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Abstract

We show that the oriented Chow groups of Barge-Morel appear in the E_2-term of the coniveau spectral sequence for hermitian K-theory. This includes a localization theorem and the Gersten conjecture (over infinite base fields) for hermitian K-theory. We also discuss the conjectural relationship between oriented and higher oriented Chow groups and Levine's homotopy coniveau spectral sequence when applied to hermitian K-theory.

Introduction

Oriented Chow groups (some people prefer to call them "Chow-Witt groups") have been invented by Barge and Morel [BM], and have been further studied by Morel [M2], Fasel [Fa], Fasel-Srinivas [FS1] and probably others. The main application of these groups so far is a theorem of Morel [M2, Theorem 10 and Remark 11] which states that these oriented Chow groups contain a certain Euler class which detects whether or not a d-dimensional vector bundle over a smooth affine base scheme of dimension $d \geq 4$ splits off a trivial one-dimensional subbundle (the case $d = 2$ was established in [BM]).

Let F be a field and X be an object of Sm/F, that is a smooth variety over F. It is known ("Bloch's formula") that the Chow groups $CH^p(X)$ appear in the Brown-Gersten-Quillen spectral sequence [Qu] (also called coniveau spectral sequence for algebraic K-theory). More precisely, writing $X^{(p)}$ for the set of points of X having codimension p and $k(x)$ for the residue field of a point x, there is a differential in the E_1-term $d : E_1^{p-1,-p} \cong \oplus_{x \in X^{(p-1)}} K_1(k(x)) \to E_1^{p,-p} \cong \oplus_{x \in X^{(p)}} K_0(k(x))$ inducing an isomorphism $E_2^{p,-p} \cong CH^p(X)$.

In section 1, we prove that the oriented Chow groups $\widetilde{CH}^p(X)$ of Barge and Morel appear in a similar fashion in the E_2-term of the coniveau spectral sequence for hermitian K-theory (or for other related theories if $p \neq 0$ $(mod\ 4)$). The proof (see Theorem 1.7) relies amomg other things on real motivic Bott periodicity and a localization theorem for hermitian K-theory

(Proposition 1.4). In particular, we see that the Gersten conjecture for hermitian K-theory is true for essentially smooth local algebras over an inifite field.

Section 2 discusses higher oriented Chow groups and the homotopy coniveau spectral sequence as established by Levine in [Le] when applied to hermitian K-theory. We also include some comments on the slice filtration as introduced by Voevodsky [Vo2], as well as on the relationship between the coniveau and the homotopy coniveau spectral sequence. Our understanding so far of what is going on is rather limited, and further investigations remain to be carried out.

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After a first version of this text was written, J. Fasel informed me that he too had considered the Gersten complex for hermitian K-theory, which he will use as a tool in a joint work with V. Srinivas [FS2] in order to extend the above theorem of Morel to the case $d = 3$.

1 The coniveau spectral sequence for hermitian K-theory

We fix a base field F of characteristic $\neq 2$. Let us recall the definition of oriented Chow groups [BM]. Some statements in this article, in particular those involving differentials in all kind of Gersten complexes rely on choices of local parameters or dualizing objects. In order to make these functorial, for any point $x \in X$ one should consider the maximal exterior power of the dual of m_x/m_x^2 as dualizing object on $k(x)$-vector spaces, see e.g. [BW, p. 129]. Similar to loc. cit., we have suppressed these canonical dualities from our notation.

Definition 1.1 *(1) For $n \geq 0$, define $J^n(F)$ by the following cartesian square*

$$\begin{array}{ccc} J^n(F) & \longrightarrow & K_n^M(F) \\ \downarrow & & \downarrow \\ I(F)^n & \longrightarrow & I(F)^n/I(F)^{n+1} \cong K_n^M(F)/2 \end{array}$$

where $I(F)$ denotes the fundamental ideal in the Witt ring $W(F)$. The isomorphism in the diagram is given by Voevodsky's theorem [Vo1], induced by mapping a unit $a \in F^\times$ to the quadratic form $< 1, -a >$. The differentials *in the Gersten complexes of* I^* *and* K_*^M *induce a differential* d^J *on the Gersten complex of* J^**. For* $n < 0$ *we set* $J^n(F) := W(F)$ *and extend* d^J *accordingly.*

(2) For X *a smooth variety over* F *and* $p \geq 0$ *a non-negative integer, we define the oriented Chow groups by* $\widetilde{CH}^p(X) := Ker(\oplus_{x\in X^{(p)}} J^0(k(x)) \xrightarrow{d^J} \oplus_{x\in X^{(p+1)}} J^{-1}(k(x)))/Im(\oplus_{x\in X^{(p-1)}} J^1(k(x)) \xrightarrow{d^J} \oplus_{x\in X^{(p)}} J^0(k(x)))$.

