Universität Regensburg Mathematik

Hermitian vector bundles and extension groups on arithmetic schemes. I. Geometry of numbers

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Preprint Nr. 02/2007

HERMITIAN VECTOR BUNDLES AND EXTENSION GROUPS ON ARITHMETIC SCHEMES. I. GEOMETRY OF NUMBERS

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Abstract. We define and investigate extension groups in the context of Arakelov geometry. The "arithmetic extension groups" $\widehat{\mathrm{Ext}}^i_X(F,G)$ we introduce are extensions by groups of analytic types of the usual extension groups $\mathrm{Ext}^i_X(F,G)$ attached to $\mathcal{O}_X$-modules F and G over an arithmetic scheme X. In this paper, we focus on the first arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F,G)$ — the elements of which may be described in terms of admissible short exact sequences of hermitian vector bundles over X — and we especially consider the case when X is an "arithmetic curve", namely the spectrum $\mathrm{Spec}\,\mathcal{O}_K$ of the ring of integers in some number field K. Then the study of arithmetic extensions over X is related to old and new problems concerning lattices and the geometry of numbers.

Namely, for any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over $X := \mathrm{Spec}\,\mathcal{O}_K$, we attach a logarithmic *size* $\mathfrak{s}_{\overline{F},\overline{G}}(\alpha)$ to any element α of $\widehat{\mathrm{Ext}}^1_X(F,G)$, and we give an upper bound on $\mathfrak{s}_{\overline{F},\overline{G}}(\alpha)$ in terms of slope invariants of $\overline{F}$ and $\overline{G}$. We further illustrate this notion by relating the sizes of restrictions to points in $\mathbb{P}^1(\mathbb{Z})$ of the universal extension over $\mathbb{P}^1_{\mathbb{Z}}$ to the geometry of $PSL_2(\mathbb{Z})$ acting on Poincaré's upper half-plane, and by deducing some quantitative results in reduction theory from our previous upper bound on sizes. Finally, we investigate the behaviour of size by base change (*i.e.*, under extension of the ground field K to a larger number field K'): when the base field K is $\mathbb{Q}$, we establish that the size, which cannot increase under base change, is actually invariant when the field K' is an abelian extension of K, or when $\overline{F}^\vee \otimes \overline{G}$ is a direct sum of root lattices and of lattices of Voronoi's first kind.

The appendices contain results concerning extensions in categories of sheaves on ringed spaces, and lattices of Voronoi's first kind which might also be of independent interest.

MSC: Primary 14G40; Secondary 11H31, 11H55, 14F05, 18G15.

Contents

Date: January 11,2007.

0. Introduction

The aim of this paper is to introduce and to study *arithmetic extensions* and the *extension groups* they define in the framework of Arakelov geometry.

0.1. Arithmetic extensions are objects which arise naturally at various places in arithmetic geometry. Let X be an arithmetic scheme – namely a separated scheme of finite type over $\mathbb{Z}$ – such that $X_{\mathbb{C}}$ is smooth, and let $X(\mathbb{C})$ be the complex manifold of its complex points. By definition, for any two locally free coherent $\mathcal{O}_X$-modules F and G, an arithmetic extension $(\mathcal{E}, s)$ of F by G is given by an extension of $\mathcal{O}_X$-modules

$$\mathcal{E} : 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

together with a $\mathcal{C}^\infty$-splitting over $X(\mathbb{C})$

$$s : F_{\mathbb{C}} \longrightarrow E_{\mathbb{C}},$$

invariant under complex conjugation, of the extension of complex vector bundles over $X(\mathbb{C})$

$$\mathcal{E}_{\mathbb{C}} : 0 \longrightarrow G_{\mathbb{C}} \longrightarrow E_{\mathbb{C}} \longrightarrow F_{\mathbb{C}} \longrightarrow 0$$

deduced from $\mathcal{E}$ by extending the scalars from $\mathbb{Z}$ to $\mathbb{C}$.

Recall that an *hermitian vector bundle* $\overline{V} := (V, \|.\|)$ over X is the data of a locally free coherent sheaf V over X, together with a $\mathcal{C}^\infty$-hermitian metric $\|.\|$ on the attached vector bundle $V_{\mathbb{C}}$ on $X(\mathbb{C})$ that is invariant under complex conjugation. Arithmetic extensions arise for instance from *admissible extensions*

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow 0, \tag{0.1}$$

of hermitian vector bundles over X, namely from the data of an extension

$$\mathcal{E} : 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

of the underlying $\mathcal{O}_X$-modules such that the hermitian metrics $\|.\|_{\overline{F}}$ and $\|.\|_{\overline{G}}$ on $F_{\mathbb{C}}$ and $G_{\mathbb{C}}$ are induced (by restriction and quotients) by the metric $\|.\|_{\overline{E}}$ on $E_{\mathbb{C}}$. In this case, orthogonal projection determines a $\mathcal{C}^\infty$-splitting $s^\perp : F_{\mathbb{C}} \to E_{\mathbb{C}}$ of $\mathcal{E}_{\mathbb{C}}$, and $(\mathcal{E}, s^\perp)$ is an arithmetic extensions of F by G.

It turns out that, by means of the Baer sum construction, one may define an addition law on the set $\widehat{\operatorname{Ext}}^1_X(F,G)$ of isomorphism classes of arithmetic extensions of F by G, which in this way is endowed with a natural structure of an abelian group. Moreover, in analogy to the arithmetic Chow groups, the arithmetic extension group $\widehat{\operatorname{Ext}}^1_X(F,G)$ is an extension of the "classical" extension group $\operatorname{Ext}^1_{\mathcal{O}_X}(F,G)$, defined in the context of sheaves of $\mathcal{O}_X$-modules, by a group of analytic type. More precisely, it fits into an exact sequence

$$\operatorname{Hom}_{\mathcal{O}_X}(F,G) \longrightarrow \operatorname{Hom}_{\mathcal{C}^\infty_X}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} \xrightarrow{\ b\ } \widehat{\operatorname{Ext}}^1_X(F,G) \xrightarrow{\ \nu\ } \operatorname{Ext}^1_{\mathcal{O}_X}(F,G) \longrightarrow 0, \tag{0.2}$$

where F_∞ acts on $X(\mathbb{C})$, $F_{\mathbb{C}}$, and $G_{\mathbb{C}}$ by complex conjugation. We may also define an homomorphism

$$\Psi : \widehat{\operatorname{Ext}}^1_X(F,G) \longrightarrow Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)$$

to the group $Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)$ of F_∞-invariant $\overline{\partial}$-closed forms of type $(0,1)$ on $X(\mathbb{C})$ with coefficients in $F^\vee_{\mathbb{C}} \otimes G_{\mathbb{C}}$, by sending the class of an arithmetic extension $(\mathcal{E}, s)$ to its "second fundamental form" $\overline{\partial} s$.

The arithmetic extension group $\widehat{\mathrm{Ext}}^1(F,G)$ actually admits an interpretation in terms of homological algebra, in the spirit of the well-known identification of the "classical" extension group $\mathrm{Ext}^1_X(F,G)$, originally defined by classes of 1-extensions equipped with the Baer sum, with the "cohomological" extension group $\mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(F,G[1])$, defined as a group of morphisms in the derived category of (sheaves of) $\mathcal{O}_X$-modules over X . Indeed, if $(X_\mathbb{R},\mathcal{C}^\infty_\mathbb{R})$ denotes the ringed space quotient of $(X(\mathbb{C}),\mathcal{C}^\infty_{X(\mathbb{C})})$ by the action of complex conjugation (acting both on $X(\mathbb{C})$ and on values of C^∞-functions), and if

$$\rho : (X_\mathbb{R},\mathcal{C}^\infty_\mathbb{R}) \longrightarrow (X,\mathcal{O}_X)$$

is the natural map of ringed spaces, then, for any $\mathcal{O}_X$-module G on X, we may consider the adjunction map

$$\mathrm{ad}_G : G \longrightarrow \rho_*\rho^*G \tag{0.3}$$

— it maps any local section g of G to the section $g_\mathbb{C}$, seen as a $\mathcal{C}^\infty$-section of $G_\mathbb{C}$, invariant under the complex conjugation F_∞ — and its cone $C(\mathrm{ad}_G)$, namely (0.3) seen as complex of length 2, with G (resp. $\rho_*\rho^*G$) sitting in degree -1 (resp. 0). Then, for any two locally free coherent sheaves F and G on X, we have a natural isomorphism of abelian groups :

$$\widehat{\mathrm{Ext}}^1_X(F,G) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}\big(F,C(\mathrm{ad}_G)\big)$$

between our arithmetic extension group and the group of morphisms from F to $C(\mathrm{ad}_G)$ in the derived category $D(\mathcal{O}_X-\mathbf{mod})$ of the abelian category of sheaves of $\mathcal{O}_X$-modules over X (see 2.5 *infra*).

In a forthcoming part of this work, the above cohomological interpretation of $\widehat{\mathrm{Ext}}^1(F,G)$ will motivate us to consider higher arithmetic extension groups $\widehat{\mathrm{Ext}}^i(F,G)$, defined for any integer $i \geq 1$ by means of the Dolbeault complex $(A^{0,\cdot}_{X(\mathbb{C})},\overline{\partial})$ on $X(\mathbb{C})$ and its subcomplex $(A^{0,\cdot}_{X_\mathbb{R}},\overline{\partial})$ of conjugation invariant forms, which defines a complex of sheaves of modules on the ringed space $(X_\mathbb{R},\mathcal{C}^\infty_\mathbb{R})$.

For any $\mathcal{C}^\infty_\mathbb{R}$-module F on $X_\mathbb{R}$, we get the "Dolbeault resolution" $\mathcal{D}olb(F)$ of F by applying the functor $F\otimes_{\mathcal{C}^\infty_\mathbb{R}}$. to this complex. In particular, for any sheaf G of $\mathcal{O}_X$-modules, we may consider the associated sheaf ρ^*G of $\mathcal{C}^\infty_\mathbb{R}$-modules over $X_\mathbb{R}$, and the naive truncation $\mathcal{D}olb(\rho^*G)_{\leq i-1}$ of its Dolbeault resolution. The adjunction map (0.3) extends to a morphism of complexes

$$\mathrm{ad}^{i-1}_G : G \to \rho_*(\mathcal{D}olb(\rho^*G)_{\leq i-1}),$$

and its cone $C(\mathrm{ad}^{i-1}_G)$ is a complex of (sheaves of) $\mathcal{O}_X$-modules.

For any two $\mathcal{O}_X$-modules F and G, we shall define

$$\widehat{\mathrm{Ext}}^i_X(F,G) := \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}\big(F,C(\mathrm{ad}^{i-1}_G)[i-1]\big).$$

This group may be interpreted as an "hyper-extension group":

$$\mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}\big(F,C(\mathrm{ad}^{i-1}_G)[i-1]\big) \simeq \mathrm{Ext}^i_X\big(F,C(\mathrm{ad}^{i-1}_G)[-1]\big),$$

where, by the very definitions of the Dolbeault resolution and of a cone, the "shifted cone" $C(\mathrm{ad}^{i-1}_G)[-1]$ is the following complex of length $i+1$ of sheaves of $\mathcal{O}_X$-modules, with G sitting in degree 0:

$$0 \longrightarrow G \xrightarrow{-\mathrm{ad}_G} \rho_*\rho^*G \xrightarrow{-\overline{\partial}_G} \rho_*(\rho^*G\otimes_{\mathcal{C}^\infty_\mathbb{R}} A^{0,1}_{X_\mathbb{R}}) \xrightarrow{-\overline{\partial}_G} \cdots \xrightarrow{-\overline{\partial}_G} \rho_*(\rho^*G\otimes_{\mathcal{C}^\infty_\mathbb{R}} A^{0,i-1}_{X_\mathbb{R}}) \longrightarrow 0.$$

0.2. Classical constructions in algebraic and differential geometry provide natural instances of admissible and arithmetic extensions. In the second part of this paper [BK], we shall discuss three of these constructions, which give rise to the *arithmetic Atiyah extension*, the *arithmetic Hodge extension*, and the *arithmetic Schwarz extension*. To advocate the investigation of arithmetic extensions, we want to indicate briefly their constructions:

(i) Let $\overline{E}$ be an hermitian vector bundle on an arithmetic scheme X. The bundle of 1-jets of E induces an extension of $\mathcal{O}_X$-modules

$$0 \longrightarrow \Omega^1_{X/\mathbb{Z}} \otimes E \longrightarrow \mathcal{J}^1_{X/\mathbb{Z}}(E) \longrightarrow E \longrightarrow 0,$$

the Atiyah extension of E. The holomorphic vector bundle $E_{\mathbb{C}}$ carries a unique $\mathcal{C}^\infty$-connection which is compatible with the metric and the complex structure, its so-called Chern connection, which induces a $\mathcal{C}^\infty$-splitting s of the Atiyah extension and yields a canonical arithmetic extension class

$$\widehat{\mathrm{at}}(\overline{E}) \in \widehat{\mathrm{Ext}}^1_X(E, \Omega^1_{X/\mathbb{Z}} \otimes E).$$

It is a refinement both of the algebraic Atiyah class $\mathrm{at}(E) = \nu(\widehat{\mathrm{at}}(\overline{E}))$ in $\mathrm{Ext}^1(E, \Omega^1_{X/\mathbb{Z}} \otimes E)$ and of the curvature form of the Chern connection of $\overline{E}_{\mathbb{C}}$, which coincides with $\Psi(\widehat{\mathrm{at}}(\overline{E}))$ (up to some normalization factor). Applying a trace map to $\widehat{\mathrm{at}}(\overline{E})$, we get an arithmetic first Chern class in "arithmetic Hodge cohomology":

$$\hat{c}_1^H(\overline{E}) \in \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/\mathbb{Z}}).$$

(ii) Let $f : X \to Y$ be a smooth proper morphism of arithmetic schemes such that the Hodge to de Rham spectral sequence $E_1^{p,q} = R^p f_* \Omega^q_{X/Y} \Rightarrow R^{p+q} f_* \Omega^{\cdot}_{X/Y}$ degenerates at E_1. The spectral sequence defines the so-called Hodge extension

$$\mathrm{Hdg(X/Y)} : 0 \longrightarrow f_*\Omega^1_{X/Y} \longrightarrow R^1 f_* \Omega^{\cdot}_{X/Y} \longrightarrow R^1 f_* \mathcal{O}_X \longrightarrow 0 \tag{0.4}$$

whose interest was already advocated by Grothendieck in [Gro66]. Complex Hodge theory equips $\mathrm{Hdg}(X/Y)$ with a canonical structure of an arithmetic extension[1]. We thus obtain the class of the arithmetic Hodge extension

$$\widehat{\mathrm{Hdg}}(X/Y) \in \widehat{\mathrm{Ext}}^1_Y(R^1 f_* \mathcal{O}_X, f_*\Omega^1_{X/Y}).$$

(iii) Let $f : C \to X$ be a smooth, projective curve of genus $g \geq 2$ over an arithmetic scheme X. Using Deligne's definition of the torsor of projective connections on relative curves in [Del70], I.5[2], one obtains a canonical extension of $\mathcal{O}_X$-modules

$$\mathcal{S}_{C/X} \, : \, 0 \longrightarrow f_*\Omega^{\otimes 2}_{C/X} \longrightarrow S_{C/X} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

the splittings of which correspond to projective connections on C/X. Complex uniformization by the upper half-plane $\mathbb{H}$ induces a $\mathcal{C}^\infty$ projective connection on $C(\mathbb{C})/X(\mathbb{C})$ that is

[1] Namely, the vector bundle over $Y(\mathbb{C})$ defined by the relative algebraic de Rham cohomology $R^1 f_* \Omega^\bullet_{X/Y}$ may be identified with the relative first Betti cohomology with complex coefficients of $X(\mathbb{C})/Y(\mathbb{C})$; the complex conjugation on coefficients acts on Betti cohomology and maps the $\mathbb{C}$-analytic sub-vector bundle $(f_*\Omega^1_{X/Y})_{\mathbb{C}}$ of $(R^1 f_* \Omega^\bullet_{X/Y})_{\mathbb{C}}$ onto a $\mathcal{C}^\infty$ direct summand of $(f_*\Omega^1_{X/Y})_{\mathbb{C}}$, which provides a $\mathcal{C}^\infty$-splitting of the extension of $\mathbb{C}$-analytic vector bundles over $Y(\mathbb{C})$ defined by $\mathrm{Hdg}(X/Y)$.

[2] Strictly speaking, the definition in *loc. cit.* is stated in the framework of complex analytic spaces. However, it is formulated in a general geometric language, which makes it meaningful in the context of smooth relative curves over an arbitrary scheme.

holomorphic along the fibers — hence a $\mathcal{C}^\infty$-splitting of $\mathcal{S}_{C/X}$ over $Y(\mathbb{C})$ — and allows one to define from $\mathcal{S}_{C/X}$ the arithmetic Schwarz extension and its class

$$\widehat{\mathcal{S}}_{C/X} \in \widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, f_*\Omega^{\otimes 2}_{C/X}).$$

The non-vanishing of each of the above classes $\hat{c}_1^H(\overline{E})$, $\widehat{\mathrm{Hdg}}(X/Y)$, or $\widehat{\mathcal{S}}_{C/X}$ is an intriguing issue, related to deep problems in Diophantine geometry and transcendence theory.

0.3. In this paper, after introducing the arithmetic extension groups $\widehat{\mathrm{Ext}}^1_X(F,G)$ and discussing their basic properties in Section 2, we concentrate on the case where X is an "arithmetic curve", namely the spectrum $\operatorname{Spec}\mathcal{O}_K$ of the ring of integers in some number field K. It turns out that the study of arithmetic extensions over X is related to old and new problems concerning lattices and the geometry of numbers.

Namely, if F and G are vector bundles over $X := \operatorname{Spec}\mathcal{O}_K$ (*i.e.*, projective $\mathcal{O}_K$-modules), we obtain from the basic exact sequence (0.2) a canonical isomorphism

$$\widehat{\mathrm{Ext}}^1_X(F,G) \simeq \frac{\mathrm{Hom}_{\mathcal{O}_K}(F,G)\otimes_{\mathbb{Z}}\mathbb{R}}{\mathrm{Hom}_{\mathcal{O}_K}(F,G)}. \tag{0.5}$$

Consequently the arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F,G)$ carries a canonical structure of a real torus. Moreover, if F and G are equipped with hermitian metrics, which makes them hermitian vector bundles $\overline{F}$ and $\overline{G}$, we get an induced Riemannian metric on this real torus. In Section 3, we define the *size* $\mathfrak{s}_{\overline{F},\overline{G}}(\mathcal{E},s)$ of an arithmetic extension of F by G as the logarithm (in $[-\infty,+\infty[$) of the distance to zero of the corresponding point in the torus (0.5). Let $\overline{\mathcal{E}}$ be an admissible extension (0.1) with associated arithmetic extension $(\mathcal{E}, s^\perp)$ as above, and let

$$\varphi : E \xrightarrow{\sim} G\oplus F$$

be an isomorphism of $\mathcal{O}_K$-modules compatible with the extension $\mathcal{E}$ (that is, such that $\varphi^{-1}\circ(Id_G,0): G\to E$ and $\mathrm{pr}_2\circ\varphi : E\to F$ coincide with the morphisms defining $\mathcal{E}$). Then

$$\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\|\varphi_\sigma\|^2_{\overline{E}^\vee\otimes(\overline{G}\oplus\overline{F}),\sigma} = \frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\|\varphi_\sigma^{-1}\|^2_{(\overline{G}\oplus\overline{F})^\vee\otimes\overline{E},\sigma} \geq \mathrm{rk}_{\,\mathcal{O}_K}E,$$

and the minimum value achieved by the left-hand side when φ runs over all the isomorphisms of $\mathcal{O}_K$-modules as above is precisely

$$\mathrm{rk}_{\,\mathcal{O}_K}E + \exp(\,2\mathfrak{s}_{\overline{F},\overline{G}}(\mathcal{E},s^\perp))$$

(see Proposition 3.5.3 and Corollary 3.5.5 *infra*).

Motivated by analogous results concerning vector bundles on projective curves over a field, we show that the size of arithmetic extensions satisfies the following upper bound:

$$\mathfrak{s}_{\overline{F},\overline{G}}(\mathcal{E},s) \leq \widehat{\mu}_{\max}(\overline{F}) - \widehat{\mu}_{\min}(\overline{G}) + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log\frac{\mathrm{rk}_K F_K\cdot\mathrm{rk}_K G_K}{2}, \tag{0.6}$$

where $\widehat{\mu}_{\max}(\overline{F})$ and $\widehat{\mu}_{\min}(\overline{G})$ denote the maximal and minimal normalized slopes of $\overline{F}$ and $\overline{G}$ (see 3.1, *infra*), and Δ_K the discriminant of the number field K. To establish (0.6), we rely on (i) some upper bound on the Arakelov degree of a sub-line bundle in the tensor product of two hermitian vector bundles over $\operatorname{Spec}\mathcal{O}_K$, and (ii) some "transference theorem" from the geometry of numbers, which relates the inhomogeneous minimum (also called the covering radius) of a lattice in a euclidean vector space to the first of the successive minima of the dual lattice.

Section 4 is devoted to further examples and applications of the notion of size. In particular, using the inequality (0.6), we derive an avatar, in the framework of Arakelov geometry over arithmetic curves, of the main result of the classical reduction theory of positive quadratic forms. It claims the existence of some "almost-splitting" for any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathcal{O}_K$, namely the existence of $n := \operatorname{rk} E$ hermitian lines bundles $\overline{L}_1, \ldots,$ $\overline{L}_n$ over $\operatorname{Spec}\mathcal{O}_K$, and of an isomorphism of $\mathcal{O}_K$-modules

$$\phi : E \xrightarrow{\sim} \bigoplus_{i=1}^{n} L_i$$

such that the archimedean norms of ϕ and ϕ^{-1}, computed by using the hermitian structures on $\overline{E}$ and on the orthogonal direct sum $\bigoplus_{i=1}^{n} \overline{L}_i$, are bounded in terms of K and n only (Theorem 4.3.1 *infra*).

Besides, for any rational point $P \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$, we calculate the size of the inverse image $P^*\overline{\mathcal{E}}$ of the universal extension

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{S} \longrightarrow \overline{\mathcal{O}_X^{\oplus 2}} \longrightarrow \overline{\mathcal{O}_X(1)} \longrightarrow 0$$

over the projective line $X = \mathbb{P}^1_{\mathbb{Z}}$ equipped with its natural structure of an admissible extension. The extension class of $P^*\overline{\mathcal{E}}$ is trivial iff $P \in \{0, \infty\}$. For $P \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, \infty\}$, we show that the size of $P^*\overline{\mathcal{E}}$ is related to the usual height $h(P)$ of P by the inequalities

$$-\frac{1}{2}\log 2 + h(P) \leq \mathfrak{s}(P^*\overline{\mathcal{E}}) \leq -\log 2 + 2\,h(P).$$

We also give a geometric description of the size $\mathfrak{s}(P^*\overline{\mathcal{E}})$ by means of so-called Ford circles (namely the images under elements in $SL_2(\mathbb{Z})$ of the horocycles $\{\operatorname{Im} z = 1\}$ in Poincaré's upper half-plane).

The final section of this paper is devoted to the intriguing question of the invariance of size under base change. Recall that an extension of number fields K'/K defines a morphism $g : \operatorname{Spec}\mathcal{O}_{K'} \to \operatorname{Spec}\mathcal{O}_K$ of "arithmetic curves". For hermitian vector bundles $\overline{F}$ and $\overline{G}$ over S, there is an induced morphism

$$g^* : \widehat{\operatorname{Ext}}^1_S(F, G) \longrightarrow \widehat{\operatorname{Ext}}^1_{S'}(g^*F, g^*G).$$

It is easy to see that the inequality

$$\mathfrak{s}_{g^*\overline{F}, g^*\overline{G}}(g^*e) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e) \tag{0.7}$$

holds for every extension class $e \in \widehat{\operatorname{Ext}}^1_S(F, G)$. Motivated again by geometric considerations, we ask – at least if K is the field $\mathbb{Q}$ – wether the size of extensions of $\overline{F}$ by $\overline{G}$ is invariant under the base change g, namely whether the inequality (0.7) is indeed an equality for any extension class $e \in \widehat{\operatorname{Ext}}^1_S(F, G)$.

Let $\overline{E}$ denote the hermitian vector bundle $\overline{F}^\vee \otimes \overline{G}$ over $\operatorname{Spec}\mathcal{O}_K$. The extension of scalars $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$ defines a natural $\mathbb{R}$-linear map

$$\Delta : E_{\mathbb{R}} = E \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow (g^*E)_{\mathbb{R}} = (E \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\mathcal{V}(\overline{E}) \subseteq E_{\mathbb{R}}$ denote the Voronoi cell of the euclidean lattice $E \subseteq E_{\mathbb{R}}$ underlying $\overline{E}$. Then the size of extensions of $\overline{F}$ by $\overline{G}$ is invariant under the base change g if and only if

$$\Delta\big(\mathcal{V}(\overline{E})\big) \in \mathcal{V}(g^*\overline{E}). \tag{0.8}$$

Clearly (0.8) holds iff Δ maps the set of vertices of the polytope $\mathcal{V}(\overline{E})$ to $\mathcal{V}(g^*\overline{E})$.

Here are some results which point towards a positive answer to our question in the case where the base field is $\mathbb{Q}$. Hence assume $K = \mathbb{Q}$, put $L = K'$, and define $\overline{E}$ as above. Then we show that (0.7) is an equality for any extension class $e \in \widehat{\mathrm{Ext}}^1_S(F, G)$ if either

(i) $L/\mathbb{Q}$ is an abelian extension, or

(ii) $\overline{E}$ is an orthogonal direct sum of hermitian line bundles, or

(iii) $\overline{E}$ is a root lattice, or

(iv) $\overline{E}$ is a lattice of Voronoi's first kind (hence in particular if $\mathrm{rk}_{\mathbb{Z}} E \leq 3$).

We use condition (0.8) to prove these results. For abelian extensions, we reduce to the cyclotomic case and use some auxiliary results of Kitaoka, which he established when investigating minimal vectors in tensor products of euclidean lattices. Using the elementary inequality

$$\sum_{\sigma: L \hookrightarrow \mathbb{C}} |\sigma(\alpha)|^2 - \sum_{\sigma: L \hookrightarrow \mathbb{C}} \mathrm{Re}\, \sigma(\alpha) \geq \sum_{\sigma: L \hookrightarrow \mathbb{C}} |\sigma(\alpha)|^2 - \sum_{\sigma: L \hookrightarrow \mathbb{C}} |\sigma(\alpha)| \geq 0$$

satisfied by any integral element $\alpha \in \mathcal{O}_L$, we show that (0.8) holds when $\overline{E}$ has rank one, and consequently when it splits as a direct sum of hermitian line bundles. Our proof for root lattices relies on the computation of the vertices of the Voronoi cells of the irreducible root lattices A_n, D_n, E_6, E_7, and E_8 by Conway and Sloane ([CS99], Chapter 21). Our treatment of lattices of Voronoi's first kind uses the description of the Voronoi cell of an euclidean lattice with strictly obtuse superbase which is given in Appendix B.

Finally, as a consequence of our "reduction theorem" and of case (ii), we show, in the case where the base field K is $\mathbb{Q}$, that equality holds in (0.7) "up to some constant". Namely, we derive the existence of a non-negative real constant $c(\mathrm{rk}\, F, \mathrm{rk}\, G)$ — depending on the ranks of F and G only — such that the inequality

$$\mathfrak{s}_{\overline{F},\overline{G}}(e) \leq \mathfrak{s}_{g^*\overline{F}, g^*\overline{G}}(g^* e) + c(\mathrm{rk}\, F, \mathrm{rk}\, G)$$

holds for any class $e \in \widehat{\mathrm{Ext}}^1_S(F, G)$.

Appendix A gathers "well known" facts concerning extension groups of sheaves of modules. In particular, it specifies sign conventions which enter in the construction of canonical isomorphisms between variously defined extension groups.

Appendix B contains a self-contained presentation of lattices of Voronoi's first kind, a description of their Voronoi cells, and various facts concerning these lattices which might be of independent interest.

0.4. The starting points of this paper have been, in 1998, (i) the observation that, for any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over an arithmetic curve X, the set of isomorphism classes of admissible extensions of $\overline{F}$ by $\overline{G}$ becomes an abelian group when the Baer sum of two admissible extensions is equipped with the hermitian structure defined by formula (2.37) *infra*, and (ii) Grothendieck's remark in [Gro66] on the non-trivial information encoded in the extension class of the Hodge extension (0.4).

Related ideas have been investigated in Mochizuki's preprints [Moc99]. Let us emphasize a major difference between his approach and ours: Mochizuki thinks of the Hodge extension in the context of Arakelov geometry as some kind of *non-linear* geometric object, while we see it as an element of some naturally defined *abelian* extension *group*. Moreover, Mochizuki's earlier work [Moc96] has been an inspiration for considering the arithmetic Schwarz extension.

Let us finally indicate that, in [CLT01], Chambert-Loir and Tschinkel have defined and investigated "arithmetic torsors" under some group scheme G on an arithmetic scheme X, at least when G is deduced by base change from a group scheme over an "arithmetic curve". Their definition easily extends to the case of general smooth affine group schemes over X, and specialized to vector groups of the form $\check{E} \otimes F$, where E and F are vector bundles over X, is equivalent to our definition of arithmetic extensions of E by F (see 2.7, *infra*).

It is a pleasure to thank E. Bayer for helpful remarks on euclidean lattices, and R. Bost for his help in the preparation of the figure. We are grateful to the TMR network 'Arithmetic geometry' and the DFG-Forschergruppe 'Algebraische Zykel und L-Funktionen' for their support and to the universities of Regensburg and Paris (6,7,11,13) for their hospitality.

1. Preliminaries

1.1. **Arithmetic schemes.** We work over an *arithmetic ring* $R = (R, \Sigma, F_\infty)$ in the sense of Gillet and Soulé, [GS90, 3.1.1]. Recall that this means that R is an excellent regular noetherian integral domain, Σ is a finite nonempty set of monomorphisms from R to $\mathbb{C}$, and F_∞ is a conjugate-linear involution of $\mathbb{C}$-algebras $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ such that $F_\infty \circ \delta = \delta$ for the canonical map $\delta : R \to \mathbb{C}^\Sigma = \prod_{\sigma\in\Sigma} \mathbb{C}$.

Let S be the spectrum of an arithmetic ring R, and K its field of fractions. An *arithmetic scheme*[3] X over R is a separated S-scheme X of finite type such that each base change $X_\sigma = X \times_{R,\sigma} \mathbb{C}$, $\sigma \in \Sigma$, is smooth over $\operatorname{Spec}\mathbb{C}$ (or equivalently such that X_K is smooth over K). For σ in Σ, we write $X_\sigma = X \otimes_{R,\sigma} \mathbb{C}$. We obtain a scheme

$$X_\Sigma = X \otimes_{R,\delta} \mathbb{C}^\Sigma = \coprod_{\sigma\in\Sigma} X_\sigma$$

and a complex manifold

$$X_\Sigma(\mathbb{C}) = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C}).$$

We write $X(\mathbb{C})$ instead of $X_\Sigma(\mathbb{C})$ if $\Sigma = \{\sigma : R \hookrightarrow \mathbb{C}\}$.

1.1.1. The most prominent example of an arithmetic ring is $\mathcal{O}_K = (\mathcal{O}_K, \Sigma, F_\infty)$ where $\mathcal{O}_K$ is the ring of integers in a number field K, Σ is the set of complex embeddings $\sigma : K \hookrightarrow \mathbb{C}$, and F_∞ is given by

$$F_\infty : \mathbb{C}^\Sigma \longrightarrow \mathbb{C}^\Sigma \ , \ (z_\sigma)_{\sigma\in\Sigma} \mapsto (\overline{z_{\overline{\sigma}}})_{\sigma\in\Sigma}.$$

Then an arithmetic scheme over $\mathcal{O}_K$ is precisely a separated $\mathbb{Z}$-scheme X of finite type such that $X_\mathbb{Q}$ is smooth, equipped with a scheme morphism to $\operatorname{Spec}\mathcal{O}_K$, and $X_\Sigma(\mathbb{C})$ is the complex manifold $X(\mathbb{C})$ of all complex points of X.

[3] We use the terminology *arithmetic scheme* for what is called an *arithmetic variety* in [GS90] and subsequent papers by Gillet and Soulé, in order to avoid confusion with quotients of symmetric domains by the action of arithmetic groups.

1.1.2. There are natural morphisms of locally ringed spaces

$$j : (X_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X_\Sigma}) \longrightarrow (X, \mathcal{O}_X)$$

where $\mathcal{O}^{\mathrm{hol}}_{X_\Sigma}$ denotes the sheaf of holomorphic functions on the complex manifold $X_\Sigma(\mathbb{C})$ and

$$\kappa : (X_\Sigma(\mathbb{C}), \mathcal{C}^\infty_{X_\Sigma}) \longrightarrow (X_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X_\Sigma})$$

where $\mathcal{C}^\infty_{X_\Sigma}$ denotes the sheaf of complex valued smooth functions. The morphism j is flat by [SGA03, Exp. XII]. To any $\mathcal{O}_X$-module F on X is associated an $\mathcal{O}^{\mathrm{hol}}_{X_\Sigma}$-module $F^{\mathrm{hol}}_{\mathbb{C}} = j^*F$ on $X_\Sigma(\mathbb{C})$ and an $\mathcal{C}^\infty_{X_\Sigma}$-module $F_{\mathbb{C}} = \kappa^* j^* F$. The so-defined functor $F \longmapsto F_{\mathbb{C}}$ is exact, as a consequence of following Lemma:

Lemma 1.1.3. *The morphism κ is flat, i.e. $\mathcal{C}^\infty_{X_\Sigma,p}$ is a flat $\mathcal{O}^{\mathrm{hol}}_{X_\Sigma,p}$-module for each p in $X_\Sigma(\mathbb{C})$.*

Proof. We consider for $n \geq 0$ the $\mathbb{R}$-algebra $\mathcal{O}_{\mathbb{R}^{2n},0}$ (resp. $\mathcal{E}_{\mathbb{R}^{2n},0}$) of germs of real analytic (resp. real valued $\mathcal{C}^\infty$) functions around 0 in $\mathbb{R}^{2n}$, and the $\mathbb{C}$-algebra $\mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n,0}$ of germs of holomorphic functions around 0 in $\mathbb{C}^n$. The canonical map from $\mathcal{O}_{\mathbb{R}^{2n},0}$ to $\mathcal{E}_{\mathbb{R}^{2n},0}$ is flat by [Tou72, VI Cor. 1.3]. We have $\mathcal{E}_{\mathbb{R}^{2n},0} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{C}^\infty_{\mathbb{C}^n,0}$ under the canonical identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Therefore κ is flat if we show that $\mathcal{O}_{\mathbb{R}^{2n},0} \otimes_{\mathbb{R}} \mathbb{C}$ is flat over $\mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n,0}$. This can be checked on completions (which are faithfully flat). We have

$$\mathcal{O}_{\widehat{\mathbb{R}^{2n},0 \otimes_{\mathbb{R}}}} \mathbb{C} = \mathbb{C}[[z_1, .., z_n, \overline{z}_1, .., \overline{z}_n]]$$

and $\widehat{\mathcal{O}^{\mathrm{hol}}_{\mathbb{C}^n,0}} = \mathbb{C}[[z_1, .., z_n]]$. Our claim follows. □

1.1.4. Let F_∞ denote the anti-holomorphic involution of the complex manifold $X_\Sigma(\mathbb{C})$ which maps $s : \operatorname{Spec} \mathbb{C} \to X$ to the composition of complex conjugation in $\mathbb{C}$ with s. We obtain an induced $\mathbb{C}$-antilinear involution

$$F_\infty : A^k(X_\Sigma(\mathbb{C}), \mathbb{C}) \longrightarrow A^k(X_\Sigma(\mathbb{C}), \mathbb{C}) \ , \ \alpha \mapsto \overline{F^*_\infty(\alpha)}$$

on the space of smooth complex valued k-forms on $X_\Sigma(\mathbb{C})$. One checks easily that this map is $\mathbb{C}$-anti-linear and $\Gamma(X, \mathcal{O}_X)$-linear. Furthermore it respects the (p,q)-type and commutes with d, ∂, and $\overline{\partial}$.

1.1.5. For any $\mathcal{O}_X$-module F on X, we consider the sheaf

$$A^k(\_, F) := F^{\mathrm{hol}}_{\mathbb{C}} \otimes_{\mathcal{O}^{\mathrm{hol}}_{X_\Sigma}} A^k(\_, \mathbb{C}) = F_{\mathbb{C}} \otimes_{\mathcal{C}^\infty_{X_\Sigma}} A^k(\_, \mathbb{C})$$

on $X_\Sigma(\mathbb{C})$. It may be decomposed according to types:

$$A^k(\_, F) = \bigoplus_{p+q=k} A^{p,q}(\_, F) \tag{1.1}$$

where, for any two non-negative integers p and q:

$$A^{p,q}(\_, F) := F^{\mathrm{hol}}_{\mathbb{C}} \otimes_{\mathcal{O}^{\mathrm{hol}}_{X_\Sigma}} A^{p,q}(\_, \mathbb{C}) = F_{\mathbb{C}} \otimes_{\mathcal{C}^\infty_{X_\Sigma}} A^{p,q}(\_, \mathbb{C}).$$

The space of sections $A^k(X_\Sigma(\mathbb{C}), F)$ is endowed with the $\mathbb{C}$-antilinear involution F_∞ (which specializes to the one considered above when $F = \mathcal{O}_X$), defined by complex conjugation, acting both on $X_\Sigma(\mathbb{C})$ and on the coefficients (namely, k-forms and fibers of $F_{\mathbb{C}}$).

This involution is compatible with the decomposition into types (1.1) and with the Dolbeault operator. We define

$$A^k(X_{\mathbb{R}}, F) = A^k(X_\Sigma(\mathbb{C}), F)^{F_\infty} \text{ and } A^{p,q}(X_{\mathbb{R}}, F) = A^{p,q}(X_\Sigma(\mathbb{C}), F)^{F_\infty}$$

and we obtain an induced Dolbeault operator

$$\overline{\partial}_F : A^{p,q}(X_{\mathbb{R}}, F) \longrightarrow A^{p,q+1}(X_{\mathbb{R}}, F).$$

Its kernel will be denoted $Z^{p,q}_{\overline{\partial}}(X_{\mathbb{R}}, F)$, and the p-th cohomology group

$$Z^{0,p}_{\overline{\partial}}(X_{\mathbb{R}}, F)/\overline{\partial}_F(A^{0,p-1}(X_{\mathbb{R}}, F))$$

of the Dolbeault complex $(A^{0,\cdot}(X_{\mathbb{R}}, F), \overline{\partial}_F)$, will be denoted $H^p_{\mathrm{Dolb}}(X_{\mathbb{R}}, F)$. The Dolbeault isomorphism (see Appendix A.5.1)

$$\mathrm{Dolb}_{F^{\mathrm{hol}}_{\mathbb{C}}} : H^p(X_\Sigma(\mathbb{C}), F^{\mathrm{hol}}_{\mathbb{C}}) \longrightarrow H^p_{\mathrm{Dolb}}(X_\Sigma(\mathbb{C}), F^{\mathrm{hol}}_{\mathbb{C}}) := H^p(X_\Sigma(\mathbb{C}), \mathcal{D}olb(F^{\mathrm{hol}}_{\mathbb{C}}))$$

yields an isomorphism

$$\mathrm{Dolb}_{F_{\mathbb{R}}} : H^p(X_\Sigma(\mathbb{C}), F^{\mathrm{hol}}_{\mathbb{C}})^{F_\infty} \longrightarrow H^p_{\mathrm{Dolb}}(X_{\mathbb{R}}, F)$$

between $H^p_{\mathrm{Dolb}}(X_{\mathbb{R}}, F)$ and the real vector subspace in the cohomology group $H^p(X_\sigma(\mathbb{C}), F^{\mathrm{hol}}_{\mathbb{C}})$ of elements invariant under complex conjugation (acting both on $X_\Sigma(\mathbb{C})$ and on coefficients).

We shall also denote

$$\mathrm{Dolb}_{F_{\mathbb{R}}} : H^p(X, F) \longrightarrow H^p_{\mathrm{Dolb}}(X_{\mathbb{R}}, F)$$

the composition of the above isomorphism and of the canonical map $j^* : H^p(X, F) \to H^p(X_\sigma(\mathbb{C}), F^{\mathrm{hol}}_{\mathbb{C}})^{F_\infty}$ defined by pulling back through the morphism of ringed spaces $j : (X_\Sigma(\mathbb{C}), \mathcal{O}^{\mathrm{hol}}_{X_\Sigma}) \to (X, \mathcal{O}_X)$. More generally, if E and F are $\mathcal{O}_X$-modules such that $E^{\mathrm{hol}}_{\mathbb{C}}$ is a locally free of finite rank (*i.e.*, a holomorphic vector bundle), the base change by j and the Dolbeault isomorphism $\mathrm{Dolb}_{E_{\mathbb{C}}, F_{\mathbb{C}}}$ (*cf.* A.5.1) define a map

$$\mathrm{Dolb}_{E_{\mathbb{R}}, F_{\mathbb{R}}} : \mathrm{Ext}^p_{\mathcal{O}_X}(E, F) \longrightarrow H^p_{\mathrm{Dolb}}(X_{\mathbb{R}}, E^\vee \otimes F).$$

Let $\Omega^r_{X/S}$ denote the sheaf of r-th relative Kähler differentials of X over S. Then $(\Omega^r_{X/S})^{\mathrm{hol}}_{\mathbb{C}}$ is the sheaf of holomorphic r-forms on $X_\Sigma(\mathbb{C})$. This allows a natural identification

$$\nu : A^{p,q}(X_\Sigma(\mathbb{C}), F) \xrightarrow{\sim} A^{0,q}(X_\Sigma(\mathbb{C}), F \otimes \Omega^p_{X/S}) \ , \ f \otimes (\alpha \wedge \beta) \mapsto (f \otimes \alpha) \otimes \beta$$

for differential forms α and β of type $(p, 0)$ and $(0, q)$ respectively. Observe that the space $A^{p,p}(X_{\mathbb{R}}, \mathcal{O}_X)$ does *not* coincide with the space $A^{p,p}(X_{\mathbb{R}})$, considered in [GS90, 3.2.1], of real (p, p)-forms α on $X_\Sigma(\mathbb{C})$ satisfying $F^*_\infty \alpha = (-1)^p \alpha$. Instead, we have an embedding

$$A^{p,p}(X_{\mathbb{R}}) \hookrightarrow A^{p,p}(X_{\mathbb{R}}, \mathcal{O}_X) \simeq A^{0,p}(X_{\mathbb{R}}, \Omega^p_{X/S}) \ , \ \alpha \mapsto (-2\pi i)^p \nu(\alpha),$$

the image of which is the $(-1)^p$-eigenspace of the involution on $A^{p,p}(X_{\mathbb{R}}, \mathcal{O}_X)$ defined by complex conjugation acting on coefficients only.

1.2. **Hermitian coherent sheaves.** Let X be an arithmetic scheme. A vector bundle on X is a locally free $\mathcal{O}_X$-module E of finite rank. The dual vector bundle $\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is denoted by $E^\vee$. Following [GS92, Def. 25] we define a *hermitian coherent sheaf* $\overline{E}$ on X as a pair (E, h) consisting of a coherent $\mathcal{O}_X$-module E whose restriction to the generic fiber X_F is locally free, together with a F_∞-invariant $\mathcal{C}^\infty$-hermitian metric h on the holomorphic vector bundle $E_{\mathbb{C}}^{\mathrm{hol}}$. A *hermitian vector bundle* on X is a hermitian coherent sheaf whose underlying coherent $\mathcal{O}_X$-module is locally free. There are natural hermitian structures on tensor products, exterior powers, and inverse images of hermitian coherent sheaves, and on the dual of hermitian vector bundles.

Observe also that, if $\overline{E}$ and $\overline{F}$ are two hermitian vector bundles over X, then the canonical isomorphism $\mathcal{H}om_{\mathcal{O}_X}(E, F) \simeq E^\vee \otimes F$ allows us to equip $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ with a structure of hermitian vector bundle, which makes it canonically isomorphic with $\overline{E}^\vee \otimes \overline{F}$. For any section T of $\mathcal{H}om_{\mathcal{O}_X}(E, F)_{\mathbb{C}}$ and any $x \in X_\Sigma(\mathbb{C})$, the so-defined norm $\|T(x)\|_{\overline{E}^\vee \otimes \overline{F}}$ is the Hilbert-Schmidt norm of the $\mathbb{C}$-linear map between the hermitian vector spaces $(E_x, \|.\|_{\overline{E}})$ and $(F_x, \|.\|_{\overline{F}})$. Occasionally we shall also use the operator norm of such maps, and when confusion may arise, we shall denote $\|T(x)\|_\infty$ or $\|T(x)\|^\infty$ the latter, and $\|T(x)\|_{HS}$ the former.

1.2.1. *Direct image.* Let $f : Y \to X$ be a finite flat morphism of arithmetic varieties such that $f_F : Y_F \to X_F$ is étale — or equivalently, such that $f_\Sigma : Y_\Sigma \to X_\Sigma$ is an étale covering — then, for any hermitian coherent sheaf $\overline{E}$ over Y, we may consider its direct image $f_*\overline{E}$, namely the hermitian coherent sheaf on X defined by the coherent sheaf f_*E equipped with the hermitian structure which, for any $x \in X_\Sigma(\mathbb{C})$, is given on the fiber

$$(f_*E)_x \simeq \bigoplus_{y \in f_\Sigma^{-1}(x)} E_y$$

by the direct sum of the hermitian structures on the E_y, $y \in f_\Sigma^{-1}(x)$.

1.2.2. For $f : Y \to X$ as above and hermitian coherent sheaves $\overline{E}$ on Y and $\overline{F}$ on X, adjunction defines a natural morphism of $\mathcal{O}_Y$-modules

$$f^*(f_*E \otimes_{\mathcal{O}_X} F) = (f^* f_* E) \otimes_{\mathcal{O}_Y} f^*F \longrightarrow E \otimes_{\mathcal{O}_Y} f^*F.$$

By adjunction it induces a canonical isomorphism of $\mathcal{O}_X$-modules

$$f_*(\overline{E}) \otimes_{\mathcal{O}_X} \overline{F} \xrightarrow{\sim} f_*(\overline{E} \otimes_{\mathcal{O}_Y} f^*\overline{E})$$

which is an isometry as a direct consequence of our definitions.

1.2.3. Let

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{f} & X \end{array}$$

be a cartesian square of arithmetic varieties. Let $\overline{E}$ be a hermitian coherent sheaf on Y and assume that f is as in 1.2.1. By adjunction we obtain a natural morphism of $\mathcal{O}_X$-modules

$$h'^* f^* h_* E = f'^* h^* h_* E \longrightarrow f'^* E$$

which induces a canonical isomorphism of $\mathcal{O}_X$-modules

$$f^* h_* \overline{E} \xrightarrow{\sim} h'_* f'^* \overline{E}.$$

The latter is an isometry as a direct consequence of our definitions.

1.3. **Extensions.** We briefly recall some basic facts concerning 1-extensions of sheaves of modules. For more details and references, we refer the reader to Appendix A.

Let F, G denote $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. An *extension of* F *by* G is a short exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{E} : 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0.$$

A morphism of extensions

$$(\alpha, \beta, \gamma) : \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \tag{1.2}$$

is given by a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{E}_1 : 0 & \longrightarrow & G_1 & \longrightarrow & E_1 & \longrightarrow & F_1 & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ \mathcal{E}_2 : 0 & \longrightarrow & G_2 & \longrightarrow & E_2 & \longrightarrow & F_2 & \longrightarrow & 0. \end{array}$$

Recall that two extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ of F by G are called *isomorphic* iff there exists a morphism (1.2) as above with $\alpha = \mathrm{id}_G$ and $\gamma = \mathrm{id}_F$.

