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Abstract

We consider the Cauchy problem for the massless scalar wave equation in the Kerr geometry for smooth initial data compactly supported outside the event horizon. We prove that the solutions decay in time in L^∞_{loc}. The proof is based on a representation of the solution as an infinite sum over the angular momentum modes, each of which is an integral of the energy variable ω on the real line. This integral representation involves solutions of the radial and angular ODEs which arise in the separation of variables.

Contents

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1 Introduction

In this paper we study the long-time dynamics of massless scalar waves in the Kerr geometry. We prove that solutions of the Cauchy problem with smooth initial data which is compactly supported outside the event horizon, decay in L^∞_{loc}. Our starting point is the integral representation for the propagator [5], which involves an integral over a complex contour in the energy variable ω. In order to study the long-time dynamics, we must deform the contour to the real line. To this end, we carefully analyze the solutions of the associated radial and angular ODEs which arise in the separation of variables. In particular, we show that the integrand in our representation has no poles on the real axis. We call such poles *radiant modes*, because in a dynamical situation they would lead to continuous radiation coming out of the ergosphere.

We now set up some notation and state our main result. As in [5], we choose Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, in which the Kerr metric takes the form

$$ds^2 \;=\; \frac{\Delta}{U}\,(dt \;-\; a\,\sin^2\vartheta\,d\varphi)^2 \;-\; U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \;-\; \frac{\sin^2\vartheta}{U}\,(a\,dt \;-\; (r^2+a^2)\,d\varphi)^2 \quad (1.1)$$

with

$$U(r,\vartheta) \;=\; r^2 + a^2\,\cos^2\vartheta\,, \qquad \Delta(r) \;=\; r^2 - 2Mr + a^2\,,$$

where M and aM denote the mass and the angular momentum of the black hole, respectively. We restrict attention to the *non-extreme case* $M^2 > a^2$, where the function Δ has two distinct zeros,

$$r_0 \;=\; M \;-\; \sqrt{M^2-a^2} \qquad \text{and} \qquad r_1 \;=\; M \;+\; \sqrt{M^2-a^2}\,,$$

corresponding to the Cauchy and the event horizon, respectively. We consider only the region $r > r_1$ outside the event horizon, and thus $\Delta > 0$. The ergosphere is the region where the Killing vector $\frac{\partial}{\partial t}$ is space-like, that is where

$$r^2 - 2Mr + a^2\,\cos^2\vartheta < 0\,. \qquad (1.2)$$

The ergosphere lies outside the event horizon $r = r_1$, and its boundary intersects the event horizon at the poles $\vartheta = 0, \pi$.

Theorem 1.1 *Consider the Cauchy problem for the wave equation in the Kerr geometry for smooth initial data which is compactly supported outside the event horizon and has fixed angular momentum in the direction of the rotation axis of the black hole, i.e. for some* $k \in \mathbb{Z}$,

$$(\Phi_0, \partial_t\Phi_0) \;=\; e^{-ik\varphi}\,(\Phi_0, \partial_t\Phi_0)(r,\vartheta) \;\in\; C^\infty_0((r_1,\infty)\times S^2)^2\,.$$

Then the solution decays in $L^\infty_{loc}((r_1,\infty)\times S^2)^2$ *as* $t \to \infty$.

The study of linear hyperbolic equations in a black hole geometry has a long history. Regge and Wheeler [12] considered the radial equation for metric perturbations of the Schwarzschild metric. In the late 1960s and early 1970s, Carter, Teukolsky and Chandrasekhar discovered that the equations describing scalar, Dirac, Maxwell and linearized gravitational fields in the Kerr geometry are separable into ordinary differential equations (see [2]). Much research has been done concerning the long-time behavior of the solutions of these equations, through both numerical and analytical methods. Price [10] gave arguments which indicated decay of solutions of the scalar wave equation in the Schwarzschild geometry. Press and Teukolsky [9] did a numerical study which strongly suggested the absence of unstable modes, and Whiting [11] later proved that for ω in the complex plane, such unstable modes cannot exist. This "mode stability" does not rule out that there might be unstable modes for real ω (what we call radiant modes). Furthermore, mode stability does not lead to any statement on the Cauchy problem. Finally, Kay and Wald [7] used energy estimates to prove a boundedness result for solutions of the scalar wave equation in the Schwarzschild geometry.

Unfortunately, these energy methods cannot be used in a rotating black hole geometry, because the energy density is indefinite inside the ergosphere, making it impossible to introduce a positive definite conserved scalar product. This difficulty was dealt with in [5, 6], where Whiting's mode stability result was combined with estimates for the resolvent and for the radial and angular ODEs. In [5] we established an integral representation which expresses the solution as a contour integral of an integrand involving the separated radial and angular eigenfunctions over a contour staying within a neighborhood located arbitrarily close to the real axis. This integral representation is the starting point of the present paper. After deforming the contours onto the real axis, we can prove decay using the Riemann-Lebesgue Lemma, similar to the case of the Dirac equation [4]. We remark that the decay result of our paper would not be expected to hold for a massive scalar field satisfying the Klein-Gordon equation, as indicated in [1].

Finally, we note that the problem considered here is closely related to one of the major open questions in general relativity; namely the problem of linearized stability of the Kerr metric. For the stability under metric perturbations one considers the equation for linearized gravitational waves, which can be identified with the general wave equation for spin $s = 2$ (see [2]). Thus replacing scalar waves ($s = 0$) by gravitational waves ($s = 2$), the above theorem would prove linearized stability of the Kerr metric. However, the analysis for $s = 2$ would be considerably more difficult due to the complexity of the linearized Einstein equations. Nevertheless, we regard this paper as a first step towards proving linearized stability of the Kerr metric.

2 Preliminaries

We recall a few constructions and results from [5, 6] which will be needed later on. As radial variable we usually work with the Regge-Wheeler variable $u \in \mathbb{R}$ defined by

$$\frac{du}{dr} = \frac{r^2 + a^2}{\Delta}\,; \tag{2.1}$$

then $u = -\infty$ corresponds to the event horizon. It is most convenient to write the wave equation in the Hamiltonian form

$$i\,\partial_t \Psi = H\,\Psi, \tag{2.2}$$

where $\Psi = (\Phi, i\partial_t \Phi)$. The Hamiltonian can be written as

$$H = \begin{pmatrix} 0 & 1 \\ A & \beta \end{pmatrix}, \tag{2.3}$$

where

$$A = \frac{1}{\rho}\left[-\frac{\partial}{\partial u}(r^2+a^2)\frac{\partial}{\partial u} - \frac{\Delta}{r^2+a^2}\,\Delta_{S^2} - \frac{a^2k^2}{r^2+a^2}\right] \tag{2.4}$$

$$\beta = -\frac{2ak}{\rho}\left(1-\frac{\Delta}{r^2+a^2}\right) \tag{2.5}$$

$$\rho = r^2+a^2 \;-\; a^2\,\sin^2\vartheta\,\frac{\Delta}{r^2+a^2}\,. \tag{2.6}$$

The operators A and β are symmetric on the Hilbert space $L^2(\mathbb{R}\times S^2, d\mu)^2$ with the measure

$$d\mu \;:=\; \rho\,du\,d\cos\vartheta\,. \tag{2.7}$$

It is immediately verified that the Hamiltonian is symmetric with respect to the bilinear form

$$<\Psi_1,\Psi_2> \;=\; \int_{\mathbb{R}\times S^2} \langle \Psi_1, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_2\rangle_{\mathbb{C}^2}\,d\mu\,. \tag{2.8}$$

As is worked out in detail in [5], the inner product $<\Psi,\Psi>$ is the physical energy of Ψ. Therefore, we refer to $<.,.>$ as the *energy scalar product*. The fact that the energy scalar product is not positive definite can be understood from the fact that the operator A is not positive on $L^2(\mathbb{R}\times S^2, d\mu)$.