In the sequel, we denote by KO the presheaf of (S^1-)spectra on Sm/F introduced in [Ho, Section 2] by applying a functorial version of Jouanoulou's device to the presheaf of spectra on smooth affine varieties over F introduced in [HS, Definition 3.14]. The homotopy groups of $KO(X)$ are by definition the hermitian K-groups $KO_n(X)$. As usual, we denote by ${}_-KO$ the spectrum corresponding to anti-symmetric forms and by U and ${}_-U$ the homotopy fibers of the hyperbolic functors $K \to KO$ resp. $K \to {}_-KO$. Recall also that for X an object of Sm/k the spectrum $K(X)$ is (-1)-connected whereas the four spectra $KO(X)$, ${}_-KO(X)$, $U(X)$ and ${}_-U(X)$ are not. By [Ho] the zero spaces of these four spectra fit together to a motivic $\Omega^1_{\mathbf{P}}$-spectrum which is 4-periodic, and we will encounter this 4-periodicity pattern frequently in the sequel.

We start with the following lemma. The first part is obvious and classical, whereas the second part has been proved in various disguises by various people (see e.g. [Sch1] and the references given there).

Lemma 1.2 *(1) There is a canonical isomorphism* $c_0 : KO_0(F) \xrightarrow{\cong} J^0(F)$.

(2) There is a canonical isomorphism $c_1 :\ {}_-U_1(F) \xrightarrow{\cong} J^1(F)$.

Proof: To prove (2), observe that by Karoubi's fundamental theorem [K3] ${}_-U_1(F) \cong V_0(F)$ where $V_0(F) = V(F)$ is the free abelian group generated by triples (M, ϕ, ψ) with ϕ and ψ nondegenerate symmetric bilinear forms on a finite dimensional F-vector space M modulo the relations $(M, \phi, \psi) + (M', \phi', \psi') \sim (M \oplus M', \phi \oplus \phi', \psi \oplus \psi')$ and $(M, \phi, \psi) + (M, \psi, \rho) \sim (M, \phi, \rho)$ (compare [K1, Appendice 2]). As any symmetric bilinear form over a field is diagonalisable, we may restrict to the case $M = F$, $\phi =< a >$ and $\psi =< b >$. Mapping $(F, < a >, < b >)$ to $ab^{-1} \in K_1^M(F)$ and to $< a, -b >$ in $I(F)$ induces a map $V(F) \to J^1(F)$ by the universal property of the pullback (observe that the form $< 1, -a, b, -ab >$ lies in $I(F)^2$). Now one checks by hand that the square

$$\begin{array}{ccc} V(F) & \longrightarrow & K_1^M(F) \\ \downarrow & & \downarrow \\ I(F) & \longrightarrow & I(F)/I(F)^2 \cong K_1^M(F)/2 \end{array}$$

is cartesian. □

In order to describe various 4-periodic phenomena arising from real motivic Bott periodicity in an efficient way we introduce some notation.

Definition 1.3 *For any $n \in \mathbf{Z}$, we set $L^{[4n]} := KO$, $L^{[4n+1]} := {}_{-}U$, $L^{[4n+2]} := {}_{-}KO$ and $L^{[4n+3]} := U$.*

The reader familiar with the motivic spectrum $\mathbf{KO}$ introduced in [Ho] has certainly observed that the presheaves of simplicial sets $L_0^{[i]}$ and $\mathbf{KO}_i$ are locally weakly equivalent.

We now state a result about localization in hermitian K-theory whose analogue is known in algebraic K-theory by [Qu]. For X a smooth affine curve, this is an immediate consequence of the results established in [HS, Theorem 1.15, Corollary 1.22].

