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Abstract

In this article we prove conditions under which a certain parabolic group cohomology space over a finite field $\mathbb{F}$ is a faithful module for the Hecke algebra of Katz modular forms over an algebraic closure of $\mathbb{F}$. These results can e.g. be used to compute Katz modular forms of weight one with methods of linear algebra over $\mathbb{F}$.

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1 Introduction

A theorem by Eichler and Shimura relates the space $S_k(\Gamma_1(N), \mathbb{C})$ of cuspidal holomorphic modular forms of weight $k \geq 2$ for the group $\Gamma_1(N)$ for some integer $N \geq 1$ to the $\mathbb{C}$-vector space $H^1_{\text{par}}(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$, where $\mathbb{C}[X,Y]_{k-2}$ denotes the $\mathbb{C}$-vector space of homogeneous polynomials of degree $k-2$ in two variables. For the definition of the parabolic group cohomology see Section 2. The result of Eichler and Shimura (see Theorem 7.1) is the isomorphism

$$S_k(\Gamma_1(N), \mathbb{C}) \oplus \overline{S_k(\Gamma_1(N), \mathbb{C})} \cong H^1_{\text{par}}(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2}).$$

On the modular forms, as well as on $H^1_{\text{par}}(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$, one disposes of natural Hecke operators, which are compatible with the Eichler-Shimura isomorphism. An immediate consequence is the observation that $H^1_{\text{par}}(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$ is a free module of rank 2 for the Hecke algebra $\mathbb{T}$ generated by the Hecke operators in the endomorphism ring of $S_k(\Gamma_1(N), \mathbb{C})$. This implies, in particular, that $H^1_{\text{par}}(\Gamma_1(N), \mathbb{C}[X,Y]_{k-2})$ is a faithful $\mathbb{T}$-module.

In general there is no analogue of the Eichler-Shimura isomorphism over finite fields. However, in the present article we show that the above observation generalises to Katz modular forms over finite fields in certain cases.

Let now $\mathbb{T}_{\mathbb{F}_p}$ denote the $\mathbb{F}_p$-Hecke algebra generated by the Hecke operators inside the endomorphism ring of the cuspidal Katz modular forms $S_k(\Gamma_1(N), \overline{\mathbb{F}_p})$ of weight $k \geq 2$ for $\Gamma_1(N)$ over the field $\overline{\mathbb{F}_p}$. In view of the observations from the Eichler-Shimura isomorphism, two questions arise naturally.

(a) Is $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p[X,Y]_{k-2})$ a faithful $\mathbb{T}_{\mathbb{F}_p}$-module?

(b) Is $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p[X,Y]_{k-2})$ free of rank 2 as a $\mathbb{T}_{\mathbb{F}_p}$-module?

A positive answer to Question (b) clearly implies a positive answer to Question (a).

Using p-adic Hodge theory (in particular the article [Faltings-Jordan]), Edixhoven shows in Theorem 5.2 of [EdixJussieu] that Question (a) has a positive answer in the weight range $2 \leq k \leq p-1$. Emerton, Pollack and Weston were able to deduce from the fundamental work by Wiles on Fermat's last theorem ([Wiles]) that Question (b) is true locally at primes of the Hecke algebra which are p-ordinary and p-distinguished (see [EPW], Proposition 4.1.1, and the proof of Proposition 4.1.4) for $\mathbb{Z}_p$-coefficients and the full cohomology (i.e. not the parabolic subspace), from which the result follows for $\mathbb{F}_p$-coefficients and the full cohomology. We should, however, point out that it is not at all clear whether a positive answer to Question (a) with $\mathbb{Z}_p$-coefficients implies a positive answer over $\mathbb{F}_p$.

In the present article we use completely different techniques that were inspired by the special case $p = 2$ of [EdixJussieu], Theorem 5.2. We are able to prove (see Corollary 7.9) that Question (a) is true locally at p-ordinary primes in the weight range $2 \leq k \leq p+1$.

Let us point out the following consequence, which presented the initial motivation for this study. If the answer to Question (a) is yes, we may compute $\mathbb{T}_{\mathbb{F}_p}$ via the Hecke operators on the finite dimensional $\mathbb{F}_p$-vector space $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p[X,Y]_{k-2})$. This can be done explicitly on a computer and only requires linear algebra methods. We remark that the knowledge of the Hecke algebra is very interesting. One can, for example, try to verify whether it is a Gorenstein ring. This property is often implied when the Hecke algebra is isomorphic to a deformation ring (as e.g. in [Wiles]).

Techniques from [EdixJussieu], Section 4, show how Katz modular forms of weight one over $\overline{\mathbb{F}_p}$ can be related to weight p. We observe that the corresponding local factors of the Hecke algebra in weight p are p-ordinary. Hence, a consequence of our result is that methods from $\mathbb{F}_p$-linear algebra can be used for the computation of weight one modular forms (see Theorem 9.5). The author carried out some such calculations and reported on them in [W-App].

It should be pointed out that if one is not interested in the Hecke algebra structure, but only in the systems of eigenvalues of Hecke eigenforms in $S_k(\Gamma_1(N), \overline{\mathbb{F}_p})$, one can get them directly from $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p[X,Y]_{k-2})$ for all $k \geq 2$ (see e.g. [Thesis], Proposition 4.3.1). However, there may be more systems of eigenvalues in the parabolic group cohomology than in $S_k(\Gamma_1(N), \overline{\mathbb{F}_p})$, but the extra ones can easily be identified.

This article focusses on the use of group cohomology. The author studied in his thesis [Thesis] and the article [W-MS] its relation to modular symbols and a certain cohomology group on the corresponding modular curve. For the group $\Gamma_1(N)$ with $N \geq 5$ all these agree.

We add some remarks on Katz modular forms. As a reference one can use [EdixBoston], [Katz] or [Diamond-Im]. For $R = \mathbb{C}$, the Katz cuspidal modular forms are precisely the holomorphic cusp forms. Under the assumptions $N \geq 5$ and $k \geq 2$ one has $S_k(\Gamma_1(N), \mathbb{Z}[1/N]) \otimes R \cong S_k(\Gamma_1(N), R)$

for any ring R in which N is invertible. In particular, this means that any Katz form over $\overline{\mathbb{F}_p}$ can be lifted to characteristic zero in the same weight and level. The previous statement does not hold for weight $k = 1$ in general.

Katz modular forms over $\overline{\mathbb{F}_p}$ for a prime p play an important rôle in a modified version of Serre's conjecture on modular forms (see [EdixWeight]), as their use, on the one hand, gets rid of some character lifting problems, and on the other hand, allows one to minimize the weights of the modular forms. In particular, odd, irreducible Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ which are unramified at p are conjectured to correspond to Katz eigenforms of weight one.

Finally, we should mention that William Stein has implemented $\mathbb{F}_p$-modular symbols and the Hecke operators on them in Magma and Python. One could say that this article is about what they compute.

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2 Parabolic group cohomology

Let $\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tau := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ be matrix classes in $\mathrm{PSL}_2(\mathbb{Z})$. Hence $T = \tau\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. As $\mathrm{PSL}_2(\mathbb{Z})$ is the free product $\langle\sigma\rangle * \langle\tau\rangle$, one obtains the following special case of the *Mayer-Vietoris sequence*, which we state without proof (see [HiltonStammbach], p. 221).

2.1 Proposition. *Let R be a ring and M a left $R[\mathrm{PSL}_2(\mathbb{Z})]$-module. Then the* Mayer-Vietoris sequence *gives the exact sequence*

$$0 \to M^{\mathrm{PSL}_2(\mathbb{Z})} \to M^{\langle\sigma\rangle} \oplus M^{\langle\tau\rangle} \to M \to H^1(\mathrm{PSL}_2(\mathbb{Z}), M) \to H^1(\langle\sigma\rangle, M) \oplus H^1(\langle\tau\rangle, M) \to 0$$

and for all $i \geq 2$ isomorphisms $H^i(\mathrm{PSL}_2(\mathbb{Z}), M) \cong H^i(\langle\sigma\rangle, M) \oplus H^i(\langle\tau\rangle, M)$.

2.2 Corollary. *Let R be a ring and $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup of finite index such that all the orders of all stabiliser groups Γ_x for $x \in \mathbb{H}$ are invertible in R. Then for all $R[\Gamma]$-modules V one has $H^1(\Gamma, V) = M/(M^{\langle\sigma\rangle} + M^{\langle\tau\rangle})$ with $M = \mathrm{Coind}_\Gamma^{\mathrm{PSL}_2(\mathbb{Z})}(V)$ and $H^i(\Gamma, V) = 0$ for all $i \geq 2$.*

Proof. We first note that any non-trivially stabilised point x of $\mathbb{H}$ is conjugate by an element of $\mathrm{PSL}_2(\mathbb{Z})$ to either i or ζ_3, whence all non-trivial stabiliser groups are of the form $g\langle\sigma\rangle g^{-1} \cap \Gamma$ or $g\langle\tau\rangle g^{-1} \cap \Gamma$ for some $g \in \mathrm{PSL}_2(\mathbb{Z})$. We can apply Mackey's formula (see e.g. [W-MS]) to obtain

$$H^i(\langle\sigma\rangle, \mathrm{Coind}_\Gamma^{\mathrm{PSL}_2(\mathbb{Z})} V) \cong \prod_{g \in \Gamma\backslash\mathrm{PSL}_2(\mathbb{Z})/\langle\sigma\rangle} H^i(g\langle\sigma\rangle g^{-1} \cap \Gamma, V)$$

for all i and a similar result for τ. Due to the invertibility assumption the result follows from Shapiro's lemma and Proposition 2.1. □

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$ be some subgroup of finite index. We define the *parabolic group cohomology group for the $R[\Gamma]$-module V* as the kernel of the restriction map in

$$0 \to H^1_{\mathrm{par}}(\Gamma, V) \to H^1(\Gamma, V) \xrightarrow{\mathrm{res}} \prod_{g \in \langle T \rangle \backslash \mathrm{PSL}_2(\mathbb{Z})/\Gamma} H^1(\Gamma \cap \langle gTg^{-1} \rangle, V). \tag{2.1}$$

This definition is compatible with Shapiro's lemma, i.e. Equation 2.1 is isomorphic to

$$0 \to H^1_{\mathrm{par}}(\mathrm{PSL}_2(\mathbb{Z}), M) \to H^1(\mathrm{PSL}_2(\mathbb{Z}), M) \xrightarrow{\mathrm{res}} H^1(\langle T \rangle, M) \tag{2.2}$$

with $M = \mathrm{Coind}_\Gamma^{\mathrm{PSL}_2(\mathbb{Z})} V$. This also follows from Mackey's formula (see e.g. [W-MS]).

