Milnor K-theory and motivic cohomology

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MILNOR $K$-THEORY AND MOTIVIC COHOMOLOGY

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Abstract. These are the notes of a talk given at the Oberwolfach Workshop $K$-Theory 2007. We sketch a proof of Beilinson’s conjecture relating Milnor $K$-theory and motivic cohomology. For detailed proofs see [4].

1. Introduction

$A$ — semi-local commutative ring with infinite residue fields

$k$ — field

$\mathbb{Z}(n)$ — Voevodsky’s motivic complex [8]

Definition 1.1.

$$K^M_n(A) = \bigoplus_n (A^\times)^{\otimes n} / (a \otimes (1 - a)) \quad a, 1 - a \in A^\times$$

Beilinson conjectured [1]:

Theorem 1.2. $A/k$ essentially smooth, $|k| = \infty$. Then:

$$\eta : K^M_n(A) \to H^n_{zar}(A, \mathbb{Z}(n))$$

is an isomorphism for $n > 0$.

The formerly known cases are:

Remark 1.3.

— $A=k$ a field (Nesterenko-Suslin [6], Totaro [10])

— surjectivity of $\eta$ (Gabber [3], Elbaz-Vincent/Müller-Stach [2], Kerz/Müller-Stach [5])

— $\eta \otimes \mathbb{Q}$ is isomorphic (Suslin)

— injectivity for $A$ a DVR, $n = 3$ (Suslin-Yarosh [9])

2. General idea of proof

$X = \text{Spec } A$

We have a morphism of Gersten complexes which we know to be exact except possibly at $K^M_n(A)$:

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So it suffices to prove:

**Theorem 2.1** (Main Result). \( A/k \) regular, connected, infinite residue fields, \( F = Q(A) \). Then:

\[
i: K^M_n(A) \rightarrow K^M_n(F)
\]

is (universally) injective.

### 3. Applications

**Corollary 3.1** (Equicharacteristic Gersten conjecture). \( A/k \) regular, local, \( X = \text{Spec} A \), \( |k| = \infty \). Then:

\[
0 \rightarrow K^M_n(A) \rightarrow \bigoplus_{x \in X^{(0)}} K^M_n(x) \rightarrow \bigoplus_{x \in X^{(1)}} K^M_{n-1}(x) \rightarrow \cdots
\]

is exact.

**Proof.** Case \( A/k \) essentially smooth: Use (1) + Main Result. Case \( A \) general: Use smooth case + Panin’s method [7] + Main Result. \( \square \)

\( K^M_n \) — Zariski sheaf associated to \( K^M_n \)

**Corollary 3.2** (Bloch formula). \( X/k \) regular excellent scheme, \( |k| = \infty \), \( n \geq 0 \). Then:

\[
H^n_{\text{zar}}(X, K^M_n) = CH^n(X).
\]

Levine and Kahn conjectured:

**Corollary 3.3** (Generalized Bloch-Kato conjecture). Assume the Bloch-Kato conjecture. \( A/k \), \( |k| = \infty \), \( \text{char}(k) \) prime to \( l > 0 \). Then the galois symbol

\[
\chi_n : K^M_n(A)/I \rightarrow H^n_{\text{et}}(A, \mathbb{Q}_l^\oplus)
\]

is an isomorphism for \( n > 0 \).

Idea of proof. Case \( A/k \) essentially smooth: Use Gersten resolution. Case \( A/k \) general: Use Hoobler’s trick + Gabber’s rigidity for étale cohomology. \( \square \)

**Corollary 3.4** (Generalized Milnor conjecture). \( A/k \) local, \( |k| = \infty \), \( \text{char}(k) \) prime to 2. Then there exists an isomorphism

\[
K^M_n(A)/2 \rightarrow I^n_A/I^{n+1}_A
\]

where \( I_A \subset W(A) \) is the fundamental ideal in the Witt ring of \( A \).
4. New methods in Milnor $K$-theory

The first new result for $K$-groups used in the proof of the Main Result states:

**Theorem 4.1 (COCA).** $A \subset A'$ local extension of semi-local rings, i.e. $A^\infty = A \cap A'^\infty$. $A, A'$ factorial, $f \in A$ such that $A/(f) = A'/f)$. Then:

\[
\begin{array}{ccc}
K^M_n(A) & \longrightarrow & K^M_n(A_f) \\
\downarrow & & \downarrow \\
K^M_n(A') & \longrightarrow & K^M_n(A'_f)
\end{array}
\]

is co-Cartesian.

