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The aim of class field theory is the description of the abelian étale coverings of an arithmetic scheme in terms of its arithmetic/geometric invariants. This talk will deal with global class field theory exclusively, and therefore the term arithmetic scheme will mean "scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$" here.

We start with a look at algebraic topology. Let T be a (sufficiently good) topological space and let $x \in T$ be a point. As is well known, there are two descriptions of the fundamental group of (T, x):

1) (The "outer" description): $\pi_1(T, x) = \mathrm{Aut}(F)$, where F is the fibre functor

$$\begin{array}{rccc} F: & \mathrm{Cov}(T) & \longrightarrow & \mathrm{Sets} \\ & (T' \xrightarrow{\pi} T) & \longmapsto & \pi^{-1}(x). \end{array}$$

   If $\tilde{T} \to T$ is a universal covering space of T, then $\pi_1(T, x) \cong \mathrm{Aut}(\tilde{T}/T)$, the isomorphism being canonical up to inner automorphisms.

2) (The "inner" description):

$$\begin{array}{rcl} \pi_1(T, x) & = & [(S^1, *), (T, x)] \\ & = & \text{loops modulo homotopy.} \end{array}$$

For a scheme X with a geometric base point $\bar{x} \to X$, we have the *étale fundamental group* $\pi_1(X, \bar{x}) = \pi_1^{\mathrm{et}}(X, \bar{x})$, a profinite group which is defined by the natural analogue of the outer description 1). It classifies finite étale coverings of X. The following problem occurs naturally:

> *Find an "inner" description of $\pi_1(X, \bar{x})$, i.e. a description in geometric terms of the scheme X.*

So far, there seems to be no idea to attack this problem. A naive approach is lacking an appropriate object "S^1". The considerably weaker task of describing the maximal abelian quotient of the fundamental group runs under the slogan "class field theory". Among others, one technical advantage of considering only abelian coverings is that the maximal abelian quotient $\pi_1^{ab}(X,\bar{x})$ of $\pi_1(X,\bar{x})$ is (canonically) independent of the chosen base point $\bar{x}$, which will be omitted from the notation from now on. The classical example of class field theory is

Artin-reciprocity: *Let $k|\mathbb{Q}$ be a totally imaginary algebraic number field, $\mathcal{O}_k$ its ring of integers and $X = Spec(\mathcal{O}_k)$. Then there exists a natural isomorphism of finite abelian groups*

$$rec\colon \mathrm{Pic}(X) \xrightarrow{\sim} \pi_1^{ab}(X).$$

Artin reciprocity is a particular case of "one-dimensional class field theory", which was one of the major achievements of number theory in the first half of the previous century. It describes the abelian extensions of number fields, including the ramification and decomposition behaviour of its prime ideals.

The question for a higher dimensional analogue of Artin-reciprocity occurs naturally. There are two related approaches to study the geometry of X:

1. study vector bundles on X ($\Rightarrow$ K-theory)
2. study algebraic cycles on X ($\Rightarrow$ intersections theory, cycle groups).

Both approaches are related by the concept of *motivic (co)homology*, which is, however, still not sufficiently developed in the case of schemes over a Dedekind domain. For varieties over fields, a satisfying theory exists [VSF]. In the example $X = \mathrm{Spec}(\mathcal{O}_k)$ of Artin-reciprocity, we can interpret the group $\mathrm{Pic}(X)$ not only as the group of isomorphism classes of line bundles on X, but also as the group $\mathrm{CH}_0(X)$ of zero-cycles modulo rational equivalence and also as the first filtration step $F_0K_0(X)$ of the 0th K-group of X. The question occurs which of these interpretations is the suitable one for a higher dimensional generalization of class field theory.

1 Class field theory using Milnor K-groups

A first step towards a higher dimensional generalization of class field theory was made by K. Kato in 1982. We recall the following concepts:

Higher dimensional local fields are defined by induction. A 0-dimensional local field is a finite field. For $n \geq 1$, an n-dimensional local field is a field which is complete with respect to a discrete valuation and whose residue field is an $(n-1)$-dimensional local field. One-dimensional local fields are the usual locally compact local fields.