2.3 Proposition. *Let R be a ring and $\Gamma \leq \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup of finite index such that all the orders of all stabiliser groups Γ_x for $x \in \mathbb{H}$ are invertible in R. Then for all $R[\Gamma]$-modules V the sequence*

$$0 \to H^1_{\mathrm{par}}(\Gamma, V) \to H^1(\Gamma, V) \xrightarrow{\mathrm{res}} \prod_{g \in \langle T \rangle \backslash \mathrm{PSL}_2(\mathbb{Z})/\Gamma} H^1(\Gamma \cap \langle gTg^{-1} \rangle, V) \to V_\Gamma \to 0$$

is exact.

Proof. Due to the assumptions we may apply Corollary 2.2. The restriction map in Equation 2.2 thus becomes

$$M/(M^{\langle \sigma \rangle} + M^{\langle \tau \rangle}) \xrightarrow{m \mapsto (1-\sigma)m} M/(1-T)M,$$

since $H^1(\langle T \rangle, M) \cong M/(1-T)M$. Using the isomorphism $M \cong (R[\mathrm{PSL}_2(\mathbb{Z})] \otimes_R V)_\Gamma$, it is easy to compute that the cokernel of this map is V_Γ, the Γ-coinvariants. □

3 Hecke action

Hecke operators conceptually come from Hecke correspondences on modular curves resp. modular stacks. They are best described on the moduli interpretation (see e.g. [Diamond-Im], 3.2 and 7.3). All Hecke operators that we will encounter come from these correspondences. However, we will only present in detail what the Hecke operators on group cohomology look like (formally they can be obtained from the Hecke correspondences on the moduli interpretation using the cohomology of modular stacks with locally constant coefficients). For that we follow [Diamond-Im], 12.4.

Hecke operators on group cohomology

Denote by $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$ the semi-group of integral 2×2-matrices with non-zero determinant. Let R be a ring, $\alpha \in \mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$ and Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We use the notations $\Gamma_\alpha :=$

$\Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma^\alpha := \Gamma \cap \alpha\Gamma\alpha^{-1}$, where we consider α^{-1} as an element of $\mathrm{GL}_2(\mathbb{Q})$. Both groups are commensurable with Γ.

Suppose that V is an R-module with a $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-(semi-group)-action. The *Hecke operator* T_α acting on group cohomology is the composite

$$H^1(\Gamma, V) \xrightarrow{\text{res}} H^1(\Gamma^\alpha, V) \xrightarrow{\text{conj}_\alpha} H^1(\Gamma_\alpha, V) \xrightarrow{\text{cores}} H^1(\Gamma, V).$$

We have a similar description on the parabolic subspace and notice that the two are compatible. The first map is the usual *restriction*, and the third one is the so-called *corestriction*, which one also finds in the literature under the name *transfer* (cf. [Brown]). We explicitly describe the second map on non-homogeneous cocycles (cf. [Diamond-Im], p. 116):

$$\text{conj}_\alpha : H^1(\Gamma^\alpha, V) \to H^1(\Gamma_\alpha, V), \;\; c \mapsto \big(g_\alpha \mapsto \alpha^\iota.c(\alpha g_\alpha \alpha^{-1})\big).$$

Here ι is Shimura's main involution, i.e. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. The following formula can also be found in [Diamond-Im], p. 116, and [Shimura], Section 8.3.

3.1 Proposition. *Suppose that $\Gamma\alpha\Gamma = \bigcup_{i=1}^n \Gamma\delta_i$ is a disjoint union. Then the Hecke operator T_α acts on $H^1(\Gamma, V)$ amd $H^1_{\text{par}}(\Gamma, V)$ by sending the non-homogeneous cocyle c to $T_\alpha c$ defined by*

$$(T_\alpha c)(g) = \sum_{i=1}^n \delta_i^\iota c(\delta_i g \delta_{j(i)}^{-1})$$

for $g \in \Gamma$. Here $j(i)$ is the index such that $\delta_i g \delta_{j(i)}^{-1} \in \Gamma$.

Proof. We only have to describe the corestriction explicitly. For that we notice that one has $\Gamma = \bigcup_{i=1}^n \Gamma_\alpha g_i$ with $\alpha g_i = \delta_i$. Furthermore the corestriction of a non-homogeneous cocycle $u \in H^1(\Gamma_\alpha, V)$ is the cocycle $\text{cores}(u)$ uniquely given by

$$\text{cores}(u)(g) = \sum_{i=1}^n g_i^{-1} u(g_i g g_{j(i)}^{-1})$$

for $g \in \Gamma$. Combining with the explicit description of the map conj_α yields the result. □

For a positive integer n, the *Hecke operator* T_n is T_α with $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$.

If $\Gamma_1(N) \subseteq \Gamma \subset \Gamma_0(N)$ and the integer d is coprime to N, the *diamond operator* $\langle d \rangle$ is T_α for any matrix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ whose reduction modulo N is $\left(\begin{smallmatrix} d^{-1} & 0 \\ 0 & d \end{smallmatrix}\right)$. The diamond operator gives a group action by $(\mathbb{Z}/N\mathbb{Z})^*$. If the level is NM with $(N, M) = 1$, then we can separate the diamond operator into two parts $\langle d \rangle = \langle d \rangle_M \times \langle d \rangle_N$, corresponding to $\mathbb{Z}/NM\mathbb{Z} \cong \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Hecke operators and Shapiro's lemma

For later use we need to extend the $\Gamma_1(N)$-action on $\mathrm{Coind}_{\Gamma_1(NM)}^{\Gamma_1(N)}(V)$ to a $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-semi-group action "in the right way". This can be achieved by the following lemma.

3.2 Lemma. *Let N, M be coprime positive integers, and let V be an $R[\Gamma_1(N)]$-module. Define the R-module*

$$\mathcal{W}(M,V) := \{f \in \mathrm{Hom}_R(R[(\mathbb{Z}/M\mathbb{Z})^2], V) \mid f((u,v)) = 0 \; \forall (u,v) \text{ s.t. } \langle u,v\rangle \neq \mathbb{Z}/M\mathbb{Z}\}.$$

We equip it with the left $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-(semi-group)-action $(g.f)((u,v)) = gf((u,v)g)$.

Then the homomorphism

$$\mathcal{W}(M,V) \to \mathrm{Hom}_{R[\Gamma_1(NM)]}(R[\Gamma_1(N)], V), \quad f \mapsto \big(g \mapsto (g.f)((0,1))\big)$$

is an isomorphism of left $\Gamma_1(N)$-modules (by restricting the action on $\mathcal{W}(M,V)$). In particular, $\mathcal{W}(M,V)$ is isomorphic to $\mathrm{Coind}_{\Gamma_1(NM)}^{\Gamma_1(N)}(V)$ as a left $\Gamma_1(N)$-module.

Proof. As N and M are coprime, reduction modulo M defines a surjection from $\Gamma_1(N)$ onto $\mathrm{SL}_2(\mathbb{Z}/M\mathbb{Z})$. This implies that the map

$$\Gamma_1(NM)\backslash\Gamma_1(N) \xrightarrow{A \mapsto (0,1)A \bmod M} (\mathbb{Z}/M\mathbb{Z})^2$$

is injective, and its image is the set of the (u,v) with $\mathbb{Z}/M\mathbb{Z} = \langle u,v\rangle$. From this the claimed isomorphism follows directly. □

3.3 Lemma. *Let N be a positive integer and l a prime. There is the coset decomposition*

$$\Gamma_1(N)\begin{pmatrix}1&0\\0&l\end{pmatrix}\Gamma_1(N) = \bigcup_a \bigcup_b \Gamma_1(N)\sigma_a\begin{pmatrix}a&b\\0&d\end{pmatrix}$$

when a runs through the integers such that $a > 0$, $(a,N) = 1$, $ad = l$ and b through a system of representatives of $\mathbb{Z}/d\mathbb{Z}$. Here $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ is a matrix reducing to $\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}$ modulo N.

Proof. This is [Shimura], Proposition 3.36. □

We can now prove the compatibility of the Hecke operators with the isomorphism from Shapiro's lemma when we take the $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-action on the coinduced module from Lemma 3.2. A proof of this fact in the more general, but rather heavy language of weakly compatible Hecke pairs can be found in [Ash-Stevens] (Lemma 2.2(b)).

The Shapiro map is the isomorphism on cohomology groups

$$\mathrm{Sh} : H^1(\Gamma_1(N), \mathcal{W}(M,V)) \to H^1(\Gamma_1(NM), V)$$

induced by the homomorphism

$$\mathcal{W}(M,V) \to V, \quad f \mapsto f((0,1)).$$

3.4 Proposition. *Let N, M be coprime positive integers, and let V be an $R[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-module. For all primes l and all integers $d \geq 1$ with $(d,N) = 1$ we have*

$$T_l \circ \mathrm{Sh} = \mathrm{Sh} \circ T_l \quad \text{and} \quad \langle d\rangle_N \circ \mathrm{Sh} = \mathrm{Sh} \circ \langle d\rangle_N.$$

Proof. We prove the statement for T_l. The proof for the diamond operator is similar.

For every integer $a > 0$ dividing l such that $(a, N) = 1$ we choose a matrix σ_a such that it reduces to $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ modulo N. If $(a, M) = 1$, then we also impose that σ_a reduces to $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ modulo M. If $(a, M) \neq 1$, then we want $\sigma_a \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo M. A simple calculation shows that $(0,1)\big(\sigma_a \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\big)^{\iota}$ equals $(0, 1)$ if $(a, M) = 1$ resp. $(0, a)$ if $(a, M) \neq 1$.

Lemma 3.3 gives explicit coset representatives of $\Gamma_1(NM)\backslash\Gamma_1(NM)\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_1(NM)$ as a subset of coset representatives of $\Gamma_1(N)\backslash\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)\Gamma_1(N)$, namely of those with $(a, M) = 1$.