**Remark 4.2.** COCA was proposed by Gabber who proved the surjectivity part at the lower right corner.

**Theorem 4.3 (Local Milnor Theorem).** $q \in A[t]$ monic. There is a split short exact sequence

\[
0 \longrightarrow K^M_n(A) \longrightarrow K^I_n(A, q) \longrightarrow \oplus_{(\pi,q)=1} K^M_{n-1}(A[t]/(\pi)) \longrightarrow 0
\]

**Explanation:** The abelian group $K^I_n(A, q)$ is generated by symbols $\{p_1, \ldots, p_n\}$ with $p_1, \ldots, p_n \in A[t]$ pairwise coprime, $(p_i, q) = 1$ and highest non-vanishing coefficients invertible. For $A = k$ a field $K^I_n(k, 1) = K^M_n(k(t))$.

The standard technique gives:

**Theorem 4.4 (Norm Theorem).** Assume $A$ has big residue fields (depending on $n$). $A \subset B$ finite, étale. Then there exists a norm

\[
N_{B/A} : K^M_n(B) \longrightarrow K^M_n(A)
\]

satisfying projection formula, base change.

5. Proof of Main Result

1st step: Reduce to $A$ semi-local with respect to closed points $y_1, \ldots, y_l \in Y/k$ smooth, $|k| = \infty$ and $k$ perfect. For this use Norm Theorem + Popescu desingularization.

2nd step: Induction on $d = \dim A$ for all $n$ at once.

\[
i : K^M_n(A) \longrightarrow K^M_n(F)
\]

If $x \in K^M_n(A)$ with $i(x) = 0$ then there exists $f \in A$ such that $i_f(x) = 0$ where

\[
i_f : K^M_n(A) \longrightarrow K^M_n(A_f)
\]

Gabber’s presentation theorem produces $A' \subset A$ a local extension and $f' \in A'$ such that $f'/f \in A^\ast$ and $A'/f') = A/(f)$. Here $A'$ is a semi-local ring with respect to closed points $y_1, \ldots, y_l \in \mathbb{A}^d_k$. 

The COCA Theorem gives that
\[ K^M_n(A') \longrightarrow K^M_n(A'_f) \]
is co-Cartesian. So it suffices to prove that
\[ i' : K^M_n(A') \longrightarrow K^M_n(k(t_1, \ldots, t_d)) \]
is injective. Let \( x \in \ker(i') \) and \( p_1, \ldots, p_m \in k[t_1, \ldots, t_d] \) be the irreducible, different polynomials appearing in \( x \), \( p_i \in A^{\times} \).

Let \( W = \bigcup_i \text{sing. loc.}(V(p_i)) \cup \bigcup_{i,j} V(p_i) \cap V(p_j) \)

Then \( \dim(W) < d - 1 \) since \( k \) is perfect.

There exists a linear projection
\[ p_{A^d_k} \longrightarrow A^{d-1}_k \]
such that \( p|_{V(p_i)} \) is finite and \( p(y_i) \not\in p(W) \) for all \( i \).

Let now \( A'' \) be the semi-local ring with respect to \( p(y_1), \ldots, p(y_l) \in A^{d-1}_k \)
\[ A'' \subset A''[t] \subset A' \]

Let \( q \in A''[t] \) be monic such that
\[ V(q) \cap p^{-1}(p(y_i)) = \{y_1, \ldots, y_l\} \cap p^{-1}(p(y_i)) \]

Under the natural map \( K^t_n(A'', q) \rightarrow K^M_n(A') \) there exists a preimage \( x' \in K^t_n(A'', q) \) of \( x \).

We have a commutative diagram, \( F = Q(A'') \):
\[
\begin{array}{c}
0 \longrightarrow K^M_n(A'') \longrightarrow K^M_n(A'', q) \longrightarrow \oplus \pi K^M_{n-1}(A''[t]/(\pi)) \longrightarrow 0 \\
\downarrow \hspace{1cm} \hspace{1cm} \downarrow \hspace{1cm} \hspace{1cm} \downarrow \\
0 \longrightarrow K^M_n(F) \longrightarrow K^M_n(F(t)) \longrightarrow \oplus \pi K^M_{n-1}(F[t]/(\pi)) \longrightarrow 0
\end{array}
\]

But since the important summands in the right vertical arrow are injective, a simple diagram chase gives \( x' = 0 \) and finally \( x = 0 \).

\[ \square \]

REFERENCES


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