Universität Regensburg
Mathematik

Witt Motives, Transfers and Dévissage

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Preprint Nr. 9/2005

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Introduction

In this paper we define transfer maps between Witt groups with respect to proper morphisms and establish the base change and projection formulae for those. Then we use this to define the cateory of *Witt motives*. We also deduce a dévissage theorem. In forthcoming work, these results together with the computations of Grothendieck-Witt and Witt groups of representation categories of split reductive algebraic groups (see [5, section2]) will hopefully lead to the computation of Witt groups of twisted flag varietes.

Let us fix a base field F of characteristic different from 2. In his paper [16], Panin computes the K-groups of twisted flag varietes, generalizing results of Quillen on Brauer-Severi varieties and of Swan on quadrics. To this purpose, he constructs a category of K_0-motives with nice properties. This allows him to reduce the computations to K-groups of finite-dimensional separable F-algebras and to K_0 of representation categories of split reductive algebraic groups.

This paper is part of an attempt to apply the techniques of Panin to Grothendieck-Witt groups GW and Witt groups W instead of the Grothendieck group and higher K-groups. As usual, the Grothendieck-Witt group $\mathrm{GW}(\mathcal{A})$ of an abelian category with duality $(\mathcal{A}, *)$ is defined as the Grothendieck group of isomorphism classes of symmetric spaces (and identifying metabolic spaces with the associated hyperbolic spaces if $\mathcal{A}$ is not semi-simple. Identifying the hyperbolic spaces with zero yields the Witt group $\mathrm{W}(\mathcal{A})$. Two examples we are interested in are the categories of vector bundles over a smooth F-schemes $(Vect(X), Hom_{\mathcal{O}_X}(\ \ , \mathcal{O}_X))$ and of finite-dimensional representations $(Rep(G), Hom_F(\ \ , F))$ of a reductive algebraic group G. If $X = Spec(F) = G$, these to examples coincide and yield the classical Witt group of the field F.

The situation for Witt groups is more complicated than for K-groups. In particular, it is much harder to construct reasonable *transfer* (or *norm*) morphisms between Witt groups of schemes with a given dualizing complex and with respect to proper morphisms. This is what this paper is about. Some very special cases of morphisms have been treated in [8] and [22]. Once these transfers along with some basic properties are established, we are able to construct the category of *Witt motives*. More precisely, the second section contains a construction of W-motives reminiscent to Panin's K_0-motives. As for K_0, the construction of this category relies heavily on the existence on transfers having good properties such as base change and the projection formula. For coherent Witt groups, we prove (see Lemma 2.15, Corollary 2.22 and Proposition 2.24)

Theorem 0.1. *Let $f : X \to Y$ be a proper map of relative dimension d between smooth varieties and L a line bundle on Y. Then we can construct a transfer map of degree $-d$*

$$f_* : W^{*+d}(X, f^*L \otimes_{\mathcal{O}_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$$

which satisfies the base change and projection formula with respect to flat morphisms.

This is a consequence of a more general result (see Definition 2.12 and Theorem 2.21). Observe the twists and shifts that show up. The construction of the transfer map is more tricky than one might expect as one has to keep track carefully of the dualites and isomorphisms between objects and their biduals involved. Moreover, one has to keep track of the signs and and all kind of compabilities between the triangulated structure, the duality and the monoidal structure. This is carried out in the first section. We show that the natural isomorphism from the identity to the bidual and various other isomorphisms can be constructed from an internal Hom adjoint to some tensor product. We also show that this and other constructions related to the adjointness of $\mathbf{L}f^*$, $\mathbf{R}f_*$ and $f^!$ can be carried out in a compatible way, and moreover compatible with the various sign conditions of Balmer and Gille. These verifications - though not very surprising except maybe that there is a nice choice of sign conventions - are rather long, but there is no way of avoiding them. We have presented them in a general framework. As long as possible - namely the entire section 1 - we stay in this general framework rather than appealing to known results or arguments related to varieties, dualizing complexes and Witt groups. This has at least two advantages. First, it emphasizes which of the results are formal and which depend on the special case of Witt groups and varieties. Second, section 1 may be applied to other areas of mathematics, for instance other "motivic" categories or stable homotopy theory.

Theorem 0.1 allows us to construct the category of W-motives (see section 4.3) which are more complicated but similar in spirit to Panin's K_0-motives (and Manin's classical motives). We then construct a graph functor and explain the usual structures (pseudoabelian completion, tensor product) as well as an involution on this category of W-motives.

In the last section, we use the transfers and the base change theorem to prove a devissage theorem (Theorem 3.1) for Witt groups. As a corollary, we obtain a localization exact sequence (Corollary 3.2)

$$... \to W^{n-1}(X{-}Z, j^*L) \xrightarrow{\partial} W^n(Z, f^!L) \xrightarrow{f_*} W^n(X, L) \xrightarrow{j^*} W^n(X{-}Z, j^*L) \xrightarrow{\partial} W^{n+1}(Z, f^!L) \to ...$$

which is useful for computations, of course. We believe there will be other applications of the transfers constructed in this paper.

Except the short section on dévissage which is new, this paper contains essentially section 4 and the appendix of the long preprint [5] which contains more results in the first three sections aiming at the computation of Witt groups of twisted flag varieties and a discussion of remaining difficulties. For instance, it remains to be checked that the category of Witt motives generalizes well to the H-equivariant setting for H an algebraic group, and that one can enlarge this category with respect to semisimple algebras as Panin does for K_0 (compare Remark 2.31). We hope to settle these issues in forthcoming work.

Ch. Walter has computed the (Grothendieck-)Witt groups of projective bundles in [21] by different methods, and there is work in progress by him on quadrics. Pumplün [17] has some partial results about the classical Witt group of Brauer-Severi varieties, and very recently Nenashev [15] obtained some partial results about the Witt group of the standard hyperbolic quadric. It seems that the methods of Pumplün and Nenashev do not generalize to obtain results about Witt groups of twisted flag varieties in general.

We thank Bruno Kahn for discussions about and around Grothendieck-Verdier duality and Stefan Gille for useful comments on earlier drafts.

1. Duality formalism and classical adjunctions

In this section, we obtain formal consequences of adjunctions of the type $\otimes$-$\mathcal{H}$, f^*-f_* and f_*-$f^!$ in tensor-triangulated categories that are useful for Balmer's theory of higher Witt groups. They will be applied in section 2. We present them in a general axiomatic framework which allows to distinguish between the geometric input and the formal arguments concerning adjunctions in triangulated categories. Moreover, we believe that the abstract frameworks and the results we prove about it do apply to different examples (e.g., other motivic categories or the stable homotopy category). For this reason, we have presented some aspects in slighty greater generality and provided some more results than we actually need for our applications in section 2.

Our philosophy is to exhibit a minimal axiomatic setting which can be verified without too much work in the examples of interest and from which everything can be deduced in a formal way. The example of a triangulated category to keep in mind for this paper is of course the derived category $D^b_c(X)$ of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology on a separated noetherian scheme X.

This section is rather long, and reading it might seem very unpleasant at first glance. Don't get discouraged: writing it and checking all the details one is tempted to believe anyway was even more unpleasant. Of course, you can trust us (in the spirit of [10, pp. 117-119]) and stop reading this section now. If you don't, here is a survey of its subsections. After reviewing some generalities on triangulated categories (1.1), we refine the notion of a triangulated category with duality in 1.2. In 1.3, we axiomatize the derived tensor product, the derived Hom and the adjunction between them. Then we use this to construct dualities on triangulated categories in 1.4. Section 1.5 studies functors f^*, f_* and $f^!$ with adjunctions f^*-f_* and f_*-$f^!$ and discusses in which sense these have to be compatible with the tensor-triangulated structure. The projection formula appears in 1.6. Combining everything yields to the general Theorems 1.31 and 1.32 in section 1.7 about the existence of exact functors between triangulated categories which we then may apply in section 2 (see Proposition 2.7 and Theorem 2.10). In fact, before applying this to Witt groups some details remain to be checked (see 1.8.2). In particular, we prove that there is a particular nice choice of signs. Don't forget that choosing correct signs is very important, as one wrong sign transforms the Witt group of symmetric forms to the Witt group of skew-symmetric forms.

1.1. **Generalities in triangulated categories.** We recall a few basic notions that we need in triangulated categories. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories, with translation functors $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$.

Definition 1.1. (see for example [9, § 1.1]) Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant (resp. contravariant) functor. Let $\theta : FT_{\mathcal{C}} \to T_{\mathcal{D}}F$ (resp. $\theta : T^{-1}F \to FT$) be an isomorphism of functors. We say that the pair (F, θ) is δ-exact ($\delta = \pm 1$) if for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

the triangle

$$FA \xrightarrow{Fu} FB \xrightarrow{Fv} FC \xrightarrow{\delta\theta_A \circ Fw} TFA$$

respectively

$$FC \xrightarrow{Fv} FB \xrightarrow{Fu} FA \xrightarrow{\delta T_{\mathcal{D}}(Fw\circ\theta_A)} TFC$$

is exact.

The following is a well-known lemma, but we include here a complete proof for lack of reference.

Lemma 1.2. *Let (F, f) be a covariant (resp. contravariant) δ-exact functor from $\mathcal{C}$ to $\mathcal{D}$, such that F admits a right adjoint R on the level of the underlying additive categories. Then there is a canonical way to define an isomorphism of functors $r : RT_{\mathcal{D}} \to T_{\mathcal{C}}R$ (resp. $r : T^{-1}R \to RT$) such that (R, r) is δ-exact. The same is true for a left adjoint.*

Proof: We prove the lemma in the contravariant case, for a right adjoint and for $\delta = 1$. The other cases are proved alike. The morphism $r_A : RTA \to TRA$ is the image of Id_{RTA} by the chain of isomorphisms

$$\mathrm{Hom}(RTA, RTA) \to \mathrm{Hom}(FRTA, TA) \to \mathrm{Hom}(FTT^{-1}RTA, TA)$$

$$\overset{(f^{-1})^\sharp}{\to} \mathrm{Hom}(TFT^{-1}RTA, TA) \to \mathrm{Hom}(FT^{-1}RTA, A) \to \mathrm{Hom}(T^{-1}RTA, RA)$$
$$\to \mathrm{Hom}(RTA, TRA).$$

Its inverse is obtained from Id_{RA} by the chain

$$\mathrm{Hom}(RA, RA) \to \mathrm{Hom}(FRA, A) \to \mathrm{Hom}(TFRA, TA)$$

$$\overset{f^\sharp}{\to} \mathrm{Hom}(FTRA, TA) \to \mathrm{Hom}(TRA, RTA).$$

It is easy to check (using the standard procedure as e.g. in Proposition 1.14) that these two elements are inverse to each other. We now have to show that the pair (R, r) is exact. Let

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

be an exact triangle. We want to prove that the triangle

$$RA \xrightarrow{u} RB \xrightarrow{v} RC \overset{r_A \circ Rw}{\longrightarrow} TRA$$

is exact. We first complete $RA \xrightarrow{u} RB$ as an exact triangle

$$RA \xrightarrow{u} RB \xrightarrow{v'} C' \xrightarrow{w'} TRA$$

and we prove that this triangle is in fact isomorphic to the previous one. To do so, one completes the uncomplete morphism of triangles

$$\begin{array}{ccccccc} FRA & \xrightarrow{FRu} & FRB & \xrightarrow{FRv} & FC' & \xrightarrow{f_{RA} \circ FRw} & TFRA \\ \downarrow & & \downarrow & & \vdots\, h & & \downarrow \\ A & \xrightarrow[u]{} & B & \xrightarrow[v]{} & C & \xrightarrow[w]{} & TA \end{array}.$$

Looking at the adjoint diagram, we see that $ad(h) : C' \to RC$ is an isomorphism by the five lemma for triangulated categories. □

1.2. **Weak duality.** We now explain the notion of triangulated category with weak duality. It is obtained from Balmer's definition of a triangulated category with duality by weakening the axiom $DT = T^{-1}D$ for the contravariant endofunctor (called *duality functor*) D on $\mathcal{C}$ of [3, definition 2.2]. Namely, we just assume that we have an isomorphism of functors

$$d : T^{-1}D \to DT.$$

such that the (contravariant) pair (D, d) is δ-exact, for some $\delta \in \pm 1$. By the definition of a morphism of functors and composition of those, we have the formula

$$dd = (dT^{-1}D \circ T^{-1}Dd) = (DTd \circ dDT)$$

for the natural isomorphism $dd : T^{-1}DDT \to DTT^{-1}D$. As in [9, Remark 1.1], we also get iterated versions ($d^{(2)} \neq dd$)

$$d^{(i)} : T^{-i}D \to DT^i$$

for all $i \in \mathbf{Z}$ which e. g. for $i > 0$ is given by $d^{(i)} = dT^{i-1} \circ T^{-1}d^{(i-1)}$ (or equivalently $d^{(i)} = d^{(i-1)}T \circ T^{-(i-1)}d$). One easily checks that if (D, d) is δ-exact, then (T^iDT^j, T^idT^j) is $(-1)^{i+j}\delta$-exact.

We also assume that we have an isomorphism of functors

$$\varpi : Id \to D^2$$

with the usual compatibility formula

$$D\varpi \circ \varpi D = id_D$$

and the *modified* usual compatibility formula

$$TdT^{-1}D \circ Dd \circ \varpi T = T\varpi.$$

Definition 1.3. We say that $(\mathcal{C}, D, d, \varpi)$ is a triangulated category with weak δ-duality if all the above conditions are satisfied. If $T^{-1}D = DT$ and $d = id$, we recover Balmer's usual definition (subsequently called strict duality, not to be confused with the condition $\varpi = id$).

Definition 1.4. Let $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ (resp. $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \varpi_\mathcal{D})$) be a triangulated category with weak $\delta_\mathcal{C}$-(resp. $\delta_\mathcal{D}$-) duality. Let (F, θ) be a 1-exact pair from $\mathcal{C}$ to $\mathcal{D}$ and let $\rho : FD_\mathcal{C} \to D_\mathcal{D}F$ be an isomorphism of functors. We then say that the triple (F, θ, ρ) is a duality preserving functor if the following diagrams are commutative:

(1)
$$\begin{array}{ccc} F & \xrightarrow{F\varpi_\mathcal{C}} & FD_\mathcal{C}D_\mathcal{C} \\ {\scriptstyle \varpi_\mathcal{D}F}\downarrow & & \downarrow{\scriptstyle \rho D_\mathcal{C}} \\ D_\mathcal{D}D_\mathcal{D}F & \xrightarrow[D_\mathcal{D}\rho]{} & D_\mathcal{D}FD_\mathcal{C} \end{array}$$

(2)
$$\begin{array}{ccccc} FT_\mathcal{C}D_\mathcal{C} & \xleftarrow{FT_\mathcal{C}d_\mathcal{C}T_\mathcal{C}^{-1}} & FD_\mathcal{C}T_\mathcal{C}^{-1} & \xrightarrow{\rho T_\mathcal{C}^{-1}} & D_\mathcal{D}FT_\mathcal{C}^{-1} \\ {\scriptstyle (\delta_\mathcal{C}\delta_\mathcal{D})\theta D_\mathcal{C}}\downarrow & & & & \downarrow{\scriptstyle D_\mathcal{D}T_\mathcal{D}^{-1}\theta T_\mathcal{C}^{-1}} \\ T_\mathcal{D}FD_\mathcal{C} & \xrightarrow[T_\mathcal{D}\rho]{} & T_\mathcal{D}D_\mathcal{D}F & \xleftarrow[T_\mathcal{D}d_\mathcal{D}T_\mathcal{D}^{-1}F]{} & D_\mathcal{D}T_\mathcal{D}^{-1}F \end{array}$$

Note that the first condition is the classical one (see for example [9, Definition 1.8, 1.]), and that the second is just a refinement of [9, Definition 1.8, 2.] where the special case of a strict duality is considered. The proof of the following proposition is straightforward.