Let now $c \in H^1(\Gamma_1(N), \mathcal{W}(M, V))$ be a cocycle. Then by Proposition 3.1 and the definition of the $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-action on $\mathcal{W}(M, V)$ we have for $g \in \Gamma_1(NM)$

$$(\mathrm{Sh}(T_n c))(g) = \sum_{\delta} \delta^{\iota}\big(c(\delta g \widetilde{\delta}^{-1})((0,1)\delta^{\iota})\big),$$

where the sum runs over the above coset representatives for $\Gamma_1(N)$ and $\widetilde{\delta}$ is chosen among these representatives such that $\delta g \widetilde{\delta}^{-1} \in \Gamma_1(NM)$. Moreover, we have

$$(T_n(\mathrm{Sh}(c)))(g) = \sum_{\delta} \delta^{\iota}(c(\delta g \widetilde{\delta}^{-1})((0,1))),$$

where now the sum only runs through the subset described above. By what we have remarked right above $(0,1)\delta^{\iota}$ equals $(0, 1)$ if and only if $(a, M) = 1$. If $(a, M) \neq 1$, then $\langle a \rangle \neq \mathbb{Z}/M\mathbb{Z}$, but we have $(0,1)\delta^{\iota} = (0, a)$. This proves the compatibility for T_l. □

3.5 Proposition. *Let N, M be coprime positive integers, and let V be an $R[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-module. For $(n, M) = 1$ we define the $R[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-isomorphism*

$$\mathrm{mult}_n : \mathcal{W}(M, V) \to \mathcal{W}(M, V), \quad f \mapsto ((u, v) \mapsto f\big((nu, nv)\big)).$$

Then we have

$$\langle n \rangle_M \circ \mathrm{Sh} = \mathrm{Sh} \circ \mathrm{mult}_n.$$

Proof. Let $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix reducing to $\left(\begin{smallmatrix} n^{-1} & 0 \\ 0 & n \end{smallmatrix}\right)$ modulo M and to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. This means in particular that $\sigma \in \Gamma_1(N)$. Hence, for a cocycle $c \in H^1(\Gamma_1(N), \mathcal{W}(M, V))$ we have

$$\sigma^{-1} c(\sigma g \sigma^{-1}) = c(g) + (g - 1)c(\sigma^{-1}),$$

so that the equality $c(\sigma g \sigma^{-1}) = \sigma c(g)$ holds in $H^1(\Gamma_1(N), \mathcal{W}(M, V))$.

We can now check the claim. First we have

$$(\langle n \rangle_M \circ \mathrm{Sh})(c)(g) = \sigma^{\iota}\big((\sigma.c(g))((0,1))\big) = c(g)((0,1)\sigma).$$

This agrees with $(\mathrm{Sh} \circ \mathrm{mult}_n)(c)(g) = c(g)((0, n))$. □

4 The module $V_{k-2}(R)$

For a ring R and an integer $k \geq 2$ we let

$$V_{k-2}(R) := \mathrm{Sym}^{k-2}(R^2)$$

which carries the natural left $\mathrm{SL}_2(\mathbb{Z})$-action. In the sequel we will often use the following different description of $V_{k-2}(R)$.

4.1 Lemma. *Let $R[X,Y]_n$ denote the R-module of homogeneous polynomials of degree n in the variables X and Y over R. The map*

$$\mathrm{Sym}^n(R^2) \to R[X,Y]_n, \ \ \left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right) \otimes \cdots \otimes \left(\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}\right) \mapsto (a_1X + b_1Y)\ldots(a_nX + b_nY)$$

defines an isomorphism of left $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-modules, when we equip the polynomials with the action $(M.P)(X,Y) = P\big((X,Y)M\big)$.

Proof. The map is well defined and every monomial is obviously hit. As $\mathrm{Sym}^n(R^2)$ is freely generated by the classes of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \otimes \cdots \otimes \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \otimes \cdots \otimes \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, the map is in fact an isomorphism. □

4.2 Proposition. *Suppose that $n!$ is invertible in R. Then there is a perfect pairing*

$$V_n(R) \times V_n(R) \to R$$

of R-modules, which induces an isomorphism $V_n(R) \to V_n(R)^\vee$ of $R[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-modules, if we equip $V_n(R)^\vee$ with the left action $(M.\phi)(w) = \phi(M^\iota w)$. If M is invertible, we then $(M.\phi)(w)$ equals $\det(M)^n \phi(M^{-1}w)$.

Proof. One defines the perfect pairing on $V_n(R)$ by first constructing a perfect pairing on R^2, which we consider as column vectors. We set

$$R^2 \times R^2 \to R, \ \ \langle v, w \rangle := \det(v|w) = v_1w_2 - v_2w_1.$$

If M is a matrix in $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$, one checks easily that $\langle Mv, w \rangle = \langle v, M^\iota w \rangle$. This pairing extends to a pairing on the n-th tensor power of R^2 by letting

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle = \langle v_1, w_1 \rangle \cdot \cdots \cdot \langle v_n, w_n \rangle.$$

Due to our assumption on the invertibility of $n!$, we may view $\mathrm{Sym}^n(R^2)$ as a submodule in the n-th tensor power, and hence obtain the desired pairing. Consequently, one has the isomorphism of R-modules

$$V_n(R) \to V_n(R)^\vee, \ \ v \mapsto (w \mapsto \langle v, w \rangle),$$

which is $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-equivariant for the actions considered. □

4.3 Lemma. *Let $n \geq 1$ be an integer. We suppose that $n!N$ is not a zero divisor in R. The t-invariants are $V_n(R)^{\langle t \rangle} = \langle X^n \rangle$ for $t = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ and the t'-invariants are $V_n(R)^{\langle t' \rangle} = \langle Y^n \rangle$ for $t' = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$.*

Proof. The action of t gives $t.(X^{n-i}Y^i) = X^{n-i}(NX+Y)^i$ and consequently

$$(t-1).(X^{n-i}Y^i) = \sum_{j=0}^{i-1} r_{i,j} X^{n-j} Y^j$$

with $r_{i,j} = N^{i-j}\binom{i}{j}$, which is not a zero divisor by assumption. For $x = \sum_{i=0}^{n} a_i X^{n-i} Y^i$ we have

$$(t-1).x = \sum_{j=0}^{n-1} X^{n-j} Y^j (\sum_{i=j+1}^{n} a_i r_{i,j}).$$

If $(t-1).x = 0$, we conclude for $j = n-1$ that $a_n = 0$. Next, for $j = n-2$ it follows that $a_{n-1} = 0$, and so on, until $a_1 = 0$. This proves the first part. The second follows from symmetry. $\square$

4.4 Proposition. *Let $n \geq 1$ be an integer.*

(a) If $n!N$ is not a zero divisor in R, then the R-module of $\Gamma(N)$-invariants $V_n(R)^{\Gamma(N)}$ is zero.

(b) If $n!$ is invertible in R and N is not a zero divisor in R, then the R-module of $\Gamma(N)$-coinvariants $V_n(R)_{\Gamma(N)}$ is zero.

(c) Suppose that Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that reduction modulo p defines a surjection $\Gamma \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_p)$ (e.g. $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$ for $p \nmid N$). Suppose moreover that $1 \leq n \leq p$ if $p > 2$, and $n = 1$ if $p = 2$. Then one has $V_n(\mathbb{F}_p)^{\Gamma} = 0 = V_n(\mathbb{F}_p)_{\Gamma}$.

Proof. As $\Gamma(N)$ contains the matrices t and t', Lemma 4.3 already finishes Part (a). Under the assumptions of Part (b) Proposition 4.2 implies a self-duality, so that (b) follows from (a). The only part of (c) that is not yet covered is when the degree is $n = p > 2$. In that case we have an exact sequence of $\Gamma(N)$-modules

$$0 \to V_1(\mathbb{F}_p) \to V_p(\mathbb{F}_p) \to V_{p-2}(\mathbb{F}_p) \to 0.$$

In fact, $V_p(\mathbb{F}_p)$ is naturally isomorphic with the space U_1 considered on p. 11, so one can proceed as there. It suffices to take (co-)invariants to obtain the result. $\square$

Torsion-freeness and base change properties

Herremans has computed a torsion-freeness result like the following proposition in [Herremans], Proposition 9. Here we give a short and conceptual proof of a slightly more general statement. The way of approach was suggested to the author by Bas Edixhoven.

4.5 Proposition. *Assume that R is an integral domain of characteristic 0 such that $R/pR \cong \mathbb{F}_p$ for a prime p. Let $N \geq 5$ and $k \geq 2$ be integers and let $\Gamma = \Gamma_1(N)$. Then the following statements hold:*

(a) $H^1(\Gamma, V_{k-2}(R)) \otimes_R \mathbb{F}_p \cong H^1(\Gamma, V_{k-2}(\mathbb{F}_p))$.

(b) If $k = 2$, then $H^1(\Gamma, V_{k-2}(R))[p] = 0$. If $k \geq 3$, then $H^1(\Gamma, V_{k-2}(R))[p] = V_{k-2}(\mathbb{F}_p)^\Gamma$. In particular, if $p \nmid N$, then $H^1(\Gamma, V_{k-2}(R))[p] = 0$ for all $k \in \{2, \ldots, p+2\}$.

(c) If $k = 2$, or if $k \in \{3, \ldots, p+2\}$ and $p \nmid N$, then $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(R)) \otimes_R \mathbb{F}_p \cong H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{F}_p))$.

Proof. Let us first notice that the sequence

$$0 \to V_{k-2}(R) \xrightarrow{\cdot p} V_{k-2}(R) \to V_{k-2}(\mathbb{F}_p) \to 0$$

of $R[\Gamma]$-modules is exact. The associated long exact sequence of $R[\Gamma]$-cohomology gives rise to the short exact sequence

$$0 \to H^i(\Gamma, V_{k-2}(R)) \otimes \mathbb{F}_p \to H^i(\Gamma, V_{k-2}(\mathbb{F}_p)) \to H^{i+1}(\Gamma, V_{k-2}(R))[p] \to 0$$

for every $i \geq 0$. Exploiting this sequence for $i = 1$ immediately yields Part (a), since any H^2 of Γ is zero by Corollary 2.2. Part (b) is a direct consequence of the case $i = 0$ and Proposition 4.4.