Using the ansatz

$$\Phi(t,r,\vartheta,\varphi) = e^{-i\omega t - ik\varphi}\,R(r)\,\Theta(\vartheta), \tag{2.9}$$

the wave equation can be separated into an angular and a radial ODE,

$$\mathcal{R}_{\omega,k}\,R_\lambda \;=\; -\lambda\,R_\lambda, \qquad \mathcal{A}_{\omega,k}\,\Theta_\lambda \;=\; \lambda\,\Theta_\lambda\,. \tag{2.10}$$

Here the angular operator $\mathcal{A}_{\omega,k}$ is also called the *spheroidal wave operator*. The separation constant λ is an eigenvalue of $\mathcal{A}_{\omega,k}$ and can thus be regarded as an angular quantum number. In [6] it was shown that if ω is in a small neighborhood of the real line, more precisely if

$$\omega \;\in\; U_\varepsilon \;:=\; \left\{\omega\in\mathbb{C} \mid |\mathrm{Im}\,\omega| < \frac{\varepsilon}{1+|\mathrm{Re}\,\omega|}\right\},$$

then for sufficiently small $\varepsilon>0$ the angular operator $\mathcal{A}_{\omega,k}$ has a purely discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$ with corresponding one-dimensional eigenspaces which span the Hilbert space $L^2(S^2)$. We denote the projections onto the eigenspaces by $Q_n(k,\omega)$. These projections as well as the corresponding eigenvalues λ_n are holomorphic in $\omega\in U_\varepsilon$. In analogy to the eigenvalues $l(l+1)$ of the Laplacian on the sphere, the angular eigenvalues λ_n grow quadratically for large n in the sense that there is a constant $C(k,\omega)>0$ such that

$$|\lambda_n(k,\omega)| \;\geq\; \frac{n^2}{C(k,\omega)} \qquad \text{for all } n\in\mathbb{N}\,. \tag{2.11}$$

We set

$$\omega_0 \;=\; -\frac{ak}{r_1^2+a^2} \tag{2.12}$$

with r_1 the event horizon and use the notation

$$\Omega(\omega) \;=\; \omega-\omega_0\,. \tag{2.13}$$

In order to bring the radial equation into a convenient form, we introduce a new radial function $\phi(r)$ by

$$\phi(r) \;=\; \sqrt{r^2+a^2}\, R(r) \;.$$

Then in the Regge-Wheeler variable, the radial equation can be written as the "Schrödinger equation"

$$\left(-\frac{d^2}{du^2} + V(u) \right) \phi(u) \;=\; 0 \tag{2.14}$$

with the potential

$$V(u) \;=\; -\left(\omega + \frac{ak}{r^2+a^2}\right)^2 \;+\; \frac{\lambda_n(\omega)\,\Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\, \partial_u^2 \sqrt{r^2+a^2}\,. \tag{2.15}$$

In [5] we derived an integral representation for the solution of the Cauchy problem of the following form,

$$\begin{aligned} &\Psi(t,r,\vartheta,\varphi) \\ &\quad=\; -\frac{1}{2\pi i} \sum_{k\in\mathbf{Z}} e^{-ik\varphi} \sum_{n\in\mathbb{N}} \lim_{\varepsilon\searrow 0} \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}} \right) d\omega\; e^{-i\omega t}\, (Q_{k,n}(\omega)\, S_\infty(\omega)\, \Psi_0^k)(r,\vartheta)\,. \end{aligned} \tag{2.16}$$

Here the integration contour C_ε must lie inside the set U_ε.

3 Asymptotic Estimates for the Radial Equation

3.1 Holomorphic Families of Radial Solutions

In this section we fix the angular quantum numbers k, n and consider solutions $\acute{\phi}$ and $\grave{\phi}$ of the Schrödinger equation (2.14) which satisfy the following asymptotic boundary conditions on the event horizon and at infinity, respectively,

$$\lim_{u\to-\infty} e^{-i\Omega u}\, \acute{\phi}(u) \;=\; 1\,, \qquad \lim_{u\to-\infty} \left(e^{-i\Omega u}\, \acute{\phi}(u)\right)' \;=\; 0 \tag{3.1}$$

$$\lim_{u\to\infty} e^{i\omega u}\, \grave{\phi}(u) \;=\; 1\,, \qquad \lim_{u\to\infty} \left(e^{i\omega u}\, \grave{\phi}(u)\right)' \;=\; 0\,. \tag{3.2}$$

These solutions were introduced in [5] for ω in the lower complex half plane intersected with U_ε. Here we will show that they are holomorphic in ω, and we will extend their definition to a larger ω-domain. More precisely, we prove the following two theorems.

Theorem 3.1 *The solutions $\acute{\phi}$ are well-defined on the domain*

$$D \;=\; U_\varepsilon \cap \left\{ \omega\in\mathbb{C} \,|\, \mathrm{Im}\,\omega \;\leq\; \frac{r_1-r_0}{2(r_1^2+a^2)} \right\}.$$

They form a holomorphic family of solutions in the sense that for every fixed $u \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $\acute{\phi}(u)$ is holomorphic in $\omega \in D$.

Theorem 3.2 *For every angular momentum number n there is an open set E containing the real line except for the origin,*

$$E \;\supset\; E_0 \;:=\; U_\varepsilon \cap \{\omega\in\mathbb{C} \,|\, \mathrm{Im}\,\omega \leq 0 \text{ and } \omega\neq 0\}\,, \tag{3.3}$$

such that the solutions $\grave{\phi}$ are well-defined for all $\omega \in E$ and form a holomorphic family on E.

For the proofs we will rewrite the Schrödinger equation with boundary conditions (3.1, 3.2) as an integral equation (which in different contexts is called Lipman-Schwinger or Jost equation). Then we will perform a perturbation expansion and get estimates for all the terms of the expansion. To introduce the method, we begin with the solutions $\acute{\phi}$; the solutions $\grave{\phi}$ will be treated later with a similar technique. First we write the Schrödinger equation (2.14) in the form

$$\left(-\frac{d^2}{du^2} - \Omega^2\right) \acute{\phi}(u) \;=\; -W(u)\, \acute{\phi}(u) \tag{3.4}$$

with a potential $W = \Omega^2 + V(u)$ which vanishes at $u = -\infty$. We define the *Green's function* of the differential operator $-\partial_u^2 - \Omega^2$ by the distributional equation

$$(-\partial_v^2 - \Omega^2)\, S(u,v) \;=\; \delta(u-v)\,. \tag{3.5}$$

The Green's function is not unique; we choose it such that its support is contained in the region $v \leq u$; i.e.

$$S(u,v) \;=\; \Theta(u-v) \times \left\{ \begin{array}{cl} \frac{1}{2i\Omega} \left(e^{-i\Omega(u-v)} - e^{i\Omega(u-v)}\right) & \text{if } \Omega \neq 0 \\ v-u & \text{if } \Omega = 0\,. \end{array} \right. \tag{3.6}$$

(Here Θ denotes the Heaviside function defined by $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise.) We multiply (3.4) by the Green's function and integrate,

$$\int_{-\infty}^{\infty} S(u,v) \left((-\partial_v^2 - \Omega^2)\, (\acute{\phi}(v) - e^{i\Omega u})\right) dv \;=\; -\int_{-\infty}^{\infty} S(u,v)\, W(v)\, \acute{\phi}(v)\, dv\,.$$