If R is a commutative ring with 1, the *Milnor K-groups* $K_n^M(R)$, $n \geq 0$, are defined by

$$K_n^M(R) = \overbrace{R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times}^{n-\text{times}} / (\cdots \otimes a \otimes \cdots \otimes 1 - a \otimes \dots).$$

If $R = k$ is a field, we have $K_n^M(k) = K_n(K)$ for $n \leq 2$ by Matsumoto's theorem.

Theorem 1 (K. Kato, [K1]) *If k is an n-dimensional local field, then there exists a natural reciprocity map*

$$rec\colon K_n^M(k) \longrightarrow G(k^{ab}|k).$$

For any finite Galois extension $\ell|k$, the reciprocity map induces an isomorphism

$$K_n^M(k)/Norm_{\ell/k}(K_n^M(\ell)) \xrightarrow{\sim} G(\ell|k)^{ab}.$$

Remark: The description of the norm groups is difficult (in dimension ≥ 3), see [K2].

The natural idea to describe the abelian extensions of an arbitrary regular arithmetic scheme is to consider the various higher dimensional local fields attached to it. Let $\bar{X}$ be a normal, connected scheme, projective and of finite type over $\mathrm{Spec}(\mathbb{Z})$ and let $X \subset \bar{X}$ be a non-empty open regular subscheme. Let $d = \dim(X)$ and assume for simplicity that $X(\mathbb{R}) = \varnothing$. We sheafify the notion of Milnor K-groups in order to obtain the Milnor K-sheaf $\mathcal{K}_d^M(\mathcal{O}_{\bar{X}})$ on $\bar{X}$. For a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ we have the closed immersion $i : Y := \mathcal{S}pec(\mathcal{O}_{\bar{X}}/\mathcal{I}) \longrightarrow \bar{X}$ and we define the relative Milnor K-sheaf by

$$\mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I}) = \ker\left(\mathcal{K}_d^M(\mathcal{O}_{\bar{X}}) \longrightarrow i_*\mathcal{K}_d^M(\mathcal{O}_Y)\right).$$

Finally, recall the 'completely decomposed' (c.d.) topology, a Grothendieck topology which lies between Zariski and étale topology and which is often also called Nisnevich topology (cf. [Ni]).

Theorem 2 (K. Kato and S. Saito, [KS2], see also [Ra]) *Let $\bar{X}$ be a normal connected scheme, projective and of finite type over Spec$(\mathbb{Z})$ and let $X \subset \bar{X}$ be a non-empty open regular subscheme. Let $d = \dim(X)$ and assume (for simplicity) that $X(\mathbb{R}) = \varnothing$. Then there exists a natural reciprocity map*

$$rec\colon \quad \varprojlim_{\substack{\mathcal{I}\subset\mathcal{O}_{\bar{X}} \\ \mathcal{I}_{|X}=\mathcal{O}_X}} H^d_{c.d.}(\bar{X}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})) \longrightarrow \pi_1^{ab}(X).$$

If X is flat over $\mathbb{Z}$, then rec is an isomorphism. If X is a variety over a finite field, then rec is injective and coker$(rec) \cong \hat{\mathbb{Z}}/\mathbb{Z}$.

This solves the problem of describing the abelianized fundamental group $\pi_1(X)^{ab}$ in terms of geometric data attached to X (if $X(\mathbb{R}) \neq \varnothing$, only a minor modification is necessary). Unfortunately, the left hand side of the reciprocity map is difficult to understand and, in particular, contains a cohomology group. It is therefore desirable to find a more direct description.

2 Class field theory using algebraic cycles - the compact case

Let us return to the topological considerations of the introduction and look for an 'algebraic S^1'. The easiest example of an arithmetic scheme is a point $\mathrm{Spec}(\mathbb{F})$, where $\mathbb{F}$ is a finite field. The fundamental group $\pi_1(\mathrm{Spec}(\mathbb{F})) = \mathrm{Gal}(\bar{\mathbb{F}}|\mathbb{F})$ is isomorphic to $\hat{\mathbb{Z}}$, a canonical generator is given by the Frobenius automorphism. Moreover, the higher étale homotopy groups (cf. [AM]) of $\mathrm{Spec}(\mathbb{F})$ vanish, i.e., we have

$$\pi_i^{\mathrm{et}}(\mathrm{Spec}(\mathbb{F})) = \begin{cases} \hat{\mathbb{Z}} & i = 1, \\ 0 & i \neq 1 \end{cases}$$

Therefore, from a homotopical point of view, finite fields can be considered as 'algebraic circles'. An arithmetic scheme contains many 'loops', namely its closed points, and we can try to exhaust $\pi_1(X)^{ab}$ by such 'loops'. Due to the problem of base points, this method is only applicable to the abelianized fundamental group.