Proposition 1.5. *If $(\mathcal{C}, D, d, \varpi)$ is a triangulated category with weak δ-duality, then $(\mathcal{C}, TD, Td, -\delta(TdD) \circ \varpi)$ is a triangulated category with weak $(-\delta)$-duality. Iterating, we get that $(\mathcal{C}, T^iDT^j, T^idT^j, (-1)^{(i-j)(i-j+1)/2}\delta^{i-j}(T^id^{(i)}T^jd^{(j)}) \circ \varpi)$ is a category with weak $(-1)^{i-j}\delta$-duality.*

Definition 1.6. Following Balmer's convention (see [3, Definition 2.8]), given a triangulated category with weak δ-duality $(\mathcal{C}, D, d, \varpi)$, we define a triangulated category with shifted (or translated) duality by

$$T(\mathcal{C}, D, d, \varpi) = (\mathcal{C}, TD, Td, -\delta TdD \circ \varpi).$$

Definition 1.7. For convenience, when (F, ρ) is a duality preserving functor from $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ to $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \epsilon\varpi_\mathcal{D})$, $\epsilon = \pm 1$, we say that it is a duality ϵ-preserving functor from $(\mathcal{C}, D_\mathcal{C}, d_\mathcal{C}, \varpi_\mathcal{C})$ to $(\mathcal{D}, D_\mathcal{D}, d_\mathcal{D}, \varpi_\mathcal{D})$. The composition of such functors trivially multiplies the signs.

Remark 1.8. As in the strict case, a duality ϵ-preserving functor induces duality ϵ-preserving functors on the translated categories (same ϵ).

It is possible to define symmetric spaces as usual, and to extend Balmer's definition of Witt groups to this more general setting with weak dualities.

1.3. **The functors $\mathcal{H}$ and $\otimes$.** We assume that the triangulated category $\mathcal{C}$ is endowed with an internal Hom functor, denoted by $\mathcal{H}$, and an internal tensor product denoted $\otimes$. This will be used below to make $\mathcal{C}$ into a triangulated category with duality.

We assume the $\mathcal{H}$ and $\otimes$ functors satisfies the following axioms.

Compatibility of $\mathcal{H}$ with the translation T.

(TH1) There is a functorial (in both variables A and B) isomorphism $th_{1,A,B}$: $\mathcal{H}(TA,B) \to T^{-1}\mathcal{H}(A,B)$.

(TH2) There is a functorial (in both variables A and B) isomorphism $th_{2,A,B}$: $\mathcal{H}(A,TB) \to T\mathcal{H}(A,B)$.

(TH1TH2) The following diagram is **anti**commutative.

$$\begin{array}{ccc} \mathcal{H}(TA,TB) & \xrightarrow{th_{1,A,TB}} & T^{-1}\mathcal{H}(A,TB) \\ \downarrow{\scriptstyle th_{2,TA,B}} & & \downarrow{\scriptstyle T^{-1}th_{2,A,B}} \\ T\mathcal{H}(TA,B) & \xrightarrow[Tth_{1,A,B}]{} & \mathcal{H}(A,B) \end{array}$$

Compatibility of $\otimes$ with the translation T.

(TP1) There is a functorial (in both variables A and B) isomorphism $tp_{1,A,B}$: $TA\otimes B \to T(A\otimes B)$.

(TP2) There is a functorial (in both variables A and B) isomorphism $tp_{2,A,B}$: $A\otimes TB \to T(A\otimes B)$.

(TP1TP2) The following diagram is **anti**commutative.

$$\begin{array}{ccc} TA\otimes TB & \xrightarrow{tp_{1,A,TB}} & T(A\otimes TB) \\ \downarrow{\scriptstyle tp_{2,TA,B}} & & \downarrow{\scriptstyle Ttp_{2,A,B}} \\ T(TA\otimes B) & \xrightarrow[Ttp_{1,A,B}]{} & T^2(A\otimes B) \end{array}$$

Adjunction of $\otimes$ and $\mathcal{H}$.

(ATH) We have a functorial (in A, B and C) bijection $ath_{A,B,C}$: $\mathrm{Hom}(A\otimes B,C) \to \mathrm{Hom}(A,\mathcal{H}(B,C))$.

Let $m : A \to A'$ be a morphism. For simplicity, and until Proposition 1.12, we also denote by m the induced application from $\mathrm{Hom}(A',B)$ to $\mathrm{Hom}(A,B)$ (resp. from $\mathrm{Hom}(B,A)$ to $\mathrm{Hom}(B,A')$).

Compatibility of the adjunction ath and the translation T.

(TATH12) The following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}(TA\otimes B,C) & \xrightarrow{ath} & \mathrm{Hom}(TA,\mathcal{H}(B,C)) \\ \uparrow{\scriptstyle tp} & & \uparrow{\scriptstyle T} \\ \mathrm{Hom}(T(A\otimes B),C) & & \mathrm{Hom}(A,T^{-1}\mathcal{H}(B,C)) \\ \downarrow{\scriptstyle tp} & & \uparrow{\scriptstyle th} \\ \mathrm{Hom}(A\otimes TB,C) & \xrightarrow[ath]{} & \mathrm{Hom}(A,\mathcal{H}(TB,C)) \end{array}$$

(TATH23) The following diagram is commutative.

$$\begin{CD}
\mathrm{Hom}(A\otimes TB, C) @>{ath}>> \mathrm{Hom}(A, \mathcal{H}(TB, C)) \\
@A{tp}AA @VV{th}V \\
\mathrm{Hom}(T(A\otimes B), C) @. \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \\
@A{T}AA @VV{T^{-1}th}V \\
\mathrm{Hom}(A\otimes B, T^{-1}C) @>>{ath}> \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{CD}$$

The following are consequences of the previous axioms.

(TATH13) The following diagram is commutative (combine (TATH12) and (TATH23)).

$$\begin{CD}
\mathrm{Hom}(TA\otimes B, C) @>{ath}>> \mathrm{Hom}(TA, \mathcal{H}(B, C)) \\
@A{tp}AA @AA{T}A \\
\mathrm{Hom}(T(A\otimes B), C) @. \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \\
@A{T}AA @VV{T^{-1}th}V \\
\mathrm{Hom}(A\otimes B, T^{-1}C) @>>{ath}> \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{CD}$$

(TATH12b) The following diagram is **anti**commutative (glue the diagram induced by (TP1P2) on the left of (TATH12)).

$$\begin{CD}
\mathrm{Hom}(TA\otimes B, C) @>{ath}>> \mathrm{Hom}(TA, \mathcal{H}(B, C)) \\
@V{T^{-1}tp}VV @AA{T}A \\
\mathrm{Hom}(T^{-1}(TA\otimes TB), C) @. \mathrm{Hom}(A, T^{-1}\mathcal{H}(B, C)) \\
@A{T^{-1}tp}AA @AA{th}A \\
\mathrm{Hom}(A\otimes TB, C) @>>{ath}> \mathrm{Hom}(A, \mathcal{H}(TB, C))
\end{CD}$$

(TATH23b) The following diagram is **anti**commutative (glue the diagram induced by (TH1H2) on the right of (TATH23)).

$$\begin{CD}
\mathrm{Hom}(A\otimes TB, C) @>{ath}>> \mathrm{Hom}(A, \mathcal{H}(TB, C)) \\
@A{tp}AA @VV{th}V \\
\mathrm{Hom}(T(A\otimes B), C) @. \mathrm{Hom}(A, T\mathcal{H}(TB, T^{-1}C)) \\
@A{T}AA @VV{th}V \\
\mathrm{Hom}(A\otimes B, T^{-1}C) @>>{ath}> \mathrm{Hom}(A, \mathcal{H}(B, T^{-1}C))
\end{CD}$$

Symmetric commutativity constraint of $\otimes$.

We assume that the following holds.

(CP) There is a functorial isomorphism $c_{A,B} : A\otimes B \to B\otimes A$.
(SCP) The isomorphism c satisfies $c_{B,A} \circ c_{A,B} = Id_{A\otimes B}$.

(TCP) The following diagram is commutative.

$$\begin{array}{ccc} TA\otimes B & \xrightarrow{c_{TA,B}} & B\otimes TA \\ {\scriptstyle tp_{1,A,B}}\downarrow & & \downarrow{\scriptstyle tp_{2,B,A}} \\ T(A\otimes B) & \xrightarrow[Tc_{A,B}]{} & T(B\otimes A) \end{array}$$

1.3.1. *Another possible definition of all this.* It would have been possible to define the morphisms above in a different way, which would make some axioms become definitions. Suppose that the tensor product $\otimes$, tp_1 and tp_2 are given and that they satisfy (TP1TP2). Assume $\mathcal{H}$ is also given, but not necessarily th_1 and th_2. Suppose an adjunction ath is given without any compatibility property. Then, one can define $th1$ using the diagram (TATH12) in the following way. Replace A in the diagram by $\mathcal{H}(TB,C)$, and start with $Id_{\mathcal{H}(TB,C)}$ in the lower right set. All the morphisms but the lower right one are defined, we therefore get the image of the identity in the middle right group by circling clockwise. This defines an element $th_{1,B,C}$, and by definition, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(T\mathcal{H}(TB,C)\otimes B,C) & \xrightarrow{ath} & \mathrm{Hom}(T\mathcal{H}(TB,C),\mathcal{H}(B,C)) \\ {\scriptstyle tp}\uparrow & & \uparrow{\scriptstyle T} \\ \mathrm{Hom}(T(\mathcal{H}(TB,C)\otimes B),C) & & \mathrm{Hom}(\mathcal{H}(TB,C),T^{-1}\mathcal{H}(B,C)) \\ {\scriptstyle tp}\downarrow & & \uparrow{\scriptstyle th} \\ \mathrm{Hom}(\mathcal{H}(TB,C)\otimes TB,C) & \xrightarrow[ath]{} & \mathrm{Hom}(\mathcal{H}(TB,C),\mathcal{H}(TB,C)) \end{array}$$

is commutative. Now it is easy to prove that (TATH12) is commutative for any A; to do this, we use the following easy trick. We have to show that any element f in $\mathrm{Hom}(A,\mathcal{H}(TB,C))$ is sent to the same element using both sides of the diagram. Putting the above diagram under (TATH12) and sending $Id_{\mathcal{H}(TB,C)}$ to f by the map composing by f we are done by functoriality.

Trick 1.9. Since the above proof can be applied to other commutative diagrams involving morphisms obtained by adjunctions and can be adapted to all sorts of similar versions (exchanging left and right, adding isomorphisms of functors, etc...), each time we will need such a diagram, we will just refer to the previous proof, and leave it to the reader to make the suitable modifications.

Of course, one can define th_2 by a similar technique using (TATH23) and that tp and th_1 are already defined, considering $A = T^{-1}\mathcal{H}(B,C)$. Thus the commutativity of (TATH12) and (TATH23) is true by definition and trick 1.9. It is then easy to show that th_1 and th_2 satisfy (TH1TH2), using (TP1TP2).

One can use the same trick to define tp_2 starting with tp_1 using c (symmetric) and (TCP). This shows that one may just start with

- $\otimes$, tp_1
- c
- $\mathcal{H}$
- ath

and use them to define

- tp_2
- th_1
- th_2

such that all the constraints are satisfied.

1.3.2. *Evaluation morphism.* Let A and K be objects of $\mathcal{C}$. We define the evaluation morphism

$$\mathrm{ev}_{A,K} : \mathcal{H}(A,K) \otimes A \to K$$

as the image of the identity by the adjunction (ATH).

$$\mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(A,K)) \simeq \mathrm{Hom}(\mathcal{H}(A,K)\otimes A, K)$$

Proposition 1.10. *The evaluation satisfies the equality*

(EVT1) $$\mathrm{ev}_{A,TK} = T(\mathrm{ev}_{A,K}) \circ tp_{1,\mathcal{H}(A,K),A} \circ (th_{2,A,K} \otimes Id_A)$$

Proof: Consider the following diagram of isomorphisms

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{H}(A,TK),\mathcal{H}(A,TK)) & \xleftarrow{ath} & \mathrm{Hom}(\mathcal{H}(A,TK)\otimes A, TK) \\
\uparrow{\scriptstyle th} & & \uparrow{\scriptstyle th} \\
\mathrm{Hom}(T\mathcal{H}(A,K),\mathcal{H}(A,TK)) & \xleftarrow[ath]{} & \mathrm{Hom}(T\mathcal{H}(A,K)\otimes A, TK) \\
\downarrow{\scriptstyle th} & & \uparrow{\scriptstyle tp} \\
\mathrm{Hom}(T\mathcal{H}(A,K),T\mathcal{H}(A,K)) & & \mathrm{Hom}(T(\mathcal{H}(A,K)\otimes A), TK) \\
\uparrow{\scriptstyle T} & & \uparrow{\scriptstyle T} \\
\mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(A,K)) & \xleftarrow[ath]{} & \mathrm{Hom}(\mathcal{H}(A,K)\otimes A, K)
\end{array}$$

which is commutative by the functoriality of the adjunction and (TATH13) applied to $\mathcal{H}(A,K)$, A and TK.

Start with the identity in the top left set. It is sent to the identity in the bottom left set (this is completely formal, solely due to the fact that T is a functor). These identities are respectively map to $\mathrm{ev}_{A,TK}$ in the upper left corner and $\mathrm{ev}_{A,K}$ in the lower left corner. Now the result follows from the right column. $\square$

Proposition 1.11. *The evaluation satisfies the equality*

(EVT2) $$\mathrm{ev}_{TA,K} = \mathrm{ev}_{A,K} \circ (tp_{1,T^{-1}\mathcal{H}(A,K),A})^{-1} \circ tp_{2,T^{-1}\mathcal{H}(A,K),A} \circ (th_{1,A,K} \otimes Id_{TA})$$

Proof: Similar to the previous proof, but replace (TATH13) by (TATH12) applied to $T^{-1}\mathcal{H}(A,K)$, A and K. $\square$

1.3.3. *Bidual morphism.* We now define what we call the bidual morphism

$$\varpi_{A,K} : A \to \mathcal{H}(\mathcal{H}(A,K),K)$$

as the image of the evaluation $\mathrm{ev}_{A,K}$ under the chain of bijections

$$\mathrm{Hom}(\mathcal{H}(A,K)\otimes A, K) \simeq \mathrm{Hom}(A \otimes \mathcal{H}(A,K), K) \simeq \mathrm{Hom}(A, \mathcal{H}(\mathcal{H}(A,K),K)$$

where the first one is induced by $c_{\mathcal{H}(A,K),A}$ and the second one is the adjunction.

In most applications, K will be chosen so that this morphisms is an isomorphism for all A, but formally, it has no reason to be so.

Proposition 1.12. *The bidual morphism $\varpi_{A,K}$ satisfies the formula*

($\varpi T1$) $$\varpi_{TA,K} = \mathcal{H}(th_{1,A,K},K) \circ Tth_{1,T^{-1}\mathcal{H}(A,K),K} \circ T\varpi_{A,K}.$$

Proof: We consider the following diagram.

$$
\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{H}(TA,K)\otimes TA,K) & \xrightarrow{c} & \mathrm{Hom}(TA\otimes\mathcal{H}(TA,K),K) & \xrightarrow{ath} & \mathrm{Hom}(TA,\mathcal{H}(\mathcal{H}(TA,K),K)) \\
{\scriptstyle th}\uparrow & & \uparrow{\scriptstyle th} & & \uparrow{\scriptstyle th} \\
\mathrm{Hom}(T^{-1}\mathcal{H}(A,K)\otimes TA,K) & \xrightarrow{c} & \mathrm{Hom}(TA\otimes T^{-1}\mathcal{H}(A,K),K) & \xrightarrow{ath} & \mathrm{Hom}(TA,\mathcal{H}(T^{-1}\mathcal{H}(A,K),K)) \\
{\scriptstyle tp}\uparrow & & \uparrow{\scriptstyle tp} & & \uparrow{\scriptstyle Tth} \\
\mathrm{Hom}(T(T^{-1}\mathcal{H}(A,K)\otimes A),K) & \xrightarrow{c} & \mathrm{Hom}(T(A\otimes T^{-1}\mathcal{H}(A,K)),K) & & \mathrm{Hom}(TA,T\mathcal{H}(\mathcal{H}(A,K),K)) \\
{\scriptstyle tp}\downarrow & & \downarrow{\scriptstyle tp} & & \uparrow{\scriptstyle T} \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) & \xrightarrow[c]{} & \mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) & \xrightarrow[ath]{} & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K))
\end{array}
$$

We have already seen in the proof of Proposition 1.11 that the left column send $\mathrm{ev}_{A,K}$ to $\mathrm{ev}_{TA,K}$. The top and bottom rows send respectively $\mathrm{ev}_{TA,K}$ and $\mathrm{ev}_{A,K}$ to $\varpi_{TA,K}$ and $\varpi_{A,K}$, by definition. Following what happens in the right column, we thus just have to show that the outer diagram is commutative. In fact, each inner diagram is commutative:

- the two bottom left squares because of (TCP),
- the bottom right rectangle because of (TATH12),
- the top left square because of the functoriality of the morphism c,
- the top right square because of the functoriality of the adjunction ath.