If we assume for the moment that $\acute{\phi}$ satisfies the desired boundary conditions (3.1), we can integrate by parts on the left and use (3.5). This gives the Lipman-Schwinger equation

$$\acute{\phi}(u) \;=\; e^{i\Omega u} - \int_{-\infty}^{u} S(u,v)\, W(v)\, \acute{\phi}(v)\, dv\,,$$

which in the context of potential scattering is also called *Jost equation* (see e.g. [3]). Its significance lies in the fact that we can now easily perform a perturbation expansion in the potential W. Namely, taking for $\acute{\phi}$ the ansatz as the perturbation series

$$\acute{\phi} \;=\; \sum_{l=0}^{\infty} \phi^{(l)}\,, \tag{3.7}$$

we are led to the iteration scheme

$$\left. \begin{array}{rcl} \phi^{(0)}(u) &=& e^{i\Omega u} \\ \phi^{(l+1)}(u) &=& -\displaystyle\int_{-\infty}^{u} S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\,. \end{array} \right\} \tag{3.8}$$

This iteration scheme can be used for constructing solutions of the Jost equation, and this will give us the functions $\acute{\phi}$ with the desired properties.

Proof of Theorem 3.1. Fix $\omega \in D$. As the potential W is smooth in r and vanishes on the event horizon, we know that W has near r_1 the asymptotics $W = \mathcal{O}(r - r_1)$. This means in the Regge-Wheeler variable (2.1) that W decays exponentially as $u \to -\infty$. More precisely, there is a constant $c > 0$ such that

$$|W(u)| \;\leq\; c\, e^{\gamma u} \qquad \text{with} \qquad \gamma \;:=\; \frac{r_1 - r_0}{r_1^2 + a^2}\,. \tag{3.9}$$

Let us show inductively that

$$|\phi^{(l)}(u)| \;\leq\; \mu^l\, e^{-\mathrm{Im}\,\Omega\, u} \qquad \text{with} \qquad \mu \;:=\; \frac{c\, e^{\gamma u}}{(\gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|)^2}\,. \tag{3.10}$$

In the case $l = 0$, the claim is obvious from (3.8). Thus assume that (3.10) holds for a given l. Then, estimating the integral equation in (3.8) using (3.9), we obtain

$$|\phi^{(l+1)}(u)| \;\leq\; c\, \mu^l \int_{-\infty}^{u} |S(u,v)|\, e^{(\gamma - \mathrm{Im}\,\Omega)\, v}\, dv\,. \tag{3.11}$$

The Green's function (3.6) can be estimated in the case $v \leq u$ by

$$|S(u,v)| \;=\; \frac{u-v}{2} \left| \int_0^1 e^{-i\Omega(u-v)\,\tau}\, d\tau \right| \;\leq\; (u-v)\, e^{|\mathrm{Im}\,\Omega|\,(u-v)}\,.$$

Substituting this inequality in (3.11) gives

$$|\phi^{(l+1)}(u)| \;\leq\; c\, \mu^l\, e^{|\mathrm{Im}\,\Omega|\, u} \int_{-\infty}^{u} (u-v)\, e^{(\gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|)\, v}\, dv\,.$$

Since the parameter $\alpha := \gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|$ is positive according to the definition of D, we can carry out the integral as follows,

$$\int_{-\infty}^{u} (u-v)\, e^{\alpha v}\, dv \;=\; \left(u - \frac{d}{d\alpha}\right) \int_{-\infty}^{u} e^{\alpha v}\, dv \;=\; \left(u - \frac{d}{d\alpha}\right) \frac{e^{\alpha u}}{\alpha} \;=\; \frac{e^{\alpha u}}{\alpha^2}\,.$$

This gives (3.10) with l replaced by $l+1$.

Since for u on a compact interval, the analytic dependence of the solutions in ω from the coefficients and the initial conditions follows immediately from the Picard-Lindelöf Theorem, it suffices to consider the region $u < u_0$ for any $u_0 \in \mathbb{R}$. By choosing u_0 sufficiently small, we can arrange that $\mu < 1/2$ for all $u < u_0$. Then the estimate (3.10) shows that the perturbation series (3.7) converges absolutely, uniformly in $u \in (-\infty, u_0)$. Using similar estimates for the u-derivatives of $\phi^{(l)}$, one sees furthermore that the perturbation series (3.10) can be differentiated term by term, and using (3.5) we find that $\acute{\phi}$ is indeed a solution of (3.4). Furthermore,

$$\acute{\phi}(u) - e^{i\Omega u} \;=\; \sum_{l=1}^{\infty} \phi^{(l)}(u)\,,$$

and taking the limit $u \to \infty$ and using (3.10) we find that the right side goes to zero. Using the same argument for the first derivatives, we obtain (3.1).

In order to prove that $\acute{\phi}$ is analytic in ω, we first note that if $\Omega \neq 0$, we can differentiate the perturbation series (3.7) term by term and verify that the Cauchy-Riemann equations are satisfied (note that λ_n is holomorphic in ω according to [6]). Since $\acute{\phi}$ is bounded near $\Omega = 0$, it is also analytic at $\Omega = 0$. ■

We turn to the solutions $\acute{\phi}$. In analogy to (3.4), we now write the Schrödinger equation as

$$\left(-\frac{d^2}{du^2} - \omega^2 \right) \acute{\phi}(u) \;=\; -W(u)\, \acute{\phi}(u) \tag{3.12}$$

with

$$W(u) \;=\; -\omega \frac{2ak}{r^2+a^2} \;-\; \frac{(ak)^2}{(r^2+a^2)^2} \;+\; \frac{\lambda_n\, \Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\, \partial_u^2 \sqrt{r^2+a^2}\,. \tag{3.13}$$

Assuming that $\omega \neq 0$, we choose the Green's function as

$$S(u,v) \;=\; \frac{1}{2i\omega} \left(e^{-i\omega(v-u)} - e^{i\omega(v-u)} \right) \, \Theta(v-u)\,. \tag{3.14}$$

The corresponding Jost equation is

$$\acute{\phi}(u) \;=\; e^{-i\omega u} - \int_u^\infty S(u,v)\, W(v)\, \acute{\phi}(v)\, dv\,.$$

The perturbation series ansatz

$$\acute{\phi} \;=\; \sum_{l=0}^\infty \phi^{(l)} \tag{3.15}$$

leads to the iteration scheme

$$\left.\begin{aligned} \phi^{(0)}(u) &\;=\; e^{-i\omega u} \\ \phi^{(l+1)}(u) &\;=\; -\int_u^\infty S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\,. \end{aligned}\right\} \tag{3.16}$$

Note that, in contrast to the exponential decay (3.9), now the potential W, (3.13), has only polynomial decay. As a consequence, the iteration scheme allows us to construct $\acute{\phi}$ only inside the set E_0 as defined in (3.3).