Given an extension $\mathcal{E}$ as above, we consider the boundary map

$$\mathrm{Hom}_{\mathcal{O}_X}(F, F) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{O}_X}(F, G)$$

where $\mathrm{Ext}^p_{\mathcal{O}_X}(F, \,.\,)$ denotes the p-th right derived functor of $\mathrm{Hom}_{\mathcal{O}_X}(F, \,.\,)$. It is well known (compare [Wei94, Th. 3.4.3], [Har77, Ex. III 6.1], or Proposition A.4.5) that

$$\left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{extensions of } F \text{ by } G \end{matrix} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(F, G) \ , \ [\mathcal{E}] \mapsto \partial(\mathrm{id}_F) \tag{1.3}$$

defines a group isomorphism if we equip the left-hand side with the group structure induced by the Baer sum of extensions. Recall that the Baer sum of two extensions

$$\mathcal{E}_j : 0 \longrightarrow G \xrightarrow{i_j} E_j \xrightarrow{p_j} F \longrightarrow 0 \ \ (j = 1, 2) \tag{1.4}$$

is the extension

$$\mathcal{E} : \ 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

where

$$E = \frac{\mathrm{Ker}(p_1 - p_2 : E_1 \oplus E_2 \longrightarrow F)}{\mathrm{Im}\,((i_1, -i_2) : G \longrightarrow E_1 \oplus E_2)}, \tag{1.5}$$

and p and i are given as $p(e_1, e_2) = p_1(e_1) = p_2(e_2)$ and $i(g) = (i_1(g), 0)) = (0, i_2(g))$.

2. The arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F, G)$

This section is devoted to the basic definitions and properties of arithmetic extensions and of the corresponding groups of 1-extensions on an arithmetic scheme X. We associate a canonical differential form $\Psi(\mathcal{E}, s)$ with an arithmetic extension $(\mathcal{E}, s)$, namely its "second fundamental form" $\overline{\partial} s$. The arithmetic extension group fits into two exact sequences which are formally similar to corresponding sequences for arithmetic Chow groups. We discuss functorial properties of our extension groups and relate arithmetic extensions to admissible extensions of hermitian coherent sheaves and arithmetic torsors in the sense of Chambert-Loir and Tschinkel. We also discuss an interpretation of the group of arithmetic extensions $\widehat{\mathrm{Ext}}^1_X(F,G)$ as suitable group of morphisms in the derived category of $\mathcal{O}_X$-modules.

In this section, (R,Σ) denotes an arithmetic ring, K the field of fractions of R, and X an arithmetic scheme over $S := \operatorname{Spec} R$.

2.1. **Basic definitions.** Let F and G be $\mathcal{O}_X$-modules. An *arithmetic extension* $(\mathcal{E},s)$ *of* F *by* G is by definition an extension

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0 \tag{2.1}$$

of $\mathcal{O}_X$-modules together with an F_∞-invariant $\mathcal{C}^\infty$-splitting

$$s : F_{\mathbb{C}} \longrightarrow E_{\mathbb{C}} \tag{2.2}$$

of the associated extension of $\mathcal{C}^\infty_{X_\Sigma}$-modules

$$\mathcal{E}_{\mathbb{C}} : 0 \longrightarrow G_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} E_{\mathbb{C}} \xrightarrow{p_{\mathbb{C}}} F_{\mathbb{C}} \longrightarrow 0.$$

There exists a unique map such that the relation

$$\mathrm{Id}_{E_{\mathbb{C}}} = s \circ p_{\mathbb{C}} + i_{\mathbb{C}} \circ t \tag{2.3}$$

holds. It is $\mathcal{C}^\infty$ and F_∞-invariant, and the sections s and t determine each other uniquely. We sometimes write $(\mathcal{E},s,t)$ to emphasize that t is defined by (2.3).

A *morphism between arithmetic extensions* $(\mathcal{E}_1,s_1,t_1)$ and $(\mathcal{E}_2,s_2,t_2)$ is given by a morphism $(\alpha,\beta,\gamma) : \mathcal{E}_1 \to \mathcal{E}_2$ of extensions such that $\beta_{\mathbb{C}} \circ s_1 = s_2 \circ \gamma_{\mathbb{C}}$ holds. Observe that this condition implies already $t_2 \circ \beta_{\mathbb{C}} = \alpha_{\mathbb{C}} \circ t_1$.

Two arithmetic extensions $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$ of F by G are called *isomorphic* iff there exists a morphism (α,β,γ) from $(\mathcal{E}_1,s_1)$ to $(\mathcal{E}_2,s_2)$ such that $\alpha = \mathrm{id}_G$ and $\gamma = \mathrm{id}_F$. Any such morphism is automatically an isomorphism and defines an isomorphism between the arithmetic extensions $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$. We denote the set of isomorphism classes of arithmetic extensions of F by G by $\widehat{\mathrm{Ext}}^1_X(F,G)$.

Let $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$ be two arithmetic extensions of F by G on X. Let $\mathcal{E}$ denote the algebraic Baer sum of $\mathcal{E}_1$ and $\mathcal{E}_2$. The $\mathcal{C}^\infty$-splittings s_1 and s_2 induce an F_∞-invariant $\mathcal{C}^\infty$-splittings of $\mathcal{E}$ which maps a section f of $F_{\mathbb{C}}$ to the class $s(f)$ of $(s_1(f),s_2(f))$ in the quotient (1.5). We obtain an arithmetic extension $(\mathcal{E},s)$ which we call the *arithmetic Baer sum* of $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$.

It is a straightforward exercise to check that the arithmetic Baer sum defines on $\widehat{\mathrm{Ext}}^1_X(F,G)$ the structure of an abelian group. Actually, the opposite of the class of $(\mathcal{E},s)$ is the one of $(\tilde{\mathcal{E}},-s)$, if we let

$$\tilde{\mathcal{E}} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{-p} F \longrightarrow 0.$$

Moreover, with the notation of A.4.3, if $(\mathcal{E}_1,s_1),\ldots,(\mathcal{E}_k,s_k)$ are arithmetic extensions over X, one defines a F_∞-invariant $\mathcal{C}^\infty$-splitting s of $\mathcal{E}_1 + \ldots + \mathcal{E}_k$ by sending a section f of $F_{\mathbb{C}}$ to the class of $(s_1(f),\ldots,s_k(f))$, and the class of $(\mathcal{E}_1 + \ldots + \mathcal{E}_k, s)$ in $\widehat{\mathrm{Ext}}^1_X(F,G)$ is the sum of the classes $[(\mathcal{E}_1,s_1)]$,…,$[(\mathcal{E}_k,s_k)]$.

2.2. **The first exact sequence.** Let F and G be $\mathcal{O}_X$-modules on the arithmetic scheme X. We consider the group $\mathrm{Hom}_{\mathcal{O}_X}(F,G)$ of homomorphisms of $\mathcal{O}_X$-modules from F to G and the space

$$\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty} = \{T=(T_\sigma)_\sigma \in \bigoplus_{\sigma\in\Sigma} \mathrm{Hom}_{\mathcal{C}^\infty_{X_\sigma(\mathbb{C})}}(F_\sigma,G_\sigma)\,|\,\overline{T}_\sigma = T_{\overline{\sigma}}\}.$$

of F_∞-invariant $\mathcal{C}^\infty$-homomorphisms from $F_\mathbb{C}$ to $G_\mathbb{C}$. There is a canonical map

$$b : \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty} \longrightarrow \widehat{\mathrm{Ext}}^1_X(F,G)$$

which sends any T in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty}$ to the class of the arithmetic extension $(\mathcal{E},s)$ where $\mathcal{E}$ is the trivial algebraic extension, defined by (2.1) with $E := G\oplus F$ and i and p the obvious injection and projection morphisms, and s is given by $s(f)=(T(f),f))$.

Theorem 2.2.1. *The map b is a group homomorphism. It fits into an exact sequence*

$$\mathrm{Hom}_{\mathcal{O}_X}(F,G) \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty} \xrightarrow{b} \widehat{\mathrm{Ext}}^1_X(F,G) \xrightarrow{\nu} \mathrm{Ext}^1_{\mathcal{O}_X}(F,G) \xrightarrow{\digamma} \mathrm{Ext}^1_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C}) \tag{2.4}$$

where ι is the canonical map, ν maps the class of an arithmetic extension $(\mathcal{E},s)$ to the class of the underlying extension $\mathcal{E}$ of $\mathcal{O}_X$-modules, and $\digamma$ maps the class of an extension of $\mathcal{O}_X$-modules to the class of the associated extension of $\mathcal{C}^\infty_{X_\Sigma}$-modules. Furthermore

$$\mathrm{Ext}^1_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C}) = 0 \tag{2.5}$$

if $F_\mathbb{C}$ is a vector bundle.

Observe that, when this last assumption is satisfied, we may also identify the real vector space $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty}$ with $A^0(X_\mathbb{R},F^\vee\otimes G)$.

Proof. We first show that b is a group homomorphism. Consider T_1,T_2 in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty}$, and the associated arithmetic extension classes $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$ that define $b(T_1)$ and $b(T_2)$ respectively. Let $(\mathcal{E},s)$ be the arithmetic Baer sum of $(\mathcal{E}_1,s_1)$ and $(\mathcal{E}_2,s_2)$ With self-explanatory notation, we get

$$E_1\oplus E_2 = G\oplus F\oplus G\oplus F.$$

and obtain an isomorphism

$$E = \frac{\mathrm{Ker}\,(p_1-p_2 : E_1\oplus E_2 \longrightarrow F)}{\mathrm{Im}\,((i_1,-i_2) : G \longrightarrow E_1\oplus E_2)} \longrightarrow G\oplus F.$$

which maps the class of (g_1,f_1,g_2,f_2) to (g_1+g_2,f_1). This isomorphism defines an isomorphism of arithmetic extensions from $(\mathcal{E},s)$ to the arithmetic extension of F by G that defines $b(T_1+T_2)$. Consequently $b(T_1)+b(T_2)$ and $b(T_1+T_2)$ coincide.

Let us check the exactness at $\widehat{\mathrm{Ext}}^1_X(F,G)$. We clearly have $\nu\circ b=0$. Conversely, let $\mathrm{cl}(\mathcal{E},s)$ be an arithmetic extension class in the kernel of ν. One can chose an isomorphism $\tau : E \xrightarrow{\sim} G\oplus F$ between $\mathcal{E}$ and the trivial algebraic extension of F by G. Then the class of $(\mathcal{E},s)$ is precisely the image of $\mathrm{pr}_1\circ\tau_\mathbb{C}\circ s \in \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C},G_\mathbb{C})^{F_\infty}$ under b.

Let us show that $\ker b = \mathrm{Im}\,\iota$. Given $T\in\mathrm{Hom}_{\mathcal{O}_X}(F,G)$, the map

$$G\oplus F\longrightarrow G\oplus F \ \ , \ \ (g,f)\mapsto (g+T(f),f)$$

defines an isomorphism of arithmetic extensions which gives $b(0) = b(T)$. Hence we have $b \circ \iota = 0$. Conversely, let $T \in \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$ be in the kernel of b. One gets an isomorphism between the arithmetic extensions representing $b(0)$ and $b(T)$ which is given by a matrix

$$\begin{pmatrix} 1 & \tilde{T} \\ 0 & 1 \end{pmatrix}.$$

for some $\tilde{T} \in \mathrm{Hom}_{\mathcal{O}_X}(F, G)$. It follows that T equals $\iota(\tilde{T})$, and consequently belongs to the image of ι.

The image of ν is the kernel of F by the very definition of an arithmetic extension class. This establishes the exactness at $\mathrm{Ext}^1_{\mathcal{O}_X}(F, G)$. Finally (2.5) follows from the existence of partitions of unity. □

Corollary 2.2.2. *When X is an affine scheme, F is a vector bundle* [4]*, and G quasi-coherent, the map b induces an isomorphism of abelian groups*

$$\widehat{\mathrm{Ext}}^1_X(F, G) \simeq \frac{\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F, G)^{F_\infty}}{\iota(\mathrm{Hom}_{\mathcal{O}_X}(F, G))}.$$

Indeed, under these assumptions on X and F, the groups $\mathrm{Ext}^1_X(F, G)$ and $\mathrm{Ext}^1_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})$ vanish.

Example 2.2.3. Let K be a number field. We work over the arithmetic ring $\mathcal{O}_K$ (as defined in 1.1.1). Let $\mathbb{A}_K$ denote the ring of adèles of K. Let E and F be vector bundles on $S = \mathrm{Spec}\ \mathcal{O}_K$. Consider the vector bundle $G = \mathcal{H}om_{\mathcal{O}_S}(E, F)$ as a vector group scheme on S. There is a canonical isomorphism

$$\widehat{\mathrm{Ext}}^1_S(E, F) = G(K)\backslash G(\mathbb{A}_K)/\mathbb{K} \tag{2.6}$$

where

$$\mathbb{K} = \prod_{v \in \mathrm{Spec}\, \mathcal{O}_K} G(\hat{\mathcal{O}}_{K,v}) \times \prod_{v|\infty} \{0\}$$

is the standard compact subgroup of the commutative group $G(\mathbb{A}_K^{F_\infty})$. In order to see this, we observe that Corollary 2.2.2 gives a canonical isomorphism

$$\widehat{\mathrm{Ext}}^1_S(E, F) \simeq \frac{\left[\prod_\sigma \mathrm{Hom}_{\mathbb{C}}(E_\sigma, F_\sigma)\right]^{F_\infty}}{\mathrm{Hom}_{\mathcal{O}_K}(E, F)} = G(\mathcal{O}_K)\backslash\Big[\prod_\sigma G_\sigma(\mathbb{C})\Big]^{F_\infty}.$$

Moreover $\Big[\prod_\sigma G_\sigma(\mathbb{C})\Big]^{F_\infty}$ may be identified with $\prod_{v|\infty} G(K_v)$, and the canonical map

$$G(\mathcal{O}_K)\backslash\Big[\prod_\sigma G_\sigma(\mathbb{C})\Big]^{F_\infty} \longrightarrow G(K)\backslash G(\mathbb{A}_K^{F_\infty})/\mathbb{K}$$

is an isomorphism by the strong approximation theorem (see for example [Cas67, Chapter II.15]).

[4] *i.e.*, locally free of finite rank.

2.3. **The second exact sequence.** In this paragraph, we consider $\mathcal{O}_X$-modules F and G over X such that $F_{\mathbb{C}}^{\rm hol}$ and $G_{\mathbb{C}}^{\rm hol}$ are *holomorphic vector bundles* over $X_\Sigma(\mathbb{C})$. This is for instance the case when F and G are coherent $\mathcal{O}_X$-modules, and F_K and G_K are locally free.

We use the notations introduced in 1.1.5, i.e. $Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)$ denotes the subspace of forms in $A^{0,1}(X_{\mathbb{R}}, F^\vee \otimes G)$ which are $\overline{\partial}_{F^\vee\otimes G}$-closed. For any arithmetic extension $(\mathcal{E}, s)$ of the $\mathcal{O}_X$-modules F by G as in 2.1, the Dolbeault operator

$$\overline{\partial}_E : A^0(X_{\mathbb{R}}, E) \longrightarrow A^{0,1}(X_{\mathbb{R}}, E)$$

has a matrix representation

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_G & \alpha \\ 0 & \overline{\partial}_F \end{pmatrix} \tag{2.7}$$

with respect to the direct sum decomposition $E_{\mathbb{C}} \cong G_{\mathbb{C}} \oplus F_{\mathbb{C}}$ induced by s, for some form α in $Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)$. In other words, the form α is defined by the equality in $A^{0,1}(X_{\mathbb{R}}, F^\vee \otimes G)$:

$$i_{\mathbb{C}} \circ \alpha = \overline{\partial}_{F^\vee\otimes G} s,$$

or, equivalently, by the following identity, valid for any local $\mathcal{C}^\infty$ section f of $F_{\mathbb{C}}$ over $X_\Sigma(\mathbb{C})$:

$$\overline{\partial}_E(s \circ f) = s \circ \overline{\partial}_F f + i_{\mathbb{C}} \circ \alpha \circ f,$$

where, with a slight abuse of notation, we have denoted $s \circ .$ and $i_{\mathbb{C}} \circ .$ the action by composition of s and $i_{\mathbb{C}}$ on sections of $F_{\mathbb{C}}$ and $E_{\mathbb{C}}$ with form coefficients. If, as before, t is defined by (2.3), we also have:

$$\overline{\partial}_{E^\vee\otimes G} t = -\alpha \circ p_{\mathbb{C}}.$$

Following the terminology discussed in Appendix A.5.2, we call α the *second fundamental form* of the arithmetic extension $(\mathcal{E}, s)$. We shall denote it $\Psi(\mathcal{E}, s)$.

Let us denote

$$p : Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G) \longrightarrow H^1_{\rm Dolb}(X_{\mathbb{R}}, F^\vee \otimes G) := Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)/\overline{\partial}_{F^\vee\otimes G}(A^0(X_{\mathbb{R}}, F^\vee \otimes G))$$

the canonical quotient map.

Proposition 2.3.1. *There is a well defined group homomorphism*

$$\Psi : \widehat{\rm Ext}^1_X(F, G) \longrightarrow Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G)$$

which associates the second fundamental form $\Psi(\mathcal{E}, s)$ to the class of an arithmetic extension $(\mathcal{E}, s)$. The map Ψ fits into the commutative diagrams

$$\begin{array}{ccc} {\rm Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} & \stackrel{b}{\longrightarrow} & \widehat{\rm Ext}^1_X(F, G) \\ \| & & \downarrow \Psi \\ A^0(X_{\mathbb{R}}, F^\vee \otimes G) & \stackrel{\overline{\partial}_{F^\vee\otimes G}}{\longrightarrow} & Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G). \end{array} \tag{2.8}$$

and

$$\begin{array}{ccc} \widehat{\rm Ext}^1_X(F, G) & \stackrel{\Psi}{\longrightarrow} & Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G) \\ \downarrow \nu & & \downarrow p \\ {\rm Ext}^1_{\mathcal{O}_X}(F, G) & \stackrel{{\rm Dolb}_{F_{\mathbb{R}}, G_{\mathbb{R}}}}{\longrightarrow} & H^1_{\rm Dolb}(X_{\mathbb{R}}, F^\vee \otimes G). \end{array} \tag{2.9}$$

Proof. Two isomorphic arithmetic extensions yield obviously the same form α. Consider a morphism T in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty}$, and the associated extension $(\mathcal{E}, s)$ that defines $b(T)$. The identity

$$\overline{\partial}_G(T \circ f) = \overline{\partial}_{\mathrm{Hom}(F,G)}(T) \circ f + T \circ \overline{\partial}_F f,$$

valid for any local $\mathcal{C}^\infty$ section f of $F_\mathbb{C}$ over $X_\Sigma(\mathbb{C})$, shows that $\Psi(\mathcal{E}, s) = \overline{\partial}(T)$. This proves the commutativity of (2.8). We can check locally on X_Σ that Ψ is a homomorphism. Hence it suffices to consider elements in the image of b. In this case the commutativity of (2.8) implies that Ψ is a homomorphism.

The commutativity of (2.8) follows from the classical fact that the "second fundamental form" provides a representative in Dolbeault cohomology of the extension class of a 1-extension of holomorphic vector bundles (*cf.* [Gri66] and Appendix A, Proposition A.5.3). □

As a consequence of Theorem 2.2.1 and Proposition 2.3.1, we have:

Theorem 2.3.2. *Let F and G be two $\mathcal{O}_X$-modules such that X such that are $F^{\mathrm{hol}}_\mathbb{C}$ and $G^{\mathrm{hol}}_\mathbb{C}$ are holomorphic vector bundles. We have an exact sequence*

$$(2.10)\quad \mathrm{Hom}_{\mathcal{O}_X}(F,G) \xrightarrow{\iota} H^0(X_\mathbb{R}, F^\vee \otimes G) \xrightarrow{b} \widehat{\mathrm{Ext}}^1_X(F,G) \longrightarrow$$
$$\xrightarrow{\binom{\nu}{\Psi}} \mathrm{Ext}^1_{\mathcal{O}_X}(F,G) \oplus Z^{0,1}_{\overline{\partial}}(X_\mathbb{R}, F^\vee \otimes G) \xrightarrow{(\mathrm{Dolb}_{F^\vee \otimes G_\mathbb{R}}, -p)} H^1_{\mathrm{Dolb}}(X_\mathbb{R}, F^\vee \otimes G) \longrightarrow 0.$$

Proof. To establish the exactness at $H^0(X_\mathbb{R}, F^\vee \otimes G)$ and at $\widehat{\mathrm{Ext}}^1_X(F,G)$, we use Theorem 2.2.1: according to the commutativity of (2.8), we may "replace" $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty}$ by

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty} \cap \mathrm{Ker}\, \overline{\partial}_{F^\vee \otimes G} &= \mathrm{Hom}_{\mathcal{O}^{\mathrm{hol}}_{X_\Sigma}}(F^{\mathrm{hol}}_\mathbb{C}, G^{\mathrm{hol}}_C)^{F_\infty} \\ &= H^0(X_\mathbb{R}, F^\vee \otimes G).\end{aligned}$$

The exactness at $H^1_{\mathrm{Dolb}}(X_\mathbb{R}, F^\vee \otimes G)$ is clear. It remains to show exactness at $\mathrm{Ext}^1_{\mathcal{O}_X}(F,G) \oplus Z^{0,1}_{\overline{\partial}}(X_\mathbb{R}, F^\vee \otimes G)$. The fact that the composition of $\binom{\nu}{\Psi}$ and $(\mathrm{Dolb}_{F^\vee \otimes G_\mathbb{R}}, -p)$ vanishes is precisely the commutativity of (2.9). Conversely, consider $\mathrm{cl}(\mathcal{E})$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(F,G)$ and α in $Z^{0,1}_{\overline{\partial}}(X_\mathbb{R}, F^\vee \otimes G)$ such that

$$\mathrm{Dolb}_{F^\vee \otimes G_\mathbb{R}}(\mathrm{cl}(\mathcal{E})) - p(\alpha) = 0.$$

Choose s such that $(\mathcal{E}, s)$ becomes an arithmetic extension. Then the commutativity of (2.9) implies that

$$\Psi(\mathcal{E}, s) = \alpha + \overline{\partial} T$$

for some $T \in \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_\mathbb{C}, G_\mathbb{C})^{F_\infty}$, and the commutativity of (2.8) that $(\mathrm{cl}(\mathcal{E}), \alpha)$ is the image of $\mathrm{cl}(\mathcal{E}, s) - b(T)$. □

2.4. **Pushout, pullback, and inverse image.** Arithmetic extensions have the expected functorial properties and behave well with respect to the maps ι, b, ν, and Ψ introduced above.

2.4.1. *Pushout.* Let $g : G \to G'$ a morphism of $\mathcal{O}_X$-modules on an arithmetic scheme X. For an arithmetic extension $(\mathcal{E}, s)$ of an $\mathcal{O}_X$-module F by G, we obtain a pushout diagram

$$\begin{array}{ccccccccc} \mathcal{E} : 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow q & & \| & & \\ \mathcal{E}' : 0 & \longrightarrow & G' & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

The section $s' = q_{\mathbb{C}} \circ s$ turns $(\mathcal{E}', s')$ into an arithmetic extension which we denote as $g \circ (\mathcal{E}, s)$. We obtain a morphism of arithmetic extensions

$$(g, q, \mathrm{id}_F) : (\mathcal{E}, s) \longrightarrow (\mathcal{E}', s')$$

which is universal in the sense that every morphism of arithmetic extensions

$$(\alpha, \beta, \gamma) : (\mathcal{E}, s) \longrightarrow (\mathcal{E}'', s'')$$

with $\alpha = g$ factors uniquely through (g, q, id_F). Our construction defines a canonical $\mathbb{Z}$-bilinear pairing

$$\widehat{\mathrm{Ext}}^1_X(F, G) \times \mathrm{Hom}_{\mathcal{O}_X}(G, G') \longrightarrow \widehat{\mathrm{Ext}}^1_X(F, G') \ , \ (\alpha = \mathrm{cl}(\mathcal{E}, s), g) \mapsto g \circ \alpha = \mathrm{cl}(\mathcal{E}', s').$$

The formation of $g \circ \alpha$ is functorial in g. It is compatible with the maps ι, b, ν, and Ψ in the sense that we have the equalities

$$\nu(g \circ \alpha) = g \circ \nu(\alpha) \ , \ g \circ b(f) = b(\iota(g) \circ f) \ , \ \Psi(g \circ \alpha) = g_{\mathbb{C}} \circ \Psi(\alpha) \tag{2.11}$$

for all α, g as above and f in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$. These are direct consequences of our definitions.

2.4.2. *Pullback.* Let $h : F' \to F$ a morphism of $\mathcal{O}_X$-modules on an arithmetic scheme X. For an arithmetic extension $(\mathcal{E}, s, t)$ of F by an $\mathcal{O}_X$-module G, we obtain a pullback diagram

$$\begin{array}{ccccccccc} \mathcal{E}' : 0 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & 0 \\ & & \| & & \downarrow p & & \downarrow h & & \\ \mathcal{E} : 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

The section $t' = t \circ p_{\mathbb{C}}$ induces a section s' of $\mathcal{E}_{\mathbb{C}}$ (as in (2.3)) which turns $(\mathcal{E}', s', t')$ into an arithmetic extension which we denote as $\mathrm{cl}(\mathcal{E}, s) \circ h$. We obtain a morphism of arithmetic extensions

$$(\mathrm{id}_G, p, h) : (\mathcal{E}', s') \longrightarrow (\mathcal{E}, s)$$

which is universal in the sense that every morphism of arithmetic extensions

$$(\alpha, \beta, \gamma) : (\mathcal{E}'', s'') \longrightarrow (\mathcal{E}, s)$$

with $\gamma = h$ factors uniquely through (id_G, p, h). Our construction defines a canonical $\mathbb{Z}$-bilinear pairing

$$\mathrm{Hom}_{\mathcal{O}_X}(F', F) \times \widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow \widehat{\mathrm{Ext}}^1_X(F', G) \ , \ (h, \alpha = \mathrm{cl}(\mathcal{E}, s)) \mapsto \alpha \circ h = \mathrm{cl}(\mathcal{E}', s').$$

The formation of $\alpha \circ h$ is functorial in h. It is compatible with the maps ι, b, ν, and Ψ in the sense that we have the equalities

$$\nu(\alpha \circ h) = \nu(\alpha) \circ h \ , \ b(f) \circ h = b(f \circ \iota(h)) \ , \ \Psi(\alpha \circ h) = \Psi(\alpha) \circ h_{\mathbb{C}} \tag{2.12}$$

for all α, h as above and f in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$. These are again a direct consequences of our definitions.

2.4.3. *Compatibilities.* Given an arithmetic extension $(\mathcal{E}, s)$ and morphisms $g : G \to G'$ and $f : F' \to F$ as above, there exists a natural isomorphism

$$(g \circ (\mathcal{E}, s)) \circ h \xrightarrow{\sim} g \circ ((\mathcal{E}, s) \circ h). \tag{2.13}$$

Given an arithmetic extension $(\mathcal{E}, s)$ as in (2.1), there exists a natural isomorphism

$$i \circ (\mathcal{E}, s) \xrightarrow{\sim} (\mathcal{E}, s) \circ p. \tag{2.14}$$

Every morphism of arithmetic extensions $(\alpha, \beta, \gamma) : (\mathcal{E}, s) \to (\mathcal{E}', s')$ determines a natural isomorphism

$$\alpha \circ (\mathcal{E}, s) \xrightarrow{\sim} (\mathcal{E}', s') \circ \gamma. \tag{2.15}$$

The isomorphisms (2.13), (2.14), and (2.15) are constructed precisely as in the algebraic case using the universal properties of pushout and pullback (compare for example [ML95, III.1]).

2.4.4. *Inverse image.* Let $f : Y \to X$ be a morphism of arithmetic varieties. Let F and G be $\mathcal{O}_X$-modules. We assume either that f is flat or that F is a flat $\mathcal{O}_X$-module. The pullback along f defines a canonical homomorphism

$$f^* : \widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow \widehat{\mathrm{Ext}}^1_Y(f^*F, f^*G)$$

and the formation of f^* is functorial in f and compatible with ι, b, v and Ψ in the expected way.

2.4.5. *An application.* Given an arithmetic extension $(\mathcal{E}, s)$ of F by G and a vector bundle E on the arithmetic scheme X, we obtain a natural arithmetic extension

$$(\mathcal{E}, s) \otimes E := (\mathcal{E} \otimes E, s \otimes \mathrm{id}_{E_{\mathbb{C}}}).$$

We obtain an induced map

$$\widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow \widehat{\mathrm{Ext}}^1_X(F \otimes E, G \otimes E)\ ,\ \mathrm{cl}(\mathcal{E}, s) \mapsto \mathrm{cl}\big((\mathcal{E}, s) \otimes E\big) \tag{2.16}$$

which is easily seen to be a group homomorphism. The pushout and pullback constructions above allow with the previous remark to construct the following canonical isomorphism.

Proposition 2.4.6. *For any $\mathcal{O}_X$-modules F, G and any vector bundle E on an arithmetic variety X, there is a canonical isomorphism*

$$\widehat{\mathrm{Ext}}^1_X(F, G \otimes E) \xrightarrow{\sim} \widehat{\mathrm{Ext}}^1_X(F \otimes E^\vee, G).$$

which maps the class of an arithmetic extension $(\mathcal{M}, s_M)$ of F by $G \otimes E$ to the pushout of $\mathcal{M} \otimes E^\vee$ along the evaluation (or trace) map $\mathrm{id}_G \otimes \mathrm{ev} : G \otimes E \otimes E^\vee \to G$. The inverse map sends the class of an arithmetic extension $(\mathcal{N}, s_N)$ to the pullback of $(\mathcal{N}, s_N) \otimes E$ along the canonical morphism $\mathrm{id}_F \otimes \Delta : F \to F \otimes E^\vee \otimes E$, defined by the "identity" section Δ of $E^\vee \otimes E$.

Proof. Let $(\mathcal{M}, s_M)$ be an arithmetic extension with underlying algebraic extension

$$\mathcal{M} : 0 \longrightarrow G \otimes E \longrightarrow M \xrightarrow{p} F \longrightarrow 0.$$

Consider the pushout

$$(\mathcal{N}, s_N) = \big((\mathcal{M}, s_M) \otimes E^\vee\big) \circ (\mathrm{id}_G \otimes \mathrm{ev})$$

with associated morphism $f : M \otimes E^\vee \to N$ and section $s_N = f_{\mathbb{C}} \circ (s_M \otimes \mathrm{id}_{E^\vee_{\mathbb{C}}})$. Define g as the composition

$$g = (f \otimes \mathrm{id}_E) \circ (\mathrm{id}_M \otimes \Delta) : M \longrightarrow M \otimes E^\vee \otimes E \longrightarrow N \otimes E.$$

We claim that

$$(\mathrm{id}_{G\otimes E}, g, \mathrm{id}_F \otimes \Delta) : (\mathcal{M}, s_M) \longrightarrow (\mathcal{N}, s_N) \otimes E \tag{2.17}$$

is a morphism of arithmetic extensions. It is easily seen that (2.17) is a morphism of the underlying algebraic extensions. Hence $\mathcal{M}$ is the pullback of $\mathcal{N} \otimes E$ along $\mathrm{id}_F \otimes \Delta$ and we obtain an induced section $\tilde{s}_M$ of $\mathcal{M}_{\mathbb{C}}$ as in 2.4.2. We have to show that s_M equals $\tilde{s}_M$. We conclude from

$$\begin{aligned} g_{\mathbb{C}} \circ \tilde{s}_M \circ p_{\mathbb{C}} &= (s_N \otimes \mathrm{id}_{E_{\mathbb{C}}}) \circ (\mathrm{id}_{F_{\mathbb{C}}} \otimes \Delta_{\mathbb{C}}) \circ p_{\mathbb{C}} \\ &= (f_{\mathbb{C}} \otimes \mathrm{id}_{E_{\mathbb{C}}}) \circ (s_M \otimes \mathrm{id}_{E^\vee_{\mathbb{C}} \times E_{\mathbb{C}}}) \circ (\mathrm{id}_{F_{\mathbb{C}}} \otimes \Delta_{\mathbb{C}}) \circ p_{\mathbb{C}} \\ &= (f_{\mathbb{C}} \otimes \mathrm{id}_{E_{\mathbb{C}}}) \circ (\mathrm{id}_{M_{\mathbb{C}}} \otimes \Delta_{\mathbb{C}}) \circ s_M \circ p_{\mathbb{C}} \\ &= g_{\mathbb{C}} \circ s_M \circ p_{\mathbb{C}} \end{aligned}$$

that $g_{\mathbb{C}} \circ (\tilde{s}_M - s_M) \circ p_{\mathbb{C}} = 0$ which proves our claim. Hence $(\mathcal{M}, s_M)$ is the pullback of $(\mathcal{N}, s_N) \otimes E$ along $\mathrm{id}_F \otimes \Delta$. This proves one half of our proposition. The other half follows by a dual argument which we leave to the reader. □

2.5. **Arithmetic extensions as homomorphisms in the derived category.** In this paragraph, we present an interpretation of the arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F, G)$ in terms of homological algebra, in the spirit of the well-known identification of the "classical" extension group $\mathrm{Ext}^1_{\mathcal{O}_X}(F, G)$, defined by classes of 1-extensions of $\mathcal{O}_X$-modules equipped with the Baer sum, with the "cohomological" extension group $\mathrm{Hom}_{D(\mathcal{O}_X\text{-mod}))}(F, G[1])$ (see Appendix A.4.4).

We follow the notation and conventions concerning derived categories discussed in Appendix A.

To any $\mathcal{O}_X$-module E is naturally attached the adjunction map

$$\mathrm{ad}_E \colon E \longrightarrow (\rho_* E_{\mathbb{C}})^{F_\infty}$$

with respect to the flat map

$$\rho = j \circ \kappa \colon (X_\Sigma(\mathbb{C}), \mathcal{C}^\infty_{X_\Sigma}) \longrightarrow (X, \mathcal{O}_X)$$

from 1.1.2. It maps any section e of E over an open subscheme U of X to the (analytic, hence $\mathcal{C}^\infty$)-section $s_{\mathbb{C}}$ of $E_{\mathbb{C}}$ over $U_\Sigma(\mathbb{C})$, which is indeed a section over U of $(\rho_* E_{\mathbb{C}})^{F_\infty}$. Moreover, for any two $\mathcal{O}_X$-modules F and G, the map

$$\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}\big(F, (\rho_* G_{\mathbb{C}})^{F_\infty}\big)$$

which maps an F_∞-invariant $\mathcal{C}^\infty$-morphism $\varphi : F_{\mathbb{C}} \to G_{\mathbb{C}}$ to the composition

$$\tilde{\varphi} \colon F \xrightarrow{\mathrm{ad}_F} (\rho_* F_{\mathbb{C}})^{F_\infty} \xrightarrow{\rho_*(\varphi)} (\rho_* G_{\mathbb{C}})^{F_\infty}$$

is an isomorphism. (Its inverse maps a morphism of $\mathcal{O}_X$-modules $\psi : F \to (\rho_* G_{\mathbb{C}})^{F_\infty}$ to the composition

$$F_{\mathbb{C}} = \rho^* F \xrightarrow{\rho^*(\psi)} \rho^* \rho_* G_{\mathbb{C}} \xrightarrow{\mathrm{ad}^{G_{\mathbb{C}}}} G_{\mathbb{C}}.)$$

Let $(\mathcal{E}, s)$ be an arithmetic extension, *i.e.*, an extension of $\mathcal{O}_X$-modules

$$\mathcal{E}\colon 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \to 0$$

together with a F_∞-invariant $\mathcal{C}^\infty$-splitting $s : F_{\mathbb{C}} \to E_{\mathbb{C}}$. Let $t : G_{\mathbb{C}} \to E_{\mathbb{C}}$ be the F_∞-invariant $\mathcal{C}^\infty$-section of $i_{\mathbb{C}}$ such that such that $\mathrm{Id}_{E_{\mathbb{C}}} = s \circ p_{\mathbb{C}} + i_{\mathbb{C}} \circ t$. Let us consider the following diagram of complexes (written horizontally, with morphisms written vertically) of $\mathcal{O}_X$-modules:

$$\begin{array}{ccc} & & F \\ & & \uparrow p \\ G & \xrightarrow{i} & E \\ \downarrow \mathrm{Id}_G & & \downarrow \tilde{t} \\ G & \xrightarrow{\mathrm{ad}_G} & (\rho_* G_{\mathbb{C}})^{F_\infty} \end{array} \tag{2.18}$$

where F, E, and $(\rho_* G_{\mathbb{C}})^{F_\infty}$ sit in degree zero, and G in degree -1. The last two lines are precisely the cones $C(i)$ and $C(\mathrm{ad}_G)$ of i and ad_G, and the map p defines a quasi-isomorphism $\mathbf{p} : C(i) \to F$. Instead of (2.18), we may write as well

$$F \xleftarrow{\mathbf{p}} C(i) \xrightarrow{(\mathrm{Id}_G, \tilde{t})} C(\mathrm{ad}_G).$$

This diagram defines a morphism

$$\partial_{(\mathcal{E},s)} := (\mathrm{Id}_G,\, \tilde{t}) \circ \mathbf{p}^{-1} \colon F \longrightarrow C(\mathrm{ad}_G)$$

in the derived category $D^+(\mathcal{O}_X\text{-mod})$ of bounded below complexes of $\mathcal{O}_X$-modules. Clearly, if $(\mathcal{E}, s)$ and $(\mathcal{E}', s')$ are isomorphic arithmetic extensions of F by G, the associated morphisms $\partial_{(\mathcal{E},s)}$ and $\partial_{(\mathcal{E}',s')}$ coincide.

Proposition 2.5.1. *For any two $\mathcal{O}_X$-modules F and G, the mapping*

$$\widehat{\mathrm{cl}}_{F,G} \colon \widehat{\mathrm{Ext}}^1_X(F,G) \longrightarrow \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, C(\mathrm{ad}_G)\big)\,,\ \big[(\mathcal{E}, s)\big] \mapsto \partial_{(\mathcal{E},s)}.$$

is an isomorphism of abelian groups.

As hinted to above, the map $\widehat{\mathrm{cl}}_{F,G}$ is an avatar of the classical isomorphism

$$\mathrm{cl}_{F,G} \colon \mathrm{Ext}^1_{\mathcal{O}_X}(F,G) \xrightarrow{\sim} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, G[1]\big)\,,\ [\mathcal{E}] \mapsto \partial_{\mathcal{E}} \tag{2.19}$$

described in Appendix A.4.4. Recall that, to the class $[\mathcal{E}]$ of a 1-extension of $\mathcal{O}_X$-modules

$$\mathcal{E}\colon 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0,$$

it associates its "boundary operator" $\partial_{\mathcal{E}}$ in $\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, G[1]\big)$ that is defined as follows. We may consider the diagram of complexes of $\mathcal{O}_X$-modules (written horizontally)

$$\begin{array}{ccc} & & F \\ & & \uparrow p \\ G & \xrightarrow{i} & E \\ \downarrow -\mathrm{Id}_G & & \\ G & & \end{array} \tag{2.20}$$

where E and F sit in degree 0, and G in degree -1. Its last two lines are $C(i)$ and $G[1]$, the upper map defines a quasi-isomorphism $\mathbf{p} : C(i) \to F$, and (2.20) may be also written

$$F \xleftarrow{\mathbf{p}} C(i) \xrightarrow{(-\mathrm{Id}_G, 0)} G[-1].$$

Then, by definition, $\partial_{\mathcal{E}}$ is the morphism in $D^+(\mathcal{O}_X\text{-mod})$ defined by this last diagram:

$$\partial_{\mathcal{E}} := (-\mathrm{Id}_G, 0) \circ \mathbf{p}^{-1} \colon F \to G[1].$$

Observe that, if we are given two $\mathcal{O}_X$-modules F and G, we may consider the distinguished triangle in $D^+(\mathcal{O}_X\text{-mod})$ attached to ad_G:

$$G \xrightarrow{\mathrm{ad}_G} (\rho_* G_{\mathbb{C}})^{F_\infty} \xrightarrow{\alpha} C(\mathrm{ad}_G) \xrightarrow{\beta} G[1] \tag{2.21}$$

— we use the sign conventions discussed in Appendix A.1; thus α (resp. β) is the morphism defined by $\mathrm{Id}_{(\rho_* G_{\mathbb{C}})^{F_\infty}}$ (resp. by $-\mathrm{Id}_G$) — and apply the functor $\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}(F, .)$. This yields the following long exact sequence:

(2.22)

$$\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}(F, G) \xrightarrow{\mathrm{ad}_G \circ .} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, (\rho_* G_{\mathbb{C}})^{F_\infty}\big) \xrightarrow{\alpha \circ .} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, C(\mathrm{ad}_G)\big)$$
$$\xrightarrow{\beta \circ .} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, G[1]\big) \xrightarrow{\mathrm{ad}_G[1] \circ .} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, (\rho_* G_{\mathbb{C}})^{F_\infty}[1]\big).$$

We shall show simultaneously that $\widehat{\mathrm{cl}}_{F,G}$ is an isomorphism and that the exact sequence (2.22) is naturally isomorphic with the following variant of the long exact sequence in Theorem 2.2.1:

$$\mathrm{Hom}_{\mathcal{O}_X}(F, G) \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} \xrightarrow{b} \widehat{\mathrm{Ext}}^1_X(F, G) \xrightarrow{\nu} \mathrm{Ext}^1_{\mathcal{O}_X}(F, G) \tag{2.23}$$
$$\xrightarrow{\mathrm{ad}_G \circ .} \mathrm{Ext}^1_{\mathcal{O}_X}(F_{\mathbb{C}}, (\rho_* G_{\mathbb{C}})^{F_\infty}).$$

The exactness of (2.23) follows from Theorem 2.2.1 combined with the following observation:

Lemma 2.5.2. *For any extension of $\mathcal{O}_X$-modules*

$$\mathcal{E} \colon 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0,$$

an F_∞-invariant $\mathcal{C}^\infty$ splitting of $\mathcal{E}_{\mathbb{C}}$ may be equivalently described by:

- *a morphism t in $\mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(E_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$ such that $t \circ i_{\mathbb{C}} = \mathrm{id}_{G_{\mathbb{C}}}$;*
- *a morphism $\tilde{t}$ in $\mathrm{Hom}_{\mathcal{O}_X}\big(E, (\rho_* G_{\mathbb{C}})^{F_\infty}\big)$ such that $\tilde{t} \circ i = \mathrm{ad}_G$;*
- *a splitting of the extension of $\mathcal{O}_X$-modules defined as the pushout of $\mathcal{E}$ by ad_G :*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{i} & E & \xrightarrow{p} & F & \longrightarrow & 0 \\ & & \downarrow \mathrm{ad}_G & & \downarrow & & \| & & \\ 0 & \longrightarrow & (\rho_* G_{\mathbb{C}})^{F_\infty} & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

Consequently, the homomorphisms

$$F \; : \mathrm{Ext}^1_{\mathcal{O}_X}(F, G) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}}, G_{\mathbb{C}})$$

and

$$\mathrm{ad}_G \circ . : \mathrm{Ext}^1_{\mathcal{O}_X}(F, G) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(F_{\mathbb{C}}, (\rho_* G_{\mathbb{C}})^{F_\infty})$$

have the same kernel.

□

Actually, the five lemma shows that Proposition 2.5.1 follows from the long exact sequences (2.4) and (2.23), and Lemma 2.5.3 and Proposition 2.5.4 below.

Lemma 2.5.3. *The mapping* $\widehat{\mathrm{cl}}_{F,G}$ *in Proposition 2.5.1 is a homomorphism of groups.*

Proof. Consider arithmetic extensions $(\mathcal{E}_j, s_j)$, $j = 1, 2$ with underlying extensions

$$\mathcal{E}_j \colon 0 \longrightarrow G \xrightarrow{i_j} E_j \xrightarrow{p_j} F \longrightarrow 0,$$

and F_∞-invariant $\mathcal{C}^\infty$ splittings $s_j : F_{\mathbb{C}} \to E_{j,\mathbb{C}}$. Let $(\mathcal{E}, s)$ denote the arithmetic Baer sum of $(\mathcal{E}_1, s_1)$ and $(\mathcal{E}_2, s_2)$ as defined in 2.1. We have to show

$$\partial_{(\mathcal{E},s)} = \partial_{(\mathcal{E}_1,s_1)} + \partial_{(\mathcal{E}_2,s_2)}. \tag{2.24}$$

Consider the following diagrams

$$\begin{array}{ccc} G \oplus G & \xrightarrow{w} & \ker(E_1 \oplus E_2 \xrightarrow{p_1 - p_2} F) \\ \downarrow q_j & & \downarrow q_j \\ G & \xrightarrow{i_j} & E_j \end{array} \tag{2.25}$$

for $j = 1, 2$, and

$$\begin{array}{ccc} G \oplus G & \xrightarrow{w} & \ker(E_1 \oplus E_2 \xrightarrow{p_1 - p_2} F) \\ \downarrow + & & \downarrow \mathrm{can} \\ G & \xrightarrow{i} & E \end{array} \tag{2.26}$$

where q_j denotes the j-th projection, w the direct sum $i_1 \oplus i_2$, and 'can' the canonical projection. These diagrams are commutative and induce quasi-isomorphisms

$$\mathbf{q}_j : C(w) \to C(i_j) \text{ and } a : C(w) \to C(i).$$

As before, let us introduce $t : G_{\mathbb{C}} \to E_{\mathbb{C}}$ (resp. $t_j : G_{\mathbb{C}} \to E_{j,\mathbb{C}}$) such that $\mathrm{Id}_{E_{\mathbb{C}}} = s \circ p_{\mathbb{C}} + i_{\mathbb{C}} \circ t$ (resp. $\mathrm{Id}_{E_{j,\mathbb{C}}} = s_j \circ p_{j,\mathbb{C}} + i_{j,\mathbb{C}} \circ t_j$). According to the very definition of the arithmetic Baer sum,

$$t \circ \mathrm{can}_{\mathbb{C}} = t_1 \oplus t_2.$$

With obvious notation, we have:

$$\partial_{(\mathcal{E},s)} \colon F \xleftarrow{\mathbf{p}} C(i) \xleftarrow{a} C(w) \xrightarrow{a} C(i) \xrightarrow{(\mathrm{Id}_G, \tilde{t})} C(\mathrm{ad}_G)$$

and

$$\partial_{(\mathcal{E}_j,s_j)} \colon F \xleftarrow{\mathbf{p}_j} C(i_j) \xleftarrow{\mathbf{q}_j} C(w) \xrightarrow{\mathbf{q}_j} C(i) \xrightarrow{(\mathrm{id}_G, \tilde{t_j})} C(\mathrm{ad}_G).$$

We finally obtain (2.24) from the straightforward equalities

$$\mathbf{p}_1 \circ \mathbf{q}_1 = \mathbf{p} \circ a = \mathbf{p}_2 \circ \mathbf{q}_2 \colon C(w) \longrightarrow F$$

and

$$(\mathrm{Id}_G, \tilde{t_1}) \circ \mathbf{q}_1 + (\mathrm{Id}_G, \tilde{t_2}) \circ \mathbf{q}_2 = (\mathrm{Id}_G, \tilde{t}) \circ a \colon C(w) \longrightarrow C(\mathrm{ad}_G)$$

of homomorphisms of complexes of $\mathcal{O}_X$-modules. □

Recall that the category of $\mathcal{O}_X$-modules may be identified with a full subcategory of its derived category, and that consequently, for any two $\mathcal{O}_X$-modules F_1 and F_2, we may identify $\mathrm{Hom}_{\mathcal{O}_X}(F_1, F_2)$ and $\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}(F_1, F_2)$.