This proves the stated equality. □

Proposition 1.13. *The bidual morphism $\varpi_{A,K}$ satisfies the formulas:*

($\varpi T2a$) $$\varpi_{A,TK} = \mathcal{H}(th_{2,A,K},TK)\circ th^{-1}_{1,\mathcal{H}(A,K),TK}\circ T^{-1}th^{-1}_{2,\mathcal{H}(A,K),K}\circ\varpi_{A,K}$$

($\varpi T2b$) $$\varpi_{A,TK} = -\mathcal{H}(th_{2,A,K},TK)\circ th^{-1}_{2,T\mathcal{H}(A,K),K}\circ Tth^{-1}_{1,\mathcal{H}(A,K),K}\circ\varpi_{A,K}$$

Proof: To get the first formula, we proceed as for the previous proposition with the following diagram of isomorphism. The commutative rectangle involved is (TATH23).

$$
\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{H}(A,TK)\otimes A,TK) & \xrightarrow{c} & \mathrm{Hom}(A\otimes\mathcal{H}(A,TK),TK) & \xrightarrow{ath} & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,TK),TK)) \\
{\scriptstyle th}\uparrow & & {\scriptstyle th}\uparrow & & \uparrow{\scriptstyle th} \\
\mathrm{Hom}(T\mathcal{H}(A,K)\otimes A,TK) & \xrightarrow{c} & \mathrm{Hom}(A\otimes T\mathcal{H}(A,K),TK) & \xrightarrow{ath} & \mathrm{Hom}(A,\mathcal{H}(T\mathcal{H}(A,K),TK)) \\
{\scriptstyle tp}\uparrow & & {\scriptstyle tp}\uparrow & & \downarrow{\scriptstyle th} \\
\mathrm{Hom}(T(\mathcal{H}(A,K)\otimes A),TK) & \xrightarrow{c} & \mathrm{Hom}(T(A\otimes\mathcal{H}(A,K)),TK) & & \mathrm{Hom}(A,T^{-1}\mathcal{H}(\mathcal{H}(A,K),TK)) \\
{\scriptstyle T}\uparrow & & {\scriptstyle T}\uparrow & & \downarrow{\scriptstyle T^{-1}th} \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) & \xrightarrow[c]{} & \mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) & \xrightarrow[ath]{} & \mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K))
\end{array}
$$

The second formula is then a trivial consequence of the first one, using (TH1TH2). □

Proposition 1.14. *The bidual morphism $\varpi_{A,K}$ satisfies the formula:*

(ϖD) $$\mathcal{H}(\varpi_{A,K},K)\circ\varpi_{\mathcal{H}(A,K),K} = Id_{\mathcal{H}(A,K)}$$

Proof: Consider the following diagram, in which all vertical maps are isomorphisms. Let $f : F \to F'$. We use the notation $f^\sharp : \mathrm{Hom}(F', G) \to \mathrm{Hom}(F, G)$ and $f_\sharp : \mathrm{Hom}(G, F) \to \mathrm{Hom}(G, F')$ for the maps iduced by f.

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(A,K)) & \xleftarrow{(\mathcal{H}(\varpi_{A,K},K))_\sharp} & \mathrm{Hom}(\mathcal{H}(A,K),\mathcal{H}(\mathcal{H}(\mathcal{H}(A,K),K),K)) \\
\uparrow{\scriptstyle ath_{\mathcal{H}(A,K),A,K}} & & \uparrow{\scriptstyle ath_{\mathcal{H}(A,K),\mathcal{H}(\mathcal{H}(A,K),K),K}} \\
\mathrm{Hom}(\mathcal{H}(A,K)\otimes A,K) & \xleftarrow[(\mathcal{H}(A,K)\otimes\varpi_{A,K})^\sharp]{} & \mathrm{Hom}(\mathcal{H}(A,K)\otimes\mathcal{H}(\mathcal{H}(A,K),K),K) \\
\uparrow{\scriptstyle c_{A,\mathcal{H}(A,K)}} & & \uparrow{\scriptstyle c_{\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(A,K)}} \\
\mathrm{Hom}(A\otimes\mathcal{H}(A,K),K) & \xleftarrow[(\varpi_{A,K}\otimes\mathcal{H}(A,K))^\sharp]{} & \mathrm{Hom}(\mathcal{H}(\mathcal{H}(A,K),K)\otimes\mathcal{H}(A,K),K) \\
\downarrow{\scriptstyle ath_{A,\mathcal{H}(A,K),K}} & & \downarrow{\scriptstyle ath_{\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(A,K),K}} \\
\mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,K),K)) & \xleftarrow[(\varpi_{A,K})^\sharp]{} & \mathrm{Hom}(\mathcal{H}(\mathcal{H}(A,K),K),\mathcal{H}(\mathcal{H}(A,K),K))
\end{array}$$

Everything commutes by functoriality of ath and c. Now $Id_{\mathcal{H}(\mathcal{H}(A,K),K)}$ in the lower right set is sent to $\varpi_{\mathcal{H}(A,K),K}$ in the upper right set, which is in turn sent to $\mathcal{H}(\varpi_{A,K}, K) \circ \varpi_{\mathcal{H}(A,K),K}$ in the upper left set. But $Id_{\mathcal{H}(\mathcal{H}(A,K),K)}$ is also sent to $\varpi_{A,K}$ in the lower left set, which is sent to $Id_{\mathcal{H}(A,K)}$ in the upper left set by definition of $\varpi_{A,K}$. This proves the formula. □

1.4. **Natural structures of triangulated categories with dualities.** We now assume that K is an object such that $(\mathcal{H}(-, K), \varpi_{-,K})$ is δ_K-exact. In this assumption is included the fact that ϖ_K is an isomorphism. We introduce some notations for the following functors and morphisms of functors.

$$\begin{array}{rlcl}
D_K : & \mathcal{C} & \to & \mathcal{C} \\
& A & \mapsto & \mathcal{H}(A,K) \\
\varpi_K : & Id & \to & D_K \circ D_K \\
& (\varpi_K)_A & = & \varpi_{A,K} \\
d_K : & T^{-1}D_K & \to & D_K T \\
& (d_K)_A & = & th^{-1}_{1,A,K} \\
\rho_K : & TD_K & \to & D_{TK} \\
& (\rho_K)_A & = & th^{-1}_{2,A,K}
\end{array}$$

Theorem 1.15. *The 4-tuple $(\mathcal{C}, D_K, d_K, \varpi_K)$ is a triangulated category with weak duality. If furthermore $T^{-1}\mathcal{H}(-,K) = \mathcal{H}(T(-),K)$ and $th_{1,-,K} = Id_{\mathcal{H}(T(-),K)}$, then $(\mathcal{C}, D_K, \varpi_K)$ is a triangulated category with strict duality.*

Proof: We have to prove the relations $D_K\varpi_K \circ \varpi_K D_K = id_{D_K}$ and $Td_KT^{-1}D_K \circ D_Kd_K \circ \varpi_KT = T\varpi_K$. The first one is Proposition 1.14 and the second one is Proposition 1.12. □

Proposition 1.16. *$(\mathcal{C}, D_{TK}, d_{TK}, \varpi_{TK})$ is a triangulated category with weak δ_K-duality. If furthermore $T^{-1}\mathcal{H}(-,K) = \mathcal{H}(T(-),K)$ and $th_{1,A,K} = Id_{\mathcal{H}(TA,K)}$, then $(\mathcal{C}, D_{TK}, \varpi_{TK})$ is a triangulated category with strict duality.*

Proof: All the relations required are obtained by replacing K by TK in the previous theorem, so we just have to prove that (D_{TK}, d_{TK}) is δ_K-exact. Recall that $(\mathcal{C}, TD_K, Td_K, (Td_KD_K) \circ \varpi_K)$ is a triangulated category with duality by Proposition 1.5, so if

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

is an exact triangle, then the triangle

$$D_K(C) \xrightarrow{D_K(v)} D_K(B) \xrightarrow{D_K(u)} D_K(A) \xrightarrow{\delta_K T(D_K(w)\circ(d_K)_A)} TD_K(C)$$

is exact. Applying the functor T, we get that the triangle

$$TD_K(C) \overset{TD_K(v)}{\longrightarrow} TD_K(B) \overset{TD_K(u)}{\longrightarrow} TD_K(A) \overset{-\delta_K T^2(D_K(w)\circ(d_K)_A)}{\longrightarrow} T^2D_K(C)$$

is exact. We now use the isomorphism of triangles

$$\begin{array}{ccccccc} TD_K(C) & \xrightarrow{TD_K(v)} & TD_K(B) & \xrightarrow{TD_K(u)} & TD_K(A) & \xrightarrow{-\delta_K T^2(D_K(w)\circ(d_K)_A)} & T^2D_K(C) \\ \downarrow{\scriptstyle (\rho_K)_C} & & \downarrow{\scriptstyle (\rho_K)_B} & & \downarrow{\scriptstyle (\rho_K)_A} & & \downarrow{\scriptstyle T(\rho_K)_C} \\ D_{TK}(C) & \xrightarrow[D_{TK}(v)]{} & D_{TK}(B) & \xrightarrow[D_{TK}(u)]{} & D_{TK}(A) & \xrightarrow[\delta_K T(D_{TK}(w)\circ(d_{TK})_A)]{} & TD_{TK}(C) \end{array}$$

The two squares on the left are commutative by simple functoriality, and the square on the right is in fact T applied to the diagram

$$\begin{array}{ccccc} \mathcal{H}(A,K) & \xrightarrow{(-\delta_K)Tth^{-1}_{1,A,K}} & T\mathcal{H}(TA,K) & \xrightarrow{T\mathcal{H}(w,K)} & T\mathcal{H}(C,K) \\ \downarrow{\scriptstyle T^{-1}th^{-1}_{2,A,K}} & & \downarrow{\scriptstyle th^{-1}_{2,TA,K}} & & \downarrow{\scriptstyle th^{-1}_{2,C,K}} \\ T^{-1}\mathcal{H}(A,TK) & \xrightarrow[\delta_K th^{-1}_{1,A,TK}]{} & \mathcal{H}(TA,TK) & \xrightarrow[\mathcal{H}(w,TK)]{} & \mathcal{H}(C,TK) \end{array}$$

in which the first square is commutative by (TH1TH2) and the second is commutative by functoriality of th. □

Proposition 1.17. *The pair $(Id_{\mathcal{C}}, id, \rho_K)$ defines an isomorphism of triangulated categories with dualities from $(\mathcal{C}, TD_K, Td_K, -(Td_KD_K)\circ\varpi_K)$ to $(\mathcal{C}, D_{TK}, d_{TK}, \varpi_{TK})$.*

Proof: Let us prove the relations of Definition 1.4:

1. $\rho_K TD_K \circ (-Td_KD_K) \circ \varpi_K = D_{TK}\rho_K \circ \varpi_{TK}$
2. $T\rho_K \circ T^2 d_K T^{-1} = -Td_{TK}T^{-1} \circ \rho_K T^{-1}$ (since TD_K has sign $-\delta_K$ and D_{TK} has sign δ_K)

The first one is exactly equality ($\varpi T2b$) in Proposition 1.13 and the second one is, after translation, (TH1TH2) applied to $T^{-1}A$ and K □

Note that the first category in the previous proposition is (resp. is not) $T(\mathcal{C}, D_K, d_K, \varpi_K)$ of Definition 1.6 when $\delta_K = 1$ (resp. when $\delta_K = -1$). For convenience, we now set $\mathcal{C}_K = (\mathcal{C}, D_K, d_K, \varpi_K)$. With this notation, (Id, id, ρ_K) is a duality δ_K-preserving functor (recall Definition 1.7) from $T\mathcal{C}_K$ to $\mathcal{C}_{TK}$.

Corollary 1.18. *By induction on i, we obtain higher versions of $(Id_{\mathcal{C}}, \rho_K)$*

$$\Gamma^{(i)}_K : T^i\mathcal{C}_K \to \mathcal{C}_{T^iK}$$

by setting $\Gamma^{(1)}_K = (Id_{\mathcal{C}}, \rho_K)$ and $\Gamma^i_K = \Gamma_{T^{i-1}K} \circ T(\Gamma^{(i-1)}_K)$. By multiplications of signs, $\Gamma^{(i)}_K$ is a duality δ^i_K-preserving functor.

Proof: Follows from the previous discussion and Remark 1.8. □

Of course, all this only affects the duality transformation ρ (the underlying functor is always $Id_{\mathcal{C}}$). So these duality transformations induce isomorphisms from Witt groups to Witt groups or to skew Witt groups (changing the bidual isomorphism by (-1)) according to the sign of Γ.

Remark 1.19. If with start with a K such that $\mathcal{C}_K$ is 1-exact, then no duality (-1)-preserving functor can appear in the higher versions.

1.5. **The functors f^*, f_* and $f^!$.** We now assume that we have two categories as above, $\mathcal{C}$ and $\mathcal{D}$, that each of them is equipped with internal Hom and $\otimes$ satisfying the axioms of section 1.3. Whenever possible, we use the same notation for this data in both categories.

We assume furthermore that we have additive functors $E, G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{C}$ such that E is a left adjoint to F and G is a right adjoint to F. Of course, we are interested in the example corresponding to Corollary 2.3, that is $E = \mathbf{L}f^*$, $F = \mathbf{R}f_*$ and $G = f^!$ (which - contrary to the adjunction between the tensor product and the internal Hom - does not exist on the level of additive categories already, but only when passing to derived categories). These adjunctions are denoted aef and afg. We also assume the following.

(TE) There is a functorial isomorphism $te_A : ETA \to TEA$, such that (E, te) is a 1-exact functor.

Using Lemma 1.2, one can define an isomorphism of functors $tf_A : FTA \to TFA$ (resp. $tg_A : GTA \to TGA$) such that (F, tf) (resp. (G, tg)) is a 1-exact functor. It is easy to show using trick 1.9 that the two following diagrams are commutative, because they are exactly the ones used to find the inverse of tf (resp. tg) with $A = FB$ (resp. $A = GB$).

(AEF)

$$\begin{array}{ccccc}
\mathrm{Hom}(EA,B) & \xrightarrow{T} & \mathrm{Hom}(TEA,TB) & \xleftarrow{(te_A)^\sharp} & \mathrm{Hom}(ETA,TB) \\
\downarrow{\scriptstyle aef_{A,B}} & & & & \downarrow{\scriptstyle aef_{TA,TB}} \\
\mathrm{Hom}(A,FB) & \xrightarrow[T]{} & \mathrm{Hom}(TA,TFB) & \xrightarrow[(tf_B)_\sharp]{} & \mathrm{Hom}(TA,FTB)
\end{array}$$

(AFG)

$$\begin{array}{ccccc}
\mathrm{Hom}(FA,B) & \xrightarrow{T} & \mathrm{Hom}(TFA,TB) & \xleftarrow{(tf_A)^\sharp} & \mathrm{Hom}(FTA,TB) \\
\downarrow{\scriptstyle afg_{A,B}} & & & & \downarrow{\scriptstyle afg_{TA,TB}} \\
\mathrm{Hom}(A,GB) & \xrightarrow[T]{} & \mathrm{Hom}(TA,TGB) & \xrightarrow[(tg_B)_\sharp]{} & \mathrm{Hom}(TA,GTB)
\end{array}$$

We then assume the following.

(EP) There is a functorial (in both variables) isomorphism $ep_{A,B} : EA \otimes EB \to E(A \otimes B)$.

