \leq \frac{4u^2}{1+|\omega u|}e^{u|\,\mathrm{Im}\,\omega| + v\,\mathrm{Im}\,\omega} + \frac{4v^2}{1+|\omega v|}e^{v|\,\mathrm{Im}\,\omega| + u\,\mathrm{Im}\,\omega} \,,
\end{aligned}$$

and obtain due to the assumption $v \geq u > 0$

$$\leq \frac{8v^2}{1+|\omega v|}e^{v|\,\mathrm{Im}\,\omega| + u\,\mathrm{Im}\,\omega} \,.$$

Thus, we have shown (3.10) for $n=1$. We proceed inductively to conclude the proof. $\square$

Note that the estimate (3.10) remains valid in the limit $0 \neq \omega \to 0$ for all n, because

$$\lim_{\omega\to 0}\partial_\omega^n\left(\frac{1}{\omega}\sin(\omega(u-v))\right) = \begin{cases} (-1)^{n/2}\dfrac{1}{n+1}(u-v)^{n+1} & , \quad \text{if } n \text{ even,} \\ 0 & , \quad \text{if } n \text{ odd.} \end{cases}$$

Proof of Lemma 3.4: First, remember that the solution $\grave{\phi}_\omega(u)$ is given by the perturbation series

$$\grave{\phi}_\omega(u) = \sum_{k=0}^{\infty} \phi_\omega^{(k)}(u) \ ,$$

where the summands follow the iteration scheme

$$\phi_\omega^{(0)}(u) = e^{-i\omega u} \ , \ \phi_\omega^{(k+1)}(u) = -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \phi_\omega^{(k)}(v) \, dv \ , \quad (3.14)$$

with potential $V_0(u) = \left(1 - \frac{2M}{r(u)}\right) \frac{2M}{r(u)^3}$. According to Lemma 3.3, this potential can be represented for large u as $V_0(u) = \frac{c_{30}}{u^3} + h(u)$, with $h(u) = \mathcal{O}\left(\frac{\log u}{u^4}\right)$. Next, we split this iteration scheme up. To this end, we define

$$\tilde{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) h(v) e^{-i\omega v} \, dv \ , \quad (3.15)$$

and analogously,

$$\hat{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} e^{-i\omega v} \, dv \ . \quad (3.16)$$

Thus, obviously $\phi_\omega^{(1)}(u) = \hat{\phi}_\omega^{(1)}(u) + \tilde{\phi}_\omega^{(1)}(u)$. Now we iterate these two functions

$$\tilde{\phi}_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) V_0(v) \tilde{\phi}_\omega^{(k)}(v) \, dv \ , \ k \geq 1 \ ,$$

analogously for $\hat{\phi}_\omega^{(k+1)}(u)$. Hence, we have the formal decomposition

$$\grave{\phi}_\omega(u) = e^{-i\omega u} + \sum_{k=1}^{\infty} \hat{\phi}_\omega^{(k)}(u) + \sum_{k=1}^{\infty} \tilde{\phi}_\omega^{(k)}(u) \ . \quad (3.17)$$

Both series are well-defined. In order to show this, we use the bound

$$\left| \frac{1}{\omega} \sin(\omega(u-v)) \right| \ \leq \ \frac{4|v|}{1+|\omega v|} \ , \quad (3.18)$$

from Lemma 3.5 for real ω [Note that this estimate is also valid for the case $v \geq u > 0$]. Hence, we get inductively the estimates

$$\left| \hat{\phi}_\omega^{(k+1)}(u) \right| \leq \hat{R}_\omega(u) \frac{P_\omega(u)^k}{k!} \ ,$$
$$\left| \tilde{\phi}_\omega^{(k+1)}(u) \right| \leq \tilde{R}_\omega(u) \frac{P_\omega(u)^k}{k!} \ ,$$

for all $k \geq 0$, where the functions $\hat{R}, \tilde{R}$ and P are given by

$$\hat{R}_\omega(u) \ := \ \int_u^\infty \frac{4v}{1+|\omega|v} \left| \frac{c_{30}}{v^3} \right| dv \ ,$$
$$\tilde{R}_\omega(u) \ := \ \int_u^\infty \frac{4v}{1+|\omega|v} |h(v)| \, dv \ ,$$

$$P_\omega(u) \quad := \quad \int_u^\infty \frac{4v}{1+|\omega|v} |V_0(v)| \, dv \; .$$

Thus, the series $\sum \hat{\phi}_\omega^{(k)}(u)$ as well as $\sum \tilde{\phi}_\omega^{(k)}(u)$ converge locally uniformly with respect to u and ω. In the next step we show that, for fixed $u > 0$, $\sum \tilde{\phi}_\omega^{(k)}(u)$ is $C^1(\mathbb{R})$ with respect to ω. To this end, it suffices to prove that each summand $\tilde{\phi}_\omega^{(k)}$, $k \geq 1$, is C^1 and that the series $\sum \partial_\omega \tilde{\phi}_\omega^{(k)}$ converges locally uniformly in ω. Due to the estimates (3.10),(3.18), we have the inequality

$$\begin{aligned} &\left| \partial_\omega \left[\frac{1}{\omega} \sin(\omega(u-v)) h(v) e^{-i\omega v} \right] \right| \leq \\ &\qquad \leq \left| \frac{12\, v^2}{1+|\omega|v} h(v) \right| + \left| \frac{4v^2}{1+|\omega|v} h(v) \right| = \frac{16v^2}{1+|\omega|v} |h(v)| \; . \end{aligned} \tag{3.19}$$

Hence, the second term is an integrable bound, uniformly in ω, for the first derivative of the integrand. It follows that $\tilde{\phi}_\omega^{(1)}(u)$ is C^1 with respect to ω, bounded by

$$\left| \partial_\omega \tilde{\phi}_\omega^{(1)}(u) \right| \leq \int_u^\infty \frac{16v^2}{1+|\omega|v} |h(v)| \, dv =: \tilde{R}_\omega^{(1)}(u) \; .$$

Together with the estimate

$$\tilde{R}_\omega(u) \leq \frac{1}{4u} \int_u^\infty \frac{16v^2}{1+|\omega|v} |h(v)| \, dv \leq \frac{1}{u} \tilde{R}_\omega^{(1)}(u) \; ,$$

one shows inductively that $\tilde{\phi}_\omega^{(k+1)}(u)$ is C^1 with respect to ω, bounded by

$$\left| \partial_\omega \tilde{\phi}_\omega^{(k+1)}(u) \right| \leq \tilde{R}_\omega^{(1)}(u) \frac{(4P_\omega(u))^k}{k!} \; .$$

This yields that the sum $\sum \partial_\omega \tilde{\phi}_\omega^{(k)}$ converges locally uniformly in ω. Hence, the sum $\sum \tilde{\phi}_\omega^{(k)}(u)$ is $C^1(\mathbb{R})$ with respect to ω. According to the decomposition (3.17), it remains to analyze the $\omega-$dependence of $\sum \hat{\phi}_\omega^{(k)}(u)$. To this end, we compute the first summand:

$$\begin{aligned} \hat{\phi}_\omega^{(1)}(u) &= \frac{1}{2i\omega} \int_u^\infty \left(e^{-i\omega(u-v)} - e^{i\omega(u-v)} \right) e^{-i\omega v} \frac{c_{30}}{v^3} \, dv \\ &= \frac{1}{2i\omega} e^{-i\omega u} \int_u^\infty \frac{c_{30}}{v^3} \, dv - \frac{1}{2i\omega} e^{i\omega u} \int_u^\infty \frac{c_{30}}{v^3} e^{-2i\omega v} \, dv \; . \end{aligned}$$

Integrating the second term by parts, we obtain

$$\begin{aligned} &= \frac{1}{2i\omega} \left(e^{-i\omega u} \frac{c_{30}}{2u^2} - e^{i\omega u} \frac{c_{30}}{2u^2} e^{-2i\omega u} + e^{i\omega u} \int_u^\infty \frac{c_{30}}{-2v^2} (-2i\omega) e^{-2i\omega v} \, dv \right) \\ &= e^{i\omega u} \int_u^\infty \frac{c_{30}}{2v^2} \, e^{-2i\omega v} \, dv \; . \end{aligned}$$

The series expansion of Lemma 3.1 in the limit $\varepsilon \to 0$ yields

$$\hat{\phi}_\omega^{(1)}(u) = \frac{c_{30}}{2} e^{i\omega u} \Big\{ 2i\omega \big(\log(2i\omega u) + c_0 \big)$$

$$-u^{-1}\sum_{k=0,k\neq 1}^{\infty}\frac{(-1)^k}{(k-1)k!}\,(2i\omega u)^k\Bigg\}\,. \tag{3.20}$$

Intuitively, the only term which is not C^1 is the term involving $2i\omega\log(2i\omega u)$. More precisely, defining

$$\hat{\psi}^{(1)}_\omega(u) := \hat{\phi}^{(1)}_\omega(u) - c_{30}e^{i\omega u} i\omega \log(2i\omega u)\ , \tag{3.21}$$

and iterating this by

$$\hat{\psi}^{(k+1)}_\omega(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\hat{\psi}^{(k)}_\omega(v)\,dv\ ,\ k\geq 1,$$

we show next that the sum $\sum \hat{\psi}^{(k)}_\omega(u)$ is C^1 with respect to ω. By definition this holds for the initial function $\hat{\psi}^{(1)}_\omega(u)$. In order to prove this for the sum, we apply the same method as above. To this end, we need good estimates for the initial functions $\hat{\psi}^{(1)}_\omega(u)$ and $\partial_\omega\hat{\psi}^{(1)}_\omega(u)$. Estimating the integral representation of $\hat{\phi}^{(1)}_\omega(u)$, we obtain for arbitrary $u>0$ and $\omega\in\mathbb{R}$,

$$\left|\hat{\psi}^{(1)}_\omega(u) + c_{30}e^{i\omega u} i\omega\log(2i\omega u)\right| = \left|\hat{\phi}^{(1)}_\omega(u)\right| \leq \int_u^\infty \left|\frac{c_{30}}{2v^2}\right| dv = \frac{c_{30}}{2u}\ .$$