Besides, as a special instance of the canonical isomorphism $\mathrm{cl}_{F,G}$ (with $(\rho_* G_{\mathbb{C}})^{F_\infty}$ instead of G), we may consider the isomorphism

$$\mathrm{cl}_{F,(\rho_* G_{\mathbb{C}})^{F_\infty}} : \mathrm{Ext}^1_{\mathcal{O}_X}(F, (\rho_* G_{\mathbb{C}})^{F_\infty}) \xrightarrow{\sim} \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, (\rho_* G_{\mathbb{C}})^{F_\infty}[1]\big).$$

Proposition 2.5.4. *The following four diagrams are commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(F,G) & \xrightarrow{\iota} & \mathrm{Hom}_{\mathcal{C}^\infty_X}(F_{\mathbb{C}},G_{\mathbb{C}})^{F_\infty} \\ \| & & \sim\downarrow T\mapsto\tilde{T} \\ \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}(F,G) & \xrightarrow{\mathrm{ad}_G\circ.} & \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,(\rho_*G_{\mathbb{C}})^{F_\infty}\big) \end{array} \tag{2.27}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}^\infty_X}(F_{\mathbb{C}},G_{\mathbb{C}})^{F_\infty} & \xrightarrow{-b} & \widehat{\mathrm{Ext}}^1_X(F,G) \\ \sim\downarrow T\mapsto\tilde{T} & & \downarrow \widehat{\mathrm{cl}}_{F,G} \\ \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,(\rho_*G_{\mathbb{C}})^{F_\infty}\big) & \xrightarrow{\alpha\circ.} & \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,C(\mathrm{ad}_G)\big) \end{array} \tag{2.28}$$

$$\begin{array}{ccc} \widehat{\mathrm{Ext}}^1_X(F,G) & \xrightarrow{\nu} & \mathrm{Ext}^1_{\mathcal{O}_X}(F,G) \\ \downarrow \widehat{\mathrm{cl}}_{F,G} & & \sim\downarrow \mathrm{cl}_{F,G} \\ \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,C(\mathrm{ad}_G)\big) & \xrightarrow{\beta\circ.} & \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,G[1]\big) \end{array} \tag{2.29}$$

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_X}(F,G) & \xrightarrow{\mathrm{ad}_G\circ.} & \mathrm{Ext}^1_{\mathcal{O}_X}\big(F,(\rho_*G_{\mathbb{C}})^{F_\infty}\big) \\ \sim\downarrow \mathrm{cl}_{F,G} & & \sim\downarrow \mathrm{cl}_{F,(\rho_*G_{\mathbb{C}})^{F_\infty}} \\ \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,G[1]\big) & \xrightarrow{\mathrm{ad}_G[1]\circ.} & \mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,(\rho_*G_{\mathbb{C}})^{F_\infty}[1]\big). \end{array} \tag{2.30}$$

Proof. The commutativity of (2.27) is immediate. Let us check the one of (2.28). Given

$$T\in \mathrm{Hom}_{\mathcal{C}^\infty_{X_\Sigma}}(F_{\mathbb{C}},G_{\mathbb{C}})^{F_\infty},$$

we equip the trivial extension

$$\mathcal{E}\colon 0\longrightarrow G\xrightarrow{\binom{\mathrm{id}_G}{0}} G\oplus F\xrightarrow{(0\,\mathrm{id}_F)} F\longrightarrow 0$$

with the $\mathcal{C}^\infty$-splitting defined by

$$s=\begin{pmatrix}-T\\ \mathrm{id}_{F_{\mathbb{C}}}\end{pmatrix}\text{ and } t=(\mathrm{id}_{G_{\mathbb{C}}}\ T).$$

Then $\tilde{t}=(\mathrm{ad}_G\ \tilde{T})$, the arithmetic extension $(\mathcal{E},s)$ admits $-b(T)$ as class in $\widehat{\mathrm{Ext}}^1_X(F,G)$, and $\partial_{(\mathcal{E},s)}$ is the morphism in $\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F,G[1]\big)$ defined by the diagram:

$$\begin{array}{ccc} & & F \\ & & \uparrow (0\,\mathrm{id}_F) \\ G & \xrightarrow{\binom{\mathrm{id}_G}{0}} & G\oplus F \\ \downarrow \mathrm{id}_G & & \downarrow \tilde{t} \\ G & \xrightarrow{\mathrm{ad}_G} & (\rho_*G_{\mathbb{C}})^{F_\infty}. \end{array}$$

A quasi-inverse of $(0\,\mathrm{id}_F)$ in this diagram is given by

$$\begin{array}{ccc} & & F \\ & & \downarrow \binom{0}{\mathrm{id}_F} \\ G & \xrightarrow{\binom{\mathrm{id}_G}{0}} & G\oplus F. \end{array}$$

Consequently $\partial_{(\mathcal{E},s)}$ is also defined by the map of complexes:

$$\begin{array}{ccc} & & F \\ & & \downarrow \tilde{T} \\ G & \xrightarrow{\mathrm{ad}_G} & [\rho_* G_{\mathbb{C}}]^{F_\infty}. \end{array}$$

This shows that $\widehat{\mathrm{cl}}_{F,G}(-b(T)) = \alpha \circ \tilde{T}$, as required.

The commutativity of (2.29) is a direct consequence of our definitions and sign conventions.

Finally the diagram (2.30) commutes since pushout on extension groups corresponds under the isomorphism (2.19) to the natural functoriality of the $\mathrm{Hom}_{D^+(\mathcal{O}_X\text{-mod})}$ functor in the second argument (compare Appendix A.4.4). □

2.5.5. We give a homological description of the operator Ψ which computes the second fundamental form.

For any $\mathcal{O}_X$-module G and any non-negative integer j, we shall denote

$$A^{0,j}(-_{\mathbb{R}}, G) = \big(\rho_* A^{0,j}(-, G)\big)^{F_\infty}$$

as in 1.1.5. We have in particular $A^{0,0}(-_{\mathbb{R}}, G) = (\rho_* G_{\mathbb{C}})^{F_\infty}$, and, for any coherent $\mathcal{O}_X$-module F, $\mathrm{Hom}_{\mathcal{O}_X}(F, A^{0,j}(-_{\mathbb{R}}, G))$ may be identified with $A^{0,j}(X_{\mathbb{R}}, \mathcal{H}om_{\mathcal{O}_X}(F, G))$. The Dolbeault operator $\overline{\partial}$ induces a complex of $\mathcal{O}_X$-modules

$$(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G)\colon 0 \longrightarrow A^{0,0}(-_{\mathbb{R}}, G) \xrightarrow{\overline{\partial}_G} A^{0,1}(-_{\mathbb{R}}, G) \xrightarrow{\overline{\partial}_G} A^{0,2}(-_{\mathbb{R}}, G) \xrightarrow{\overline{\partial}_G} \ldots.$$

The diagram

$$\begin{array}{ccccc} G & \xrightarrow{\mathrm{ad}_G} & A^{0,0}(-_{\mathbb{R}}, G) & & \\ & & \downarrow \overline{\partial}_G & & \\ & & A^{0,1}(-_{\mathbb{R}}, G) & \xrightarrow{-\overline{\partial}_G} & A^{0,2}(-_{\mathbb{R}}, G) \xrightarrow{-\overline{\partial}_G} \ldots \end{array}$$

defines a morphism of complexes

$$\Psi'\colon C(\mathrm{ad}_G) \longrightarrow (\sigma_{\geq 1}(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G))[1]$$

where $\sigma_{\geq 1}$ denotes the naive truncation.

Let F and G be two $\mathcal{O}_X$-modules, with F coherent. Then the identification of

$$A^{0,1}(X_{\mathbb{R}}, \mathcal{H}om_{\mathcal{O}_X}(F, G))$$

and

$$\mathrm{Hom}_{\mathcal{O}_X}(F, A^{0,1}(-_{\mathbb{R}}, G))$$

defines, by restriction, a canonical isomorphism

$$Z^{0,1}_{\overline{\partial}}\big(X_{\mathbb{R}}, \mathcal{H}om_{\mathcal{O}_X}(F, G)\big) = \mathrm{Hom}_{\mathcal{O}_X}\Big(F, \mathrm{Ker}\,\big(\overline{\partial}_G : A^{0,1}(-_{\mathbb{R}}, G) \to A^{0,2}(-_{\mathbb{R}}, G)\big)\Big), \tag{2.31}$$

hence a morphism of abelian groups

$$z : Z^{0,1}_{\overline{\partial}}\big(X_{\mathbb{R}}, \mathcal{H}om_{\mathcal{O}_X}(F, G)\big) \to \mathrm{Hom}_{D^+(\mathcal{O}_X-\mathrm{mod})}\big(F, (\sigma_{\geq 1}(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G))[1]\big). \tag{2.32}$$

Lemma 2.5.6. *i) The sheaves $A^{0,j}(\_, G)$ over $X_\Sigma(\mathbb{C})$ are fine. For any $k > 0$, we have*

$$R^k \rho_* A^{0,j}(\_, G) = 0 \tag{2.33}$$

and

$$R^k \rho_* \sigma_{\geq 1}(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G) = 0. \tag{2.34}$$

ii) The morphism z in (2.32) is an isomorphism if F is a vector bundle over X.

Proof. i) The sheaves $A^{0,j}(-, \mathbb{C})$ on the paracompact Hausdorff space $X_\Sigma(\mathbb{C})$ are fine. It follows that the tensor products $A^{0,j}(-, G)$ are fine. We can compute $R^k \rho_* A^{0,j}(-, G)$ as the sheaf associated with the presheaf

$$U \mapsto H^k\big((U_\Sigma(\mathbb{C}), A^{0,j}(\_, G)\big).$$

The latter groups vanish as fine sheaves are $\Gamma\big(U_\Sigma(\mathbb{C}), -\big)$-acyclic. This proves (2.33), which in turn immediately implies (2.34).

ii) To prove that z is an isomorphism when F is a vector bundle, we may replace F by $\mathcal{O}_X$ and G by $F^\vee \otimes G$. Hence we may assume that F is $\mathcal{O}_X$. To simplify notations, let us write $\sigma_{\geq 1} A^{0,\cdot}(X_{\mathbb{R}}, G)$ for $\sigma_{\geq 1}(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G$. The Leray spectral sequence

$$E_2^{j,k} = H^j\big(X, R^k \rho_* \sigma_{\geq 1} A^{0,\cdot}(-, G)\big) \Rightarrow H^{j+k}\big(X_\Sigma(\mathbb{C}), \sigma_{\geq 1} A^{0,\cdot}(-, G)\big)$$

degenerates at E_2 by i). Hence

$$H^1\big(X, \sigma_{\geq 1} \rho_* A^{0,\cdot}(-, G)\big) = H^1\big(X_\Sigma(\mathbb{C}), \sigma_{\geq 1} A^{0,\cdot}(-, G)\big),$$

and consequently,

$$\begin{aligned} \operatorname{Hom}_{D^+(\mathcal{O}_X-\mathrm{mod})}\big(\mathcal{O}_X, (\sigma_{\geq 1} A^{0,\cdot}(-_{\mathbb{R}}, G))[1]\big) &= H^1\big(X, \sigma_{\geq 1} \rho_* A^{0,\cdot}(-, G)\big)^{F_\infty} \\ &= H^1\big(X_\Sigma(\mathbb{C}), \sigma_{\geq 1} A^{0,\cdot}(-, G)\big)^{F_\infty} \\ &= Z^{0,1}_{\overline{\partial}}(X_\Sigma(\mathbb{C}), G_{\mathbb{C}})^{F_\infty} \\ &= Z^{0,1}_{\overline{\partial}}\big(X_{\mathbb{R}}, G\big) \end{aligned}$$

as the sheaves $A^{0,\cdot}(-, G)$ are $\Gamma\big(X_\Sigma(\mathbb{C}), -\big)$-acyclic. □

Proposition 2.5.7. *For any two vector bundles F and G on X, the following diagram is commutative:*

$$\begin{array}{ccc} \widehat{\operatorname{Ext}}^1_X(F, G) & \xrightarrow{\Psi} & Z^{0,1}_{\overline{\partial}}\big(X_{\mathbb{R}}, \mathcal{H}om_{\mathcal{O}_X}(F, G)\big) \\ \downarrow \widehat{\operatorname{cl}}_{F,G} & & \sim \downarrow -z \\ \operatorname{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, C(\mathrm{ad}_G)\big) & \xrightarrow{\Psi' \circ \cdot} & \operatorname{Hom}_{D^+(\mathcal{O}_X\text{-mod})}\big(F, (\sigma_{\geq 1}(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G))[1]\big) \end{array}$$

is commutative.

Proof. Let $(\mathcal{E}, s)$ be an arithmetic extension and, as before, define t by (2.3). The composite map $\Psi' \circ \widehat{\operatorname{cl}}_{F,G}$ sends the class of $(\mathcal{E}, s)$ in $\widehat{\operatorname{Ext}}^1_X(F, G)$ to the morphism $\Psi' \circ \partial_{(\mathcal{E},s)}$ in

$D^+(\mathcal{O}_X\text{-mod})$, which is defined by the diagram

$$\begin{array}{ccccccc}
 & & F & & & & \\
 & & \uparrow p & & & & \\
G & \xrightarrow{i} & E & & & & \\
\downarrow \mathrm{Id}_G & & \downarrow \tilde{t} & & & & \\
G & \xrightarrow{\mathrm{ad}_G} & (\rho_* G_{\mathbb{C}})^{F_\infty} & & & & \\
 & & \downarrow \overline{\partial}_G & & & & \\
 & & A^{0,1}(-_{\mathbb{R}}, G) & \xrightarrow{-\overline{\partial}_G} & A^{0,2}(-_{\mathbb{R}}, G) & \xrightarrow{-\overline{\partial}_G} & \dots
\end{array}$$

It may also be written as

$$\Psi' \circ \partial_{(\mathcal{E},s)} : F \xleftarrow{\mathbf{p}} C(i) \xrightarrow{(\mathrm{Id}_G, \tilde{t})} C(\mathrm{ad}_G) \xrightarrow{\overline{\partial}_G} (\sigma_{\geq 1} A^{0,\cdot}(-_{\mathbb{R}}, G))[1]. \tag{2.35}$$

Let us denote $\alpha := \Psi(\mathcal{E}, s)$. This is the element of $Z^{0,1}_{\overline{\partial}}\big(X_{\mathbb{R}}, F^\vee \otimes G\big)$ characterized by the relation

$$\overline{\partial}_E(s.f) = s.\overline{\partial}_F + i_{\mathbb{C}}(\alpha.f),$$

for any local section f of $A^{0,\cdot}(-, F_{\mathbb{C}})$ over $X_\Sigma(\mathbb{C})$. We also have

$$\overline{\partial}_{E^\vee \otimes G} t = -\alpha \circ p_{\mathbb{C}}. \tag{2.36}$$

To any section β of $A^{0,j}(U_{\mathbb{R}}, F)$ over some open U in X, we may attach the element

$$w(\beta) := (-\alpha(\beta), s(\beta))$$

of degree j in the cone of the morphism $A^{0,\cdot}(-_{\mathbb{R}}, i)$ from $\big(A^{0,\cdot}(-_{\mathbb{R}}, G), \overline{\partial}_G\big)$ to $\big(A^{0,\cdot}(-_{\mathbb{R}}, E), \overline{\partial}_E\big)$ defined by $i_{\mathbb{C}}$. The differential of $w(\beta)$ in the complex $C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big)$ is

$$\begin{aligned}
d(-\alpha.\beta, s.\beta) &= \big(-\overline{\partial}_G(-\alpha.\beta), i_{\mathbb{C}}(-\alpha.\beta) + \overline{\partial}_E(s.\beta)\big) \\
&= \big(-\alpha.(\overline{\partial}_F \beta), s.(\overline{\partial}_F.\beta)\big).
\end{aligned}$$

Consequently w defines a homomorphism of complexes

$$w\colon A^{0,\cdot}(-_{\mathbb{R}}, F) \longrightarrow C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big),$$

where, as before, we write $A^{0,\cdot}(-_{\mathbb{R}}, F)$ for $(A^{0,\cdot}(-_{\mathbb{R}}, F), \overline{\partial}_F)$. It is straightforward that it is a right inverse of the morphism of complexes

$$A^{0,\cdot}(-_{\mathbb{R}}, \mathbf{p})\colon C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big) \longrightarrow A^{0,\cdot}(-_{\mathbb{R}}, F)$$

deduced from the quasi-isomorphism $\mathbf{p} : C(i) \to F$ by considering the associated Dolbeault complexes.

Observe that — since α is $\overline{\partial}$ closed — one defines a morphism of complexes of $\mathcal{O}_X$-modules

$$A : C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big) \longrightarrow (\sigma_{\geq 1} A^{0,\cdot}(-_{\mathbb{R}}, G))[1]$$

by mapping a sections (γ, δ) of

$$C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big)^k := A^{0,k+1}(-_{\mathbb{R}}, G) \oplus A^{0,k}(-_{\mathbb{R}}, E)$$

to the section $\alpha.p(\delta)$ of $A^{0,k+1}(-_{\mathbb{R}}, G)$. Moreover the relation (2.36) shows that the right-hand square of the following diagram is commutative:

$$\begin{array}{ccccc} F & \stackrel{\mathbf{p}}{\longleftarrow} & C(i) & \stackrel{(\mathrm{Id}_G,\tilde{t})}{\longrightarrow} & C(\mathrm{ad}_G) \\ \downarrow \mathrm{ad}_F & & \downarrow (\mathrm{ad}_G,\mathrm{ad}_E) & & \downarrow \overline{\partial}_G \\ A^{0,\cdot}(-_{\mathbb{R}}, F) & \stackrel{A^{0,\cdot}(-_{\mathbb{R}},\mathbf{p})}{\longleftarrow} & C\big(A^{0,\cdot}(-_{\mathbb{R}}, i)\big) & \stackrel{-A}{\longrightarrow} & (\sigma_{\geq 1}A^{0,\cdot}(-_{\mathbb{R}}, G))[1]. \end{array}$$

The commutativity of the left-hand square is straightforward.

Finally, together with (2.35) and the relation $\mathbf{p} \circ w = \mathrm{Id}_{A^{0,\cdot}(-_{\mathbb{R}},F)}$, the commutativity of the last diagram shows the equality of morphisms in $D^+(\mathcal{O}_X - \mathrm{mod})$:

$$\Psi' \circ \partial_{(\mathcal{E},s)} = -A \circ w \circ \mathrm{ad}_F.$$

It is straightforward that this is precisely $-z(\alpha)$. □

2.6. **Admissible extensions.** Let $\overline{F}$ and $\overline{G}$ be hermitian coherent sheaves over an arithmetic scheme X.

Given a hermitian coherent sheaf $\overline{E}$ on X and an extension

$$\mathcal{E} : 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0,$$

we write

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow 0$$

and call $\overline{\mathcal{E}}$ an *admissible* extension of hermitian coherent sheaves if $\overline{F}$ and $\overline{G}$ carry the induced hermitian metrics from $\overline{E}$. In this case, orthogonal projection on the orthogonal complement of $G_{\mathbb{C}}$ in $E_{\mathbb{C}}$ determines an F_∞-invariant $\mathcal{C}^\infty$-splitting $s^\perp : F_{\mathbb{C}} \to E_{\mathbb{C}}$ of the extension $\mathcal{E}_{\mathbb{C}}$ on $X_\Sigma(\mathbb{C})$, which we shall call the *orthogonal splitting* of $\mathcal{E}$. Amongst the splittings of $\mathcal{E}_{\mathbb{C}}$ over $X_\Sigma(\mathbb{C})$, it is characterized as being fiberwise isometric. In this way, each admissible extension $\overline{\mathcal{E}}$ determines an arithmetic extension $(\mathcal{E}, s^\perp)$ of F by G.

Conversely, if

$$\mathcal{E} : 0 \longrightarrow G \stackrel{i}{\longrightarrow} E \stackrel{p}{\longrightarrow} F \longrightarrow 0$$

is an extension of $\mathcal{O}_X$-modules, then any F_∞-invariant $\mathcal{C}^\infty$-splitting $s : F_{\mathbb{C}} \to E_{\mathbb{C}}$ of the extension $\mathcal{E}_{\mathbb{C}}$ determines an hermitian structure on E. Namely, the map

$$\varphi := (i_{\mathbb{C}}, s) : G_{\mathbb{C}} \oplus F_{\mathbb{C}} \longmapsto E_{\mathbb{C}}$$

is a F_∞-invariant $\mathcal{C}^\infty$-isomorphism of vector bundles over $X_\Sigma(\mathbb{C})$, and the hermitian metric on $G_{\mathbb{C}} \oplus F_{\mathbb{C}}$ defined as the orthogonal direct sum of the ones defining $\overline{G}$ and $\overline{F}$ may be transported by φ to a F_∞-invariant $\mathcal{C}^\infty$-metric $\|.\|$ on E. If we let $\overline{E} := (E, \|.\|)$, then

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \stackrel{i}{\longrightarrow} \overline{E} \stackrel{p}{\longrightarrow} \overline{F} \longrightarrow 0$$

is an admissible extension, the orthogonal splitting of which is precisely s.

An isomorphism of admissible extensions $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ of $\overline{F}$ by $\overline{G}$ is, by definition, an isomorphism from $\overline{E}_1$ to $\overline{E}_2$ which induces the identity on F and G. The constructions above induces a one to one correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes of admissible} \\ \text{extensions of } \overline{F} \text{ by } \overline{G} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphim classes of arithmetic} \\ \text{extensions of } F \text{ by } G \end{array} \right\},$$

and we obtain a new interpretation of the group $\widehat{\operatorname{Ext}}^1_X(F,G)$ as group of isomorphism classes of admissible extensions of $\overline{F}$ by $\overline{G}$. The corresponding "admissible Baer sum" of two admissible extensions

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \xrightarrow{i_j} \overline{E}_j \xrightarrow{p_j} \overline{F} \longrightarrow 0\ ,\ (j=1,2)$$

has the following explicit description. We define a hermitian metric on

$$\ker(p_1 - p_2 : E_1 \oplus E_2 \longrightarrow F) \subseteq E_1 \oplus E_2$$

by the formula

$$\|(e_1,e_2)\|^2_\sigma = 2(\|e_1\|^2_{\overline{E}_1,\sigma} + \|e_2\|^2_{\overline{E}_2,\sigma}) - 3\|p_1(e_1)\|^2_{\overline{F},\sigma} \tag{2.37}$$

for σ in Σ and (e_1,e_2) a vector in a fiber of $E_{1,\sigma} \oplus E_{2,\sigma}$ such that $p_1(e_1) = p_2(e_2)$. We equip the algebraic Baer sum E in (1.4)(=(A.12)) with the quotient metric. It is straightforward to check that the resulting extension

$$\overline{\mathcal{E}_j} : 0 \longrightarrow \overline{G} \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow 0$$

is admissible. It corresponds to the arithmetic Baer sum. Indeed we have

$$\|[(s_1^\perp(f), s_2^\perp(f))]\|_{\overline{E},\sigma} = \|f\|_{\overline{F},\sigma}$$

for any vector f in a fiber of F_σ, if $s_j^\perp$ denotes the orthogonal splitting of $\overline{\mathcal{E}}_j$.

We leave the details to the interested reader, and just want to emphasize that this correspondence would *not* hold with $\overline{E}$ defined as E equipped with the hermitian structure it naively inherits as a subquotient of the orthogonal direct sum $\overline{E}_1 \oplus \overline{E}_2$.

2.7. **Arithmetic torsors.** Let X be an arithmetic scheme over some arithmetic ring R. Let G be a connected flat linear group scheme over $S = \operatorname{Spec} R$ of finite type. For each σ in Σ, we fix a maximal compact subgroup K_σ of $G_\sigma(\mathbb{C})$ such that the family $K_\infty = (K_\sigma)_{\sigma\in\Sigma}$ is invariant under F_∞. Chambert-Loir and Tschinkel define an arithmetic (G,K_∞)-torsor on X as a pair $(\mathcal{T}, s)$ given by a G-torsor $\mathcal{T}$ on X together with a family of sections $s = (s_\sigma)_{\sigma\in\Sigma}$ where s_σ is a $\mathcal{C}^\infty$-section of the $K_\sigma\backslash G_\sigma(\mathbb{C})$-fibration on $X_\sigma(\mathbb{C})$ obtained as the quotient of $\mathcal{T}_\sigma(\mathbb{C})$ by the action of K_σ. The set of isomorphism classes of arithmetic (G,K_∞)-torsors on X is denoted by $\widehat{H}^1\big(X,(E,K_\infty)\big)$ [CLT01, 1.1].

Let V be a vector bundle on S. Let $(\mathcal{E}, s)$ be an arithmetic extension over X with underlying extension

$$\mathcal{E}\colon\ 0 \longrightarrow f^*V \longrightarrow E \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0.$$

where $f : X \to S$. We consider E, f^*V and $\mathcal{O}_X = \mathbb{G}_a$ as vector group schemes and denote by $1\colon\ X \to \mathbb{G}_a$ the section of $\mathbb{G}_a$ associated to the unit in $\mathcal{O}_X$. The scheme $\mathcal{T} = \pi^{-1}(1) = E \times_{\mathbb{G}_a} X$ is the V-torsor defined by the splittings of the extension $\mathcal{E}$. The $\mathcal{C}^\infty$ splitting s of $\mathcal{E}_\mathbb{C}$ induces a $\mathcal{C}^\infty$ section of the $V_\mathbb{C}$-torsor $\mathcal{T}_\mathbb{C}$ on $X_\mathbb{C}$. Let $K_\sigma = \{0\}$ be the maximal compact subgroup of $G_\sigma(\mathbb{C})$. The pair $(\mathcal{T}, s)$ is an arithmetic $\big(G, K_\infty = (K_\sigma)_\sigma\big)$-torsor and determines an element in $\widehat{H}^1\big(X,(G,K_\infty)\big)$. One checks easily that the construction which associates $\big(\mathcal{T} = \pi^{-1}(1), s\big)$ to $(\mathcal{E}, s)$ induces an isomorphism of groups:

$$\widehat{\operatorname{Ext}}^1_X(\mathcal{O}_X, f^*V) \longrightarrow \widehat{H}^1(X,(G,K_\infty)). \tag{2.38}$$

One recovers the first exact sequence 2.2.1 when $F = \mathcal{O}_X$ and $G = f^*V$ as a special case of the exact sequence [CLT01, 1.2.1 and 1.2.3], and the adelic description 2.2.3 as a special case of [CLT01, 1.2.6].

3. Slopes of hermitian vector bundles and splittings of extensions over arithmetic curves

Consider an extension

$$\mathcal{E}\colon\ 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

of a line bundle F by a line bundle G on a smooth projective geometrically connected curve C of genus g over a field k. A straightforward application of Serre duality shows that *if $\mathcal{E}$ does not split, the following inequality holds:*

$$\deg G - \deg F \leq 2g - 2. \tag{3.1}$$

Indeed, in this situation, the class of $\mathcal{E}$ provides a non-zero element in

$$\mathrm{Ext}^1_C(F, G) \simeq H^1(C,\ F^\vee \otimes G) \simeq H^0(C,\ \Omega^1_{C/k} \otimes F \otimes G^\vee)^\vee.$$

Hence $\Omega^1_{C/k} \otimes F \otimes G^\vee$ has a non-trivial regular section on C, and its degree is consequently non-negative; this yields (3.1).

In this section, our aim is primarily to establish an arithmetic analog of the inequality (3.1), valid for an admissible extension $\overline{\mathcal{E}}$ of hermitian vector bundles $\overline{F}$ and $\overline{G}$ of arbitrary ranks over an "arithmetic curve" $S := \operatorname{Spec} \mathcal{O}_K$ defined by some number field K.

In order to formulate this analog, we need to introduce some quantitative measure for the non-triviality of an arithmetic extension $\overline{\mathcal{E}}$: it will be given by its *size* $\mathfrak{s}(\overline{\mathcal{E}})$, defined as the logarithm of the distance to zero of the corresponding point on the real torus

$$\widehat{\mathrm{Ext}}^1_S(F, G) = \frac{\mathrm{Hom}_{\mathcal{O}_K}(F, G) \otimes_{\mathbb{Z}} \mathbb{R}}{\mathrm{Hom}_{\mathcal{O}_K}(F, G)}.$$

Indeed, the hermitian structures of $\overline{F}$ and $\overline{G}$ induce an euclidean norm on the real vector space $\mathrm{Hom}_{\mathcal{O}_K}(F, G) \otimes_{\mathbb{Z}} \mathbb{R}$, which may be seen as a flat Riemannian metric on the torus $\widehat{\mathrm{Ext}}^1_S(F, G)$ and defines a distance on it (see 3.5 *infra* for the formal definition of $\mathfrak{s}(\overline{\mathcal{E}})$).

Our arithmetic analog of the inequality (3.1) will take the form:

$$\widehat{\mu}_{\min}(\overline{G}) - \widehat{\mu}_{\max}(\overline{F}) + \mathfrak{s}(\overline{\mathcal{E}}) \leq \frac{\log |\Delta_K|}{[K : \mathbb{Q}]} + \log \frac{\mathrm{rk}_K F_K \cdot \mathrm{rk}_K G_K}{2}. \tag{3.2}$$

Here, $\widehat{\mu}_{\max}(\overline{F})$ (resp. $\widehat{\mu}_{\min}(\overline{G})$) denotes the maximal (resp. minimal) normalized slope of $\overline{F}$ (resp. of $\overline{G}$) (*cf.* [Stu76], [Gra84], and 3.1, *infra*), and Δ_K the discriminant of the number field K. Our proof of (3.2) will rely on (i) some upper bound on the Arakelov degree of a sub-line bundle in the tensor product of two hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$ (Proposition 3.4.1), and (ii) on some "transference" result from the geometry of numbers, relating minima of an euclidean lattice and of its dual lattice, in the precise form obtained by Banaszczyk ([Ban93]).

3.1. **Arithmetic degree and slopes.**

We now discuss a few results concerning hermitian vector bundles on "arithmetic curves" and their arithmetic degree and slopes. We refer the reader to [Szp85],[Lan88],[Stu76],[Gra84], and [Neu99] for more extensive discussions of these notions.

Let K be a number field, and Σ the set of its fields embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Let $\overline{L} = (L, (\|.\|_\sigma)_{\sigma \in \Sigma})$ be a hermitian line bundle on the arithmetic curve $S = \operatorname{Spec} \mathcal{O}_K$. The expression

$$\widehat{\deg}\,\overline{L} = \log \# \left(L/\mathcal{O}_K \cdot s\right) - \sum_{\sigma\in\Sigma} \log \|s\|_\sigma$$

does not depend on the choice of a non-zero section $s \in \Gamma(S, L)$ and defines the arithmetic degree of $\overline{L}$.

For an arbitrary hermitian coherent sheaf $\overline{E} = (E, (\|.\|_\sigma)_{\sigma\in\Sigma})$ over S with $\mathcal{O}_K$-torsion subsheaf E_{tors}, the quotient E/E_{tors} equipped with $(\|.\|_\sigma)_{\sigma\in\Sigma}$ defines a hermitian vector bundle $\overline{E/E_{\mathrm{tors}}}$ on S. We define the *arithmetic degree* of the hermitian coherent sheaf $\overline{E}$ by the formula [GS91, 2.4.1]

$$\widehat{\deg}\,\overline{E} = \widehat{\deg}\,(\wedge^{\max}\overline{E/E_{\mathrm{tors}}}) + \log \# E_{\mathrm{tors}}.$$

This is also the arithmetic degree $\widehat{\deg}\det\overline{E}$ of the determinant line of $\overline{E}$.

For every extension of hermitian coherent sheaves on S

$$0 \longrightarrow \overline{G} \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow 0$$

which is admissible in the sense of Section 2.6, we have

$$\widehat{\deg}\,\overline{E} = \widehat{\deg}\,\overline{G} + \widehat{\deg}\,\overline{F} \tag{3.3}$$

(see for instance [Lan88, V 2.1]).

It is also convenient to introduce the *normalized arithmetic degree* of an hermitian coherent sheaf $\overline{E}$ over S, namely

$$\widehat{\deg}_n\,\overline{E} := \frac{1}{[K:\mathbb{Q}]}\,\widehat{\deg}\,\overline{E}.$$

Then the (normalized) *arithmetic slope* of an hermitian coherent sheaf $\overline{E}$ of positive rank $\mathrm{rk}\,E := \mathrm{rk}_K\,E_K$ is defined as

$$\widehat{\mu}(\overline{E}) = \frac{\widehat{\deg}_n\,(\overline{E})}{\mathrm{rk}\,E} = \frac{1}{[K:\mathbb{Q}]} \cdot \frac{\widehat{\deg}(\overline{E})}{\mathrm{rk}\,E}. \tag{3.4}$$

We define the *maximal slope* $\widehat{\mu}_{\max}(\overline{E})$ of an hermitian vector bundle $\overline{E}$ of positive rank as the maximal arithmetic slope of a subbundle of positive rank in E equipped with the induced metric, and its *minimal slope* as

$$\widehat{\mu}_{\min}(\overline{E}) := -\widehat{\mu}_{\max}(\overline{E}^\vee). \tag{3.5}$$

Observe that $\widehat{\mu}_{\max}(\overline{E})$ (resp. $\widehat{\mu}_{\min}(\overline{E})$) coincides with the maximal (resp. minimal) slope of a saturated subbundle (resp. of a quotient coherent sheaf) of E, equipped with the hermitian structure induced by restriction (resp. by quotient) from the one of $\overline{E}$.

It is easily seen that, for any hermitian line bundle $\overline{L}$ on S, we have

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{L}) = \widehat{\mu}_{\max}(\overline{E}) + \widehat{\deg}_n\,\overline{L}$$

and

$$\widehat{\mu}_{\min}(\overline{E}\otimes\overline{L}) = \widehat{\mu}_{\min}(\overline{E}) + \widehat{\deg}_n\,\overline{L},$$

and that, if $\overline{F}$ is any hermitian vector bundle of positive rank on S,

$$\widehat{\mu}(\overline{E}\otimes\overline{F}) = \widehat{\mu}(\overline{E}) + \widehat{\mu}(\overline{F}),$$

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{F}) \geq \widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}), \tag{3.6}$$

and

$$\widehat{\mu}_{\min}(\overline{E}\otimes\overline{F}) \leq \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}). \tag{3.7}$$

Finally, recall that the normalized degree is invariant under field extension. Namely, if K' is a number field containing K and if

$$g : S' := \operatorname{Spec}\mathcal{O}_{K'} \longrightarrow S := \operatorname{Spec}\mathcal{O}_K$$

is the morphism of schemes defined by the inclusion $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$, then for any hermitian vector bundle $\overline{E}$ over S, the hermitian vector bundle $g^*\overline{E}$ over S' satisfies

$$\widehat{\deg}_n\, g^*\overline{E} = \widehat{\deg}_n\, \overline{E}.$$

Consequently, when E has positive rank:

$$\widehat{\mu}(g^*\overline{E}) = \widehat{\mu}(\overline{E}).$$

A simple descent argument, using the Harder-Narasimhan filtrations of $\overline{E}$ and $g^*\overline{E}$, shows that this also holds for the maximal and minimal slopes:

$$\widehat{\mu}_{\max}(g^*\overline{E}) = \widehat{\mu}_{\max}(\overline{E}), \tag{3.8}$$

$$\widehat{\mu}_{\min}(g^*\overline{E}) = \widehat{\mu}_{\min}(\overline{E}). \tag{3.9}$$

Lemma 3.1.1. *Let $\overline{E}$ be a hermitian vector bundle over S. Let $\overline{F}$ be an hermitian coherent subsheaf of E, equipped with the metrics of $\overline{E}$, such that the quotient Q of E by F is torsion. Let I be the annihilator ideal of Q in $\mathcal{O}_K$, and let $N(I) := \#\mathcal{O}_K/I$ denote its norm. Then*

$$\begin{aligned}\widehat{\mu}_{\min}(\overline{E}) &\leq \widehat{\mu}_{\min}(\overline{F}) + \frac{1}{[K:\mathbb{Q}]\cdot \mathrm{rk}E}\log \#Q \\ &\leq \widehat{\mu}_{\min}(\overline{F}) + \frac{1}{[K:\mathbb{Q}]}\log \#N(I).\end{aligned}$$

Proof. The various coherent sheaves on S we shall consider will be equipped with the hermitian structures deduced from the one of $\overline{E}$ by quotient.

Let F' be a quotient vector bundle of F such that

$$\widehat{\mu}(\overline{F'}) = \widehat{\mu}_{\min}(\overline{F}),$$

and let us form the pushout diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & Q & \longrightarrow & 0. \end{array}$$

Observe that E' is a quotient coherent sheaf of E.

As a special case of (3.3) we get:

$$\widehat{\deg}\,\overline{E'} = \widehat{\deg}\,\overline{F'} + \log\#Q. \tag{3.10}$$

Besides, since the $\mathcal{O}_K$-module Q is killed by I, and is a quotient of the locally free $\mathcal{O}_K$-module E, we have:

$$\log\#Q \leq \mathrm{rk}\,E\log N(I). \tag{3.11}$$

From (3.10), we get the first of the desired inequalities:

$$\widehat{\mu}_{\min}(\overline{E}) \leq \widehat{\mu}(\overline{E'}) = \widehat{\mu}(\overline{F'}) + \frac{1}{[K:\mathbb{Q}]\cdot \mathrm{rk}E}\log \#Q.$$

The second one follows from (3.11). □

Lemma 3.1.2. *For any admissible extension*

$$0 \longrightarrow \overline{N} \longrightarrow \overline{E} \longrightarrow \overline{Q} \longrightarrow 0$$

of hermitian vector bundles of positive ranks over S, the following inequality holds:

$$\widehat{\mu}_{\max}(\overline{E}) \geq \frac{\operatorname{rk} N}{1+\operatorname{rk} N}\, \widehat{\mu}(\overline{N}) + \frac{1}{1+\operatorname{rk} N}\, \widehat{\mu}_{\max}(\overline{Q}). \tag{3.12}$$

Proof. let F be any $\mathcal{O}_K$-submodule of positive rank in Q, and F' its inverse image in E. Consider the hermitian vector bundles $\overline{F}$ and $\overline{F'}$ defined by F and F' equipped with the restrictions of the hermitian metrics of $\overline{Q}$ and $\overline{E}$ respectively. We have:

$$\operatorname{rk} F' = \operatorname{rk} N + \operatorname{rk} F$$

and

$$\widehat{\deg}_n \overline{F'} = \widehat{\deg}_n \overline{N} + \widehat{\deg}_n \overline{F}.$$

Therefore

$$\begin{aligned}
\widehat{\mu}(\overline{F}) &= \frac{1}{\operatorname{rk} F}\left(\widehat{\deg}_n \overline{F'} - \widehat{\deg}_n \overline{N}\right) \\
&\leq \frac{1}{\operatorname{rk} F}\left[\operatorname{rk} F' \cdot \widehat{\mu}_{\max}(\overline{E}) - \operatorname{rk} N \cdot \widehat{\mu}(\overline{N})\right] \\
&= \widehat{\mu}_{\max}(\overline{E}) + \frac{\operatorname{rk} N}{\operatorname{rk} F} \cdot \left(\widehat{\mu}_{\max}(\overline{E}) - \widehat{\mu}(\overline{N})\right).
\end{aligned}$$

Since $\operatorname{rk} F \geq 1$ and $\widehat{\mu}_{\max}(\overline{E}) - \widehat{\mu}(\overline{N}) \geq 0$, this shows:

$$\widehat{\mu}_{\max}(\overline{Q}) \leq \widehat{\mu}_{\max}(\overline{E}) + \operatorname{rk} N \cdot \left(\widehat{\mu}_{\max}(\overline{E}) - \widehat{\mu}(\overline{N})\right).$$

This inequality is equivalent to (3.12). □

3.2. **Euclidean lattices and direct images.** By an *euclidean lattice*, we shall mean a pair $(\Gamma, \|.\|)$ where Γ is a free $\mathbb{Z}$-module of finite rank and $\|.\|$ a euclidean norm on $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. such a norm uniquely extends to an hermitian norm, invariant under complex conjugation, on $\Gamma_{\mathbb{C}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{C} \simeq \Gamma_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Accordingly, euclidean lattices may be identified with hermitian vector bundles over $\operatorname{Spec} \mathbb{Z}$, and the classical invariants of the former, such as their successive minima, may be interpreted as invariant of the latter.

For instance, if $\overline{V} := (V, \|.\|)$ is any hermitian vector bundle of positive rank over $\operatorname{Spec} \mathbb{Z}$, the first of its successive minima is by definition:

$$\lambda_1(\overline{V}) := \min \left\{\|v\|, v \in V \setminus \{0\}\right\}.$$

Recall also that, if $\operatorname{covol}(\overline{V})$ denotes the covolume of this euclidean lattice, then

$$\widehat{\deg}\, \overline{V} = -\log \operatorname{covol}(\overline{V}).$$

For any positive integer r, let

$$v_r := \frac{\pi^{r/2}}{\Gamma(\frac{r}{2}+1)}$$

be the volume of the unit euclidean ball in $\mathbb{R}^r$. Minkowski's First Theorem on euclidean lattices may be reformulated as follows:

Proposition 3.2.1. *For any hermitian vector bundle $\overline{V}$ of positive rank r over* $\operatorname{Spec}\mathbb{Z}$, *we have*

$$\log \lambda_1(\overline{V}) \leq -\widehat{\mu}(\overline{V}) - \frac{1}{r}\log v_r + \log 2. \tag{3.13}$$

Observe that

$$\psi(r) := -\frac{1}{r}\log v_r + \log 2 = \frac{1}{r}\log\Gamma(\frac{r}{2}+1) - \frac{1}{2}\log\frac{\pi}{4}$$

considered as a function of $r \in]0,+\infty[$, is increasing (since $\log\Gamma$ is convex on $]0,+\infty[$ and vanishes at 1). Moreover, for any $x \in]0,+\infty[$, we have

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\theta(x)/12x},$$

where $\theta(x)$ belongs to $]0,1[$. This shows that, for positive r,

$$\psi(r) = \frac{1}{2}\log r - \frac{1}{2}\log\frac{e\pi}{2} + \varepsilon(r)$$

where

$$\varepsilon(r) := \frac{1}{2r}\log(\pi r) + \frac{1}{6r^2}\theta(r/2)$$

goes to 0 when r increases to infinity. Finally, since the unit euclidean ball in $\mathbb{R}^r$ contains the "cube" $[-1/\sqrt{r}, 1/\sqrt{r}]^r$ for any positive integer r, we have

$$v_r \geq \left(\frac{2}{\sqrt{r}}\right)^r,$$

and consequently

$$\psi(r) \leq \frac{1}{2}\log r.$$

Moreover this inequality is strict if $r > 1$.

If K is a number field and $\overline{E} = (E, (\|.\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ an hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$, we have defined its direct image $\pi_*\overline{E}$ — where π denotes the morphism $\pi : \operatorname{Spec}\mathcal{O}_K \to \operatorname{Spec}\mathbb{Z}$ — in 1.2.1 as the euclidean lattice $(\pi_*E, \|.\|)$, where π_*E denotes E seen as a $\mathbb{Z}$-module, and $\|.\|$ the euclidean norm on $E\otimes_{\mathbb{Z}}\mathbb{R}$ restriction of the hermitian scalar product $\langle .,.\rangle$ on $E\otimes_{\mathbb{Z}}\mathbb{C} \simeq \bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} E_\sigma$ defined by the direct sum of the hermitian scalar products $\langle .,.\rangle_\sigma$ on the E_σ's attached to the norms $\|.\|_\sigma$. In other words, for any v, w in $\pi_*E = E$,

$$\langle v,w\rangle := \sum_{\sigma:K\hookrightarrow\mathbb{C}} \langle v,w\rangle_\sigma.$$

If Δ_K denotes the discriminant of the number field K, then the arithmetic degree of $\pi_*\overline{E}$ is given by:

$$\widehat{\deg}\,\pi_*\overline{E} = \widehat{\deg}\,\overline{E} - \frac{\operatorname{rk}_{\mathcal{O}_K}E}{2}\log|\Delta_K|.$$

(see for instance [Neu99, III.7-8] or [BGS94, (2.1.13)]). Consequently, if E has positive rank,

$$\widehat{\mu}(\pi_*\overline{E}) = \widehat{\mu}(\overline{E}) - \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}. \tag{3.14}$$

Let $\omega_{\mathcal{O}_K} := \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$ denote the canonical module, or inverse of the different, of the number field K. The formula

$$(af)(b) = f(ab) \tag{3.15}$$

defines a $\mathcal{O}_K$-module structure on $\omega_{\mathcal{O}_K}$. It is an invertible $\mathcal{O}_K$-module which is generated up to torsion by the trace map $\mathrm{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$. It becomes an hermitian line bundle $\overline{\omega}_{\mathcal{O}_K} := (\omega_{\mathcal{O}_K}, (\|.\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ over $\operatorname{Spec}\mathcal{O}_K$ if we equip it with the hermitian norms defined by $\|\mathrm{tr}_{K/\mathbb{Q}}\|_\sigma = 1$. Then the arithmetic degree of $\overline{\omega}_{\mathcal{O}_K}$ is given by the well-known formula

$$\widehat{\deg}\,\overline{\omega}_{\mathcal{O}_K} = \log \#\left(\omega_{\mathcal{O}_K}/\mathcal{O}_K \cdot \mathrm{tr}_{K/\mathbb{Q}}\right) = \log|\Delta_K|. \tag{3.16}$$

Moreover, the hermitian line bundle $\overline{\omega}_{\mathcal{O}_K}$ satisfies the following duality property:

Proposition 3.2.2. *For any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathcal{O}_K$, the $\mathbb{Z}$-linear map*

$$\begin{array}{rrcl} I: & E^\vee \otimes_{\mathcal{O}_K} \omega_{\mathcal{O}_K} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(E,\mathbb{Z}) \\ & \xi\otimes\lambda & \longmapsto & \lambda\circ\xi \end{array}$$

defines an isometric isomorphism of hermitian vector bundles over $\operatorname{Spec}\mathbb{Z}$:

$$\pi_*(\overline{E}^\vee \otimes \overline{\omega}_{\mathcal{O}_K}) \xrightarrow{\sim} (\pi_*\overline{E})^\vee. \tag{3.17}$$

Proof. To check that I is an isomorphism of $\mathbb{Z}$-modules, we may work locally over $\operatorname{Spec}\mathbb{Z}$. Then we may assume that E is a trivial vector bundle, and the assertion is clear.

Moreover, we have canonical isomorphisms:

$$\pi_*(E^\vee\otimes\omega_{\mathcal{O}_K})_{\mathbb{C}} \simeq \bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} (E_\sigma^\vee \otimes_{\mathbb{C}} \omega_{\mathcal{O}_K,\sigma}) \tag{3.18}$$

and

$$(\pi_*E)^\vee_{\mathbb{C}} \simeq \mathrm{Hom}_{\mathbb{C}}\left(\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} E_\sigma, \mathbb{C}\right) \simeq \bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} E_\sigma^\vee. \tag{3.19}$$

Since $\omega_{\mathcal{O}_K}\otimes_{\mathcal{O}_K} K = K\cdot \mathrm{tr}_{K/\mathbb{Q}}$, for any embedding $\sigma\hookrightarrow\mathbb{C}$, the $\mathbb{C}$-vector space $\omega_{\mathcal{O}_K,\sigma}$ is one dimensional with basis the image $\mathrm{tr}_{K/\mathbb{Q},\sigma}$ of $\mathrm{tr}_{K/\mathbb{Q}}$, and a vector in $E_\sigma^\vee\otimes_{\mathbb{C}}\omega_{\mathcal{O}_K,\sigma}$ may be written uniquely as $\xi\otimes\mathrm{tr}_{K/\mathbb{Q},\sigma}$ for some ξ in $E_\sigma^\vee$.