We have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\Gamma, V_{k-2}(R)) & \xrightarrow{\cdot p} & H^1(\Gamma, V_{k-2}(R)) & \longrightarrow & H^1(\Gamma, V_{k-2}(\mathbb{F}_p)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_g H^1(D_g, V_{k-2}(R)) & \xrightarrow{\cdot p} & \prod_g H^1(D_g, V_{k-2}(R)) & \longrightarrow & \prod_g H^1(D_g, V_{k-2}(\mathbb{F}_p)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & (V_{k-2}(R))_\Gamma & \xrightarrow{\cdot p} & (V_{k-2}(R))_\Gamma, & & & &
\end{array}$$

where the products are taken over $g \in \Gamma\backslash \mathrm{PSL}_2(\mathbb{Z})/\langle T \rangle$, and $D_g = \Gamma \cap \langle gTg^{-1} \rangle$. The exactness of the first row is the contents of Parts (a) and (b). That the columns are exact follows from Proposition 2.3. The zero on the right of the second row is due to the fact that D_g is free on one generator. That generator is of the form $g \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix} \right)$ with $r \mid N$, so that r is invertible in $\mathbb{F}_p$. The zero on the left is trivial for $k = 2$ and for $3 \leq k \leq p+2$ it is a consequence of Lemma 4.3. Part (c) now follows from the snake lemma and Proposition 4.4, which implies that the bottom map is an injection. □

We mention without proof that the statements of the proposition also hold for any congruence subgroup Γ of level N if the orders of the stabiliser subgroups for the action of Γ on $\mathbb{H}$ are coprime to p. This is for example always the case if $p \geq 5$.

5 Level raising for parabolic group cohomology

The contents of this section is already partly present in [Ash-Stevens]. However, in that paper the parabolic subspace is not treated.

Decomposition of $\mathcal{W}(p,\mathbb{F}_p)$ as $\mathbb{F}_p[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-module

We will now relate the $\mathbb{F}_p[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-modules $\mathcal{W}(p,\mathbb{F}_p)$ and $V_d(\mathbb{F}_p)$ for $0 \leq d \leq p-1$. The latter are in fact precisely the simple $\mathbb{F}_p[\mathrm{SL}_2(\mathbb{F}_p)]$-modules.

5.1 Lemma. *Evaluation of polynomials on $\mathbb{F}_p^2$ induces an isomorphism of $\mathbb{F}_p[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$-modules*

$$\mathbb{F}_p[X,Y]/(X^p - X, Y^p - Y) \cong \mathbb{F}_p^{\mathbb{F}_p^2}.$$

Proof. We first notice that $(X^p - X, Y^p - Y)$ is preserved by the $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-action. The map is well-defined because of Fermat's little theorem and the compatibility for the natural action is clear. As the dimensions on both sides agree, it suffices to prove injectivity. Let $f \in \mathbb{F}_p[X,Y]$ be a polynomial of degree $\leq p-1$ in both variables such that $f(a,b) = 0$ for all $a,b \in \mathbb{F}_p$. Then for fixed a the polynomial $f(a,Y)$ is identically zero, as it is zero for all the p specialisations of Y. Hence, considering f as a polynomial in Y with coefficients in $\mathbb{F}_p[X]$, it follows that all those coefficients are identically zero for the same argument. Consequently, the polynomial f is zero as an element of $\mathbb{F}_p[X,Y]$ proving the claim. $\square$

We can thus identify $\mathcal{W}(p,\mathbb{F}_p)$ with $\{f \in \mathbb{F}_p[X,Y]/(X^p - X, Y^p - Y) \mid f((0,0)) = 0\}$. Let $U_d(\mathbb{F}_p)$ be the subspace consisting of polynomial classes of degree $d \in \{0,\dots,p-2\}$, i.e. those that satisfy $f(lx,ly) = l^d f(x,y)$ for all $l \in \mathbb{F}_p$. Note that the degree is naturally defined modulo $p-1$. It is clear that the natural $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-action respects the degree. By collecting the monomials we obtain

$$\mathcal{W}(p,\mathbb{F}_p) = \bigoplus_{d=0}^{p-2} U_d(\mathbb{F}_p).$$

Furthermore, we dispose of the perfect bilinear pairing

$$\mathcal{W}(p,\mathbb{F}_p) \times \mathcal{W}(p,\mathbb{F}_p) \to \mathbb{F}_p, \quad \langle f,g \rangle = \sum_{(a,b)\in\mathbb{F}_p^2} f(a,b)g(a,b).$$

5.2 Lemma. *Let $d,e \geq 0$ be integers. With $(p-1) \nmid d$ or $(p-1) \nmid e$ we have*

$$\sum_{(a,b)\in\mathbb{F}_p^2} a^d b^e = 0.$$

Proof. As the statement is symmetric in d and e, we may suppose that $(p-1) \nmid e$ and in particular $e \neq 0$. Then $\sum_{(a,b)\in\mathbb{F}_p^2} a^d b^e = \sum_{a=0}^{p} a^d (\sum_{b=1}^{p-1} b^e)$. The latter sum, however, is zero, as one can for instance see by choosing a generator σ of $\mathbb{F}_p^*$ and rewriting $\sum_{b=1}^{p-1} b^e = \sum_{i=1}^{p-1} (\sigma^e)^i$. As σ^e clearly is a zero of the polynomial $X^{p-1} - 1$, it is a zero of the polynomial $\sum_{i=1}^{p-1} X^i$, since $\sigma^e \neq 1$ using $(p-1) \nmid e$. $\square$

If $(p-1) \nmid (d+e)$, Lemma 5.2 implies that $U_d(\mathbb{F}_p)$ pairs to zero with $U_e(\mathbb{F}_p)$. Hence, the restricted pairing $U_d(\mathbb{F}_p) \times U_{p-1-d}(\mathbb{F}_p) \to \mathbb{F}_p$ is perfect for $0 \leq d \leq p-1$, as the dimensions of $U_{p-1-d}(\mathbb{F}_p)$

and $U_d(\mathbb{F}_p)$ are equal. Furthermore, $\mathbb{F}_p[X,Y]_d$ pairs to zero with $\mathbb{F}_p[X,Y]_{p-1-d}$. This follows from Lemma 5.2 and an easy calculation. Consequently the induced pairing

$$U_d(\mathbb{F}_p)/V_d(\mathbb{F}_p) \times V_{p-1-d}(\mathbb{F}_p) \to \mathbb{F}_p$$

is perfect. Let $M \in \mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$ such that its reduction modulo p is invertible. Then it is clear that the above pairing respects the action of M, i.e. $\langle Mf, Mg\rangle = \langle f, g\rangle$. Consequently, we receive an isomorphism of $\mathbb{F}_p$-vector spaces $U_d(\mathbb{F}_p)/V_d(\mathbb{F}_p) \to V_{p-1-d}(\mathbb{F}_p)^\vee$. Composing with the map from Proposition 4.2, we obtain an isomorphism

$$U_d(\mathbb{F}_p)/V_d(\mathbb{F}_p) \to V_{p-1-d}(\mathbb{F}_p).$$

Weight $k \in \{2, \ldots, p+1\}$ in weight 2

We now study how the behaviour of the $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$-action.

5.3 Lemma. *Let $0 < d \leq p-1$ and let $M \in \mathrm{Mat}_2(\mathbb{Z})_{\neq 0}$ such that its reduction modulo p is in $GL_2(\mathbb{F}_p)$. Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \to & V_d(\mathbb{F}_p) & \to & U_d(\mathbb{F}_p) & \to & V_{p-1-d}(\mathbb{F}_p) & \to & 0 \\ & & \downarrow {\scriptstyle M.} & & \downarrow {\scriptstyle M.} & & \downarrow {\scriptstyle \det(M)^d M.} & & \\ 0 & \to & V_d(\mathbb{F}_p) & \to & U_d(\mathbb{F}_p) & \to & V_{p-1-d}(\mathbb{F}_p) & \to & 0. \end{array}$$

Proof. This follows from the compatibilities of the two pairings with the group actions described above. □

5.4 Lemma. *Let $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ and $0 < d \leq p-1$. Then the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \to & V_d(\mathbb{F}_p) & \to & U_d(\mathbb{F}_p) & \to & V_{p-1-d}(\mathbb{F}_p) & \to & 0 \\ & & \downarrow {\scriptstyle M^\iota.} & & \downarrow {\scriptstyle M^\iota.} & & \downarrow {\scriptstyle 0} & & \\ 0 & \to & V_d(\mathbb{F}_p) & \to & U_d(\mathbb{F}_p) & \to & V_{p-1-d}(\mathbb{F}_p) & \to & 0. \end{array}$$

Proof. We have $M^\iota = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. A basis of $U_d(\mathbb{F}_p)$ is given by the monomials of degree d, which correspond to the embedding of $V_d(\mathbb{F}_p)$, together with the monomials $X^i Y^{p-1+d-i}$ for $d \leq i \leq p-1$. As the latter monomials all contain at least one factor of X, they are killed by applying the matrix. □

We hence find formulae similar to those that hold in a comparable situation for modular forms of level Np (see Proposition 6.1, resp. [Gross], p. 475). The following Proposition, except for the parabolic part, is also [Ash-Stevens], Theorem 3.4.

We introduce the following notation. Let M be any $\mathbb{F}_p$-vector space on which the Hecke operators T_l and the p-part of the diamond operators $\langle \cdot \rangle_p$ act. By $M[d]$ we mean M with the action of the Hecke operator T_l "twisted" to be $l^d T_l$ (in particular T_p acts as zero). Furthermore, by $M(d)$ be denote the subspace on which $\langle l \rangle_p$ acts as $l^d = \chi_p(l)^d$ with χ_p the mod p cyclotomic character.

5.5 Proposition. *Let p be a prime, $N \geq 5$ and $0 < d \leq p-1$ integers such that $p \nmid N$. We have isomorphisms respecting the Hecke operators*

$$H^1(\Gamma_1(Np), \mathbb{F}_p)(d) \cong H^1(\Gamma_1(N), U_d(\mathbb{F}_p)) \quad \text{and}$$

$$H^1_{\mathrm{par}}(\Gamma_1(Np), \mathbb{F}_p)(d) \cong H^1_{\mathrm{par}}(\Gamma_1(N), U_d(\mathbb{F}_p)).$$

Moreover, there are the exact sequences

$$0 \to H^1(\Gamma_1(N), V_d(\mathbb{F}_p)) \to H^1(\Gamma_1(N), U_d(\mathbb{F}_p)) \to H^1(\Gamma_1(N), V_{p-1-d}(\mathbb{F}_p))[d] \to 0$$

and

$$0 \to H^1_{\mathrm{par}}(\Gamma_1(N), V_d(\mathbb{F}_p)) \to H^1_{\mathrm{par}}(\Gamma_1(N), U_d(\mathbb{F}_p)) \to H^1_{\mathrm{par}}(\Gamma_1(N), V_{p-1-d}(\mathbb{F}_p))[d] \to 0,$$

which respect the Hecke operators.