Let X be a connected arithmetic scheme and let $x \in X$ be a closed point. We define $\mathrm{Frob}_x \in \pi_1^{ab}(X)$ as the image of $\mathrm{Frob} \in \mathrm{Gal}(\overline{k(x)}|k(x)) = \pi_1(x)^{ab}$ under the natural map $\pi_1^{ab}(x) \to \pi_1^{ab}(X)$. Let $Z_0(X)$ be the group of zero-cycles on X, i.e. the free abelian group generated by the closed points of X. We consider the map

$$r\colon Z_0(X) \longrightarrow \pi_1^{ab}(X), \quad 1_x \longmapsto \mathrm{Frob}_x.$$

Theorem 3 (S. Lang, [La]) *If the reduced subscheme of X is normal, then r has dense image.*

This means that, under a mild technical restriction, we can exhaust $\pi_1(X)^{ab}$ by 'algebraic loops'. Our next task is to find the appropriate homotopy relation among these loops. That means to find elements in $\ker(r)$, and most preferably a geometric description of these elements. Having found 'many' relations and dividing them out, we can hope to obtain a map which 'almost' an isomorphism. Note that the source of r is a discrete group and that its target carries a natural compact topology. Thus we cannot expect to find an actual isomorphism between a quotient of $Z_0(X)$ and $\pi_1^{ab}(X)$, unless the abelianized fundamental group is finite. In the general case, the map should induce an isomorphism on profinite completions.

The task of finding the right equivalence relation on $Z_0(X)$ was first solved in the case when X is projective. More precisely, the required equivalence relation in the compact case is nothing else but rational equivalence. The quotient of $Z_0(X)$ by this relation is called $\mathrm{CH}_0(X)$, the Chow group of zero-cycles on X. The next theorem was first proved by S. Bloch [Bl] for smooth arithmetic surfaces and then by K. Kato and S. Saito [KS1], [Sa] in the general case.

Theorem 4 (S. Bloch, K. Kato, S. Saito) *Let X be a regular, connected and projective scheme over $\mathbb{Z}$. Assume (for simplicity) that $X(\mathbb{R}) = \varnothing$. Then r factors through rational equivalence, inducing a reciprocity map*

$$rec\colon \mathrm{CH}_0(X) \longrightarrow \pi_1^{ab}(X).$$

If X is flat over $\mathbb{Z}$, then rec is an isomorphism of finite abelian groups. If X is a variety over a finite field, then rec is injective and coker(rec) $\cong \hat{\mathbb{Z}}/\mathbb{Z}$.

Remark: The nontrivial cokernel in the geometric case occurs because the image of *rec* contains only integral powers of the global Frobenius automorphism in $\pi_1^{ab}(X)$. We have degree maps $\mathrm{CH}_0(X) \to \mathbb{Z}$ and $\pi_1^{ab}(X) \to \hat{\mathbb{Z}}$ and *rec* induces an isomorphism of finite abelian groups $rec_0\colon \mathrm{CH}_0(X)^0 \xrightarrow{\sim} \pi_1^{ab}(X)^0$ on the degree-zero parts.