For any ξ in $E^\vee$, the image of the element $\xi\otimes\mathrm{tr}_{K/\mathbb{Q}}$ of $\pi_*(E^\vee\otimes\omega_{\mathcal{O}_K})_{\mathbb{C}}$ by (3.17) is $(\xi_\sigma\otimes\mathrm{tr}_{K/\mathbb{Q},\sigma})_\sigma$, and the image of the element $\mathrm{tr}_{K/\mathbb{Q}}\circ\xi$ of $\pi_*E^\vee$ by (3.19) is $(\xi_\sigma)_\sigma$. Indeed, the $\mathbb{C}$-vector space $(\pi_*E)_{\mathbb{C}} := E\otimes_{\mathbb{Z}}\mathbb{C}$ is generated by the image of the inclusion

$$E \hookrightarrow E\otimes_{\mathbb{Z}}\mathbb{C} \simeq \bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} E_\sigma,$$

and for any v in E,

$$\begin{aligned} (\mathrm{tr}_{K/\mathbb{Q}}\circ\xi)(v) &= \mathrm{tr}_{K/\mathbb{Q}}(\xi((v)) \\ &= \sum_{\sigma:K\hookrightarrow\mathbb{C}} \sigma(\xi(v)) \\ &= \sum_{\sigma:K\hookrightarrow\mathbb{C}} \xi_\sigma(v_\sigma). \end{aligned}$$

Since $\pi_*(E^\vee\otimes\omega_{\mathcal{O}_K})_{\mathbb{C}}$ is generated, as a $\mathbb{C}$-vector space, by its $\mathcal{O}_K$-submodule $E^\vee\otimes\mathrm{tr}_{K/\mathbb{Q}}$, this shows that, using the identifications (3.18) and 3.19), the $\mathbb{C}$-linear map

$$I_{\mathbb{C}} : \pi_*(E^\vee\otimes\omega_{\mathcal{O}_K})_{\mathbb{C}} \longmapsto (\pi_*E)^\vee_{\mathbb{C}}$$

maps an arbitrary element $(\xi(\sigma)\otimes \mathrm{tr}_{K/\mathbb{Q},\sigma})_\sigma$ of $\pi_*(E^\vee\otimes\omega_{\mathcal{O}_K})_\mathbb{C}$ to $(\xi(\sigma))_\sigma$. This makes clear that $I_\mathbb{C}$ is an isometry with respect to the hermitian structures defining $\pi_*(\overline{E}^\vee\otimes\overline{\omega}_{\mathcal{O}_K})$ and $(\pi_*\overline{E})^\vee$. □

3.3. **First minimum, upper degree, and maximum slope.** If K is a number field and $\overline{E}=(E,(\|.\|_{\overline{E},\sigma})_{\sigma:K\hookrightarrow\mathbb{C}})$ an hermitian vector bundle of positive rank over $\operatorname{Spec}\mathcal{O}_K$, we may define its (normalized) *upper arithmetic degree* $\widehat{\mathrm{udeg}}_n\,\overline{E}$ as the maximum of the normalized degree $\widehat{\deg}_n\,\overline{L}$ of a sub-line bundle L of E equipped with the hermitian structure induced by the one of $\overline{E}$. Clearly, in this definition, we may restrict to saturated sub-line bundles. Moreover, if for any prime ideal $\mathbf{p}\neq(0)$ in $\mathcal{O}_K$, we denote $\|.\|_{E,\mathbf{p}}$ the $\mathbf{p}$-adic norm on $E_{K_\mathbf{p}}:=E\otimes_{\mathcal{O}_K}K_\mathbf{p}$ attached to its $\mathcal{O}_\mathbf{p}$-lattice $E\otimes_{\mathcal{O}_k}\mathcal{O}_\mathbf{p}$, we have:

$$\widehat{\mathrm{udeg}}_n\,\overline{E}=-\min_{v\in E_K\setminus\{0\}}\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\mathbf{p}\in\operatorname{Spec}\mathcal{O}_K\setminus\{(0)\}}\log\|v\|_{E,\mathbf{p}}+\sum_{\sigma:K\hookrightarrow\mathbb{C}}\log\|v\|_{\overline{E},\sigma}\right). \tag{3.20}$$

Observe also that, for any hermitian line bundle $\overline{L}$ over $\operatorname{Spec}\mathcal{O}_K$,

$$\widehat{\mathrm{udeg}}_n\,\overline{E}\otimes\overline{L}=\widehat{\mathrm{udeg}}_n\,\overline{E}+\widehat{\deg}_n\,\overline{L}. \tag{3.21}$$

In this paragraph, we show that, for any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathcal{O}_K$ as above, the differences between $\widehat{\mu}_{\max}(\overline{E})$, $\widehat{\mathrm{udeg}}_n\,(\overline{E})$, and $-\log\lambda_1(\pi_*\overline{E})$ may be bounded in terms of K and $\mathrm{rk}_K E_K$ only. We refer the reader to [Sou97], 1.1-2, and [Bor05] for related results concerning the successive minima of hermitian vector bundles over $\operatorname{Spec}\mathcal{O}_K$.

Proposition 3.3.1. *Let K be a number field and π the morphism from $\operatorname{Spec}\mathcal{O}_K$ to $\operatorname{Spec}\mathbb{Z}$. For any hermitian vector bundle $\overline{E}$ of positive rank r over $\operatorname{Spec}\mathcal{O}_K$, the following inequalities hold:*

$$\widehat{\mathrm{udeg}}_n\,\overline{E}\leq\widehat{\mu}_{\max}(\overline{E}), \tag{3.22}$$

$$-\log\lambda_1(\pi_*\overline{E})\leq\widehat{\mathrm{udeg}}_n\,\overline{E}-\frac{1}{2}\log[K:\mathbb{Q}], \tag{3.23}$$

and

$$\widehat{\mu}_{\max}(\overline{E})\leq-\log\lambda_1(\pi_*\overline{E})+\psi([K:\mathbb{Q}]\cdot\mathrm{rk}_{\mathcal{O}_K}E)+\frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}. \tag{3.24}$$

Observe that the compatibility of these three estimates for hermitian vector bundles of rank one imply the following lower bound on $|\Delta_K|$, *à la* Hermite-Minkowski:

$$\frac{\log|\Delta_K|}{[K:\mathbb{Q}]}\geq\log[K:\mathbb{Q}]-2\psi([K:\mathbb{Q}]). \tag{3.25}$$

The right-hand side of (3.25) is positive if $[K:\mathbb{Q}]>1$ and has the positive limit $\log\frac{e\pi}{2}$ when $[K:\mathbb{Q}]$ goes to infinity.

In the sequel, we shall use the following consequence of (3.24) and (3.23):

$$\widehat{\mu}_{\max}(\overline{E})\leq\widehat{\mathrm{udeg}}_n\,\overline{E}-\frac{1}{2}\log[K:\mathbb{Q}]+\psi([K:\mathbb{Q}]\cdot\mathrm{rk}_{\mathcal{O}_K}E)+\frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}. \tag{3.26}$$

Using the inequality

$$\log([K:\mathbb{Q}]\cdot \operatorname{rk}_{\mathcal{O}_K} E) - 2\psi([K:\mathbb{Q}]\cdot \operatorname{rk} E) \geq 0,$$

we also obtain a slightly weaker version of (3.26):

$$\widehat{\mu}_{\max}(\overline{E}) \leq \widehat{\operatorname{udeg}}_n \overline{E} + \frac{1}{2}\log \operatorname{rk} E + \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}. \tag{3.27}$$

Proof. The estimate (3.22) is a trivial consequence of the definitions of $\widehat{\operatorname{udeg}}_n \overline{E}$ and $\widehat{\mu}_{\max}(\overline{E})$.

To prove (3.23), consider a non-zero element v of $E(=\pi_* E)$ such that

$$\|v\|^2_{\pi_*\overline{E}} = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \|v\|^2_\sigma$$

is minimal, and the rank one sub-bundle $L := \mathcal{O}_K\cdot v$ of E it generates. We have:

$$-\log\lambda_1(\pi_*\overline{E}) = \frac{1}{2}\log\|v\|^2_{\pi_*\overline{E}} = -\frac{1}{2}\log\left(\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\|v\|^2_\sigma\right) - \frac{1}{2}\log[K:\mathbb{Q}], \tag{3.28}$$

and

$$-\frac{1}{2[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\log\|v\|^2_\sigma = \widehat{\deg}_n \overline{L} \leq \widehat{\operatorname{udeg}}_n \overline{E}. \tag{3.29}$$

The inequality (3.23) follows from (3.28), (3.29), and the convexity of the function $-\log$.

To prove (3.24), consider an arbitrary sub-vector bundle F of positive rank in E. Clearly,

$$\lambda_1(\pi_*\overline{E}) \leq \lambda_1(\pi_*\overline{F}). \tag{3.30}$$

Moreover, Minkowski's theorem (3.13) shows that

$$\log\lambda_1(\pi_*\overline{F}) \leq -\widehat{\mu}(\pi_*\overline{F}) + \psi([K:\mathbb{Q}]\cdot \operatorname{rk}_{\mathcal{O}_K} F). \tag{3.31}$$

Using (3.14) and the fact that ψ is increasing, this shows:

$$\log\lambda_1(\pi_*\overline{F}) \leq -\widehat{\mu}(\overline{F}) + \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]} + \psi([K:\mathbb{Q}]\cdot \operatorname{rk}_{\mathcal{O}_K} E). \tag{3.32}$$

The estimate (3.24) follows from (3.30) and (3.32). □

Actually, the proof above establishes a stronger form of (3.23). Namely, it shows that, if v is an element of $E\setminus\{0\}$ such that $\|v\|_{\pi_*\overline{E}}$ is minimal, then

$$\begin{aligned} -\log\lambda_1(\pi_*\overline{E}) = -\log\|v\|_{\pi_*\overline{E}} &\leq \widehat{\operatorname{udeg}}_n \overline{\mathcal{O}_K\cdot v} - \frac{1}{2}\log[K:\mathbb{Q}] \\ &\leq \widehat{\operatorname{udeg}}_n \overline{E} - \frac{1}{2}\log[K:\mathbb{Q}]. \end{aligned} \tag{3.33}$$

When $\overline{E}$ has rank one, (3.24) may be written:

$$\widehat{\deg}_n(\overline{E}) \leq -\log\|v\|_{\pi_*\overline{E}} + \psi([K:\mathbb{Q}]) + \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}, \tag{3.34}$$

and the conjunction of (3.33) and (3.34) may be seen as a "quantitative version" of the fundamental theorems of Dirichlet about the rings of algebraic integers — the finiteness of the ideal class group, and the "unit theorem".

Indeed, these theorems easily lead to (3.34), with

$$\psi([K:\mathbb{Q}]) + \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}$$

replaced by a constant depending on K only: if E is the trivial line bundle over $\mathcal{O}_K$, this follows from the unit theorem; the general case follows using the finiteness of the ideal class group.

Conversely, (3.33) and (3.34) easily imply these two theorems (compare [Szp85]). To show this, let us define

$$A(K) := -\frac{1}{2}\log[K:\mathbb{Q}] + \psi([K:\mathbb{Q}]) + \frac{\log|\Delta_K|}{2\,[K:\mathbb{Q}]}$$

and

$$B(K) = [K:\mathbb{Q}]\cdot A(K).$$

(One might observe that $A(K) \leq (\log|\Delta_K|)/(2[K:\mathbb{Q}])$ and $B(K) \leq (\log|\Delta_K|)/2$.) If, for every $\mathbf{p} \in \operatorname{Spec}\mathcal{O}_K \setminus \{(0)\}$, we denote $n_{\mathbf{p}}$ the $\mathbf{p}$-adic valuation of v as a section of E, we get from (3.33) and (3.5):

$$\begin{aligned}\sum_{\mathbf{p}} n_{\mathbf{p}} \log N_{\mathbf{p}} = \widehat{\deg}\,\overline{E} - \widehat{\deg}\,\overline{\mathcal{O}_K\cdot v} \quad &\leq \quad [K:\mathbb{Q}](\widehat{\deg}\,\overline{E} + \log\|v\|_{\pi_*\overline{E}} - \frac{1}{2}\log[K:\mathbb{Q}]) \\ &\leq \quad B(K).\end{aligned}$$

In other words, the divisor

$$\operatorname{div} v := \sum_{\mathbf{p}} n_{\mathbf{p}}\mathbf{p}$$

of v belongs to the finite set of effective divisors on $\operatorname{Spec}\mathcal{O}_K$:

$$\mathcal{D} := \left\{ \sum_{\mathbf{p}} n_{\mathbf{p}}\mathbf{p} \mid n_{\mathbf{p}} \geq 0 \wedge \sum_{\mathbf{p}} n_{\mathbf{p}} \log N_{\mathbf{p}} \leq B(K) \right\}.$$

This already shows that any element of the ideal class group $Cl_K := \operatorname{Pic}(\operatorname{Spec}\mathcal{O}_K)$ of K is the class of some divisor in $\mathcal{D}$ — in particular, Cl_K is finite.

Moreover, if $\overline{L}$ is any hermitian line bundle over $\operatorname{Spec}\mathcal{O}_K$ such that $L = \mathcal{O}_K$ and $\widehat{\deg}\,\overline{L} = 0$, then we may consider the hermitian line bundles $\overline{L}^{\otimes n}$, $n \in \mathbb{N}$, and choose non-zero elements v_n in $L^{\otimes n} \simeq \mathcal{O}_K$ such that $\|v_n\|_{\pi_*\overline{L}^{\otimes n}}$ is minimal. Since the divisors $\operatorname{div} v_n$ lie in the finite set $\mathcal{D}$, there exists an increasing sequence (n_i) in $\mathbb{N}$ such that the divisors $\operatorname{div} v_{n_i}$ coincide. Then the elements $u_i := v_{n_0}^{-1}\cdot v_{n_i}$ of $L^{\otimes(n_i-n_0)} \simeq \mathcal{O}_K$ are units of $\mathcal{O}_K$, and their norms $\|u_i\|_{\pi_*\overline{L}^{\otimes(n_i-n_0)}}$ stay bounded when i goes to infinity. If we let $t_\sigma := \log\|1\|_{\overline{L},\sigma}^{-1}$ for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, and $m_i := n_i - n_0$, this shows that the sequences $(\log|\sigma(u_i)| - m_i.t_\sigma)_i$ are bounded from above. Since $\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log|\sigma(u_i)|$ and $\sum_{\sigma:K\hookrightarrow\mathbb{C}} t_\sigma$ vanish, these sequences are also bounded from below, and consequently, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$\lim_{i\to+\infty} \frac{1}{m_i}\log|\sigma(u_i)| = t_\sigma.$$

As any family $(t_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$ of $[K:\mathbb{Q}]$ real numbers satisfying $t_{\overline{\sigma}} = t_\sigma$ and $\sum_\sigma t_\sigma = 0$ arises in this construction, this implies Dirichlet's unit theorem.

3.4. **The upper degree of a tensor product.**

Proposition 3.4.1. *For any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ of positive rank over* $\operatorname{Spec}\mathcal{O}_K$*, we have:*

$$\widehat{\operatorname{udeg}}_n\,(\overline{F}\otimes\overline{G}^\vee) \leq \widehat{\mu}_{\max}(\overline{F}) - \widehat{\mu}_{\min}(\overline{G}) \tag{3.35}$$

and

$$\widehat{\mathrm{udeg}}_n\,(\overline{F}\otimes\overline{G}) \leq \widehat{\mu}_{\max}(\overline{F}) + \widehat{\mu}_{\max}(\overline{G}). \tag{3.36}$$

Observe that, from this proposition and the comparison of the upper degree and the maximal slope in Proposition 3.3.1, we derive estimates on the maximum and minimum slopes of tensor products which complement (3.6) and (3.7), namely:

$$\widehat{\mu}_{\max}(\overline{E}\otimes\overline{F}) \leq \widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}) + \frac{1}{2}\log(\mathrm{rk}\,E\cdot\mathrm{rk}\,F) + \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}, \tag{3.37}$$

and

$$\widehat{\mu}_{\min}(\overline{E}\otimes\overline{F}) \geq \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - \frac{1}{2}\log(\mathrm{rk}\,E\cdot\mathrm{rk}\,F) - \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}. \tag{3.38}$$

Actually, it is possible to establish similar estimates where the term

$$\frac{1}{2}\log(\mathrm{rk}\,E\cdot\mathrm{rk}\,F) + \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}$$

is replaced by a constant depending on $\mathrm{rk}\,E$ and $\mathrm{rk}\,F$ only (see [Bos96], Propositions A.4 and A.5, and also [Gra01], appendix), which however is larger than this term when $K=\mathbb{Q}$.

Proof. According to the expression (3.24) of $\widehat{\mu}_{\min}$ in term of $\widehat{\mu}_{\max}$, its is enough to establish (3.35). The expression (3.20) for the upper degree shows that, to achieve this, we need to prove that any non-zero element ϕ of $(F\otimes G^\vee)_K \simeq \mathrm{Hom}_K(G_K,F_K)$ satisfies

(3.39)

$$-\frac{1}{[K:\mathbb{Q}]}\left(\sum_{\mathbf{p}\in\mathrm{Spec}\,\mathcal{O}_K\setminus\{(0)\}}\log\|\phi\|_{F\otimes G^\vee,\mathbf{p}} + \sum_{\sigma:K\hookrightarrow\mathbb{C}}\log\|\phi\|_{\overline{F}\otimes\overline{G}^\vee,\sigma}\right) \leq \widehat{\mu}_{\max}(\overline{F}) - \widehat{\mu}_{\min}(\overline{G}).$$

For any such ϕ, let $I\subset F$ (resp. $J\subset G$) be the saturated subbundle of F (resp. of G) such that $I_K = \mathrm{Im}\,(\phi)$ (resp. $J_K = \mathrm{Ker}(\phi)$), and consider the canonical factorization of ϕ:

$$\begin{array}{ccc} G_K & \xrightarrow{\phi} & F_K \\ \downarrow & & \uparrow \\ (G/J)_K & \xrightarrow{\tilde{\phi}} & I_K. \end{array}$$

By construction, $\tilde{\phi}$ is an isomorphism of K-vector spaces. We shall denote r the rank of ϕ and $\tilde{\phi}$; it is also the rank of I and of G/J.

The determinant $\det\tilde{\phi}$ of $\tilde{\phi}$ may be seen as a non-zero meromorphic section of $(\Lambda^r G/J)^\vee\otimes\Lambda^r I$ over $\mathrm{Spec}\,\mathcal{O}_K$. As such, it has a well defined valuation $v_{\mathbf{p}}(\det\tilde{\phi})$ at every $\mathbf{p}$ in $\mathrm{Spec}\,\mathcal{O}_K\setminus\{(0)\}$, which vanishes for almost every $\mathbf{p}$. We may equip G/J and I with the hermitian structures defined by quotient and restriction from the ones of $\overline{G}$ and $\overline{F}$. Then we have:

$$\begin{aligned} -\widehat{\deg}\,\overline{G/J} + \widehat{\deg}\,\overline{I} &= \widehat{\deg}\,\Lambda^r\overline{G/J}^\vee\otimes\Lambda^r\overline{I} \\ &= \sum_{\mathbf{p}} v_{\mathbf{p}}(\det\tilde{\phi})\log N(\mathbf{p}) - \sum_\sigma \log\|\det\tilde{\phi}\|_{\Lambda^r\overline{G/J}^\vee\otimes\Lambda^r\overline{I},\sigma}. \end{aligned} \tag{3.40}$$

Observe that, if $v_{\mathbf{p}}(\phi)$ denotes the valuation at $\mathbf{p}$ of ϕ seen as a meromorphic section of $F\otimes G^\vee$ over $\mathrm{Spec}\,\mathcal{O}_K$, we have:

$$v_{\mathbf{p}}(\det\tilde{\phi}) \geq r v_{\mathbf{p}}(\phi),$$

or, equivalently:

$$-\log \| \det \tilde{\phi} \|_{\Lambda^r G/J^\vee \otimes \Lambda^r I, \mathbf{p}} \geq -r \log \|\phi\|_{F \otimes G^\vee, \mathbf{p}}. \tag{3.41}$$

This is a straightforward consequence of the theory of elementary divisors applied to $\phi_{K_\mathbf{p}} : G_{K_\mathbf{p}} \to F_{K_\mathbf{p}}$.

Similarly, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, by considering the polar decomposition of ϕ_σ we see that

$$\| \det \tilde{\phi} \|_{\Lambda^r \overline{G/J}^\vee \otimes \Lambda^r \overline{I}, \sigma} \leq \|\phi\|^r_{\overline{F} \otimes \overline{G}^\vee, \sigma},$$

or, equivalently,

$$-\log \| \det \tilde{\phi} \|_{\Lambda^r \overline{G/J}^\vee \otimes \Lambda^r \overline{I}, \sigma} \geq -r \log \|\phi\|_{\overline{F} \otimes \overline{G}^\vee, \sigma}. \tag{3.42}$$

From, (3.40), (3.41) and (3.42), we deduce:

$$-\sum_{\mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{(0)\}} \log \|\phi\|_{F \otimes G^\vee, \mathbf{p}} - \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \|\phi\|_{\overline{F} \otimes \overline{G}^\vee, \sigma} \leq -\widehat{\mu}(\overline{G/J}) + \widehat{\mu}(\overline{I}).$$

Since $\widehat{\mu}(\overline{G/J}) \geq \widehat{\mu}_{\min}(\overline{G})$ and $\widehat{\mu}(\overline{I}) \leq \widehat{\mu}_{\max}(\overline{F})$, this proves (3.39).

□

3.5. **The size of an arithmetic extension.**

3.5.1. *Definition of the size.* Consider a number field K with ring of integers $\mathcal{O}_K$ and denote the spectrum of $\mathcal{O}_K$ by S. Let F and G two vector bundles on the arithmetic curve S, defined by projective $\mathcal{O}_K$-modules we shall also F and G. The morphism

$$b : \pi_*(F^\vee \otimes G)_\mathbb{R} = \operatorname{Hom}_{\mathcal{O}_K}(F, G) \otimes_\mathbb{Z} \mathbb{R} \longrightarrow \widehat{\operatorname{Ext}}^1_S (F, G)$$

induces an isomorphism of abelian groups (compare Corollary 2.2.2):

$$\widehat{\operatorname{Ext}}^1_S (F, G) \simeq \frac{\operatorname{Hom}_{\mathcal{O}_K}(F, G) \otimes_\mathbb{Z} \mathbb{R}}{\operatorname{Hom}_{\mathcal{O}_K}(F, G)} \tag{3.43}$$

Moreover, as an abelian group, the right-hand side of (3.43) may be identified with the real torus

$$\frac{\pi_*(F^\vee \otimes G)_\mathbb{R}}{\pi_*(F^\vee \otimes G)}.$$

From the hermitian structures on $\overline{F}$ and $\overline{G}$, we deduce an hermitian structure on $\overline{F}^\vee \otimes \overline{G}$, that is, for every field embedding $\sigma : K \hookrightarrow \mathbb{C}$, an hermitian structure on the $\mathbb{C}$-vector space

$$(F^\vee \otimes G)_\sigma \simeq \operatorname{Hom}_\mathbb{C}(F_\sigma, G_\sigma).$$

It is given by the "Hilbert-Schmidt" hermitian scalar product $\langle ., . \rangle_\sigma$, defined by

$$\langle T_1, T_2 \rangle_\sigma := \operatorname{tr}(T_2^* T_1)$$

for any $T_1, T_2 \in \operatorname{Hom}_\mathbb{C}(F_\sigma, G_\sigma)$, where the adjoint T_2^* is taken with respect to the hermitian norms $\|.\|_{\overline{F}, \sigma}$ and $\|.\|_{\overline{G}, \sigma}$ on F_σ and G_σ.

We may use these metrics to define the size of an arithmetic extension $(\mathcal{E}, s)$ over S with underlying algebraic extension

$$\mathcal{E} : 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0.$$

Namely, we define the *size of* (the class $\mathrm{cl}\,(\mathcal{E},s)$ in $\widehat{\mathrm{Ext}}^1_S\,(F,G)$ of) *the arithmetic extension* $(\mathcal{E},s)$ as

$$(3.44)\quad \mathfrak{s}(\mathcal{E},s) := \log\,\inf\left\{ \sqrt{\frac{1}{[K:\mathbb{Q}]}\sum_{\sigma:K\hookrightarrow\mathbb{C}} \|h_\sigma\|^2_{\overline{F}^\vee\otimes\overline{G},\sigma}} \;\middle|\; \begin{array}{c}(h_\sigma)\in\big[\bigoplus_\sigma \mathrm{Hom}_{\mathbb{C}}(F_\sigma,G_\sigma)\big]^{F_\infty}\\ \wedge\ b\big((h_\sigma)\big)=\mathrm{cl}(\mathcal{E},s)\end{array}\right\}.$$

It takes values in $\mathbb{R}\cup\{-\infty\}$ and the equality $\mathfrak{s}(\mathcal{E},s)=-\infty$ holds if and only $\mathrm{cl}\,(\mathcal{E},s)$ vanishes. Clearly, the size of $(\mathcal{E},s)$ is the logarithm of the distance from 0 to $\mathrm{cl}\,(\mathcal{E},s)$ in the real torus

$$\widehat{\mathrm{Ext}}^1_S\,(F,G) = \frac{\pi_*(F^\vee\otimes G)_{\mathbb{R}}}{\pi_*(F^\vee\otimes G)}$$

equipped with the translation invariant Riemannian metric defined by the euclidean norm $[K:\mathbb{Q}]^{-1/2}\|.\|_{\pi_*(\overline{F}^\vee\otimes\overline{G})}$ on $\pi_*(F^\vee\otimes G)_{\mathbb{R}}$. Moreover, the infimum in the right-hand side of (3.44) is actually a minimum.

Let us emphasize that, to define the size $\mathfrak{s}(\mathcal{E},s)$, we need to have chosen hermitian structures on F and G. It will be therefore more appropriate to call (3.44) the *size of* $(\mathcal{E},s)$ *with respect to* $\overline{F}$ *and* $\overline{G}$ and to denote it

$$\mathfrak{s}_{\overline{F},\overline{G}}(\mathcal{E},s)$$

when some ambiguity may arise.

3.5.2. *Size of admissible extensions.* We define the *size* $\mathfrak{s}(\overline{\mathcal{E}})$ *of an admissible extension*

$$(3.45)\qquad \overline{\mathcal{E}}:0\longrightarrow\overline{G}\longrightarrow\overline{E}\longrightarrow\overline{F}\longrightarrow 0$$

of hermitian vector bundles over S as the size with respect to $\overline{F}$ and $\overline{G}$ of its class in $\widehat{\mathrm{Ext}}^1_S\,(F,G)$. If

$$s^\perp\in\mathrm{Hom}_{\mathcal{O}_K}(F,E)\otimes_{\mathbb{Z}}\mathbb{R}\simeq[\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}}\mathrm{Hom}(F_\sigma,E_\sigma)]^{F_\infty}$$

denotes the orthogonal splitting of (3.45), we have:

$$\mathfrak{s}(\overline{\mathcal{E}}) = \log\min\left\{[K:\mathbb{Q}]^{-1/2}\cdot\|s-s^\perp\|_{\pi_*(\overline{F}^\vee\otimes\overline{G})}\;\middle|\; s:F\to E \text{ an } \mathcal{O}_K\text{-linear splitting of } \mathcal{E}\right\}.$$

Let us consider the set $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$ of trivializations of $\mathcal{E}$ over $\mathrm{Spec}\,\mathcal{O}_K$, namely of $\mathcal{O}_K$-modules isomorphisms

$$\varphi: E\xrightarrow{\sim} G\oplus F$$

such that $\mathrm{pr}_2\circ\phi = p$ and $\phi\circ i=(Id_G,0_F)$. The map which sends $\varphi\in\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$ to $\varphi^{-1}\circ(0,Id_F)\in\mathrm{Hom}_{\mathcal{O}_K}(F,G)$ defines a bijection from $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$ onto the set of splittings of the extension $\mathcal{E}$ of $\mathcal{O}_K$-modules. In particular, $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$ is non-empty, and becomes a torsor under the abelian group $\mathrm{Hom}_{\mathcal{O}_K}(F,G)$ thanks to the action defined by letting, for any ψ in $\mathrm{Hom}_{\mathcal{O}_K}(F,G)$ and any φ in $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$:

$$\psi+\varphi := i\circ\psi\circ p+\varphi.$$

For any φ in $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$, the difference

$$s^\perp-((\varphi^{-1}\circ(0,Id_F))_\sigma)\in[\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}}\mathrm{Hom}(F_\sigma,E_\sigma)]^{F_\infty}$$

factorizes through the morphism $(i_\sigma) \in [\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}(G_\sigma, E_\sigma)]^{F_\infty}$, and consequently may be considered as an element of $[\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}(F_\sigma, G_\sigma)]^{F_\infty}$. Moreover its image under the map

$$b : [\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}(F_\sigma, G_\sigma)]^{F_\infty} \to \widehat{\mathrm{Ext}}^1_S(F, G)$$

is precisely the class $[\overline{\mathcal{E}}]$ of the admissible extension $\overline{\mathcal{E}}$.

In the sequel, for any embedding $\sigma \hookrightarrow \mathbb{C}$, the $\mathbb{C}$-vector spaces E_σ, F_σ, and G_σ are equipped with the hermitian structures defining $\overline{E}$, $\overline{F}$ and $\overline{G}$, and $F_\sigma \oplus G_\sigma$ with the direct sum of the hermitian structures of F_σ and G_σ. We shall also denote $\|.\|_{\infty,\sigma}$ (resp. $\|.\|_{HS,\sigma}$) the operator norm (resp. the Hilbert-Schmidt norm) on $\mathbb{C}$-linear maps between some of the hermitian spaces E_σ, F_σ, G_σ, or $F_\sigma \oplus G_\sigma$. Finally we define

$$\theta_\sigma : E_\sigma \simeq F_\sigma \oplus G_\sigma$$

as the orthogonal trivialization over $\mathbb{C}$ of the extension $\mathcal{E}_\sigma$ of $\mathbb{C}$-vector spaces. By definition, it is unitary when E_σ and $F_\sigma \oplus G_\sigma$ are equipped with the above hermitian structures, and we have:

$$s^\perp = (\theta_\sigma^{-1} \circ (0, Id_{F_\sigma})).$$

Proposition 3.5.3. *1) The map*

$$\begin{array}{rccc} \beta : & \mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E}) & \longrightarrow & b^{-1}([\overline{\mathcal{E}}]) \\ & \varphi & \longmapsto & s^\perp - ((\varphi^{-1} \circ (0, Id_F))_\sigma) \end{array}$$

is a bijection.

2) For any φ in $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$, the norms of (φ_σ) and of its image $(h_\sigma) := \beta(\varphi)$ satisfy the relations:

$$\|\varphi_\sigma\|^2_{HS,\sigma} = \|\varphi_\sigma^{-1}\|^2_{HS,\sigma} = \mathrm{rk}\, E + \|h_\sigma\|^2_{HS,\sigma},$$
$$\|\varphi_\sigma - \theta_\sigma\|^2_{HS,\sigma} = \|\varphi_\sigma^{-1} - \theta_\sigma^{-1}\|^2_{HS,\sigma} = \|h_\sigma\|^2_{HS,\sigma},$$

and

$$\|\varphi_\sigma - \theta_\sigma\|_{\infty,\sigma} = \|\varphi_\sigma^{-1} - \theta_\sigma^{-1}\|_{\infty,\sigma} \leq \|h_\sigma\|_{\infty,\sigma} \leq \|h_\sigma\|_{HS,\sigma}.$$

Proof. This is a straightforward consequence of the definitions and of the following elementary lemma:

Lemma 3.5.4. *Let p and q be non-negative integers, and $n := p + q$. For any matrix A in $M_{p,q}(\mathbb{C})$, the matrix*

$$\tilde{A} := \begin{pmatrix} I_p & A \\ 0 & I_q \end{pmatrix}$$

belongs to $\mathrm{GL}_n(\mathbb{C})$, its inverse is

$$\tilde{A}^{-1} := \begin{pmatrix} I_p & -A \\ 0 & I_q \end{pmatrix},$$

and the Hilbert-Schmidt and operator norms of A, $\tilde{A}$, and $\tilde{A}^{-1}$ satisfy:

$$\|\tilde{A}\|^2_{HS} = \|\tilde{A}^{-1}\|^2_{HS} = n + \|A\|^2_{HS},$$
$$\|\tilde{A} - I_n\|^2_{HS} = \|\tilde{A}^{-1} - I_n\|^2_{HS} = \|A\|^2_{HS},$$

and

$$\|\tilde{A} - I_n\|_\infty = \|\tilde{A}^{-1} - I_n\|_\infty \leq \|A\|_\infty \leq \|A\|_{HS}.$$

□

Corollary 3.5.5. *For any* φ *in* $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$*, we have:*

$$
\begin{aligned}
[K:\mathbb{Q}]^{-1} \sum_{\sigma:k\hookrightarrow\mathbb{C}} \|\varphi_\sigma - \theta_\sigma\|^2_{HS,\sigma} &= [K:\mathbb{Q}]^{-1} \sum_{\sigma:k\hookrightarrow\mathbb{C}} \|\varphi_\sigma^{-1} - \theta_\sigma^{-1}\|^2_{HS,\sigma} \\
&= [K:\mathbb{Q}]^{-1} \sum_{\sigma:k\hookrightarrow\mathbb{C}} \|h_\sigma\|^2_{HS,\sigma}.
\end{aligned} \tag{3.46}
$$

When φ *runs over* $\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$*, this quantity achieves* $\exp(2\mathfrak{s}(\overline{\mathcal{E}}))$ *as minimal value. Moreover any* $\varphi \in \mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E})$ *which achieves it also satisfies:*

$$
\begin{aligned}
[K:\mathbb{Q}]^{-1} \sum_{\sigma:k\hookrightarrow\mathbb{C}} \|\varphi_\sigma - \theta_\sigma\|^2_{\infty,\sigma} &= [K:\mathbb{Q}]^{-1} \sum_{\sigma:k\hookrightarrow\mathbb{C}} \|\varphi_\sigma^{-1} - \theta_\sigma^{-1}\|^2_{\infty,\sigma} \\
&\leq \exp(2\mathfrak{s}(\overline{\mathcal{E}})).
\end{aligned} \tag{3.47}
$$

Observe that, from any admissible extension over $\mathrm{Spec}\,\mathcal{O}_K$ as above

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \xrightarrow{\ i\ } \overline{E} \xrightarrow{\ p\ } \overline{F} \longrightarrow 0,$$

we derive an other one by duality:

$$\overline{\mathcal{E}}^\vee : 0 \longrightarrow \overline{F}^\vee \xrightarrow{\ p^t\ } \overline{E}^\vee \xrightarrow{\ i^t\ } \overline{G}^\vee \longrightarrow 0.$$

The sets of trivializations of $\mathcal{E}$ and $\mathcal{E}^\vee$ are in bijection *via* the map

$$
\begin{array}{ccc}
\mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E}) & \longrightarrow & \mathrm{Triv}_{\mathcal{O}_K}(\mathcal{E}^\vee) \\
\varphi & \longmapsto & (\varphi^t)^{-1}.
\end{array}
$$

As a consequence of Proposition 3.5.3, we derive:

Corollary 3.5.6. *For any admissible extension* $\overline{\mathcal{E}}$ *over* $\mathrm{Spec}\,\mathcal{O}_K$*, we have:*

$$\mathfrak{s}(\overline{\mathcal{E}}^\vee) = \mathfrak{s}(\overline{\mathcal{E}}).$$

Proposition 3.5.7. *For any admissible extension* $\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow 0$ *over* $\mathrm{Spec}\,\mathcal{O}_K$ *such that* $m := \min(\mathrm{rk}\,F, \mathrm{rk}\,G)$ *is positive, there exists a submodule* F' *of the* $\mathcal{O}_K$*-module* E *satisfying* $E = F' \oplus G$ *and*

$$\widehat{\deg}_n \overline{F}' \geq \widehat{\deg}_n \overline{F} - \frac{m}{2} \log(1 + e^{2\mathfrak{s}(\overline{\mathcal{E}})/m}).$$

As usual, $\overline{F}'$ denotes the hermitian vector bundle over $\mathrm{Spec}\,\mathcal{O}_K$ defined by F' equipped with the restriction of the hermitian structure of $\overline{E}$.

Proof. Actually the image $F' := s(F)$ of any splitting s of $\mathcal{E}$ over $\mathcal{O}_K$ such that

$$\mathfrak{s}(\overline{\mathcal{E}}) = \log([K:\mathbb{Q}]^{-1/2} \cdot \|s - s^\perp\|_{\pi_*(\overline{F}^\vee \otimes \overline{G})})$$

satisfies the above conditions. This follows from the concavity of the logarithm, combined with the following simple consequence of the polar decomposition of linear maps between finite dimensional hermitian vector spaces:

Lemma 3.5.8. *Let* p *and* q *be positive integers,* $n := p + q$*, and* $m := \min(p, q)$*. For any matrix* S *in* $M_{p,q}(\mathbb{C})$*, the matrix*

$$\tilde{S} := \begin{pmatrix} S \\ I_q \end{pmatrix}$$

in $M_{n,q}(\mathbb{C})$ defines a $\mathbb{C}$-linear map from $\mathbb{C}^q$ to $\mathbb{C}^n$ whose (operator) norm of the q-th exterior power satisfy the following upper bound:

$$\log \| \wedge^q \tilde{S}\|^2 \leq m \log(1 + \|S\|^2_{HS}/m),$$

when $\mathbb{C}^q$, $\mathbb{C}^n$, and there q-th exterior powers are equipped with their standard hermitian structures.

□

3.6. **Size and operations on extensions.** As observed in 2.4, the description (3.43) of the group $\widehat{\mathrm{Ext}}^1_S(F, G)$ is compatible with various functorial operations on arithmetic extensions. This allows us to study the behavior of sizes of extensions under these operations.

3.6.1. *Pushout and Pullback.* For instance, if $\overline{F'}$ and $\overline{G'}$ are hermitian vector bundles over $\mathrm{Spec}\, \mathcal{O}_K$, and if

$$\alpha : F' \longrightarrow F \quad \text{and} \quad \beta : G \longrightarrow G'$$

are morphisms of $\mathcal{O}_K$-modules, we easily get, for any extension class e in $\widehat{\mathrm{Ext}}^1_S(F, G)$:

$$\mathfrak{s}_{\overline{F'},\overline{G}}(e \circ \alpha) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e) + \log \max_{\sigma:K\hookrightarrow\mathbb{C}} \|\alpha\|^{\infty}_{\sigma}, \tag{3.48}$$

and

$$\mathfrak{s}_{\overline{F},\overline{G'}}(\beta \circ e) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e) + \log \max_{\sigma:K\hookrightarrow\mathbb{C}} \|\beta\|^{\infty}_{\sigma}, \tag{3.49}$$

where $\|.\|^{\infty}_{\sigma}$ denotes the operator norm on $\mathrm{Hom}_{\mathbb{C}}(F'_{\sigma}, F_{\sigma})$ (resp. on $\mathrm{Hom}_{\mathbb{C}}(G_{\sigma}, G'_{\sigma})$) deduced from the hermitian norms $\|.\|_{\overline{F'},\sigma}$ and $\|.\|_{\overline{F},\sigma}$ (resp. from $\|.\|_{\overline{G},\sigma}$ and $\|.\|_{\overline{G'},\sigma}$). Applied to $F' = F$ and $G' = G$, (3.48) and (3.49) give a control on the variation of $\mathfrak{s}_{\overline{F},\overline{G}}(e)$ when the hermitian structures on $\overline{F}$ and $\overline{G}$ are modified.

3.6.2. *Inverse image.* Consider now K' a number field containing K and

$$g : S' = \mathrm{Spec}\ \mathcal{O}_{K'} \longrightarrow S = \mathrm{Spec}\ \mathcal{O}_K$$

the associated morphism of arithmetic curves. Define $\overline{F'} := g^*\overline{F}$ and $\overline{G'} := g^*\overline{G}$. Then the pullback morphism

$$g^* : \widehat{\mathrm{Ext}}^1_S(F, G) \longrightarrow \widehat{\mathrm{Ext}}^1_{S'}(F', G') \tag{3.50}$$

gets identified with the morphism of real tori

$$\frac{\left(\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(F_{\sigma}, G_{\sigma})\right)^{F_{\infty}}}{\mathrm{Hom}_{\mathcal{O}_K}(F, G)} \longrightarrow \frac{\left(\bigoplus_{\sigma':K'\hookrightarrow\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(F'_{\sigma'}, G'_{\sigma'})\right)^{F_{\infty}}}{\mathrm{Hom}_{\mathcal{O}_{K'}}(F', G')}$$

deduced from the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_K}(F, G) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{K'}}(F', G') \\ \downarrow \iota & & \downarrow \iota \\ \left(\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(F_{\sigma}, G_{\sigma})\right)^{F_{\infty}} & \longrightarrow & \left(\bigoplus_{\sigma':K'\hookrightarrow\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(F'_{\sigma'}, G'_{\sigma'})\right)^{F_{\infty}} \end{array} \tag{3.51}$$

where the upper horizontal arrow is the "extension of scalars" from $\mathcal{O}_K$ to $\mathcal{O}_{K'}$, and the lower one, the "diagonal" $\mathbb{R}$-linear map

$$(T_{\sigma})_{\sigma:K\hookrightarrow\mathbb{C}} \longmapsto (T_{\sigma'_{|K}})_{\sigma':K'\hookrightarrow\mathbb{C}}.$$

Observe that an $\mathcal{O}_{K'}$-linear map $T' : F' \to G'$ descends to an $\mathcal{O}_K$-linear map $T : F \to G$ iff the K'-linear map $T'_{K'} : F'_{K'} \to G'_{K'}$ descends to a K-linear map $T_K : F_K \to G_K$. Together with the separability of K' over K, this shows that the diagram (3.51) is cartesian. Moreover, its upper horizontal arrow is an isometry, in the sense that, for any $(T_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$ in $\bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(F_\sigma, G_\sigma)$,

$$[K' : \mathbb{Q}]^{-1} \sum_{\sigma':K'\hookrightarrow\mathbb{C}} \|T_{\sigma'_{|K}}\|^2_{\overline{F'}^\vee\otimes\overline{G'},\sigma'} = [K : \mathbb{Q}]^{-1} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \|T_\sigma\|^2_{\overline{F}^\vee\otimes\overline{G},\sigma}.$$

From these remarks, we get:

Proposition 3.6.3. *With the above notation, the base change morphism (3.50) is injective, and for any e in $\widehat{\mathrm{Ext}}^1_S(F, G)$, we have:*

$$\mathfrak{s}_{\overline{F'},\overline{G'}}(g^*e) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e). \tag{3.52}$$

By considering a geometric analogue of the above notion of size, concerning extensions of vector bundles over a smooth projective curve over a field, one is led to wonder whether the inequality (3.52) would not actually be an equality (see 5.1 *infra*). We shall investigate this issue more closely in Section 5 below.

3.7. **Covering radius and size of admissible extensions.** Recall that, if Γ is a lattice and B a convex, symmetric body in some finite dimensional real vector space V, the corresponding *inhomogeneous minimum*, or *covering radius*, is defined as[5]

$$\rho(\Gamma, B) := \min\{r \in \mathbb{R}^*_+ \mid \Gamma + rB = V\}.$$

In particular, if $\overline{E} = (E, \|.\|)$ is an hermitian vector bundle over $\mathrm{Spec}\,\mathbb{Z}$ — or equivalently an euclidean lattice — we may consider the covering radius attached to the lattice E and to the unit ball $B_{\overline{E}} := \{v \in E_{\mathbb{R}} \mid \|v\| \leq 1\}$ in $E_{\mathbb{R}}$. We shall denote it $\rho(\overline{E})$.

This invariant of euclidean lattices is relevant for estimating the sizes of arithmetic extensions over arithmetic curves. Indeed, by the very definitions of these sizes and of the covering radius of an euclidean lattice, we have:

Proposition 3.7.1. *For any number field K and any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over $S := \mathrm{Spec}\,\mathcal{O}_K$, we have*[6]*:*

$$\max\left\{\mathfrak{s}(\overline{\mathcal{E}}) \;\middle|\; \begin{array}{c}\overline{\mathcal{E}} \text{ admissible extension} \\ \text{of } \overline{F} \text{ by } \overline{G}\end{array}\right\} = \log\left([K : \mathbb{Q}]^{-1/2} \cdot \rho(\pi_*(\overline{F}^\vee \otimes \overline{G}))\right). \tag{3.53}$$

The so-called "transference theorems" of the geometry of numbers relate minima of various kinds attached to some lattice to minima of the dual lattice. To get upper bounds on sizes of arithmetic extensions, we shall use the following transference result of Banaszczyk ([Ban93, Theorem 2.2]):

Theorem 3.7.2. *For any hermitian vector bundle $\overline{E}$ of rank n over $\mathrm{Spec}\,\mathbb{Z}$, we have:*

$$\rho(\overline{E}) \cdot \lambda_1(\overline{E}^\vee) \leq \frac{n}{2}. \tag{3.54}$$

[5]In [Ban93], it is denoted $\mu(\Gamma, B)$. We depart from this notation to avoid confusion with slopes.

[6]As before, we denote π the morphism from $\mathrm{Spec}\,\mathcal{O}_K$ to $\mathrm{Spec}\,\mathbb{Z}$.

The covering radius of an hermitian vector bundle $\overline{E}$ of positive rank n over $\operatorname{Spec}\mathbb{Z}$ satisfies also the following elementary lower bounds:

$$\rho(\overline{E}) \cdot \lambda_1(\overline{E}^{\vee}) \geq \frac{1}{2} \tag{3.55}$$

and

$$v_n \rho(\overline{E})^n \geq \operatorname{covol}(\overline{E}). \tag{3.56}$$

Indeed, (3.55) follows from the surjectivity of the map

$$\begin{array}{rcl} E_{\mathbb{R}}/E & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ [v] & \longmapsto & \xi(v) \bmod 1 , \end{array}$$

where $\xi \in E^{\vee} \setminus \{0\}$ is an element of minimal norm, and (3.56) from the surjectivity of the composition

$$B_{\overline{E}} \hookrightarrow E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}/E.$$

Let us us emphasize that, as pointed out in [Ban93], p. 633, Banaszczyk's upper bound (3.54) is basically optimal. Indeed, as explained in [MH80] (Chapter II, Theorem 9.5), by using Siegel's formula on integral quadratic forms in a given genus, Conway and Thompson have shown that one may construct a sequence $\overline{CT}_n$ of rank n euclidean lattices, $n \in \mathbb{N}$, such that there exist (symmetric) isometric isomorphisms

$$\overline{CT}_n \simeq \overline{CT}_n^{\vee}, \tag{3.57}$$

and their first minima satisfy

$$\lambda_1(\overline{CT}_n) \geq \sqrt{\frac{n}{2\pi e}}(1 + o(1)) \quad \text{as } n \text{ goes to infinity.} \tag{3.58}$$

From (3.57), it follows that

$$\operatorname{covol}(\overline{CT}_n) = 1,$$

and, by (3.56), that

$$\rho(\overline{CT}_n) \geq v_n^{-\frac{1}{n}} \sim \sqrt{\frac{n}{2\pi e}}(1 + o(1)) \quad \text{as } n \text{ goes to infinity.} \tag{3.59}$$

Finally,

$$\rho(\overline{CT}_n)\lambda_1(\overline{CT}_n^{\vee}) = \rho(\overline{CT}_n)\lambda_1(\overline{CT}_n) \sim \frac{n}{2\pi e}(1 + o(1)) \quad \text{as } n \text{ goes to infinity,} \tag{3.60}$$

and for the euclidean lattices $\overline{CT}_n$, Banaszczyk's transference upper bound (3.54) is an equality up to a bounded multiplicative factor.

With the notation of Proposition (3.7.1), when $\overline{F}$ and $\overline{G}$ have positive rank,we obtain from Banaszczyk's upper bound (3.54):

$$\log \rho(\pi_*(\overline{F}^{\vee} \otimes \overline{G})) \leq -\log \lambda_1(\pi_*(\overline{F}^{\vee} \otimes \overline{G})^{\vee}) + \log(\frac{1}{2} \cdot [K : \mathbb{Q}] \cdot \operatorname{rk} F \cdot \operatorname{rk} G). \tag{3.61}$$

Moreover, using successively Propositions 3.2.2 and 3.3.1 and (3.21), we derive:

$$
\begin{aligned}
-\log \lambda_1(\pi_*(\overline{F}^\vee \otimes \overline{G})^\vee) &= -\log \lambda_1(\pi_*(\overline{F} \otimes \overline{G}^\vee \otimes \overline{\omega}_K)) \\
&\leq \widehat{\mathrm{udeg}}_n\,(\overline{F} \otimes \overline{G}^\vee \otimes \overline{\omega}_K) - \frac{1}{2}\log[K:\mathbb{Q}] \\
&= \widehat{\mathrm{udeg}}_n\,(\overline{F} \otimes \overline{G}^\vee) + \widehat{\deg}_n\,\overline{\omega}_K - \frac{1}{2}\log[K:\mathbb{Q}].
\end{aligned} \tag{3.62}
$$

From (3.53), (3.61), and (3.62), using (3.16) and Proposition 3.4.1, we finally obtain the upper bound on sizes of arithmetic extensions announced in (3.2):

Theorem 3.7.3. *For any number field K, for any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ of positive rank over $S := \operatorname{Spec} \mathcal{O}_K$, and for any admissible extension $\overline{\mathcal{E}}$ of $\overline{F}$ by $\overline{G}$, we have:*

$$
\begin{aligned}
\mathfrak{s}(\overline{\mathcal{E}}) &\leq \widehat{\mathrm{udeg}}_n\,(\overline{F} \otimes \overline{G}^\vee) + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log \frac{\mathrm{rk}\, F \cdot \mathrm{rk}\, G}{2} \\
&\leq \widehat{\mu}_{\max}(\overline{F}) - \widehat{\mu}_{\min}(\overline{G}) + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log \frac{\mathrm{rk}\, F \cdot \mathrm{rk}\, G}{2}.
\end{aligned} \tag{3.63}
$$

Observe that, when S is $\operatorname{Spec} \mathbb{Z}$ and $\overline{F}$ the trivial hermitian line bundle, the first upper bound in (3.63) is equivalent (by taking the logarithm) to Banaszczyk's bound (3.54). In particular, when moreover $\overline{G} = \overline{CT}_n$, it becomes an equality, up to a bounded additive error term. Observe also that the right hand side of the second upper bound is invariant by unramified extension of the number field K.

Similarly, the lower bounds (3.55) and (3.56) lead to lower bounds on the maximal size of admissible extensions of $\overline{F}$ by $\overline{G}$. We leave this to the interested reader.