Proposition 1.4 *Let X be a smooth variety and $Z \subset X$ be a regular closed subvariety of codimension d. Assume that the normal bundle of the closed embedding $Z \to X$ is trivial. Then for all $i \in \mathbf{Z}$ we have a homotopy fiber sequence of spectra*

$$L^{[i-d]}(Z) \to L^{[i]}(X) \to L^{[i]}(X - Z).$$

For example, if $i = 0$ and $d = 1$, one obtains a homotopy fibration

$$U(Z) \to KO(X) \to KO(X - Z).$$

Proof: We just give the proof for $i = 0$, the other cases are similar. Using [Ho, Corollary 6.3] we obtain a homotopy fiber sequence $L^{[-d]}(Z) \to KO(X) \to KO(X - Z)$. Now apply [MV, Proposition 3.2.17] and real motivic Bott periodicity [Ho, Section 5]. □

We expect that work in progress of Schlichting [Sch2] will establish more general localization theorems for hermitian K-theory. Observe that if the normal bundle isn't trivial the statement in the proposition might fail. Indeed, looking at the negative homotopy groups of $KO(-)$ which are Balmer

Witt groups if X and Z are regular [HS, Corollary 7.5], one knows from the general localization and dévissage theorem for Witt groups [CH, Section 3] that to describe the objects corresponding to $L^{[-d]}(Z)$ in the associated long exact sequence one has not only to shift by the codimension but moreover to twist the duality by a suitable line bundle.

Next, we look at the coniveau spectral sequence for hermitian K-theory. As we do not have a duality functor on coherent O_X-modules, we apply the general framework of [CTHK] to KO (and $_-KO$, U and $_-U$) rather than trying to apply [Qu] directly to KO. This is why we assume that F is infinite from now on, although we believe that this assumption isn't necessary. We set $KO_{n,x}(X) := colim_{U \ni x} KO_{n,\bar{x}\cap U}(U)$, where $K_{n,Z}(X) := \pi_n(KO_Z(X))$ and $KO_Z(X)$ is the homotopy fiber of $KO(X) \to KO(X - Z)$.

Proposition 1.5 *Assume that X is a smooth variety over an infinite field F. Then using [CTHK, Remark 5.1.3 (3)] we obtain a strongly convergent coniveau spectral squence*

$$E_1^{p,q} = \oplus_{x \in X^{(p)}} KO_{-p-q,x}(X) \Rightarrow KO_{-p-q}(X)$$

with differentials $d_n : E_n^{pq} \to E^{p+n,q-n+1}$. This result holds more generally when replacing KO by $L^{[i]}$ for any $i \in \mathbf{Z}$.

Proof: According to [CTHK] we have to show that KO satisfies homotopy invariance and Nisnevich excision, which is true by [Ho, section 2]. □

The E_1-term of this spectral sequence consists of Cousin complexes, and by [Ho, Corollary 2.9]. we have the following.

Corollary 1.6 *The Cousin complex (see e.g. [CTHK, section 1]) in hermitian K-theory yields a resolution of the Zariski sheaf associated to $X \mapsto KO_n(X)$. In particular, if $X = Spec(R)$ is local and essentially smooth over F, then the augmented Cousin complex*

$$0 \to KO_n(R) \to KO_n(Frac(R)) \to \oplus_{x \in X^{(1)}} KO_{n-1,x}(X) \to ...$$

is exact. This result holds more generally when replacing KO by $L^{[i]}$ for any $i \in \mathbf{Z}$.

For example, the localization and dévissage theorem of [HS] for smooth affine curves mentioned above show that for a smooth affine curve X the Cousin complex for $_-U$-theory becomes $_-U_1(k(X)) \to \oplus_{x \in X^{(1)}} KO_0(k(x))$.

We will now establish a generalization of this to higher dimensional smooth varieties.

In order to identify the objects in the E_1-term of the coniveau spectral sequence with various $\oplus L_*^{[i]}(k(x))$, one might want to use the equivalence $M^p(X)/M^{p+1}(X) = \coprod_{x\in X^{(p)}} O_{X,x} - mod^{fl}$ of categories where $M^p(X)$ denotes coherent O_X-modules with support in codimension $\geq p$ and mod^{fl} means modules of finite length. Unfortunately, there is no duality on coherent O_X-modules, so one should consider complexes of those instead. For $n = 0$ this is possible even if $dim(X) > 1$ as in this case the triangulated machinery of Balmer applies, and some of the results of [BW] carry over from Witt groups to Grothendieck-Witt groups. For $n > 0$ we hope that this will follow from the work in progress of Schlichting mentioned above. Fur our purposes, the results above will be sufficient.