(TEP1) The following diagram is commutative.

$$\begin{array}{ccccc}
EA\otimes ETB & \xrightarrow{id_{EA}\otimes te_B} & EA\otimes TEB & \xrightarrow{tp_{2,EA,EB}} & T(EA\otimes EB) \\
\downarrow{\scriptstyle ep_{A,TB}} & & & & \downarrow{\scriptstyle Tep_{A,B}} \\
E(A\otimes TB) & \xrightarrow[Etp_{2,A,B}]{} & ET(A\otimes B) & \xrightarrow[te_{A\otimes B}]{} & TE(A\otimes B)
\end{array}$$

(TEP2) The following diagram is commutative.

$$\begin{array}{ccccc}
ETA\otimes EB & \xrightarrow{te_A\otimes id_{EB}} & TEA\otimes EB & \xrightarrow{tp_{1,EA,EB}} & T(EA\otimes EB) \\
\downarrow{\scriptstyle ep_{TA,B}} & & & & \downarrow{\scriptstyle Tep_{A,B}} \\
E(TA\otimes B) & \xrightarrow[Etp_{1,A,B}]{} & ET(A\otimes B) & \xrightarrow[te_{A\otimes B}]{} & TE(A\otimes B)
\end{array}$$

(EPC) The following diagram is commutative.

$$\begin{array}{ccc} EA\otimes EB & \xrightarrow{ep_{A,B}} & E(A\otimes B) \\ {\scriptstyle c_{EA,EB}}\downarrow & & \downarrow{\scriptstyle Ec_{A,B}} \\ EB\otimes EA & \xrightarrow[ep_{B,A}]{} & E(B\otimes A) \end{array}$$

Remark 1.20. Note that (TEP1) and (EPC) imply (TEP2), using (TCP).

Definition 1.21. Let x_A be the counit of aef, that is the image of Id_{FA} by the adjunction $\mathrm{Hom}(FA, FA) \to \mathrm{Hom}(EFA, A)$. Let $x_{A,B}$ be the element $x_A\otimes x_B$ in $\mathrm{Hom}(EFA\otimes EFB, A\otimes B)$. We denote by $fp_{A,B}$ the image of $x_{A,B}$ by the chain of morphisms

$$\mathrm{Hom}(EFA\otimes EFB, A\otimes B) \xrightarrow{(ep^{-1}_{FA,FB})^\sharp} \mathrm{Hom}(E(FA\otimes FB), A\otimes B) \xrightarrow{aef} \mathrm{Hom}(FA\otimes FB, F(A\otimes B))$$

Notice that we have used the fact that ep is an isomorphism, so we cannot go on with the same procedure to define a similar morphism from $GA\otimes GB$ to $G(A\otimes B)$ since there is no reason for fp to be an isomorphism (and it is of course not an isomorphism in the classical examples).

The following is a consequence of (EPC) and the adjunction of E and F:

(FPC) The following diagram is commutative.

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{fp_{A,B}} & F(A\otimes B) \\ {\scriptstyle c_{FA,FB}}\downarrow & & \downarrow{\scriptstyle Fc_{A,B}} \\ FB\otimes FA & \xrightarrow[fp_{B,A}]{} & F(B\otimes A) \end{array}$$

We will see below that this definition also implies that the same diagrams as (TEP1) and (TEP2) are still commutative after having replaced E by F, te by tf and ep by fp. Explicitly, we get the following commutative diagrams.

(TFP1) The following diagram is commutative.

$$\begin{array}{ccccc} FA\otimes FTB & \xrightarrow{id_{FA}\otimes tf_B} & FA\otimes TFB & \xrightarrow{tp_{2,FA,FB}} & T(FA\otimes FB) \\ {\scriptstyle fp_{A,TB}}\downarrow & & & & \downarrow{\scriptstyle Tfp_{A,B}} \\ F(A\otimes TB) & \xrightarrow[Ftp_{2,A,B}]{} & FT(A\otimes B) & \xrightarrow[tf_{A\otimes B}]{} & TF(A\otimes B) \end{array}$$

(TFP2) The following diagram is commutative.

$$\begin{array}{ccccc} FTA\otimes FB & \xrightarrow{tf_A\otimes id_{FB}} & TFA\otimes FB & \xrightarrow{tp_{1,FA,FB}} & T(FA\otimes FB) \\ {\scriptstyle fp_{TA,B}}\downarrow & & & & \downarrow{\scriptstyle Tfp_{A,B}} \\ F(TA\otimes B) & \xrightarrow[Ftp_{1,A,B}]{} & FT(A\otimes B) & \xrightarrow[tf_{A\otimes B}]{} & TF(A\otimes B) \end{array}$$

To establish the commutativity of (TFP1) is equivalent to showing that $fp_{TA,B} \in Hom(FA\otimes FTB, F(A\otimes TB))$ resp. $fp_{A,B} \in Hom(FA\otimes FB, F(A\otimes B))$ are mapped to the same element in $Hom(FA\otimes FTB, TF(A\otimes B))$ under $tf_\sharp\circ tp_\sharp$ resp. $T\circ tf^\sharp\circ tp^\sharp$. By definition we have that $aef\circ(ep^{-1})^\sharp(x_{A,TB}) = fp_{A,TB}$ and $aef\circ(ep^{-1})^\sharp(x_{A,B}) = fp_{A,B}$. Using all kind of functorialities and (AEF) and (TEP1), we are reduced to show that $x_{A,B}$ maps to $x_{A,TB}$ under $T\circ tf^\sharp\circ te^\sharp\circ tp^\sharp\circ(tp^{-1})_\sharp$. This follows using the definition of tf and again functorialities. The commutativity of (TFP2) can be proved in a similar way.

Definition 1.22. Let $eh_{A,B} : E\mathcal{H}(A,B) \to \mathcal{H}(EA,EB)$ denote the image of $Id_{\mathcal{H}(A,B)}$ by the chain of morphisms

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{H}(A,B),\mathcal{H}(A,B)) & & & & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(A,B)\otimes A,B) & \xrightarrow[E]{} & \mathrm{Hom}(E(\mathcal{H}(A,B)\otimes A),EB) & \xrightarrow[(ep_{\mathcal{H}(A,B),A})^\sharp]{} & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB)
\end{array}$$

We define $fh_{A,B} : F\mathcal{H}(A,B) \to \mathcal{H}(FA,FB)$ the same way, replacing E by F.

Applying trick 1.9, we immediately get that the following diagram is commutative

(EHP)

$$\begin{array}{ccccc}
\mathrm{Hom}(A,\mathcal{H}(B,C)) & \xrightarrow{E} & \mathrm{Hom}(EA,E\mathcal{H}(B,C)) & \xrightarrow{(eh_{B,C})_\sharp} & \mathrm{Hom}(EA,\mathcal{H}(EB,EC)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(A\otimes B,C) & \xrightarrow[E]{} & \mathrm{Hom}(E(A\otimes B),EC) & \xrightarrow[(ep_{A,B})^\sharp]{} & \mathrm{Hom}(EA\otimes EB,EC)
\end{array}$$

We also get the same commutative diagram with E, eh and ep replaced by F, fh and fp.

(FHP)

$$\begin{array}{ccccc}
\mathrm{Hom}(A,\mathcal{H}(B,C)) & \xrightarrow{F} & \mathrm{Hom}(FA,F\mathcal{H}(B,C)) & \xrightarrow{(fh_{B,C})_\sharp} & \mathrm{Hom}(FA,\mathcal{H}(FB,FC)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(A\otimes B,C) & \xrightarrow[F]{} & \mathrm{Hom}(F(A\otimes B),FC) & \xrightarrow[(fp_{A,B})^\sharp]{} & \mathrm{Hom}(FA\otimes FB,FC)
\end{array}$$

It is also easy to show that the following diagrams are commutative, using (TEP1) (resp. (TEP2)) and (ATH) and proceeding similar to the argument that establishes (TFP1) using (TEP1) and (AEF).

(TEH1)

$$\begin{array}{ccccc}
\mathcal{H}(ET^{-1}A,EB) & \longrightarrow & \mathcal{H}(T^{-1}EA,EB) & \longrightarrow & T\mathcal{H}(EA,EB) \\
\uparrow & & & & \uparrow \\
E\mathcal{H}(T^{-1}A,B) & \longrightarrow & ET\mathcal{H}(A,B) & \longrightarrow & TE\mathcal{H}(A,B)
\end{array}$$

(TEH2)

$$\begin{array}{ccccc}
\mathcal{H}(EA,ETB) & \longrightarrow & \mathcal{H}(EA,TEB) & \longrightarrow & T\mathcal{H}(EA,EB) \\
\uparrow & & & & \uparrow \\
E\mathcal{H}(A,TB) & \longrightarrow & ET\mathcal{H}(A,B) & \longrightarrow & TE\mathcal{H}(A,B)
\end{array}$$

Similarly, using (TFP1) and (TFP2) we obtain

(TFH1)

$$\begin{array}{ccccc}
\mathcal{H}(FT^{-1}A,FB) & \longrightarrow & \mathcal{H}(T^{-1}FA,FB) & \longrightarrow & T\mathcal{H}(FA,FB) \\
\uparrow & & & & \uparrow \\
F\mathcal{H}(T^{-1}A,B) & \longrightarrow & FT\mathcal{H}(A,B) & \longrightarrow & TF\mathcal{H}(A,B)
\end{array}$$

(TFH2)

$$\begin{array}{ccccc}
\mathcal{H}(FA,FTB) & \longrightarrow & \mathcal{H}(FA,TFB) & \longrightarrow & T\mathcal{H}(FA,FB) \\
\uparrow & & & & \uparrow \\
F\mathcal{H}(A,TB) & \longrightarrow & FT\mathcal{H}(A,B) & \longrightarrow & TF\mathcal{H}(A,B)
\end{array}$$

Proposition 1.23. *The morphisms $eh_{A,B}$ make the diagram*

$$\begin{array}{ccc}
EA & \xrightarrow{E\varpi_B} & E\mathcal{H}(\mathcal{H}(A,B),B) \\
{\scriptstyle \varpi_{EB}}\downarrow & & \downarrow{\scriptstyle eh_{\mathcal{H}(A,B),B}} \\
\mathcal{H}(\mathcal{H}(EA,EB),EB) & \xrightarrow[\mathcal{H}(eh_{A,B},EB)]{} & \mathcal{H}(E\mathcal{H}(A,B),EB)
\end{array}$$

commutative, in other words

$$eh_{\mathcal{H}(A,B),B} \circ E\varpi_B = \mathcal{H}(eh_{A,B},B) \circ \varpi_{EB}$$

Proof: This amounts, after applying the functor $\mathcal{H}(EA,-)$, to proving that Id_{EA} is sent to the same element by the two paths in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(EA,EA) & \xrightarrow{(E\varpi_B)_\sharp} & \mathrm{Hom}(EA,E\mathcal{H}(\mathcal{H}(A,B),B)) \\
{\scriptstyle (\varpi_{EB})_\sharp}\downarrow & & \downarrow{\scriptstyle (eh_{\mathcal{H}(A,B),B})_\sharp} \\
\mathrm{Hom}(EA,\mathcal{H}(\mathcal{H}(EA,EB),EB)) & \xrightarrow[\mathcal{H}(eh_{A,B},B)_\sharp]{} & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB))
\end{array}$$

Glueing under this last diagram the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(EA,\mathcal{H}(\mathcal{H}(EA,EB),EB)) & \xrightarrow{eh} & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB)) \\
{\scriptstyle ath}\uparrow & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(EA\otimes\mathcal{H}(EA,EB),EB) & \xrightarrow{eh} & \mathrm{Hom}(EA\otimes E\mathcal{H}(A,B),EB) \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c} \\
\mathrm{Hom}(\mathcal{H}(EA,EB)\otimes EA,EB) & \xrightarrow{eh} & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB) \\
{\scriptstyle ath}\downarrow & & \downarrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(EA,EB),\mathcal{H}(EA,EB)) & \xrightarrow[eh]{} & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB))
\end{array}$$

in which all vertical maps are isomorphisms, one sees that id_{EA} is sent to $eh_{A,B}$ in the lower right set by definition of ϖ_{EB}. We now prove that id_{EA} is again sent to this element $eh_{A,B}$ using the other path in the first diagram. Consider the diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}(A,\mathcal{H}(\mathcal{H}(A,B),B)) & \xrightarrow{E} & \mathrm{Hom}(EA,E\mathcal{H}(\mathcal{H}(A,B),B)) & \xrightarrow{(eh_{\mathcal{H}(A,B),B})^\sharp} & \mathrm{Hom}(EA,\mathcal{H}(E\mathcal{H}(A,B),EB)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(A\otimes\mathcal{H}(A,B),B) & \xrightarrow{E} & \mathrm{Hom}(E(A\otimes\mathcal{H}(A,B)),EB) & \xrightarrow{(ep_{A,\mathcal{H}(A,B)})^\sharp} & \mathrm{Hom}(EA\otimes E\mathcal{H}(A,B),EB) \\
\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} \\
\mathrm{Hom}(\mathcal{H}(A,B)\otimes A,B) & \xrightarrow{E} & \mathrm{Hom}(E(\mathcal{H}(A,B)\otimes A),EB) & \xrightarrow{(ep_{\mathcal{H}(A,B),A})^\sharp} & \mathrm{Hom}(E\mathcal{H}(A,B)\otimes EA,EB) \\
\downarrow{\scriptstyle ath} & & & & \downarrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(A,B),\mathcal{H}(A,B)) & \xrightarrow[E]{} & \mathrm{Hom}(E\mathcal{H}(A,B),E\mathcal{H}(A,B)) & \xrightarrow[(eh_{A,B})_\sharp]{} & \mathrm{Hom}(E\mathcal{H}(A,B),\mathcal{H}(EA,EB))
\end{array}
$$

in which the top and bottom rectangles are commutative because of (EHP), the middle left one because of the functoriality of E and the middle right one because of (EPC). Starting with $id_{\mathcal{H}(A,B)}$ in the lower left corner, we end up with $\varpi_{A,B}$ in the upper left corner, with $eh_{A,B}$ in the lower right corner, wich proves our claim. □

Definition 1.24. We define $\alpha_{A,B} : F\mathcal{H}(A,GB) \to \mathcal{H}(FA,B)$ as the image of $id_{\mathcal{H}(A,GB)}$ by the chain of morphisms

$$
\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) & & & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) & \xrightarrow{afg^{-1}} & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),B) & \xrightarrow{(fp_{\mathcal{H}(A,GB),A})^\sharp} & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B)
\end{array}
$$

Verdier [20, Proposition 3] defines a natural isomorphism with similar source and target. In our setting, his definition becomes the following.

Definition 1.25. We define $\tilde{\alpha}$ by the composition

$$\tilde{\alpha} : F\mathcal{H}(A,GB) \xrightarrow{fh} \mathcal{H}(FA,FGB)) \xrightarrow{Tr} \mathcal{H}(FA,B)$$

where $Tr : FG \to id$ is the counit of the adjunction afg between F and G.

Verdier uses the projection formula to show that $\tilde{\alpha}$ is an isomorphism. Below, we will use the projection formula to show that α is also an isomorphism. But anyway, we have the following.

Proposition 1.26. *The two natural isomorphisms $\alpha = \tilde{\alpha}$ are equal.*

Proof: We have to prove that the following diagram commutes.

$$
\begin{array}{ccccc}
\mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) & & \xrightarrow{\qquad fh \qquad} & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,FGB)) \\
\uparrow{\scriptstyle F} & & & & \downarrow{\scriptstyle Tr} \\
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) & & & & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) \\
\uparrow{\scriptstyle ath} & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) & \xrightarrow[afg]{} & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),B) & \xrightarrow[fp]{} & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B)
\end{array}
$$

Using that $Tr \circ F = afg$ and that Tr and fp are applied to different variables and thus commute, we are reduced to show the commutativity of

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{H}(A,GB),\mathcal{H}(A,GB)) & \xrightarrow{F} & \mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) & \xrightarrow{fh} & \mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,FGB)) \\
{\scriptstyle ath}\uparrow & & & & \downarrow{\scriptstyle ath} \\
\mathrm{Hom}(\mathcal{H}(A,GB)\otimes A,GB) & \xrightarrow[F]{} & \mathrm{Hom}(F(\mathcal{H}(A,GB)\otimes A),FGB) & \xrightarrow[fp]{} & \mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,FGB)
\end{array}$$

which is (FHP) applied to $\mathcal{H}(A,GB)$, A and GB. □

Thus we no longer distinguish between α and $\tilde{\alpha}$ Applying trick 1.9, we immediately get the commutative diagram from Definition 1.24

(GHP)

$$\begin{array}{ccccc}
\mathrm{Hom}(A,\mathcal{H}(B,GC)) & \xrightarrow{F} & \mathrm{Hom}(FA,F\mathcal{H}(B,GC)) & \xrightarrow{(\alpha_{B,C})_\sharp} & \mathrm{Hom}(FA,\mathcal{H}(FB,C)) \\
{\scriptstyle ath}\uparrow & & & & \uparrow{\scriptstyle ath} \\
\mathrm{Hom}(A\otimes B,GC) & \xrightarrow[afg]{} & \mathrm{Hom}(F(A\otimes B),C) & \xrightarrow[(fp_{A,B})^\sharp]{} & \mathrm{Hom}(FA\otimes FB,C)
\end{array}$$

Diagram (TFH1) and Proposition 1.26 imply that the following diagram is commutative.