3.8. **The geometric case I.** The results in the previous sections admit (simpler) analogs in the geometric case where the number field K is replaced by the function field $k(C)$ defined by a smooth projective geometrically connected curve C over a field k, and hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$ by vector bundles over C.

Recall that the slope of a such vector bundle E of positive rank is defined as the quotient

$$
\mu(E) := \frac{\deg E}{\mathrm{rk}\, E}, \tag{3.64}
$$

its maximal slope $\mu_{\max}(E)$ as the maximum of the slope of a subvector bundle of positive rank in E, and its minimal slope as

$$
\mu_{\min}(E) := -\mu_{\max}(E^\vee).
$$

We may also introduce the upper degree of E:

$$
\mathrm{udeg}(E) := \max\{\deg L \mid L \text{ sub-line bundle of } E\}.
$$

For any line bundle M on C, we have:

$$
\mathrm{udeg}(E \otimes M) = \mathrm{udeg}(E) + \deg M. \tag{3.65}
$$

Observe that we are considering "un-normalized" slopes: their behavior under some base extension involves the degree of the latter. Namely, if C' is another smooth projective geometrically connected curve C over k, and $f : C' \to C$ a dominant k-morphism, then

$$
\deg f^*E = \deg f \cdot \deg E,
$$

and consequently,

$$\mu(f^*E) = \deg f \cdot \mu(E).$$

The maximal and minimal slopes satisfy similar formulae when f is a separable morphism (in particular when $\mathrm{char}(k) = 0$).

A (simplified) variant of the proof of Proposition 3.4.1 establishes:

Proposition 3.8.1. *For any two vector bundles F and G of positive rank over C, we have:*

$$\mathrm{udeg}(F \otimes G^\vee) \leq \mu_{\max}(F) - \mu_{\min}(G). \tag{3.66}$$

and

$$\mathrm{udeg}(F \otimes G) \leq \mu_{\max}(F) + \mu_{\max}(G). \tag{3.67}$$

In characteristic zero, (3.67) and consequently (3.66) follows form the trivial upper bound

$$\mathrm{udeg}(F \otimes G) \leq \mu_{\max}(F \otimes G)$$

combined with the equality

$$\mu_{\max}(F \otimes G) = \mu_{\max}(F) + \mu_{\max}(G),$$

which is nothing but a reformulation of the classical fact that the tensor product of two semi-stable vector bundles is semi-stable. However, in positive characteristic, this equality does not hold in general.

Using Proposition 3.8.1, we may establish a generalization of the upper bound (3.1) concerning non-trivial extensions of vector bundles of positive ranks over C:

Proposition 3.8.2. *Let g denote the genus of C. For any extension*

$$\mathcal{E}\colon\ 0 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 0$$

of vector bundles of positive ranks over C which does not split, we have:

$$\mu_{\min}(G) - \mu_{\max}(F) \leq 2g - 2.$$

Proof. The class of $\mathcal{E}$ provides a non-zero element in

$$\mathrm{Ext}^1_C(F, G) \simeq H^1(C,\, F^\vee \otimes G) \simeq H^0(C,\, \Omega^1_{C/k} \otimes F \otimes G^\vee)^\vee.$$

Hence $\Omega^1_{C/k} \otimes F \otimes G^\vee$ has a non-trivial regular section on C, and its upper degree is consequently non-negative. As (3.65) and Proposition (3.8.1) show that

$$\mathrm{udeg}(\Omega^1_{C/k} \otimes F \otimes G^\vee) = \mathrm{udeg}(F \otimes G^\vee) + 2g - 2 \leq \mu_{\max}(F) - \mu_{\min}(G) + 2g - 2,$$

this proves (3.8.2). □

3.9. **The geometric case II.** It turns out that the geometric analogues of our arithmetic results discussed in the previous sections possess some "refined versions" that more closely parallel our investigation of the arithmetic case, where *archimedean places* of number fields play a special role.

To formulate it, let C be as above a smooth geometrically connected projective curve over some field k, and let

$$D = \sum_{P \in |D|} n_P . P$$

be an effective divisor on C with non-empty support $|D|$, and let $\dot{C}$ be the affine curve $C \setminus |D|$.

The function field $K := k(C) = k(\dot{C})$ is a geometric analogue of a number field — in this analogy, the curve $\dot{C}$ plays the role of the affine scheme $\operatorname{Spec}\mathcal{O}_K$ where K is a number field, and the points of $|D|$ of its archimedean places. Moreover a vector bundle over C (resp., its restriction $\dot{E}$ to $\dot{C}$) is the counterpart of an hermitian vector bundle $(E, (\|.\|_\sigma)_{\sigma: K\to \mathbb{C}})$ (resp. of the vector bundle E) over $\operatorname{Spec}\mathcal{O}_K$. In this section, we want to extend this dictionary by describing the analogues, in this geometric setting, of our arithmetic extension groups and of the size function on them.

For any closed point P of C, let $\hat{\mathcal{O}}_{C,P}$ the completion of the local ring $\mathcal{O}_{C,P}$ of C at P, $\mathfrak{m}_P$ its maximal ideal, and $\hat{K}_P$ its field of fractions.

It is natural to define an analogue of the arithmetic extension group over an arithmetic curve by mimicking its description in Proposition 2.2.2. Namely, if$F$ and G are two vector bundles over C, and $\dot{F}$ and $\dot{G}$ their restrictions to $\dot{C}$, we let

$$\widehat{\operatorname{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G}) := \frac{\bigoplus_{P\in|D|}\operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}) \otimes_{\mathcal{O}(\dot{C})} \hat{K}_P}{\iota(\operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}))}, \tag{3.68}$$

where

$$\iota : \operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}) \longrightarrow \bigoplus_{P\in|D|} \operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}) \otimes_{\mathcal{O}(\dot{C})} \hat{K}_P$$

denotes the diagonal embedding, defined by

$$\iota(x) := (x \otimes 1_{\hat{K}_P})_{P\in|D|}.$$

Observe that its image is discrete in $\bigoplus_{P\in|D|}\operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}) \otimes_{\mathcal{O}(\dot{C})} \hat{K}_P$ equipped with the product topology deduced from the $\mathfrak{m}_P$-adic topologies on the finite dimensional $\hat{K}_P$-vector spaces

$$\operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G}) \otimes_{\mathcal{O}(\dot{C})} \hat{K}_P \simeq (\check{F} \otimes G)_K \otimes_K \hat{K}_P \simeq (\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \hat{K}_P.$$

Consequently the quotient topology on the abelian group $\widehat{\operatorname{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$ is separated and complete.

Besides, a neighbourhood basis of zero in $\widehat{\operatorname{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$ equipped with this topology is formed by the images in $\widehat{\operatorname{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$ of the subgroups

$$\bigoplus_{P\in|D|} (\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \mathfrak{m}_P^{N.n_P}, \ \ N \in \mathbb{N} \tag{3.69}$$

of

$$\bigoplus_{P\in|D|} (\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \hat{K}_P,$$

and the quotient of $\widehat{\operatorname{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$ by the image of (3.69) may be identified with

$$\frac{\bigoplus_{P\in|D|}(\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \hat{K}_P}{\iota(\operatorname{Hom}_{\dot{C}}(\dot{F}, \dot{G})) + \bigoplus_{P\in|D|}(\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \mathfrak{m}_P^{N.n_P}}.$$

In turn, this space is canonically isomorphic to

$$H^1(C, \check{F} \otimes G(-N.D)) \simeq \mathrm{Ext}^1_C(F, G(-N.D)).$$

This follows from the long exact sequence of cohomology groups associated to the "localization" short exact sequence of sheaves of $\mathcal{O}_C$-modules:

$$(3.70) \quad 0 \longrightarrow \check{F} \otimes G(-N.D) \longrightarrow j_* j^*(\check{F} \otimes G) \oplus \bigoplus_{P \in |D|} i_{P*}((\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \mathfrak{m}_P^{N.n_P}) \longrightarrow \bigoplus_{P \in |D|} i_{P*}((\check{F} \otimes G)_{\mathcal{O}_{X,P}} \otimes_{\mathcal{O}_{X,P}} \hat{K}_P) \longrightarrow 0,$$

where j (resp. i_P) denotes the open (resp. closed) immersion $\dot{C} \hookrightarrow C$ (resp. $\{P\} \hookrightarrow C$).

Finally, the topological group $\widehat{\mathrm{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$ is canonically isomorphic to the projective limit

$$\varprojlim_{N \in \mathbb{Z}} \mathrm{Ext}^1_C(F, G(-N.D))$$

of the finite dimensional k-vector spaces $\mathrm{Ext}^1_C(F, G(-N.D))$ equipped with the discrete topology. In particular, it is a linearly compact topological k-vector space. (This is similar to the compactness of $\widehat{\mathrm{Ext}}^1_{\mathrm{Spec}\, \mathcal{O}_K}(F, G)$ in the number field case.)

For any integer $N \in \mathbb{Z}$, let

$$\pi_N : \widehat{\mathrm{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G}) \simeq \varprojlim_{M \in \mathbb{Z}} \mathrm{Ext}^1_C(F, G(-M.D)) \longrightarrow \mathrm{Ext}^1_C(F, G(-N.D))$$

be the projection on the N-th component. To define a geometric counterpart of the size of arithmetic extensions, we let, for every $e \in \widehat{\mathrm{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$:

$$\mathfrak{s}_{F,G}(e) := -\inf\{N \in \mathbb{Z} \mid \pi_N(e) \neq 0\}.$$

It is straightforward that it is an element of $\mathbb{Z} \cup \{-\infty\}$, finite for any $e \neq 0$, and that Proposition 3.8.2 applied to the vector bundles F and $G(-N.D)$ yields the following geometric analogue of Theorem 3.7.3:

$$(3.71) \qquad \deg D . \mathfrak{s}_{F,G}(e) \leq \mu_{\max}(F) - \mu_{\min}(G) + 2g - 2,$$

when F and G have positive rank[7].

4. Sizes of admissible extensions: explicit computations and an application to reduction theory

In this section, we want to show how evaluating the size of admissible extensions is related to basic questions in the geometry of lattices. Firstly we compute it explicitly in some elementary examples — notably in the most basic case of extension of hermitian line bundles over $\mathrm{Spec}\, \mathbb{Z}$. Then we consider the size of the restriction of the universal admissible extension over $\mathbb{P}^1_{\mathbb{Z}}$ at rational points in $\mathbb{P}^{(}\mathbb{Q})$ ($\simeq \mathbb{P}^1(\mathbb{Z})$) and relate it to the usual logarithmic height of these points, and to the geometry of the Ford circles[8] in the upper half-plane. Finally, using the upper bound (3.2) on the sizes of admissible extensions over $\mathrm{Spec}\, \mathcal{O}_K$,

[7] The occurrence of $\deg D$ in the right hand side of (3.71) is related to the use of "non-normalized" slopes (3.64) instead of normalized slopes (3.4) in the arithmetic case.

[8] Namely, the horocycles image of $\{\mathrm{Im}\, z = 1\}$ under the action of $SL^2(\mathbb{Z})$.

we derive the existence of some "almost-splitting" for any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathcal{O}_K$, namely the existence of $n := \operatorname{rk} E$ hermitian lines bundles $\overline{L}_1, \dots, \overline{L}_n$, and of an isomorphism of $\mathcal{O}_K$-modules

$$\phi : E \xrightarrow{\sim} \bigoplus_{i=1}^{n} L_i$$

such that the archimedean norms of ϕ and ϕ^{-1} (defined by means of the hermitian metrics on $\overline{E}$ and $\bigoplus_{i=1}^{n} \overline{L}_i$) are bounded in terms of K and n only. When $K = \mathbb{Q}$, this is basically the main result of the classical reduction theory of positive quadratic forms. For general number fields K, our method yields explicit bounds on the norms of ϕ and ϕ^{-1}, which improve on the qualitative results which may be derived from the general reduction theory for reductive algebraic groups over number fields.

4.1. **Some explicit computations of size.** We discuss various examples of admissible extensions over arithmetic curves, the sizes of which can be "explicitly" computed.

Example 4.1.1. For any positive integer n, the morphism of $\mathcal{O}_K$-modules

$$\begin{array}{cccc} p: & \mathcal{O}_K^{n+1} & \longrightarrow & \mathcal{O}_K \\ & (x_i)_{0\leq i\leq n} & \longmapsto & \sum_{i=0}^{n} x_i \end{array}$$

defines an admissible extension:

$$\overline{\mathcal{A}}_{n,K} : 0 \longrightarrow \overline{\ker p} \longrightarrow \overline{\mathcal{O}_S}^{\oplus(n+1)} \xrightarrow{p} \overline{L}_{n,S} \longrightarrow 0,$$

where $\overline{\mathcal{O}_S}^{\oplus(n+1)}$ denotes the trivial hermitian vector bundle of rank $n+1$ over S, and $\overline{L}_{n,S}$ the hermitian line bundle $(\mathcal{O}_K, (\|.\|_{n,\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}))$ defined by the hermitian metrics:

$$\|1\|_{n,\sigma} := \frac{1}{\sqrt{n+1}}.$$

This admissible extension is the base change to S of the admissible extension over $\operatorname{Spec}\mathbb{Z}$ defining the root lattice A_n (see 5.6 below).

The orthogonal splitting $s^{\perp}$ of $\overline{\mathcal{A}}_{n,K}$ satisfies

$$s^{\perp}_{\sigma}(1) = (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \quad \text{for every embedding } \sigma : L \hookrightarrow \mathbb{C}.$$

An algebraic splitting s of $\mathcal{A}_{n,K}$ over $\mathcal{O}_K$ is given by

$$s(1) := (1, 0, \dots, 0).$$

Since the euclidean norm of

$$s^{\perp}_{\sigma}(1) - s(1) = (-\frac{n}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1})$$

is $\sqrt{n/n+1}$, and $\|1\|_{n,\sigma} = 1/\sqrt{n+1}$, this shows that

$$\mathfrak{s}(\overline{\mathcal{A}}_{n,K}) \leq \frac{1}{2}\log n.$$

It is easy to check that this inequality is an equality when $K = \mathbb{Q}$. Our results on the invariance of size by base change in section 5.4 below will show that, for any number field K,

$$\mathfrak{s}(\overline{\mathcal{A}}_{n,K}) = \frac{1}{2}\log n.$$

Example 4.1.2. Let $S = \operatorname{Spec}\mathbb{Z}$, and let $\overline{F}$ and $\overline{G}$ be hermitian line bundles. We may choose generators f and g for F and G over $\mathbb{Z}$; they are well defined up-to-sign, and determine a group isomorphism:

$$\begin{array}{ccc} \widehat{\operatorname{Ext}}^1_S(F,G) & \longrightarrow & \widehat{\operatorname{Ext}}^1_{\operatorname{Spec}\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z} \\ e & \longmapsto & \tilde{e} := g^{-1} \circ e \circ f^{-1}. \end{array}$$

Then, if we let

$$\begin{array}{rcl} \delta : \mathbb{R}/\mathbb{Z} & \longrightarrow & [0,1/2] \\ {}[x] & \longmapsto & \min_{k\in\mathbb{Z}} |x-k| \ (= |x| \text{ if } |x| \leq 1/2), \end{array}$$

the size of an extension class $e \in \widehat{\operatorname{Ext}}^1_S(F,G)$ may be expressed as

$$\begin{aligned} \mathfrak{s}_{\overline{F},\overline{G}}(e) &= \log \delta(\tilde{e}) + \log \|g\| - \log \|f\| \\ &= \log \delta(\tilde{e}) + \widehat{\deg}\,\overline{F} - \widehat{\deg}\,\overline{G}. \end{aligned} \tag{4.1}$$

This clearly implies (3.2) in this special situation.

Example 4.1.3. Let $\overline{E}$ be an hermitian vector bundle over $S := \operatorname{Spec}\mathbb{Z}$. Consider the attached projective space over S:

$$\pi : \mathbb{P}(E) := \mathbf{Proj}\,\operatorname{Sym}(E) \longrightarrow S,$$

and the tautological exact sequence of vector bundles over $\mathbb{P}(E)$:

$$0 \longrightarrow V \longrightarrow \pi^* E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0.$$

We may equip V (resp. $\mathcal{O}_E(1)$) with the induced (resp. quotient) hermitian structure deduced from $\pi^*\overline{E}$. In such a way, we define the *tautological admissible extension* over $\mathbb{P}(E)$:

$$\overline{\mathcal{E}} :\ 0 \longrightarrow \overline{V} \longrightarrow \pi^*\overline{E} \longrightarrow \overline{\mathcal{O}}_E(1) \longrightarrow 0. \tag{4.2}$$

The function

$$\begin{array}{rcl} \mathbb{P}(E)(\mathbb{Q}) \simeq \mathbb{P}(E)(\mathbb{Z}) & \longrightarrow & [-\infty, +\infty[\\ P & \longmapsto & \mathfrak{s}(P^*\overline{\mathcal{E}}) \end{array}$$

may be described as follows.

For any $P \in \mathbb{P}(E)(\mathbb{Z})$, the line bundle $P^*\overline{\mathcal{O}}_E(1)$ over $\operatorname{Spec}\mathbb{Z}$ is trivial, and $P^*\overline{\mathcal{E}}$ is isomorphic to an admissible extension of the form

$$0 \longrightarrow \overline{\ker a} \longrightarrow \overline{E} \xrightarrow{\ a\ } (\mathbb{Z}, \|.\|_P) \longrightarrow 0, \tag{4.3}$$

where a denotes a surjective morphism in $E^\vee := \operatorname{Hom}_\mathbb{Z}(E,\mathbb{Z})$, and $\|.\|_P$ the quotient norm on $\mathbb{Z}_\mathbb{R} \simeq \mathbb{R}$. Let $\tilde{a}$ denote the image of a by the isomorphism $E^\vee_\mathbb{R} \simeq E_\mathbb{R}$ determined by the euclidean structure $\|.\|_{\overline{E}}$ on $E_\mathbb{R}$. The orthogonal splitting $s^\perp : \mathbb{R} \to E_\mathbb{R}$ of (4.3) satisfies

$$s^\perp(1) = \frac{\tilde{a}}{\|\tilde{a}\|^2_{\overline{E}}},$$

and therefore

$$\|1\|_P = \|s^\perp(1)\|_{\overline{E}} = \|\tilde{a}\|^{-1}_{\overline{E}}.$$

Consequently,

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) = \log\min\left\{ \|m - \|\tilde{a}\|^{-2}_{\overline{E}}\tilde{a}\|_{\overline{E}} \cdot \|\tilde{a}\|_{\overline{E}}\ ,\ m \in E \text{ such that } a(m) = 1 \right\}. \tag{4.4}$$

Observe that, for any $v \in E_{\mathbb{R}}$,

$$\|v\|^2_{\overline{E}} \cdot \|\tilde{a}\|^2_{\overline{E}} = \langle v, \tilde{a}\rangle^2_{\overline{E}} + \|v \wedge \tilde{a}\|^2_{\Lambda^2\overline{E}}.$$

For any m in E such that $a(m) = 1$, the vector $m - \|\tilde{a}\|^{-2}_{\overline{E}}\tilde{a}$ is orthogonal to $\tilde{a}$, and the identity above shows that

$$\|m - \|\tilde{a}\|^{-2}_{\overline{E}}\tilde{a}\|_{\overline{E}} \cdot \|\tilde{a}\|_{\overline{E}} = \|m \wedge \tilde{a}\|_{\Lambda^2\overline{E}}.$$

This shows that the expression (4.4) for the size of $P^*\overline{\mathcal{E}}$ may also be written:

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) = \log \min \left\{ \|m \wedge \tilde{a}\|_{\Lambda^2\overline{E}} \text{ , } m \in E \text{ such that } a(m) = 1 \right\}. \tag{4.5}$$

4.2. **Universal extensions and heights over $\mathbb{P}^1_{\mathbb{Z}}$.** Let us specialize the previous example 4.1.3 to the situation where $\overline{E}$ is the trivial hermitian vector bundle of rank 2 over $\operatorname{Spec}\mathbb{Z}$. In other words, $\overline{E} = (E, \|.\|)$ where $\|.\|$ denotes the "standard" hermitian metric on $\mathbb{Z}^2 \otimes \mathbb{C} = \mathbb{C}^2$, and $\mathbb{P}(E)$ is the "projective line" $\mathbb{P}^1_{\mathbb{Z}}$.

For any P in $\mathbb{P}(E)(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$, we may choose homogeneous coordinates $(a_0 : a_1)$ such that a_0 and a_1 are coprime integers, and m_0 and m_1 be coprime integers such that $a_0 m_0 + a_1 m_1 = 1$. Then the class in $\mathbb{R}/\mathbb{Z}$ of

$$\frac{m_0 a_1 - m_1 a_0}{a_0^2 + a_1^2}$$

depends only on P, and the discussion above applied to $a = (a_0, a_1)$ and $\tilde{a} = \binom{a_0}{a_1}$ shows that

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) = \log \delta \left(\frac{m_0 a_1 - m_1 a_0}{a_0^2 + a_1^2} \right) + \log(a_0^2 + a_1^2).$$

Observe that the second term in the right-hand side is the *height of* P with respect to $\overline{\mathcal{O}}_E(1)$:

$$\log \sqrt{a_0^2 + a_1^2} = h_{\overline{\mathcal{O}}_E(1)}(P) := \widehat{\deg}\, P^*\overline{\mathcal{O}}_E(1) = -\widehat{\deg}\, P^*\overline{V}.$$

Therefore, if we let

$$s(P) := \left[\frac{m_0 a_1 - m_1 a_0}{a_0^2 + a_1^2} \right] \ (\in \mathbb{R}/\mathbb{Z}),$$

we have

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) = \log \delta(s(P)) + 2h_{\overline{\mathcal{O}}_E(1)}(P). \tag{4.6}$$

The matrix

$$\gamma := \begin{pmatrix} a_1 & -m_0 \\ a_0 & m_1 \end{pmatrix}$$

belongs to $SL_2(\mathbb{Z})$and we have [9]

$$\gamma \cdot \infty = \frac{a_1}{a_0}.$$

The inverse matrix of γ,

$$\gamma^{-1} := \begin{pmatrix} m_1 & m_0 \\ -a_0 & a_1 \end{pmatrix},$$

[9] We let $SL_2(\mathbb{Z})$ act on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ and the upper half-plane by the usual formula $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z := (az + b)/(cz + d)$.

satisfies:

$$\mathrm{Re}(\gamma^{-1}\cdot i) = \frac{m_0 a_1 - m_1 a_0}{a_0^2 + a_1^2}.$$

Consequently, the map $s : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{R}/\mathbb{Z}$ admits the following description in terms of the actions of $\Gamma := SL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ and the upper half-plane: if

$$\Gamma_\infty := \{\gamma \in \Gamma \mid \gamma\cdot\infty = \infty\} = \{\left(\begin{smallmatrix}\epsilon & n\\ 0 & \epsilon\end{smallmatrix}\right), \epsilon = \pm 1, n \in \mathbb{Z}\},$$

then the map s is characterized by the commutativity of the following diagram:

$$\begin{array}{cccc} & [\gamma] & \longmapsto & \gamma\cdot\infty \\ [\gamma] & \Gamma/\Gamma_\infty & \xrightarrow{\sim} & \mathbb{P}^1(\mathbb{Q}) \\ \big\downarrow & \big\downarrow & & \big\downarrow s \\ [\,\mathrm{Re}\,(\gamma^{-1}\cdot i)] & \mathbb{R}/\mathbb{Z} & = & \mathbb{R}/\mathbb{Z}. \end{array} \tag{4.7}$$

Proposition 4.2.1. *With the above notation, for any $P \in \mathbb{P}^1(\mathbb{Z})$, the extension class of the admissible extension $P^*\overline{\mathcal{E}}$ vanishes iff P is $0 := (1:0)$ or $\infty := (0:1)$.*

Moreover, for any P in $\mathbb{P}^1(\mathbb{Z}) \setminus \{0,\infty\}$, we have:

$$-\frac{1}{2}\log 2 + h_{\overline{\mathcal{O}}_E(1)}(P) \leq \mathfrak{s}(P^*\overline{\mathcal{E}}) \leq -\log 2 + 2h_{\overline{\mathcal{O}}_E(1)}(P). \tag{4.8}$$

Proof. As above, let us choose a_0 and a_1 prime together such that $P = (a_0 : a_1)$.

To establish the first assertion, observe that the following conditions are successively equivalent:

- the admissible extension $P^*\overline{\mathcal{E}}$ is split;
- there exists (m_0, m_1) in $\mathbb{Z}^2$ such that $a_0m_0 + a_1m_1 = 1$ and $m_0a_1 - m_1a_0 = 0$;
- there exists $k \in \mathbb{Z}$ such that $(m_0, m_1) = k(a_0, a_1)$ and $a_0m_0 + a_1m_1 = 1$;
- $a_0^2 + a_1^2 = 1$;
- (a_0, a_1) belongs to $\{(1,0), (-1,0), (0,1), (0,-1)\}$.

The second estimate in (4.8) follows from (4.6), since the values of δ lie in $[0, 1/2]$.

To derive the first one, without loss of generality, we may assume a_0 and a_1 positive. Let us choose b_0 and b_1 two integers satisfying $a_0b_1 - a_1b_0 = 1$ such that b_0 is non-negative and minimal, or equivalently, in $\{0, \dots, a_0 - 1\}$. Then $a_0b_1 = 1 + a_1b_0$ is positive and $\leq 1 + a_1(a_0 - 1) \leq a_0a_1$. Consequently b_1 belongs to $\{1, \dots, a_1\}$, and the following inequality holds:

$$0 \leq \frac{b_0}{a_0} \leq \frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2} < \frac{b_1}{a_1} \leq 1.$$

If $b_0 \neq 0$ and $b_1 \neq a_1$, this implies

$$\frac{1}{a_0} \leq \frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2} \leq 1 - \frac{1}{a_1},$$

and consequently,

$$\delta\left(\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2}\right) \geq \min(\frac{1}{a_0}, \frac{1}{a_1}) \geq \frac{1}{\sqrt{a_0^2 + a_1^2}}.$$

If $b_0 = 0$, then $a_0b_1 = 1$, and necessarily $a_0 = b_1 = 1$, hence

$$\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2} = \frac{a_1}{a_1^2 + 1}.$$

This is a number in $[0, 1/2]$, and

$$\delta\left(\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2}\right) = \frac{a_1}{a_1^2 + 1} \geq \frac{1}{\sqrt{2}\sqrt{a_1^2 + 1}} = \frac{1}{\sqrt{2}\sqrt{a_0^2 + a_1^2}}.$$

If $b_1 = a_1$, then $a_1(b_0 - a_0) = -1$, and necessarily $a_1 = 1$ and $b_0 - a_0 = -1$. Then

$$\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2} = \frac{a_0(a_0 - 1) + 1}{a_0^2 + 1} = 1 - \frac{a_0}{a_0^2 + 1}.$$

Since $a_0/(a_0^2 + 1)$ belongs to $[0, 1/2]$, we obtain:

$$\delta\left(\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2}\right) = \frac{a_0}{a_0^2 + 1} \geq \frac{1}{\sqrt{2}\sqrt{a_0^2 + a_1^2}}.$$

We have shown that the lower bound

$$\delta\left(\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2}\right) \geq \frac{1}{\sqrt{2}\sqrt{a_0^2 + a_1^2}}$$

always holds. This may also be written

$$\log s(P) \geq -h_{\overline{\mathcal{O}}_E(1)}(P) - \frac{1}{2}\log 2,$$

and, according to (4.6), is equivalent to the first inequality in (4.8). □

Observe also that the estimates (4.8) are basically optimal.

Indeed, if for any positive integer n, we let $P_n := (1 : n)$, then

$$h_{\overline{\mathcal{O}}_E(1)}(P_n) = \log\sqrt{n^2 + 1} = \log n + O(1/n^2),$$

and, since $\begin{pmatrix} 1 & 1 \\ n & n+1 \end{pmatrix}$ belongs to $SL_2(\mathbb{Z})$, we have

$$s(P_n) = \frac{n^2 + n + 1}{n^2 + 1} = \frac{n}{n^2 + 1} \mod \mathbb{Z}$$

and

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) - 2h_{\overline{\mathcal{O}}_E(1)}(P_n) = \log\frac{n}{n^2 + 1} = -\log n + O(1/n^2).$$

In particular, if $\mathcal{F}$ denotes the filter of complements in $\mathbb{P}^1(\mathbb{Q})$ of finite subsets, we have:

$$\liminf_{\mathcal{F}} \left(\mathfrak{s}(P^*\overline{\mathcal{E}}) - h_{\overline{\mathcal{O}}_E(1)}(P)\right) \leq 0.$$

Moreover, the interpretation of s given by the diagram (4.7) shows that $s(\mathbb{P}^1(\mathbb{Q}))$ is dense in $\mathbb{R}/\mathbb{Z}$. Indeed, the closure of $\Gamma \cdot i$ in $\mathbb{P}^1(\mathbb{C})$ contains the limit set $\mathbb{P}^1(\mathbb{R})$ for the action of Γ in $\mathbb{P}^1(\mathbb{C})$; therefore, $\operatorname{Re}(\Gamma \cdot i)$ is dense in $\mathbb{R}$. This implies that the set of values of

$$\mathfrak{s}(P^*\overline{\mathcal{E}}) - 2h_{\overline{\mathcal{O}}_E(1)}(P) = \log\delta(s(P))$$

when P runs over $\mathbb{P}^1(\mathbb{Q})$ is dense in $\log \delta(\mathbb{R}/\mathbb{Z}) = [-\infty, -\log 2]$. In particular, with $\mathcal{F}$ as above,

$$\limsup_{\mathcal{F}} \left(\mathfrak{s}(P^*)\overline{\mathcal{E}} - 2h_{\overline{\mathcal{O}}_E(1)}(P) \right) \geq -\log 2.$$

4.2.2. *A geometric interpretation by means of Ford circles.* It may be worth noting that the function $s : \mathbb{P}^1(\mathbb{Q}) \mapsto \mathbb{R}/\mathbb{Z}$ in terms of which we have expressed $\mathfrak{s}(P^*\overline{\mathcal{E}})$ admits a geometric interpretation by means of the so-called *Ford circles.*[10]

Recall these are circles $C(q)$ in the upper half-plane $\{\operatorname{Re} z \geq 0\}$ in $\mathbb{C}$ associated to rational numbers $q \in \mathbb{Q}$: if $q = b/a$, where a and b are integers which are prime together, $C(q)$ is defined as the circle of center $q + i/2a^2$ and radius $1/2a^2$. It meets tangentially the line $\mathbb{R}$ at q. It is convenient to define also $C(\infty)$ as the subset $(i + \mathbb{R}) \cup \{\infty\}$ in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Then, for any $q \in \mathbb{P}^1(\mathbb{Q})$, $C(q) \setminus \{q\}$ is an horocycle in the Poincaré upper half-plane, and a straightforward computation shows that, for any γ in $SL_2(\mathbb{Z})$,

$$\gamma \cdot C(q) = C(\gamma \cdot q).$$

This easily implies that for any two distinct points $q_0 = b_0/a_0$ and $q_1 = b_1/a_1$ in $\mathbb{P}^1(\mathbb{Q})$ — with a_0 and b_0 (resp. a_1 and b_1) prime together — the "circles" $C(q_0)$ and $C(q_1)$ are disjoint if $|a_0b_1 - a_1b_0| \neq 1$, and are externally tangent if $|a_0b_1 - a_1b_0| = 1$. Moreover, when the latter possibility arises, the abscissa of their tangency point is

$$\frac{a_0b_0 + a_1b_1}{a_0^2 + a_1^2}$$

(see Figure 1).

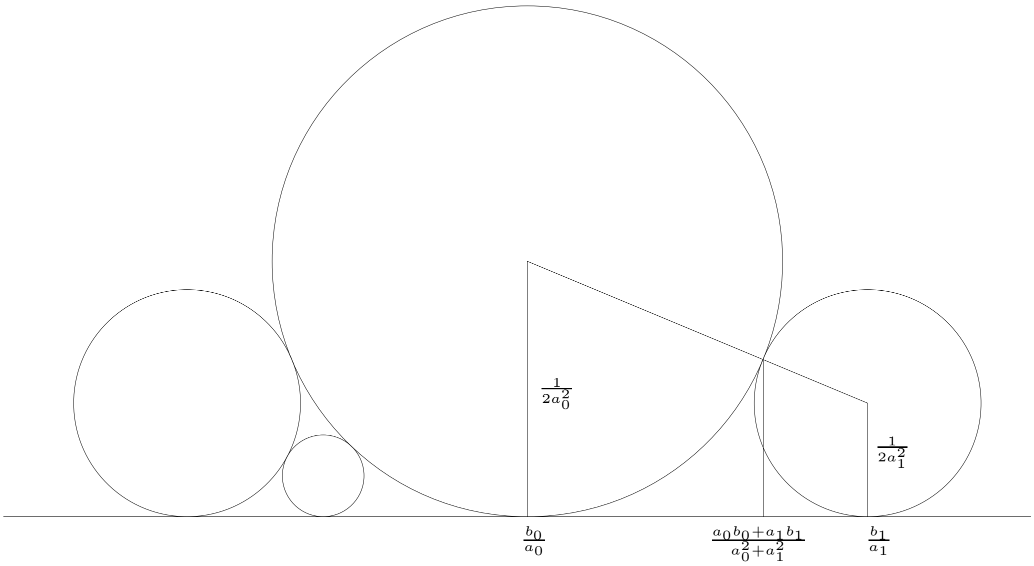

FIGURE 1. The tangency point of two adjacent Ford circles

[10] These circles also appear in the literature under the names of Farey or Speiser circles. We refer the reader to [For38] and [Rad64], chapter 6, for more information and references about their properties.

Consequently, for any P in $\mathbb{P}^1(\mathbb{Q}) \setminus \{0, \infty\}$, its image $s(P)$ in $\mathbb{R}/\mathbb{Z}$ may be constructed as follows. Let a_0 and a_1 be two integers prime together such that $P = (a_0 : a_1)$, and let $C(q_0)$ and $C(q_1)$ be two tangent Ford circles such that $q_0 < q_1$, attached to rational points $q_0 = b_0/a_0$ and $q_1 = b_1/a_1$ with b_0 (resp. b_1) an integer prime to a_0 (resp. a_1). Denote by ε the sign of $a_0 a_1$. Then $\varepsilon.s(P)$ is the abscissa of the tangency point of $C(q_0)$ and $C(q_1)$ in $\mathbb{R}/\mathbb{Z}$. Observe that this construction makes geometrically obvious the density of $s(\mathbb{P}^1(\mathbb{Q}))$ in $\mathbb{R}/\mathbb{Z}$.

4.3. **An application to reduction theory.** In this section, we want to discuss the relation between the classical reduction theory and our results concerning extensions of hermitian vector bundles over arithmetic curves and their sizes (see for instance [vdW56] and [Bor69] for classical expositions and references, and [LLS90] for more recent applications to lattice geometry).

In one direction, observe that, prior to Banaszczyk's contribution — which is based on properties of "Gaussian-like" measures on lattices ([Ban95], Section1) — transference results similar to Theorem 3.7.2 were classically established by means of reduction theory (see for instance [LLS90], Sections 3 and 5).

Conversely, using our basic upper bound on sizes in Theorem 3.7.3, we are going to establish the following

Theorem 4.3.1. *For any number field K and any positive integer n, there exists a constant $r(n, K)$ satisfying the following property: for every hermitian vector bundle $\overline{E}$ of rank n over $\operatorname{Spec} \mathcal{O}_K$, there exist hermitian line bundles $\overline{L}_1, \ldots, \overline{L}_n$, and an isomorphism of $\mathcal{O}_K$-modules*

$$\phi : E \xrightarrow{\sim} \bigoplus_{i=1}^{n} L_i$$

such that, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$\log \|\phi_\sigma\|^{\infty}_{\overline{E}, \bigoplus_{i=1}^n \overline{L}_{i,\sigma}} \leq r(n, K) \quad \text{and} \quad \log \|\phi_\sigma^{-1}\|^{\infty}_{\bigoplus_{i=1}^n \overline{L}_i, \overline{E}, \sigma} \leq r(n, K). \tag{4.9}$$

As above $\|.\|^{\infty}_{\overline{E}, \bigoplus_{i=1}^n \overline{L}_{i,\sigma}}$ (resp. $\|.\|^{\infty}_{\bigoplus_{i=1}^n \overline{L}_i, \overline{E}, \sigma}$) denotes the operator norm on the space $\operatorname{Hom}_{\mathbb{C}}(E_\sigma, \bigoplus_{i=1}^n L_{i,\sigma})$ (resp. on $\operatorname{Hom}_{\mathbb{C}}(\bigoplus_{i=1}^n L_{i,\sigma}, (E_\sigma))$ deduced from the hermitian norms $\|.\|_{\overline{E},\sigma}$ and $\|.\|_{\bigoplus_{i=1}^n \overline{L}_{i,\sigma}}$.

When $K = \mathbb{Q}$, this follows from the classical reduction theory of quadratic forms of Lagrange, Gauß ($n = 2$), Hermite, and Korkin-Zolotarev (n arbitrary). For arbitrary number fields, Proposition (4.9) follows from the general theory of fundamental domains for arithmetic groups (for instance, from [Bor69], Théorème 13.1, applied to the Weil restriction from K to $\mathbb{Q}$ of $GL_{n,K}$).

An effective control of the constants $r(n, K)$ does not seem to follow simply from results in the literature, except when $K = \mathbb{Q}$ (in which case the best available estimates would follow from the results in [LLS90] concerning bases of euclidean lattices which are "Korkin-Zolotarev reduced"). An interesting feature of the proof below is the explicit values of $r(n, K)$ it provides for arbitrary number fields.

Actually we will establish a significantly more precise version of Theorem 4.3.1. Before we state it, we need to introduce some preliminary definitions.

Consider an hermitian vector bundle $\overline{E}$ of rank n over $S := \operatorname{Spec} \mathcal{O}_K$ as above, and a *complete flag*

$$E_\bullet : E_0 = \{0\} \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

namely a filtration by saturated $\mathcal{O}_K$-submodules of ranks $\operatorname{rk} E_i = i$, $0 \leq i \leq n$. To these data, we may attach the vector bundle of rank n over S

$$\mathrm{Gr} E_\bullet := \bigoplus_{1 \leq i \leq n} E_i / E_{i-1}.$$

We define a *splitting over* $\mathcal{O}_K$ of the flag $E_\bullet$ as an isomorphism of $\mathcal{O}_K$-modules

$$\varphi : E \xrightarrow{\sim} \mathrm{Gr} E_\bullet$$

such that, for any $k \in \{0, \ldots, n\}$,

$$\varphi(E_k) = \bigoplus_{1 \leq i \leq k} (E_i / E_{i-1}).$$

For any such splitting φ and any $i \in \{1, \ldots, n\}$, the inverse image

$$L_i := \varphi^{-1}((E_i / E_{i-1})) \tag{4.10}$$

is a direct summand of rank 1 in E such that

$$E_i = E_{i-1} \oplus L_i.$$

Conversely any family $(L_i)_{1 \leq i \leq n}$ of $\mathcal{O}_K$-submodules of E that satisfies these conditions is deduced by the above construction from a unique splitting φ of $E_\bullet$ over $\mathcal{O}_K$. Observe also that to any complete flag $E_\bullet$ in E is attached by duality a complete flag $E_\bullet^\perp := ((E/E_{n-i})^\vee)_{0 \leq i \leq n}$ in $E^\vee$, and that this construction establishes a one-to-one correspondence between complete flags in E and $E^\vee$.

For any hermitian vector bundle $\overline{E}$ as above, we may perform the following inductive construction of a complete flag $E_\bullet$: we let $E_0 := \{0\}$, and for any integer $i \in \{0, \ldots, n-1\}$, we choose E_{i+1} as a (necessarily saturated) $\mathcal{O}_K$-submodule of rank $i+1$ in E containing E_i that satisfies

$$\widehat{\deg}_n \overline{E_{i+1}/E_i} = \widehat{\mathrm{udeg}}_n \overline{E/E_i}.$$

We shall call a complete flag $E_\bullet$ obtained by the above construction a *reduced complete flag* associated to $\overline{E}$. This terminology is meant to emphasize the analogy with classical reduction theory *à la* Hermite-Korkin-Zolotarev. Actually, when $S = \operatorname{Spec} \mathbb{Z}$, if $(b_1, \ldots, b_n)$ is a basis of the euclidean lattice $\overline{E}$ which is "Korkin-Zolotarev reduced" (see for instance [LLS90], Section 2), then the complete flag $(\bigoplus_{1 \leq i \leq k} \mathbb{Z} b_i)_{0 \leq k \leq n}$ of E is reduced in our sense.

We may now state a refined version of Theorem 4.3.1.

Theorem 4.3.2. *Let K be a number field, and let $(c(n,K))_{n>0}$ be a sequences in $\mathbb{R}_+$ that satisfy, for any $n > 0$,*

$$c(n,K) \geq \frac{1}{2n} \sum_{1 \leq i \leq n-1} \log(1 + |\Delta_K|^{4/[K:\mathbb{Q}]}(n-1)^2) \tag{4.11}$$

For any hermitian vector bundle $\overline{E}$ of positive rank n over $\operatorname{Spec} \mathcal{O}_K$ *and any complete flag $E_\bullet$ in E such that the dual flag $E_\bullet^\perp$ is a reduced flag of $\overline{E}^\vee$, there exists a splitting over*

$\mathcal{O}_K$ of $E_\bullet$ such that the associated $\mathcal{O}_K$-submodules $(L_i)_{1\leq i\leq n}$ satisfy

$$\delta(\overline{E}; L_1, \ldots, L_n) := \widehat{\mu}(\overline{E}) - \frac{1}{n}\sum_{i=1}^{n} \widehat{\deg}_n \overline{L}_i \leq c(n, K). \tag{4.12}$$

Moreover, the archimedean operator norms of the tautological "sum map"

$$\Sigma : \bigoplus_{1\leq i\leq n} L_i \xrightarrow{\sim} E$$

and of its inverse, computed with respect to the hermitian structures of $\bigoplus_{1\leq i\leq n} \overline{L}_i$ and $\overline{E}$, satisfy the following bounds:

$$\|\Sigma\|_\sigma \leq \sqrt{n} \quad \text{for any embedding } \sigma : K \hookrightarrow \mathbb{C}, \tag{4.13}$$

and

$$[K:\mathbb{Q}]^{-1} \sum_{\sigma : k\hookrightarrow\mathbb{C}} \log \|\Sigma^{-1}\|_\sigma \leq \frac{n-1}{2}\log n + n.c(n, K). \tag{4.14}$$

Condition (4.11) is satisfied by

$$c(n, K) := 2\frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \frac{1}{n}\log n!,$$

and consequently also by

$$c(n, K) := 2\frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log n. \tag{4.15}$$

Observe that, for any family $(L_i)_{1\leq i\leq n}$ of $\mathcal{O}_K$-submodules of rank 1 in E such that E_K is the direct sum $\bigoplus_{1\leq i\leq n} L_{i,K}$, we may attach the real number

$$\delta(\overline{E}; L_1, \ldots, L_n) := \widehat{\mu}(\overline{E}) - \frac{1}{n}\sum_{i=1}^{n} \widehat{\deg}_n \overline{L}_i.$$

It is easily checked to be non-negative and to vanish iff the tautological "sum map" defines an isomorphism of hermitian vector bundles from $\bigoplus_{1\leq i\leq n} \overline{L}_i$ to $\overline{E}$.

When $K = \mathbb{Q}$ and $(b_1, \ldots, b_n)$ is a $\mathbb{Z}$-basis of the euclidean lattice $\overline{E}$, then $\delta(\overline{E}; \mathbb{Z}b_1, \ldots, \mathbb{Z}b_n)$ is the logarithm of the "orthogonality defect" $\prod_{1\leq i\leq n} \|b_i\| / \mathrm{covol}(\overline{E})$ of the basis $(b_1, \ldots, b_n)$ (compare [LLS90], p.336). Lagarias and his coworkers establish in *loc. cit.* that the submodules $L_i := \mathbb{Z}.b_i$ generated by the vectors of a Korkin-Zolotarev reduced base $(b_1, \ldots, b_n)$ of $\overline{E}$ satisfy the upper bound (4.12) with, in place of $c(n, \mathbb{Q})$, a function of n which, like $c(n, \mathbb{Q})$, grows like $\log n + O(1)$ when n goes to infinity.

The existence of a splitting of $E_\bullet$ such that the associated family $(L_i)_{1\leq i\leq n}$ satisfies (4.12) follows by induction on n from Corollary 3.5.6, Proposition 3.5.7, and the following lemma applied to the dual of $\overline{E}$ and the submodule $(E/E_{n-1})^\vee$ of rank 1 in $E^\vee$:

Lemma 4.3.3. *Let $\overline{E}$ be an hermitian vector bundle of rank $n > 1$, and L a saturated $\mathcal{O}_K$-submodule of rank 1 in E such that $\widehat{\deg}_n \overline{L}$ $(= \widehat{\mu}(\overline{L}))$ equals $\widehat{\mathrm{udeg}}_n \overline{E}$.*

We have:

$$\widehat{\mathrm{udeg}}_n \overline{E/L} - \widehat{\deg}_n \overline{L} \leq \log 2 + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]}. \tag{4.16}$$

Moreover the size of the admissible extension

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{L} \longrightarrow \overline{E} \longrightarrow \overline{E/L} \longrightarrow 0 \tag{4.17}$$

satisfies

$$\mathfrak{s}(\overline{\mathcal{E}}) \leq 2\frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log(n-1). \tag{4.18}$$

Proof. Consider a saturated $\mathcal{O}_K$-submodule M of rank 1 in E/L such that

$$\widehat{\deg}_n \overline{M} = \widehat{\mathrm{udeg}}_n \overline{E/L}, \tag{4.19}$$

and let p be its inverse image in E. It a saturated $\mathcal{O}_K$-submodule of rank 2 in E, which contains L and defines an admissible extension over $\operatorname{Spec}\mathcal{O}_K$:

$$0 \longrightarrow \overline{L} \longrightarrow \overline{P} \longrightarrow \overline{M} \longrightarrow 0.$$

In particular,

$$\widehat{\deg}_n \overline{P} = \widehat{\deg}_n \overline{L} + \widehat{\deg}_n \overline{M}. \tag{4.20}$$

Moreover, by the very definition of L, the inequalities

$$\widehat{\deg}_n \overline{L} \leq \widehat{\mathrm{udeg}}_n \overline{P} \leq \widehat{\mathrm{udeg}}_n \overline{E}$$

are equalities. Together with ((3.27), this shows that:

$$\widehat{\deg}_n \overline{L} = \widehat{\mathrm{udeg}}_n \overline{P} \geq \widehat{\mu}(\overline{P}) - \frac{1}{2}\log 2 - \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}. \tag{4.21}$$

Inequality (4.16) follows from (4.19), (4.20), and (4.21).

According to the upper bound (3.63) in Theorem 3.7.3 on the size of $\overline{\mathcal{E}}$, we have:

$$\mathfrak{s}(\overline{\mathcal{E}}) \leq \widehat{\mathrm{udeg}}_n (\overline{E/L} \otimes \overline{L}^{\vee}) + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]} + \log\big((n-1)/2\big). \tag{4.22}$$

Inequality (4.17) follows from (4.22), ((4.16), and the equality

$$\widehat{\mathrm{udeg}}_n (\overline{E/L} \otimes \overline{L}^{\vee}) = \widehat{\mathrm{udeg}}_n (\overline{E/L}) - \widehat{\deg}_n \overline{L},$$

which is a straightforward consequence of (3.21).

□

The upper bound (4.13) on the archimedean norms of Σ is obvious. The upper bounds (4.14) on its inverse map then follow from the next observation applied to Σ.

Lemma 4.3.4. *For any two hermitian vector bundles $\overline{E}_1$ and $\overline{E}_2$ of positive rank n over* $\operatorname{Spec}\mathcal{O}_K$ *and any isomorphism of $\mathcal{O}_K$-modules $\psi : E_1 \xrightarrow{\sim} E_2$, we have:*

$$\begin{aligned}
&[K:\mathbb{Q}]^{-1} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log\|\psi^{-1}\|_{\overline{E}_2,\overline{E}_1,\sigma} \\
&\quad = [K:\mathbb{Q}]^{-1} \sum_{\sigma:K\hookrightarrow\mathbb{C}} (\log\|\wedge^{n-1}\psi\|_{\wedge^{n-1}\overline{E}_1,\wedge^{n-1}\overline{E}_2,\sigma} - \log\|\wedge^{n}\psi\|_{\wedge^{n}\overline{E}_1,\wedge^{n}\overline{E}_2,\sigma}) \\
&\quad \leq (n-1)[K:\mathbb{Q}]^{-1} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log\|\psi\|_{\overline{E}_1,\overline{E}_2,\sigma} + n(\widehat{\mu}(\overline{E}_2) - \widehat{\mu}(\overline{E}_1)).
\end{aligned} \tag{4.23}$$

Theorem 4.3.2 — or rather a variant, involving different constants — may also be deduced by induction from Lemma 4.3.3 by means of the bounds (3.46) and (3.47) on the norms of trivializations of admissible extensions. The above proof, which emphasizes the role of the invariant $\delta(\overline{E}; L_1, \ldots, L_n)$ attached to a linearly independent n-tuple of rank one subbundles, is computationally simpler.