On the other hand, looking at the series in (3.20), we obtain for all $u\leq \frac{1}{|\omega|}$ the estimate

$$\left|\hat{\psi}^{(1)}_\omega(u)\right| = \left|\frac{c_{30}}{2}e^{i\omega u}\Bigg\{2i\omega c_0 - \frac{1}{u}\sum_{k=0,k\neq 1}^{\infty}\frac{(-1)^k}{(k-1)k!}\,(2i\omega u)^k\Bigg\}\right| \leq \frac{\tilde{c}}{u}\ , \tag{3.22}$$

with a suitable constant $\tilde{c}$. Thus, we get for all $u>0$ and $\omega\in\mathbb{R}$ the estimate

$$\left|\hat{\psi}^{(1)}_\omega(u)\right| \leq \frac{c}{u} + c|\omega||\log(2i\omega u)|\,1_{[\frac{1}{|\omega|},\infty)}(u)\ , \tag{3.23}$$

where c is chosen suitably and $1_.(.)$ denotes the characteristic function. In order to estimate the derivative $\partial_\omega\hat{\psi}^{(1)}_\omega(u)$, we use in the domain $u\geq\frac{1}{|\omega|}$, $|\omega|\neq 0$, the following bound for $\partial_\omega\hat{\phi}^{(1)}_\omega(u)$ [see also (3.19)],

$$\begin{aligned}\left|\partial_\omega\left(\hat{\phi}^{(1)}_\omega(u)\right)\right| &\leq \int_u^\infty \frac{16v^2}{1+|\omega|v}\left|\frac{c_{30}}{v^3}\right| dv \\ &\leq \frac{16}{|\omega|}\int_u^\infty \frac{|c_{30}|}{v^2}\,dv \leq \frac{16c_{30}}{|\omega|u} \leq 16\,c_{30}\ .\end{aligned}$$

Together with the analogon to estimate (3.22) in the region $u\leq\frac{1}{|\omega|}$, we obtain the bound

$$\left|\partial_\omega\hat{\psi}^{(1)}_\omega(u)\right| \leq \tilde{c} + \tilde{c}(1+u|\omega|)|\log(2i\omega u)|\,1_{[\frac{1}{|\omega|},\infty)}(u)\ , \tag{3.24}$$

where $u>0,\omega\in\mathbb{R}$ and $\tilde{c}$ is an appropriate constant. For reasons of simplicity, we choose $c=\tilde{c}$ such that both inequalities (3.23),(3.24) hold. Using these inequalities, we show by induction, in the same way as above, that $\hat{\psi}^{(k)}_\omega(u)$ is C^1 with respect to ω and obeys the estimates

$$\left|\hat{\psi}^{(k)}_\omega(u)\right| \quad\leq\quad \frac{c}{u}\frac{P_\omega(u)^{k-1}}{(k-1)!} + \frac{c}{u}\,r(|\omega|)\frac{P_\omega(u)^{k-2}}{(k-2)!}\ , \tag{3.25}$$

$$\left|\partial_\omega\big(\hat\psi^{(k)}_\omega(u)\big)\right| \;\;\leq\;\; c\frac{\big(4P_\omega(u)\big)^{k-1}}{(k-1)!} + 5c\, r(|\omega|)\frac{\big(4P_\omega(u)\big)^{k-2}}{(k-2)!}\;, \qquad (3.26)$$

for all $k \geq 2, u > 0$ and $\omega \in \mathbb{R}$, where r is given by

$$r(|\omega|) := \int_{\frac{1}{|\omega|}}^\infty \frac{4|\omega|v^2}{1+|\omega|v}\,|V_0(v)||\log(2i\omega v)|\,dv\;.$$

Due to (3.25),(3.26), the sums $\sum\hat\psi^{(k)}_\omega(u)$ and $\sum\partial_\omega\hat\psi^{(k)}_\omega(u)$ converge locally uniformly in ω. Hence, we conclude that $\sum\hat\psi^{(k)}_\omega(u)$ is well defined and continuously differentiable with respect to ω.

Thus, it remains to look at the term we get by the iteration of

$$\vartheta^{(1)}_\omega(u) := c_{30}e^{i\omega u}i\omega\log(2i\omega u) = ic_{30}\,\omega\log(2i\omega)e^{i\omega u} + ic_{30}\,\omega\log(u)e^{i\omega u}\;.$$

To this end, we split up the iteration, exactly as we did for the iteration of $\phi^{(k)}_\omega(u)$, i.e. we define

$$\begin{aligned}\tilde\vartheta^{(2)}_\omega(u) &:= -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))h(v)\vartheta^{(1)}_\omega(v)\,dv\;,\\ \hat\vartheta^{(2)}_\omega(u) &:= -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\vartheta^{(1)}_\omega(v)\,dv\;,\end{aligned}$$

and iterate these functions,

$$\tilde\vartheta^{(k+1)}_\omega(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\tilde\vartheta^{(k)}_\omega(v)\,dv\;,\;k\geq 2\;,$$

analogously for $\hat\vartheta^{(k+1)}_\omega(u)$. Next, in exactly the same way as for $\tilde\phi^{(k)}$, one sees that

$$\sum_{k=2}^\infty \tilde\vartheta^{(k)}_\omega(u) = 2i\omega\log(2i\omega)\,f_1(\omega,u) + 2i\omega f_2(\omega,u)\;,$$

where $f_1(.,u)$ and $f_2(.,u)$ are C^1 with respect to ω. Finally, by an exact calculation

$$\begin{aligned}\hat\vartheta^{(2)}_\omega(u) = ic_{30}^2\omega\log(2i\omega)e^{-i\omega u}\int_u^\infty e^{2i\omega v}\frac{1}{2v^2}\,dv\\ +ic_{30}^2\omega e^{-i\omega u}\int_u^\infty e^{2i\omega v}\left(\frac{1}{4v^2}+\frac{\log v}{2v^2}\right)\,dv\;,\end{aligned}$$

together with the series expansion of Lemma 3.1 in the limit $\varepsilon\to 0$ we obtain

$$\begin{aligned}\hat\vartheta^{(2)}_\omega(u) = \frac{1}{4}ic_{30}^2\,\omega(1+2\log(2i\omega))e^{-i\omega u}\\ \times\Bigg[(-2i\omega)\big(\log(-2i\omega u)+c_0\big) - u^{-1}\sum_{k=0,k\neq 1}^\infty \frac{(-1)^k}{(k-1)k!}\,(-2i\omega u)^k\Bigg]\\ +\frac{1}{2}ic_{30}^2\,\omega e^{-i\omega u}\Bigg[\sum_{m=0}^1\binom{1}{m}\log^{1-m}(u)\Big\{(-2i\omega)\times\end{aligned}$$

$$\left(\frac{(-1)^{m+2}}{m+1}\log^{m+1}(-2i\omega u)+\sum_{k=0}^{m}c_k\log^k(-2i\omega u)\right)$$
$$-u^{-1}\sum_{k=0,k\neq p-1}^{\infty}\frac{(-1)^k(-1)^m\, m!}{(k-1)^{m+1}k!}(-2i\omega u)^k\Bigg\}\Bigg]\ .$$

Proceeding in the same way as for $\sum \hat{\grave{\psi}}_\omega^{(k)}(u)$ [i.e. we omit the $\log\omega$-terms in the square brackets, and iterate these functions], we again get terms of the form

$$2i\omega\log(2i\omega)\, f_3(\omega,u)+2i\omega f_4(\omega,u)$$

with continuously differentiable functions $f_3(.,u), f_4(.,u)$. So after simplifications there remain terms of the form

$$(2i\omega)^2\log^s(2i\omega)\log^r(-2i\omega)\log^t(u)e^{-i\omega u}\ .$$

These are obviously C^1 with respect to ω and so is their iteration, due to the fact that the additional ω-order yields directly integrable bounds for all ω. This completes the proof. □

Note that one can apply this idea of the proof to the case $l\geq 1$. This yields a similar result but also requires much more complex calculations according to the construction of the Jost solutions $\grave{\phi}_\omega$ [cf. [15, Section 5]].

4 The decay rate for spherical symmetric initial data

According to the integral representation 2.5 the solution of the Cauchy problem for compactly supported smooth initial data $\Psi_0\in C_0^\infty(\mathbb{R})^2$ has the pointwise representation

$$\Psi(t,u)=e^{-itH}\Psi_0(u)=$$
$$-\frac{1}{\pi}\int_{\mathbb{R}}e^{-i\omega t}\left(\int_{\text{supp}\,\Psi_0}\mathrm{Im}\left(\frac{\acute{\phi}_{\omega l}(u)\grave{\phi}_{\omega l}(v)}{w(\acute{\phi}_{\omega l},\grave{\phi}_{\omega l})}\right)\begin{pmatrix}\omega & 1\\ \omega^2 & \omega\end{pmatrix}\Psi_0(v)dv\right)d\omega\ ,$$

Our goal is now to use the Fourier transform (2.5), in order to get detailed decay rates. To this end, we have to analyze the integral kernel, hence essentially

$$\mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega,\grave{\phi}_\omega)}\right)\ . \tag{4.1}$$

Since we already know that $\acute{\phi}_\omega$ is analytic on a neighborhood of the real line, it remains to understand $\grave{\phi}_\omega$ at the point $\omega=0$. To this end, we want to use an expansion as in Lemma 3.4. The problem is that this expansion is not sufficient for this purpose. Thus, we apply a similar method in order to gain

Lemma 4.1. *For $l=0$, $\omega\in\mathbb{R}\setminus\{0\}$, $n\geq 3$ and fixed $u>0$, we get for the fundamental solution $\grave{\phi}_\omega(u)$ the representation*

$$\grave{\phi}_\omega(u)=e^{-i\omega u}+g_0(\omega,u)+\sum_{i\geq j+k=1}^{n}(2i\omega)^i\log^j(2i\omega)\log^k(-2i\omega)g_{ijk}(\omega,u)\ , \tag{4.2}$$

where the functions $g_0, g_{ijk}\in C^n(\mathbb{R})$ with respect to ω.

In order to prove this, we need the following lemma.

Lemma 4.2. *Let $u > 0$, $n \in \mathbb{N}$ and $h \in C^\infty(\mathbb{R}_+)$ be a smooth function satisfying $\int_u^\infty v^{n+1}|h(v)|\, dv < \infty$.*
Then:

(i)

$$f_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))h(v)e^{-i\omega v}\, dv$$

is $C^n(\mathbb{R})$ with respect to ω.

(ii) For all $k \geq 1$

$$f_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)f_\omega^{(k)}(v)\, dv\ ,$$

are $C^n(\mathbb{R})$ with respect to ω and the series $\sum_{k\geq 1}\partial_\omega^m f_\omega^{(k)}(u)$, $m \leq n$, converge locally uniformly.

In particular, $\sum f_\omega^{(k)}(u)$ is $C^n(\mathbb{R})$ with respect to ω.