Theorem 1.7 *Let X be a smooth irreducible variety over an infinite field F. Then $E_1^{pq} \cong \oplus_{x\in X^{(p)}} L_{-p-q}^{[-p]}(X)$ in the coniveau spectral sequence for $KO(X)$. As n varies, the groups $E_1^{p-n,-p}$ together with the E_1-differentials assemble to a sequence*

$$KO_n(k(X)) \to \oplus_{x\in X^{(1)}} U_{n-1}(k(x)) \to \oplus_{x\in X^{(2)}}\ {}_{-}KO_{n-2}(k(x)) \to ...$$

which might be called the "hermitian Gersten complex". The Gersten conjecture is true for hermitian K-theory if X is local and essentially smooth over F, which means by definition that the augmented hermitian Gersten complex is exact in this case. Analoguous statements are true when replacing KO by ${}_{-}KO$, U or ${}_{-}U$. Moreover, the hermitian variant of Bloch's formula holds, namely $E_2^{p,-p} \cong \widetilde{CH}^p(X)$ in the coniveau spectral sequence for $L^{[p]}(X)$.

Proof: The first claim follows by combining Proposition 1.5 and Proposition 1.4 as the normal bundles are locally trivial. Using Corollary 1.6 one deduces that the Gersten conjecture holds. For the last claim, one has to show that the differentials d^J in the Gersten complex for J^* that are used in Definition 1.1(2) are compatible via the isomorphisms of Lemma 1.2 with the corresponding two boundary maps of the above hermitian Gersten complex. In both cases one may reduce to the one-dimensional local case by an argument similar to the one in the proof of [Qu, Proposition 5.14]. Now for the differentials from J^0 to J^{-1}, the claim follows from an explicit verification similar to [HS, 6.13, 6.15]. For the differentials from J^1 to J^0 one uses the explicit description of the boundary maps as in the proof of Proposition 1.8 below, which thus becomes a key lemma for this proof. □

One might want to consider the groups $E_2^{p,-p}$ also in the spectral sequence for $L^{[i]}$ if $i \neq p \; mod \; 4$ and call them "twisted oriented Chow groups".

In the case of smooth affine curves, on may work with the localization theorem of [HS], or simply with the localization exact sequences in low degrees of [K1], [K2].

Proposition 1.8 *Let X be a smooth affine curve over F. Then the term $E_2^{1,-1}$ of the coniveau spectral sequence for ${}_{-}U(X)$ is isomorphic to $\widetilde{CH}^1(X)$, and E_2^{00} of the coniveau spectral sequence for $KO(X)$ is isomorphic to $\widetilde{CH}^0(X)$.*

Proof: One has to check the condition in Theorem 1.7 on the compatibility of the differentials. By the universal property of J^0, this amounts to check that two maps from $V_0(k(X))$ to $\oplus W(k(x))$ resp. to $\oplus K_0(k(x))$ coincide. This follows as all these maps are essentially compositions of forgetful functors and boundary maps that may be described using projective resolutions of length 1. See [K2, p. 102] for an explicit description of the boundary map $d_K : V_0(k(X)) \to \oplus KO_0(k(x))$, and [BW, Section 8] for a proof that the differentials in the localization sequence for triangular Witt groups are in this case indeed given by locally the same construction, which is also known as the second residue map for Witt groups. Finally, one has to check that Karoubi's boundary map d_K does indeed coincide via the periodicity isomorphism $V_0 \cong {}_{-}U_1$ with the boundary map of Proposition 1.4 for $i = 1$. But this periodicity isomorphism is built in the motivic periodicity theorem used in the proof of Proposition 1.4, so we are reduced to compare d_K with the topological boundary map $V_0(k(X)) \to KO_0(k(x))$. That these are equal follows from the corresponding comparison result between the explicit and the topological boundary morphism $KO_0(k(X)) \to W(k(x))$ and from the commutative diagram of homotopy fibrations

$$\begin{array}{ccccc} V(k(X)) & \longrightarrow & KO(k(X)) & \longrightarrow & K(k(X)) \\ \downarrow & & \downarrow & & \downarrow \\ KO(k(X)) & \longrightarrow & BW(k(x)) & \longrightarrow & BQ(k(x)). \end{array}$$

Here BQ and BW are deloopings of K- resp. U-theory (see [Qu] resp. [HS]), the two right horizontal arrows are forgetful maps and the vertical arrows are the topological boundary morphisms considered above. □

We expect that there is a more elegant way to compare the differentials using the forthcoming results of [Sch2], which hopefully include a more general comparison statement between the "topological" boundary morphisms for hermitian K-theory or (Grothendieck-) Witt groups arising from localization homotopy fibrations and the boundary morphisms of Balmer's localization sequence [Ba] which use the cone construction in the triangular setting.