Lemma 3.3 *The solutions $\acute{\phi}$ are well-defined for every $\omega \in E_0$. They form a holomorphic family in the interior of E_0.*

Proof. Fix $\omega \in E_0$. Then $\omega \neq 0$ and $\mathrm{Im}\,\omega \leq 0$, and this allows us to estimate the potential (3.13) and the Green's function (3.14) for $u, v > u_0$ and some $u_0 > 0$ by

$$|W(v)| \;\leq\; \frac{c}{v^2}\,, \qquad\qquad |S(u,v)| \;\leq\; \frac{1}{|\omega|}\, e^{\mathrm{Im}\,\omega\,(u-v)}\,. \tag{3.17}$$

Let us show by induction that

$$|\phi^{(l)}(u)| \;\leq\; \frac{1}{l!} \left(\frac{c}{|\omega|\, u} \right)^l e^{\mathrm{Im}\,\omega\, u}\,.$$

For $l = 0$ this is obvious from (3.16), whereas the induction step follows by estimating the integral equation in (3.16) with (3.17),

$$\begin{aligned} |\phi^{(l+1)}(u)| &\;\leq\; \frac{1}{l!} \left(\frac{c}{|\omega|} \right)^n \int_u^\infty \frac{1}{|\omega|}\, e^{\mathrm{Im}\,\omega\,(u-v)}\, \frac{c}{v^{2+l}}\, e^{\mathrm{Im}\,\omega\, v} \\ &\;=\; \frac{1}{(l+1)!} \left(\frac{c}{|\omega|\, u} \right)^{l+1} e^{\mathrm{Im}\,\omega\, u}\,. \end{aligned}$$

Hence the perturbation series (3.15) converges absolutely, locally uniformly in u. It is straightforward to check that $\grave{\phi}$ satisfies the Schrödinger equation (3.12) with the correct boundary values (3.2). If $\mathrm{Im}\,\omega < 0$, one can differentiate the series (3.15) term by term with respect to ω and verify that the Cauchy-Riemann equations are satisfied. ∎

It remains to analytically extend the solutions $\grave{\phi}$ for fixed n to a neighborhood of any point $\omega_0 \in \mathbb{R} \setminus \{0\}$. To this end, we need good estimates of the derivatives of $\grave{\phi}$ with respect to ω and u. It is most convenient to work with the functions

$$\psi^{(l)}(u) \;:=\; (2i\omega)^l \, e^{i\omega u} \, \phi^{(l)}(u) \,, \tag{3.18}$$

for which the iteration scheme (3.16) can be written as $\psi^{(0)} = 1$ and

$$\psi^{(l+1)}(u) \;=\; \int_u^\infty (e^{-2i\omega(v-u)} - 1)\, W(v)\, \psi^{(l)}(v)\, dv \,. \tag{3.19}$$

Lemma 3.4 *For every $\omega_0 \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, there are positive constants c, K, δ, such that for all $\omega \in E_0 \cap B_\delta(\omega_0)$ with $\mathrm{Im}\,\omega < 0$ and all $p, q, n \in \mathbb{N}$ the following inequality holds,*

$$\left| \left(\frac{\partial}{\partial\omega}\right)^p \left(\frac{\partial}{\partial u}\right)^q \psi^{(l)}(u) \right| \;\leq\; c^{1+l+p}\, K^q \, \frac{p!\, q!}{l!} \, \frac{1}{u^{l+q}} \,. \tag{3.20}$$

Proof. According to [6], λ_n is holomorphic in a neighborhood of ω_0, and thus (for example using the Cauchy integral formula) its derivatives can be bounded in $B_\delta(\omega_0)$ by

$$|\partial_\omega^p \lambda_n(\omega)| \;\leq\; \left(\frac{K}{2}\right)^{1+p} p!$$

for suitable $K > 0$. Since the potential W, (3.13), is also holomorphic in r (in a suitable neighborhood of the positive real axis) and has quadratic decay, its derivatives can be estimated by

$$|\partial_\omega^p \partial_u^q W(u)| \;\leq\; \left(\frac{K}{2}\right)^{1+p+q} \frac{p!\, q!}{u^{2+q}} \,. \tag{3.21}$$

We choose c so large that the following conditions hold,

$$c \;>\; 16\, K \,, \qquad \frac{K\, e^{\frac{1}{K}}}{(\omega_0 - \delta)\, c} \;\leq\; \frac{1}{2} \,. \tag{3.22}$$

We proceed to prove (3.20) by induction in l. For $l = 0$ there is nothing to prove. Thus assume that (3.20) holds for a given l. Using the induction hypothesis together with (3.21), we can then estimate the derivatives of the product $W\psi^{(l)}$ as follows,

$$\begin{aligned} |\partial_\omega^p \partial_u^q (W\psi^{(l)})| \;&\leq\; \sum_{a=0}^p \binom{p}{a} \sum_{b=0}^q \binom{q}{b} \left(\frac{K}{2}\right)^{1+a+b} \frac{a!\, b!}{u^{2+b}} \, \frac{c^{1+l+p-a}\, K^{q-b}}{u^{l+q-b}} \, \frac{(p-a)!\, (q-b)!}{l!} \\ &=\; \frac{c^{1+l+p}\, K^{1+q}}{u^{2+l+q}} \, \frac{p!\, q!}{l!} \sum_{a=0}^p \left(\frac{K}{2c}\right)^a \sum_{b=0}^q \left(\frac{1}{2}\right)^b \,. \end{aligned}$$

According to (3.22), the two remaining sums can be bounded by the geometric series $\sum_{m=0}^{\infty} 2^{-m} = 2$, and thus

$$|\partial_\omega^p \partial_u^q (W\psi^{(l)})| \;\leq\; 4\, \frac{c^{1+l+p}\, K^{1+q}}{u^{2+l+q}}\, \frac{p!\, q!}{l!}\,. \tag{3.23}$$

Next we differentiate the integral equation (3.19),

$$\partial_\omega^p \partial_u^q \psi^{(l+1)}(u) \;=\; \sum_{r=0}^{p} \binom{p}{r} \int_{-\infty}^{\infty} \partial_\omega^r \partial_u^q \left[\Theta(v-u)\,(e^{-2i\omega(v-u)}-1)\right]\, \partial_\omega^{p-r} \left(W\psi^{(l)}(v)\right)\, dv$$

(note that, since $\mathrm{Im}\,\omega < 0$, the factor $e^{-2i\omega v}$ gives an exponential decay of the integrand as $v \to \infty$). After manipulating the partial derivatives as follows,

$$\partial_\omega^r \partial_u^q \left[\Theta(v-u)\,(e^{-2i\omega(v-u)}-1)\right] \;=\; (-\partial_v)^q \left[\Theta(v-u) \left(\frac{v-u}{\omega}\, \partial_v\right)^r (e^{-2i\omega(v-u)}-1)\right],$$

the resulting v-derivatives can all be integrated by parts. The boundary terms drop out, and we obtain

$$\partial_\omega^p \partial_u^q \psi^{(l+1)}(u) \;=\; \sum_{r=0}^{p} \binom{p}{r} \int_u^{\infty} (e^{-2i\omega(v-u)}-1) \left(\partial_v\, \frac{u-v}{\omega}\right)^r \partial_v^q \partial_\omega^{p-r} \left(W\psi^{(l)}(v)\right) dv\,.$$

Since ω is in the lower half plane, we have the inequality $|e^{-2i\omega(v-u)}| \leq 1$. We conclude that

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 2 \sum_{r=0}^{p} \binom{p}{r} \int_u^{\infty} \left| \left\{\partial_v\, \frac{u-v}{\omega}\right\}^r \partial_v^q \partial_\omega^{p-r} \left(W\psi^{(l)}(v)\right) \right| dv\,. \tag{3.24}$$

The v-derivatives in the curly brackets can act either on one of the factors $(u-v)$ or on the function $W\psi^{(l)}$. Taking into account the combinatorics, we obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 2 \sum_{r=0}^{p} \binom{p}{r} \frac{1}{\omega^r} \sum_{s=0}^{r} \binom{r}{s}\, r^{r-s} \int_u^{\infty} (v-u)^s \left|\partial_\omega^{p-r} \partial_v^{q+s} \left(W\psi^{(l)}\right)\right| dv\,.$$