Proof. This is shown in exactly the same way as the statement that the functions $\tilde{\phi}_\omega^{(k)}$ in the proof of Lemma 3.4 as well as the series are C^1 with respect to ω. In order to show the differentiability up to the n-th order, we use the estimates of Lemma 3.5. ☐

Proof of Lemma 4.1. Because of complex calculations we show this at first in the case $n = 3$. To this end, we split up the iteration scheme (3.14) of the fundamental solutions in the following way. According to Lemma 3.3, we can write the potential V_0 as

$$V_0(v) = \sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p} + r_6(v)\ ,$$

where r_6 is a smooth function for $v \geq u$ behaving asymptotically at infinity as $\mathcal{O}\left(\frac{\log^3(v)}{v^6}\right)$. Thus, defining

$$\tilde{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))r_6(v)e^{-i\omega v}\, dv$$

and for all $k \geq 1$

$$\tilde{\phi}_\omega^{(k+1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\tilde{\phi}_\omega^{(k)}(v)\, dv\ ,$$

Lemma 4.2 yields that $\sum \tilde{\phi}_\omega^{(k)} \in C^3(\mathbb{R})$ with respect to ω and is a contribution to $g_0(\omega, u)$ in the statement of the lemma. Thus, we have to compute the remaining term

$$\hat{\phi}_\omega^{(1)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}e^{-i\omega v}\, dv.$$

We do this essentially in the same way as we computed the terms $\hat{\phi}^{(1)}, \hat{\vartheta}^{(2)}$ in the proof of Lemma 3.4. We split up the $\sin(\omega(u-v))$ with Euler's formula and integrate by parts and obtain

$$= -e^{i\omega u} \int_u^\infty \left(\frac{c_{30}}{-2v^2} + \sum_{p=3}^{4} \sum_{q=0}^{p-2} \tilde{c}_{pq} \frac{\log^q v}{v^p} \right) e^{-2i\omega v} \, dv \, , \tag{4.3}$$

where the coefficients $\tilde{c}_{pq}$ depend on the integral functions of the terms $\log^r v / v^s$. Now, we apply Lemma 3.1 in the limit $\varepsilon \searrow 0$ and get

$$\begin{aligned} = \quad & e^{i\omega u} \frac{c_{30}}{2} \left\{ 2i\omega \log(2i\omega u) - \frac{1}{u} \sum_{k=0}^{\infty} d_k (2i\omega u)^k \right\} \\ + \quad & e^{i\omega u} \frac{c_{41}}{3} \left\{ (2i\omega)^2 \left(\frac{1}{4} \log^2(2i\omega u) + \log(2i\omega u)(c - \frac{1}{2} \log u) \right) \right. \\ & \qquad \left. + \; (c + c\log u) \frac{1}{u^2} \sum_{k=0}^{\infty} d_k (2i\omega u)^k \right\} \\ + \quad & e^{i\omega u} \left\{ (2i\omega)^3 \sum_{s+t=1}^{3} c \log^s(2i\omega u) \log^t u \right. \\ & \qquad \left. + \sum_{m=0}^{2} c \log^m u \frac{1}{u^3} \sum_{k=0}^{\infty} d_k (2i\omega u)^k \right\} \end{aligned} \tag{4.4}$$

with appropriate constants c and d_k, which are of the form

$$d_k = \frac{(-1)^k (-1)^m \, m!}{(k-p+1)^{m+1} k!} \, . \qquad \text{[cf. Lemma 3.1]}$$

Since the series-terms are obviously $C^3(\mathbb{R})$ with respect to ω, this expression of $\hat{\phi}^{(1)}_\omega(u)$ fits into the desired expansion (4.2). In the next step we have to iterate (4.4). To this end, we treat each term in the curly brackets separately. We show this exemplarily for the first term which we denote by

$$\begin{aligned} \alpha^{(1)}(u) \quad := \quad & e^{i\omega u} \frac{c_{30}}{2} \left\{ 2i\omega \log(2i\omega u) - \frac{1}{u} \sum_{k=0}^{\infty} d_k (2i\omega u)^k \right\} \\ = \quad & e^{i\omega u} \int_u^\infty \frac{c_{30}}{2v^2} e^{-2i\omega v} \, dv \, . \end{aligned} \tag{4.5}$$

In order to derive sufficient bounds for all $u > 0$, we use different methods for the regions $|\omega| u \geq 1$ and $|\omega| u < 1$. First, let u be such that $|\omega| u \geq 1$, and by integrating by parts we get:

$$\begin{aligned} \alpha^{(1)}(u) \quad = \quad & e^{i\omega u} \int_u^\infty \frac{c_{30}}{2v^2} \frac{1}{(-2i\omega)^3} \partial_v^3 e^{-2i\omega v} \, dv \\ = \quad & \frac{c_{30}}{4i\omega} e^{-i\omega u} \frac{1}{u^2} - \frac{c_{30}}{(2i\omega)^2} e^{-i\omega u} \frac{1}{u^3} + \frac{3c_{30}}{(2i\omega)^3} e^{-i\omega u} \frac{1}{u^4} \\ & - e^{i\omega u} \int_u^\infty \frac{12}{v^5} \frac{1}{(2i\omega)^3} e^{-2i\omega v} \, dv \, . \end{aligned} \tag{4.6}$$

Using this expression and elementary integral estimates, we get for all $u > 0$ satisfying $|\omega| u \geq 1$ the bounds

$$\begin{aligned}
\left|\alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega| u^2}\,, \\
\left|\partial_\omega \alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega| u}\,, \\
\left|\partial_\omega^2 \alpha^{(1)}(u)\right| &\leq c\frac{1}{|\omega|}\,, \\
\left|\partial_\omega^3 \alpha^{(1)}(u)\right| &\leq c\frac{u}{|\omega|}\,,
\end{aligned} \tag{4.7}$$

with suitable constants c. Moreover, comparing the infinite sum of (4.5) with the exponential function, one directly sees that it is C^3 with respect to ω. It satisfies for all $u > 0$ with $|\omega| u < 1$ the bounds

$$\begin{aligned}
\left|\frac{1}{u}\sum_{k=0}^{\infty} d_k (2i\omega u)^k\right| &\leq \frac{c}{u} \\
\left|\partial_\omega \left(\frac{1}{u}\sum_{k=0}^{\infty} d_k (2i\omega u)^k\right)\right| &\leq c \\
\left|\partial_\omega^2 \left(\frac{1}{u}\sum_{k=0}^{\infty} d_k (2i\omega u)^k\right)\right| &\leq cu \\
\left|\partial_\omega^3 \left(\frac{1}{u}\sum_{k=0}^{\infty} d_k (2i\omega u)^k\right)\right| &\leq cu^2\,.
\end{aligned} \tag{4.8}$$

Using (4.7),(4.8), we verify that iterating the sum first with r_6 followed by the full iteration with the potential V_0, we obtain a C^3-function. For the remaining term $c_{30}/2e^{i\omega u} 2i\omega \log(2i\omega u)$ we use the identity $\log(2i\omega u) = \log(2i\omega) + \log u$ together with Lemma 4.2 to show that the first iteration with r_6 followed by the full iteration with the potential V_0 yields a term of the form $2i\omega \log(2i\omega) f_{110}(\omega, u) + \omega f_{100}(\omega, u)$, $f_{110}, f_{100} \in C^3(\mathbb{R})$ with respect to ω, fitting into the expansion (4.2). Thus, it remains to compute the integral

$$-\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \sum_{p=3}^{5} \sum_{q=0}^{p-3} c_{pq} \frac{\log^q(v)}{v^p} \alpha^{(1)}(v)\, dv\,.$$

We do this exemplarily for the term

$$\beta^{(2)}(u) := -\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} \alpha^{(1)}(v)\, dv\,. \tag{4.9}$$

A complex calculation, using Lemma 3.1 and Lemma 4.3, which will be stated and proven afterwards, yields

$$\begin{aligned}
\beta^{(2)}(u) = \quad & 2i\omega \log(2i\omega) e^{-i\omega u} c \Big\{ (-2i\omega) \log(-2i\omega u) - \frac{1}{u} \sum_{k=0}^{\infty} d_k (-2i\omega u)^k \Big\} \\
+ \; & 2i\omega e^{-i\omega u} \frac{c_{30}^2}{4} \Big\{ (-2i\omega) \big(- \frac{1}{2} \log^2(-2i\omega u)
\end{aligned}$$

$$
\left.+\log(-2i\omega u)(c+\log u)\big) - (c + c\log u)\frac{1}{u}\sum_{k=0}^{\infty} d_k(-2i\omega u)^k\right\}
$$

$$
+\ e^{-i\omega u}\left\{c(2i\omega)^2\log(-2i\omega u) + \frac{1}{u^2}\sum_{k=0}^{\infty} d_k(-2i\omega u)^k\right\}, \qquad (4.10)
$$

with suitable constants c, d_k. Hence $\beta^{(2)}(u)$ goes with (4.2). So far, we cannot finish this scheme, but if one has a close look, one sees that the most irregular term at $\omega = 0$, namely $2i\omega\log(2i\omega)$, now appears with a $1/u$ decay, while the other irregularities appear with an additional ω-power. Furthermore, due to the bounds (4.7) together with direct integral estimates, we obtain for all u with $|\omega|u \geq 1$ the bounds

$$
\begin{aligned}
\left|\beta^{(2)}(u)\right| &\leq c\int_u^\infty \frac{v}{1+|\omega|v}\frac{1}{v^3}\frac{1}{v^2|\omega|}\,dv \leq c\frac{1}{u^4|\omega|} \\
\left|\partial_\omega\beta^{(2)}(u)\right| &\leq c\frac{1}{u^3|\omega|} \\
\left|\partial_\omega^2\beta^{(2)}(u)\right| &\leq c\frac{1}{u^2|\omega|} \\
\left|\partial_\omega^3\beta^{(2)}(u)\right| &\leq c\frac{1}{u|\omega|}\ .
\end{aligned} \qquad (4.11)
$$

Using in the region $|\omega|u < 1$ for the sum-terms in (4.10) estimates analog to (4.8), we conclude in the same way as before, that iterating these first with r_6 followed by the full iteration with the potential V_0 and summing up, we obtain C^3-terms. We split up the remaining log-terms by $\log(-2i\omega u) = \log(-2i\omega) + \log(u)$ and use Lemma 4.2 to show that applying the same procedure yields terms that go with (4.2). Hence, we have to analyze the integral

$$
-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}\beta^{(2)}(v)\,dv\ ,
$$

exemplarily we treat the term

$$
\gamma^{(3)}(u) := -\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\beta^{(2)}(v)\,dv\ . \qquad (4.12)
$$