(TFG)

$$\begin{array}{ccccc}
FT\mathcal{H}(A,GB) & \xleftarrow{FTth^{-1}T^{-1}} & F\mathcal{H}(T^{-1}A,GB) & \xrightarrow{\alpha} & \mathcal{H}(FT^{-1}A,B) \\
{\scriptstyle tf}\downarrow & & & & \downarrow{\scriptstyle T^{-1}tfT^{-1}} \\
TF\mathcal{H}(A,GB) & \xrightarrow[\alpha]{} & T\mathcal{H}(FA,B) & \xleftarrow[Tth^{-1}T^{-1}]{} & \mathcal{H}(T^{-1}FA,B)
\end{array}$$

Proposition 1.27. *The morphisms $\alpha_{A,B}$ make the diagram*

$$\begin{array}{ccc}
FA & \xrightarrow{F\varpi_{A,GB}} & F\mathcal{H}(\mathcal{H}(A,GB),GB) \\
{\scriptstyle \varpi_{FA,B}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{H}(A,GB),B}} \\
\mathcal{H}(\mathcal{H}(FA,B),B) & \xrightarrow[\mathcal{H}(\alpha_{A,B},B)]{} & \mathcal{H}(F\mathcal{H}(A,GB),B)
\end{array}$$

commutative.

Proof: The proof is the same as for Proposition 1.23, but replacing E by F, B by GB, eh by $\alpha = Tr \circ fh$, (EHP) by (GHP), (EPC) by (FPC) and functoriality of E by functoriality of afg. □

1.6. **The projection formula and its consequences.** Let u be the unit of the adjunction aef. We now assume that the morphism $q_{A,B} = u \circ fp : A \otimes FB \to F(EA \otimes B)$ obtained by adjunction from $id_{EA\otimes B}$ is an isomorphism for all A and B.

Definition 1.28. We define $\beta_{A,B} : \mathcal{H}(FA,B) \to F\mathcal{H}(A,GB)$ as the image of $id_{\mathcal{H}(A,GB)}$ by the chain of morphisms

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{H}(FA,B),\mathcal{H}(FA,B)) & & \mathrm{Hom}(\mathcal{H}(FA,B),F\mathcal{H}(A,GB)) \\
\Big\downarrow{\scriptstyle ath^{-1}} & & \Big\uparrow{\scriptstyle aef} \\
\mathrm{Hom}(\mathcal{H}(FA,B)\otimes FA,B) & & \mathrm{Hom}(E\mathcal{H}(FA,B),\mathcal{H}(A,GB)) \\
\Big\downarrow{\scriptstyle (q^{-1}_{\mathcal{H}(FA,B),A})^\sharp} & & \Big\uparrow{\scriptstyle ath} \\
\mathrm{Hom}(F(E\mathcal{H}(FA,B)\otimes A),B) & \xrightarrow{afg} & \mathrm{Hom}(E\mathcal{H}(FA,B)\otimes A,GB)
\end{array}
$$

Proposition 1.29. *The morphisms $\alpha_{A,B}$ and $\beta_{A,B}$ are inverse to each other.*

Proof: The proof that $\alpha_{A,B} \circ \beta_{A,B} = id_{\mathcal{H}(FA,B)}$ follows as in the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(F\mathcal{H}(A,GB),F\mathcal{H}(A,GB)) & \xrightarrow{(\beta_{A,B})^\sharp} & \mathrm{Hom}(\mathcal{H}(FA,B),F\mathcal{H}(A,GB)) \\
\Big\downarrow{\scriptstyle aef^{-1}} & & \Big\downarrow{\scriptstyle aef^{-1}} \\
\mathrm{Hom}(EF\mathcal{H}(A,GB),\mathcal{H}(A,GB)) & \xrightarrow{\beta} & \mathrm{Hom}(E\mathcal{H}(FA,B),\mathcal{H}(A,GB)) \\
\Big\downarrow{\scriptstyle ath^{-1}} & & \Big\downarrow{\scriptstyle ath^{-1}} \\
\mathrm{Hom}(EF\mathcal{H}(A,GB)\otimes A,GB) & \xrightarrow{\beta} & \mathrm{Hom}(E\mathcal{H}(FA,B)\otimes A,GB) \\
\Big\downarrow{\scriptstyle afg^{-1}} & & \Big\downarrow{\scriptstyle afg^{-1}} \\
\mathrm{Hom}(F(EF\mathcal{H}(A,GB)\otimes A),B) & \xrightarrow{\beta} & \mathrm{Hom}(F(E\mathcal{H}(FA,B)\otimes A),B) \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\
\mathrm{Hom}(F\mathcal{H}(A,GB)\otimes FA,B) & \xrightarrow{\beta} & \mathrm{Hom}(\mathcal{H}(FA,B)\otimes FA,B) \\
\Big\downarrow{\scriptstyle ath} & & \Big\downarrow{\scriptstyle ath} \\
\mathrm{Hom}(F\mathcal{H}(A,GB),\mathcal{H}(FA,B)) & \xrightarrow[(\beta_{A,B})^\sharp]{} & \mathrm{Hom}(\mathcal{H}(FA,B),\mathcal{H}(FA,B))
\end{array}
$$

the element id in the upper left corner is mapped to α in the lower left (since $u^\sharp = aef^{-1} \circ F$ and all morphism are natural with respect to u) and to β in the upper right corner, which are both mapped to id in the lower right corner. The fact that $\beta_{A,B} \circ \alpha_{A,B} = id_{F\mathcal{H}(A,GB)}$ comes again from the same kind of reasoning applied to the same diagram with β instead of α.

□

Remark 1.30. Looking at the proof of the previous proposition, one can in fact weaken the assumptions, and just require that $q_{F\mathcal{H}(A,GB),A}$ and $q_{\mathcal{H}(FA,B),A}$ are isomorphisms for a certain A and B, and the proposition will still hold for these particular A and B.

1.7. **Natural functors of triangulated categories with duality.**

Recall that (E, te) is a 1-exact functor from $\mathcal{C}$ to $\mathcal{D}$ and that (F, tf) is a 1-exact functor from $\mathcal{D}$ to $\mathcal{C}$. We denote by $\lambda_K = e_{,K}$ the morphism of functors

$$
\begin{array}{rccc}
\lambda_K : & ED_K & \to & D_{EK}E \\
eh_{A,K} : & E\mathcal{H}(A,K) & \mapsto & \mathcal{H}(EA,EK).
\end{array}
$$

Name	context
$th_{1,A,B}$	$\mathcal{H}(TA,B) \to T^{-1}\mathcal{H}(A,B)$
$th_{2,A,B}$	$\mathcal{H}(A,TB) \to T\mathcal{H}(A,B)$
$tp_{1,A,B}$	$TA\otimes B \to T(A\otimes B)$
$tp_{2,A,B}$	$A\otimes TB \to T(A\otimes B)$
$ath_{A,B,C}$	$\mathrm{Hom}(A\otimes B,C) \to \mathrm{Hom}(A,\mathcal{H}(B,C))$
$c_{A,B}$	$A\otimes B \to B\otimes A$
$\mathrm{ev}_{A,K}$	$\mathcal{H}(A,K)\otimes K \to K$
$\varpi_{A,K}$	$A \to \mathcal{H}(\mathcal{H}(A,K),K)$
$aef_{A,B}$	$\mathrm{Hom}(EA,B) \to \mathrm{Hom}(A,FB)$
$afg_{A,B}$	$\mathrm{Hom}(FA,B) \to \mathrm{Hom}(A,GB)$
te_A	$ETA \to TEA$
tf_A	$FTA \to TFA$
tg_A	$GTA \to TGA$
$ep_{A,B}$	$EA\otimes EB \to E(A\otimes B)$
$fp_{A,B}$	$FA\otimes FB \to F(A\otimes B)$
$eh_{A,B}$	$E\mathcal{H}(A,B) \to \mathcal{H}(EA,EB)$
$fh_{A,B}$	$F\mathcal{H}(A,B) \to \mathcal{H}(FA,FB)$
$\alpha_{A,B}$	$F\mathcal{H}(A,GB) \to \mathcal{H}(FA,B)$
$\beta_{A,B}$	$\mathcal{H}(FA,B) \to F\mathcal{H}(A,GB)$
$q_{A,B}$	$A\otimes FB \to F(EA\otimes B)$

TABLE 1. Summary of the different definitions

Theorem 1.31. *Let K be an object of $\mathcal{C}$ such that ϖ_K and λ_K are isomorphisms of functors, and such that (D_K, d_K) and (D_{EK}, d_{EK}) are δ-exact (same δ). Then the triple (E, te, λ_K) is a functor of triangulated categories with dualities from $(\mathcal{C}, D_K, d_K, \varpi_K)$ to $(\mathcal{D}, D_{EK}, d_{EK}, \varpi_{EK})$.*

Proof: We have to check that λ_K satisfies the diagrams of Definition 1.4. The first one is Proposition 1.23 and the second one is (TEH1). □

Theorem 1.32. *Let L be an object of $\mathcal{D}$ such that ϖ_{GL} and ϖ_L are isomorphisms, (D_L, d_L) and $(D_{GL}, d_G L)$ are δ-exact (same δ), and such that $q_{FD_{GL}(A),A}$ and $q_{D_L F(A),A}$ are isomorphism for all A and set $\alpha_L = \alpha_{\ ,L}$. Then the triple (F, tf, α_L) is a functor of triangulated categories with dualities from $(\mathcal{D}, D_{GL}, d_{GL}, \varpi_{GL})$ to $(\mathcal{C}, D_L, d_L, \varpi_L)$.*

Proof: Proposition 1.29 and Remark 1.30 ensure that α_L is an isomorphism of functors. We then have to check that α_L satisfies the diagrams of Definition 1.4. The first one is Proposition 1.27 and the second one is (TFG). □

1.8. **An example for section 1.3.**

1.8.1. *Category of complexes.* Let $\mathcal{A}$ be an additive category with an internal Hom (denoted by h) and an internal tensor product (denoted by $\bullet$) with a commutativity constraint and an adjunction between $\bullet$ and h additive and functorial in the three variables. We now show that the both the categories of bounded and unbounded complexes of objects of $\mathcal{A}$ can be equipped with an internal Hom $\mathcal{H}$ and an internal tensor product $\otimes$ satisfying the axioms of section 1.3. This is essentially a problem of choosing some signs, and as explained in section 1.3.1, some choices determine the others.

We work with homological differentials, *i.e.*

$$d_i^A : A_i \to A_{i-1}.$$

The groups defining the translation functor are

$$(TA)_n = A_{n-1}.$$

The groups in the tensor product and the internal Hom are given by

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \bullet B_j$$

and

$$\mathcal{H}(A,B)_n = \prod_{j-i=n} h(A_i, B_j).$$

In table 2 can be read where we put which sign in our definitions of the different groups and morphisms. As a general rule, the indices in the sign correspond to the groups from which the morphism with this sign starts.

Definition of	Sign	Locus
TA	ϵ_i^T	$d_{i+1}^{TA} = \epsilon_i^T d_i A$
$A \otimes B$	$\epsilon_{i,j}^{1\otimes}$	$\epsilon_{i,j}^{1\otimes} d_i^A \bullet id_{B_j}$
	$\epsilon_{i,j}^{2\otimes}$	$\epsilon_{i,j}^{2\otimes} id_{A_i} \bullet d_j^B$
$\mathcal{H}(A,B)$	$\epsilon_{i,j}^{1\mathcal{H}}$	$\epsilon_{i,j}^{1\mathcal{H}} (d_{i+1}^A)^\sharp$
	$\epsilon_{i,j}^{2\mathcal{H}}$	$\epsilon_{i,j}^{2\mathcal{H}} (d_j^B)_\sharp$
$tp_{1,A,B}$	$\epsilon_{i,j}^{tp1}$	$\epsilon_{i,j}^{tp1} id_{A_i \bullet B_j}$
$tp_{2,A,B}$	$\epsilon_{i,j}^{tp2}$	$\epsilon_{i,j}^{tp2} id_{A_i \bullet B_j}$
$th_{1,A,B}$	$\epsilon_{i,j}^{th1}$	$\epsilon_{i,j}^{th1} id_{h(A_i,B_j)}$
$th_{2,A,B}$	$\epsilon_{i,j}^{th2}$	$\epsilon_{i,j}^{th2} id_{h(A_i,B_j)}$
$ath_{A,B,C}$	$\epsilon_{i,j}^{ath}$	$\epsilon_{i,j}^{ath}(\mathrm{Hom}(A_i \bullet B_j, C_{i+j}) \to \mathrm{Hom}(A_i, h(B_j, C_{i+j})))$
$c_{A,B}$	$\epsilon_{i,j}^{c}$	$\epsilon_{i,j}^{c}(A_i \bullet B_j \to B_j \bullet A_i)$

TABLE 2. Sign definitions

In table 3, we state the compatibility that these signs must satisfy for the axioms to be true.

As the discussion in section 1.3.1 suggests, some equalities are consequences of other ones. It is also easy to see that $(1,4,6) \Rightarrow 7$ and $(2,10,12) \Rightarrow 8$. If you assume 19, then 17 and 18 are equivalent.

Balmer, Gille and Nenashev [3], [4],[6], [9] always consider strict dualities, that is $\epsilon^{th1} = 1$. The signs choosen in [4, §2.6] imply that $\epsilon_{i,0}^{1\mathcal{H}} = 1$. The choices made by [9, Example 1.4] are $\epsilon_{i,j}^{1\otimes} = 1$ and $\epsilon_{i,j}^{2\otimes} = (-1)^i$. In [6, p. 111] the signs $\epsilon_{i,j}^{1\mathcal{H}} = 1$ and $\epsilon_{i,j}^{2\mathcal{H}} = (-1)^{i+j+1}$ are choosen. Finally, the sign choosen for ϖ in [6, p. 112] corresponds via our definition of ϖ to the equality $\epsilon_{j-i,i}^{ath}\epsilon_{i,j-i}^{ath}\epsilon_{j-i,i}^{c} = (-1)^{j(j-1)/2}$. It is possible to choose the signs in a way compatible with all these choices and our formalism.

	compatibility	reason
1	$\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j-1}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i-1,j}=-1$	$A\otimes B$ is a complex
2	$\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i,j-1}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i+1,j}=-1$	$\mathcal{H}(A,B)$ is a complex
3	$\epsilon^{T}_{i}\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i-1,j}=1$	$tp_{1,A,B}$ is a morphism
4	$\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i+1,j}\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j-1}=1$	
5	$\epsilon^{T}_{j}\epsilon^{T}_{i+j}\epsilon^{2\otimes}_{i,j}\epsilon^{2\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i,j-1}=1$	$tp_{2,A,B}$ is a morphism
6	$\epsilon^{T}_{i+j}\epsilon^{1\otimes}_{i,j}\epsilon^{1\otimes}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i-1,j}=1$	
7	$\epsilon^{tp1}_{i,j}\epsilon^{tp1}_{i,j+1}\epsilon^{tp2}_{i,j}\epsilon^{tp2}_{i+1,j}=-1$	(TP1TP2) is true
8	$\epsilon^{th1}_{i,j}\epsilon^{th1}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i+1,j}=-1$	(TH1TH2) is true
9	$\epsilon^{T}_{i+1}\epsilon^{T}_{j-i-1}\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i+1,j}\epsilon^{th1}_{i,j}\epsilon^{th1}_{i+1,j}=1$	$th_{1,A,B}$ is a morphism
10	$\epsilon^{T}_{j-i-1}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i+1,j}\epsilon^{th1}_{i,j}\epsilon^{th1}_{i,j-1}=1$	
11	$\epsilon^{T}_{j}\epsilon^{T}_{j-i}\epsilon^{2\mathcal{H}}_{i,j}\epsilon^{2\mathcal{H}}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i,j-1}=1$	$th_{2,A,B}$ is a morphism
12	$\epsilon^{T}_{j-i}\epsilon^{1\mathcal{H}}_{i,j}\epsilon^{1\mathcal{H}}_{i,j+1}\epsilon^{th2}_{i,j}\epsilon^{th2}_{i+1,j}=1$	
13	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{i,j}\epsilon^{1\mathcal{H}}_{j-1,i+j-1}\epsilon^{ath}_{i,j-1}\epsilon^{ath}_{i-1,j}=-1$	ath is well defined
14	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\mathcal{H}}_{j,i+j}\epsilon^{ath}_{i-1,j}\epsilon^{ath}_{i,j}=1$	
15	$\epsilon^{tp1}_{i,j}\epsilon^{tp2}_{i,j}\epsilon^{th1}_{j,i+j+1}\epsilon^{ath}_{i,j+1}\epsilon^{ath}_{i+1,j}=1$	(TATH12) is true
16	$\epsilon^{tp2}_{i,j}\epsilon^{th1}_{j,i+j+1}\epsilon^{th2}_{j,i+j}\epsilon^{ath}_{i,j}\epsilon^{ath}_{i,j+1}=1$	(TATH23) is true
17	$\epsilon^{1\otimes}_{i,j}\epsilon^{2\otimes}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i-1,j}=1$	$c_{A,B}$ is a morphism
18	$\epsilon^{1\otimes}_{j,i}\epsilon^{2\otimes}_{i,j}\epsilon^{c}_{i,j}\epsilon^{c}_{i,j-1}=1$	
19	$\epsilon^{c}_{i,j}\epsilon^{c}_{j,i}=1$	(SCP) is true
20	$\epsilon^{tp1}_{i,j}\epsilon^{tp2}_{j,i}\epsilon^{c}_{i,j}\epsilon^{c}_{i+1,j}=1$	(TCP) is true