Using (3.23), we get

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8 \sum_{r=0}^{p} \sum_{s=0}^{r} \frac{p!\,(q+s)!}{s!\,(r-s)!\,l!}\, \omega^{-r}\, r^{r-s}\, c^{1+l+p-r}\, K^{1+q+s} \int_u^{\infty} \frac{(v-u)^s}{v^{2+l+q+s}}\, dv\,.$$

Introducing the new variable $\tau = \frac{u}{v}$, the integral can be computed with iterative integrations by parts,

$$\begin{aligned}
\int_u^{\infty} \frac{(v-u)^s}{v^{2+l+q+s}}\, dv \;&=\; \frac{1}{u^{1+l+q}} \int_0^1 (1-\tau)^s\, \tau^{l+q}\, d\tau \\
&=\; \frac{1}{u^{1+l+q}}\, \frac{(l+q)!}{(l+q+s)!} \int_0^1 (1-\tau)^s\, \frac{d^s}{d\tau^s} \tau^{l+q+s}\, d\tau \\
&=\; \frac{1}{u^{1+l+q}}\, \frac{(l+q)!\, s!}{(l+q+s)!} \int_0^1 \tau^{l+q+s}\, d\tau \;=\; \frac{1}{u^{1+l+q}}\, \frac{(l+q)!\, s!}{(1+l+q+s)!}\,.
\end{aligned}$$

We thus obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8\, \frac{c^{1+l+p}\, K^{1+q}}{u^{1+l+q}} \sum_{r=0}^{p} \sum_{s=0}^{r} \frac{p!\,(q+s)!}{(r-s)!\,l!}\, \frac{K^s\, r^{r-s}}{(\omega c)^r}\, \frac{(l+q)!}{(1+l+q+s)!}\,.$$

Using the elementary estimate

$$\frac{(q+s)!\,(l+q)!}{(1+l+q+s)!} \;=\; \frac{q!}{q+l+1}\cdot\frac{q+1}{q+l+2}\cdots\frac{q+s}{q+l+s+1} \;\leq\; \frac{q!}{l+1}\,,$$

we obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8\, \frac{c^{1+l+p}\, K^{1+q}}{u^{1+l+q}}\, \frac{p!\,q!}{(l+1)!} \sum_{r=0}^{p} \left(\frac{K}{\omega c}\right)^r \sum_{s=0}^{r} \frac{1}{(r-s)!} \left(\frac{r}{K}\right)^{r-s}.$$

The last sum can be estimated by an exponential,

$$\sum_{s=0}^{r} \frac{1}{(r-s)!} \left(\frac{r}{K}\right)^{r-s} \;\leq\; \sum_{a=0}^{r} \frac{1}{a!} \left(\frac{r}{K}\right)^{a} \;\leq\; \sum_{a=0}^{\infty} \frac{1}{a!} \left(\frac{r}{K}\right)^{a} \;=\; \exp\left(\frac{r}{K}\right).$$

According to (3.22), we can now estimate the remaining sum over r by a geometric series,

$$\sum_{r=0}^{p} \left(\frac{K}{\omega c}\right)^r \exp\left(\frac{r}{K}\right) \;\leq\; \sum_{r=0}^{\infty} \left(\frac{K\, e^{\frac{1}{K}}}{\omega c}\right)^r \;\leq\; 2\,.$$

We thus obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 16\, \frac{c^{1+l+p}\, K^{1+q}}{u^{1+l+q}}\, \frac{p!\,q!}{(l+1)!} \;\leq\; \frac{c^{2+l+p}\, K^{q}}{u^{1+l+q}}\, \frac{p!\,q!}{(l+1)!}\,,$$

where in the last step we again used (3.22). ∎

Proof of Theorem 3.2. According to (3.15, 3.18),

$$\grave{\phi}(\omega, u) \;=\; e^{-i\omega u} \sum_{l=0}^{\infty} \frac{1}{(2i\omega)^l}\, \psi^{(l)}(\omega, u)\,.$$

Expanding $\psi^{(l)}$ in a Taylor series in ω, we obtain the formal expansion

$$\grave{\phi}(\omega+\zeta, u) \;=\; e^{-i\omega u} \sum_{l=0}^{\infty} \frac{1}{(2i(\omega+\zeta))^l} \sum_{p=0}^{\infty} \frac{\zeta^p}{p!}\, \partial_\omega^p \psi^{(l)}(\omega, u)\,.$$

Lemma 3.4 allows us to estimate this expansion for every $\omega \in E_0 \cap B_\delta(\omega_0)$ with $\mathrm{Im}\,\omega < 0$ as follows,

$$|\grave{\phi}(\omega+\zeta, u)| \;\leq\; c \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{c}{|\omega+\zeta|\, u}\right)^l \sum_{p=0}^{\infty} (c|\zeta|)^p\,.$$

This expansion converges uniformly for $|\zeta| < \frac{c}{2}$. Similarly, one can show that the series of ζ-derivatives also converge uniformly. Hence we can interchange differentiation with summation, and a straightforward calculation shows that the Cauchy-Riemann equations are satisfied. Thus the above expansion allows us to extend $\grave{\phi}$ analytically to the ball $|\zeta| < \frac{c}{2}$. Since the constant c is independent of $\mathrm{Im}\,\omega$, we thus obtain an analytic extension of $\grave{\phi}$ across the real line. ∎

3.2 A Continuous Family of Solutions near $\omega = 0$

In Theorem 3.2 we made no statement about the behavior of the fundamental solutions $\grave{\phi}$ at $\omega = 0$. Indeed, we cannot expect the solutions to have a holomorphic extension in a neighborhood of $\omega = 0$. But at least, after suitable rescaling, these solutions have a well-defined limit at $\omega = 0$:

Theorem 3.5 *For every angular momentum number n, there is a real solution ϕ_0 of the Schrödinger equation (2.14) for $\omega = 0$ with the asymptotics*

$$\lim_{u\to\infty} u^{\mu-\frac{1}{2}}\,\phi_0(u) \;=\; \frac{\Gamma(\mu)}{\sqrt{\pi}} \qquad \text{with} \qquad \mu \;:=\; \sqrt{\lambda_n(0)+\frac{1}{4}}\,. \tag{3.25}$$

This solution can be obtained as a limit of the solutions from Theorem 3.2, in the sense that for all $u \in \mathbb{R}$,

$$\phi_0(u) \;=\; \lim_{E_0\ni\omega\to 0} \omega^{\mu-\frac{1}{2}}\,\grave{\phi}(u) \qquad \text{and} \qquad \phi_0'(u) \;=\; \lim_{E_0\ni\omega\to 0} \omega^{\mu-\frac{1}{2}}\,\grave{\phi}'(u)\,.$$

Note that the λ_n are the eigenvalues of the Laplacian on the sphere. They are clearly non-negative, and thus the parameter μ in (3.25) is positive.