Computing this expression with the same methods one sees that the term with $2i\omega\log(2i\omega)$ decays as $1/u^2$ and the $\omega^2\log^s(\pm 2i\omega)$-terms decay as $\log^t(u)/u$. With bounds analog to (4.11),(4.8) the same procedure applies and yields terms that match with (4.2). Once again it remains to analyze

$$
-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\sum_{p=3}^{5}\sum_{q=0}^{p-3} c_{pq}\frac{\log^q(v)}{v^p}\gamma^{(3)}(v)\,dv\ ,
$$

and exemplarily

$$
-\int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))\frac{c_{30}}{v^3}\gamma^{(3)}(v)\,dv\ .
$$

Calculating this, one checks that the $2i\omega\log(2i\omega)$-term decays as $1/u^3$, the $\omega^2\log^s(\pm 2i\omega)$-terms decay as $\log^t(u)/u^2$ and the $\omega^3\log^m(\pm 2i\omega)$-terms decay as

$\log^n(u)/u$. Applying this scheme two times more, all terms which are not C^3 with respect to ω decay at least as $\log^s(u)/u^3$. Subtracting these terms from the full term, we obtain a C^3-term which is decaying at least as $\log^s(u)/u^3$, according to estimates analog to (4.11),(4.8) and estimating $|\omega|$ by $1/u$ in the region $|\omega|u < 1$. So Lemma 4.2 applies for the full iteration with the potential V_0 and we get a C^3-term. Due to their decay, we are able to iterate the subtracted $\log\omega$-terms also with the full potential V_0 and get terms that match with (4.2). Thus, the scheme can be stopped after finitely many calculations and the lemma is proven for $n = 3$. For $n \geq 4$ we split the potential in the way

$$V_0(v) = \sum_{p=3}^{n+2} \sum_{q=0}^{p-3} c_{pq} \frac{\log^q(v)}{v^p} + r_{n+3}(v) \ ,$$

and proceed with the same calculations. In (4.6) we have to integrate by parts up to the n-th order, in order to obtain as analogon to estimate (4.7)

$$\left|\partial_\omega^m \alpha^{(1)}(u)\right| \leq \frac{c}{|\omega|} u^{2-l} \ , \quad m \leq n \ .$$

The next difference appears in the estimates (4.11). These cannot be done for $n \geq 4$ by simple integral estimates as a matter of convergence. Thus, we have to subtract from the result of the analog calculation to (4.6) for $\alpha^{(1)}(u)$ the first $n-3$ exact terms of the form

$$\frac{c}{\omega u^2} e^{-i\omega u} + \ldots + \frac{c}{\omega^{n-3} u^{n-2}} e^{-i\omega u} =: \rho^{(1)}(u) \ ,$$

and get for $m \leq n$

$$\begin{aligned} \left|\partial_\omega^m \beta^{(2)}(u)\right| \quad \leq & \quad \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} \left(\alpha^{(1)}(u) - \rho^{(1)}(u)\right)\right| \\ & + \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{c_{30}}{v^3} \rho^{(1)}(u)\right| \\ \leq & \quad \frac{c}{|\omega|} u^{m-4} \ , \end{aligned}$$

where for the first integral this can be done by elementary integral estimates, and for the second integral we have to integrate the subtracted terms by parts, as we did to obtain the estimates for $\alpha^{(1)}(u)$. Keeping these differences in mind, we can conclude exactly in the same way as for $n = 3$, which yields the claim for arbitrary n.

□

We now state the missing lemma.

Lemma 4.3. *Let $u > 0$ and $\omega \in \mathbb{R} \setminus \{0\}$. For the calculation of the iteration of the infinite sums that appear in the integration in Lemma 3.1 with an arbitrary part of the potential, $\log^q u/u^p$, cf. Lemma 3.3, we obtain the identity*

$$-\int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{\log^q v}{v^p} \frac{\log^s v}{v^t} e^{\pm i\omega v} \sum_{k=0}^\infty d_k (\pm 2i\omega v)^k \, dv$$

$$= \frac{1}{u^{p+t-2}} e^{\mp i\omega u} \sum_{l=0}^{s+q} c \log^l(u) \sum_{k=0}^{\infty} d_{kl} (\mp 2i\omega u)^k \qquad (4.13)$$

$$+(2i\omega)^{p+t-2} e^{\mp i\omega u} \sum_{m=0}^{s+q} \sum_{r=1}^{m+1} c \log^r(\mp 2i\omega u) \log^{s+q-m}(u) \, , \qquad (4.14)$$

for suitable constants d_{kl}, c.

Proof. Let us denote $m = q + s \geq 0$ and $n = p + t \geq 4$. In order to compute the integral on the left hand side in the lemma, we insert a convergence generating factor

$$- \int_u^\infty \frac{1}{\omega} \sin(\omega(u-v)) \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^{\infty} d_k (\pm 2i\omega v)^k \, dv$$

$$= \lim_{\varepsilon \searrow 0} \int_u^\infty e^{-\varepsilon v} \frac{1}{\omega} \sin(\omega(v-u)) \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^{\infty} d_k (\pm 2i\omega v)^k \, dv. \qquad (4.15)$$

In the next step we interchange the integral and the infinite sum. This can be done for any $\varepsilon > 2|\omega|$ by a dominating convergence argument, if one estimates the modulus of the sum very roughly by $\exp(2|\omega|v)$. Thus, the two expressions

$$\int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^n} e^{\pm i\omega v} \sum_{k=0}^{\infty} d_k (\pm 2i\omega v)^k \, dv$$

and

$$\sum_{k=0}^{\infty} d_k (\pm 2i\omega)^k \int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^n} e^{\pm i\omega v} v^k \, dv$$

coincide for any $\varepsilon > 2|\omega|$. Moreover, both expressions are analytic in ε for $\mathrm{Re}\, \varepsilon > 0$. So by the identity theorem for analytic functions both expressions coincide for any $\varepsilon > 0$. So (4.15) is equal to

$$\lim_{\varepsilon \searrow 0} \sum_{k=0}^{\infty} d_k (\pm 2i\omega)^k \int_u^\infty \frac{1}{\omega} \sin(\omega(v-u)) e^{-\varepsilon v} \frac{\log^m v}{v^{n-k}} e^{\pm i\omega v} \, dv \, .$$

Once again we rewrite $\sin(\omega(v-u))$ with Eulers formula and integrate by parts [note that one has to be careful with the ε-terms that are generated by this integration by parts, but in the limit $\varepsilon \searrow 0$ they vanish] to obtain

$$= \lim_{\varepsilon \searrow 0} \sum_{k=0}^{\infty} d_k (\pm 2i\omega)^k e^{\mp i\omega u} \int_u^\infty e^{\pm 2i\omega v - \varepsilon v} \sum_{l=0}^{m} c \frac{\log^l v}{v^{n-k-1}} \, dv \, ,$$

with suitable constants c arising from the integral function of $\log^m v / v^{n-k}$. Now we apply Lemma 3.1, Lemma 3.2, take the limit $\varepsilon \searrow 0$ and get

$$= \sum_{l=0}^{m} \sum_{k=0}^{\infty} d_k (\pm 2i\omega)^k e^{\mp i\omega u} \sum_{i=0}^{l} c \binom{l}{i} \log^{l-i}(u) \times$$
$$\left\{ (\mp 2i\omega)^{n-k-2} \sum_{j=0}^{i+1} c \log^j [\mp 2i\omega u] - \frac{1}{u^{n-k-2}} \sum_{r=0, r \neq n-k-2}^{\infty} d_r (\mp 2i\omega u)^r \right\}.$$

We reorder the two infinite sums to one infinite sum, which can be done because of the structure of the coefficients d_k, d_r of the exponential integral that lets us compare the new coefficients to the coefficients of the exponential series, and get the expression (4.14). $\square$

Next, we need a similar expansion for the derivative $\grave{\phi}'_\omega(u)$.

Lemma 4.4. *For $l = 0$, $\omega \in \mathbb{R}\backslash\{0\}$, $n \geq 3$ and fixed $u > 0$, the first u-derivative of $\grave{\phi}_\omega(u)$ satisfies the expansion*

$$\grave{\phi}'_\omega(u) = -i\omega e^{-i\omega u} + h_0(\omega, u) + \sum_{i \geq j+k=1}^{n} (2i\omega)^i \log^j(2i\omega) \log^k(-2i\omega) h_{ijk}(\omega, u) \;, \tag{4.16}$$

where the functions $h_0, h_{ijk} \in C^n(\mathbb{R})$ with respect to ω.

Proof. In order to prove this, we use the fact that $\grave{\phi}'_\omega(u)$ satisfies for $u > 0$ an integral equation analog to (2.10)

$$\grave{\phi}'_\omega(u) = -i\omega e^{-i\omega u} - \int_u^\infty \cos(\omega(u-v)) V_0(v) \grave{\phi}_\omega(v)\, dv \;.$$

We estimate $\cos(\omega(u-v))$ and its $\omega-$derivatives for real ω and $v \geq u > 0$ by

$$\left|\partial_\omega^n \cos(\omega(u-v))\right| \leq (v-u)^n \leq (2v)^n \;, \quad n \in \mathbb{N}_0 \;. \tag{4.17}$$

Thus, using this estimate for $n = 0$ together with the iteration scheme (3.14) for $\grave{\phi}_\omega(u)$, we obtain a well defined iteration scheme for the u-derivative:

$$\begin{aligned} \grave{\phi}'_\omega(u) &= \sum_{k=0}^\infty \psi_\omega^{(k)}(u) \;, \quad \text{where} \\ \psi_\omega^{(0)}(u) &= -i\omega e^{-i\omega u} = \left(\phi_\omega^{(0)}\right)'(u) \;, \\ \psi_\omega^{(k+1)}(u) &= -\int_u^\infty \cos(\omega(u-v)) V_0(v) \phi_\omega^{(k)}(v)\, dv = \left(\phi_\omega^{(k+1)}\right)'(u) \;, \end{aligned} \tag{4.18}$$

with $k \geq 0$. Due to this iteration scheme together with the estimates (4.17) that replace the bounds (3.10) and the identity $\cos(\omega(u-v)) = 1/2(e^{i\omega(u-v)} + e^{i\omega(v-u)})$, we can use the decompositions of the $\phi_\omega^{(k)}$, which we have made in the proof of Lemma 4.1. In particular, we apply the procedure of this proof, in order to show the claim. $\square$

Now, we use the expansions (4.2),(4.16), in order to analyze the ω-dependence of the essential part of the integral kernel

$$\mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\right) \;.$$

At this stage, it is enough to set $n = 4$ in (4.2),(4.16) for our purposes. Looking at the integral representation (2.5) of the solution, we see that $u \in \mathbb{R}$ is fixed while $v \in \mathbb{R}$ varies in a compact set, the support of our initial data Ψ_0. Due to the Picard-Lindelöf theorem and the analytical dependence in ω of the Schrödinger equation from the coefficients, the expansions (4.2),(4.16) extend to any u, and v, respectively, on compact sets. Moreover, the following properties follow directly by the construction of the expansions.