TABLE 3. Sign definitions

Theorem 1.33. *Let $a,b\in\{+1,-1\}$. Then*

$$\begin{array}{ll}
\epsilon^{1\otimes}_{i,j}=1 & \epsilon^{tp1}_{i,j}=a\\
\epsilon^{2\otimes}_{i,j}=(-1)^{i} & \epsilon^{tp2}_{i,j}=a(-1)^{i}\\
\epsilon^{1\mathcal{H}}_{i,j}=1 & \epsilon^{th1}_{i,j}=1\\
\epsilon^{2\mathcal{H}}_{i,j}=(-1)^{i+j+1} & \epsilon^{th2}_{i,j}=a(-1)^{i+j}\\
\epsilon^{ath}_{i,j}=b(-1)^{i(i-1)/2} & \epsilon^{c}_{i,j}=(-1)^{ij}\\
\epsilon^{T}_{i}=-1 &
\end{array}$$

satisfies all equalities of Table 3 as well as $\epsilon^{ath}_{j-i,i}\epsilon^{ath}_{i,j-i}\epsilon^{c}_{j-i,i}=(-1)^{j(j-1)/2}$ and is compatible with all the above sign choices of Balmer, Gille and Nenashev.

Proof: Straightforward. □

1.8.2. *Derived category.* Assume now that $\mathcal{A}$ is as in 1.8.1 and furthermore is an abelian (or more generally exact) category with enough injectives and projectives (or flat objects). Then we obtain a (right) derived internal Hom and a (left) derived tensor product. In general, these bifunctors are defined on categories of complexes with cohomology bounded above or below. In the case of the derived category of $\mathcal{O}_X$-modules for X a regular noetherian scheme as will be studied in section 2, these bifuctors reduce to complexes in D^b (as we have finite resolutions) and even to complexes with coherent cohomology (see e. g. [10, Proposition II.3.3 and Proposition II.4.3]). In particular, we obtain an adjunction between $\mathbf{R}\underline{Hom}$ and $\otimes^{\mathbf{L}}$ in $D^b_c(X)$ satisfying all the formulae of subsection 1.3.

2. Witt motives

In this section, we will construct transfer maps between (Grothendieck-)Witt groups with respect to proper morphisms and establish some properties such as the base change and the projection formula. In contrast to K_0, the transfer maps for (Grothendieck-)Witt groups will shift the degree and twist the duality. Using section 1, it seems straigthforward to generalize the construction of transfer maps to the H-equivariant setting for an algebraic group H, but we have not checked this in full detail, so the careful reader should assume $H = \{1\}$. These transfer maps and their properties are then used for the construction of the categories $\mathbf{GW}^H$ and $\mathbf{W}^H$ of Grothendieck-Witt motives and Witt motives with respect to an algebraic group H. This category is the analogue of the category $\mathbf{K}^H$ of [16, section 6] which is the crucial construction for Panin's computations.

Everything in the sequel is true both for GW and W, so we just state everything for W.

We observe that in some very special cases there are already constructions that deserve the name transfer map. In particular, for any projection map $\pi : \mathbf{P}^n \times X \to X$, Walter establishes maps $W^i(\mathbf{P}^n \times X, \pi^*L(-n-1)) \to W^{i-n}(X, L)$ [21, p. 24] which using in particular Theorem 5.6, Proposition 5.11 and p.23/24 of *loc. cit.* can be seen to be natural with respect to X. Also, there seems to be work in progress by C. Walter on the construction of transfer maps in a very general setting (which should presumably yield the same transfer maps as those we constructed). There are also transfer constructions for Witt groups with respect to certain finite maps and closed embeddings in the affine space [8], [22], but not for other projective morphisms which is what we need.

2.1. **Some derived categories.** In order to define transfers, it will be necessary to consider larger categories than just $D^b(Vect(X))$. We denote $D^b(X)$ the bounded derived category of sheaves of $\mathcal{O}_X$-modules. Recall that for any exact category $\mathcal{E}$ (e.g. $\mathcal{E} = \mathcal{O}_X$-modules on a given scheme X), the canonical functor from the derived category of bounded complexes $D^b(\mathcal{E})$ to the subcategory of the unbounded derived category $D(\mathcal{E})$ of complexes with bounded cohomology is an equivalence of categories (see e.g. [11, Lemma 11.7]). Thus, we shall use the same symbol for this latter category as well, and it is this variant we work with when using the previous section. We further denote by $D^b_c(X)$ and $D^b_{qc}(X)$ the full subcategories of complexes with coherent resp. quasi-coherent cohomology. For X noetherian regular of finite Krull dimension, the inclusion $D^b(Vect(X)) \to D^b_c(X)$ is an equivalence of triangulated categories.

The inclusion $D^b_c(X) \to D^b_{qc}(X)$ is fully faithful by definition, but never an equivalence. Recall that for X locally noetherian, the functor $D^b(QCoh(X)) \to D^b_{qc}(X)$ is an equivalence, see e.g. [10, Corollary II.7.19]. Under suitable assumptions (see section 1, and in particular 1.8.2), the category $D^b(X)$ has an internal Hom denoted by $\mathbf{R}\underline{Hom}_X$ or just $\mathbf{R}\underline{Hom}$ which is right adjoint to $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$ and restricts to internal Homs in $D^b_c(X)$ and $D^b_{qc}(X)$.

Remark 2.1. *Sign conventions.* First of all, we use chain complexes, as in Balmer's work (i.e. the differential in degree n is $d_n : A_n \to A_{n-1}$). The sign conventions that we then use are discussed in 1.8.1.

2.2. **Transfers.** If $f : X \to Y$ is proper and Y locally noetherian, then there is a functor $\mathbf{R}f_* : D^+_{qc}(X) \to D^+_{qc}(Y)$ and similar for D^+_c (see [10, p. 88-89]). The construction of transfers for Witt groups along $\mathbf{R}f_*$ will rely on the following duality theorem due to Grothendieck-Verdier (-Hartshorne-Deligne) (see [20, Proposition 3, p. 404]):

Theorem 2.2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes of finite Krull dimension. Then there is a functor $f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$ and a natural transformation Tr_f such that for all $F \in D^-_{qc}(X)$, $G \in D^+_{qc}(Y)$ the composition*

$$\tilde{\alpha} : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, f^!G) \overset{a\mathbf{R}f_*}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, \mathbf{R}f_*f^!G) \overset{Tr_f}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$$

is an isomorphism in $D^+_{qc}(Y)$.

Applying the global section functor $\mathbf{R}\Gamma(Y, \ )$ and using the isomorphism of functors $\mathbf{R}\Gamma(X, \ ) \overset{\simeq}{\to} \mathbf{R}\Gamma(Y, \mathbf{R}f_*(\ ))$ (see [10, II.Proposition 5.2]), the isomorphism of the theorem becomes an isomorphism $\mathbf{R}\underline{Hom}(F, f^!G) \overset{\simeq}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ in $D^b(\mathbf{Ab})$. Observe also that since f is proper, the above statement remains true after replacing qc by c everywhere by [10, Proposition II.2.2] and [20, p. 396].

Applying H^0, we obtain (see [20, Theorem 1]):

Corollary 2.3. *In the above situation, the functors $\mathbf{R}f_* : D^+_{qc}(X) \to D^+_{qc}(Y)$ and $f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$ form an adjoint pair.*

Proof: Apply H^0 to the isomorphism $\mathbf{R}\underline{Hom}(F, f^!G) \overset{\simeq}{\to} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ in $D^b(\mathbf{Ab})$. □

Remark 2.4. In fact, Verdier proceeds in the other direction. That is, he first states Corollary 2.3 and then deduces Theorem 2.2 using the projection formula. We will also use Corollary 2.3 and the projection formula to construct a natural isomorphism $\alpha : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, f^!G) \to \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, G)$ (see Theorem 1.24), but in a different way than Verdier.

Being part of an adjoint pair, the functor $f^! : D^+_{qc}(X) \to D^+_{qc}(Y)$ and the natural transformation Tr_f are unique up to unique isomorphism, see [20, p. 394]. There are at least two different ways to construct them and to prove the isomorphism of the theorem (see also [14] for still another approach). One is to use residual complexes as Hartshorne [10] does. The other is to use apply the techniques of [20] as done by Deligne in the appendix of [10]. We will use this second construction. Although $f^!$ and Tr_f are unique up to unique isomorphism, this does not automatically mean that we can say explicitely how the isomorphism between the constructions of Hartshorne and Deligne looks like.

We now explain why this theorem is useful to define transfers. Given a line bundle L on a Gorenstein scheme X (for instance $L = O_Y$, or $L = \omega_Y := \omega_{Y/F}$ the canonical sheaf if X is smooth over F), the functor $* := *_L := \underline{Hom}_{\mathcal{O}_X}(\ , L)$ is a duality functor on $Vect(X)$ with the natural isomorphism $\varpi : Id \overset{\simeq}{\to} **$ defined below. This induces a duality on the triangulated category $\mathbf{R}\underline{Hom}(\ , L)$ on $D^b(Vect(X))$ where L is considered as a complex concentrated in degree 0. We will work with the larger category $D^b_c(X)$ instead on which $\mathbf{R}\underline{Hom}(\ , L)$ is still a duality (see [6, 2.5.3 p. 115] and [10, Theorem V.3.1]) and gives rise to the so-called *coherent Witt groups* (compare [6, Definition 2.16]). For the precise definition of the ϖ with respect to $\mathbf{R}\underline{Hom}(\ , L)$ on $D^b_c(X)$ we use and for a comparison with the signs choosen in [6, p. 112] see sections 1.8.1 and 1.8.2. As we always work with coherent Witt groups (instead of derived categories of vector bundles), we denote these simply by $W^*(X)$:

Definition 2.5. Let L be a line bundle on a scheme X which is Gorenstein noetherian of finite Krull dimension. Then we define

$$W^i(X, L) := W(D^b_c(X), \mathbf{R}\underline{Hom}(\ \ , L), \varpi_L).$$

Beware that following Hartshorne, in the notation $\mathbf{R}\underline{Hom}(\ \ , L)$ the $\mathbf{R}$ means $\mathbf{R}_I\mathbf{R}_{II}$, so we replace the line bundle L by an injective resolution. We also have locally free resolutions if X is quasiprojective. Moreover, the derived functor of Hom using projective resolutions if those exist (denoted by $\mathbf{R}_{II}\mathbf{R}_I$ in Hartshorne) is canonically isomorphic to the one defined via injective resolutions (see [10, p. 65/66, 91]), but we will not use this in the sequel. *From now on, we assume that all schemes are noetherian of finite Krull dimension and Gorenstein.*

Remark 2.6. First, we have a natural transformation

$$\varpi_{Y,K} : Id \to \mathbf{R}\underline{Hom}(\mathbf{R}\underline{Hom}(\ \ , K), K))$$

for any bounded complex K (see 1.3.2). We say that K is a dualizing complex if $\varpi_{Y,K}$ is an isomorphism. The fact that X is Gorenstein ensures that $\mathcal{O}_X$ is a dualizing complex. Moreover, [10, Theorem V.3.1] implies that any dualizing complex of finite injective dimension is quasi-isomorphic to a shifted line bundle, provided X is connected. Of course, Definition 2.5 generalizes to dualizing complexes.

Recall that as X is noetherian regular of finite Krull dimension, the inclusion $(D^b(Vect(X)), *_L) \to (D^b_c(X), *_L)$ is an equivalence of triangulated categories with duality, inducing a non-canonical isomorphism between the associated Witt groups (the proof of [6, Corollary 2.17.2] for $\mathcal{O}_X$ carries over to arbitrary line bundles L). We also obtain a map $f^* : W^*(Y, O_Y) \to W^*(X, \mathcal{O}_X)$ between coherent Witt groups for $f : X \to Y$ a flat morphism [7, p. 221].

The techniques of section 1 yield such maps f^* for other dualities provided X and Y are regular.

Proposition 2.7. *Let $f : X \to Y$ be a flat morphism of regular schemes and M a dualizing complex on Y. Then there is a natural morphism $f^* : W^*(Y, M) \to W^*(X, f^*M)$ induced by an exact functor of categories with dualities.*

Proof: This follows from Theorem 1.31 which hypotheses are satisfied by the arguments of the proof of Theorem 2.10 and [10, Proposition II.5.8]. □

Now if we have a proper morphism $f : X \to Y$, we want to find dualizing complexes M on X (i.e., $M \in D^b_c(X)$ such that $D_M := \mathbf{R}\underline{Hom}(\ \ , M)$ is a duality on $D^b_c(X)$) such that $\mathbf{R}f_*$ can be extended to a functor of triangulated categories with duality $(\mathbf{R}f_*, \alpha) : (D^b_c(X), D_M, \varpi) \to (D^b_c(Y), *_L, \varpi)$. By definition, α must be a natural isomorphism $\alpha : \mathbf{R}f_*\mathbf{R}\underline{Hom}(F, M) \xrightarrow{\simeq} \mathbf{R}\underline{Hom}(\mathbf{R}f_*F, L)$. The duality theorem above tells us that this might be possible if we choose M to be isomorphic to $f^!L$. We also have the following:

Lemma 2.8. *If $f : X \to Y$ is a smooth proper morphism of relative dimension d and L a line bundle on Y, then there is a natural isomorphism $\beta : f^!L \xrightarrow{\simeq} f^*L \otimes \omega_{X/Y}[d]$. If moreover $g : Y \to Z$ is also smooth, we have an isomorphism $\omega_{X/Z} \simeq f^*(\omega_{Y/Z}) \otimes \omega_{X/Y}$. If f as above and $h : V \to Y$ arbitrary, then $\tilde{h}^*(\omega_{X/Y}) \cong \omega_{X \times_Y V/V}$.*

Proof: See [10, p. 143, p. 419-421] or [20, Theorem 3] for the first claim and [10, p. 142, p. 141] for the second and third one. □

If f is a closed embedding of codimension d which is locally complete intersection (e. g. the graph of a morphism), then it is still possible to define $\omega_{X/Y}$ (see [10, p. 141]). and one may establish the second and third isomorphisms again using [10,

p. 142, p. 141]. The description of $f^!L$ can be generalized easily to non-smooth morphisms when using absolute rather that relative canonical sheafs, which turns out to be more natural for our later purposes anyway.

Lemma 2.9. *(B. Kahn) Let $f : X \to Y$ be a proper morphism of relative dimension d between smooth varieties over a field F and L a line bundle on Y. Then there is a natural isomorphism $\beta : f^!L \xrightarrow{\cong} f^*L \otimes f^*\omega_Y^{-1} \otimes \omega_X[d]$.*

Proof: Recall that $f^!L \cong f^*L \otimes f^!\mathcal{O}_Y$ [10, p. 419-420]. Let p_X and p_Y be the projections from X and Y to Spec(F). We have $p_X = p_Y \circ f$, so, by adjunction $p_X^! = f^! \circ p_Y^!$, hence, by Lemma 2.8 and the above isomorphism

$$\omega_X[\dim X] \cong p_X^!\mathcal{O}_F \cong f^!p_Y^!\mathcal{O}_F \cong f^!\omega_Y[\dim Y] \cong f^*\omega_Y[\dim Y] \otimes f^!O_Y$$

which yields the formula for $L = O_Y$. Now apply the above isomorphism again for the general case. □

Observe that f_* has finite cohomological dimension (see e.g. [10, p. 87]), hence $\mathbf{R}f_*$ restricts to a functor $\mathbf{R}f_* : D_c^b(X) \to D_c^b(Y)$.