Unfortunately, the function ϕ_0 cannot be constructed with the iteration scheme (3.16) because if we put in the Green's function for $\omega = 0$ (which is obtained from (3.14) by taking the limit $\omega \to 0$), we get the for $\phi^{(1)}$ the equation

$$\phi^{(1)}(u) \;=\; \int_u^\infty (v-u)\,W(v)\,dv\,,$$

and since W decays at infinity only quadratically, the integral diverges. To overcome this problem, we combine the quadratically decaying part of the potential with the unperturbed operator. More precisely, for any ω in the set

$$F \;:=\; \left\{\omega \in \mathbb{C} \mid \operatorname{Im}\omega \leq 0 \text{ and } |\omega| \leq (16ak)^{-1}\right\}\,,$$

we write the Schrödinger equation as

$$\left(-\frac{d^2}{du^2} + \frac{\mu^2-\frac{1}{4}}{u^2} - \omega^2\right)\phi(u) \;=\; -W(u)\,\phi(u)\,,$$

where $\mu(\omega) = (\lambda_n(\omega) - 2ak\omega + \frac{1}{4})^{\frac{1}{2}}$. The potential W is continuous in ω and bounded by

$$|W(u)| \;\leq\; \frac{c}{u^3} \qquad \text{for all } \omega \in F\,. \tag{3.26}$$

The solutions of the unperturbed Schrödinger equation can be expressed with Bessel functions,

$$h_1(u) \;=\; \sqrt{\frac{\pi u\omega}{2}}\,J_\mu(\omega u)\,, \qquad h_2(u) \;=\; \sqrt{\frac{\pi u\omega}{2}}\,Y_\mu(\omega u)\,.$$

They have the following asymptotics,

$$\left\{\begin{array}{llll} h_1(u) \sim \cos(\omega u) & , & h_2(u) \sim \sin(\omega u) & \text{if } \omega u \gg 1 \\ h_1(u) \sim \dfrac{\sqrt{\pi}\,\omega^{\mu+\frac{1}{2}}}{\Gamma(\mu+1)\,2^{\mu+\frac{1}{2}}}\,u^{\mu+\frac{1}{2}} & , & h_2(u) \sim \dfrac{\Gamma(\mu)\,2^{\mu-\frac{1}{2}}}{\sqrt{\pi}\,\omega^{\mu+\frac{1}{2}}}\,u^{-\mu+\frac{1}{2}} & \text{if } \omega u \ll 1. \end{array}\right.$$

The Green's function can be expressed in terms of the two fundamental solutions by the standard formula

$$S(u,v) \;=\; \Theta(v-u)\, \frac{h_1(u)\, h_2(v) - h_1(v)\, h_2(u)}{w(h_1, h_2)}\;,$$

where $w(h_1, h_2) = h_1' h_2 - h_1 h_2' = -\omega$ is the Wronskian. The perturbation series ansatz

$$\phi \;=\; \sum_{l=1}^\infty \phi^{(l)} \tag{3.27}$$

now leads to the integral equation

$$\phi^{(l+1)}(u) \;=\; \int_u^\infty S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\;. \tag{3.28}$$

We choose the function $\phi^{(0)}$ such that its asymptotics at infinity is a multiple times the plane wave $e^{-i\omega u}$, whereas for $\omega = 0$, it has the asymptotics (3.25),

$$\phi^{(0)}(u) \;=\; \omega^\mu\, (h_1 - i h_2)(u)\;. \tag{3.29}$$

Lemma 3.6 *For any fixed n there is $u_0 \in \mathbb{R}$ such that the iteration scheme (3.29, 3.28) converges uniformly for all $u > u_0$ and $\omega \in F$. The functions ϕ defined by (3.27) are solutions of the Schrödinger equation (2.14) with the asymptotics*

$$\left| \frac{\phi(u)}{\phi^{(0)}(u)} - 1 \right| \;\leq\; \frac{c}{u}$$

and a constant $c = c(n)$.

Proof. Using the asymptotic formulas for the Bessel functions, one sees (similar to the estimate [3, eqn. (4.4)] for $\mu = l + \frac{1}{2}$ and integer l) that for all $v \geq u$ and $\omega \in F$, the Green's function is bounded by

$$|S(u,v)| \;\leq\; C\, e^{\mathrm{Im}\,\omega\,(v-u)} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{v}{1+|\omega|\,v} \right)^{\mu+\frac{1}{2}}\;. \tag{3.30}$$

Similarly, we can bound the Bessel functions in (3.29) to get

$$\frac{1}{C} \;\leq\; |\phi^{(0)}|\, e^{-\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{\mu-\frac{1}{2}} \;\leq\; C\;. \tag{3.31}$$

Let us show inductively that

$$|\phi^{(l)}| \;\leq\; C\, e^{\mathrm{Im}\,\omega\,u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{Cc}{u} \right)^l\;. \tag{3.32}$$

For $l = 0$ there is nothing to prove. The induction step follows from (3.28, 3.26, 3.30)

$$\begin{aligned}
|\phi^{(l+1)}| \;&\leq\; C\, e^{-\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \int_u^\infty e^{2\,\mathrm{Im}\,\omega v} \left(\frac{v}{1+|\omega|\,v} \right) \frac{cC}{v^3} \left(\frac{Cc}{v} \right)^l dv \\
&\leq\; C\, e^{\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{Cc}{u} \right)^l \int_u^\infty \frac{cC}{v^2}\, dv
\end{aligned}$$

The lemma now follows immediately from (3.32, 3.31) and by differentiating the series (3.27) with respect to u. ∎

Proof of Theorem 3.5. ¿¿From the asymptotics at infinity, it is clear that

$$\phi \;=\; \left\{ \begin{array}{cl} \omega^\mu \,\grave{\phi} & \text{if } \omega \neq 0 \\ \phi_0 & \text{if } \omega = 0\,. \end{array} \right.$$

Denoting the ω-dependence of ϕ by a subscript, we thus need to prove that for all $u \in \mathbb{R}$,

$$\lim_{F \ni \omega \to 0} \phi_\omega(u) \;=\; \phi_0(u)\,, \qquad \lim_{F \ni \omega \to 0} \phi'_\omega(u) \;=\; \phi'_0(u)\,. \tag{3.33}$$

To simplify the problem, we first note that for u on a compact intervals, the continuous dependence on ω follows immediately from the Picard-Lindelöf Theorem (i.e. the continuous dependence of solutions of ODEs on the coefficients and initial values). Thus it suffices to prove (3.33) for large u. Furthermore, writing the Schrödinger equation as

$$(\partial_u - i\omega)(\partial_u + i\omega)\, \phi_\omega \;=\; -U\, \phi\,,$$

the potential U has quadratic decay at infinity. Thus, after the substitution $(\partial_u - i\omega) = e^{i\omega u} \partial_u e^{-i\omega u}$, we can multiply the above equation by $e^{-i\omega u}$ and integrate to obtain

$$e^{-i\omega u}\, (\partial_u + i\omega)\, \phi_\omega(u) \;=\; \int_u^\infty e^{-i\omega v}\, U(v)\, \phi_\omega(v)\, dv\,.$$

Here we emphasized the ω-dependence by a subscript; note also that the integral is well-defined in view of the asymptotics of ϕ_ω at infinity. This equation shows that ϕ'_ω converges pointwise once we know that $\phi_\omega(u)$ converges uniformly in u. Hence it remains to show that for every $\epsilon > 0$ there is u_0 and $\delta > 0$ such that for all $\omega \in F$ with $|\omega| < \delta$,

$$|\phi_\omega(u) - \phi_0(u)| \;<\; \varepsilon \qquad \text{for all } u > u_0. \tag{3.34}$$

To prove (3.34) we use the uniform convergence of the functions $\phi^{(0)}_\omega$, (3.29), to choose δ such that for all $\omega \in F$ with $|\omega| < \delta$,

$$|\phi^{(0)}_\omega(u) - \phi^{(0)}_0(u)| \;<\; \frac{\varepsilon}{3} \qquad \text{for all } u > u_0.$$

According to Lemma 3.6, we can by choosing u_0 sufficiently large arrange that

$$|\phi^{(0)}_\omega(u) - \phi_\omega(u)| \;<\; \frac{\varepsilon}{3} \qquad \text{for all } u > u_0 \text{ and } \omega \in F.$$