Corollary 4.5. *For $4 \geq i = j + k \geq 1$ the function g_{ijk}, h_{ijk} can be constructed such that they obey the equalities*

$$\begin{array}{rcll} g_{ijk}(\omega,u) + o(\omega^\kappa) &=& c_{ijk}\left(e^{-i\omega u} + g_0(\omega,u)\right) & \text{and} \\ h_{ijk}(\omega,u) + o(\omega^\kappa) &=& c_{ijk}\, h_0(\omega,u)\,, & \textit{for } i \text{ even} \\ g_{ijk}(\omega,u) + o(\omega^\kappa) &=& c_{ijk}\overline{(e^{-i\omega u} + g_0(\omega,u))} & \text{and} \\ h_{ijk}(\omega,u) + o(\omega^\kappa) &=& c_{ijk}\,\overline{h_0(\omega,u)}\,, & \textit{for } i \text{ odd.} \end{array} \tag{4.19}$$

where κ is an arbitrary integer and the c_{ijk} are real constants, in particular not depending on u.

Proof. We show this exemplarily for the first terms g_{110}, h_{110}. In this situation, (4.19) holds because the first term, where $(2i\omega)\log(2i\omega)$ appears, appears with $c_{30}/2\, e^{i\omega u}$ and there are no other terms with this ω-dependence except the terms that are generated by this [cf. the calculations (4.4),(4.10)]. Thus, $g_{110}(\omega,u)$ is generated by $e^{i\omega u}$, which is just the complex conjugate of $e^{-i\omega u}$, and this behavior is kept by the iteration scheme. So any C^4-term that is generated is the complex conjugate of a corresponding term of g_0. This is valid, until one finishes the iteration scheme with the arguments at the end of the proof of Lemma 4.1, by what the $o(\omega^\kappa)$-term arises. Since one can do arbitrary many calculations and in each iteration at least a $\pm 2i\omega\log(\pm 2i\omega)$ is generated, the κ can be chosen arbitrary. Moreover, looking at the iteration scheme (4.18), the equalities for $h_{110}(\omega,u)$ are a consequence of the arguments for $g_{110}(\omega,u)$, because of the fact that by the calculations concerning this scheme no additional highest order log-terms, i.e. $i = j + k$, are generated. □

In the following assume that $\kappa = 5$. We expand the functions $g_{ijk}(\omega,u)$ and $h_{ijk}(\omega,u)$ in their Taylor polynom with respect to ω at $\omega = 0$ up to the fourth order:

$$g_{ijk}(\omega,u) = \sum_{m=0}^{4} \frac{1}{m!}\partial_\omega^m g_{ijk}\big|_{(0,u)}\omega^m + r_{ijk}(\omega,u)\,,$$

$$h_{ijk}(\omega,u) = \sum_{m=0}^{4} \frac{1}{m!}\partial_\omega^m h_{ijk}\big|_{(0,u)}\omega^m + q_{ijk}(\omega,u)\,,$$

where the remaining terms $r_{ijk}(\omega,u), q_{ijk}(\omega,u) \in C^4(\mathbb{R})$ behave for small ω as $o(|\omega|^4)$. Note that, due to this fact, any logarithmic irregularity multiplied with r_{ijk}, q_{ijk} yields a C^4-term with respect to ω. Moreover, we expand for fixed u the fundamental solution $\acute{\phi}_\omega(u)$ and its $u-$derivative $\acute{\phi}'_\omega(u)$

$$\acute{\phi}_\omega(u) = \sum_{k=0}^{\infty} c_k(u)\omega^k\,, \quad \acute{\phi}'_\omega(u) = \sum_{k=0}^{\infty} d_k(u)\omega^k\,,$$

which exist, because these are analytic in ω for fixed u. Since the fundamental solutions $\acute{\phi}, \grave{\phi}$ are real for $\omega = 0$, the coefficients $g_0(0,u), h_0(0,u), c_0(u)$ and $d_0(u)$ are real for all $u \in \mathbb{R}$. Using all these properties, we expand

$$\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi},\grave{\phi})}\,, \tag{4.20}$$

with the ansatz of a geometrical series with respect to ω. Note that, according to a result in [15, Section 6] the Wronskian does not vanish for $\omega = 0$. By a straightforward calculation it is shown that, essentially using (4.19), the terms with the highest logarithmic order, i.e. $(2i\omega)^i \log(2i\omega)^j \log(-2i\omega)^k, i = j + k$, vanish. Thus, we have to pick out the terms $(2i\omega)^2 \log(2i\omega)^j \log(-2i\omega)^k$ with $j+k = 1$, in order to get the lowest regularity. Looking at the calculations (4.4) and (4.10) [Note that these are the only possible terms, where a term with this irregularity appears the first time, according to our construction. The others are just a consequence out of these and hence a contribution to functions g_{2jk}], the desired terms appear in $\grave{\phi}$ the first time as

$$e^{-i\omega u} \left((2i\omega)^2 \log(2i\omega)(c + c\log u) - c(2i\omega)^2 \log(-2i\omega)\right) \ , \tag{4.21}$$

where a $(2i\omega)^2 \log(2i\omega) \log u$ shows up in the first line of (4.10), if one separates $\log(-2i\omega u) = \log(-2i\omega) + \log u$. All other such terms appearing in the second line of (4.10) as well as in the second line of (4.4) vanish because of their coefficients. Applying the same arguments as before, it follows that $g_{201}(\omega, u) + o(\omega^\kappa) = c(e^{-i\omega u} + g_0(\omega, u))$ and $h_{201}(\omega, u) + o(\omega^\kappa) = ch_0(\omega, u)$. Hence, the terms with $(2i\omega)^2 \log(-2i\omega)$ cancel in the ω-expansion of (4.20). Because of the additional $\log u$-term, we get

$$g_{210}(\omega, u) + o(\omega^\kappa) = c_1(e^{-i\omega u} + g_0(\omega, u)) + c_2(e^{-i\omega u} \log u + g(\omega, u)) \ ,$$

and

$$h_{210}(\omega, u) + o(\omega^\kappa) = c_1 h_0(\omega, u) + c_2 h(\omega, u) + \frac{1}{4} c_{30} e^{i\omega u} \ ,$$

with appropriate real constants c_1, c_2, where the last term appears by a direct calculation of $\psi^{(1)}(u)$ with the part c_{30}/v^3 of the potential $V_0(v)$. Furthermore, $g(\omega, u), h(\omega, u)$ are C^4-functions with respect to ω, where $g(\omega, u)$ is generated by the iteration of $e^{-i\omega u} \log u$ and $h(\omega, u)$ the consequence out of this in (4.18). One directly verifies that $g(0, u), h(0, u)$ are real, in general non-vanishing. Putting all these informations together, one sees that there appears a term with $(2i\omega)^2 \log(2i\omega)$ in the ω-expansion of (4.20), which is generated on the one hand by the $g(0, u), h(0, u)$, and on the other hand by the $2i\omega \log(2i\omega)$-part multiplied with the ω-contribution of first order of $\acute{\phi}, \acute{\phi}'$. This represents the part with the highest irregularity with respect to ω. Moreover, the related coefficients are purely *real*, depending on u, v and in general non-vanishing. Using the identity

$$\log(2i\omega) = i\frac{\pi}{2} \operatorname{sign}(\omega) + \log(2|\omega|) \ ,$$

and taking the imaginary part of (4.20), which is just the essential part of our integral kernel, we obtain as the lowest regular ω-term in the expansion of (4.1) at $\omega = 0$

$$c_0(u) g_{20}(v)\, \omega^2 \operatorname{sign}(\omega) \ , \tag{4.22}$$

where the function $g_{20}(v)$ arises out of the foregoing calculation. The symmetry of (4.1) with respect to u, v yields immediately $g_{20}(v) = kc_0(v)$ with an appropriate constant $k \neq 0$.

In the next step we want to use (4.22), in order to derive the decay of the solution $\Psi(t, u)$ given by (2.5). To this end, first we have to analyze the behavior of the ω-derivatives of the integrand up to the fourth order for large $|\omega|$.

Lemma 4.6. *For $u \in \mathbb{R}$ and compactly supported smooth initial data $\Psi_0 \in C_0^\infty(\mathbb{R})^2$ of the Cauchy problem, the ω-derivatives of the integrand in the integral representation (2.5)*

$$\partial_\omega^m \left(\int_{\mathrm{supp}\ \Psi_0} \mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)} \right) \begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix} \Psi_0(v)dv \right) , \quad m \in \{0, ..., 4\} , \tag{4.23}$$

have arbitrary polynomial decay in ω for $|\omega| \to \infty$.