Of course, we now have to show that this α indeed defines a functor of triangulated categories with duality. See Proposition 1.26 for the comparison of this construction of α with the $\tilde{\alpha}$ of Verdier.

Theorem 2.10. *If in addition to the hypothesis of Theorem 2.2 X and Y are smooth over F, then the functor*

$$(\mathbf{R}f_*, \alpha) : (D_c^b(X), *_{f^!L}, \varpi_X) \to (D_c^b(Y), *_L, \varpi_Y)$$

is a functor of triangulated categories with duality.

Proof: We want to apply Theorem 1.32 with $\mathcal{D} = D_c^b(X)$, $\mathcal{C} = D_c^b(Y)$ (see also subsection 1.8.2), $E = \mathbf{L}f^*$ (as defined in [10, Proposition II.4.4]), $F = \mathbf{R}f_*$ and $G = f^!$. Observe that $\mathbf{L}f^*$ and $f^!$ restrict to D_c^b as X and Y are regular noetherian and Lemma 2.9 applies to vector bundles as well. We know that $\varpi_{f^!L}$, ϖ_L are isomorphisms because L is a line bundle and therefore the complex $f^!L$ is quasi-isomorphic to a shifted line bundle. The projection formula morphisms

$$(\mathbf{R}f_*D_{f^!L}A)\otimes\mathbf{R}f_*A \to \mathbf{R}f_*((f^*\mathbf{R}f_*D_{f^!L}A)\otimes A)$$

and

$$(D_L\mathbf{R}f_*A)\otimes\mathbf{R}f_*A \to \mathbf{R}f_*((f^*D_L\mathbf{R}f_*A)\otimes A)$$

are isomorphisms by [10, Proposition II.5.6]. For the adjunctions aef and afg, we use [10, Corollary II.5.11] and Corollary 2.3. The isomorphism ep is provided by [10, Proposition II.5.9], and the isomorphisms c, tp_1 and tp_2 are the obvious ones given by subsections 1.8.1 and 1.8.2. It remains to define te and to check that **TE**,**TEP1**, **TEP2** and **EPC** hold. Replacing all complexes by flat ones, we are reduced to study chain complexes. Thus we see that we may choose $te = Id : Tf^* \to f^*T$ as f^* is defined degreewise, and the commutativity of the three squares is immediatly checked degreewise using that in each diagram the only sign that appears (namely ϵ^{tp2}, resp. ϵ^{tp1}, resp. ϵ^c) does appear twice. □

Remark 2.11. We may replace the condition "smooth over F" by "regular" if we know (for some other reason than Lemma 2.9) that $f^!L$ is a dualizing complex on X.

Having done all this, we can finally define the desired transfer maps between Witt groups.

Definition 2.12. Let $f : X \to Y$ be a proper map between smooth varieties over F and L a line bundle on Y. Then we define the transfer map

$$f_* : W^*(X, f^! L) \to W^*(Y, L)$$

to be the map induced by the triangulated duality preserving functor $(\mathbf{R}f_*, \alpha)$ above.

The transfer respects compositions.

Lemma 2.13. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two proper maps and N a line bundle on Z. Then we have $(g \circ f)_* = g_* \circ f_* : \mathrm{W}^*(X, (g \circ f)^! N) \to \mathrm{W}^*(Z, N)$.*

Proof: One has to check $(\mathbf{R}(g \circ f)_*, \alpha_{g \circ f}) = (\mathbf{R}g_*, \alpha_g) \circ (\mathbf{R}f_*, \alpha_f)$ where the right hand side is defined by 2.20 below. This immediately follows using among others that $Tr_g \circ (\mathbf{R}g_* Tr_f g^!) = Tr_{g \circ f}$. □

Remark 2.14. If f is a smooth finite morphism, then $\omega_{X/Y} = \mathcal{O}_X$, so by Lemma 2.8 the above transfer map becomes $f_* : W^*(X, f^* L) \to W^*(Y, L)$. In particular, we get some version of the classical Scharlau transfer [18] $f_* : W^*(X) \to W^*(Y)$ if $L = O_Y$.

For $L = \omega_Y$, the transfer map becomes

$$f_* : W^*(X, f^! \omega_Y) \to W^*(Y, \omega_Y).$$

Using the isomorphism of Lemma 2.8, Lemma 2.9 and the fact that an isomorphism of dualizing complexes induces an isomorphism of categories with dualities and thus of Witt groups, we deduce from Theorem 2.10 a transfer map $f_* : W^*(X, \omega_X[d]) \to W^*(Y, \omega_Y)$ (or $f_* : W^*(X, f^* L \otimes_{\mathcal{O}_X} \omega_X[d]) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$ for some line bundle L on Y).

Applying 1.17, we therefore have:

Lemma 2.15. *Under the assumptions of Lemma 2.9, the above transfer map induces a transfer map of degree $-d$*

$$f_* : W^{*+d}(X, f^* L \otimes_{\mathcal{O}_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y).$$

Lemma 2.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two proper maps and N a line bundle on Z. Then we have $(g \circ f)_* = g_* \circ f_* : \mathrm{W}^*(X, (g \circ f)^* N \otimes \omega_X) \to \mathrm{W}^*(Z, N \otimes \omega_Z)$.*

Proof: We want to reduce this to Lemma 2.13. Recall that from Lemma 2.9, we have isomorphisms

$$\beta_f : f^! M \xrightarrow{\simeq} f^* M \otimes f^* \omega_Y^{-1} \otimes \omega_X[d]$$

and

$$\beta_g : g^! N \xrightarrow{\simeq} g^* N \otimes g^* \omega_Z^{-1} \otimes \omega_Y[d']$$

Fix an isomorphism $\lambda : N \otimes \omega_Z^{-1} \simeq N'$. Then the proof of Lemma 2.15 shows that starting with the isomorphism

$$\lambda_* : \mathrm{W}^*(Z, N) \to \mathrm{W}^*(Z, N' \otimes \omega_Z)$$

we obtain two isomorphisms

$$\mathrm{W}^*(X, f^! g^! N) \to \mathrm{W}^*(X, f^* g^* N' \otimes \omega_X)$$

applying either first g^* and then f^* or directly $(g \circ f)^*$. The lemma follows as one can show that these two coincide. To check this, one uses among others that

$$\beta_{g \circ f} : (g \circ f)^! N \otimes \omega_X^{-1} \to f^* g^* N \otimes f^* g^* \omega_Z^{-1} \otimes f^* \omega_Y \otimes f^* \omega_Y^{-1} \otimes \omega_X \otimes \omega_X^{-1}$$

and $f^*(\beta_g \otimes Id) \circ \beta_f$ are equal. □

2.3. **Another category.** Before we prove the related properties of transfers and pull-backs for Witt groups, we introduce a new category in which those properties can be expressed nicely.

Let L, L', M, M' and N, N' be vector bundles over X, Y and Z, respectively, and assume we have morphisms $p_{X/Z} : X \to Z$ and $p_{Y/Z} : Y \to Z$. Let $V = X \times_Z Y$ be the cartesian product of X by Y over Z. We denote $L \boxtimes_N M$ the vector bundle $p^*_{V/X}(L) \otimes (p_{V/X} \circ p_{X/Z})^*(N) \otimes p^*_{V/Y}(M)$ over $X \times_Z Y$. When we write $L \boxtimes M$, we mean that Z is the point and that N is trivial. We therefore get a vector bundle over $X \times Y$. We identify

- $(L \boxtimes_N M) \otimes (L' \boxtimes_{N'} M') = (L \otimes L') \boxtimes_{N \otimes N'} (M \otimes M')$
- $\omega_{X \times_Z Y} = \omega_X \boxtimes_{\omega_Z^{-1}} \omega_Y$

where the last equality follows from Lemma 2.8 provided everything is smooth. When $f : X' \to X$ and $g : Y' \to Y$, we also identify

- $(f \times g)^*(L \boxtimes M) = f^* L \boxtimes g^* M$.

Now, let $f : X \to Y$ and g be two composable morphism and P and P' be line bundles over the target of g. We identify

- $f^*(\mathcal{O}_Y) = \mathcal{O}_X$
- $f^* \circ g^*(P) = (g \circ f)^*(P)$
- $f^*(P) \otimes f^*(P') = f^*(P \otimes P')$

Finally, we denote L^{-1} the dual line bundle of L and we identify

- $L \otimes L^{-1} = \mathcal{O}_X$

Of course, we could avoid all those identifications by working with the canonical isomorphism involved, but the proofs would become completely unreadable.

Definition 2.17. Let $\mathcal{L}$ denote the category whose objects are pairs (X, L) where X is a smooth variety and L is a line bundle over X. A morphism from (X, L) to (Y, M) is a pair (f, ϕ) where f is a morphism from X to Y and $\phi : f^*(M) \simeq L$ is an isomorphism of vector bundles. The composition is defined by $(g, \psi) \circ (f, \phi) = (g \circ f, \phi \circ f^*(\psi))$.

Associativity in $\mathcal{L}$ is clear. There is an obvious faithful functor from the category of smooth schemes to this category sending X to $(X, \mathcal{O}_X)$ and $f : X \to Y$ to $(f, Id_{\mathcal{O}_X})$. To keep notations concise, we denote by X and f the images of X and f by this functor. We denote pt the object $(SpecF, \mathcal{O}_{SpecF})$. The reader discouraged by all these notations might want to restrict his attention to the case where all the L and M are just the structure sheafs.

There is a well defined contravariant functor W^i from the subcategory of flat morphisms of $\mathcal{L}$ to the category of abelian groups that send an object (X, L) on the corresponding Witt group $W^i(X, L)$. The morphism (f, ϕ) is sent to the composition $W^i(Y, M) \to W^i(X, f^*(M)) \to W^i(X, L)$, where the second map is induced by ϕ and the first is the classical pull-back on Witt groups. For obvious reasons, we denote this morphism $(f, \phi)^*$.

We can also define the push-forwards (or transfer) for a morphism (f, ϕ) if f is as in Definition 2.12. This is a morphism $(f, \phi)_*$ from $W^{i+d}(X, M \otimes \omega_X)$ to $W^i(Y, L \otimes \omega_Y)$ when f is of dimension d. It is given by the isomorphism $W^{i+d}(X, f^*(L) \otimes \omega_X) \simeq W^{i+d}(X, M \otimes \omega_X)$ induced by ϕ composed with the morphism of 2.15. One can check that $((g, \psi) \circ (f, \phi))_* = (g, \psi)_* \circ (f, \phi)_*$.

For reasons that will be clear later, we define the twist of an object $c(X, L) = (X, L \otimes \omega_X)$. We can therefore interpret the transfer for a morphism (f, ϕ) as a morphism from $W^{i+d}(c(X, L))$ to $W^i(c(Y, M))$.

Next, we prove the base change formula for Witt groups. We need to study the following technical condition.

Definition 2.18. Let (F, α_F) and (G, α_G) be two exact fuctors $(A, D_A, \varpi_A) \to (B, D_B, \varpi_B)$ between exact categories with duality. We say that $\sigma : F \Rightarrow G$ is a natural transformation (resp. isomorphism) between duality preserving functors if $\sigma : F \Rightarrow G$ is a natural transformation (resp. isomorphism) between functors and the square

$$\begin{array}{ccc} FD_A & \xrightarrow{\alpha_F} & D_B F \\ {\scriptstyle \sigma_{D_A}}\downarrow & & \uparrow{\scriptstyle D_B\sigma} \\ GD_A & \xrightarrow[\alpha_G]{} & D_B G \end{array}$$

is commutative. If F and G are exact functors between triangulated categories, we say that $\sigma : F \Rightarrow G$ is a triangulated natural transformation (resp. isomorphism) between duality preserving functors if moreover $\sigma T = T\sigma$.

Lemma 2.19. *If σ is a natural isomorphism between duality preserving functors as above, then the two maps $W(A) \to W(B)$ induced by F and G coincide. If moreover σ is a triangulated isomorphism, then the two induced maps between graded Witt groups $W^*(A) \to W^*(B)$ coincide.*

Proof: It is straightforward to check that the images with respect to F and G of some symmetric space in A are isomorphic as symmetric spaces in B, and similar for the shifted dualities in the triangulated setting. □

It is possible to compose duality preserving functors between triangulated categories with dualities.

Definition 2.20. Let

$$(F, \eta) : (A, D_A, \varpi_A) \to (B, D_B, \varpi_B)$$

and

$$(G, \rho) : (B, D_B, \varpi_B) \to (C, D_C, \varpi_C)$$

be two duality preserving functors of triangulated categories with duality. Then we define their composition $(GF, \rho\eta)$ by

$$(GF, (\rho F) \circ (G\eta)) : (A, D_A, \varpi_A) \to (C, D_C, \varpi_C).$$

It is straightforward to check that $(GF, \rho\eta)$ is a duality preserving functor.

Theorem 2.21. *Assume that we have a cartesian square of smooth varieties*

$$\begin{array}{ccc} V & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

where g and g' are flat and f and f' are proper and satisfy the hypotheses of Definition 2.12. Let N be a line bundle on X. Then we have a commutative square of Witt groups

$$\begin{array}{ccc} W^*(V, g'^* f^! N) & \xleftarrow{g'^*} & W^*(X, f^! N) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ W^*(Y, g^* N) & \xleftarrow[g^*]{} & W^*(Z, N) \end{array}$$

Proof: The square of Witt groups is induced by the following diagram of cateories with duality

$$\begin{array}{ccc}
(D^b_c(V), \mathbf{R}\underline{Hom}(\ \ , g'^* f^! N)) & \xleftarrow{(g'^*, id)} & (D^b_c(X), \mathbf{R}\underline{Hom}(\ \ , f^! N)) \\
\downarrow{\scriptstyle (Id,c)} & & \\
(D^b_c(V), \mathbf{R}\underline{Hom}(\ \ , f'^! g^* N)) & & \downarrow{\scriptstyle (\mathbf{R}f_*, \alpha)} \\
\downarrow{\scriptstyle (\mathbf{R}f'_*, \alpha')} & & \\
(D^b_c(Y), \mathbf{R}\underline{Hom}(\ \ , g^* N)) & \xleftarrow[(g^*, id)]{} & (D^b_c(Z), \mathbf{R}\underline{Hom}(\ \ , N))
\end{array}$$

where c is the canonical isomorphism of [20, Theorem 2]. We may now apply Lemma 2.19 to the functors $F = g^* \circ \mathbf{R}f_*$ and $G = \mathbf{R}f'_* \circ Id \circ g'^*$. The required natural isomorphism σ is given by [2, p. 84, p. 285] and [1, p. 290]. The hypothesis in Definition 2.19 is then precisely the commutativity of the square of functors and natural isomorphisms

$$\begin{array}{ccc}
g^*\mathbf{R}f_*\mathbf{R}\underline{Hom}(\ \ , f^! N) & \xrightarrow{id \circ \alpha} & \mathbf{R}\underline{Hom}(\ \ , g^* N) g^* \mathbf{R}f_* \\
{\scriptstyle \sigma \circ \mathbf{R}\underline{Hom}(\ \ , f^! N)}\downarrow & & \uparrow{\scriptstyle \mathbf{R}\underline{Hom}(\ \ , g^* N) \circ \sigma} \\
\mathbf{R}f'_* g'^* \mathbf{R}\underline{Hom}(\ \ , f^! N) & \xrightarrow[\alpha' \circ c \circ id]{} & \mathbf{R}\underline{Hom}(\ \ , g^* N) \mathbf{R}f'_* g'^*
\end{array}$$

This can be shown using adjunctions and their standard properties, in particular the fact that the two different definitions of c in [20, p. 401] coincide. □

Corollary 2.22. *Let (X, L), (Y, M), (Z, N) and (V, P) in $\mathcal{L}$ be such that X, Y, Z, V and the morphisms f, f', g and g' between them are as in Theorem 2.21 and f, f' satisfy the hypotheses of Lemma 2.9. Let (f, ϕ), (g, ψ), (f', ϕ') and (g', ψ') have sources and targets as follows:*

$$(V, P) \xrightarrow{(f', \phi')} (Y, M)$$

$$c(V, P) \xrightarrow{\ } \ \ \ \ c(Y, M)$$

$$\begin{array}{ccc}
c(V, P) & \qquad\qquad & c(Y, M) \\
{\scriptstyle (g', \psi')}\downarrow & & \downarrow{\scriptstyle (g, \psi)} \\
c(X, L) & & c(Z, N)
\end{array}$$

$$(X, L) \xrightarrow[(f, \phi)]{} (Z, N)$$

Assume that $(\phi' \otimes Id) \circ (f')^*(\psi) = (\psi' \otimes Id) \circ (g')^*(\phi)$ (the source of these morphisms is $(g \circ f')^*(N \otimes \omega_Z) = (f \circ g')^*(N \otimes \omega_Z)$ and their target is $P \otimes (\mathcal{O}_X \boxtimes_{\mathcal{O}_Z} \omega_Y)$. Then the two morphism $(g, \psi)^* \circ (f, \phi)_* = (f', \phi')_* \circ (g', \psi')^* : W^*(X, L \otimes \omega_X) \to W^{*-d}(Y, M \otimes \omega_Y)$ coincide.