Proof. The first statement follows from Propositions 3.4 and 3.5 together with the definition of $U_d(\mathbb{F}_p)$. The twisting of the Hecke action in the exact sequences is clear from the definition of the Hecke operators on group cohomology using Lemmas 5.3 and 5.4.

For $d = p-1$ we have $U_0(\mathbb{F}_p) = V_0(\mathbb{F}_p) \oplus V_{p-1}(\mathbb{F}_p)$, from which the statements follow. So we now assume $d < p-1$, in particular $p \neq 2$. For the top sequence we only need to check that it is exact on the left and on the right. By Proposition 4.4 we have $H^0(\Gamma_1(N), V_{p-1-d}(\mathbb{F}_p)) = 0$. The H^2-terms are trivial by Corollary 2.2.

The exactness of the second sequence follows from the snake lemma, once we have established the exactness of

$$0 \to \prod_{c \text{ cusps}} H^1(D_c, V_d(\mathbb{F}_p)) \to \prod_{c \text{ cusps}} H^1(D_c, U_d(\mathbb{F}_p)) \to \prod_{c \text{ cusps}} H^1(D_c, V_{p-1-d}(\mathbb{F}_p)) \to 0,$$

where D_c is the stabiliser group of the cusp $c = g\infty$ with $g \in \mathrm{SL}_2(\mathbb{Z})$. Hence, $D_c = g\langle \pm T \rangle g^{-1} \cap \Gamma_1(N)$. This group is infinite cyclic generated by $g \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) g^{-1}$ for some $r \in \mathbb{Z}$ dividing N (see also the proof of Proposition 4.5). Hence, we have $H^2(D_c, V_d(\mathbb{F}_p)) = 0$. We claim that the sequence

$$0 \to \prod_{c \text{ cusps}} H^0(D_c, V_d(\mathbb{F}_p)) \to \prod_{c \text{ cusps}} H^0(D_c, U_d(\mathbb{F}_p)) \to \prod_{c \text{ cusps}} H^0(D_c, V_{p-1-d}(\mathbb{F}_p)) \to 0$$

is exact. It follows from Lemma 4.3 that both $H^0(D_c, V_d(\mathbb{F}_p))$ and $H^0(D_c, V_{p-1-d}(\mathbb{F}_p))$ are one-dimensional. To finish the proof, it thus suffices to prove that $H^0(D_c, U_d(\mathbb{F}_p))$ is (at least) of dimension 2. The elements $X^d \in U_d(\mathbb{F}_p)$ and $Y^d(1-X^{p-1}) \in U_d(\mathbb{F}_p)$ are invariant under T. Indeed,

$$\begin{aligned} T.Y^d(1-X^{p-1}) &= (X+Y)^d(1-X^{p-1}) \\ &= Y^d(1-X^{p-1}) + \sum_{i=1}^{d} \binom{d}{i} Y^{d-i}X^i(1-X^{p-1}) = Y^d(1-X^{p-1}), \end{aligned}$$

as in $U_d(\mathbb{F}_p)$ we have $X^i(1-X^{p-1}) = X^{i-1}(X - X^p) = 0$ for $i > 0$. □

6 Modular forms of weight 2 and level Np

We recall some work of Serre as explained in [Gross], Sections 7 and 8, cf. also [EdixWeight], Section 6.

Let us now introduce notation that is used throughout the sequel of this paper. We consider the modular curve $X_1(Np)$ over $\mathbb{Q}_p(\zeta_p)$ for a prime $p > 2$ not dividing $N \geq 5$. It has a regular stable model X over the ring $\mathbb{Z}_p[\zeta_p]$, see e.g. [Katz-Mazur]. Let J denote the Néron model over $\mathbb{Z}_p[\zeta_p]$ of $J_1(Np)$, the Jacobian of $X_1(Np)$ over $\mathbb{Q}_p(\zeta_p)$. We let, following [Gross], Section 8,

$$L = H^0(X, \Omega_{X/\mathbb{Z}_p[\zeta_p]}),$$

where $\Omega_{X/\mathbb{Z}_p[\zeta_p]}$ is the *dualising sheaf of* X of [Deligne-Rapoport], Section I.2. By [Gross], Equation 8.2, we have for the special fibre $X_{\mathbb{F}_p}$ that

$$\overline{L} := H^0(X_{\mathbb{F}_p}, \Omega_{X_{\mathbb{F}_p}/\mathbb{F}_p}) = L \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{F}_p.$$

On L and $\overline{L}$ the p-part $\langle\cdot\rangle_p$ of the diamond operator acts. The principal result on $\overline{L}$ that we will need is the following (see Propositions 8.13 and 8.18 of [Gross]). The notation is as in Proposition 5.5.

6.1 Proposition. (Serre) *Assume $3 \leq k \leq p$, $N \geq 5$ and $p \nmid N$. Then there is an isomorphism of $\mathbb{F}_p$-vector spaces*

$$\overline{L}(k-2) \cong S_k(\Gamma_1(N), \mathbb{F}_p) \oplus S_{p+3-k}(\Gamma_1(N), \mathbb{F}_p)[k-2]$$

respecting the Hecke operators. Moreover, the sequence of Hecke modules

$$0 \to S_2(\Gamma_1(N), \mathbb{F}_p)[p-1] \to \overline{L}(p-1) \to S_{p+1}(\Gamma_1(N), \mathbb{F}_p) \to 0$$

is exact.

In our attempt to compare Hecke algebras of modular forms with those of modular symbols in characteristic p, we generalise the strategy of the second part of the proof of [EdixJussieu], Theorem 5.2. Hence, we wish to bring the Jacobian into the play, since it will enable us to pass from characteristic zero geometry to characteristic p.

6.2 Lemma. *Under the assumptions and notations above we have isomorphisms*

$$\overline{L} \cong \mathrm{Cot}_0(J^0_{\mathbb{F}_p}) \cong \mathrm{Cot}_0(J^0_{\mathbb{F}_p}[p])$$

respecting the Hecke operators.

Proof. The first isomorphism is e.g. [EdixWeight], Equation $6.7.2$. The second one follows from the fact that multiplication by p on $J^0_{\mathbb{F}_p}$ induces multiplication by p on the tangent space at 0, which is the zero map. Hence, the tangent space at 0 of $J^0_{\mathbb{F}_p}[p]$ is equal to the one of $J^0_{\mathbb{F}_p}$. □

Parabolic cohomology and the p-torsion of the Jacobian

To establish an explicit link between parabolic cohomology and modular forms, we identify the parabolic cohomology group for $\Gamma_1(N)$ with $\mathbb{F}_p$-coefficients as the p-torsion of the Jacobian of the corresponding modular curve. Here we may view the Jacobian as a complex abelian variety. As in the other cases, the Hecke correspondences on the modular curves give rise to Hecke operators on the Jacobian.

6.3 Proposition. *Let $N \geq 3$ be an integer, and p a prime. Then there are isomorphisms of $\mathbb{F}_p$-vector spaces*

$$H^1_{\mathrm{par}}(\Gamma_1(Np), \mathbb{F}_p) \cong J(\mathbb{C})[p] = J(\overline{\mathbb{Q}}_p)[p]$$

respecting the Hecke operators.

Proof. The second equality follows from the fact that torsion points are algebraic. We start with the exact *Kummer sequence* of analytic sheaves over $X_1(Np)$

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 0.$$

Its long exact sequence in analytic cohomology yields

$$0 \to H^1(X_1(Np), \mu_p) \to H^1(X_1(Np), \mathbb{G}_m) \xrightarrow{p} H^1(X_1(Np), \mathbb{G}_m).$$

Using that $H^1(X_1(Np), \mathbb{G}_m) \cong J(\mathbb{C})$, we already obtain that $H^1(X_1(Np), \mu_p) \cong J(\mathbb{C})[p]$. As $\mathbb{C}$ contains the p-th roots of unity, we may replace the sheaf μ_p by the constant sheaf $\mathbb{F}_p$. Moreover, the group $H^1(X_1(Np), \mathbb{F}_p)$ coincides with $H^1_{\mathrm{par}}(\Gamma_1(Np), \mathbb{F}_p)$. This follows for example from the Leray spectral sequence for the open immersion $Y_1(Np) \hookrightarrow X_1(Np)$ (see e.g. [W-MS]). It can be checked that the action of the Hecke operators is compatible. □

7 Hecke algebras

In this section we will compare the Hecke algebra of modular forms to that of modular symbols and establish isomorphisms in certain cases. Whenever for a ring R we have an R-module M, on which Hecke operators T_n act for all n, we let

$$\mathbb{T}_R(M) := R[T_n \mid n \in \mathbb{N}] \subseteq \mathrm{End}_R(M),$$

i.e. the R-subalgebra of the endomorphism algebra generated by the Hecke operators.

The Hecke algebra of modular forms and Eichler-Shimura

We start by stating the Eichler-Shimura-Theorem.

7.1 Theorem. (Eichler-Shimura) *For $k \geq 2$ and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, there is an isomorphism of $\mathbb{T}_{\mathbb{Z}}(S_k(\Gamma, \mathbb{C}))$-modules, the* Eichler-Shimura isomorphism,

$$H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C})) \cong S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})}.$$

Proof. [Diamond-Im], Theorem 12.2.2. □

We draw the following conclusion from the Eichler-Shimura-Theorem.

7.2 Corollary. *In the situation of Theorem 7.1 we have natural ring isomorphisms*

$$\mathbb{T}_{\mathbb{Z}}\big(S_k(\Gamma, \mathbb{C})\big) \cong \mathbb{T}_{\mathbb{Z}}\big(H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{Z}))/\mathrm{torsion}\big).$$

Proof. The free $\mathbb{Z}$-module $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{Z}))/\mathrm{torsion}$ is a $\mathbb{Z}$-structure in the $\mathbb{C}$-vector space $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$. Any $\mathbb{Z}$-structure gives an isomorphic Hecke algebra. Finally, Theorem 7.1 implies that the Hecke algebra of $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathbb{C}))$ is isomorphic to the Hecke algebra of $S_k(\Gamma, \mathbb{C})$. □

The formula in this corollary is the reason why many people prefer to factor out the torsion of modular symbols.