3 Class field theory using algebraic cycles - the tame open case in positive characteristic

Obviously, one wants to extend the geometric approach of the last section to the quasi-projective case. Let $\bar{X}$ be a regular, connected and projective scheme over $\mathbb{Z}$ and let $X \subset \bar{X}$ be a non-empty open subscheme. We still have the homomorphism with dense image $r\colon Z_0(X) \to \pi_1^{ab}(X)$, $1_x \mapsto \mathrm{Frob}_x$, of Theorem 3 and we want to determine its kernel. In other words, we have to find an appropriate equivalence relation on the group $Z_0(X)$. This cannot be rational equivalence by variance reasons: CH_0 becomes smaller for open subschemes and π_1^{ab} becomes bigger. More precisely, the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0(\bar{X}) & \xrightarrow{rec_{\bar{X}}} & \pi_1^{ab}(\bar{X}) \\ \downarrow & & \uparrow \\ \mathrm{CH}_0(X) & & \pi_1^{ab}(X) \end{array}$$

destroys all hope for the existence of a natural map $rec_X\colon \mathrm{CH}_0(X) \to \pi_1^{ab}(X)$ which is 'almost' an isomorphism. Another problem is that 'good' cycle theories are homotopy invariant (i.e. give the same result on a scheme X and on the affine line $\mathbb{A}^1_X$ over X). But this is not true for the abelianized fundamental group (already the fundamental group of the affine line over an algebraically closed field of positive characteristic is huge, cf. [Ry]). As a first step towards class field theory in the open case, we therefore restrict to the maximal *tame* quotient $\pi_1^{ab,t}(\bar{X}, \bar{X}{-}X)$, which is homotopy invariant. It classifies finite abelian (possibly ramified) coverings of $\bar{X}$ which are étale over X and have at most tame ramification along the boundary $\bar{X}{-}X$. This group only depends on the scheme X (see [S2]) and we will also use the shorter notation $\pi_1^{ab,t}(X)$ for it.

We will deal with the case of smooth varieties over finite fields first. The cycle theory we will need is the (abstract) singular homology defined by A. Suslin [SV]. We recall its definition. Let k be a field and let $\Delta^\bullet$ be the standard cosimplicial object in the category of smooth schemes over k, i.e. Δ^n is the n-dimensional simplex given as a subscheme in $\mathbb{A}_k^{n+1} = \mathrm{Spec}(k[T_0, \ldots, T_n])$ by the equation $\Sigma\, T_i = 1$ and the simplicial structure is given by the obvious face and degeneracy morphisms.

Let X be a scheme of finite type over k. Denote by $C_n(X)$ the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^n$ such that the projection $Z \to \Delta^n$ is finite and surjective. One verifies immediately that, if Z is as above, then for each face map $\delta^i : \Delta^{n-1} \to \Delta^n$ each component of $(\delta^i)^{-1}(Z) \subset X \times \Delta^{n-1}$ is finite and surjective over Δ^{n-1} and hence has the 'correct' dimension. So the cycle theoretic inverse image $(\delta^i)^*(Z)$ is well-defined and lies in $C_{n-1}(X)$. This gives us face operators

$$\partial_i = (\delta^i)^* : C_n(X) \longrightarrow C_{n-1}(X).$$

The homology groups of the complex

$$(C_\bullet(X), d), \qquad d = \sum (-1)^i \partial_i$$

will be denoted by $H_*^{sing}(X, \mathbb{Z})$ and are called the (integral) singular homology groups of X. Singular homology is covariantly functorial in the scheme X. By definition, $H_0^{sing}(X, \mathbb{Z})$ is the quotient of $C_0(X) = Z_0(X)$ by some equivalence relation. This equivalence relation is in general finer than rational equivalence.

Theorem 5 (A. S. & M. Spieß, [SS]) *Let $\bar{X}$ be a smooth connected variety over a finite field and let $X \subset \bar{X}$ be a nonempty open subscheme. Then r induces a reciprocity map*

$$rec\colon H_0^{sing}(X, \mathbb{Z}) \longrightarrow \pi_1^{ab,t}(X).$$

rec is injective, $coker(rec) \cong \hat{\mathbb{Z}}/\mathbb{Z}$ and the induced map on the degree-zero parts $rec_0\colon \mathrm{CH}_0(X)^0 \xrightarrow{\sim} \pi_1^{ab}(X)^0$ is an isomorphism of finite abelian groups.