Now (3.34) follows immediately from the estimate

$$|\phi_\omega - \phi_0| \;\leq\; |\phi_\omega - \phi^{(0)}_\omega| + |\phi^{(0)}_\omega - \phi^{(0)}_0| + |\phi^{(0)}_0 - \phi_0|\,.$$

∎

4 Global Estimates for the Radial Equation

Let Y_1 and Y_2 be two real fundamental solutions of the Schrödinger equation (2.14) for a general real and smooth potential V. Then their Wronskian

$$w \;:=\; Y_1'(u)\, Y_2(u) - Y_1(u)\, Y_2'(u) \tag{4.1}$$

is a constant. By flipping the sign of Y_2, we can always arrange that $w < 0$. We combine the two real solutions into the complex function

$$z \;=\; Y_1 + i Y_2 \;,$$

and denote its polar decomposition by

$$z \;=\; \rho\, e^{i\varphi} \tag{4.2}$$

with real functions $\rho(u) \geq 0$ and $\varphi(u)$. By linearity, z is a solution of the complex Schrödinger equation

$$z'' \;=\; V\, z \;. \tag{4.3}$$

Note that z has no zeros because at every u at least one of the fundamental solutions does not vanish.

4.1 The Complex Riccati Equation

We introduce the function y by

$$y \;=\; \frac{z'}{z} \;. \tag{4.4}$$

Since z has no zeros, the function y is smooth. Moreover, it satisfies the complex Riccati equation

$$y' + y^2 \;=\; V \;. \tag{4.5}$$

The fact that the solutions of the complex Riccati equation are smooth will be helpful for getting estimates. Conversely, from a solution of the Riccati equation one obtains the corresponding solution of the Schrödinger equation by integration,

$$\log z|_u^v \;=\; \int_u^v y \;. \tag{4.6}$$

Using (4.2) in (4.4) gives separate equations for the amplitude and phase of z,

$$\rho' \;=\; \rho\, \mathrm{Re}\, y \;, \qquad\qquad \varphi' \;=\; \mathrm{Im}\, y \;,$$

and integration gives

$$\log \rho|_u^v \;=\; \int_u^v \mathrm{Re}\; y \tag{4.7}$$

$$\varphi|_u^v \;=\; \int_u^v \mathrm{Im}\; y \;. \tag{4.8}$$

Furthermore, the Wronskian (4.1) gives a simple algebraic relation between ρ and y. Namely, w can be expressed by $w = -\mathrm{Im}\,(\overline{z}\, z') = \rho^2\, \mathrm{Im}\, y$ and thus

$$\rho^2 \;=\; -\frac{w}{\mathrm{Im}\, y} \;. \tag{4.9}$$

Since ρ^2 is positive and w is negative, we see that

$$\mathrm{Im}\, y(u) > 0 \qquad\qquad \text{for all } u. \tag{4.10}$$

4.2 Invariant Disk Estimates

We now explain a method for getting estimates for the complex Riccati equation. This method was first used in [6] for estimates in the case where the potential is negative (Lemma 4.1). Here we extend the method to the situation when the potential is positive (Lemma 4.2). For sake of clarity, we develop the method again from the beginning, but we point out that the proof of Lemma 4.1 is taken from [6]. Let $y(u)$ be a solution of the complex Riccati equation (4.5). We want to estimate the Euclidean distance of y to a given curve $m(u) = \alpha + i\beta$ in the complex plane. A direct calculation using (4.5) gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{du}|y-m|^2 \;&=\; (\mathrm{Re}\, y - \alpha)\,(\mathrm{Re}\, y-\alpha)' + (\mathrm{Im}\, y - \beta)\,(\mathrm{Im}\, y - \beta)' \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y)^2 + (\mathrm{Im}\, y)^2 - \alpha'\right] - (\mathrm{Im}\, y - \beta)\left[2\,\mathrm{Re}\, y\,\mathrm{Im}\, y + \beta'\right] \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y)^2 - (\mathrm{Im}\, y)^2 + 2\beta\,\mathrm{Im}\, y - \alpha'\right] \;+\; (\mathrm{Re}\, y - \alpha)\, 2(\mathrm{Im}\, y - \beta)\,\mathrm{Im}\, y \\
&\quad -(\mathrm{Im}\, y - \beta)\left[\beta' + 2\alpha\,\mathrm{Im}\, y\right] \;-\; (\mathrm{Im}\, y - \beta)\, 2(\mathrm{Re}\, y - \alpha)\,\mathrm{Im}\, y \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y - \alpha)^2 - (\mathrm{Im}\, y - \beta)^2 - \alpha^2 + \beta^2 - \alpha'\right] \\
&\quad -(\mathrm{Im}\, y - \beta)\left[\beta' + 2\alpha\beta\right] - 2\alpha\left((\mathrm{Re}\, y - \alpha)^2 + (\mathrm{Im}\, y - \beta)^2\right).
\end{aligned}$$

Choosing polar coordinates centered at m,

$$y \;=\; m + Re^{i\varphi}\,, \qquad\qquad R \;:=\; |y-m|\,,$$

we obtain the following differential equation for R,

$$R' + 2\alpha R \;=\; \cos\varphi\left[V - R^2 - \alpha^2 + \beta^2 - \alpha'\right] - \sin\varphi\left[\beta' + 2\alpha\beta\right]. \tag{4.11}$$

In order to use this equation for estimates, we assume that α is a given function (to be determined later). With the abbreviations

$$U \;=\; V - \alpha^2 - \alpha' \qquad\text{and}\qquad \sigma(u) \;=\; \exp\left(2\int_0^u \alpha\right), \tag{4.12}$$

the ODE (4.11) can then be written as

$$(\sigma R)' \;=\; \sigma\left[U - R^2 + \beta^2\right]\cos\varphi - (\sigma\beta)'\,\sin\varphi\,.$$

To further simplify the equation, we want to arrange that the square bracket vanishes. If U is negative, this can be achieved by the ansatz

$$\beta \;=\; \frac{\sqrt{|U|}}{2}\left(T + \frac{1}{T}\right), \qquad R \;=\; \frac{\sqrt{|U|}}{2}\left(T - \frac{1}{T}\right) \qquad (U<0), \tag{4.13}$$

with $T>1$ a free function. In the case $U>0$, we make similarly the ansatz

$$\beta \;=\; \frac{\sqrt{U}}{2}\left(T - \frac{1}{T}\right), \qquad R \;=\; \frac{\sqrt{U}}{2}\left(T + \frac{1}{T}\right) \qquad (U>0) \tag{4.14}$$

with a function $T>0$. Using (4.13, 4.14), the ODE (4.11) reduces to the simple equation $(\sigma R)' = -(\sigma\beta)'\sin\varphi$. If we now replace this equation by a strict inequality,

$$(\sigma R)' \;>\; -(\sigma\beta)'\sin\varphi\,, \tag{4.15}$$

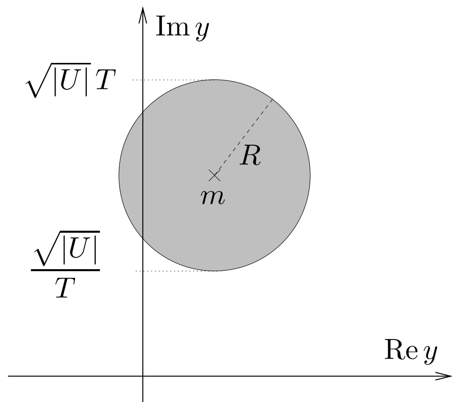


Figure 1: Invariant disk estimate for $U < 0$.

with R a general positive function, the inequality $|y - m| \leq R$ will be preserved as u increases. In other words, the disk $\overline{B_R}(m)$ will be an invariant region for the flow of y. In the next two lemmas we specify the function T in the cases $U < 0$ and $U > 0$, respectively. To avoid confusion, we note that it is only a matter of convenience to state the lemmas on the interval $[0, u_{\max}]$; by translation we can later immediately apply the lemmas on any closed interval.