7.3 Proposition. *Let $N \geq 5$, $k \geq 2$ integers and $p \nmid N$ a prime. Then we have*

$$\mathbb{T}_{\mathbb{Z}}\big(S_k(\Gamma_1(N), \mathbb{C})\big) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N), \mathbb{F}_p)\big).$$

Proof. By [Diamond-Im], Theorem 12.3.2, there is no difference between Katz modular forms over $\mathbb{F}_p$ and those that are reductions of classical modular forms whose q-expansion is in $\mathbb{Z}[1/N]$, i.e.

$$S_k(\Gamma_1(N), \mathbb{Z}[1/N]) \otimes_{\mathbb{Z}[1/N]} \mathbb{F}_p \cong S_k(\Gamma_1(N), \mathbb{F}_p).$$

Hence, the q-expansion principle gives the two perfect pairings

$$\mathbb{T}_{\mathbb{Z}}\big(S_k(\Gamma_1(N), \mathbb{C})\big) \otimes_{\mathbb{Z}} \mathbb{Z}[1/N] \times S_k(\Gamma_1(N), \mathbb{Z}[1/N]) \to \mathbb{Z}[1/N], \quad (T, f) \mapsto a_1(Tf)$$

and

$$\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N), \mathbb{F}_p)\big) \times S_k(\Gamma_1(N), \mathbb{F}_p) \to \mathbb{F}_p, \quad (T, f) \mapsto a_1(Tf).$$

Tensoring the first one with $\mathbb{F}_p$ allows us to compare it to the second one, from which the proposition follows. □

7.4 Corollary. *Let p be a prime and $N \geq 5$, $2 \leq k \leq p+2$ integers s.t. $p \nmid N$. Then sending the operator T_l to T_l for all primes l defines a surjective $\mathbb{F}_p$-algebra homomorphism*

$$\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N), \mathbb{F}_p)\big) \twoheadrightarrow \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{F}_p))\big).$$

Proof. From Corollary 7.2 we obtain because of p-torsion-freeness (Proposition 4.5) an isomorphism of $\mathbb{F}_p$-algebras

$$\mathbb{T}_{\mathbb{Z}}\big(S_k(\Gamma_1(N),\mathbb{C})\big) \otimes \mathbb{F}_p \cong \mathbb{T}_{\mathbb{Z}}\big(H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{Z}))\big) \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

By Proposition 7.3 the term on the left hand side is equal to $\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N),\mathbb{F}_p)\big)$ so that it suffices to have a surjection

$$\mathbb{T}_{\mathbb{Z}}\big(H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{Z}))\big) \otimes \mathbb{F}_p \twoheadrightarrow \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{F}_p))\big),$$

which follows from Proposition 4.5. Indeed, the isomorphism

$$H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{Z})) \otimes \mathbb{F}_p \cong H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{F}_p))$$

is compatible with Hecke operators, and allows one to define a homomorphism from the Hecke algebra on the left hand term to the one on the right hand term, which is automatically surjective by the definition of the Hecke algebra. □

7.5 Proposition. *Let $N \geq 1$, $k \geq 2$ be integers and K a field. If the characteristic of K is $p > 0$, then we assume $p \nmid N$. Furthermore, let $\Gamma_1(N) \leq \Gamma \lhd G \leq \mathrm{SL}_2(\mathbb{Z})$ be subgroups and $\epsilon : G \overset{\mathrm{proj}}{\twoheadrightarrow} \Gamma\backslash G \to R^*$ a character such that $\epsilon(-1) = (-1)^k$ if $-1 \in G$. Denote by $\mathbb{T}$ the K-Hecke algebra of $S_k(\Gamma, K)$ and by $\mathbb{T}_\epsilon$ the K-Hecke algebra of $S_k(G,\epsilon,K)$. Furthermore, let*

$$I = (\langle\delta\rangle - \epsilon(\delta) \mid \delta \in \Gamma\backslash G) \lhd \mathbb{T}.$$

Then $\mathbb{T}/I$ and $\mathbb{T}_\epsilon$ are isomorphic K-algebras.

Proof. As we work with Katz modular forms (for that we need the condition $p \nmid N$), we dispose of the q-expansion principle. Hence we have isomorphisms respecting the Hecke action $(\mathbb{T}_\epsilon)^\vee \cong S_k(G,\epsilon,K) \cong \mathbb{T}^\vee[I] \cong (\mathbb{T}/I)^\vee$, whence the proposition follows. □

Comparing Hecke algebras over $\mathbb{F}_p$

7.6 Proposition. *Let $N \geq 5$ be an integer, $p \nmid N$ a prime and $0 \leq d \leq p-1$ an integer. There exists a surjection $\mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{F}_p)(d)\big) \twoheadrightarrow \mathbb{T}_{\mathbb{F}_p}(\overline{L}(d))$ such that the diagram of $\mathbb{F}_p$-algebras*

$$\begin{array}{ccc} & & \mathbb{T}_{\mathbb{F}_p}(\overline{L}(d)) \\ & \nearrow\!\!\!\!\nearrow & \uparrow\!\!\!\uparrow \\ \mathbb{T}_{\mathbb{Z}}\big(S_2(\Gamma_1(Np),\mathbb{C})(d)\big) \otimes \mathbb{F}_p & & \\ & \searrow\!\!\!\!\searrow & \\ & & \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{F}_p)(d)\big) \end{array}$$

commutes. All maps are uniquely determined by sending the Hecke operator T_l to T_l for all primes l.

Proof. Let us first remark how the diagonal arrows are made. The lower one comes from the isomorphism (see Proposition 4.5)

$$H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{Z})\otimes\mathbb{F}_p \cong H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{F}_p).$$

The upper one is due to the fact that L is a lattice in $S_2(\Gamma_1(Np,\mathbb{C}))$, using arguments as in Corollary 7.2. As the order of $\mathbb{F}_p^*$ is invertible in $\mathbb{F}_p$, we can everywhere pass to the eigencomponents of the action of the p-part of the diamond operator $\langle\cdot\rangle_p$.

We obtain the vertical arrow by showing that the kernel of the lower diagonal map is contained in the kernel of the upper diagonal map. In other words, we will show that if a Hecke operator T in $\mathbb{T}_{\mathbb{Z}}\big(S_2(\Gamma_1(Np),\mathbb{C})(d)\big)\otimes\mathbb{F}_p$ acts as zero on $H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{F}_p)(d)$, then it acts as zero on $\overline{L}(d)$.

So assume that T acts as zero on $H^1_{\mathrm{par}}(\Gamma_1(Np),\mathbb{F}_p)(d)$. By Proposition 6.3, it acts as zero on $J_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p)[p](d)$, hence on $J_{\overline{\mathbb{Q}}_p}[p](d)$, as $J_{\mathbb{Q}_p}[p]$ is reduced. But then it also acts as zero on $J_{\mathbb{Z}_p[\zeta_p]}[p](d)$, as it acts as zero on the generic fibre using that $J[p]$ is flat over $\mathbb{Z}_p[\zeta_p]$ ([BLR], Lemma 7.3.2, as J is semi-abelian). But consequently, it also acts as zero on the special fibre $J_{\mathbb{F}_p}[p](d)$, whence also on the cotangent space $\mathrm{Cot}_0(J^0_{\mathbb{F}_p}[p])(d)$. Now Lemma 6.2 finishes the proof. $\square$

7.7 Theorem. *Let p be a prime and $2 < k \le p+1$, $N \ge 5$ integers such that $p \nmid N$. We write for short $\mathbb{T}^{\mathrm{par},N,k} := \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(N),V_{k-2}(\mathbb{F}_p))\big)$, $\mathbb{T}^{\mathrm{mod},N,k} := \mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N),\mathbb{F}_p)\big)$ and similarly for the twisted ones. Then there is the commutative diagram of $\mathbb{F}_p$-algebras*

$$\begin{array}{ccccc}
\mathbb{T}_{\mathbb{F}_p}\big(\overline{L}(k-2)\big) & \longrightarrow & \mathbb{T}^{\mathrm{mod},N,k} & \times & \mathbb{T}^{\mathrm{mod},N,p+3-k,[k-2]} \\
\uparrow & & \downarrow & & \downarrow \\
\mathbb{T}^{\mathrm{par},Np,2} & \longrightarrow & \mathbb{T}^{\mathrm{par},N,k} & \times & \mathbb{T}^{\mathrm{par},N,p+3-k,[k-2]}.
\end{array}$$

The vertical arrows are obtained from Proposition 7.6 resp. Corollary 7.4, and the horizontal ones from Proposition 6.1 and Proposition 5.5. The vertical arrows are surjective. If $2 < k \le p$, then the upper horizontal arrow is injective.

Proof. The commutativity is clear, as T_l is sent to $T_l \times T_l$ along the horizontal arrows, and T_l is sent to T_l along the vertical arrows for all primes l. The surjectivity of the vertical arrows has been proved at the places cited above. The injectivity of the upper homomorphism is the fact that $\overline{L}(k-2)$ is the direct sum of $S_k(\Gamma_1(N),\mathbb{F}_p)$ and $S_{p+3-k}(\Gamma_1(N),\mathbb{F}_p)[k-2]$, if $2 < k \le p$. $\square$

7.8 Corollary. *Let $2 < k \le p+1$, $N \ge 5$ such that $p \nmid N$. Let $\mathfrak{P}$ be a maximal ideal of the Hecke algebra $\mathbb{T}_{\mathbb{Z}}\big(S_2(\Gamma_1(Np),\mathbb{C})(d)\big)\otimes\mathbb{F}_p$ which is not in the support of $S_{p+3-k}(\Gamma_1(N),\mathbb{F}_p)$. Then we have an isomorphism*

$$\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N),\mathbb{F}_p)_{\mathfrak{P}}\big) \cong \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(N),V_{k-2}(\mathbb{F}_p))_{\mathfrak{P}}\big).$$

Proof. The assumption means that $(S_{p+3-k}(\Gamma_1(N),\mathbb{F}_p)[k-2])_{\mathfrak{P}} = 0$. Because of Corollary 7.4 we know that $\mathfrak{P}$ is not in the support of $H^1_{\mathrm{par}}(\Gamma_1(N),V_{p+1-k}(\mathbb{F}_p))[k-2]$ either, whence