If $d = dim(X)$ is a multiple of 4, then the Euler class of [BM] for a given oriented bundle E over X is an element in $E_2^{d,-d}$ in the coniveau spectral sequence for KO. If this element survives to a non-zero element in the E_∞-term $KO_0(X)$, one may wonder if that elemement is related to the bundle E in some explicit way.

2 The homotopy coniveau filtration and higher oriented Chow groups

We start by recalling some results of Levine [Le] on the homotopy coniveau tower and the associated homotopy coniveau spectral sequence, sometimes using notation from loc. cit. without recalling their definition. The homotopy coniveau tower is defined for any presheaf E of S^1-spectra on Sm/F, and Levine's main interest is the case $E = K$ which leads to the motivic Atiyah-Hirzebruch spectral sequence.

Levine defines for all non-negative integers n and p a spectrum $E^{(p)}(X, n)$ which assemble to a simplicial spectrum $E^{(p)}(X, -)$ as n varies. He then defines $E^{(p/p+1)}(X, n)$ to be the homotopy cofiber of the map $E^{(p+1)}(X, n) \to E^{(p)}(X, n)$ and proves the following.

Theorem 2.1 *(Levine) Let E be a presheaf of spectra on Sm/F which is $\mathbf{A}^1$-invariant and N-connected for some integer N. Then there is a strongly convergent homotopy coniveau spectral sequence*

$$E_1^{pq} = \pi_{-p-q}(|E^{(p/p+1)}(X, -)|) \Rightarrow E_{-p-q}(X).$$

If moreover F is perfect and infinite and E satisfies Nisnevich excision, then $|E^{(p/p+1)}(X, -)| \simeq |E_{s.l.}^{(p/p+1)}(X, -)|$ where $E_{s.l.}^{(p/p+1)}(X, -)$ is a simplicial spectrum with $E_{s.l.}^{(p/p+1)}(X, n) \simeq \coprod_{x \in X^{(p)}(n)} (\Omega_T^p E)^{(0/1)}(k(x))$.

Proof: See [Le, Corollary 2.1.4, Corollary 5.3.2]. □

Under additional assumptions, the E_1-term may be described using higher Chow groups with coefficients in E (denoted $CH^*(X, E, -)$ in loc. cit.) because the spectra $(\Omega_T^p E)^{(0/1)}(k(x))$ that appear are then Eilenberg-Mac Lane spectra. Recall [Le, Remark 6.2.2] that for $E = K$ one obtains the usual higher Chow groups of Bloch, that is $CH^*(X, K, -) = CH^*(X, -)$.

Theorem 2.2 *(Levine) Assume that E is moreover well-connected (see [Le, Definition 6.1.1]). Then $(\Omega_T^p E)^{(0/1)}(k(x))$ is a $H\pi_0((\Omega_T^p E)(k(x)))$-spectrum for all $p \geq 0$. This implies that $E_1^{pq} \cong CH^p(X, E, -p-q)$.*

Proof: The first claim follows from the previous theorem and [Le, Lemma 6.1.3] The second claim follows from [Le, Definition 6.2.1] and [Le, Theorem 6.4.2]. (For specific examples of E one might try to give a more explicit description of the simplicial structure of $z^p(X, E, -)$ as Levine does for $E = K$ in the proof of [Le, Theorem 6.4.2]. Compare [FS1] for some results in this direction for $E = KO$.) □

Observe that by definition of the homotopy coniveau tower, the inclusion of zero simplices induces a morphism of towers (as p varies) of presheaves of spectra $E^{(p)}(-, 0) \to |E^{(p)}(-, -)|$ where $E^{(p)}(-, 0)$ is the tower showing up in the coniveau filtration. Consequently, we obtain a morphism between the coniveau spectral sequence and the homotopy coniveau spectral sequence. If $E = K$, this is a map from the Brown-Gersten-Quillen spectral sequence to the motivic Atiyah-Hirzebruch spectral sequence (beware that the latter one is usually stated after reindexing, compare e.g. [Le, proof of Theorem 11.3.2]).