Proof. We proceed essentially as in the proof of [15, Theorem 6.5]. To this end, we have to investigate the behavior of $\acute{\phi}_\omega(u), \grave{\phi}_\omega(v)$ in ω for $u \in \mathbb{R}$ fixed and v in the compact set suppΨ_0. We start with $\grave{\phi}_\omega$. We assume that $|\omega| \geq 1$ and $u_0 \in \mathbb{R}$ is arbitrary. Obviously, we find for any $v \geq u \geq u_0$ and $m \in \{0, ..., 4\}$ a constant $C_1(u_0)$ such that

$$\left|\partial_\omega^m \left[\frac{1}{\omega}\sin(\omega(u-v))\right]\right| \leq \frac{1}{|\omega|} C_1(u_0)(1+|v|)^m . \tag{4.24}$$

Furthermore, splitting the potential as

$$V_0(u) = \sum_{p=3}^{5} \sum_{q=0}^{p-3} c_{pq} \frac{\log^q(v)}{v^p} + r_6(v)$$

and following an analog calculation as in (4.6), we obtain for the ω-derivatives of the first iteration $\phi_\omega^{(1)}(u)$ for all $u \geq 1$ and $m \in \{0, ..., 4\}$ the estimate

$$\left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| \leq \frac{1}{|\omega|} C_2 u^{m-2} , \tag{4.25}$$

with an appropriate constant C_2. [Note that this is just an analogue to the estimate (4.7).] For all $u < 1$ and $m \in \{0, ..., 4\}$ we get

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| &\leq \left|\partial_\omega^m \int_u^1 \frac{1}{\omega}\sin(\omega(u-v))V_0(v)e^{-i\omega v}\,dv\right| \\ &\quad + \left|\partial_\omega^m \int_1^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)e^{-i\omega v}\,dv\right| \\ &\leq \frac{1}{|\omega|} f(m,u) + C_3 \frac{1}{|\omega|} \sum_{k=0}^m |u|^k , \end{aligned}$$

where f is a continuous function with respect to u and the second term arises by the same method as we used for the estimate (4.25). Defining C_4 by

$$C_4 := \max_{m \in \{0,\dots,4\}} \max_{u \in [u_0, 1]} \left\{ \left(f(m,u) + C_3 \frac{1}{|\omega|} \sum_{k=0}^m |u|^k \right) (1+|u|)^{2-m} \right\} ,$$

and $C_5 := \max(C_2, C_4)$, we obtain for all $u \geq u_0$ and $m \in \{0, ..., 4\}$ the bound

$$\left|\partial_\omega^m \phi_\omega^{(1)}(u)\right| \leq \frac{1}{|\omega|} C_5 (1+|u|)^{m-2} . \tag{4.26}$$

In order to estimate the derivatives of the second iteration $\phi^{(2)}_\omega(u)$ up to the fourth order, we subtract the first exact term out of the integration by parts in (4.25), $\frac{c_{30}}{4i\omega u^2}e^{-i\omega u}$, from the first iteration $\phi^{(1)}_\omega(u)$ and obtain for $u \geq 1$ and $m \leq 4$ the bounds

$$\left|\partial_\omega^m \left(\phi^{(1)}_\omega(u) - \frac{c_{30}}{4i\omega u^2}e^{-i\omega u}\right)\right| \leq \frac{1}{|\omega|}Cu^{m-3}. \tag{4.27}$$

Thus, in order to estimate the ω-derivatives of the second iteration:

$$\begin{aligned}\left|\partial_\omega^m \phi^{(2)}_\omega(u))\right| \;\leq\; & \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\left(\phi^{(1)}_\omega(v) - \frac{c_{30}}{4i\omega u^2}e^{-i\omega v}\right)\,dv\right| \\ & + \left|\partial_\omega^m \int_u^\infty \frac{1}{\omega}\sin(\omega(u-v))V_0(v)\frac{c_{30}}{4i\omega u^2}e^{-i\omega v}\,dv\right| .\end{aligned}$$

Using the estimates (4.27),(4.24), and once again the method of splitting up the potential and integrating by parts for the second integral, we get for $u \geq 1$ and $m \leq 4$ the bounds

$$\left|\partial_\omega^m \phi^{(2)}_\omega(u))\right| \leq \frac{1}{|\omega|}Cu^{m-4},$$

and thus, following the foregoing arguments for all $u \geq u_0$ (after possibly enlarging C_5) the estimates

$$\left|\partial_\omega^m \phi^{(2)}_\omega(u))\right| \leq \frac{1}{|\omega|}C_5(1+|u|)^{m-4} . \tag{4.28}$$

Using (4.24) and (4.28), we obtain for the ω-derivatives of the third iteration for all $u \geq u_0$

$$\begin{aligned}\left|\partial_\omega^m \phi^{(3)}_\omega(u)\right| \;\leq\; & \left|\sum_{k=0}^m \binom{m}{k}\int_u^\infty \partial_\omega^{m-k}\left(\frac{1}{\omega}\sin(\omega(v-u))\right)V_0(v)\partial_\omega^k\phi^{(2)}_\omega(v)\,dv\right| \\ \leq\; & 16C_1(u_0)C_5\frac{1}{|\omega|}\int_u^\infty (1+|v|)^{m-4}\frac{1}{|\omega|}V_0(v)\,dv .\end{aligned} \tag{4.29}$$

Note that interchanging the integral and the ω-derviatives is permitted, because the ω-derivatives of the integrand are integrable due to the estimates (4.24),(4.28) and the $1/v^3$-decay of the potential $V_0(v)$. We show by induction in n for all $u \geq u_0$ the inequality

$$\left|\partial_\omega^m \phi^{(n)}_\omega(u)\right| \leq 16C_1(u_0)C_5\frac{1}{|\omega|}Q_\omega(m,u)\frac{1}{(n-3)!}P_\omega(u)^{n-3}, \quad \forall n \geq 3\;, \tag{4.30}$$

where the functions $Q_\omega(m,u)$ and $P_\omega(u)$ are given by the integrals

$$\begin{aligned}Q_\omega(m,u) \;&:=\; \int_u^\infty (1+|v|)^{m-4}\frac{1}{|\omega|}V_0(v)\,dv \\ P_\omega(u) \;&:=\; 16C_1(u_0)C_6\int_u^\infty \frac{1}{|\omega|}V_0(v)\,dv\;,\end{aligned}$$

where C_6 is a constant chosen such that for all $x \geq v \geq u_0$

$$(1+|x|)^{k-m} \leq C_6(1+|v|)^{k-m}\;, \quad 0 \leq k \leq m \leq 4\;.$$

The initial step is now given by (4.29). So assume that (4.30) holds for n. Then, according to the iteration scheme,

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(n+1)}(u)\right| &\leq \Big|\sum_{k=0}^m \binom{m}{k} \int_u^\infty C_1(u_0)(1+|v|)^{m-k}\frac{1}{|\omega|}V_0(v) \\ &\quad \times 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(k,v) \frac{1}{(n-3)!} P_\omega(v)^{n-3}\, dv\Big| \,. \end{aligned}$$

Using the inequality

$$Q_\omega(k,v) \leq C_6 (1+|v|)^{k-m} Q_\omega(m,v)$$

and the monotonicity of Q_ω, we obtain

$$\begin{aligned} \left|\partial_\omega^m \phi_\omega^{(n+1)}(u)\right| &\leq 16 C_1(u_0) C_5 \frac{1}{|\omega|} \int_u^\infty 16 C_1(u_0)(1+|v|)^{m-k}\frac{1}{|\omega|}V_0(v) \\ &\quad \times C_6 (1+|v|)^{k-m} Q_\omega(m,v) \frac{1}{(n-3)!} P_\omega(v)^{n-3}\, dv \\ &\leq 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(m,u) \int_u^\infty \frac{dP_\omega}{dv}(v) \frac{1}{(n-3)!} P_\omega(v)^{n-3}\, dv \\ &= 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(m,u) \frac{1}{(n-2)!} P_\omega(u)^{n-2} \,, \end{aligned}$$

and (4.30) follows. In particular, we get for all $u \geq u_0$ and $m \leq 4$ the estimate

$$\begin{aligned} \left|\partial_\omega^m \grave{\phi}_\omega(u) - \partial_\omega^m e^{-i\omega u}\right| &\leq C_5 \frac{1}{|\omega|}(1+|u|)^{m-2} + \frac{1}{|\omega|} C_5 (1+|u|)^{m-4} \\ &\quad + 16 C_1(u_0) C_5 \frac{1}{|\omega|} Q_\omega(m,u) e^{P_\omega(u)} \,, \end{aligned} \tag{4.31}$$

and the right hand side obviously tends to zero as $|\omega| \to \infty$.

In an analog way using the iteration scheme (2.12) for $\acute{\phi}$, one shows for all $u \leq u_0$ and $m \in \{0, ..., 4\}$

$$\left|\partial_\omega^m \acute{\phi}_\omega(u) - \partial_\omega^m e^{i\omega u}\right| \leq \sum_{n=1}^\infty \frac{1}{n!} M_\omega(m,u)^n = e^{M_\omega(m,u)} - 1 \,, \tag{4.32}$$

where $M_\omega(m,u)$ is given by

$$M_\omega(m,u) := \frac{C_7}{|\omega|} \int_{-\infty}^u (1+|v|)^m V_0(v)\, dv \,,$$

with a sufficiently large constant C_7. Note that this integral is well defined, and in particular the estimate is obtained easier, due to the fact that $V_0(v)$ decays exponentially as $v \to -\infty$. Moreover, the right hand side in (4.32) also goes to zero as $|\omega| \to \infty$. Thus, due to (4.31) and (4.32), the ω-derivatives of the fundamental solutions up to the fourth order $\partial_\omega^m \acute{\phi}_\omega(u)$,$\partial_\omega^m \grave{\phi}_\omega(v)$ are controlled for large $|\omega|$ by constants, which depend on u and the support of the initial data Ψ_0. One also shows with these results and applying the same arguments to $\acute{\phi}'_\omega, \grave{\phi}'_\omega$ that the Wronskian $w(\acute{\phi}, \grave{\phi})$ behaves as $\mathcal{O}(|\omega|)$ and $\partial_\omega^m w(\acute{\phi}, \grave{\phi}), m \leq 4$ is bounded by constants as $|\omega| \to \infty$. Hence, interchanging in the representation (4.23) the differentiation with respect to ω and the integral, which is no problem because of the compact support of Ψ_0, making the substitutions

$$\begin{aligned}
\grave{\phi}_\omega(v) &= \frac{1}{\omega^2}\left(-\grave{\phi}''_\omega(v) + V_0(v)\grave{\phi}_\omega(v)\right) , \\
\partial_\omega\grave{\phi}_\omega(v) &= \frac{-2}{\omega^3}\left(-\grave{\phi}''_\omega(v) + V_0(v)\grave{\phi}_\omega(v)\right) + \frac{1}{\omega^2}\left(-\partial_\omega\grave{\phi}''_\omega(v) + V_0(v)\partial_\omega\grave{\phi}_\omega(v)\right)
\end{aligned}$$

as well as the analog substitutions for the second, third and fourth ω-derivative [Note that in the region $|\omega| \geq 1$ $\grave{\phi}_\omega(v)$ is C^4 with respect to ω, cf. Lemma 4.1, $n = 4$] and integrating by parts with respect to v, one immediately has decay at least of $1/\omega^2$. Thus iterating this procedure, which can be done because V_0 and Ψ_0 are smooth, yields arbitrary decay in ω and the lemma is proven. $\square$

Remark 4.7. Since the method of the proof does not depend on the highest order ω-derivative, the statement of Lemma 4.6 can be extended to arbitrary m. The only point where one has to be careful is the derivation of (4.28), since for ω-derivatives of higher order one has to calculate and subtract more exact terms than in (4.27), due to convergence problems. If (4.28) is not sufficient, in order to start the induction, one has to iterate this procedure appropriately many times.