5. Invariance of the size under base change and Voronoi cells of euclidean lattices

An intriguing issue is the invariance property of the size under base changes $\operatorname{Spec} \mathcal{O}_{K'} \to \operatorname{Spec} \mathcal{O}_K$ associated to extensions of number fields $K \hookrightarrow K'$. It seems plausible that, defined with the precise normalization we introduce in paragraph 3.5 below, the size is invariant by any such base change, at least when $K = \mathbb{Q}$. In this section, we establish various results which support this expectation.

5.1. **A geometric consideration.** Let C and C' be two projective curves, smooth and geometrically connected over some field k, and let $f : C' \to C$ be a finite (or equivalently dominant) k-morphism.

Proposition 5.1.1. *When the degree* $\deg f$ *of* f *is prime to the characteristic exponent of* k (for instance, when k is a field of characteristic zero), *then, for any vector bundle* E *over* C, *the* k*-linear map*

$$f^* : H^1(C, E) \longrightarrow H^1(C', f^*E) \tag{5.1}$$

is injective.

Proof. Since the morphism f is affine and E is locally free, we have a canonical isomorphism

$$H^1(C', f^*E) \simeq H^1(C, E \otimes f_*\mathcal{O}_{C'})$$

and the map (5.1) may be identified with the one deduced from the canonical morphism of $\mathcal{O}_C$-modules

$$\mathcal{O}_C \longrightarrow f_*\mathcal{O}_{C'} \tag{5.2}$$

by applying the functors $E \otimes .$ and $H^1(E, .)$. Under the above hypothesis, the morphism (5.2) is split by the morphism of $\mathcal{O}_C$-modules:

$$\frac{1}{\deg f} \mathrm{Tr}_f : f_*\mathcal{O}_{C'} \longrightarrow \mathcal{O}_C, \tag{5.3}$$

where Tr_f denotes the "naive" trace morphism[11] from $f_*\mathcal{O}_{C'}$ to $\mathcal{O}_C$.

Applied to (5.3), the functors $E \otimes .$ and $H^1(C, .)$ produce a splitting of (5.1). □

However, when the characteristic p of k is positive and divides $\deg f$, the pullback map (5.1) may not be injective. Consider for instance any smooth projective, geometrically connected curve C over k and the (relative) Frobenius morphism

$$F : C \longrightarrow C^{(p)}.$$

Then

$$F^* : H^1(C^{(p)}, \mathcal{O}_{C^{(p)}}) \longrightarrow H^1(C, \mathcal{O}_C)$$

[11] For any U open in C, the map $\mathrm{Tr}_{f|U} : f_*\mathcal{O}_{C'}(U) = \mathcal{O}_{C'}(f^{-1}(U)) \to \mathcal{O}_C(U)$ is defined as the trace on the $\mathcal{O}_C(U)$-algebra $\mathcal{O}_{C'}(f^{-1}(U))$, which is a finitely generated projective $\mathcal{O}_C(U)$-module since f is finite and flat.

is injective iff the Hasse-Witt matrix of C is invertible. (This does not hold for instance when C is a supersingular elliptic curve.)

Actually, when p divides $\deg f$, (5.1) may not be injective even when f is separable, as demonstrated by the following observation, applied to an ordinary elliptic curve and its quotient by an étale subgroup of order p:

Lemma 5.1.2. *If E and E' are two elliptic curves over a field k of positive characteristic p and if $f: E' \to E$ is a separable k-isogeny of degree divisible by p, then the pullback map:*

$$f^* : H^1(E, \mathcal{O}_E) \longrightarrow H^1(E', \mathcal{O}_{E'})$$

vanishes.

Proof. Consider the "trace -map"

$$t_f : H^1(E', \mathcal{O}_{E'}) \simeq H^1(E, f_*\mathcal{O}_{E'}) \longrightarrow H^1(E, \mathcal{O}_E)$$

induced by the morphism of $\mathcal{O}_E$-modules

$$\mathrm{Tr}_f : f_*\mathcal{O}_{E'} \longrightarrow \mathcal{O}_E.$$

Using Serre duality, it may be identified with the transpose of f acting by pullback on regular 1-forms:

$$f^*_{\Omega^1} : \Omega^1(E) \longrightarrow \Omega^1(E').$$

Since f is separable, $f^*_{\Omega^1}$ and consequently t_f are isomorphisms. Besides, we have:

$$t_f \circ f^* = \deg f . Id_{H^1(E,\mathcal{O}_E)}.$$

Since p divides $\deg f$, this vanishes, and consequently f^* vanishes too. □

Consider now D a non-zero effective divisor on C, $D' := f^*(D)$ its inverse image in C', and the affine curves $\dot{C} := C \setminus |D|$ and $\dot{C}' := C' \setminus |D'|$. Let also F and G be two vector bundles on C, $\dot{F}$ and $\dot{G}$ their restrictions to $\dot{C}$, F' and G' their pullback to C', and $\dot{F}'$ and $\dot{G}'$ the restrictions of the latter to $\dot{C}'$, or equivalently, the pullback of $\dot{F}$ and $\dot{G}$ by $f_{|\dot{C}'}$. With the notation of Section 3.9, pulling back extensions by f defines a natural k-linear map:

$$f^* : \widehat{\mathrm{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G}) \longrightarrow \widehat{\mathrm{Ext}}^1_{\dot{C}'}(\dot{F}', \dot{G}').$$

By applying Proposition 5.1.1 to the vector bundles $E := \check{F} \otimes G(-N.D)$, $N \in \mathbb{Z}$, we obtain:

Corollary 5.1.3. *When the degree of f is prime to the characteristic exponent of k, then, for any $e \in \widehat{\mathrm{Ext}}^1_{\dot{C}}(\dot{F}, \dot{G})$,*

$$\mathfrak{s}_{F',G'}(f^*(e)) = \mathfrak{s}_{F,G}(e).$$

5.2. **Size and base change.** We give two equivalent formulation of the problem of the invariance of size under base change.

5.2.1. *The condition* $\mathbf{P}(K'/K, \overline{E})$. As before, consider a number field K, a number field K' containing K, and

$$g : S' = \mathrm{Spec}\ \mathcal{O}_{K'} \longrightarrow S = \mathrm{Spec}\ \mathcal{O}_K$$

the associated morphism of arithmetic curves. Let us also denote by π (resp. π') the morphism from $\mathrm{Spec}\,\mathcal{O}_K$ (resp. $\mathrm{Spec}\,\mathcal{O}_{K'}$) to $\mathrm{Spec}\,\mathbb{Z}$.

Problem 1. *Let $\overline{F}$ and $\overline{G}$ be two hermitian vector bundles over S. Is it true that, for any extension class e in $\widehat{\mathrm{Ext}}^1_S(F, G)$, the inequality*

$$\mathfrak{s}_{g^*\overline{F}, g^*\overline{G}}\, g^*(e) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e)$$

is indeed an equality ?

Let $\overline{E}$ be an hermitian vector bundle over S. The map of "extension of scalars" from $\mathcal{O}_K$ to $\mathcal{O}_{K'}$

$$\begin{array}{ccc} E & \longrightarrow & E \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \\ v & \longmapsto & v \otimes 1, \end{array}$$

seen as a $\mathbb{Z}$-linear map, defines a morphism of S-vector bundles

$$\Delta : \pi_* E \longrightarrow \pi'_* g^* E.$$

This is also the morphism deduced from the natural morphism

$$\delta : E \longrightarrow g_* g^* E$$

by taking its direct image by π.

The linear maps

$$\delta_\sigma : E_\sigma \longrightarrow (g_* g^* E)_\sigma$$

(where σ denote an embedding of K in $\mathbb{C}$) and

$$\Delta = \Delta_{\mathbb{R}} : (\pi_* E)_{\mathbb{R}} \simeq E \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow (\pi'_* g^* E)_{\mathbb{R}} \simeq E \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \otimes_{\mathbb{Z}} \mathbb{R}$$

are isometric up to a factor $[K' : K]^{1/2}$ when we equip these vector spaces with the norms defining respectively the hermitian structures of $\overline{E}$, $g_* g^* \overline{E}$, $\pi_* \overline{E}$, and $\pi'_* g^* \overline{E}$. Namely, for any v (resp. w) in E_σ (resp. $(\pi_* E)_{\mathbb{R}}$), we have:

$$(5.4) \qquad \|\delta_\sigma(v)\|_{g_* g^* \overline{E}, \sigma} = [K' : K]^{1/2} \|v\|_{\overline{E}, \sigma} \quad (\text{resp.} \quad \|\Delta(w)\|_{\pi'_* g^* \overline{E}} = [K' : K]^{1/2} \|w\|_{\pi_* \overline{E}}.)$$

Problem 2. *With the notation above, is it true that, for any $w \in (\pi_* E)_{\mathbb{R}}$, the inequality*

$$(5.5) \quad \min_{\beta \in g^* E} \|\Delta(w) - \beta\|_{\pi'_* g^* \overline{E}} \leq \min_{\alpha \in E} \|\Delta(w) - \Delta(\alpha)\|_{\pi'_* g^* \overline{E}} = [K' : K]^{1/2} \min_{\alpha \in E} \|w - \alpha\|_{\pi_* \overline{E}}$$

is actually an equality ?

These two problems are equivalent. More precisely, Problem 1 for some pair $(\overline{F}, \overline{G})$ of hermitian vector bundles is equivalent to Problem 2 for $\overline{E} = \overline{F}^\vee \otimes \overline{G}$.

We shall say that *Condition* $\mathbf{P}(K'/K, \overline{E})$ *holds* when Problem 2 has an affirmative answer.

Observe that, as a straightforward consequence of definitions, this condition satisfies the following compatibility with the operation of direct sum:

Lemma 5.2.2. *For any finite family $(\overline{E}_i)_{i \in I}$ of hermitian vector bundles over* $\operatorname{Spec} \mathcal{O}_K$*, the condition $\mathbf{P}(K'/K, \bigoplus_{i \in I} \overline{E}_i)$ holds iff the condition $\mathbf{P}(K'/K, \overline{E}_i)$ holds for any $i \in I$.*

The following property also is immediate:

Lemma 5.2.3. *Let $\overline{E} := (E, (\|.\|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$ be an hermitian vector bundle over* $\operatorname{Spec} \mathcal{O}_K$*, and λ a positive real number, and let us consider $\overline{E}' := (E, (\lambda \|.\|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$. Then $\mathbf{P}(K'/K, \overline{E})$ holds iff $\mathbf{P}(K'/K, \overline{E}')$ holds.*

Lemma 5.2.4. *Let L/K and K'/K be two extensions of number fields which are disjoint (i.e. $L' = L \otimes_K K'$ is again a number field) and have coprime discriminant ideals. Let h denote the natural map from Spec $\mathcal{O}_L$ to Spec $\mathcal{O}_K$. Let $\overline{E}$ be an hermitian vector bundle over Spec $\mathcal{O}_L$. Then $\mathbf{P}(K'/K, h_* \overline{E})$ implies $\mathbf{P}(L'/L, \overline{E})$.*

Proof. The condition on discriminants assures that the middle square in the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Spec}\mathbb{Z} & \stackrel{\tau'}{\longleftarrow} & \operatorname{Spec}\mathcal{O}_{L'} & \stackrel{g'}{\longrightarrow} & \operatorname{Spec}\mathcal{O}_L & \stackrel{\tau}{\longrightarrow} & \operatorname{Spec}\mathbb{Z} \\ \| & & \downarrow h' & & \downarrow h & & \| \\ \operatorname{Spec}\mathbb{Z} & \stackrel{\pi'}{\longleftarrow} & \operatorname{Spec}\mathcal{O}_{K'} & \stackrel{g}{\longrightarrow} & \operatorname{Spec}\mathcal{O}_K & \stackrel{\pi}{\longrightarrow} & \operatorname{Spec}\mathbb{Z} \end{array}$$

is cartesian (see for example [Neu99, I 2.11]). The base change isomorphism $g^* h_* \overline{E} \xrightarrow{\sim} h'_* g'^* \overline{E}$ is an isometry by 1.2.3. For each element $w \in (\tau_* \overline{E})_{\mathbb{R}} = (\pi_* h_* \overline{E})_{\mathbb{R}}$, we get

$$\min_{\beta \in g'^* E} \|\Delta(w) - \beta\|_{\tau'_* g'^* \overline{E}} = \min_{\beta \in g^* h_* E} \|\Delta(w) - \beta\|_{\pi'_* g^* h_* \overline{E}}.$$

The condition $\mathbf{P}(K'/K,\, h_* \overline{E})$ implies that the last term equals

$$[K' : K]^{1/2} \min_{\alpha \in E} \|w - \alpha\|_{\pi_* h_* \overline{E}} = [L' : L]^{1/2} \min_{\alpha \in E} \|w - \alpha\|_{\tau_* \overline{E}}.$$

Hence $\mathbf{P}(L'/L,\, \overline{E})$ holds. □

5.3. **The condition $\mathbf{P}(K'/K,\, \overline{E})$ and Voronoi cells.** Let us now turn to a reformulation of condition $\mathbf{P}(K'/K,\, \overline{E})$ involving the geometry of Voronoi cells of euclidean lattices.

Recall that the *Voronoi cell* $\mathcal{V}(\overline{F})$ of an euclidean lattice $\overline{F} := (F, \|.\|)$ consists of those points of $F_{\mathbb{R}}$ that are at least as close to the origin as to any element e in F:

$$\mathcal{V}(\overline{F}) := \{x \in F_{\mathbb{R}} \mid \|x\| \leq \|x - e\| \text{ for all } e \in F\}.$$

If $\langle .,. \rangle$ denotes the euclidean scalar product on $F_{\mathbb{R}}$ associated to $\|.\|$, the condition

$$\|x\| \leq \|x - e\| \tag{5.6}$$

may also be expressed as

$$\langle x - \frac{e}{2}, e \rangle \leq 0. \tag{5.7}$$

This shows that the Voronoi cell $\mathcal{V}(\overline{F})$ is a compact, convex, symmetric neighborhood of the origin. Actually, it is defined by a finite number of the conditions (5.7). In other words, it is a polytope, and consequently posses a finite number of extremal points, its *vertices*. Moreover, we have:

$$F_{\mathbb{R}} = \bigcup_{e \in F} (e + \mathcal{V}(\overline{F})). \tag{5.8}$$

Lemma 5.3.1. *With the above notation, condition $\mathbf{P}(K'/K,\, \overline{E})$ holds iff, for any vertex P of $\mathcal{V}(\pi_* \overline{E})$ and any $\beta \in g^* E$, the following equivalent conditions holds:*

$$\|\Delta(P) - \beta\|_{\pi'_* g^* \overline{E}} \geq \|\Delta(P)\|_{\pi'_* g^* \overline{E}} \ (= [K' : K]^{1/2} \|P\|_{\pi_* \overline{E}}); \tag{5.9}$$

$$2\langle \Delta(P), \beta \rangle_{\pi'_* g^* \overline{E}} \ \leq \ \|\beta\|^2_{\pi'_* g^* \overline{E}}. \tag{5.10}$$

Proof. For any P in $(\pi_* E)_{\mathbb{R}}$ $(= E \otimes \mathbb{R})$, the inequalities (5.9) and (5.10) are equivalent (this is the special case of the equivalence of (5.6) and (5.7) when $\|.\| = \|.\|_{\pi'_* g^* \overline{E}}$, $v = \Delta(P)$, and $e = \beta$).

Since $(\pi_* E)_{\mathbb{R}}$ is the union of the translates of $\mathcal{V}(\pi_* \overline{E})$ by elements of $\pi_* E\, (= E)$, the validity of (5.5) for all $v \in (\pi_* E)_{\mathbb{R}}$ is equivalent to its validity when v is a point P of $\mathcal{V}(\pi_* \overline{E})$. Consequently, $\mathbf{P}(K'/K,\, \overline{E})$ holds iff (5.9), or (5.10), holds for any P in $\mathcal{V}(\pi_* \overline{E})$

and any β in g^*E. Observe finally that, for any given β, (5.10) holds for all P in the polytope $\mathcal{V}(\pi_*\overline{E})$ iff it holds for its extremal points. □

Observe that (5.10) is fulfilled as soon as

$$\|\beta\|_{\pi'_*g^*\overline{E}} \geq 2\|\Delta(P)\|_{\pi'_*g^*\overline{E}} \ (= 2[K':K]^{1/2}\|P\|_{\pi_*\overline{E}}).$$

Consequently, Lemma 5.3.1 reduces the proof of $\mathbf{P}(K'/K, \overline{E})$ to checking the *finite set* of inequalities (5.10) for P a vertex of $\mathcal{V}(\pi_*\overline{E})$ and β a point in g^*E such that $\|\beta\|_{\pi'_*g^*\overline{E}}$ is at most

$$2[K':K]^{1/2} \max_{P\in\mathcal{V}(\pi_*\overline{E})} \|P\|_{\pi_*\overline{E}} = 2[K':K]^{1/2}\rho(\pi_*\overline{E}).$$

This shows that there are algorithms that would theoretically enable one to check the validity of $\mathbf{P}(K'/K, \overline{E})$ for any "explicitly given" K, K', and $\overline{E}$. However, one should be aware that determining the vertices of the Voronoi cell of a euclidean lattice is a "hard" problem.

5.4. **Base change from euclidean lattices.** Let L be a number field, and $\overline{E}$ an hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$. Lemma 5.3.1 may be used to reformulate the condition $\mathbf{P}(K'/K, \overline{E})$ for $K = \mathbb{Q}$ and $K' = L$.

Indeed, for any embedding $\sigma : L \hookrightarrow \mathbb{C}$, we may define

$$\begin{array}{rccc} \sigma_E := id_E \otimes \sigma : & g^*E = E\otimes_{\mathbb{Z}}\mathcal{O}_L & \longrightarrow & E_{\mathbb{C}} = E\otimes_{\mathbb{Z}}\mathbb{C} \\ & e\otimes\lambda & \longmapsto & e\otimes\sigma(\lambda). \end{array}$$

Then the morphism of $\mathbb{Z}$-modules

$$(\sigma_E)_{\sigma:L\hookrightarrow\mathbb{C}} : g^*E \longmapsto E_{\mathbb{C}}^{(\operatorname{Spec} L)(\mathbb{C})} = \oplus_{\sigma:L\hookrightarrow\mathbb{C}}E_{\mathbb{C}}$$

extends to an isomorphism of $\mathbb{C}$-vector spaces:

$$g^*E \xrightarrow{\sim} E_{\mathbb{C}}^{(\operatorname{Spec} L)(\mathbb{C})},$$

and condition (5.10) may be written

$$2\sum_{\sigma:L\hookrightarrow\mathbb{C}} \operatorname{Re}\langle P, \sigma_E(\beta)\rangle_{\overline{E}} \leq \sum_{\sigma:L\hookrightarrow\mathbb{C}} \|\sigma_E(\beta)\|^2_{\overline{E}}. \tag{5.11}$$

We shall establish that this holds for any vertex P of $\mathcal{V}(\overline{E})$ and any β in $E\otimes_{\mathbb{Z}}\mathcal{O}_L$ — in other words that condition $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ holds — for various special lattices.

Our proof will rely on the following observation:

Lemma 5.4.1. *For any number field L and any element α of its ring of integers $\mathcal{O}_L$, we have:*

$$\sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)|^2 - \sum_{\sigma:L\hookrightarrow\mathbb{C}} \operatorname{Re}\sigma(\alpha) \geq \sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)|^2 - \sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)| \geq 0.$$

Proof. We may assume that $\alpha \neq 0$. Then

$$\prod_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)| = |N_{L/\mathbb{Q}}(\alpha)| \geq 1,$$

and therefore

$$\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)| \geq 1. \tag{5.12}$$

Moreover, by Cauchy-Schwarz inequality,

$$\left(\sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)|\right)^2 \leq [K:\mathbb{Q}] \sum_{\sigma:L\hookrightarrow\mathbb{C}} |\sigma(\alpha)|^2. \tag{5.13}$$

The second inequality in Lemma 5.4.1 follows from (5.12) and (5.13). The first one is obvious.

□

Proposition 5.4.2. *For any hermitian vector bundle $\overline{E}$ over* $\operatorname{Spec}\mathbb{Z}$ *which is a direct sum of hermitian line bundles, condition* $\mathbf{P}(L/\mathbb{Q},\,\overline{E})$ *holds for any number field L.*

Proof. Proposition 5.2.2 shows that we may suppose that $\overline{E}$ has rank one.

Let e_0 denote a generator of the rank one free $\mathbb{Z}$-module E. Then the Voronoi cell of $\overline{E}$ is the "interval"

$$\mathcal{V}(\overline{E}) = [-1/2, 1/2]\cdot e_0,$$

and its vertices P are the points $e_0/2$ and $-e_0/2$. The elements β of $E\otimes\mathcal{O}_L$ may be written $e_0\otimes\alpha$ with $\alpha\in\mathcal{O}_L$, and condition (5.11) reduces to the inequality in Lemma 5.4.1 applied to α and $-\alpha$. □

The last proposition leads to ask the following:

Question 5.4.3. *Is it true that condition* $\mathbf{P}(L/\mathbb{Q},\,\overline{E})$ *holds for any hermitian vector bundle $\overline{E}$ over* $\operatorname{Spec}\mathbb{Z}$ *and any number field L ?*

A positive answer would imply that the size of any admissible extension over $\operatorname{Spec}\mathbb{Z}$ is invariant by the base change from $\mathbb{Z}$ to $\mathcal{O}_L$ for any number field L.

In the next sections, we shall prove the following theorem which, together with Proposition 5.4.2, points towards a positive answer to question 5.4.3.

Theorem 5.4.4. *For any hermitian vector bundle $\overline{E}$ over Spec $\mathbb{Z}$ and any number field L, the condition $P(L/\mathbb{Q},\overline{E})$ holds in the following cases:*

i) *If $L/\mathbb{Q}$ is an abelian extension.*
ii) *If $\overline{E}$ is a root lattice.*
iii) *If $\overline{E}$ is a lattice of Voronoi's first kind.*

Proof. Case i) is treated in Section 5.5. Recall that an integral euclidean lattice $\overline{E}$ over Spec $\mathbb{Z}$ is called a *root lattice* if E is generated by its subset $\{e\in E \mid \|e\|^2\in\{1,2\}\}$. The proof of ii) is given in Section 5.6. The definition of a lattice of Voronoi's first kind is recalled in the appendix. The proof of iii) can be found in Section 5.7. □

5.4.5. *An application of reduction theory.* Observe that, by combining Proposition 5.4.2 and our "reduction" Theorem 4.3.1, we obtain that Question 5.4.3 has a positive answer up to an additive error term bounded in terms of the ranks of the considered hermitian vector bundles:

Proposition 5.4.6. *There exist non-negative real numbers $s(n)$, $n\in\mathbb{N}$, which satisfy the following property.*

Let $\overline{E}$ be an hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$*, and n its rank. Let L be a number field, and g the unique morphism* $\operatorname{Spec}\mathcal{O}_L\to\operatorname{Spec}\mathbb{Z}$*, .*

For any $w \in E_{\mathbb{R}}$, the following inequalities hold

$$e^{-s(n)}[L:\mathbb{Q}]^{1/2} \min_{\alpha \in E} \|w-\alpha\|_{\overline{E}} \leq \min_{\beta \in g^*E} \|\Delta(w)-\beta\|_{g_*g^*\overline{E}} \leq [L:\mathbb{Q}]^{1/2} \min_{\alpha \in E} \|w-\alpha\|_{\overline{E}} \tag{5.14}$$

Proof. The second inequality in (5.14) is clear, and, according to Proposition 5.4.2, is an equality when $\overline{E}$ is an orthogonal direct sum of hermitian line bundles over $\operatorname{Spec}\mathbb{Z}$.

Let $\overline{E}_0$ be such a direct sum and $\varphi : E \to E_0$ an isomorphism of $\mathbb{Z}$-modules, and denote $\Delta_0 : E_0 \to g^*E_0 := E_0 \otimes_{\mathbb{Z}} \mathcal{O}_L$ the map $. \otimes 1$, and $\|\varphi\|_\infty$ (resp. $\left\|\varphi^{-1}\right\|_\infty$) the operator norm of $\varphi_{\mathbb{C}}$, or equivalently of $\varphi_{\mathbb{R}}$ (resp. of its inverse). Then, for any w in $E_{0,\mathbb{R}}$, we have

$$\min_{\alpha \in E} \|w-\alpha\|_{\overline{E}} \leq \left\|\varphi^{-1}\right\|_\infty \min_{\alpha_0 \in E_0} \|\varphi_{\mathbb{R}}(w)-\alpha_0\|_{\overline{E}_0}$$

and

$$\|\varphi\|_\infty^{-1} \min_{\beta_0 \in g^*E_0} \|\Delta_0(\varphi_{\mathbb{R}}(w))-\beta_0\|_{g_*g^*\overline{E}_0} \leq \min_{\beta \in g^*E} \|\Delta(w)-\beta\|_{g_*g^*\overline{E}}.$$

Besides, since $\overline{E}_0$ is an orthogonal direct sum of hermitian line bundles, we have:

$$\min_{\beta_0 \in g^*E_0} \|\Delta_0(\varphi_{\mathbb{R}}(w))-\beta_0\|_{g_*g^*\overline{E}_0} = [L:\mathbb{Q}]^{1/2} \min_{\alpha_0 \in E_0} \|\varphi_{\mathbb{R}}(w)-\alpha_0\|_{\overline{E}_0}.$$

Consequently,

$$\|\varphi\|_\infty^{-1} \left\|\varphi^{-1}\right\|_\infty^{-1} \min_{\alpha \in E} \|w-\alpha\|_{\overline{E}} \leq \min_{\beta \in g^*E} \|\Delta(w)-\beta\|_{g^*\overline{E}}.$$

To complete the proof, just recall that, as shown in Theorem 4.3.1, we may find $\overline{E}_0$ and φ as above with $\|\varphi\|_\infty$ and $\left\|\varphi^{-1}\right\|_\infty$ bounded in terms of n alone.

□

The above proof establishes that, with the notation of Theorem 4.3.1 and Theorem 4.3.2, Proposition 5.4.6 holds with

$$s(n) = 2r(n,\mathbb{Q}),$$

or

$$s(n) = n(\frac{1}{2}\log n + c(n,\mathbb{Q})).$$

Using for instance (4.15), we may take:

$$s(n) = \frac{3}{2} n \log n.$$

Using the numbers $s(n)$, $n \in \mathbb{N}$, from the previous proposition, we get:

Corollary 5.4.7. *Let L be any number field, and let g be the unique morphism $\operatorname{Spec}\mathcal{O}_L \to \operatorname{Spec}\mathbb{Z}$. For any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over $S := \operatorname{Spec}\mathbb{Z}$, and any extension class e in $\widehat{\operatorname{Ext}}^1_S(F,G)$, we have:*

$$\mathfrak{s}_{\overline{F},\overline{G}}(e) - s(\operatorname{rk} F \cdot \operatorname{rk} G) \leq \mathfrak{s}_{g^*\overline{F},g^*\overline{G}}(g^*(e)) \leq \mathfrak{s}_{\overline{F},\overline{G}}(e).$$

5.5. **Cyclotomic base change.** Let $\overline{V}$ be an hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$ and ϵ a non-zero-vector in V. We shall say that ϵ is *$\mathcal{V}$-small* in $\overline{V}$ if, for every hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathbb{Z}$, the map

$$\begin{array}{rccc}\Delta : & E_{\mathbb{R}} & \longrightarrow & E_{\mathbb{R}} \otimes_{\mathbb{R}} V_{\mathbb{R}} \\ & v & \longmapsto & v \otimes \epsilon\end{array}$$

satisfies the following compatibility conditions with the Voronoi cells of $\overline{E}$ and $\overline{E} \otimes \overline{V}$:

$$\Delta(\mathcal{V}(\overline{E})) \subset \mathcal{V}(\overline{E} \otimes \overline{V}). \tag{5.15}$$

Observe that, for any $v \in E_{\mathbb{R}}$,

$$\|\Delta(v)\|_{\overline{E} \otimes \overline{V}} = \|v \otimes \epsilon\|_{\overline{E} \otimes \overline{V}} = \|\epsilon\|_{\overline{V}} \cdot \|v\|_{\overline{E}}$$

and that $\Delta(E) \subset E \otimes V$. Therefore

$$\min_{\beta \in E \otimes V} \|\Delta(v) - \beta\|_{\overline{E} \otimes \overline{V}} \leq \|\epsilon\|_{\overline{V}} \cdot \min_{\alpha \in E} \|v - \alpha\|_{\overline{E}}, \tag{5.16}$$

and condition (5.15) holds iff this inequality is actually an equality. Observe also that (5.16) precisely means that

$$\Delta^{-1}(\mathcal{V}(\overline{E} \otimes \overline{V})) \subset \mathcal{V}(\overline{E}), \tag{5.17}$$

and that consequently (5.15) is equivalent to

$$\Delta^{-1}(\mathcal{V}(\overline{E} \otimes \overline{V})) = \mathcal{V}(\overline{E}). \tag{5.18}$$

Lemma 5.5.1. *1) For any euclidean norm $\|.\|$ on $\mathbb{R}$, the element 1 is $\mathcal{V}$-small $(\mathbb{Z}, \|.\|)$.*

2) If ϵ is $\mathcal{V}$-small in $\overline{V}$, then, for any hermitian vector bundle $\overline{W}$, $\epsilon \oplus 0$ is $\mathcal{V}$-small in $\overline{V} \oplus \overline{W}$.

3) Let V be a free $\mathbb{Z}$-module of finite rank, ϵ a non-zero element in V, and $\|.\|$ and $\|.\|'$ two euclidean norms on $V_{\mathbb{R}}$, and let us assume that

$$\|.\| \leq \|.\|' \quad \text{and } \|\epsilon\| = \|\epsilon\|'.$$

Then, if ϵ is $\mathcal{V}$-small in $\overline{V} := (V, \|.\|)$, it is $\mathcal{V}$-small in $\overline{V}' := (V, \|.\|')$.

4) If ϵ (resp. ϵ') is $\mathcal{V}$-small in $\overline{V}$ (resp. $\overline{V}'$), then $\epsilon \otimes \epsilon'$ is $\mathcal{V}$-small in $\overline{V} \otimes \overline{V}'$.

Proof. Assertions 1), 2), and 4) are straightforward consequences of the definition of $\mathcal{V}$-small vectors. To prove 3), observe that, since $\|.\| \leq \|.\|'$, for any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathbb{Z}$, we have $\|.\|_{\overline{E} \otimes \overline{V}} \leq \|.\|_{\overline{E} \otimes \overline{V}'}$. This is clear by decomposing $V_{\mathbb{R}}$ as a direct sum of rank-one $\mathbb{R}$-vector spaces which is orthogonal with respect to both $\|.\|$ and $\|.\|'$. □

Let L be a number field, π the morphism from $\operatorname{Spec}\mathcal{O}_L$ to $\operatorname{Spec}\mathbb{Z}$, and $\pi_*\overline{\mathcal{O}_L}$ the direct image of the trivial hermitian line bundle over $\operatorname{Spec}\mathcal{O}_L$. For any hermitian vector bundle $\overline{E}$ over $\operatorname{Spec}\mathbb{Z}$, we have a canonical isomorphism of hermitian vector bundles over $\operatorname{Spec}\mathbb{Z}$

$$\overline{E} \otimes \pi_*\overline{\mathcal{O}_L} \simeq \pi_*\pi^*\overline{E},$$

under which the canonical map

$$E \hookrightarrow \pi_*\pi^*E \simeq E \otimes_{\mathbb{Z}} \mathcal{O}_L$$

gets identified to $v \mapsto v \otimes 1$ by 1.2.2.

Consequently, *Condition* $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ *holds for any hermitian vector bundle* $\overline{E}$ *over* $\operatorname{Spec}\mathbb{Z}$ *iff* 1 *is* $\mathcal{V}$*-small in* $\pi_*\overline{\mathcal{O}_L}$.

For any positive integer n, let $\zeta_n := e^{2\pi i/n}$ and $K_n := \mathbb{Q}(\zeta_n)$ the number field generated by the n-th roots of unity. Its ring of integers $\mathcal{O}_{K_n}$ is

$$\mathbb{Z}[\zeta_n] = \bigoplus_{i=0}^{\varphi(n)-1} \mathbb{Z}\cdot\zeta_n^i.$$

We shall denote π_n the morphism from $\operatorname{Spec}\mathcal{O}_{K_n}$ to $\operatorname{Spec}\mathbb{Z}$. Observe that, for any two elements x and y in $\mathcal{O}_{K_n}$, their scalar product with respect to the euclidean structure of $\pi_{n*}\overline{\mathcal{O}_{K_n}}$ is

$$\langle x, y\rangle_{\pi_{n*}\overline{\mathcal{O}_{K_n}}} = \sum_{\sigma: K_n\hookrightarrow\mathbb{C}} \sigma(x)\overline{\sigma(y)} = \operatorname{Tr}_{K_n/\mathbb{Q}} x\overline{y}. \tag{5.19}$$

In particular, for any pair of integers (i,j),

$$\langle \zeta_n^i, \zeta_n^j\rangle_{\pi_{n*}\overline{\mathcal{O}_{K_n}}} = \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^*} \zeta_n^{d(i-j)}. \tag{5.20}$$

Proposition 5.5.2. *For any positive integer* n*,* 1 *is a* $\mathcal{V}$*-small vector in* $\pi_{n*}\overline{\mathcal{O}_{K_n}}$.

Thanks to the theorem of Kronecker-Weber, this establishes that *Condition* $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ *holds for any* abelian *extension* L *of* $\mathbb{Q}$ *and any hermitian vector bundle* $\overline{E}$ *over* $\operatorname{Spec}\mathbb{Z}$. Using Lemma 5.2.4, we also obtain that, *for any number field* L *and any positive integer* n *prime to the absolute discriminant of* L*, the cyclotomic extension* $L_n := L(\zeta_n)$ *satisfies* $\mathbf{P}(L_n/L, \overline{E})$ *for any hermitian vector bundle* $\overline{E}$ *over* $\operatorname{Spec} L$.

The proof will rely on some auxiliary results if Kitaoka ([Kit93], Section 7.1 and Theorem 7.1.3), which he established when investigating minimal vectors in tensor products of euclidean lattices.

When $n = p^k$ is a prime power, Proposition (5.5.2) follows from the following two lemma:

Lemma 5.5.3. *For any prime number* p*,* 1 *is* $\mathcal{V}$*-small in* $\pi_*\overline{\mathcal{O}_{K_p}}$.

Lemma 5.5.4. ([Kit93], p. 196, Lemma 7.1.7) *For any prime number* p *and any positive integer* k*, consider the* $\mathbb{Z}$*-submodule*

$$E_{p^k} := \bigoplus_{i=0}^{p^{k-1}-1} \mathbb{Z}\cdot\zeta_{p^k}^i$$

in $\mathbb{Z}[\zeta_{p^k}] = \mathcal{O}_{K_{p^k}}$*, and let* $\overline{E}_{p^k}$ *be the hermitian vector bundle over* $\operatorname{Spec}\mathbb{Z}$ *defined by* E_{p^k} *equipped with the restriction of the euclidean structure of* $\pi_*\overline{\mathcal{O}_{K_{p^k}}}$.

The vectors in the basis $(\zeta_{p^k}^i)_{0\leq i\leq p^{k-1}-1}$ *of* $E_{p^k,\mathbb{R}}$ *are pairwise orthogonal with respect to this euclidean structure.*

Moreover the morphism of $\mathbb{Z}$*-modules*

$$\psi_{p,k} : E_{p^k}\otimes\mathbb{Z}[\zeta_p] \longrightarrow \mathbb{Z}[\zeta_{p^k}]$$

which maps $a\otimes b$ *to* ab *defines an isomorphism of hermitian vector bundles over* $\operatorname{Spec}\mathbb{Z}$*:*

$$\overline{E}_{p^k}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}} \xrightarrow{\sim} \pi_{p^k*}\overline{\mathcal{O}_{K_{p^k}}}. \tag{5.21}$$

Indeed, according to Lemma 5.5.1, 1) and 2), the first assertion in Lemma 5.5.4 shows that 1 is a $\mathcal{V}$-small vector in $\overline{E}_{p^k}$. Using Lemma 5.5.3 and Lemma 5.5.1, 4), we deduce that $1\otimes 1$ is $\mathcal{V}$-small in $\overline{E}_{p^k}\otimes \pi_{p*}\overline{\mathcal{O}_{K_p}}$. Since $\psi_{p,k}$ maps $1\otimes 1$ to 1, the second assertion in Lemma (5.5.4) finally establishes that 1 is $\mathcal{V}$-small in $\pi_{p^k *}\overline{\mathcal{O}_{K_{p^k}}}$.

Taking the above two lemma for granted, Proposition 5.5.2 then follows — by writing n as a product of prime powers — from Lemma 5.5.1, 4), and from

Lemma 5.5.5. ([Kit93], p. 197, Proof of Theorem 7.1.3) *Let n_1 and n_2 be two positive integers that are prime together, and let $n := n_1 \cdot n_2$.*

The morphism of rings

$$\begin{array}{ccc} \mathbb{Z}[\zeta_{n_1}]\otimes\mathbb{Z}[\zeta_{n_2}] & \longrightarrow & \mathbb{Z}[\zeta_n] \\ a\otimes b & \longmapsto & ab \end{array} \tag{5.22}$$

is an isomorphism and defines an isomorphism of hermitian vector bundles over $\operatorname{Spec}\mathbb{Z}$*:*

$$\pi_{n_1 *}\overline{\mathcal{O}_{K_{n_1}}}\otimes \pi_{n_2 *}\overline{\mathcal{O}_{K_{n_2}}}\simeq \pi_{n*}\overline{\mathcal{O}_{K_n}}.$$

Proof of the Lemma. That the morphism $\psi_{p,k}$ in Lemma 5.5.4 is an isomorphism is straightforward by considering the bases $(\zeta_{p^k}^i)_{0\leq i\leq p^{k-1}-1}$ of E_{p^k}, $(\zeta_p^i)_{0\leq i\leq p-2}$ of $\mathbb{Z}[\zeta_p]$, and $(\zeta_{p^k}^i)_{0\leq i\leq (p-1)p^{k-1}-1})$ of $\mathbb{Z}[\zeta_{p^k}]$. Moreover, from (5.20), one obtain that, for any pair of integers (i,j),

$$\begin{array}{rcll} \langle\zeta_{p^k}^i,\zeta_{p^k}^j\rangle_{\pi_{p^k *}\overline{\mathcal{O}_{K_{p^k}}}} & = & (p-1)p^{k-1} & \text{if } i=j \mod p^k \\ & = & -p^{k-1} & \text{if } i=j \mod p^{k-1} \text{ and } i\neq j \mod p^k \\ & = & 0 & \text{if } i=j \mod p^{k-1}. \end{array} \tag{5.23}$$

This establishes in particular the first assertion in Lemma 5.5.4. The isometry property (5.21) of $\psi_{p,k}$ also is easily derived from (5.23) and its special case where $k=0$, which shows that

$$\begin{array}{rcll} \langle\zeta_p^i,\zeta_p^j\rangle_{\pi_{p*}\overline{\mathcal{O}_{K_p}}} & = & (p-1) & \text{if } i=j \mod p \\ & = & -1 & \text{if } i\neq j \mod p. \end{array} \tag{5.24}$$

Similarly, by considering the bases $(\zeta_{n_1}^i)_{0\leq i<\phi(n_1)}$ of $\mathbb{Z}[\zeta_{n_1}]$, $(\zeta_{n_2}^i)_{0\leq i<\phi(n_2)}$ of $\mathbb{Z}[\zeta_{n_2}]$, and $(\zeta_n^i)_{0\leq i<\phi(n)}$ of $\mathbb{Z}[\zeta_n]$, one proves that the morphism (5.22) is an isomorphism. Its compatibility with the euclidean structures on $\pi_{n_1*}\overline{\mathcal{O}_{K_{n_1}}}$, $\pi_{n_2*}\overline{\mathcal{O}_{K_{n_2}}}$, and $\pi_{n*}\overline{\mathcal{O}_{K_n}}$ directly follows from the expressions (5.19) for the associated scalar products.

Let us finally establish Lemma 5.5.3. Let $\overline{E}$ be any hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$ and v an element of $\mathcal{V}(\overline{E})$. To prove that $v\otimes 1$ belongs to $\mathcal{V}(\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}})$, we need to show that, for any β in $E\otimes\mathcal{O}_{K_p}$,

$$\|v\otimes 1-\beta\|_{\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}}}\geq \|v\otimes 1\|_{\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}}}. \tag{5.25}$$

To achieve this, observe that a vector u in $E_{\mathbb{R}}\otimes_{\mathbb{R}}\mathcal{O}_{K_p}$ may be uniquely written as

$$u=\sum_{i=0}^{p-2}u_i\otimes\zeta_p^i,$$

where $u_0, \dots, u_{p-2}$ are some vectors in $E_{\mathbb{R}}$. Moreover, we have (compare [Kit93], p. 196):

$$\begin{aligned}\|u\|^2_{\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}}} &= \sum_{0\leq i,j\leq p-2} \langle \zeta_p^i, \zeta_p^j\rangle_{\pi_{p*}\overline{\mathcal{O}_{K_p}}} \cdot \langle u_i, u_j\rangle_{\overline{E}} \\ &= (p-1)\cdot \sum_{0\leq i\leq p-2} \|u_i\|^2_{\overline{E}} - \sum_{0\leq i\neq j\leq p-2} \langle u_i, u_j\rangle_{\overline{E}} \quad \big(\text{according to (5.24)}\big) \\ &= \sum_{0\leq i\leq p-2} \|u_i\|^2_{\overline{E}} + \sum_{0\leq i<j\leq p-2} \|u_i - u_j\|^2_{\overline{E}}\end{aligned}$$

by an elementary computation.

In particular, we get:

$$\|v\otimes 1\|^2_{\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}}} = (p-1)\|v\|^2_{\overline{E}},$$

and, if $\beta_0, \dots, \beta_{p-2}$ are the elements of E such that

$$\beta = \sum_{i=0}^{p-2} \beta_i \otimes \zeta_p^i,$$

we also have:

$$\|v\otimes 1-\beta\|^2_{\overline{E}\otimes\pi_{p*}\overline{\mathcal{O}_{K_p}}} = \|v-\beta_0\|^2_{\overline{E}} + \sum_{1\leq i\leq p-2} \|\beta_i\|^2_{\overline{E}} + \sum_{0<j\leq p-2} \|v-\beta_0+\beta_j\|^2_{\overline{E}} + \sum_{1\leq i\neq j\leq p-2} \|\beta_i-\beta_j\|^2_{\overline{E}}.$$

Since v belongs to the Voronoi cell $\mathcal{V}(\overline{E})$, the $p-1$ terms

$$\|v-\beta_0\|^2_{\overline{E}},\ \|v-\beta_0+\beta_1\|^2_{\overline{E}}, \dots,\ \|v-\beta_0+\beta_{p-2}\|^2_{\overline{E}}$$

are all $\geq \|v\|^2_{\overline{E}}$. This establishes (5.25). □

5.6. **Base change from root lattices.** The aim of this section is to establish the following:

Proposition 5.6.1. *If $\overline{E}$ is the hermitian vector bundle over* $\operatorname{Spec}\mathbb{Z}$ *defined by any one of the root lattices A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, or E_8, then condition $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ holds for any number field L.*

The definitions of these root lattices are recalled in the proof below. We refer the reader to [Mar03], chapter 4, for more informations concerning them. Let us only recall that, according to a theorem of Witt ([Wit41]), the integral lattices[12] which are generated by vectors of square norms 1 or 2 are precisely the orthogonal sums of lattices isometric to one of the lattices $(\mathbb{Z}, |.|)$, A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, or E_8. Hence Proposition 5.6.1, Proposition 5.4.2, and Lemma 5.2.2 yield a proof of Theorem 5.4.4 ii).

Our proof of Proposition 5.6.1 relies on the description of the vertices of the Voronoi cell of such euclidean lattices $\overline{E}$ appearing in the work of Conway and Sloane ([CS99] Chapter 21, and [CS91]): these vertices are the images under the Weyl group associated to the root lattice of a "small" set of vertices $\mathcal{F}(\overline{E})$, whose coordinates are explicitly given in *loc. cit.*. Observe that, to prove that $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ holds, it is enough to establish the validity of (5.9) (or equivalently of (5.10) or (5.11)) for any P in $\mathcal{F}(\overline{E})$ and any β in $E \otimes_{\mathbb{Z}} \mathcal{O}_L$. Indeed the action of the Weyl group extends, by base change, to an isometric action on $g^*\overline{E}$.

To check these conditions, we shall use Lemma 5.4.1 and the following related inequalities, valid for any number field L:

[12] *i.e.*, the euclidean lattices $(\Gamma, \|.\|)$ such that the scalar product of any two vectors in Γ is an integer.

(i) *For any non-zero α in $\mathcal{O}_L$,*

$$\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma:L\hookrightarrow\mathbb{C}}|\sigma(\alpha)|^2\geq 1. \tag{5.26}$$

This follows for instance from (5.12) and (5.13).

(ii) *For any $\alpha\neq 1/2$ in $\frac{1}{2}\mathcal{O}_L$,*

$$\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\alpha)|^2-\operatorname{Re}\sigma(\alpha))\geq 0, \tag{5.27}$$

and

$$\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma:L\hookrightarrow\mathbb{C}}|\sigma(\alpha)-1/3|^2\geq 1/9. \tag{5.28}$$

The lower bound (5.27) follows from (5.26) applied to $2\alpha-1$. To prove (5.28), observe that, for any embedding $\sigma:L\hookrightarrow\mathbb{C}$,

$$\begin{aligned}
4|\sigma(\alpha)-\frac{1}{3}|^2 &= |\sigma(2\alpha-1)+\frac{1}{3}|^2\\
&= |\sigma(2\alpha-1)|^2+\frac{2}{3}\operatorname{Re}\sigma(2\alpha-1)+\frac{1}{9}\\
&\geq |\sigma(2\alpha-1)|^2-\frac{2}{3}|\sigma(2\alpha-1)|+\frac{1}{9}\\
&= |\sigma(2\alpha-1)|^2-|\sigma(2\alpha-1)|+\frac{1}{3}|\sigma(2\alpha-1)|+\frac{1}{9},
\end{aligned}$$

and apply Lemma 5.4.1 and (5.12) to $2\alpha-1$.

For any two elements a and b of some set $\mathcal{E}$ and any two positive integers i and j, we shall denote

$$(a^{\times i},b^{\times j}):=(\underbrace{a,\ldots,a}_{i\text{ times}},\underbrace{b,\ldots,b}_{j\text{ times}})\ (\in\mathcal{E}^{i+j}).$$

Proof for A_n. The lattice A_n is the lattice $\mathbb{Z}^{n+1}\cap A_{n,\mathbb{R}}$ in the hyperplane

$$A_{n,\mathbb{R}}:=\{(x_k)_{0\leq k\leq n}\in\mathbb{R}^{n+1}\mid\sum_{k=0}^{n}x_k=0\}$$

of $\mathbb{R}^{n+1}$ equipped with the standard euclidean norm. The corresponding hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$ coincides with the hermitian vector bundle $\overline{\ker p}$ considered in Example 4.1.1 above when $S=\operatorname{Spec}\mathbb{Z}$.