Proposition 2.3 *There is a natural map between the Brown-Gersten-Quillen spectral sequence for algebraic K-theory [Qu] and the motivic Atiyah-Hirzebruch spectral sequence of [Le]. This map is nontrivial on the E_1-terms. In particular, one has the canonical projection $Z^p(X) \to CH^p(X)$ between the $E_1^{p,-p}$-terms.*

Proof: This follows from the discussion above. □

Now when replacing K by $L^{[0]} = KO$ (or more generally by $L^{[i]}$), the ideal picture would be the following: All the above theorems hold in the hermitian setting. In particular, there is a hermitian motivic Atiyah-Hirzebruch spectral sequence converging to KO in which the groups $E_1^{p,-p}$ are (twisted if $p \neq 0$ *mod* 4) oriented Chow groups. Consequently, the other groups

$CH^*(X, L^{[i]}, -)$ in the E_1-term deserve to be called "higher oriented Chow groups".

We do not know yet precisely to which extend this ideal picture reflects the reality. What we do know is that the presheaf of S^1-spectra KO satisfies both homotopy invariance and Nisnevich excision by [Ho, section 2], so Theorem 2.1 applies to the (-1)-connected version KO^c of hermitian K-theory. Moreover, the hyperbolic and forgetful functor induce maps between the corresponding spectral sequences for algebraic and hermitian K-theory.

What goes wrong when applying Levine's arguments to KO or KO^c is that KO isn't N-connected for any N, so [Le, Proposition 2.1.3(2)] concerning the strong convergence to $E_*(X)$ does not apply. One only has weak convergence to $\pi_*(holim \ldots \to E^{(0/p+1)}(X) \to E^{(0/p)}(X) \to \ldots)$ by (1) of loc. cit. On the other hand, KO^c does not satisfy Nisnevich descent. To see this, use e.g. a cover of $\mathbf{P}^1$ by two copies of $\mathbf{A}^1$ and that $KO_{-1}(\mathbf{P}^1) = W(k)$ and $KO_{-1}(\mathbf{A}^1) = 0$. In any case, neither KO nor KO^c is well-connected. To see this for KO^c, set $X = \mathbf{A}^1$ and $Z = \{0\}$ in [Le, Definition 6.1.1(1)] and use that $coker(KO_0(\mathbf{A}^1) \to KO_0(\mathbf{G}_m)) \cong W(k)$. Nevertheless, we have at least the following.

Lemma 2.4 *We have a natural map of presheaves of S^1-spectra $\Omega_T^p KO \to (\Omega_T KO)^{(0/1)}$. This map is surjective on π_0 for any finitely generated field extension of F, e. g. for all fields $k(x)$ showing up in the above spectral sequence. It induces a natural map $CH^p(X, KO, -p-q) \to E_1^{pq}$ from oriented higher Chow groups to the E_1-term of the homotopy coniveau spectral sequence for hermitian K-theory.*

Proof: The π_0-surjectivity follows from the fact that $\Omega_T^p KO \simeq (\Omega_T^p KO)^{(0)}$ by homotopy invariance of KO and from a variant of the localization argument of [Le, Lemma 6.1.3] together with the observation that $KO_{-1}(R) = W^1(R)$ for regular rings A by [HS, Corollary 7.5] and the fact that the latter group vanishes for semi-local R by [BP]. □

We now review some results of Levine relating the homotopy coniveau filtration to the slice filtration. As in loc. cit., for any abelian group A we denote the associated Eilenberg-MacLane spectrum (and the associated constant presheaf on Sm/F) by HA, and we denote by $\mathcal{H}\mathbf{Z}$ the motivic Eilenberg-Mac Lane spectrum representing motivic cohomology with integral coefficients. We consider this and other motivic spectra either as a $\mathbf{P}^1$-spectrum or as a $\mathbf{P}^1 - \Omega$-spectrum, compare the discussion in [Le, sections 8.1 and 8.2]. Levine proves the following.