Proof: This follows from Theorem 2.21 and Lemma 2.15. □

Products. Observe that there is a product μ on $\oplus_L W^*(\ \ , L)$ induced by the (left) product of [9, Theorem 3.1] (see also [21, p.7/8]). Using the fact that for any vector bundles V and W over X one has $\Delta^*_X(V \times W) = V \times_X W$, one sees that the product μ factors through the exterior product (with $X = Y$)

$$\lambda_{(X,L),(Y,M)} : W^*(X, L) \times W^*(Y, M) \xrightarrow{\lambda} W^*(X \times Y, L \boxtimes M)$$

namely as $W^*(X, L) \times W^*(X, M) \xrightarrow{\lambda} W^*(X \times X, L \boxtimes M) \xrightarrow{\Delta_X^*} W^*(X, L \otimes M)$. The factorization follows as the pairing of exact categories with duality

$$(Vect(X), \mathrm{Hom}(\ \ , L)) \times (Vect(X), \mathrm{Hom}(\ \ , M)) \to (Vect(X), \mathrm{Hom}(\ \ , L \otimes M))$$

is dualizing in the sense of [9, Definition 1.11] and factors as a dualizing pairing through $(Vect(X \times X), \mathrm{Hom}(\ \ , L \boxtimes M))$ which induces a factorization of the dualizing pairing for the corresponding $D^b(Vect(\ \ ))$. The very same construction applies to D^b_{qc} and D^b_c.

Lemma 2.23. *The exterior product λ commutes with pull-backs and push-forwards. This means that if $f : X \to X'$ and $g : Y \to Y'$ are morphisms of schemes which are flat (resp. proper of pure dimension), we have the equalities*

$$\lambda(f^*(x'), g^*(y')) = (f \times g)^* \circ \lambda(x', y')$$

and

$$\lambda(f_*(x), g_*(y')) = (f \times g)_* \circ \lambda(x, y).$$

Proof: The first equality is a slight improvement of [9, Theorem 3.2] but is not more difficult to show. For the second equality, pick two complexes F and G on X and Y with forms. The claim then reduces essentially to the "Künneth" isomorphism $(f \times g)_*(F \boxtimes G) \simeq f_*(F) \boxtimes g_*(G)$ which can be checked by an explicit computation. □

Proposition 2.24. *For $(f, \phi) : (X, L) \to (Y, M)$ and $(f, \phi') : (X, L') \to (Y, M')$ with f as in Definition 2.12, the projection formula $(f, \phi \otimes \phi')_*(a.(f, \phi')^*(b)) = (f, \phi)_*(a).b$ holds for any $a \in W^i(X, L \otimes \omega_X)$ and $b \in W^j(Y, M')$.*

Proof: The result follows from the previous lemma and Corollary 2.22 applied to the following cartesian diagram.

$$\begin{array}{ccc}
(X, L \otimes L') & \xrightarrow{((Id \times f) \circ \Delta_X, Id \otimes \phi')} & (X \times Y, L \boxtimes M') \\
\big\downarrow {\scriptstyle (f, \phi \otimes \phi')} & & \big\downarrow {\scriptstyle (f \times Id, \phi \boxtimes Id)} \\
(Y, M \otimes M') & \xrightarrow[(\Delta_Y, Id)]{} & (Y \times Y, M \boxtimes M').
\end{array}$$

□

Let H be an algebraic group and X an H-variety. By this we mean a morphism $H \times X \to X$ satisfying the standard properties. Let $\mathrm{Vect}^H(X)$ be the category of H-equivariant vector bundles over X. This is a full subcategory of the category of H-equivariant-$\mathcal{O}_X$-modules. We say that an H-equivariant-$\mathcal{O}_X$-module is coherent if the underlying $\mathcal{O}_X$-module is, and we denote the corresponding category by $Coh^H(X)$. See [19] and [13, section 3] for the precise definitions and basic properties. The functor $Coh^H(\ \ )$ is contravariant for flat H-maps and covariant for H-projective morphisms (see [19, p. 543]). It is a non-trivial task to extend everything we did in this section so far to the equivariant setting. For instance, the existence of injective resolutions is not clear (that one has enough projectives follows from [19, Corollary 5.3]). *From now on we make the assumption that all the previous definitions and results in this section carry over to the H-equivariant setting.* We will hopefully discuss the details of this in forthcoming work. Therefore, the following is true unconditionally only for the non-equivariant setting ($H = 1$), otherwise the assumption has to be used.

Given an H-scheme X and and H-line bundle L on it, we write $W^{*,H}(X,L)$ for the Witt group of the derived category $D_c^{b,H}(X)$ of H-equivariant $\mathcal{O}_X$-modules with coherent cohomology with respect to the duality induced by $*_L$.

2.4. **Categories of motives.** Now we are ready to define the category $\mathbf{W}^H$ of H-Witt motives.

Definition 2.25. Let $\mathcal{PL}$ be the full subcategory of $\mathcal{L}$ whose objects are pairs (X,L) with X projective. Fix an algebraic group H. By definition, the category $\mathbf{W}^H$ has as objects couples (V,L) where V is endowed with an H-action and L a line bundle on V equipped with a left equivariant H-action. The set of morphisms (or *W-correspondances*) between two objects is a graded abelian group and is defined by $\mathrm{Hom}^i_{\mathbf{W}^H}((X,L),(Y,M)) = W^{i+dim X,H}(X\times Y,(L^{-1}\otimes\omega_X)\boxtimes M)$. For $a\in \mathrm{Hom}_{\mathbf{W}^H}((X,L),(Y,M))$ and $b\in\mathrm{Hom}_{\mathbf{W}^H}((Y,M),(Z,N))$ the composition ba is defined as

$$(\pi_{XZ}, Id_{L^{-1}\boxtimes\mathcal{O}_Y\boxtimes(N\otimes\omega_Z^{-1})})_*$$
$$(\mu((\pi_{XY}, Id_{(L^{-1}\otimes\omega_X)\boxtimes M\boxtimes\mathcal{O}_Z})^*(a),(\pi_{YZ}, Id_{\mathcal{O}_X\boxtimes(M^{-1}\otimes\omega_Y)\boxtimes N})^*(b))).$$

Proposition 2.26. *The above composition law in $\mathbf{W}^H$ is associative and any object admits an identity automorphism, so $\mathbf{W}^H$ really is a category.*

Proof: The proof of associativity is the usual proof of the associativity of correspondences, as in [12, §2, Lemma p. 446]. It just uses the composition of the pull-backs and push-forwards, the base change formula (Corollary 2.22) and the projection formula (Proposition 2.24). The identity of (X,L) is given by $(\Delta_X, Id_{\omega_X^{-1}})_*(1_X)$ (recall that 1_X is the class in $W_0(X,\mathcal{O}_X)$ of the one dimensional standard form $<1>$ on $\mathcal{O}_X$). Again, the proof that it is an identity is a generalization of the classical one. In fact, it is a particular case of the existence of graphs (see Proposition 2.28 below). □

Remark 2.27. There is an obvious category of Witt correspondences of degree zero defined by setting $\mathrm{Hom}_{\mathbf{W}^{0,H}}((X,L),(Y,M)) = \mathrm{Hom}^0_{\mathbf{W}^H}((X,L),(Y,M))$.

Now we can construct the graph functor.

Proposition 2.28. *There is a contravariant functor Γ from the category $\mathcal{PL}$ to $\mathbf{W}^H$. It is the identity on objects, and it sends a morphism $(f,\phi):(X,L)\to(Y,M)$ to $(\gamma_f,(\phi^\vee)^{-1}\otimes Id_L\otimes Id_{\omega_X^{-1}})_*(1_X)\in W^{dim Y}(Y\times X,(M^{-1}\otimes\omega_Y)\boxtimes L) = \mathrm{Hom}^0((Y,M),(X,L))$, where $\gamma_f: X\to Y\times X$ is the graph morphism (it is always proper as all considered varieties are separated). By $\phi^\vee$, we mean the morphism dual to ϕ, going from L^{-1} to $f^*(M)^{-1}$.*

Proof: This functor respects the composition. This follows from standard arguments, as in [12, §2, Proposition p. 447]. □

Of course, we can consider the full subcategory of $\mathbf{W}^H$ of objects of the form $(X,\mathcal{O}_X)$, but as we shall see, there are very few interesting motives that decompose in this category.

We now define a realization functor to the category of graded abelian groups.

Definition 2.29. We define the covariant functor R^H from $\mathbf{W}^H$ to the category of graded abelian groups by setting $R^H(X,L) = W^H(X,L)$ and $R^H(c) = (x\mapsto (p_Y)_*(p_X^*(x).c))$ for an element $c\in\mathrm{Hom}((X,L),(Y,M))$. For any subgroup H_1 of H, there is an obvious functor $\mathrm{Res}^H_{H_1}$ from $\mathbf{W}^H$ to $\mathbf{W}^{H_1}$ induced by the restriction of the action of H to H_1.

Remark 2.30. The functor R^H respects the composition because it coincides with the functor Hom(pt,). In particular we have thus obtained the Witt version (without twist) of [16, Key Lemma 6.5] wich is just a particular case of the fact that any motivic isomorphism induces an isomorphism on the realisations. Observe also that the composition $R^H \circ \Gamma$ sends a morphism (f, ϕ) to $(f, \phi)^*$.

The fact that we deal with categories with dualities is of course reflected by a duality already on the category of Witt motives. There is an involutive functor (of order 2) on $\mathbf{W}^H$ sending an object (X, L) to $(X, \omega_X \otimes L^{-1})$ and a morphism c in $\mathrm{Hom}^i((X, L), (Y, M)) = W_H^{i+dimX}(X \times Y, (L^{-1} \otimes \omega_X) \boxtimes M)$ to the corresponding element c^t in the group $W_H^{i+dimX}(Y \times X, M \boxtimes (\omega_X \otimes L^{-1})) = \mathrm{Hom}^{i+dimX-dimY}((Y, \omega_Y \otimes M^{-1}), (X, \omega_X \otimes L^{-1})$. Notice that it doesn't respect the graduation.

The composition $R^H \circ t \circ \Gamma$ sends a morphism (f, ϕ) to $(f, \phi)^*$ composed at the two ends by the isomorphism $W_H(X, L \otimes \omega_X) \simeq W_H(X, \omega_X \otimes L)$, and the similar one for (Y, M).

There is a pairing $W(X, M) \times W((X, M)^t) \to W(\mathrm{pt})$ given by the composition $(\pi, Id)_* \circ (\Delta_X, Id)^* \circ \lambda_{(X,M),(X,M)^t}$ where π is the structural morphism from X to the point.

Remark 2.31. Panin also constructs a category $\mathbf{A}^H$ where the objects are couples (X, B) with X smooth projective over F and B a central simple F-algebra, such that $\mathbf{K}^H$ is precisely the full subcategory of $\mathbf{A}^H$ of objects (X, F). The F-algebra B allows Panin to twist. We would like to do the same in our setting, considering of course F-algebras B with involution. In forthcoming work, we intend to settle this issue.

2.5. **Effective Witt Motives.** We now define the category $\mathbf{W}^H_{eff}$ of effective Witt motives. It is just the pseudo-abelianised completion of the previous category. For a definition of the pseudo-abelian completion, see e. g. [12, §5]. Recall that the objects are just the pairs $((X, L), p)$ where p is an idempotent in $\mathrm{End}(X, L)$ and the morphisms between $((X, L), p)$ and $((Y, L), q)$ are given by the quotient of the subgroup $\mathrm{Hom}_{\mathbf{W}^H}((X, L), (Y, L))$ given by the elements f such that $fp = qf$ by the subgroup of elements f such that $fp = qf = 0$. It contains $\mathbf{W}^H$ as the full subcategory of objects for which $p = Id$.

Remark 2.32. We don't lose the graduation on the Hom sets because an idempotent has to be of degree zero so the relation $fp = qf = 0$ is homogeneous. We can extend the realisation functors R^H and R to $\mathbf{W}^H_{eff}$ because of the universal property of the pseudo-abelian completion. More precisely, we set $R^H((X, L), p) = \ker R^H(p)$ on objects.

We can define a tensor structure on this category by setting $(X, L, p) \otimes (Y, M, q) = (X \times Y, L \boxtimes M, p \times q)$.

3. Dévissage

Assume that $f : Z \to X$ is a closed embedding of smooth varieties and L a line bundle on X. Then by Theorem 2.10,

$$(\mathbf{R}f_*, \alpha) : (D^b_c(Z), *_{f^!L}, \varpi_Z) \to (D^b_c(X), *_L, \varpi_X)$$

is a functor of triangulated categories with duality. The map $(\mathbf{R}f_*, \alpha)$ obviously factors through the full triangulated subcategory with duality $(D^b_{c,Z}(X), *_{f^!L}, \varpi_X)$ which by definition consists of complexes whose homology has support on Z. We denote its Witt groups by $W^*_Z(X, L)$. The goal of this section is to prove the following dévissage theorem for Witt groups.

Theorem 3.1. *In the above situation, the functor of triangulated categories with duality*

$$(\mathbf{R}f_*, \alpha) : (D^b_c(Z), *_{f^!L}, \varpi_Z) \to (D^b_{c,Z}(X), *_L, \varpi_X)$$

induces a map

$$f_* : W^*(Z, f^!L) \to W^*_Z(X, L)$$

which is an isomorphism.

Proof: It remains to show that f_* is an isomorphism. We roughly follow the strategy of [6, section 4]. First, replace [6, Theorem 4.2] by Theorem 2.10. Next, write down the long exact sequences arising from filtration by the codimension of support as in [6, p.130]. Of course, one needs to twist correctly the dualities (by L for X and $f^!L$ for Z) in this sequence, as well as everywhere else, but these twists don't change anything in the proof. We are thus reduced to show the claim on the top of page 131 of *loc. cit.* with B/J and B replaced by Z and X. Replace [6, Lemma 4.3] by Theorem 2.21 (closed embeddings are proper and localizations=open embeddings are flat). Now we may conclude similar to [6, 4.2.3]. □

We write $j : X - Z \to X$ for the open inclusion of the complement. As usual, dévissage implies (or improves) a localization exact sequence.

Corollary 3.2. *In the above situation, we have a long exact sequence*

$$... \to W^{n-1}(X{-}Z, j^*L) \overset{\partial}{\to} W^n(Z, f^!L) \overset{f_*}{\to} W^n(X, L) \overset{j^*}{\to} W^n(X{-}Z, j^*L) \overset{\partial}{\to} W^{n+1}(Z, f^!L) \to ...$$

Proof: By definition resp. construction, we have a short exact sequence of triangulated categories with dualities $(D^b_c(Z), *_{f^!L}, \varpi_Z) \overset{(\mathbf{R}f_*,\alpha)}{\longrightarrow} (D^b_c(X), *_L, \varpi_X) \overset{(i^*,id)}{\longrightarrow} (D^b_c(X), *_{i^*L}, \varpi_{X-Z})$. Hence Balmer's abstract localization theorem [3] and our dévissage theorem yield the claim. □

Recall from Lemma 2.9 that $f^!L$ can be described in more concrete terms.

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