Remarks: 1. If $\dim X = 1$, then (see [SV], Theorem 3.1) $H_0^{sing}(X, \mathbb{Z})$ is naturally isomorphic to the relative Picard group $\mathrm{Pic}(\bar{X}, \bar{X} - X)$. A straightforward computation identifies this relative Picard group with the ray class group of the function field $k(X)$ of X with modulus $\mathfrak{m}_X$, where $\mathfrak{m}_X$ is the (square-free) product of all primes of $\bar{X} - X$. In this case, *rec* is the reciprocity homomorphism of the classical (one-dimensional) class field theory for global fields of positive characteristic.

2. If $X = \bar{X}$ is projective, we have a natural isomorphism $H_0^{sing}(\bar{X}, \mathbb{Z}) \cong \mathrm{CH}_0(\bar{X})$. In this case, Theorem 5 is just a reformulation of the geometric case of Theorem 4 (which we use in the proof).

3. The scheme $\bar{X}$ in the theorem occurs just for technical reasons. Its existence is known in dimension ≤ 2 and any kind of desingularization theorem in positive characteristic would imply that the theorem holds for an arbitrary smooth, quasiprojective and connected variety X over a finite field.

4 Class field theory using algebraic cycles - the tame open case in mixed characteristic

Now we want to obtain a similar result in mixed characteristics. For technical reasons we restrict to the following situation:

> *$\bar{X}$ is a connected, regular scheme, flat and projective over $\mathbb{Z}$, D is a divisor on $\bar{X}$ and $X = \bar{X}$–supp(D). For simplicity, we assume that $X(\mathbb{R}) = \varnothing$.*

We have the following finiteness result:

Theorem 6 ([S1]) *Under the given assumptions, $\pi_1^{ab,t}(X)$ is finite.*

Looking for an appropriate cycle theory, the first problem we are confronted with is that Suslin's singular homology is a relative construction. For flat schemes over $\mathbb{Z}$, it does not give the right cycle theory for class field theory. Some yoga about the 'field with one element' suggests the following absolute version of singular homology groups for arithmetic schemes (cf. [S2]). We put

$C_n(X)$ = free abelian group on closed integral subschemes $Z \subset X \times \Delta^n_{\mathbb{Z}}$ such that the restriction of the projection $X \times \Delta^n_{\mathbb{Z}} \to \Delta^n_{\mathbb{Z}}$ to Z induces a finite morphism $Z \to T \subset \Delta^n_{\mathbb{Z}}$ onto a closed integral subscheme T of codimension 1 in $\Delta^n_{\mathbb{Z}}$ which intersects all faces $\Delta^m_{\mathbb{Z}} \subset \Delta^n_{\mathbb{Z}}$ properly.

We obtain a complex $(C_\bullet(X), d)$ in the usual way and denote its homology groups by $H_*^{sing}(X, \mathbb{Z})$. We call these groups the (integral) singular homology groups of X. This name is justified because for varieties over finite fields these groups coincide with those defined by Suslin. It turns out, however, that it is rather difficult to verify even basic properties of this homology theory because we are lacking good techniques of moving cycles in mixed characteristics. See [S2] for partial results. Concerning class field theory, we first note that $C_0(X)$ is nothing else but the group $Z_0(X)$ of zero-cycles on X. The following proposition can be deduced from the one-dimensional case:

Proposition 7 *The composite map*

$$Z_0(X) \xrightarrow{r} \pi_1(X)^{ab} \longrightarrow \pi_1^{ab,t}(X)$$

factors through $H_0^{sing}(X, \mathbb{Z})$, thus defining a surjective reciprocity homomorphism

$$rec : H_0^{sing}(X, \mathbb{Z}) \longrightarrow \pi_1^{ab,t}(X).$$

We conjecture that the reciprocity map $rec : H_0^{sing}(X, \mathbb{Z}) \longrightarrow \pi_1^{ab,t}(X)$ is an isomorphism. At present, we can prove this only if X is projective or if $\dim X = 1$. Therefore we change from singular homology to relative Chow groups, which are easier to deal with by using techniques from K-theory. Their definition goes as follows:

Let $G_*(X)$ denote the version of Quillen's K-theory based on the category of coherent sheaves. Recall [Qu] the usual Quillen spectral sequence for X

$$E_1^{pq}(X) = \bigoplus_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow G_{-p-q}(X),$$

which is associated to the filtration by codimension of support. If d is the dimension of X, the Chow group of zero-cycles on X and the term $E_2^{d,-d}$ of the above spectral sequence are naturally isomorphic. One can (see [S3]) construct a similar spectral sequence

$$E_1^{pq}(\bar{X}, D) \Longrightarrow G_{-p-q}(\bar{X}, D)$$

converging to relative G-theory (the E_1-terms are K-groups of certain categories) and we call $\mathrm{CH}_0(\bar{X}, D) := E_2^{d,-d}(\bar{X}, D)$ the relative Chow group of 0-cycles of $(\bar{X}, D)$. One can show that $\mathrm{CH}_0(\bar{X}, D)$ is a quotient of $Z_0(X)$ by an equivalence relation which is (a priori) coarser than the relation defining H_0. More precisely, we have a natural surjection

$$H_0^{sing}(X, \mathbb{Z}) \longrightarrow \mathrm{CH}_0(\bar{X}, D),$$

which we conjecture to be an isomorphism. A priori, it is even not obvious that $\mathrm{CH}_0(\bar{X}, D)$ only depends on the scheme X, i.e. is independent of the particular compactification $\bar{X}$. This is, however, a consequence of the following theorem, which is the main result of [S3] and provides tame class field theory in the mixed characteristic case, at least under a mild technical restriction.

Theorem 8 ([S3]) *Assume that the vertical irreducible components of D are normal schemes. Then the composite map $Z_0(X) \xrightarrow{r} \pi_1^{ab}(X) \to \pi_1^{ab,t}(X)$ factors through $\mathrm{CH}_0(\bar{X}, D)$ and induces an isomorphism of finite abelian groups*

$$rec\colon \mathrm{CH}_0(\bar{X}, D) \xrightarrow{\sim} \pi_1^{ab,t}(X).$$

Remarks. 1. In many cases (e.g. if X is semi-stable), the condition in the theorem on the vertical components of D is void. Furthermore, this condition can be weakened (see [S3] Theorem 6.5).

2. If D is zero, Theorem 8 reduces to Theorem 4 (which we use in the proof).

3. If $\dim X = 1$, then $\mathrm{CH}_0(\bar{X}, D)$ is isomorphic to the ray class group of the number field $k(X)$ with modulus $\mathfrak{m}_D$ where $\mathfrak{m}_D$ is the (square-free) product of the points in D. In this case, *rec* is the reciprocity homomorphism of classical (one-dimensional) class field theory.

Because of the easier and more direct definition, we would prefer to replace the relative Chow group in the above theorem by the 0th singular homology group. On the other side, Theorem 8 is 'better' than the conjectured version with singular homology since it detects more relations in $Z_0(X)$. However, our final goal is to show that relative Chow groups and singular homology coincide.

To shed some light on the relations on $Z_0(X)$ which define both groups, we give their explicit descriptions below. Roughly speaking, the relations on $Z_0(X)$ defining $H_0^{sing}(X,\mathbb{Z})$ are generated by those of the form $div(f)$ where the f's are functions on curves on X which are defined and $\equiv 1$ at the boundary. A relation in the relative Chow group is given as a sum $div(f_1) + \cdots + div(f_n)$, where $f_1, \dots, f_n$ are rational functions defined on curves on X whose generalized product exists and is 1 at every point of the boundary. The generalized product at a point y is defined if the zero and pole orders at y add to zero (see below).

Let us make this precise. We start with singular homology. Let C be an integral scheme of finite type over $\mathbb{Z}$ and of (Krull)dimension 1. Then to every rational function $f \neq 0$ on C, we can attach the zero-cycle $div(f) \in Z_0(C)$ (see [Fu], Ch.I,1.2). Let $\tilde{C}$ be the normalization of C in its field of functions and let $P(\tilde{C})$ the regular compactification of $\tilde{C}$, i.e. the uniquely determined regular and connected scheme of dimension 1 which is proper over $\mathbb{Z}$ and which contains $\tilde{C}$ as an open subscheme.