Theorem 2.5 *(Levine) Let $\mathcal{E}$ be an $\mathbf{P}^1 - \Omega$-spectrum with 0-spectrum E. Then $E_1^{pq} = |E^{(p/p+1)}(X,-)| \simeq H_{mot}^{2p+q}(X, \pi_p^{\mu}(\mathcal{E}))$ where the functor $\pi_p^{\mu} : SH(k) \to DM(k)$ is defined in [Le, Definition 11.3.1].*

Proof: [Le, Theorem 11.3.2]. □

If $\mathcal{E} = \mathcal{K}$ is the motivic spectrum representing algebraic K-theory, then Levine concludes that $\pi_p^{\mu}(\mathcal{K}) \cong \mathbf{Z}(p)[p]$, and thus establishes the motivic Atiyah-Hirzebruch spectral sequence $E_1^{pq} = CH^p(X, -p-q) \Rightarrow K_{-p-q}(X)$. If KO was well-connected, then we would have the hermitian analogue $E_1^{pq} = \widetilde{CH}^p(X, -p-q) \Rightarrow KO_{-p-q}(X)$ where $\widetilde{CH}^p(X, -p-q) := CH^p(X, KO, -p-q)$. But as it is not, we do not have Theorem 2.2 but only Lemma 2.4 at our disposal to describe the E_1-term.

Still, what we do know using [Le, section 11] is that for $\mathcal{E} = \mathcal{KO}$ the $\mathbf{P}^1$-spectrum representing hermitian K-theory [Ho, section 5], we have that $E_1^{pq} = H_{mot}^{2p+q}(X, \pi_p^{\mu}(\mathcal{KO}))$ where $\pi_p^{\mu}(\mathcal{KO})$ is an object in $DM(k)$. If F is perfect, this object corresponds to the $\mathcal{H}\mathbf{Z}$-module $\sigma_0(\mathcal{KO})$, where σ_0 is the zero slice in Voeovdsky's slice filtration by [Le, Theorem 9.0.3]. Using real motivic Bott periodicity [Ho, section 5] and [Le, Remark 8.3.2, Lemma 11.3.3], the computation of the coefficients in the motivic cohomology groups of the E_1-term reduces to the computation of the four objects $\pi_0^{\mu}(\mathcal{L})$ for $\mathcal{L} = \mathcal{KO}, \mathcal{U}, {}_{-}\mathcal{KO}$ and ${}_{-}\mathcal{U}$ in $DM(k)$.

Proposition 2.6 *When applying the homotopy coniveau spectral sequence for KO to a smooth variety X, we have an isomorphism*

$$E_1^{pq} \cong H_{mot}^{3p+q}(X, \pi_0^{\mu}(\mathcal{L})(p))$$

where $\mathcal{L} = \mathcal{KO}, \mathcal{U}, {}_{-}\mathcal{KO}$ or ${}_{-}\mathcal{U}$, depending on p mod 4.

Proof: This follows from the discussion above. □

For instance, if KO was well-connected then putting all the above results together would imply that $\pi_0^{\mu}(\mathcal{KO})$ was the Nisnevich sheafification aKO_0 of the Grothendieck-Witt group presheaf $KO_0 = GW$.

The ideal picture mentioned above might also include the following.

Conjecture 2.7 *The canonical isomorphism $\oplus_{x \in X^{(p)}} J^0(k(x)) \xrightarrow{\simeq} \oplus_{x \in X^{(p)}(0)} \pi_0(\Omega_T^p L^{[p]}(k(x)))$ induces an isomorphism $\widetilde{CH}^p(X) \cong CH^p(X, L^{[p]}, 0)$ if $X \in Sm/F$ is of dimension at most p.*

Observe that the condition $dim(X) \leq p$ ensures that there is no J^{-1}-part in the complex defining $\widetilde{CH}^p(X)$. The proof of the conjecture amounts to show two things: first that the square

$$\begin{array}{ccc} \oplus_{x \in X^{(p-1)}} J^1(k(x)) & \xrightarrow{d^J} & \oplus_{x \in X^{(p)}} J^0(k(x)) \\ \downarrow & & \downarrow \cong \\ \oplus_{x \in X^{(p)}(1)} \pi_0(\Omega_T^p L^{[p]}(k(x))) & \xrightarrow{\delta_0 - \delta_1} & \oplus_{x \in X^{(p)}(0)} \pi_0(\Omega_T^p L^{[p]}(k(x))) \end{array}$$

is commutative, where the left arrow will be essentially given by the isomorphism of Lemma 1.2(2) composed with the boundary morphism of the long exact localization sequence of Proposition 1.4, and second that the images of the two horizontal arrows are equal. The corresponding proof for K instead of KO is essentially given by combining [Qu, Proposition 5.14] and [Le, Theorem 6.4.2].