$(H^1_{\mathrm{par}}(\Gamma_1(N), V_{p+1-k}(\mathbb{F}_p))[k-2])_{\mathfrak{P}} = 0$. Hence, the sequences of Proposition 5.5 and 6.1 localised at $\mathfrak{P}$ give isomorphisms $\mathbb{T}_{\mathbb{F}_p}\big(\overline{L}(k-2)\big)_{\mathfrak{P}} \cong \mathbb{T}^{\mathrm{mod},N,k}_{\mathfrak{P}}$ and $\mathbb{T}^{\mathrm{par},Np,2}_{\mathfrak{P}} \cong \mathbb{T}^{\mathrm{par},N,k}_{\mathfrak{P}}$. Hence, also the vertical maps in the localisation at $\mathfrak{P}$ of the diagram of Theorem 7.7 are isomorphisms. □

7.9 Corollary. *Let $2 < k \leq p+1$, $N \geq 5$ such that $p \nmid N$. Let $\mathfrak{P}$ be a maximal ideal of $\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N), \mathbb{F}_p)\big)$ corresponding to a normalised eigenform $f \in S_k(\Gamma_1(N), \mathbb{F}_p)$ which is* ordinary, *i.e. $a_p(f) \neq 0$. Then we have an isomorphism*

$$\mathbb{T}_{\mathbb{F}_p}\big(S_k(\Gamma_1(N), \mathbb{F}_p)_{\mathfrak{P}}\big) \cong \mathbb{T}_{\mathbb{F}_p}\big(H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(\mathbb{F}_p))_{\mathfrak{P}}\big).$$

Proof. As the operator T_p always acts as zero on $S_{p+3-k}(\Gamma_1(N), \mathbb{F}_p)[k-2]$ the maximal ideal $\mathfrak{P}$ cannot be in the support of $S_{p+3-k}(\Gamma_1(N), \mathbb{F}_p)[k-2]$, whence we are in the situation of Corollary 7.8. □

7.10 Remark. *In contrast to Proposition 6.1 the exact sequence of Proposition 5.5 is in general non-split for $d = k-2$ with $2 < k \leq p$. However, it is split for $k = 2$.*

8 Action through characters

In this section we shall extend the results from $\Gamma_1(N)$ to $\Gamma_0(N)$ together with a character of the quotient group $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$.

Let us for the sequel of this section make the following assumption.

8.1 Assumption. *Let $p \geq 5$ be a prime and K a finite field of characteristic p or $K = \overline{\mathbb{F}_p}$. Suppose, moreover, that we are given integers k and N with $3 \leq k \leq p+2$ and $p \nmid N$. Let $\Delta = \Gamma_0(N)/\Gamma_1(N)$. It acts on $H^1(\Gamma_1(N), \cdot)$, the parabolic subspace and on $S_k(\Gamma_1(N), K)$ through the diamond operators. Furthermore, let ϵ be a character of the form $\epsilon : \Gamma_0(N) \overset{\mathrm{proj}}{\twoheadrightarrow} \Gamma_0(N)/\Gamma_1(N) \to K^*$. Denote by K^ϵ the $K[\Gamma_0(N)]$-module which is a copy of K with action through ϵ^{-1}. We write V^ϵ_{k-2} for the $K[\Gamma_0(N)]$-module $V_{k-2}(K) \otimes_K K^\epsilon$. Finally, let G be the group with $\Gamma_1(N) \leq G \leq \Gamma_0(N)$ such that $G/\Gamma_1(N) = \Delta_p$, the p-Sylow subgroup of Δ.*

8.2 Lemma. *Under the Assumption 8.1 both $H^1(\Gamma_1(N), V_{k-2}(K))$ and $H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(K))$ are coinduced $K[\Delta_p]$-modules.*

Proof. We write $V = V_{k-2}(K)$ and $\Gamma := \Gamma_1(N)$. The Hochschild-Serre spectral sequence (in the notation from [Milne], Appendix B, we use the exact sequence $E^2_1 \to E^{1,1}_2 \to E^{3,0}_2$) gives the exact sequence

$$\ker\big(H^2(G, V) \to H^0(\Delta_p, H^2(\Gamma, V))\big) \to H^1(\Delta_p, H^1(\Gamma, V)) \to H^3(\Delta_p, H^0(\Gamma, V)),$$

whence $H^1(\Delta_p, H^1(\Gamma, V^\epsilon_{k-2}))$ is zero by Corollary 2.2 and Proposition 4.4.

From Proposition 2.3 it follows taking Δ_p-cohomology that the sequence

$$0 \to H^1_{\mathrm{par}}(\Gamma, V)^{\Delta_p} \to H^1(\Gamma, V)^{\Delta_p} \xrightarrow{\mathrm{res}} \Big(\prod_{g \in \langle T \rangle \backslash \mathrm{PSL}_2(\mathbb{Z})/\Gamma} H^1(\Gamma \cap \langle gTg^{-1} \rangle, V) \Big)^{\Delta_p}$$
$$\to H^1(\Delta_p, H^1_{\mathrm{par}}(\Gamma, V)) \to 0$$

is exact. It can be checked (using that p does not divide the ramification indices of the cusps) that the first three terms are

$$0 \to H^1_{\mathrm{par}}(G, V) \to H^1(G, V) \xrightarrow{\mathrm{res}} \prod_{g \in \langle T \rangle \backslash \mathrm{PSL}_2(\mathbb{Z})/G} H^1(G \cap \langle gTg^{-1} \rangle, V) \to 0,$$

where the zero on the right is again a consequence of Proposition 2.3. Hence, $H^1_{\mathrm{par}}(\Delta_p, H^1(\Gamma, V^{\epsilon}_{k-2}))$ is zero.

Finally, from $H^1(\Delta_p, \cdot) = 0$ we get by [NSW], Proposition 1.7.3(ii) that $H^1(\Gamma, V_{k-2}(K))$ and its parabolic subspace are coinduced Δ_p-modules. □

8.3 Lemma. *Under the Assumption 8.1 we have that $S_k(\Gamma_1(N), K)$ is a coinduced $K[\Delta_p]$-module.*

Proof. For the notation in this proof we follow [EdixBoston], p. 209-210 and the proof of Lemma 1.9. We let $\Gamma := \Gamma_1(N)$.

The projection $\pi : X_\Gamma \twoheadrightarrow X_G$ is a Galois cover with group Δ_p of proper K-schemes. Indeed, for the open part this is [Deligne-Rapoport], VI.2.7. Moreover, the ramification index of the cusps divides N, whence the cusps are unramified in a p-extension.

The Hochschild-Serre spectral sequence gives an injection

$$0 \to H^1(\Delta_p, H^0(X_\Gamma, \pi^*\omega^{\otimes k}(-\mathrm{cusps}))) \to H^1(X_G, \omega^{\otimes k}(-\mathrm{cusps})).$$

Using Serre duality and the Kodaira-Spencer isomorphism (see [Deligne-Rapoport], VI.4.5.2) we obtain

$$H^1(X_G, \omega^{\otimes k}(-\mathrm{cusps})) \overset{\text{S-D}}{\cong} H^0(X_G, \Omega^1 \otimes (\omega^{\otimes k}(-\mathrm{cusps}))^\vee)^\vee \overset{\text{K-S}}{\cong} H^0(X_G, \omega^{\otimes 2-k})^\vee$$

which is zero, since the degree of $\omega^{\otimes 2-k}$ is negative (as $k \geq 3$). The map π is étale and we have $H^0(X_\Gamma, \pi^*\omega^{\otimes k}(-\mathrm{cusps})) \cong S_k(\Gamma, K)$, from which $H^1(\Delta_p, S_k(\Gamma, K)) = 0$ follows. The proof is finished as in the previous lemma. □

8.4 Theorem. *We keep the Assumption 8.1.*

If $H^1_{\mathrm{par}}(\Gamma_1(N), V_{k-2}(K))$ is a faithful $\mathbb{T}_K(S_k(\Gamma_1(N), K))$-module, then $H^1_{\mathrm{par}}(\Gamma_0(N), V^{\epsilon}_{k-2})$ is a faithful $\mathbb{T}_K(S_k(\Gamma_1(N), \epsilon, K))$-module.

Proof. We claim that $N_\Delta := \sum_{\delta \in \Delta} \delta \in K[\Delta]$ induces isomorphisms

$$(S_k(\Gamma, K) \otimes_K K^\epsilon)_\Delta \to (S_k(\Gamma, K) \otimes_K K^\epsilon)^\Delta \tag{8.3}$$

and

$$H^1_{\mathrm{par}}(\Gamma, V^{\epsilon}_{k-2})_\Delta \to H^1_{\mathrm{par}}(\Gamma, V^{\epsilon}_{k-2})^\Delta. \tag{8.4}$$

We note that $H^1(\Gamma, V_{k-2}(K)) \otimes_K K^\epsilon = H^1(\Gamma, V^\epsilon_{k-2})$, since the character ϵ restricted to Γ is trivial. From Lemmas 8.2 and 8.3 we obtain that $S_k(\Gamma, K)$ and $H^1_{\mathrm{par}}(\Gamma, V^\epsilon_{k-2})$ are coinduced Δ_p-modules. This implies the claim by an elementary calculation.

Dualising Equation 8.3 gives an isomorphism

$$\big(\mathbb{T}(S_k(\Gamma, K)) \otimes K^\epsilon\big)_\Delta \xrightarrow{N_\Delta} \big(\mathbb{T}(S_k(\Gamma, K)) \otimes K^\epsilon\big)^\Delta,$$

which in particular yields the implication

$$T(\sum_{\delta \in \Delta} \epsilon(\delta)^{-1} \langle \delta \rangle) = 0 \quad \Rightarrow \quad T \in I, \tag{8.5}$$

where I is the ideal defined in Proposition 7.5. In view of that proposition, we only need to show that if T acts as zero on $H^1(G, V^\epsilon_{k-2})$, then T is in I.

The Hochschild-Serre spectral sequence yields $H^1(\Gamma, V^\epsilon_{k-2})^\Delta \cong H^1(G, V^\epsilon_{k-2})$, since $(V^\epsilon_{k-2})^\Gamma$ is zero by Proposition 4.4.

Let now T be a Hecke operator. Then we have

$$\begin{aligned} T \cdot H^1(G, V^\epsilon_{k-2}) &= T \cdot H^1(\Gamma, V^\epsilon_{k-2})^\Delta \\ &= T N_\Delta \cdot H^1(\Gamma, V^\epsilon_{k-2})_\Delta = T(\sum_{\delta \in \Delta} \epsilon(\delta)^{-1} \langle \delta \rangle) \cdot H^1(\Gamma, V_{k-2}(K)) \end{aligned}$$

Suppose that this is zero. Then the assumed faithfulness implies $T(\sum_{\delta \in \Delta} \epsilon(\delta)^{-1} \langle \delta \rangle) = 0$, which by Equation 8.5 implies $T \in I$, as required. □

8.5 Remark. *If $k = 2$, then the statements hold "outside the Eisenstein" part. The Eisenstein part is the subspace on which the Hecke algebra acts via a system of eigenvalues that does not belong to an irreducible Galois representation (see e.g. [Thesis], Lemmas 4.4.3 and 3.3.16).*

9 Application to weight one modular forms

Edixhoven explains in [EdixJussieu], Section 4, how weight one Katz modular forms over finite fields of characteristic p can be computed from the knowledge of the Hecke algebra of weigth p modular forms over the same field. We shall quickly recall that.