Theorem 9 ([S2]) *The group $H_0^{sing}(X,\mathbb{Z})$ is the quotient of $Z_0(X)$ by the subgroup generated by elements of the form $div(f)$, where*

- *C is a closed integral curve on X,*
- *f is a rational function on C which, considered as a rational function on $P(\tilde{C})$, is defined and $\equiv 1$ at every point of $P(\tilde{C}) - \tilde{C}$.*

Now we describe the relations defining the relative Chow group. We first define the generalized product of functions at a point. Let y be a closed point of supp(D), D_i an irreducible component of D passing through y and π_i a uniformizer of D_i near y. Let $C = \overline{\{x\}}$ be an irreducible curve on X (i.e. $x \in X_1$) and let $f \in k(x)^\times$ be a rational function on C. We define the 'value' $f^{(i)}(y) \in k(y)^\times$ as follows: If $y \notin \bar{C}$, we put $f^{(i)}(y) = 1$. Otherwise, π_i defines an element in $k(x)^\times$ and we put $f^{(i)}(y) := \delta(f \cup \pi_i)$, where $\delta : K_2(k(x)) \to k(y)^\times$ is the boundary map induced by the Quillen spectral of $\bar{X}$ and $\cup : k(x)^\times \times k(x)^\times \to K_2(k(x))$ is the product map. The 'value' $f^{(i)}(y)$ depends on the choice of the uniformizer π_i, unless f is defined at y, in which case $f^{(i)}(y) = f(y)$. If $C_1, \dots, C_s$ are irreducible curves on X and $f_j \in k(C_j)^\times$, $j = 1, \dots, s$, are rational functions, then a straightforward computation shows that the product
$$f_1^{(i)}(y) \cdots f_s^{(i)}(y) \in k(y)^\times$$
is independent of the choice of π_i if the sum of the zero and pole orders of the f_j at y is zero. In this case, we call $f_1^{(i)}(y) \cdots f_s^{(i)}(y)$ the generalized product of the 'values' $f_j^{(i)}(y)$ relative to D_i. If all f_j are defined at y, then the generalized product is nothing else but the usual product of the values $f_j(y)$.

Let $D_1, \dots, D_r$ be the irreducible components of D and set $Y = \text{supp}(D) = D_1 \cup \cdots \cup D_r$.

Theorem 10 ([S3]) $\mathrm{CH}_0(\bar{X}, D)$ *is the quotient of* $Z_0(X)$ *by the image of the group*

$$R_{X,D} \stackrel{\mathrm{df}}{=} \ker\Big(\ker\big(\bigoplus_{x \in X_1} k(x)^\times \xrightarrow{\phi} \bigoplus_{y \in Y_0} \mathbb{Z} \big) \xrightarrow{\psi} \bigoplus_{i=1}^{r} \bigoplus_{z \in (D_i)_0} k(z)^\times \Big)$$

under the divisor map div: $\bigoplus_{x \in X_1} k(x)^\times \to Z_0(X)$. *The map* ϕ *is the composite*

$$\bigoplus_{x \in X_1} k(x)^\times \stackrel{incl}{\to} \bigoplus_{x \in \bar{X}_1} k(x)^\times \stackrel{div}{\to} Z_0(\bar{X}) \stackrel{proj}{\to} Z_0(Y)$$

and the map ψ *is given by the generalized product.*

This shows that $\mathrm{CH}_0(\bar{X}, D)$ is a quotient of $H_0^{sing}(X, \mathbb{Z})$. If we could 'move' any relation of the form $div(f_1) + \cdots + div(f_n)$, $f_j \in k(C_j)^\times$, to a relation of the form $div(f'_1) + \cdots + div(f'_m)$, $f'_j \in k(C'_j)^\times$ such that $\bar{C}'_i \cap \bar{C}'_j$ is disjoint to $Y = \mathrm{supp}(D)$ for $i \neq j$, then each of the $div(f'_j)$, $j = 1, \ldots, m$, and hence also their sum, would be a relation for $H_0^{sing}(X, \mathbb{Z})$. This would imply that $\mathrm{CH}_0(\bar{X}, D) = H_0^{sing}(X, \mathbb{Z})$. Unfortunately, at present, we have no such moving-technique available.

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