According to [CS99] Chapter 21,3.B, or [CS91], section 4, the vertices of the Voronoi cell of the root lattice A_n are the images under its Weyl group of the following n points in $A_{n,\mathbb{R}}$:

$$[i]:=\left(\left(\frac{j}{n+1}\right)^{\times i},\left(-\frac{i}{n+1}\right)^{\times j}\right),\quad 1\leq i\leq n,\ \ j:=n+1-i.$$

By considering the conditions (5.11) with $P=[i]$, $1\leq i\leq n$, we are reduced to proving the following

Lemma 5.6.2. *Let L be a number field, i an integer in $\{1,\ldots,n\}$, and $j:=n+1-i$.*

For any $(\beta_k)_{0\leq k\leq n}$ in $\mathcal{O}_L^{n+1}$ such that $\sum_{k=0}^{n}\beta_k=0$, we have:

$$2\sum_{\sigma:L\hookrightarrow\mathbb{C}}\left[\sum_{0\leq k\leq i-1}\frac{j}{n+1}\operatorname{Re}\sigma(\beta_k)-\sum_{i\leq k\leq n}\frac{i}{n+1}\operatorname{Re}\sigma(\beta_k)\right]\leq\sum_{\sigma:L\hookrightarrow\mathbb{C}}\sum_{0\leq k\leq n}|\sigma(\beta_k)|^2. \tag{5.29}$$

To prove this lemma, observe that, for any embedding $\sigma : L \hookrightarrow \mathbb{C}$,

$$\sum_{\sigma : L \hookrightarrow \mathbb{C}} \operatorname{Re} \sigma(\beta_k) = 0.$$

Consequently the left-hand side of (5.29) is equal to:

$$\begin{aligned} 2 \sum_{\sigma : L \hookrightarrow \mathbb{C}} \left(\frac{j}{n+1} + \frac{i}{n+1} \right) \sum_{0 \leq k \leq i-1} \operatorname{Re} \sigma(\beta_k) &= 2 \sum_{\sigma : L \hookrightarrow \mathbb{C}} \sum_{0 \leq k \leq i-1} \operatorname{Re} \sigma(\beta_k) \\ &= \sum_{\sigma : L \hookrightarrow \mathbb{C}} \left(\sum_{0 \leq k \leq i-1} \operatorname{Re} \sigma(\beta_k) - \sum_{i \leq k \leq n} \operatorname{Re} \sigma(\beta_k) \right). \end{aligned}$$

The inequality (5.29) now follows from Lemma (5.4.1) applied to α in

$$\{\beta_0, \ldots, \beta_{i-1}, -\beta_i, \ldots, -\beta_n\}.$$

□

Proof for D_n. The lattice D_n is the sublattice of index 2 in the lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$ equipped with the standard euclidean norm consisting of vectors $(x_1, \ldots, x_n)$ for which the x_i are integers with an even sum.

According to [CS99] Chapter 21, 3.C, and [CS91], section 6, the vertices of the Voronoi cell of D_n are the images under its Weyl group of the points $((1/2)^{\times n})$ and $(1, 0^{\times (n-1)})$ in $\mathbb{R}^n$. By considering the condition (5.11) (resp. (5.9) with $P = ((1/2)^{\times n})$ (resp. with $P = (1, 0^{\times (n-1)})$), we are reduced to proving:

Lemma 5.6.3. *Let L be a number field. For any $(\beta_k)_{1 \leq k \leq n}$ in $\mathcal{O}_L^n$ such that $\sum_{k=1}^n \beta_k$ belongs to $2\mathcal{O}_L$, the following inequalities hold:*

$$2 \sum_{\sigma : L \hookrightarrow \mathbb{C}} \sum_{1 \leq k \leq n} \frac{1}{2} \operatorname{Re} \sigma(\beta_k) \leq \sum_{\sigma : L \hookrightarrow \mathbb{C}} \sum_{1 \leq k \leq n} |\sigma(\beta_k)|^2, \tag{5.30}$$

and

$$\frac{1}{[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} |\sigma(\beta_1) - 1|^2 + \frac{1}{[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \sum_{2 \leq k \leq n} |\sigma(\beta_k)|^2 \geq 1. \tag{5.31}$$

The first inequality (5.30) directly follows from Lemma (5.4.1). To prove (5.31), observe that that at least one of the algebraic integers $\beta_1 - 1$, β_2, …, β_n is not zero, and use (5.26). □

Proof for E_8. The lattice E_8 is the lattice in $\mathbb{R}^n$ equipped with the standard euclidean norm consisting of the vectors $(x_1, \ldots, x_8)$ for which the x_i are all in $\mathbb{Z}$ or all in $1/2 + \mathbb{Z}$, and have an even sum.

According to [CS91], section 8, the vertices of the Voronoi cell of E_8 are the images under its Weyl group of the points $((0)^{\times 7}, 1)$ and $((1/3)^{\times 7}, -1/3)$ in $\mathbb{R}^8$. By considering the condition (5.9) with $P = ((0)^{\times 7}, 1)$ and with $P = ((1/3)^{\times 7}, -1/3)$, we are reduced to proving:

Lemma 5.6.4. *Let L be a number field. If $\beta_1, \ldots, \beta_8$ in $\frac{1}{2}\mathcal{O}_L$ have the same class γ in $(\frac{1}{2}\mathcal{O}_L)/\mathcal{O}_L$ and if their sum $\beta_1 + \ldots + \beta_8$ belongs to $2\mathcal{O}_L$, then the following inequalities hold:*

$$\frac{1}{[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \left(\sum_{1 \leq k \leq 7} |\sigma(\beta_k)|^2 + |\sigma(\beta_8) - 1|^2 \right) \geq 1, \tag{5.32}$$

and

$$\frac{1}{[L : \mathbb{Q}]} \sum_{\sigma : L \hookrightarrow \mathbb{C}} \left(\sum_{1 \leq k \leq 7} |\sigma(\beta_k) - 1/3|^2 + |\sigma(\beta_8) + 1/3|^2 \right) \geq 8/9. \tag{5.33}$$

When $\gamma \neq [0]$, each of the expressions

$$\frac{1}{[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} |\sigma(\beta_k)|^2, \quad 1 \leq k \leq 7$$

and

$$\frac{1}{[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} |\sigma(\beta_8) - 1|^2$$

is $\geq 1/4$, by (5.26) applied to α in $\{2\beta_1, \ldots, 2\beta_7, 2\beta_8 - 2\}$, and (5.32) follows. When $\gamma = [0]$, these expressions are ≥ 1, unless $\beta_k = 0$ (when $1 \leq k \leq 7$), or $\beta_8 = 1$. Since $\beta_1 + \ldots + \beta_8 \neq 1$, the vector $(\beta_1, \ldots, \beta_8)$ does not coincide with $((0)^{\times 7}, 1)$, and at least one of them is ≥ 1, which implies (5.32).

If $\beta_1, \ldots, \beta_7$ and $-\beta_8$ are all distinct from $1/2$, then (5.33) directly follows from (5.28). Otherwise, γ is $[1/2]$, and the vector $(\beta_k)_{1 \leq k \leq 8} - (1/2)^{\times 8}$ belongs to $D_8 \otimes \mathcal{O}_L$. The validity of $\mathbf{P}(L/\mathbb{Q}, D_8)$, applied to $v = ((1/3)^{\times 7}, -1/3) - (1/2)^{\times 8}$, shows that the left-hand side of (5.33) is greater or equal to the minimum of the expression

$$\sum_{1 \leq k \leq 7} (\alpha_k - 1/3)^2 + (\alpha_8 + 1/3)^2 \tag{5.34}$$

when $(\alpha_k)_{1 \leq k \leq 8}$ varies in $D_8 + (1/2)^{\times 8}$. This minimum is clearly greater or equal than the one of (5.34) when $(\alpha_k)_{1 \leq k \leq 8}$ varies in E_8 — which is $\|((1/3)^{\times 7}, -1/3)\|^2 = 8/9$ since $((1/3)^{\times 7}, -1/3)$ belongs to the Voronoi cell of E_8. □

Proof for E_7. The lattice E_7 may be realized as the lattice $E_8 \cap E_{7,\mathbb{R}}$ in the hyperplane

$$E_{7,\mathbb{R}} := \{(x_k)_{1 \leq k \leq 8} \in \mathbb{R}^8 \mid \sum_{k=1}^{8} x_k = 0\}$$

of $\mathbb{R}^8$ equipped with the standard euclidean.

According to [CS91], section 9, the vertices of the Voronoi cell of E_7 are the images under its Weyl group of the points $(7/8, (-1/8)^{\times 7})$ and $((3/4)^{\times 2}, (-1/4)^{\times 6})$ in $E_{7,\mathbb{R}}$. By considering the condition (5.11) with $P = (7/8, (-1/8)^{\times 7})$ and $P = ((3/4)^{\times 2}, (-1/4)^{\times 6})$, we are reduced to proving:

Lemma 5.6.5. *Let L be a number field. If $\beta_1, \ldots, \beta_8$ in $\frac{1}{2}\mathcal{O}_L$ have the same class γ in $(\frac{1}{2}\mathcal{O}_L)/\mathcal{O}_L$ and if $\beta_1 + \ldots + \beta_8 = 0$, then the following inequalities hold:*

$$\sum_{\sigma: L \hookrightarrow \mathbb{C}} \left(|\sigma(\beta_1)|^2 - \frac{7}{4} \operatorname{Re} \sigma(\beta_1)) + \sum_{2 \leq k \leq 8} (|\sigma(\beta_k)|^2 + \frac{1}{4} \operatorname{Re} \sigma(\beta_k)) \right) \geq 0, \tag{5.35}$$

and

$$\sum_{\sigma: L \hookrightarrow \mathbb{C}} \left(\sum_{1 \leq k \leq 2} (|\sigma(\beta_k)|^2 - \frac{3}{2} \operatorname{Re} \sigma(\beta_k) + \sum_{3 \leq k \leq 8} (|\sigma(\beta_k)|^2 + \frac{1}{2} \operatorname{Re} \sigma(\beta_k)) \right) \geq 0. \tag{5.36}$$

Observe that, with the notation of the proof for A_n with $n = 8$, we have

$$(7/8, (-1/8)^{\times 7}) = [1],$$

and

$$((3/4)^{\times 2}, (-1/4)^{\times 6}) = [2].$$

Therefore the computation in the proof of Lemma 5.6.2, with $n = 8$, shows that (5.35) and (5.36) may also be written

$$\sum_{\sigma: L \hookrightarrow \mathbb{C}} (|\sigma(\beta_1)|^2 - \operatorname{Re} \sigma(\beta_1)) + \sum_{2 \leq k \leq 8} \sum_{\sigma: L \hookrightarrow \mathbb{C}} (|\sigma(\beta_k)|^2 + \operatorname{Re} \sigma(\beta_k)) \geq 0, \tag{5.37}$$

and

$$\sum_{1\leq k\leq 2}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\beta_k)|^2-\operatorname{Re}\sigma(\beta_k))+\sum_{3\leq k\leq 8}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\beta_k)|^2+\operatorname{Re}\sigma(\beta_k))\geq 0. \tag{5.38}$$

These inequalities directly follow from (5.27) when $\gamma\neq[1/2]$.

If $\gamma=[1/2]$, we may introduce

$$(\tilde\beta_k)_{1\leq k\leq 8}:=(\beta_k)_{1\leq k\leq 8}-((1/2)^{\times 4},(-1/2)^{\times 4}). \tag{5.39}$$

Then $\tilde\beta_1,\ldots,\tilde\beta_8$ belong to $\mathcal{O}_L$, and their sum vanishes. Moreover, expressed in terms of $\tilde\beta_1,\ldots,\tilde\beta_8$, (5.37) takes the form:

$$\sum_{1\leq k\leq 8}\frac{1}{[L:\mathbb{Q}]}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\tilde\beta_k)|^2-\epsilon_k\operatorname{Re}\sigma(\tilde\beta_k))+1\geq 0, \tag{5.40}$$

where $\epsilon_k:=1$ if $k\in\{1,5,6,7,8\}$ and $\epsilon_k:=-1$ if $k\in\{2,3,4\}$. Similarly, (5.38) is equivalent to:

$$\sum_{1\leq k\leq 8}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\tilde\beta_k)|^2-\epsilon'_k\operatorname{Re}\sigma(\tilde\beta_k))\geq 0, \tag{5.41}$$

where $\epsilon'_k:=1$ if $k\in\{1,2,5,6,7,8\}$ and $\epsilon'_k:=-1$ if $k\in\{3,4\}$.

The inequalities (5.40) and (5.41) directly follow from Lemma 5.4.1. □

Proof for E_6. The lattice E_6 may be realized as the lattice $E_8\cap E_{6,\mathbb{R}}$ in the codimension 2 subspace

$$E_{6,\mathbb{R}}:=\{(x_k)_{1\leq k\leq 8}\in\mathbb{R}^8\mid x_1+x_8=x_2+\ldots+x_7=0\}$$

of $\mathbb{R}^8$ equipped with the standard euclidean norm.

According to [CS91], section 10, the vertices of the Voronoi cell of E_6 are the images under its Weyl group of the point $(0,(2/3)^{\times 2},(-1/3)^{\times 4},0)$ in $E_{6,\mathbb{R}}$. By considering the condition (5.11) with $P=(0,(2/3)^{\times 2},(-1/3)^{\times 4},0)$, we are reduced to proving:

Lemma 5.6.6. *Let L be a number field. If $\beta_1,\ldots,\beta_8$ in $\frac{1}{2}\mathcal{O}_L$ have the same class γ in $(\frac{1}{2}\mathcal{O}_L)/\mathcal{O}_L$ and if $\beta_1+\beta_8=0$, and $\beta_2+\ldots+\beta_6=0$, then the following inequality holds:*

$$\sum_{\sigma:L\hookrightarrow\mathbb{C}}\left(|\sigma(\beta_1)|^2)+\sum_{2\leq k\leq 3}(|\sigma(\beta_k)|^2-\frac{4}{3}\operatorname{Re}\sigma(\beta_k))+\sum_{4\leq k\leq 7}(|\sigma(\beta_k)|^2+\frac{2}{3}\operatorname{Re}\sigma(\beta_k))+|\sigma(\beta_8)|^2\right)\geq 0. \tag{5.42}$$

Again the computation in the proof of Lemma 5.6.2, now with $n=8$, $i=2$ and $j=4$, allows us to replace (5.42) by the equivalent condition:

$$\sum_{1\leq k\leq 8}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\beta_k)|^2-\eta_k\operatorname{Re}\sigma(\beta_k))\geq 0, \tag{5.43}$$

where $\eta_1=\eta_8:=0$, $\eta_2=\eta_3:=1$, and $\eta_4=\eta_5=\eta_6=\eta_7=-1$. According to (5.27), this holds if $\beta_k\neq\eta_k/2$ for $2\leq k\leq 7$, in particular when $\gamma\neq[1/2]$.

When $\gamma=[1/2]$, we introduce again $\tilde\beta_1,\ldots,\tilde\beta_8$ defined by (5.39). They belong to $\mathcal{O}_L$, the sum $\tilde\beta_2+\ldots+\tilde\beta_7$ vanishes, and (5.43) takes the form:

$$\sum_{1\leq k\leq 8}\sum_{\sigma:L\hookrightarrow\mathbb{C}}(|\sigma(\beta_k)|^2-\eta'_k\operatorname{Re}\sigma(\beta_k))\geq 0, \tag{5.44}$$

where $\eta'_k:=1$ if $k\in\{2,3,5,6,7,8\}$ and $\eta'_k:=-1$ if $k\in\{1,4\}$. Finally, (5.44) follows from Lemma 5.4.1 applied to $\alpha=\eta'_k\tilde\beta_k$, $1\leq k\leq 8$. □

5.7. **Base change from lattices of Voronoi's first kind.** From the description of the Voronoi cell of $\overline{V}(p),\ p \in \left(\mathbb{R}_+^*\right)^{\frac{n(n+1)}{2}}$ in the appendix, we may easily derive:

Proposition 5.7.1. *If $\overline{E}$ is the hermitian vector bundle over* Spec $\mathbb{Z}$ *defined by an euclidean lattice of Voronoi's first kind, then the condition $\mathbf{P}(L/\mathbb{Q}, \overline{E})$ holds for any number field L.*

Proof. A straightforward perturbation argument shows that we may restrict to the case where $\overline{E}$ possesses a strictly obtuse superbase, that is, to the case where $\overline{E} = \overline{V}(p)$ for some $p \in \left(\mathbb{R}_+^*\right)^{\frac{n(n+1)}{2}}$, where $n := \operatorname{rk} E$. According to Proposition B.3.4 every vector of $\mathcal{V}\big(\overline{V}(p)\big)$ is of the form s_A for some $A \in \mathcal{S}(n)$. Moreover, for any such

$$A = \big\{\{i_1, \ldots, i_n\},\ \{i_2, \ldots, i_n\}, \ldots, \{i_n\}\big\}$$

and any $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ if we let

$$x := \sum_{k=0}^{n} x_k\, v_{i_k},$$

then Lemma B.3.3 shows that

$$\begin{aligned} \|x\|^2_{\overline{V}(p)} - 2\,\langle s_A, x\rangle_{\overline{V}(p)} &= \|x - s_A\|^2_{\overline{V}(p)} - \|s_A\|^2_{\overline{V}(p)} \\ &= \sum_{0\le k<\ell\le n} p_{i_k i_\ell}\big[(x_\ell - x_k)^2 - (x_\ell - x_k)\big]. \end{aligned} \tag{5.45}$$

For any complex embedding $\sigma\colon L \to \mathbb{C}$ of some number field L, consider

$$\begin{aligned} \sigma_{V(p)}\colon V(p)\otimes L &\longrightarrow V(p)_{\mathbb{C}} = V(p)\otimes\mathbb{C} \\ e\otimes\lambda &\longmapsto e\otimes\sigma(\lambda). \end{aligned}$$

As observed in Lemma 5.3.1, to complete the proof of the Proposition it is enough to check that, for any $\beta \in V(p)\otimes_{\mathbb{Z}} \mathcal{O}_L$,

$$\sum_{\sigma: L\hookrightarrow\mathbb{C}} \Big[\big\|\sigma_{V(p)}\,(\beta)\|^2_{\overline{V}(p)} - 2\ \mathrm{Re}\ \langle s_A,\, \sigma_{V(p)}\,(\beta)\rangle_{\overline{V}(p)}\Big] \ge 0. \tag{5.46}$$

Any such β may be written

$$\beta = \sum_{k=0}^{n} v_{i_k} \otimes \beta_k$$

for some $(\beta_0, \ldots, \beta_n) \in \mathcal{O}_L^{n+1}$, and then (5.45) shows that the left-hand side of (5.45) is equal to

$$\sum_{\sigma: L\hookrightarrow\mathbb{C}}\ \sum_{0\le k<\ell\le n} p_{i_k i_\ell}\,\Big[\big|\sigma(\beta_k - \beta_\ell)\big|^2 -\ \mathrm{Re}\ \sigma(\beta_\ell - \beta_k)\Big].$$

This is indeed non-negative by Lemma 5.4.1. □

Appendix A. Extension groups

In this Appendix, we discuss various definitions of extension groups of sheaves of modules (notably, of groups of 1-extensions) which are used in this paper. We pay some special attention to the sign issues which arise in defining canonical isomorphisms between various constructions of extension groups.

A.1. **Notation and sign conventions.** When dealing with categories of complexes in abelian categories and their derived categories, we follow the same conventions, notably concerning signs, as in [BBM82], [Ill96], and [Con00]. (The reader should be aware that these conventions do *not* agree with the ones in some classical references such that [Har66], [Del73], or [Wei94].)

We refer to [Con00], sections 1.2-3, for a thorough discussion of signs issues, and we just specify some basic definitions where some ambiguity may arise.

If $X = (X^k)_{k\in\mathbb{Z}}$ is a complex (with cohomological grading) in some abelian category $\mathcal{C}$, then, for any integer i, $X[i]$ is the complex defined by

$$X[i]^k := X^{k+i} \text{ and } d_{X[i]} = (-1)^i d_X.$$

For any map of complexes $f^\bullet : X^\bullet \to Y^\bullet$, the map of complexes $f^\bullet[i] : X[i]^\bullet \to Y[i]^\bullet$ is defined to be f^{k+i} in degree i. The isomorphisms

$$H^k(X[i]) \simeq H^{k+i}(X) \tag{A.1}$$

are defined without the intervention of signs.

A *triangle* in some category of complexes (such as the usual category of complexes $C(\mathcal{C})$ in some abelian category $\mathcal{C}$, or the category $K(\mathcal{C})$ of such complexes with morphisms up to homotopy, or its derived category $D(\mathcal{C})$) is a sequence of morphisms of complexes of the form

$$X \to Y \to Z \to X[1].$$

The *cone* $C(f)$ of a morphism of complexes $f : X \to Y$ is defined by

$$C(f)^k := (X[1] \oplus Y)^k = X^{k+1} \oplus Y^k \text{ and } d_{C(f)}(x, y) := (-d_X(x), f(x) + d_Y(y)).$$

The *standard triangle* associated to f is:

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} X[1],$$

where $i : Y \to C(f)$ is the natural injection, and $p : C(f) \to X[1]$ the *opposite* of the natural projection.

A *distinguished triangle* is a triangle isomorphic to such a standard triangle. For instance, a short exact sequence of complexes

$$\mathcal{E} :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

determines a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial_{\mathcal{E}}} X[1]$$

in $D(\mathcal{C})$, where $\partial_{\mathcal{E}}$ is defined as the composite of $p : C(f) \to X[1]$ with the inverse of the quasi-isomorphism $q : C(f) \to Z$ given by g on Y and 0 on $X[1]$:

$$\partial_{\mathcal{E}} : Z \xleftarrow{q} C(f) \xrightarrow{p} X[1].$$

Observe that, using these sign conventions, for any integer i, the map between cohomology groups determined by $\partial_{\mathcal{E}}$:

$$H^i(\partial_{\mathcal{E}}) : H^i(Z) \longrightarrow H^i(X[1]) \simeq H^{i+1}(X)$$

coincides with the usual[13] boundary map in the long exact sequence of cohomology groups associated to $\mathcal{E}$.

For any two complexes X and Y in $\mathcal{C}$, and any integer i, we let:

$$\mathrm{Ext}^i_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{D(\mathcal{C})}(X, Y[i]).$$

If Z is again a complex in $\mathcal{C}$, and j an integer, the composition of two elements

$$f \in \mathrm{Ext}^i_{\mathcal{C}}(X,Y) \quad \text{and} \quad g \in \mathrm{Ext}^j_{\mathcal{C}}(Y,Z)$$

is the element

$$g \circ f \in \mathrm{Ext}^{i+j}_{\mathcal{C}}(X,Z)$$

defined by composing the arrows

$$f : X \longrightarrow Y[i] \quad \text{and} \quad g[i] : Y[i] \longrightarrow Z[i+j]$$

in $D(\mathcal{C})$.

A.2. **Extension groups of sheaves of modules.** Let X be any topological space, equipped with a sheaf $\mathcal{A}$ of commutative rings. Let $\mathcal{A}-\mathbf{mod}$ be the abelian category of sheaves of $\mathcal{A}$-modules over X and $D^+(\mathcal{A}-\mathbf{mod})$, the derived category of bounded below complexes of $\mathcal{A}$-modules. It is a full triangulated subcategory of $D(\mathcal{A}-\mathbf{mod})$, the derived category of complexes of $\mathcal{A}$-modules. Moreover, since the abelian category $\mathcal{A}-\mathbf{mod}$ admits enough injectives, the derived category $D^+(\mathcal{A}-\mathbf{mod})$ is equivalent to the category of bounded below complexes of injectives sheaves of $\mathcal{A}$-modules over X, with morphisms the homotopy classes of morphisms of complexes.

Any object in $\mathcal{A}-\mathbf{mod}$ may be seen as a complex concentrated in degree 0. Accordingly, for any two objects E and F in $\mathcal{A}-\mathbf{mod}$, the associated *extension groups* are defined as

$$\mathrm{Ext}^i_{\mathcal{A}}(E,F) := \mathrm{Hom}_{D^+(\mathcal{A}-\mathbf{mod})}(E, F[i]), \quad \text{for any } i \in \mathbb{N}.$$

For any resolution

$$0 \to F \to F^0 \to F^1 \to \ldots \to F^i \to F^{i+1} \to \ldots \tag{A.2}$$

of F by injective sheaves of $\mathcal{A}$-modules, these are the cohomology group of the complex of abelian groups deduced from

$$0 \to F^0 \to F^1 \to \ldots \to F^i \to F^{i+1} \to \ldots$$

by applying the functor $\mathrm{Hom}_{\mathcal{A}}(E, .)$.

When $\mathcal{A}$ is the constant sheaf $\mathbb{Z}_X$, $\mathcal{A}-\mathbf{mod}$ is the category of sheaves of abelian groups on X, and for any such sheaf F and any integer $i \in \mathbb{N}$, the extension group $\mathrm{Ext}^i_{\mathbb{Z}_X}(\mathbb{Z}_X, F)$ coincides with the i-th cohomology group $H^i(X,F)$, by the very definition of the latter. More generally, for any sheaf of ring $\mathcal{A}$ over X, the injective objects in $\mathcal{A}-\mathbf{mod}$ are flabby sheaves of abelian groups, and, for any F in $\mathcal{A}-\mathbf{mod}$, the extension group $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{A}, F)$ coincides with the cohomology group $H^i(X,F)$ of F considered as a sheaf of abelian groups.

If the sheaf of $\mathcal{A}$-modules E is locally free of finite rank, and if

$$\check{E} := \mathcal{H}om_{\mathcal{A}}(E, \mathcal{A})$$

[13] namely, defined without intervention of signs, as in [ML95], II.4, or [Wei94], 1.3 : the graph of $H^i(\partial_{\mathcal{E}})$ is the image in $H^i(Z) \times H^{i+1}(X)$ of

$$\{(z,x) \in Z^i \times X^{i+1} \mid d_Z(z) = 0 \text{ and } \exists y \in Y^i, g^i(y) = z \text{ and } d_Y(y) = f(x)\}.$$

denotes the dual sheaf, the functorial isomorphism of sheaf of $\mathcal{A}$-modules

$$\check{E} \otimes_{\mathcal{A}} G \xrightarrow{\sim} \mathcal{H}om_{\mathcal{A}}(E, G), \tag{A.3}$$

where G denotes any sheaf in $\mathcal{A} - \mathbf{mod}$, leads to isomorphisms of extension groups

$$\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{A}, \check{E} \otimes_{\mathcal{A}} F) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}}(E, F), \tag{A.4}$$

defined by applying (A.3) to any resolution (A.2), and taking global sections and cohomology groups. Finally, for any two sheaves of $\mathcal{A}$-modules E and F, with E locally free of finite rank, we get canonical isomorphisms

$$\mathrm{Ext}^i_{\mathcal{A}}(E, F) \xrightarrow{\sim} H^i(X, \check{E} \otimes_{\mathcal{A}} F). \tag{A.5}$$

A.3. **Extension groups of quasi-coherent sheaves of modules over schemes.** When $\mathcal{X} := (X, \mathcal{A})$ is a scheme, we may consider the full subcategory $\mathcal{O}_{\mathcal{X}} - \mathbf{qc}$ of the category $\mathcal{O}_{\mathcal{X}} - \mathbf{mod}$ of sheaves of $\mathcal{O}_{\mathcal{X}} := \mathcal{A}$-modules over X defined by the quasicoherent sheaves. When $\mathcal{X}$ is the affine scheme $\operatorname{Spec} A$ defined by some commutative ring A, the functor $\Gamma(X, .)$ realizes an equivalence of category between $\mathcal{O}_{\mathcal{X}} - \mathbf{qc}$ and the category of A-modules.

When the scheme $\mathcal{X}$ is quasi-compact and quasi-separated (*e.g.*, when $\mathcal{X}$ is noetherian), the category $\mathcal{O}_{\mathcal{X}} - \mathbf{qc}$ has enough injectives. If indeed $\mathcal{X}$ is noetherian, the inclusion functor from $\mathcal{O}_{\mathcal{X}} - \mathbf{qc}$ to $\mathcal{O}_{\mathcal{X}} - \mathbf{mod}$ preserves injectives, and its extension, as a functor of triangulated category from $D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{qc})$ to $D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{mod})$ is fully faithful and its essential image is the subcategory of $D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{mod})$ defined by bounded below complexes in $\mathcal{O}_{\mathcal{X}} - \mathbf{mod}$ with quasi-coherent cohomology (see for instance [Har66], II.7, [Ill71], section 3, and [TT90], Appendix B).

We shall frequently use the following consequence of the above full faithfulness: *for any two quasicoherent sheaves E and F on a noetherian scheme $\mathcal{X}$, the morphism of abelian groups*

$$\mathrm{Hom}_{D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{qc})}(E, F[i]) \longrightarrow \mathrm{Hom}_{D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{mod})}(E, F[i])$$

is an isomorphism for any integer i. In other words, the extension groups

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}} - \mathbf{qc}}(E, F) := \mathrm{Hom}_{D^+(\mathcal{O}_{\mathcal{X}} - \mathbf{qc})}(E, F[i])$$

computed in the abelian category of quasicoherent sheaves of $\mathcal{O}_{\mathcal{X}}$-modules over $\mathcal{X}$ coincide with the extension groups $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}}}(E, F)$ computed in the category of all sheaves of $\mathcal{O}_{\mathcal{X}}$-modules.

In particular, when $\mathcal{X}$ is the affine scheme $\operatorname{Spec} A$ defined by some noetherian commutative ring A, the extension group $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{X}}}(E, F)$ may be identified with the A-module $\mathrm{Ext}^i_A(\Gamma(X, E), \Gamma(X, F))$, computed in the abelian category of A-modules.

A.4. **Groups of 1-extensions.** In this section, $(X, \mathcal{A})$ denotes as above a topological space equipped with a sheaf $\mathcal{A}$ of commutative ring, and $\mathcal{A} - \mathbf{mod}$ the abelian category of sheaves of $\mathcal{A}$-modules over X. Actually, the constructions we shall now describe would still make sense with the category $\mathcal{A} - \mathbf{mod}$ replaced by a general abelian category, satisfying suitable smallness assumptions.

Let us indicate that the approach to Ext^1 groups in terms of 1-extensions described below — which directly inspired our definition of the arithmetic extension group in terms of admissible extension of hermitian vector bundles — originates in the papers of Baer [Bae34] and Eilenberg–MacLane [EM42]. See [ML95], p.103, notes of Chapter III, for additional historical references.

A.4.1. *1-extensions.* Let F and G be two objects in $\mathcal{A}-\mathbf{mod}$. A *1-extension* in $\mathcal{A}-\mathbf{mod}$ (or shortly, when no confusion may arise, an *extension*) $\mathcal{E}$ of F by G is a short exact sequence in $\mathcal{A}-\mathbf{mod}$ of the form

$$\mathcal{E}: \; 0 \longrightarrow G \stackrel{i}{\longrightarrow} E \stackrel{p}{\longrightarrow} F \longrightarrow 0. \tag{A.6}$$

A *morphism of extensions* from the extension $\mathcal{E}$ to a second one

$$\mathcal{E}': \; 0 \longrightarrow G \stackrel{i'}{\longrightarrow} E' \stackrel{p'}{\longrightarrow} F \longrightarrow 0 \tag{A.7}$$

is a morphism of $\mathcal{A}$-modules

$$f : E \longrightarrow E'$$

such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \stackrel{i}{\longrightarrow} & E & \stackrel{p}{\longrightarrow} & F & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \| & & \\
0 & \longrightarrow & G & \stackrel{i'}{\longrightarrow} & E' & \stackrel{p'}{\longrightarrow} & F & \longrightarrow & 0.
\end{array}$$

is commutative. Such a morphism of extensions, if it exists, is an isomorphism. Moreover the isomorphism classes of extensions of F by G constitute a set, which we shall denote $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G)$. The previous observation shows that any two extensions of F by G related by a morphism of extensions define the same element in $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G)$.

An extension (A.6) of F by G is said to be *split* or *trivial* if it is isomorphic to the extension

$$0 \longrightarrow G \stackrel{\binom{\mathrm{id}_G}{0}}{\longrightarrow} G \oplus F \stackrel{(0,\mathrm{id}_F)}{\longrightarrow} F \longrightarrow 0,$$

or, equivalently, if p admits a left inverse, or if i admits a right inverse in $\mathcal{A}-\mathbf{mod}$.

A.4.2. *Pullback and pushout.* Let $\mathcal{E}$ an extension of F by G in $\mathcal{A}-\mathbf{mod}$ as above (A.6) and $u : F' \longrightarrow F$ a morphism in $\mathcal{A}-\mathbf{mod}$. From $\mathcal{E}$ and u, we may define the $\mathcal{A}$-module

$$E' := \mathrm{Ker}(E \oplus F' \stackrel{(p,-u)}{\longrightarrow} F)$$

and construct a cartesian diagram

$$\begin{array}{ccccc}
 & (e,f') & \longmapsto & f' \\
(e,f') & E' & \stackrel{\tilde{p}}{\longrightarrow} & F' \\
\big\downarrow & \tilde{u}\big\downarrow & & \big\downarrow u \\
e & E & \stackrel{p}{\longrightarrow} & F,
\end{array} \tag{A.8}$$

a morphism $\tilde{i} : G \rightarrow E'$ in $\mathcal{A}-\mathbf{mod}$ defined by $\tilde{i}(g) := (i(g), 0)$, and an extension $\mathcal{E} \circ u$ of F' by G:

$$\mathcal{E} \circ u : \; 0 \longrightarrow G \stackrel{\tilde{i}}{\longrightarrow} E' \stackrel{\tilde{p}}{\longrightarrow} F' \longrightarrow 0$$

By construction, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G & \stackrel{\tilde{i}}{\longrightarrow} & E' & \stackrel{\tilde{p}}{\longrightarrow} & F' & \longrightarrow & 0 \\
 & & \| & & \downarrow \tilde{u} & & \downarrow u & & \\
0 & \longrightarrow & G & \stackrel{i}{\longrightarrow} & E & \stackrel{p}{\longrightarrow} & F & \longrightarrow & 0
\end{array} \tag{A.9}$$

is commutative, and its right-hand square is cartesian. The pair formed by the extension $\mathcal{E} \circ u$ and the morphism $\tilde{u}$ are characterized by these properties, and defines the *pullback* of $\mathcal{E}$ by u.

For instance, the diagram

$$\text{(A.10)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\binom{\mathrm{id}_G}{0}} & G \oplus E & \xrightarrow{\tilde{p}} & F & \longrightarrow & 0 \\ & & \| & & \downarrow {\scriptstyle (i,\mathrm{id}_E)} & & \downarrow {\scriptstyle p} & & \\ 0 & \longrightarrow & G & \xrightarrow{i} & E & \xrightarrow{p} & F & \longrightarrow & 0 \end{array}$$

establishes that the pullback $\mathcal{E} \circ p$ is a split extension.

Symmetrically, if $v : G \longrightarrow G'$ is a morphism in $\mathcal{A} - \mathbf{mod}$, we may define the $\mathcal{A}$-module

$$E' := \mathrm{Coker}(G \xrightarrow{\binom{i}{-v}} E \oplus G')$$

and construct a cocartesian diagram:

$$\text{(A.11)}\qquad \begin{array}{ccccc} G & \xrightarrow{i} & E & & e \\ v \big\downarrow & & \big\downarrow \tilde{v} & & \big\downarrow \\ G' & \xrightarrow{\tilde{i}} & E' & & [(e,0)] \\ g' & \longmapsto & [(0,g')], & & \end{array}$$

a morphism $\tilde{p} : E' \to F$ in $\mathcal{A} - \mathbf{mod}$ defined by $\tilde{p}([(e,g')]) := p(e)$, and an extension $v \circ \mathcal{E}$ of F by G':

$$v \circ \mathcal{E} :\ 0 \longrightarrow G' \xrightarrow{\tilde{i}} E' \xrightarrow{\tilde{p}} F \longrightarrow 0.$$

By construction, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{i} & E & \xrightarrow{p} & F & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle v} & & \downarrow {\scriptstyle \tilde{v}} & & \| & & \\ 0 & \longrightarrow & G' & \xrightarrow{\tilde{i}} & E' & \xrightarrow{\tilde{p}} & F & \longrightarrow & 0 \end{array}$$

is commutative, and its left-hand square is cocartesian. The pair formed by the extension $v \circ \mathcal{E}$ and the morphism $\tilde{v}$ are characterized by these properties, and defines the *pushout* of $\mathcal{E}$ by v.

The constructions of pullback and pushout of extensions satisfy associativity properties. Namely, with the above notation, the extensions $v \circ (\mathcal{E} \circ u)$ and $(v \circ \mathcal{E}) \circ u$ of F' by G' are canonically isomorphic. Moreover, if $u' : F'' \to F'$ (resp. $v' : G' \to G''$) is another morphism in $\mathcal{A} - \mathbf{mod}$, the extensions $(\mathcal{E} \circ u) \circ u'$ and $\mathcal{E} \circ (u \circ u')$ (resp. $v' \circ (v \circ \mathcal{E})$ and $(v' \circ v) \circ \mathcal{E}$) are canonically isomorphic. This follows from the above characterization of pullback and pushforward extensions, and legitimates the notation $v \circ \mathcal{E} \circ u$, $\mathcal{E} \circ u \circ u'$, and $v' \circ v \circ \mathcal{E}$ for these extensions.

A.4.3. *Baer sum.* The *Baer sum* of two extensions of F by G in $\mathcal{A} - \mathbf{mod}$

$$\mathcal{E}_1 : 0 \longrightarrow G \xrightarrow{i_1} E_1 \xrightarrow{p_1} F \longrightarrow 0 \ \text{ and } \ \mathcal{E}_2 : 0 \longrightarrow G \xrightarrow{i_2} E_2 \xrightarrow{p_2} F \longrightarrow 0$$

is the extension

$$\mathcal{E}_1 + \mathcal{E}_2 :\ 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

where

$$E = \frac{\mathrm{Ker}((p_1, -p_2) : E_1 \oplus E_2 \longrightarrow F)}{\mathrm{Im}\,(\binom{i_1}{-i_2} : G \longrightarrow E_1 \oplus E_2)}, \tag{A.12}$$

and p and i are given by

$$p[(e_1, e_2)] = p_1(e_1) = p_2(e_2) \quad \text{and} \quad i(g) = [(i_1(g), 0))] = [(0, i_2(g))].$$

This construction defines a composition law $+$ on $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F, G)$. One easily checks that $(\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F, G), +)$ is a commutative group in which the opposite of the class of an extension

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

is the class of

$$\tilde{\mathcal{E}} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{-p} F \longrightarrow 0.$$

Actually, for any integer $k \geq 1$ and any k extensions $\mathcal{E}_1, \ldots \mathcal{E}_k$ of F by G in $\mathcal{A} - \mathbf{mod}$,

$$\mathcal{E}_j : 0 \longrightarrow G \xrightarrow{i_j} E_j \xrightarrow{p_j} F \longrightarrow 0, \quad 1 \leq j \leq k,$$

the sum of their classes in $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F, G)$ is the class of their Baer sum, which is defined as

$$\mathcal{E}_1 + \cdots + \mathcal{E}_k := \Sigma^k_G \circ (\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k) \circ \Delta^k_F, \tag{A.13}$$

where

$$\Sigma^k_G : G^{\oplus k} \longrightarrow G$$

denotes the “sum” morphism, and where

$$\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k : 0 \longrightarrow G^{\oplus k} \overset{i_1 \oplus \ldots \oplus i_k}{\longrightarrow} E_1 \oplus \ldots \oplus E_k \overset{p_1 \oplus \ldots \oplus p_k}{\longrightarrow} F^{\oplus k} \longrightarrow 0$$

is the direct sum of the extensions $\mathcal{E}_k, \ldots \mathcal{E}_k$, and

$$\Delta^k_F : F \longrightarrow F^{\oplus k}$$

the diagonal embedding of F into $F^{\oplus k}$. This definition is easily seen to coincide with the previous one when $k = 2$. In general, for any integer l such that $1 < l < k$, one sees that the extensions $(\mathcal{E}_1 + \cdots + \mathcal{E}_l) + (\mathcal{E}_l + \cdots + \mathcal{E}_k)$ and $\mathcal{E}_1 + \cdots + \mathcal{E}_k$ are isomorphic. This establishes in particular the associativity of the Baer sum.

Using this expression for the Baer sum, it is straightforward to derive that it is compatible with composition. Namely, if $u : F' \to F$ is a morphism in $\mathcal{A} - \mathbf{mod}$, the extensions $(\mathcal{E}_1 + \ldots + \mathcal{E}_k) \circ u$ and $\mathcal{E}_1 \circ u + \ldots + \mathcal{E}_k \circ u$ are canonically isomorphic. Similarly, if $v : G \to G'$ s a morphism in $\mathcal{A} - \mathbf{mod}$, the extensions $v \circ (\mathcal{E}_1 + \ldots + \mathcal{E}_k)$ and $v \circ \mathcal{E}_1 + \ldots + v \circ \mathcal{E}_k$ are canonically isomorphic.

A.4.4. Ext^1 *and* $\mathbb{E}\mathrm{xt}^1$. Consider a 1-extension

$$\mathcal{E} : \; 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

in $\mathcal{A}-\mathbf{mod}$ as above. It may be seen as an exact sequence of complexes in $\mathcal{A}-\mathbf{mod}$ (by identifying E, F, and G with complexes concentrated in degree 0), and defines a distinguished triangle

$$G \xrightarrow{i} E \xrightarrow{p} F \xrightarrow{\partial_{\mathcal{E}}} G[1]$$

in $D(\mathcal{A} - \mathbf{mod})$ by the construction recalled in A.1.

The arrow $\partial_{\mathcal{E}}$ in

$$\mathrm{Hom}_{D(\mathcal{A}-\mathbf{mod})}(F, G[1]) =: \mathrm{Ext}^1_{\mathcal{A}}(F, G)$$

will be denoted $\mathrm{cl}(\mathcal{E})$.

Its definition boils down to the following. From $\mathcal{E}$, we construct the cone of i — which simply is the complex in $\mathcal{A}-\mathbf{mod}$

$$C(i) := [\ldots \longrightarrow 0 \longrightarrow G \xrightarrow{i} E \longrightarrow 0 \longrightarrow \ldots],$$

where G (resp. E) is placed in degree -1 (resp. 0) — and two morphisms of complexes of $\mathcal{A}$-modules,

$$\mathbf{p} : C(i) \longrightarrow F,$$

defined by p from E to F both placed in degree 0, and

$$q : C(i) \longrightarrow G[1],$$

defined by the identity morphism of G, placed in degree -1. By the exactness of $\mathcal{E}$, $\mathbf{p}$ is a quasi-isomorphism and defines an isomorphism

$$\mathbf{p} : C(i) \xrightarrow{\sim} E \text{ in } D(\mathcal{A}-\mathbf{mod}).$$

Then we have

$$\mathrm{cl}(\mathcal{E}) = \partial_{\mathcal{E}} := -q \circ \mathbf{p}^{-1} \in \mathrm{Ext}^1_{\mathcal{A}}(F,G) := \mathrm{Hom}_{D(\mathcal{A}-\mathbf{mod})}(F, G[1]).$$

In this way, one defines a map

$$\mathrm{cl} : \mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(F,G).$$

Observe that, if $u : F' \longrightarrow F$ is a morphism in $\mathcal{A}-\mathbf{mod}$, then, from the diagram (A.9) defining $\mathcal{E} \circ u$, we derive a map of complexes

$$\tilde{u} \colon C(\tilde{i}) := [\ldots \longrightarrow 0 \longrightarrow G \xrightarrow{\tilde{i}} E' \longrightarrow 0 \longrightarrow \ldots] \longrightarrow C(i) := [\ldots \longrightarrow 0 \longrightarrow G \xrightarrow{i} E \longrightarrow 0 \longrightarrow \ldots]$$

defined by id_G (resp. $\tilde{u}$) in degree -1 (resp. 0), and a commutative diagram of complexes

$$\begin{array}{ccccc} F' & \xleftarrow{\tilde{\mathbf{p}}} & C(\tilde{i}) & \xrightarrow{\tilde{q}} & G \\ \downarrow u & & \downarrow \mathbf{u} & & \| \\ F & \xleftarrow{\mathbf{p}} & C(i) & \xrightarrow{q} & G. \end{array}$$

This shows that $\partial_{\mathcal{E}\circ u} = \partial_{\mathcal{E}} \circ u$, or in other terms, the equality $\mathrm{cl}(\mathcal{E} \circ u) = \mathrm{cl}(\mathcal{E}) \circ u$ in $\mathrm{Ext}^1_{\mathcal{A}}(F,G)$.

Similarly, one shows that, for any morphism $v : G \longrightarrow G'$ in $\mathcal{A}-\mathbf{mod}$, we have $\mathrm{cl}(v \circ \mathcal{E}) = v \circ \mathrm{cl}(\mathcal{E})$.

Using the definition (A.13) of the Baer sum, it follows that the map cl is indeed a morphism of abelian groups.

Observe also that the class in $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G)$ of the extension $\mathcal{E}$ lies in the kernel of cl precisely when $\partial_{\mathcal{E}}$ vanishes, that is when $\mathcal{E}$ splits. (Indeed, the vanishing of $\partial_{\mathcal{E}}$ implies that the map $p_* : \mathrm{Hom}_{\mathcal{A}}(F,E) \to \mathrm{Hom}_{\mathcal{A}}(F,F)$ is onto. Any element in $p_*^{-1}(\mathrm{id}_F)$ then defines a splitting of $\mathcal{E}$.) Consequently, the group morphism cl is injective.

Conversely, let F and G be any two objects in $\mathcal{A}-\mathbf{mod}$, and let

$$\iota : G \hookrightarrow I$$

be a monomorphism from G to an injective object in $\mathcal{A}-\mathbf{mod}$. Together with the quotient map

$$\pi : I \longrightarrow \mathrm{Coker}\,\iota,$$

this monomorphism defines an extension

$$\mathcal{I} :\ 0 \longrightarrow G \xrightarrow{\iota} I \xrightarrow{\pi} \mathrm{Coker}\,\iota \longrightarrow 0$$

in $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(\mathrm{Coker}\iota, G)$. The long exact sequence of $\mathrm{Ext}^{\cdot}_{\mathcal{A}}(F,.)$'s deduced from $\mathcal{I}$ starts as follows:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F,G) \xrightarrow{\iota_*} \mathrm{Hom}_{\mathcal{A}}(F,I) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathcal{A}}(F,\mathrm{Coker}\iota) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{A}}(F,G) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(F,I) = \{0\}.$$

In particular, the boundary map

$$\partial : \mathrm{Hom}_{\mathcal{A}}(F, \mathrm{Coker}\,\iota) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(F,G)$$

induces an isomorphism of abelian groups

$$j_\iota : \mathrm{Coker}\,\pi_* \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}}(F,G).$$

Since ∂ coincides with the composition with $\mathrm{cl}(\mathcal{I}) \in \mathrm{Ext}^1_{\mathcal{A}}(\mathrm{Coker}\iota, G)$, the isomorphism j_ι maps the class in $\mathrm{Coker}\,\pi_*$ of an element $u \in \mathrm{Hom}_{\mathcal{A}}(F,I)$ to

$$\mathrm{cl}(\mathcal{I}) \circ u = \mathrm{cl}(\mathcal{I} \circ u). \tag{A.14}$$

Besides, as the 1-extension $\mathcal{I} \circ \pi$ is split (*cf.* A.4.2 above), one may define a morphism of abelian groups

$$j'_\iota : \mathrm{Coker}\,\pi_* \xrightarrow{\sim} \mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G)$$

by sending the class in $\mathrm{Coker}\,\pi_*$ of an element $u \in \mathrm{Hom}_{\mathcal{A}}(F,I)$ to the class of $\mathcal{I} \circ u$, and (A.14) asserts that this morphism satisfies

$$\mathrm{cl} \circ j'_\iota = j_\iota.$$

This shows that the map cl is onto — actually this constructs an inverse of cl — and concludes the proof of the following:

Proposition A.4.5. *The map*

$$\mathrm{cl} : \mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(F,G)$$

is an isomorphism from the group $\mathbb{E}\mathrm{xt}^1_{\mathcal{A}}(F,G)$ *equipped with the Baer sum onto the "cohomological" extension group* $\mathrm{Ext}^1_{\mathcal{A}}(F,G)$.

A.5. **Extension groups of holomorphic vector bundles.** Let X be a paracompact complex analytic manifold, and $\mathcal{O}_X$ the sheaf of $\mathbb{C}$-analytic functions on X.