Let $\mathbb{F}$ be a finite field of prime characteristic p or $\overline{\mathbb{F}_p}$ and fix a level $N \geq 1$ with $p \nmid N$ and a character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{F}^*$ with $\epsilon(-1) = (-1)^k$. We have two injections of $\mathbb{F}$-vector spaces

$$F, A : S_1(\Gamma_1(N), \epsilon, \mathbb{F}) \to S_p(\Gamma_1(N), \epsilon, \mathbb{F}),$$

given on q-expansions by $a_n(Ag) = a_n(g)$ and $a_n(Fg) = a_{n/p}(g)$ (with $a_n(Fg) = 0$ if $p \nmid n$), which are compatible with all Hecke operators T_l for primes $l \neq p$. The former comes from the *Frobenius* and the latter is multiplication by the *Hasse invariant*. One has $T_p^{(p)}F = A$ and $AT_p^{(1)} = T_p^{(p)}A + \epsilon(p)F$, where we have indicated the weight as a superscript (see e.g. [EdixJussieu], Equation 4.1.2).

Let $\mathbb{T}^{(k)}$ be the Hecke algebra over $\mathbb{F}$ of weight k for a fixed level N and a fixed character ϵ. We will also indicate the weight of Hecke operators by superscripts. We denote by $A^{(p)}$ the $\mathbb{F}_p$-subalgebra of $\mathbb{T}^{(p)}$ generated by all Hecke operators $T_n^{(p)}$ for $p \nmid n$.

9.1 Proposition. *(a) There is a homomorphism Θ, called a* derivation*, which on q-expansions is given by $a_n(\Theta f) = na_n(f)$ such that the sequence*

$$0 \to S_1(\Gamma_1(N), \epsilon, \mathbb{F}) \xrightarrow{F} S_p(\Gamma_1(N), \epsilon, \mathbb{F}) \xrightarrow{\Theta} S_{p+2}(\Gamma_1(N), \epsilon, \mathbb{F})$$

is exact.

(b) Suppose $f \in S_1(\Gamma_1(N), \epsilon, \mathbb{F})$ such that $a_n(f) = 0$ for all n with $p \nmid n$. Then $f = 0$. In particular $AS_1(\Gamma_1(N), \epsilon, \mathbb{F}) \cap FS_1(\Gamma_1(N), \epsilon, \mathbb{F}) = 0$.

(c) The Hecke algebra $\mathbb{T}^{(1)}$ in weight one can be generated by all $T_l^{(1)}$, where l runs through the primes different from p.

(d) The weight one Hecke algebra $\mathbb{T}^{(1)}$ is the algebra generated by the $A^{(p)}$-action on the module $\mathbb{T}^{(p)}/A^{(p)}$.

Proof. (a) The main theorem of [KatzDerivation] gives taking Galois invariants the exact sequence

$$0 \to S_1(\Gamma_1(N), \epsilon, \mathbb{F}) \xrightarrow{F} S_p(\Gamma_1(N), \epsilon, \mathbb{F}) \xrightarrow{A\Theta} S_{2p+1}(\Gamma_1(N), \epsilon, \mathbb{F}).$$

However, as explained in [EdixJussieu], Section 4, the image $A\Theta S_p(\Gamma_1(N), \epsilon, \mathbb{F})$ in weight $2p+1$ can be divided by the Hasse invariant, whence the weight is as claimed.

(b) The condition implies by looking at q-expansions that $A\Theta f = 0$, whence by Part (3) of Katz' theorem f comes from a lower weight than 1, but below there is just the 0-form (see also [EdixJussieu], Proposition 4.4).

(c) It is enough to show that $T_p^{(1)}$ is linearly dependent on the span of all $T_n^{(1)}$ for $p \nmid n$. If it were not, then there would be a modular form of weight 1 satisfying $a_n(f) = 0$ for $p \nmid n$, but $a_p(f) \neq 0$, contradicting (b).

(d) Dualising the exact sequence in (a) yields that $\mathbb{T}^{(p)}/A^{(p)}$ and $\mathbb{T}^{(1)}$ are isomorphic as $A^{(p)}$-modules, which implies the claim. □

9.2 Proposition. *Let $N \geq 1$ and $k \geq 2$ be integers such that $p \nmid N$, $\mathbb{F}|\mathbb{F}_p$ a finite extension and let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{F}^*$ be a character with $\epsilon(-1) = (-1)^k$. Set*

$$B = \frac{N}{12} \prod_{l|N, l \text{ prime}} (1 + \frac{1}{l}).$$

(a) Then the Hecke operators $T_1^{(k)}, T_2^{(k)}, \ldots, T_{kB}^{(k)}$ generate $\mathbb{T}^{(k)}$ as an $\mathbb{F}$-vector space.

(b) The $\overline{\mathbb{F}}$-algebra $A^{(p)}$ can already be generated as an $\overline{\mathbb{F}}$-vector space by the set

$$\{ T_n^{(p)} \mid p \nmid n,\ n \leq (p+2)B \}.$$

Proof. (a) This follows from the proof of [EdixJussieu], Proposition 4.2.

(b) Assume that some $T_m^{(p)}$ for $m > (p+2)B$ and $p \nmid m$ is linearly independent of the operators in the set of the assertion. This means that there is a modular form $f \in S_p(\Gamma_1(N), \epsilon, \overline{\mathbb{F}})$ satisfying $a_n(f) = 0$ for all $n \leq (p+2)B$, but $a_m(f) \neq 0$. One gets $a_n(\Theta f) = 0$ for all $n \leq (p+2)B$, but $a_m(\Theta f) \neq 0$. This contradicts (a). □

9.3 Remark. *If we work with $\Gamma_1(N)$ and no character, the number B above has to be replaced by*

$$B' = \frac{N^2}{24} \prod_{l \mid N, l \text{ prime}} (1 - \frac{1}{l^2}).$$

9.4 Proposition. *Let $V \subset S_p(\Gamma_1(N), \epsilon, \overline{\mathbb{F}})$ be the eigenspace of a system of eigenvalues for the operators $T_l^{(p)}$ for all primes $l \neq p$*

If the system of eigenvalues does not come from a weight one form, then V is at most of dimension one. Conversely, if there is a normalised weight one eigenform g with that system of eigenvalues for $T_l^{(1)}$ for all primes $l \neq p$, then $V = \langle Ag, Fg \rangle$ and that space is 2-dimensional. On it $T_p^{(p)}$ acts with eigenvalues u and $\epsilon(p)u^{-1}$ satisfying $u + \epsilon(p)u^{-1} = a_p(g)$. In particular, the eigenforms in weight p which come from weight one are ordinary.

Proof. We choose a normalised eigenform f for all operators. If V is at least 2-dimensional, then we have $V = \mathbb{F}f \oplus \{h \mid a_n(h) = 0\ \forall p \nmid n\}$. As a form h in the right summand is annihilated by Θ, it is equal to Fg for some form g of weight one by Proposition 9.1 (a). By Part (b) of that proposition we know that $\langle Ag, Fg \rangle$ is 2-dimensional. If V were more than 2-dimensional, then there would be two different modular forms in weight 1, which are eigenforms for all $T_l^{(1)}$ with $l \neq p$. This, however, contradicts Part (c).

Assume now that V is 2-dimensional. Any normalised eigenform $f \in V$ for all Hecke operators in weight p has to be of the form $Ag + \mu Fg$ for some $\mu \in \overline{\mathbb{F}}$. The eigenvalue of $T_p^{(p)}$ on f is the p-th coefficient, hence $u = a_p(g) + \mu$, as $a_p(Fg) = a_1(g) = 1$. Now we have

$$\begin{aligned}(a_p(g)+\mu)(Ag+\mu Fg) = {} & T_p^{(p)}(Ag+\mu Fg) = & T_p^{(p)}Ag + \mu Ag \\ = {} & AT_p^{(1)}g - \epsilon(p)Fg + \mu Ag = & (a_p(g)+\mu)Ag - \epsilon(p)Fg,\end{aligned}$$

which implies $-\epsilon(p) = (a_p(g)+\mu)\mu = u^2 - ua_p(g)$ by looking at the p-th coefficient. From this one obtains the claim on u. □

9.5 Theorem. *Let $N \geq 5$ an integer and p be a prime not dividing N.*

(a) The Hecke algebra of $S_1(\Gamma_1(N), \mathbb{F}_p)$ can be computed using the first $(p+2)B'$ Hecke operators on $H^1_{\text{par}}(\Gamma_1(N), V_{p-2}(\mathbb{F}))$.

(b) Assume $p \geq 5$ and let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{F}^*$ be a character. The Hecke algebra of $S_1(\Gamma_1(N), \epsilon, \mathbb{F}_p)$ can be computed using the first $(p+2)B$ Hecke operators on $H^1_{\mathrm{par}}(\Gamma_1(N), V^{\epsilon}_{p-2}(\mathbb{F}))$.

The numbers B resp. B' were defined in Proposition 9.2 and Remark 9.3.

Proof. Corollary 7.9 implies that the ordinary part of $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p)$ is a faithful module for the ordinary part of the Hecke algebra of weight p Katz modular forms over $\mathbb{F}_p$. So that part of the Hecke algebra can be computed using the Hecke operators $T_1, \ldots, T_{pB'}$ on $H^1_{\mathrm{par}}(\Gamma_1(N), \mathbb{F}_p)$ (see Proposition 9.2(a)). From Proposition 9.4 we know that the image of the weight one forms in weight p under the Hasse invariant and Frobenius lies in the ordinary part. Proposition 9.2(b) implies that $A^{(p)}$ can be computed by the Hecke operators indicated there. Now the Hecke algebra of weight one Katz modular forms on $\Gamma_1(N)$ without a character can be computed as described in Proposition 9.1(d).

Under the extra assumption the same arguments holds also with a character using Theorem 8.4 □

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