We are now ready to state and prove our main theorem:

Theorem 4.8. *Consider the Cauchy problem of the scalar wave equation in the Schwarzschild geometry*

$$\Box\phi = 0 , \quad (\phi_0, i\partial_t\phi_0)(0, r, x) = \Phi_0(r, x)$$

for smooth spherical symmetric initial data $\Phi_0 \in C_0^\infty((2M, \infty) \times S^2)^2$ which is compactly supported outside the event horizon. Let $\Phi(t) = (\phi(t), i\partial_t\phi(t)) \in C^\infty(\mathbb{R} \times (2M, \infty) \times S^2)^2$ be the unique global solution which is compactly supported for all times t. Then for fixed r there is a constant $c = c(r, \Phi_0)$ such that for large t

$$|\phi(t)| \leq \frac{c}{t^3} . \tag{4.33}$$

Moreover, if we have initially momentarily static initial data, i.e. $\partial_t\phi_0 \equiv 0$, the solution $\phi(t)$ satisfies

$$|\phi(t)| \leq \frac{c}{t^4} . \tag{4.34}$$

Proof. First, we decompose our initial data Φ_0 into spherical harmonics. Due to the spherical symmetry we obtain $\Phi_0(r, \vartheta, \varphi) = \tilde{\Phi}_0(r)Y_{00}(\vartheta, \varphi)$, where $\tilde{\Phi}_0(r) \in C_0^\infty((2M, \infty))^2$. Introducing the Regge-Wheeler coordinate $u(r)$ and making the substitution $\Psi(t, u) = r(u)\tilde{\Phi}(t, r(u))$, our solution has the representation

$$\Phi(t, r, \vartheta, \varphi) = \frac{1}{r}\Psi(t, u(r))Y_{00}(\vartheta, \varphi) ,$$

where $\Psi(t, u)$ satisfies

$$\Psi(t, u) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\left(\int_{\text{supp}\,\Psi_0} \text{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\right)\begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix}\Psi_0(v)dv\right)d\omega, \tag{4.35}$$

with initial data $\Psi_0(u) := r(u)\tilde{\Phi}_0(u)$ and the Jost solutions $\acute{\phi}, \grave{\phi}$ in the case $l = 0$. According to the detailed analysis of (4.20) with respect to ω, the term

$$\mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\right) - c_0(u)g_{20}(v)\omega^2\,\mathrm{sign}(\omega) - c_{32}(u)g_{32}(v)\omega^3\log^2|\omega|$$
$$-c_{31}(u)g_{31}(v)\omega^3\log|\omega| - c_{30}(u)g_{30}(v)\omega^3\,\mathrm{sign}(\omega)$$

is $C^3(\mathbb{R})$ with respect to ω for fixed $u \in \mathbb{R}$, $v \in \mathrm{supp}\Psi_0$, where the $c_{ij}(u)$, $g_{ij}(v)$ denote the appropriate coefficient functions. [Note that these are linearly dependent due to the symmetry of (4.1) with respect to u, v.] Thus, defining

$$f(\omega, u) := \left(\int_{\mathrm{supp}\,\Psi_0} \mathrm{Im}\left(\frac{\acute{\phi}_\omega(u)\grave{\phi}_\omega(v)}{w(\acute{\phi}_\omega, \grave{\phi}_\omega)}\right)\begin{pmatrix} \omega & 1 \\ \omega^2 & \omega \end{pmatrix}\Psi_0(v)dv\right)_1 ,$$

where the subscript denotes the first vector component, the term

$$\tilde{f}(\omega, u) := f(\omega, u) - \Big(c_0(u)d_{20}(\psi_0^2)\,\omega^2\,\mathrm{sign}(\omega) + c_{32}(u)d_{32}(\psi_0^2)\,\omega^3\log^2|\omega|$$
$$+c_{31}(u)d_{31}(\psi_0^2)\,\omega^3\log|\omega| + c_{30}(u)d_{30}(\psi_0^2)\,\omega^3\,\mathrm{sign}(\omega)\Big)\eta(\omega)$$
$$=: f(\omega, u) - r(\omega, u) ,$$

is also $C^3(\mathbb{R})$ with respect to ω. Here, ψ_0^2 denotes the second component of the initial data Ψ_0,

$$d_{ij}(\psi_0^2) := \int_{\mathrm{supp}\,\Psi_0} g_{ij}(v)\psi_0^2(v)\,dv ,$$

and $\eta(\omega) \in C_0^\infty(\mathbb{R})$ is a smooth cutoff-function which is identically to 1 on a neighborhood of $\omega = 0$ and 0 outside a compact set. Moreover, because of Lemma 4.6 the $\partial_\omega^m \tilde{f}(\omega, u), m \in \{0, 1, 2, 3\}$ have rapid decay for large $|\omega|$ and are in particular $L^1(\mathbb{R})$ with respect to ω. Thus, due to (4.35), the first component of Ψ satisfies

$$\begin{aligned}\psi^1(t, u) &= -\frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}\tilde{f}(\omega, u)\,d\omega - \frac{1}{\pi}\int_{\mathbb{R}} e^{-i\omega t}r(\omega, u)\,d\omega \\ &= -\frac{1}{(it)^3\pi}\left(\int_{\mathbb{R}} \tilde{f}(\omega, u)\partial_\omega^3 e^{-i\omega t}\,d\omega + \int_{\mathbb{R}} r(\omega, u)\partial_\omega^3 e^{-i\omega t}\,d\omega\right) .\end{aligned}$$

We write the second integral as $\int_{-\infty}^0 + \int_0^\infty$, integrate every integral three times by parts and obtain

$$\begin{aligned}\psi^1(t, u) = \frac{1}{(it)^3\pi}\Bigg(&4c_0(u)d_{20}(\psi_0^2) + \int_{\mathbb{R}} e^{-i\omega t}\partial_\omega^3\tilde{f}(\omega, u)\,d\omega \\ &+ \int_{-\infty}^0 e^{-i\omega t}\partial_\omega^3 r(\omega, u)\,d\omega + \int_0^\infty e^{-i\omega t}\partial_\omega^3 r(\omega, u)\,d\omega\Bigg) .\end{aligned}$$

Note that the other boundary terms vanish, because the $\partial_\omega^m \tilde{f}(\omega), m \le 3$ have rapid decay and $\eta(\omega) \equiv 0$ outside of a compact set. Obviously, all integrals are well defined, and the Riemann-Lebesgue lemma shows the claim in the first case. If the initial data is initially momentarily static, all the $d_{ij}(\psi_0^2)$ vanish and the entries in the matrix in (4.35) yield an additional ω. Hence, the highest irregular term is $c_0(u)d_{20}(\psi_0^1)\omega^3\,\mathrm{sign}(\omega)$, and the same arguments as before conclude the proof. $\square$

Remark 4.9. The decay rates $1/t^3$, and $1/t^4$, respectively, are optimal in the sense that there exists initial data such that these cannot be improved. This is obvious due to the fact that $c_0(u) > 0$.

5 Discussion on the case $l \neq 0$

According to Price's Law [16], the lm-component $\Phi^{lm}(t,u) = \frac{1}{r}\Psi^{lm}(t,u)$ of a solution for the Cauchy problem for the scalar wave equation in Schwarzschild spacetime with compactly supported smooth initital data generally falls off at late times t as t^{-2l-3} and t^{-2l-4} for initially momentarily static initial data, respectively. This has been confirmed in the previous section for spherical symmetric initial data, i.e. in the case $l = 0$ [cf. Theorem 4.8]. Moreover, there is numerical evidence which lets us conjecture this to be correct [13]. We briefly discuss whether the methods of the preceding section still apply to the case when the angular mode l is non-zero.

To this end, let us reconsider the construction of the fundamental solutions $\grave{\phi}_{\omega l}$ of the Schrödinger equation (2.6). First, we make some remarks about the fundamental solutions $\omega^l \grave{\phi}_{\omega l}(u)$ (see also [15, Section 5]). The fundamental solutions were constructed as the series

$$\omega^l \grave{\phi}_\omega(u) = \sum_{m=0}^{\infty} \phi_\omega^{(m)}(u) \ , \tag{5.1}$$

where the $\phi^{(m)}$ are given by the iteration scheme

$$\phi_\omega^{(m+1)}(u) = -\int_u^\infty S_\omega(u,v) W_l(v) \phi_\omega^{(m)}(v)\, dv \ , \tag{5.2}$$

with potential, cf. also Lemma 3.3,

$$\begin{aligned} W_l(u) = V_l(u) - \frac{l(l+1)}{u^2} &= c_{31}\frac{\log u}{u^3} + \frac{c_{30}}{u^3} + h(u) \ , \\ \text{where} \ \ h(u) &= \mathcal{O}\left(\frac{\log^2 u}{u^4}\right) \end{aligned} \tag{5.3}$$

for large u, and Green's function

$$S_\omega(u,v) = \frac{(-1)^{l+1}}{\omega}\big(h_1(l,\omega v) h_2(l,\omega u) - h_1(l,\omega u) h_2(l,\omega v)\big) \ , \tag{5.4}$$

where

$$h_1(l,\omega u) = \sqrt{\frac{\pi\omega u}{2}} J_{l+1/2}(\omega u) \ , \quad h_2(l,\omega u) = \sqrt{\frac{\pi\omega u}{2}} J_{-l-1/2}(\omega u) \ , \tag{5.5}$$

and J_ν denotes the Bessel function of the first kind. As initial function $\phi_\omega^{(0)}(u)$ we have chosen