A.5.1. *Dolbeault isomorphisms.* For any $\mathcal{O}_X$-module F, we obtain a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules

$$\mathcal{D}olb_F : F \longrightarrow \mathcal{D}olb(F) := (F \otimes_{\mathcal{O}_X} A^{0,\cdot}_X, \overline{\partial}_F) \tag{A.15}$$

— the *Dolbeault resolution of* F — by applying the functor $F \otimes_{\mathcal{O}_X} .$ to the Dolbeault resolution

$$0 \to \mathcal{O}_X \hookrightarrow A^{0,0}_X \xrightarrow{\overline{\partial}} A^{0,1}_X \to \cdots \to A^{0,i}_X \xrightarrow{\overline{\partial}} A^{0,i+1}_X \to \cdots$$

of the sheaf $\mathcal{O}_X$ of holomorphic functions on X. This construction is functorial and natural: to any morphism of $\mathcal{O}_X$-modules $\phi : F_1 \to F_2$, we may attach the morphism of Dolbeault complexes:

$$\mathcal{D}olb(\phi) := \phi \otimes_{\mathcal{O}_X} \mathrm{Id}_{A^{0,\cdot}_X} := \mathcal{D}olb(F_1) \longrightarrow \mathcal{D}olb(F_2),$$

which fits into a commutative diagram:

$$\begin{array}{ccc} F_1 & \xrightarrow{\mathcal{D}olb_{F_1}} & \mathcal{D}olb(F_1) \\ \downarrow \phi & & \downarrow \mathcal{D}olb(\phi) \\ F_2 & \xrightarrow{\mathcal{D}olb_{F_2}} & \mathcal{D}olb(F_2). \end{array}$$

Actually, the functor $\mathcal{D}olb$ may be defined on bounded below complexes of $\mathcal{O}_X$-modules, and is exact (by the flatness of $A_X^{0,\cdot}$ over $\mathcal{O}_X$; *cf.* 1.1.2).

The sheaves $F \otimes_{\mathcal{O}_X} A_X^{0,i}$ are sheaves of $A_X^{0,0}$-modules, hence acyclic. Consequently the cohomology groups $H^i(X, \mathcal{D}olb(F))$ may be identified with the Dolbeault cohomology groups $H^i_{\mathrm{Dolb}}(X, F)$, defined as the cohomology groups of the complex of $\mathbb{C}$-vector spaces:

$$0 \to \Gamma(X, F \otimes_{\mathcal{O}_X} A_X^{0,0}) \xrightarrow{\overline{\partial}_F} \Gamma(X, F \otimes_{\mathcal{O}_X} A_X^{0,1}) \to \dots \to \Gamma(X, F \otimes_{\mathcal{O}_X} A_X^{0,i}) \xrightarrow{\overline{\partial}_F} \Gamma(X, F \otimes_{\mathcal{O}_X} A_X^{0,i+1}) \to \dots .$$

Thanks to this identification, the isomorphisms of $\mathbb{C}$-vector spaces

$$H^i(X, F) \longrightarrow H^i(X, \mathcal{D}olb(F))$$

deduced from the quasi-isomorphism (A.15) define the *Dolbeault isomorphisms*:

$$\mathrm{Dolb}_F : H^i(X, F) \longrightarrow H^i_{\mathrm{Dolb}}(X, F).$$

More generally, let E be a locally free $\mathcal{O}_X$-module of finite rank (or, equivalently, the sheaf of $\mathbb{C}$-analytic sections of some $\mathbb{C}$-analytic vector bundle over X). Then, for any $\mathcal{O}_X$-module F, the composition of the isomorphism (A.5) (where $\mathcal{A} = \mathcal{O}_X$) and $\mathrm{Dolb}_{\check{E} \otimes F}$ defines an isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{Dolb}_{E,F} : \mathrm{Ext}^i_{\mathcal{O}_X}(E, F) \xrightarrow{\sim} H^i_{\mathrm{Dolb}}(X, \check{E} \otimes F),$$

which coincides with Dolb_F when $E = \mathcal{O}_X$.

A.5.2. *Second fundamental form.* Consider a short exact sequence of $\mathbb{C}$-analytic vector bundles over X, or equivalently, of locally free $\mathcal{O}_X$-modules of finite rank:

$$\mathcal{E} : \quad 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0.$$

Let s be a C^∞-splitting of $\mathcal{E}$, namely a C^∞-section of p. We may see it as an element of

$$\Gamma(X, \check{F} \otimes_{\mathcal{O}_X} E \otimes_{\mathcal{O}_X} A_X^{0,0}) \simeq \mathrm{Hom}_{A_X^{0,0}}(F \otimes_{\mathcal{O}_X} A_X^{0,0}, E \otimes_{\mathcal{O}_X} A_X^{0,0}),$$

and consider its image by the Dolbeault operator

$$\overline{\partial}_{\check{F} \otimes E}\, s \in \Gamma(X, \check{F} \otimes_{\mathcal{O}_X} E \otimes_{\mathcal{O}_X} A_X^{0,1}) \simeq \mathrm{Hom}_{A_X^{0,0}}(F \otimes_{\mathcal{O}_X} A_X^{0,0}, E \otimes_{\mathcal{O}_X} A_X^{0,1}).$$

The relation $p \circ s = \mathrm{id}_E$ implies that $(p \otimes_{\mathcal{O}_X} \mathrm{id}_{A^{0,1}})\overline{\partial}_{\check{F} \otimes E} s$ vanishes. Consequently there exists a unique α in

$$\Gamma(X, \check{F} \otimes_{\mathcal{O}_X} G \otimes_{\mathcal{O}_X} A_X^{0,0}) \simeq \mathrm{Hom}_{A_X^{0,0}}(F \otimes_{\mathcal{O}_X} A_X^{0,0}, G \otimes_{\mathcal{O}_X} A_X^{0,0})$$

such that

$$\overline{\partial}_{\check{F} \otimes E}\, s = (i \otimes_{\mathcal{O}_X} \mathrm{id}_{A_X^{0,1}})\, \alpha.$$

Moreover s, and therefore α, is $\overline{\partial}$-closed, namely $\overline{\partial}_{\check{F} \otimes G}\, \alpha$ vanishes in $\Gamma(X, \check{F} \otimes_{\mathcal{O}_X} G \otimes_{\mathcal{O}_X} A_X^{0,0})$.

When s is the orthogonal splitting of $\mathcal{E}$ deduced from some C^∞-hermitian metric $\|.\|$ on E, the $(0,1)$-form α with coefficients in $\check{F} \otimes G$ is the adjoint of the so-called second fundamental form associated to the extension of hermitian vector bundles on X defined by $\mathcal{E}$ and $\|.\|$ (*cf.* [Gri66], VI.3; see also [Gri69], 2.d-e, and [GH78], pp. 72-73). In our more general context, we shall call the $(0,1)$-form α itself the *second fundamental form* attached to the extension $\mathcal{E}$ and its C^∞-splitting s.

Recall that the extension $\mathcal{E}$ of analytic vector bundles over X defines a class $\mathrm{cl}(\mathcal{E})$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(F,G)$, and that the $\overline{\partial}_{\check{F}\otimes G}$-closed $(0,1)$-form α in $A^{0,1}(X,\check{F}\otimes G) := \Gamma(X,\check{F}\otimes G\otimes_{\mathcal{O}_X} A^{0,1}_X)$ defines a class $[\alpha]$ in $H^1_{\mathrm{Dolb}}(X,\check{F}\otimes G)$.

The following proposition is classical (see for instance [Gri66], Proposition p.422), up to the precise determination of the sign.

Proposition A.5.3. *With the above notation, the following equality holds in the Dolbeault cohomology group $H^1_{\mathrm{Dolb}}(X,\check{F}\otimes G)$:*

$$\mathrm{Dolb}_{F,G}(\mathrm{cl}(\mathcal{E})) = [\alpha].$$

In other words, the image of $\mathrm{cl}(\mathcal{E})$ under the composition of canonical isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_X}(F,G) &= \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(F,G[1])\\ &\simeq \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(F,\mathcal{D}olb(G)[1])\\ &\simeq \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(\mathcal{O}_X,\check{F}\otimes_{\mathcal{O}_X}\mathcal{D}olb(G)[1])\\ &\simeq \mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(\mathcal{O}_X,\mathcal{D}olb(\check{F}\otimes_{\mathcal{O}_X}G)[1])\\ &\simeq H^1_{\mathrm{Dolb}}(X,\check{F}\otimes_{\mathcal{O}_X}G)\end{aligned}$$

which defines $\mathrm{Dolb}_{F,G}$ coincides with the class of α. This follows from the following two straightforward lemmas:

Lemma A.5.4. *By applying the functor $\mathcal{D}olb$ to $\mathcal{E}$, we obtain an exact sequence of complexes of sheaves of $\mathcal{O}_X$-modules:*

$$\mathcal{D}olb(\mathcal{E}):\ 0\longrightarrow\mathcal{D}olb(G)\stackrel{\mathcal{D}olb(i)}{\longrightarrow}\mathcal{D}olb(E)\stackrel{\mathcal{D}olb(p)}{\longrightarrow}\mathcal{D}olb(F)\longrightarrow 0.$$

Moreover the cone $C(\mathcal{D}olb(p))$ may be identified with $\mathcal{D}olb(C(p))$, and the following diagram is commutative:

$$\begin{array}{ccccccc}\partial_{\mathcal{E}}: & F & \stackrel{\mathbf{p}}{\longleftarrow} & C(i) & \stackrel{(-\mathrm{Id}_G,0)}{\longrightarrow} & G[1]\\ & \downarrow \mathcal{D}olb_F & & \downarrow \mathcal{D}olb_{C(i)} & & \downarrow \mathcal{D}olb_G[1]\\ \partial_{\mathcal{D}olb(\mathcal{E})}: & \mathcal{D}olb(F) & \stackrel{\mathcal{D}olb(\mathbf{p})}{\longleftarrow} & C(\mathcal{D}olb(i)) & \stackrel{(-\mathrm{Id}_{\mathcal{D}olb(G)},0)}{\longrightarrow} & \mathcal{D}olb(G)[1].\end{array}$$

Lemma A.5.5. *One defines a morphism of complexes of $\mathcal{O}_X$-modules*

$$w:\mathcal{D}olb(F)\to C(\mathcal{D}olb(i))$$

that is a right inverse of $\mathcal{D}olb(\mathbf{p})$ by letting, for any local section β of $\mathcal{D}olb(F)^i := F\otimes_{\mathcal{O}_X}A^{0,i}_X$,

$$w(\beta) := (-\alpha.\beta, s.\beta).$$

Indeed Lemma A.5.4 and A.5.5 show that the image in $\mathrm{Hom}_{D(\mathcal{O}_X-\mathbf{mod})}(F,\mathcal{D}olb(G)[1])$ of $\mathrm{cl}(\mathcal{E})$ is defined by the composition:

$$F\stackrel{\mathcal{D}olb_F}{\longrightarrow}\mathcal{D}olb(F)\stackrel{w}{\longrightarrow}C(\mathcal{D}olb(i))\stackrel{(-\mathrm{Id}_{\mathcal{D}olb(G)},0)}{\longrightarrow}\mathcal{D}olb(G)[1].$$

It is immediate that it corresponds in $H^1_{\mathrm{Dolb}}(X,\check{F}\otimes_{\mathcal{O}_X}G)$ to the class of α.

Appendix B. Lattices of Voronoi's first kind

B.1. **Selling parameters.** We recall the definition of lattices of Voronoi's first kind and their associated Selling parameters.

B.1.1. *Definitions.* Let $\overline{E} = \big(E, \|\cdot\|\big)$ be an euclidean lattice, of positive rank n. It is said to be a *lattice of Voronoi's first kind* if it possesses what Conway and Sloane ([CS92, §2]) call an *obtuse superbase*, namely if there exists a $n+1$-tuple $(v_0, \ldots, v_n)$ of vectors in E such that

i) $(v_1, \ldots, v_n)$ is a $\mathbb{Z}$-basis of E, and

$$v_0 + \cdots + v_n = 0$$

(this is a *superbase*);

ii) for any $(i,j) \in \{0, \ldots, n\}^2,\ i \neq j$,

$$p_{ij} := -v_i \cdot v_j \geq 0 \tag{B.1}$$

(the angle of the vectors v_i and v_j is *obtuse*).

When the inequalities (B.1) are strict, the superbase $(v_0, \ldots, v_n)$ is called *strictly obtuse*.

Observe that, for any superbase $(v_0, \ldots, v_n)$ of E, the $n(n+1)/2$-coefficients

$$p_{ij} = p_{ji},\ 0 \leq i < j \leq n,$$

defined by (B.1) uniquely determine the euclidean structure of $\overline{E}$. Indeed, for any $i \in \{0, \ldots, n\}$ we have

$$\|v_i\|^2 = -v_i \cdot \sum_{\substack{0 \leq k \leq n \\ k \neq i}} v_k = \sum_{\substack{0 \leq k \leq n \\ k \neq i}} p_{ik},$$

and consequently, for any $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$,

$$\Big\| \sum_{i=0}^{n} x_i v_i \Big\|^2 = \sum_{0 \leq i < j \leq n} p_{ij}(x_i - x_j)^2. \tag{B.2}$$

This formula goes back to Selling [Sel74], at least when $n = 3$, and the coefficients

$$(p_{ij})_{\ 0 \leq i < j \leq n}$$

will be called the *Selling parameters* attached to the superbase $(v_0, \ldots, v_n)$.

Selling's formula (B.2) shows in particular that a superbase $(v_0, \ldots, v_n)$ of an euclidean lattice $\overline{E} = (\overline{E}, \|\cdot\|)$ is obtuse iff the quadratic form $\|\cdot\|^2$ on $E_{\mathbb{R}}$ expressed in the base $(v_1, \ldots, v_n)$, takes the form

$$\sum_{k=1}^{n} \lambda_k X_k^2 + \sum_{1 \leq i < j \leq n} \lambda_{ij}(X_i - X_j)^2$$

for some $\lambda_i,\ \lambda_{ij}$ in $\mathbb{R}_+$ or equivalently, iff the matrix $(a_{ij})_{1 \leq i,j \leq n} := (v_i \cdot v_j)_{1 \leq i,j \leq n}$ of this quadratic form satisfies

$$\sum_{j=1}^{n} a_{ij} \geq 0 \ \text{ for any } \ i \in \{1, \ldots, n\}$$

and

$$a_{ij} = a_{ji} \leq 0 \ \text{ if } \ 1 \leq i < j \leq n.$$

(Indeed, with the above notation,

$$p_{ij} = \lambda_{ij} = -a_{ij} \ \text{ if } \ 1 \leq i < j \leq n$$

and

$$p_{0i} = \lambda_i = \sum_{j=1}^{n} a_{ij}.)$$

These quadratic forms are precisely the ones in the domain associated by Voronoi to the "forme parfaite principale"

$$\varphi := \sum_{1 \leq i \leq j \leq n} X_i X_j \tag{B.3}$$

(*cf.* [Vor08, 29]).

Observe also that an euclidean lattice $\overline{E}$ of rank n is of Voronoi's first kind iff there exists a basis $(\xi_1, \dots, \xi_n)$ of the $\mathbb{Z}$-module $E^\vee$ and a family

$$(p_{ij})_{0 \leq i < j \leq n} \in \mathbb{R}_+^{\frac{1}{2}\cdot n(n+1)}$$

such that, for any $x \in E_\mathbb{R}$,

$$\|x\|^2 = \sum_{0 \leq i < j \leq n} p_{ij} \big(\xi_i(x) - \xi_j(x)\big)^2 \tag{B.4}$$

where $\xi_0 := 0$. Indeed this identity is equivalent to the fact that the superbase $(v_0, v_1, \dots, v_n)$ of E, defined by the dual basis $(v_1, \dots, v_n)$ of $(\xi_1, \dots, \xi_n)$ and $v_0 := -v_1 - \dots - v_n$, satisfies (B.2). Any $\mathbb{Z}$-basis $(\xi_1, \dots, \xi_n)$ of $E^\vee$ satisfying (B.4) will be said to be *adapted* to the euclidean lattice $\overline{E}$ of Voronoi's first kind and the (unique) family $(p_{ij})_{\,0 \leq i < j \leq n}$ the *associated Selling parameters.*

For any subset S of $\big\{(i,j),\ 0 \leq i < j \leq n\big\}$, let $\gamma(S)$ be the (non-oriented) graph whose set of vertices is $\{0, \dots, n\}$, with an edge between any i and j such that $0 \leq i < j \leq n$ iff $(i,j) \in S$. The following lemma is left as an easy exercise for the reader:

Lemma B.1.2. *Let $\overline{E}$ be an euclidean lattice of Voronoi's first kind, $(\xi_1, \dots, \xi_n)$ an adapted $\mathbb{Z}$-basis of $E^\vee$ and $(p_{ij})_{0 \leq i < j \leq n}$ the corresponding Selling parameters, defined by* $(B.4)$ *where* $\xi_0 := 0$.

1) If the euclidean lattice $\overline{E}$ is indecomposable — namely, if it is not (isomorphic to) the direct sum of two euclidean lattices of positive rank — *then the graph $\gamma(S)$ attached to*

$$S := \big\{(i,j) \big| p_{ij} \neq 0\big\}$$

is connected.

2) For any subset S' of $\big\{(i,j),\ 0 \leq i < j \leq n\big\}$ such that $\gamma(S')$ is connected, we have

$$E^\vee = \sum_{(i,j) \in S'} \mathbb{Z}(\xi_i - \xi_j).$$

B.2. **Examples.** We now describe remarkable classes of euclidean lattices which are and which are not of Voronoi's first kind.

B.2.1. Any euclidean lattice of rank 2 is of Voronoi's first kind. Indeed, if (v_1, v_2) is a base of such a lattice which is obtuse (i.e., such that $v_1 \cdot v_2 \leq 0$) and reduced in the sense of Lagrange (i.e., which satisfies $\|v_1\| \leq \|v_2\| \leq \|v_2 \pm v_1\|$), then the superbase $(-v_1 - v_2, v_1, v_2)$ is obtuse.

B.2.2. Any euclidean lattice of rank 3 is of Voronoi's fist kind. This is a classical result of Selling [Sel74], which has been reproved by Voronoi [Vor08, 33] as a consequence of his theory of "Voronoi's reduction" associated to perfect forms. A direct argument appears in [CS92], and may be concisely reformulated as follows.

To any superbasis (v_0, v_1, v_2, v_3) of an euclidean lattice of rank 3, we may attach

$$\begin{aligned} N(v_0, v_1, v_2, v_3) &:= \sum_{S\subset\{0,1,2,3\}} \|\sum_{k\in S} v_k\|^2 \\ &= 2\left[\|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2 + \|v_1+v_2\|^2 + \|v_2+v_3\|^2 \right. \\ &\quad \left. + \|v_3+v_1\|^2 + \|v_1+v_2+v_3\|^2\right]. \end{aligned} \tag{B.5}$$

If we let, as above, $p_{ij} := -v_i \cdot v_j$ if $0 \leq i < j \leq 3$, a straightforward computation shows that:

$$N(v_0, v_1, v_2, v_3) = 8 \sum_{0\leq i<j\leq 3} p_{ij}.$$

There exists a superbasis (v_0, v_1, v_2, v_3) such that $N(v_0, v_1, v_2, v_3)$ is minimal — this follows from its very definition (B.5) — and any such "minimal" superbasis is obtuse. Indeed, if some p_{ij}, say p_{01}, were negative, then the superbasis

$$(v'_0, v'_1, v'_2, v'_3) := (-v_0, v_1, v_0 + v_2, v_0 + v_3)$$

would satisfy

$$\begin{aligned} N(v'_0, v'_1, v'_2, v'_3) - N(v_0, v_1, v_2, v_3) &= 2\left(\|-v_0 + v_1\|^2 - \|v_0 + v_1\|^2\right) \\ &= 8\, p_{01} < 0. \end{aligned}$$

A similar argument shows that a superbase (v_0, v_1, v_2) of an euclidean lattice of rank 2 is obtuse if it minimizes $\|v_0\|^2 + \|v_1\|^2 + \|v_2\|^2$.

B.2.3. For any positive integer n, the euclidean lattice A_n, defined in section 5.6, admits an obtuse superbase defined by

$$v_i := e_i - e_{i+1},\ 0 \leq i \leq n,$$

where $(e_0, \ldots, e_n)$ is the standard basis of $\mathbb{Z}^{n+1}$ and $e_{n+1} := e_0$.

The dual euclidean lattice A_n^* may be identified with the lattice in the real vector space

$$A^*_{n,\mathbb{R}} := A_{n,\mathbb{R}} = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \middle| \sum_{k=0}^{n} x_k = 0\right\},$$

equipped with the restriction of the standard euclidean scalar product on $\mathbb{R}^{n+1}$, that is defined by

$$\begin{aligned} A_n^* &:= \left\{x \in A_{n,\mathbb{R}} \,|\, \forall\, y \in A_n,\ x \cdot y \in \mathbb{Z}\right\} \\ &= \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \middle| \sum_{k=0}^{n} x_k = 0 \text{ and} \right. \\ &\qquad \left. \forall\, (i,j) \in \{0, \ldots, n\}^2,\ x_j - x_i \in \mathbb{Z}\right\}. \end{aligned}$$

One easily checks that the vectors

$$v_i := e_i - \frac{1}{n+1} \sum_{k=0}^{n+1} e_k\,, \quad (0 \leq i \leq n),$$

constitute a superbase of A_n^*. It is strictly obtuse, since $v_i \cdot v_j = -1/(n+1)$ if $i \neq j$.

This shows that A_n and A_n^* are euclidean lattices of Voronoi's first kind. This property of A_n^* also follows from Voronoi's theory [Vor08]. Indeed, as observed above, an euclidean lattice $\overline{E}$ is of Voronoi's first kind iff it is isometric to a lattice $(\mathbb{Z}^n, \|\cdot\|_\Psi)$ where $\|\cdot\|_\Psi := \Psi^{1/2}$ is the euclidean norm on $\mathbb{R}^n$ defined by a quadratic form Ψ on $\mathbb{R}^n$ in the polyhedral domain associated by Voronoi's "reduction of the first kind" ([Vor08, 15]) to the perfect form φ defined in (B.3). The euclidean lattice $(\mathbb{Z}^n, \|\cdot\|_\varphi)$, up to a scaling, is isomorphic to A_n. (Indeed the isomorphism

$$\begin{array}{rcccc} \mathrm{pr} : A_n & \hookrightarrow & \mathbb{Z}^{n+1} & \longrightarrow & \mathbb{Z}^n \\ & & (x_i)_{0\leq i\leq n} & \longmapsto & (x_i)_{1\leq i\leq n} \end{array}$$

satisfies $\|x\|^2_{A_n} = 2\,\|\mathrm{pr}_{\mathbb{R}}(x)\|^2_\varphi$ for any $x \in A_{n,\mathbb{R}}$.) As the perfect lattice A_n — or equivalently, the perfect form φ — is extreme [14], it is eutactic, *i.e.* the adjoint form[15] $\widetilde{\varphi}$ belongs to the interior of the polyhedral domain attached to φ (*cf.* [Vor08, 17] and [Cox51]). Since, up to a scaling, $(\mathbb{Z}^n, \|\cdot\|_{\widetilde{\varphi}})$ is isometric to A_n^*, this establishes again that A_n^* is of Voronoi's first kind. Actually, the fact the $\widetilde{\varphi}$ lies in the *interior* of the domain attached to φ shows that A_n^* admits a *strictly* obtuse superbase.

Since the Voronoi domains attached to two non-proportional perfect forms on $\mathbb{R}^n$ meet only along a common face ([Vor08, 20]), we see similarly that the only extreme form Ψ on $\mathbb{R}^n$ whose adjoint form $\widetilde{\Psi}$ belongs to the domain of φ is φ itself, or a multiple of φ. In other words, *if an euclidean lattice $\overline{E}$ of rank n is extreme and if its dual $E^\vee$ is of Voronoi's first kind, then, up to a scaling, $\overline{E}$ is isometric to A_n.*

In particular, the dual root lattices $D_n^*(n \geq 4)$, E_6^*, E_7^* and E_8^* are not of Voronoi's first kind. Consequently the euclidean lattice E_8 (resp. D_n) which is isometric to E_8^* (resp. to D_n^*, up to a scaling) is *not* of Voronoi's first kind.

B.2.4. From Voronoi's theory, one may also derive some "upper bound" on automorphism groups of euclidean lattices of Voronoi's first kind:

Proposition B.2.5. *Let $\overline{E}$ be an euclidean lattice of rank n, of Voronoi's first kind, and let $(\xi_1, \dots, \xi_n)$ be a $\mathbb{Z}$-basis of E adapted to $\overline{E}$.*

If the (finite) automorphism group $G := \mathrm{Aut}\,\overline{E}$ of $\overline{E}$ acts irreducibly on $E_\mathbb{R}$, then there exists $\lambda \in \mathbb{R}_+^*$ and a subset S of $\big\{(i,j),\ 0 \leq i < j \leq n\big\}$ such that, for any $x \in E_\mathbb{R}$,

$$\|x\|^2 = \lambda \sum_{(i,j)\in S} \big(\xi_i(x) - \xi_j(x)\big)^2,$$

where as above $\xi_0 := 0$. Moreover, the contragredient action of G on $\check{E}$ permutes transitively the pairs of vectors $\pm(\xi_i - \xi_j)$, $(i,j) \in S$.

---

[14] Recall from [Mar03, 3.4.6] that a lattice is extreme iff it is perfect and eutactic and that the irreducible root lattices A_n, D_n, E_6, E_7 and E_8 are extreme [Mar03, 4.7.2].

[15] Namely up to a scaling, the quadratic form on $\mathbb{R}^n$ whose matrix is the inverse of the one of φ.

Proof. We may assume that $\overline{E}$ is $\mathbb{Z}^n$ and $\|\cdot\|^2$ a quadratic form Ψ on $\mathbb{R}^n$ in Voronoi's principal domain — in other words, $\xi_0 = 0$, $\xi_1 = X_1, \dots, \xi_n = X_n$. This form may be written

$$\Psi = \sum_{0 \leq i < j \leq n} p_{ij}\, q_{ij},$$

where $(q_{ij})_{0 \leq i < j \leq n}$ is the basis of the space $S^2\, \mathbb{R}^{n\vee}$ of quadratic form on $\mathbb{R}^n$ defined by

$$q_{ij} := \begin{cases} X_j^2 & \text{if} \quad i = 0 \\ (X_i - X_j)^2 & \text{if} \quad i \geq 1, \end{cases}$$

and where the Selling parameters p_{ij} are non-negative. The minimal face of the first Voronoi decomposition of the cone of non-negative quadratic forms on $\mathbb{R}^n$ which contains Ψ is the simplicial cone

$$\sum_{(i,j) \in S} \mathbb{R}_+\, q_{ij}$$

where $S := \big\{(i,j) \big|\, p_{ij} \neq 0\big\}$. Any subgroup G of $GL_n(\mathbb{Z})$ preserving the form Ψ also preserves this cone, and permutes its extremal half-lines $(\mathbb{R}_+\, q_{ij})_{(i,j) \in S}$ and consequently the forms $(q_{ij})_{(i,j) \in S}$, since the action of G preserves the integral structure of quadratic forms.

When the action of G on $\mathbb{R}^n$ is irreducible, the form Ψ is — up to scaling — the unique element of $S^2\, \mathbb{R}^{n\vee}$ which is G-invariant. Consequently the non-zero Selling parameters are necessarily equal — this establishes the existence of S and λ — and the action of G permutes transitively the quadratic forms $q_{ij} := (\xi_i - \xi_j)^2$, $(i,j) \in S$, or equivalently, the pairs of vectors $\pm\, (\xi_i - \xi_j)$, $(i,j) \in S$. □

Observe that, with the notation of Proposition B.2.5, the linear forms

$$\xi_i - \xi_j, \ (i,j) \in S$$

are primitive vectors of $E^\vee$, and that their squares $(\xi_i - \xi_j)^2$, $(i,j) \in S$, are rank 1 quadratic forms which span a G-invariant subspace of $S^2 \check{E}_{\mathbb{R}}$, containing the euclidean form $\|\cdot\|^2$ defining the euclidean structure of $\overline{E}$.

Proposition B.2.5 and these observations may be used to show that some lattice with "big" automorphism groups are *not* of Voronoi's first kind. For instance, we have:

Proposition B.2.6. *For any integer $n \geq 4$, the root lattice D_n is not of Voronoi's first kind.*

Surprisingly, this statement does not seem to appear in the literature when $n \geq 5$.

Proof. We shall allow ourselves to leave a few computational details as exercises for the reader.

Recall that D_n is defined by the lattice

$$\big\{(x_i)_{1 \leq i \leq n} \in \mathbb{Z}^n \big|\, \sum_{i=1}^n x_i \in 2\,\mathbb{Z}\big\}$$

in $D_{n,\mathbb{R}} = \mathbb{R}^n$ equipped with the standard euclidean norm.

The dual lattice $D_n^\vee$ in $D_{n,\mathbb{R}}^\vee \simeq \mathbb{R}^n$ is easily seen to be $\mathbb{Z}^n \cup \big((\frac{1}{2})^{\times n} + \mathbb{Z}^n\big)$. The automorphism group of D_n contains (indeed, when $n \geq 5$, is equal to) the semi-direct product

$$G := \{\pm 1\}^n \rtimes \mathfrak{S}_n,$$

where $\{\pm 1\}^n$ acts diagonal, and $\mathfrak{S}_n$ by permutation of the coordinates. By considering the commutants of the action of G on $D_{n,\mathbb{R}}$ and on $S^2 D_{n,\mathbb{R}}^{\vee}$, it is straightforward to check that the action of G on $D_{n,\mathbb{R}}$ is irreducible, and that there are precisely four non-zero G-invariant subspaces of $S^2 D_{n,\mathbb{R}}^{\vee}$ containing the standard euclidean form $\sum_{i=1}^{n} X_i^2$, namely

$$\begin{aligned} V_1 &:= \mathbb{R} \cdot \sum_{i=1}^{n} X_i^2, \\ V_2 &:= \bigoplus_{i=1}^{n} \mathbb{R} \cdot X_i^2, \\ V_3 &:= \mathbb{R} \cdot \sum_{i=1}^{n} X_i^2 \oplus \bigoplus_{1 \le i < j \le n} \mathbb{R} \cdot X_i X_j, \\ V_4 &:= S^2 D_{n,\mathbb{R}}^{\vee}. \end{aligned}$$

Besides, by considering the vectors of minimal positive length in D_n — namely the vectors $(\pm e_i \pm e_j),\ 1 \le i < j \le n$, where $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$ — one sees that the euclidean lattice D_n is indecomposable [16].

Let us assume that D_n is of Voronoi's first kind. To derive a contradiction, consider an adapted basis $(\xi_1, \ldots, \xi_n)$ of $D_n^{\vee}$, and apply Proposition B.2.5, of which we now use the notation, and the subsequent observations. Consider in particular the $\mathbb{R}$-linear span V of $\big((\xi_i - \xi_j)^2\big)_{(i,j)\in S}$ in $S^2 D_{n,\mathbb{R}}^{\vee}$: it is necessarily one of the spaces $V_i,\ 1 \le i \le 4$ introduced above.

Observe that V cannot be V_1, which contains no quadratic form of rank 1.

Assume now that V is V_2. Then the action of $\{\pm 1\}^n$ $(\subset G)$ on V is trivial, and the quadratic forms $(\xi_i - \xi_j)$, $(i,j) \in S$, are fixed by this action. This implies that each linear form $\xi_i - \xi_j$, $(i,j) \in S$, is $\pm X_k$ for some $k \in \{1, \ldots, n\}$. Lemma B.1.2 now shows that

$$D_n^{\vee} = \sum_{(i,j)\in S} \mathbb{Z}\,(\xi_i - \xi_j) \subset \bigoplus_{k=1}^{n} \mathbb{Z}\, X_k.$$

This contradicts the fact that $D_n^{\vee}$ contains $\frac{1}{2} \sum_{k=1}^{n} X_k$.

It is straightforward to check that any quadratic form of rank 1 in V_3 may be written $\lambda \big(\sum_{i=1}^{n} \varepsilon_k X_k\big)^2$ where $\lambda \in \mathbb{R}^*$ and $(\varepsilon_k)_{1 \le k \le n} \in \{\pm 1\}^n$. Consequently, if $V = V_3$, then the linear forms $\xi_i - \xi_j$, $(i,j) \in S$, may be written either

$$\sum_{k=1}^{n} \varepsilon_k X_k, \ \text{ where } (\varepsilon_k) \in \{\pm 1\}^n \setminus \big\{1^{\times n}, (-1)^{\times n}\big\} \tag{B.6}$$

or

$$\frac{\varepsilon}{2} \sum_{k=1}^{n} X_k, \ \text{ where } \varepsilon \in \{\pm 1\}. \tag{B.7}$$

The sum of any two elements in $D_n^{\vee}$ of the form (B.6) or (B.7) is never of the form (B.6) or (B.7). This shows that, when $V = V_3$, there is no triple (i,j,k), $0 \le i < j < k \le n$, such that (i,j), (j,k) and (i,k) belong to S. In other words, the graph $\gamma(S)$ attached to S has no cycle of length 3. An

[16] Observe that, if an euclidean lattice $\overline{E}$ may be written as an orthogonal direct sum $\overline{E}_1 \oplus \overline{E}_2$, then any vector $v \in E$ of minimal positive length belongs to $E_1 \cup E_2$. Consequently, if the set M of vectors of minimal positive length of some euclidean lattice $\overline{E}$ generates the $\mathbb{R}$-vector space $E_{\mathbb{R}}$ and cannot be partitioned as $M = M_1 \coprod M_2$ where any two vectors $e_1 \in M_1$ and $e_2 \in M_2$ are orthogonal, then $\overline{E}$ is indecomposable.

elementary counting argument shows that this contradicts the fact that this graph has $n+1$ vertices and

$$\dim_{\mathbb{R}} V_3 = \frac{n(n+1)}{2} - (n-1)$$

edges.

Finally, when $V = V_4$, then

$$|S| = \dim V_4 = \frac{n(n+1)}{2},$$

and consequently

$$S = \big\{(i,j),\ 0 \leq i < j \leq n\big\}$$

and, for every $x \in D_{n,\mathbb{R}}$,

$$\|x\|^2 = \lambda \sum_{0 \leq i < j \leq n} \big(\xi_i(x) - \xi_j(x)\big)^2.$$

This implies that, up to scaling, the euclidean lattice D_n is isometric with A_n^* — this is plainly wrong (compare the cardinality of their sets of vectors of minimal length, or of their automorphism groups). □

B.3. **The Voronoi cell of an euclidean lattice with strictly obtuse superbase.** Let n be a positive integer. To any $n(n+1)/2$-tuple

$$p = (p_{ij})_{0 \leq i < j \leq n} \in (\mathbb{R}_+^*)^{\frac{n(n+1)}{2}},$$

we attach an euclidean lattice $\overline{V}(p)$ of rank n defined as follows:

$$\overline{V}(p) := (V, \langle \cdot, \cdot \rangle_p),$$

where

$$V := \mathbb{Z}^{n+1}/\mathbb{Z} \cdot 1^{\times(n+1)}$$

and where $\langle \cdot, \cdot \rangle_p$ denotes the euclidean scalar product on $V_{\mathbb{R}} \simeq \mathbb{R}^{n+1}/\mathbb{R} \cdot 1^{\times(n+1)}$ defined by

$$\Big\langle \big[(x_i)_{0 \leq i \leq n}\big], \big[(y_i)_{0 \leq i \leq n}\big] \Big\rangle_p = \sum_{0 \leq i < j \leq n} p_{ij}(x_i - x_j) \cdot (y_i - y_j),$$

for any $(x_i)_{0 \leq i \leq n}$ and $(y_i)_{0 \leq i \leq n}$ in $\mathbb{R}^{n+1}$ (where $[\,\alpha]$ denotes the image in $V_{\mathbb{R}}$ of $\alpha \in \mathbb{R}^{n+1}$).

In the sequel, we shall often omit the subscript p to simplify notations. In particular, we shall write $v \cdot w$ (resp. $\|v\|^2$) instead of $\langle v, w \rangle_p$ (resp. $\langle v, v \rangle_p$).

Let $(\varepsilon_0, \ldots, \varepsilon_n)$ be the canonical basis of $\mathbb{Z}^{n+1}$. Clearly

$$\varepsilon_0 + \cdots + \varepsilon_n = 1^{\times(n+1)}$$

and

$$(v_0, \ldots, v_n) := \big([\varepsilon_0], \ldots, [\varepsilon_n]\big)$$

is a superbase of $\overline{V}$, which is strictly obtuse since

$$v_i \cdot v_j = -p_{ij} \quad \text{if} \quad 0 \leq i < j \leq n. \tag{B.8}$$

It is convenient to define

$$p_{ij} := p_{ji} \quad \text{if} \quad 0 \leq j < i \leq n.$$

Then (B.8) holds for any $(i,j) \in \{0, \ldots, n\}^2$ such that $i \neq j$, and, for any $i \in \{0, \ldots, n\}$

$$\|v_i\|^2 = -v_i \cdot \sum_{\substack{0 \leq j \leq n \\ j \neq i}} v_j = \sum_{\substack{0 \leq j \leq n \\ j \neq i}} p_{ij}. \tag{B.9}$$

More generally, for every $S \subset \{0, \dots, n\}$ we let

$$v_S := \sum_{i \in S} v_i.$$

Observe that $v_S = 0$ iff $S = \emptyset$ or $S = \{0, \dots, n\}$. Moreover, for any two elements S_1 and S_2 in

$$\mathfrak{P}'\big(\{0, \dots, n\}\big) := \mathfrak{P}\big(\{0, \dots, n\}\big) \setminus \{\emptyset, \{0, \dots, n\}\},$$

we have

$$v_{S_1} = v_{S_2} \text{ iff } S_1 = S_2,$$

and

$$v_{S_1} + v_{S_2} = 0 \text{ iff } \{0, \dots, n\} = S_1 \coprod S_2.$$

A straightforward application of Selling's formula (B.2) shows that the $2^{n+1} - 2$ vectors v_S, $S \in \mathfrak{P}'\big(\{0, \dots, n\}\big)$, are precisely the Voronoi vectors of $\overline{V}(p)$, and indeed are strict Voronoi vectors ([CS92, Theorem 3]). For any such S, let

$$\begin{aligned} H_S &:= \Big\{x \in V_{\mathbb{R}} \,\Big|\, \|x - v_S\| = \|x\|\Big\} \\ &= \big\{x \in V_{\mathbb{R}} \,|\, 2v_S \cdot x = \|v_S\|^2\big\} \end{aligned}$$

and let

$$F_S := \mathcal{V}\big(\overline{V}(p)\big) \cap H_S$$

be the corresponding face of $\mathcal{V}\big(\overline{V}(p)\big)$. The F_S, $S \in \mathfrak{P}'\big(\{0, \dots, n\}\big)$ are precisely the $(n-1)$-dimensional faces (also known as *facets*) of $\mathcal{V}\left(\overline{V}(p)\right)$.

Lemma B.3.1. *If the faces F_S and $F_{S'}$ attached to two elements S and S' in $\mathfrak{P}'\big(\{0, \dots, n\}\big)$ are not disjoint, then $S \subset S'$ or $S' \subset S$.*

Proof. Define $I := S \cap S'$, $T := S \setminus I$, and $T' := S' \setminus I$. Then S (resp. S') is the disjoint union of I and T (resp. I and T'). Moreover, if x is some element of

$$F_S \cap F_{S'} = H_S \cap H_{S'} \cap \mathcal{V}\big(\overline{V}(p)\big),$$

then we have

$$\begin{aligned} 2\, v_S \cdot x &= \|v_S\|^2, \\ 2\, v_{S'} \cdot x &= \|v_{S'}\|^2, \\ 2\, v_{S \cup S'} \cdot x &\leq \|v_{S \cup S'}\|^2, \\ \text{and} \qquad 2\, v_I \cdot x &\leq \|v_I\|^2. \end{aligned}$$

As

$$v_S + v_{S'} = v_{S \cup S'} + v_I,$$

this implies:

$$\|v_S\|^2 + \|v_{S'}\|^2 \leq \|v_{S \cup S'}\|^2 + \|v_I\|^2.$$

However, we have

$$\begin{aligned} \|v_{S \cup S'}\|^2 + \|v_I\| - \|v_S\|^2 - \|v_{S'}\|^2 &= \|v_T + v_{T'} + v_I\|^2 + \|v_I\|^2 \\ &\quad - \|v_T + v_I\|^2 - \|v_{T'} + v_I\|^2 \\ &= 2\,\langle v_T, v_{T'} \rangle \\ &= -2 \sum_{\substack{i \in T \\ j \in T'}} p_{ij}, \end{aligned}$$

and the last sum is negative when T and T' are not empty. □

Let us consider the set $\mathcal{S}(n)$ of subsets of cardinality n of $\mathfrak{P}'\big(\{0,\ldots,n\}\big)$ which are totally ordered by inclusion. The group $\mathfrak{S}_{n+1}$ of permutations of $\{0,\ldots,n\}$ acts naturally on $\mathcal{S}(n)$, and the following lemma is straightforward:

Lemma B.3.2. *The action of $\mathfrak{S}_{n+1}$ on $\mathcal{S}(n)$ is simply transitive. In other words, the mapping*

$$\begin{array}{ccl} \mathfrak{S}_{n+1} & \longrightarrow & \mathcal{S}(n) \\ \sigma & \longmapsto & \Big\{\{\sigma(1),\ldots,\sigma(n)\},\ \{\sigma(2),\ldots,\sigma(n)\},\ldots,\ \{\sigma(n)\}\Big\} \end{array}$$

is a bijection.

For any $A \in \mathcal{S}(n)$, the vectors $(v_S)_{S\in A}$ are linearly independent, and consequently the hyperplanes $(H_S)_{S\in A}$ have an unique common point, which we shall denote s_A.

Lemma B.3.3. *If $A = \big\{\{i_1,\ldots,i_n\},\{i_2,\ldots,i_n\},\ldots,\{i_n\}\big\}$, then*

$$\|\sum_{k=0}^{n} x_k v_{i_k} - s_A\|^2 - \|s_A\|^2 = \sum_{0\le k<\ell\le n} p_{i_k i_\ell}\big[(x_\ell - x_k)^2 - (x_\ell - x_k)\big]. \tag{B.10}$$

holds for any element $(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}$.

Proof. The point s_A is defined by the n linear equations

$$2\, v_S\cdot s_A = \|v_S\|^2\,,\ S\in A,$$

which may also be written:

$$2\, v_{i_k}\cdot s_A = \|v_{i_k}+\cdots+v_{i_n}\|^2 - \|v_{i_{k+1}}+\cdots+v_{i_n}\|^2,\ 1\le k\le n-1$$

and

$$2\, v_{i_n}\cdot s_A = \|v_{i_n}\|^2.$$

Using (B.8) and (B.9), these relations take finally the form

$$2\, v_{i_k}\cdot s_A = \sum_{\ell=0}^{n}\varepsilon(\ell,k)\, p_{i_k i_\ell},\ 1\le k\le n,$$

where

$$\varepsilon(\ell,k) := \begin{cases} 1 & \text{if}\quad \ell<k \\ 0 & \text{if}\quad \ell=k \\ -1 & \text{if}\quad \ell>k. \end{cases}$$

Consequently, for any $(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}$,

$$\begin{aligned} \|\sum_{k=0}^{n} x_k v_{i_k} - s_A\|^2 - \|s_A\|^2 &= \|\sum_{k=0}^{n} x_k v_{i_k}\|^2 - 2\sum_{k=0}^{n} x_k v_{i_k}\cdot s_A \\ &= \sum_{0\le k<\ell\le n} p_{i_k i_\ell}\,(x_k - x_\ell)^2 \\ &\quad - \sum_{0\le k,\ell\le n}\varepsilon(\ell,k)\, p_{i_k i_\ell}\, x_k \\ &= \sum_{0\le k<\ell\le n} p_{i_k i_\ell}\big[(x_\ell - x_k)^2 + x_k - x_\ell\big]. \end{aligned}$$

□

When $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$, the right-hand side of (B.10) is non-negative. This shows that, for any $v \in V$,

$$\|v - s_A\|^2 \geq \|s_A\|^2, \tag{B.11}$$

or equivalently, that s_A belongs to the Voronoi cell $\mathcal{V}\left(V(p)\right)$ of $\overline{V}(p)$.

We are now in position to establish:

Proposition B.3.4. *1) The mapping $(A \mapsto s_A)$ is a bijection from $\mathcal{S}(n)$ onto the set of vertices of $\mathcal{V}\left(\overline{V}(p)\right)$.*

2) More generally, one defines an inclusion reversing bijection from the set $\mathcal{O}(n)$ of non-empty subsets of $\mathfrak{P}'\left(\{0, \dots, n\}\right)$ which are totally ordered onto the set of faces of $\mathcal{V}\left(\overline{V}(p)\right)$ by mapping $P \in \mathcal{O}(n)$ to $\Delta_P := \bigcap_{S \in P} F_S$.

Observe that Δ_P is a face of dimension $n - |P|$ of $\mathcal{V}\left(\overline{V}(p)\right)$.

When all the p_{ij}'s are equal to 1, the permutation group $\mathfrak{S}_{n+1}$ acts on $\overline{V}(p)$ by permutation of the vectors in the superbase $(v_0, \dots, v_n)$, and the vertices of $\mathcal{V}\left(\overline{V}(p)\right)$ are the $(n+1)!$ points in the orbit under $\mathfrak{S}_{n+1}$ of

$$\frac{1}{n+1}\left[(-\frac{n}{2}, -\frac{n}{2}+1, \dots, \frac{n}{2}-1, \frac{n}{2})\right] = \frac{1}{n+1}\sum_{i=0}^{n}(-\frac{n}{2}+i)\, v_i = \frac{1}{n+1}\sum_{i=0}^{n} i \cdot v_i.$$

The Voronoi cell $\mathcal{V}\left(\overline{V}(p)\right)$ is then a so-called *permutohedron*. Actually in this case, $\overline{V}(p)$ is isometric (up to a scaling) with A_n^*, and this description of its Voronoi cell is classical (see for instance [Vor09, 102 - 103], and [CS99, Chapter 21, Theorem 7]).

Proposition B.3.4 shows that the Voronoi cell of any lattice of Voronoi's first kind with positive Selling parameters still has the combinatorial type of a permutohedron. Actually, this is also a consequence of results in Voronoi's last paper (see [Vor09, 104], notably p. 147). Let us finally point out that, when all the p_{ij}'s are 1, identity (B.10) also appears in this paper (up to a permutation of the variables, it coincides with the penultimate equation in p. 140 of [Vor09, 103]).

Proof. We freely use basic facts concerning polytopes and their posets of faces, as described for instance in [Grü67, chapter 3], and [Zie95, chapter 2].

1) We have just shown in (B.11) that, for any $A \in \mathcal{S}(n)$, the point s_A belongs to $\mathcal{V}\left(\overline{V}(p)\right)$. Moreover, it is an extreme point of $\mathcal{V}\left(\overline{V}(p)\right)$, since it is the "vertex" of the intersection of n "half spaces"

$$\bigcap_{S \in A}\left\{x \in V(p)_{\mathbb{R}} \,|\, 2\, v_S \cdot x \leq \|v_S\|^2\right\}$$

which contains $\mathcal{V}\left(\overline{V}(p)\right)$. (Observe that the vectors $(v_S)_{S \in A}$ are linearly independent.)

Conversely, any vertex P of $\mathcal{V}\left(\overline{V}(p)\right)$ is contained in at least n distinct facets $F_{S_1}, \dots, F_{S_n}$ of $\mathcal{V}\left(\overline{V}(p)\right)$. Lemma B.3.1 shows that $A := \{F_{S_1}, \dots, F_{S_n}\}$ is a totally ordered subset of $\mathfrak{S}'\left(\{0, \dots, n\}\right)$. Consequently, $P = s_A$.

2) For any P in $\mathcal{O}(n)$, the intersection $\Delta_P := \bigcap_{S \in P} F_S$ is not empty, since it exists some element A of $\mathcal{S}(n)$ containing P and therefore Δ_P contains s_A. Moreover, the vectors

$(v_S)_{S\in P}$, orthogonal to the facets $(F_S)_{S\in P}$, are linearly independent, and therefore Δ_P is a face of $\mathcal{V}\left(\overline{V}(p)\right)$ of dimension $n-|P|$.

Conversely any face of dimension $n-p$ of $\mathcal{V}\left(\overline{V}(p)\right)$ is an intersection of p distinct facets of $\mathcal{V}\left(\overline{V}(p)\right)$ and may therefore be written $\bigcap_{S\in P} F_S$ where P is a subset of $\mathfrak{S}'\big(\{0,\ldots,n\}\big)$ of cardinality p. Again Lemma B.3.1 shows that P is totally ordered by inclusion.

This establishes that the map

$$\begin{array}{ccc} \mathcal{O}(n) & \longrightarrow & \Big\{\text{faces of } \mathcal{V}\big(\overline{V}(p)\big)\Big\} \\ P & \longmapsto & \Delta_P \end{array}$$

is bijective. It is clearly inclusion reversing. □

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