Let us also make the trivial observation that if $X = Spec(F)$ a field, then (contrary to the isomorphism $CH^p(Spec(F), p) \cong K_p^M(F)$) there is no isomorphism $CH^p(Spec(F), KO, p) \cong K_p^{MW}(F)$ simply because the groups on the left vanish for negative n. Still, it seems conceivable that the Milnor-Witt groups $K_*^{MW}(F)$ (which are isomorphic to J^* by [M1, Theorem 6.3.4]) show up as $E_1^{p,-2p}$ in the homotopy conivau spectral sequence for hermitian K-theory and $Spec(F)$.

When considering Witt groups rather than hermitian K-theory, the coniveau spectral sequence as discussed in section 1 applied to the presheaf of S^1-spectra KT representing Witt groups [Ho, section 4] is conjeturally isomorphic to the Gersten-Witt spectral sequence of [BW]. As Witt groups are weight invariant (that is $(1,1)$-periodic), one might wonder about the relationship between this spectral sequence and the homotopy coniveau spectral sequence when applied to KT.

References

[Ba] P. Balmer: Triangular Witt groups. I. The 12-term localization exact sequence. K-Theory 19 (2000), no. 4, 311–363.

[BM] J. Barge and F. Morel: Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels. C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), no. 4, 287–29.

[BP] P. Balmer and R. Preeti: Shifted Witt groups of semi-local rings. Manuscripta Mathematica 117, no. 1 (2005), pp. 1–27.

[BW] P. Balmer and C. Walter: A Gersten-Witt spectral sequence for regular schemes. Annales Scientifiques de l'ENS. 35, no. 1 (2002), pp.127–152.

[CH] B. Calmes and J. Hornbostel: Witt motives, transfers and dévissage. Preprint 2006, available at `http://www.math.uiuc.edu/K-theory/`

[CTHK] J.-L. Colliot-Thélène, R. T. Hoobler and B. Kahn: The Bloch-Ogus-Gabber theorem. Algebraic K-theory (Toronto, ON, 1996), 31–94, Fields Inst. Commun., 16, Amer. Math. Soc., Providence, RI, 1997.

[Fa] J. Fasel: The oriented Chow ring. Preprint 2006, available at `http://www.math.uiuc.edu/K-theory/`

[FS1] J. Fasel and V. Srinivas: A vanishing theorem for oriented intersection multiplicities. Preprint 2007, available at `http://www.math.uiuc.edu/K-theory/`

[FS2] J. Fasel and V. Srinivas: Chow-Witt groups and Grothendieck-Witt groups of triangulated categories, in preparation.

[Ho] J. Hornbostel: A^1-Representability of hermitian K-theory and Witt groups. Topology 44 (2005), 661–687.

[HS] J. Hornbostel and M. Schlichting: Localization in Hermitian K-theory of rings. J. London Math. Soc. (2) 70 (2004), no. 1, 77–124.

[K1] M. Karoubi: Localisation de formes quadratiques I. Annales Sci. Ec. Norm. Sup (1974), 359–403.

[K2] M. Karoubi: Localisation de formes quadratiques II. Annales Sci. Ec. Norm. Sup (1975), 99–155.

[K3] M. Karoubi: Le théorème fondamental de la K-théorie hermitienne. Annals of Math. 112 (1980), 259–282.

[Le] M. Levine: The homotopy coniveau filtration. Preprint (revised version) 2005, available at `http://www.math.neu.edu/∼levine/mathindex.html`

[M1] F. Morel: On the motivic π_0 of the sphere spectrum. Axiomatic, Enriched and Motivic Homotopy Theorey, Proceedings of the 2002 Conference at the Newton Institute, 2004.

[M2] F. Morel: A^1-homotopy classification of vector bundles over smooth affine schemes. Preprint 2007, available at `http://www.mathematik.uni-muenchen.de/~morel/preprint.html`

[MV] F. Morel and V. Voevodsky: A^1-homotopy theory of schemes, Publ. Math. IHES no. 90 (1999), 45–143.

[Qu] D. Quillen: Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[Sch1] M. Schlichting: Matsumoto's theorem for quadratic forms. Preliminary version, available at `http://www.math.lsu.edu/~mschlich/research/prelim.html`

[Sch2] M. Schlichting: Hermitian K-theory of schemes and derived categories, in preparation.

[Vo1] V. Voevodsky: Motivic cohomology with $Z/2$-coefficients. Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 59–104.

[Vo2] V. Voevodsky: Open problems in the motivic stable homotopy theory. I. Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), 3–34, Int. Press. Lect. Ser., 3, I, Int. Press, Sommerville, MA 2002.

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