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i(l+1)\frac{\pi}{2}} \sqrt{\frac{\pi\omega u}{2}} H^{(2)}_{l+1/2}(\omega u) \ ,$$

where $H_\nu^{(2)}$ denotes the second Hankel function. Since l is an integer, these functions are directly connected to the spherical Bessel functions and simplify significantly. Namely, h_1, h_2 have the following representations [cf. [1, Chapter 10]]

$$h_1(l, \omega u) = P(l + \frac{1}{2}, \omega u) \sin(\omega u - \frac{1}{2} l\pi) + Q(l + \frac{1}{2}, \omega u) \cos(\omega u - \frac{1}{2} l\pi) \quad (5.6)$$

$$h_2(l, \omega u) = P(l + \frac{1}{2}, \omega u) \cos(\omega u + \frac{1}{2} l\pi) - Q(l + \frac{1}{2}, \omega u) \sin(\omega u + \frac{1}{2} l\pi) \quad (5.7)$$

where P, Q are finite polynomials given by

$$\begin{aligned} P(l + \frac{1}{2}, \omega u) &= \sum_{k=0}^{[\frac{1}{2}l]} (-1)^k \frac{(l + \frac{1}{2}, 2k)}{(2\omega u)^{2k}} , \\ Q(l + \frac{1}{2}, \omega u) &= \sum_{k=0}^{[\frac{1}{2}(l-1)]} (-1)^k \frac{(l + \frac{1}{2}, 2k+1)}{(2\omega u)^{2k+1}} , \end{aligned}$$

with

$$(l + \frac{1}{2}, k) = \frac{(l+k)!}{k!\, \Gamma(l - k + 1)} .$$

And the initial function can be expressed by

$$\phi_\omega^{(0)}(u) = \omega^l e^{-i\omega u} \sum_{k=0}^{l} \frac{(l + \frac{1}{2}, k)}{(2i\omega u)^k} . \quad (5.8)$$

Due to the recurrence formulas for the derivatives of the Bessel functions, we have the identities

$$\begin{aligned} \partial_\omega h_1(l, \omega u) &= u h_1(l-1, \omega u) - \frac{l}{\omega} h_1(l, \omega u) \quad , \\ \partial_\omega h_2(l, \omega u) &= -u h_2(l-1, \omega u) - \frac{l}{\omega} h_2(l, \omega u) . \end{aligned}$$

As a consequence,

$$\begin{aligned} \partial_\omega S_\omega(u, v) = &- \frac{2l+1}{\omega} S_\omega \\ &+ v \frac{(-1)^{l+1}}{\omega} (h_1(l-1, \omega v) h_2(l, \omega u) + h_1(l, \omega u) h_2(l-1, \omega v)) \\ &+ u \frac{(-1)^l}{\omega} (h_1(l, \omega v) h_2(l-1, \omega u) + h_1(l-1, \omega u) h_2(l, \omega v)) . \end{aligned}$$

This allows us to derive the necessary estimates for the Green's function $S_\omega(u, v)$. Exploiting the asymptotics we have already seen in [15, Section 5]

$$|S_\omega(u, v)| \le C_1 \left(\frac{u}{1 + |\omega| u} \right)^{-l} \left(\frac{v}{1 + |\omega| v} \right)^{l+1} e^{v |\operatorname{Im} \omega| + u \operatorname{Im} \omega} ,$$

for $v \ge u > 0$ and an appropriate constant C_1. In order to derive an estimate for $\partial_\omega S_\omega$ and small $|\omega|$, we make use of

$$h_1(l, \omega u) \quad \sim \quad k_1 (\omega u)^{l+1} + k_2 (\omega u)^{l+3}$$

$$h_2(l,\omega u) \quad \sim \quad k_3(\omega u)^{-l} + k_4(\omega u)^{-l+2}\ , \quad \text{if } |\omega|u \ll 1\ ,$$

and certain constants $k_1, ..., k_4$ [refer to the series expansion of the Bessel functions [1, 9.1.10]] to obtain (note that $v \geq u > 0$),

$$|\partial_\omega S_\omega(u,v)| \leq C_2 \left(\frac{u}{1+|\omega|u}\right)^{-l} \left(\frac{v}{1+|\omega|v}\right)^{l+2} \quad , \quad \text{if } |\omega|v \ll 1\ .$$

For large arguments $|\omega|u \gg 1$ we use (5.6),(5.7) and get by a straightforward calculation

$$\partial_\omega S_\omega(u,v) \sim \frac{-2l}{\omega^2}\sin(\omega(u-v)) + \partial_\omega\left[\frac{1}{\omega}\sin(\omega(u-v))\right]\ , \quad \text{if } |\omega|u \gg 1.$$

Together with (3.10), we obtain

$$|S_\omega(u,v)| \leq C_3 \frac{v^2}{1+|\omega|v} e^{v|\operatorname{Im}\omega| + u\operatorname{Im}\omega}\ , \quad \text{if } |\omega|u \gg 1.$$

Combining these estimates, we find a constant C such that

$$|\partial_\omega S_\omega(u,v)| \leq C \left(\frac{u}{1+|\omega|u}\right)^{-l} \left(\frac{v}{1+|\omega|v}\right)^{l+1} v\, e^{v|\operatorname{Im}\omega| + u\operatorname{Im}\omega}\ , \tag{5.9}$$

for $v \geq u > 0$. Moreover, looking at (5.8) we get the following bounds for the initial function,

$$|\phi_\omega^{(0)}(u)| \quad \leq \quad C_4 \left(\frac{u}{1+|\omega|u}\right)^{-l} e^{u\operatorname{Im}\omega}\ , \tag{5.10}$$

$$|\partial_\omega\phi_\omega^{(0)}(u)| \quad \leq \quad C_5 \left(\frac{u}{1+|\omega|u}\right)^{-l} u\, e^{u\operatorname{Im}\omega}\ . \tag{5.11}$$

These estimates allow us to proceed in exactly the same way as in the proof of Lemma 3.4. As analogon to $\hat{\phi}_\omega^{(1)}(u)$ we obtain the term

$$-\int_u^\infty S_\omega(u,v)\left(c_{31}\frac{\log v}{v^3} + \frac{c_{30}}{v^3}\right)\phi_\omega^{(0)}(v)\, dv,$$

which we calculate using (5.6),(5.7) and (5.8). Essentially, we get integrals of the shape

$$\frac{\omega^l}{(\omega u)^n \omega^{m+k+1}}\left(C_6 e^{i\omega u}\int_u^\infty e^{-2i\omega v}\frac{\log^q v}{v^{3+k+m}}\, dv + C_7 e^{-i\omega u}\int_u^\infty \frac{\log^q v}{v^{3+k+m}}\, dv\right)\ ,$$

where $q \in \{0,1\}, 0 \leq n,m,k \leq l$. Note that the terms involving ω singularities resolve, due to the fact that $\omega^l \grave{\phi}_\omega$ is continuous with respect to ω. Computing these integrals via Lemma 3.1 (in the limit $\varepsilon \to 0$), we see (as before) that the only terms not being C^1 with respect to ω are of the form

$$e^{i\omega u}\frac{1}{\omega^{m+k+1}}(2i\omega)^{k+m+2}\left(\log^2(2i\omega u) + \log u \log(2i\omega u) + \log(2i\omega u)\right)\ , \tag{5.12}$$

modulo coefficients. Now, we apply the same iteration with analog estimates and all in all we have shown:

Lemma 5.1. *For $l \geq 1$, $\omega \in \mathbb{R} \setminus \{0\}$ and fixed $u > 0$ the fundamental solutions $\omega^l \grave{\phi}_\omega(u)$ have the representation*

$$\omega^l \grave{\phi}_\omega(u) = \phi_\omega^{(0)}(u) + g_3(\omega, u) + 2i\omega \log^2(2i\omega) g_4(\omega, u) \\ +2i\omega \log(2i\omega) g_5(\omega, u) + 2i\omega g_6(\omega, u) \;, \tag{5.13}$$

where the functions g_3, g_4, g_5 and g_6 are $C^1(\mathbb{R})$ with respect to ω.

Hence, we still have finite expressions for the Green's function $S_\omega(u, v)$ as well as for the initial function $\phi_\omega^{(0)}(v)$, which involve essentially the plane waves $e^{\pm i\omega u}, e^{\pm i\omega v}$. Expanding all these expressions and deriving estimates analog to (5.9) and (5.11) for higher order ω-derivatives, we can improve Lemma 5.1 in the same way as Lemma 3.4 following the arguments of the proof of Lemma 4.1. Also, a similar result to Corollary 4.5 seems straightforward. The problem now arises, when we have to derive an $\omega-$expansion of the essential part of the integral kernel

$$\mathrm{Im} \left(\frac{\acute{\phi}_{\omega l}(u) \grave{\phi}_{\omega l}(v)}{w(\acute{\phi}_l, \grave{\phi}_l)} \right) \;. \tag{5.14}$$

The main difficulty can be seen as follows. If we proceeded in the same way as in the case $l = 0$, the lowest regular term with respect to ω should appear with the power ω^{2l+2} [cf. proof of Theorem 4.8] in order to satisfy Price's law. But due to the fact that the first irregularity in ω looks as follows,

$$e^{i\omega u} u^{-l} 2i\omega \big(c \log^2(2i\omega u) + c \log u \log(2i\omega u) + c \log(2i\omega u) \big) \;,$$

[cf. equation (5.12)], we would have to find a systematic way in order to check that the coefficients in front of terms with lower regularity vanish. Because of the complexity of the calculations we did not succeed in this point. Thus, following the same arguments as for $l = 0$ together with the analog result to Corollary 4.5, which would involve $2i\omega \log^2(2i\omega)$ as highest irregularity, we would have to assume $\omega \log |\omega|$ as the lowest regular term in the expansion of (5.14). Except for this problem, we do not expect any further difficulties in extending Lemma 4.6 to $l \neq 0$, apart from the complexity of the calculations and the estimates. Thus, for arbitrary l it follows a similar statement to Theorem 4.8, but with the decay $|\phi(t)| \leq c/t^2$, and in the case of momentarily static initial data $|\phi(t)| \leq c/t^3$, respectively. The proof uses essentially the arguments of the proof of Theorem 4.8, with the difference that one basically has to check the inequality

$$\Big| \int_{-1}^{1} \log |\omega| e^{-i\omega t} \, d\omega \Big| \leq \frac{c}{t} \, .$$

To this end, one makes the substitution $z = \omega t$ and splits up the integrals to obtain

$$\int_{-1}^{1} \log |\omega| e^{-i\omega t} \, d\omega \;=\; \frac{1}{t} \bigg(\int_{-1}^{1} \log |z| e^{-iz} \, dz - \log t \int_{-t}^{t} e^{-iz} \, dz \\ + \int_{-t}^{-1} \log(-z) e^{-iz} \, dz + \int_{1}^{t} \log z e^{-iz} \, dz \bigg) \, .$$

Computing the second integral and integrating the last two integrals by parts yields

$$= \frac{1}{t}\left(\int_{-1}^{1} \log|z| e^{-iz}\, dz + \frac{1}{i}\int_{-t}^{-1} \frac{1}{z} e^{-iz}\, dz + \frac{1}{i}\int_{1}^{t} \frac{1}{z} e^{-iz}\, dz \right),$$

and the inequality follows, after having integrated the last two integrals once again by parts followed by standard integral estimates. However, in view of Price's law, this result is not satisfying.

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