Lemma 4.1 *Let α be a real function on $[0, u_{max}]$ which is continuous and piecewise C^1, such that the corresponding function U, (4.12), is negative,*

$$U \leq 0 \quad on \quad [0, u_{max}]\,.$$

For a constant $T_0 \geq 1$ we introduce the function T by

$$T(u) \;=\; T_0\, \exp\left(\frac{1}{2}\; TV_{[0,u)} \log|\sigma^2 U|\right)\,, \tag{4.16}$$

define the functions β and R by (4.13) and set $m = \alpha + i\beta$. If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,, \tag{4.17}$$

then this condition holds for all $u \in [0, u_{max}]$ (for illustration see Figure 1).

Proof. For $\varepsilon > 0$ we set

$$T_\varepsilon(u) \;=\; T_0\, \exp\left(\frac{1}{2}\int_0^u \left|\frac{|\sigma^2 U|'}{|\sigma^2 U|}\right| + \varepsilon(1 - e^{-u})\right) \tag{4.18}$$

and denote corresponding functions α, R, m, and σ by an additional subscript ε. Since $T_\varepsilon(0) = T(0)$ and $\lim_{\varepsilon \searrow 0} T_\varepsilon = T$, it suffices to show that for all $\varepsilon > 0$ the following statement holds,

$$|y - m_\varepsilon|(0) \leq R_\varepsilon(0) \quad\Longrightarrow\quad |y - m_\varepsilon|(u) \leq R_\varepsilon(u) \;\text{ for all } u \in [0, u_{\max}].$$

In differential form, we get the sufficient condition

$$|y - m_\varepsilon|(u) = R_\varepsilon(u) \quad\Longrightarrow\quad |y - m_\varepsilon|'(u) < R_\varepsilon'(u)\,.$$

According to (4.15), this last condition will be satisfied if

$$(\sigma_\varepsilon R_\varepsilon)' \;>\; |(\sigma_\varepsilon \beta_\varepsilon)'| \,. \tag{4.19}$$

¿¿From now on we omit the subscripts ε.

In order to prove (4.19), we first use (4.13, 4.12) to rewrite the functions $\sigma\beta$ and σR as

$$\left.\begin{aligned} \sigma\beta &= \frac{1}{2}\left(\sqrt{|\sigma^2 U|}\, T + \sqrt{|\sigma^2 U|}\, T^{-1}\right) \\ \sigma R &= \frac{1}{2}\left(\sqrt{|\sigma^2 U|}\, T - \sqrt{|\sigma^2 U|}\, T^{-1}\right). \end{aligned}\right\} \tag{4.20}$$

By definition of T_ε (4.18),

$$\frac{T'}{T} \;=\; \frac{1}{2}\left|\frac{|\sigma^2 U|'}{|\sigma^2 U|}\right| + \varepsilon e^{-u} \,.$$

It follows that

$$\left\{\begin{aligned} (\sqrt{|\sigma^2 U|}\, T^{-1})' &= -\varepsilon e^{-u}\, (\sqrt{|\sigma^2 U|}\, T^{-1}) && \text{if } |\sigma^2 U|' \geq 0 \\ (\sqrt{|\sigma^2 U|}\, T)' &= \varepsilon e^{-u}\, (\sqrt{|\sigma^2 U|}\, T) && \text{if } |\sigma^2 U|' < 0 \,. \end{aligned}\right.$$

Hence when we differentiate through (4.20) and set $\varepsilon = 0$, either the first or the second summand drops out in each equation, and we obtain $(\sigma R)' = |\sigma\beta|'$. If $\varepsilon > 0$, an inspection of the signs of the additional terms gives (4.19). ■

Lemma 4.2 *Let α be a real function on $[0, u_{max}]$ which is continuous and piecewise C^1, such that the corresponding function U, (4.12), satisfies on $[0, u_{max}]$ the conditions*

$$U \geq 0 \qquad \text{and} \qquad U' + 4U\alpha \;\geq\; 0 \,. \tag{4.21}$$

For a constant $T_0 \geq 0$ we introduce the function T by

$$T(u) \;=\; T_0 \sqrt{\frac{U(0)}{\sigma^2 U}} \,, \tag{4.22}$$

define the functions β and R by (4.14) and set $m = \alpha + i\beta$. If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R \,,$$

then this condition holds for all $u \in [0, u_{max}]$ (see Figure 2). Furthermore,

$$\mathrm{Re}\, y \;\geq\; \alpha - \sqrt{U} - T_0\, \frac{\sqrt{U(0)}}{2\sigma} \,. \tag{4.23}$$

Proof. For $\varepsilon > 0$ we set

$$T_\varepsilon \;=\; T_0\, (\sigma^2 U)^{-\frac{1}{2}}\, (1 - \varepsilon e^{-u}) \,.$$

Using (4.14, 4.12) we can write the functions $\sigma\beta$ and σR as

$$\left.\begin{aligned} \sigma\beta &= -\frac{1}{2}\left(T_0^{-1}\, \sigma^2 U\, (1 - \varepsilon e^{-u})^{-1} - T_0\, (1 - \varepsilon e^{-u})\right) \\ \sigma R &= \frac{1}{2}\left(T_0^{-1}\, \sigma^2 U\, (1 - \varepsilon e^{-u})^{-1} + T_0\, (1 - \varepsilon e^{-u})\right), \end{aligned}\right\}$$

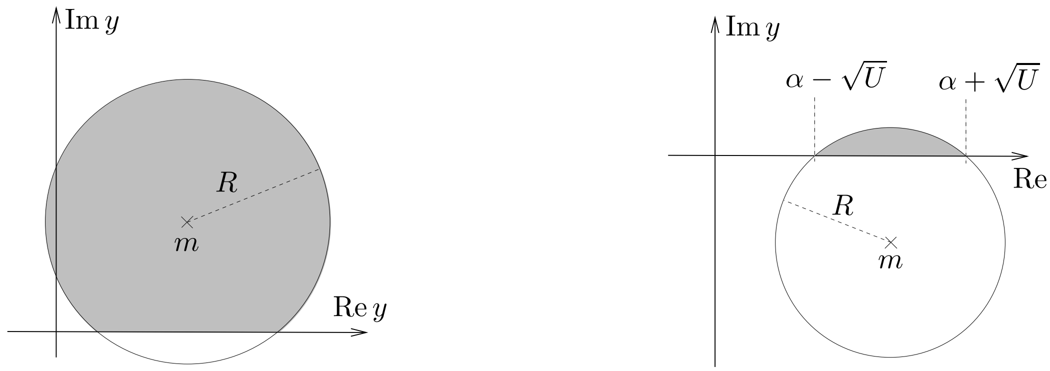


Figure 2: Invariant disk estimate for $U > 0$, in the cases $T > 1$ (left) and $T < 1$ (right).

where we again omitted the subscript ε. Differentiation gives

$$(\sigma R)' \;>\; -(\sigma\beta)' \;=\; \frac{1}{2}\, T_0^{-1} \left(\sigma^2 U \,(1-\varepsilon e^{-u})^{-1}\right)' - \frac{1}{2}\, T_0 \left(1-\varepsilon e^{-u}\right)' \,. \tag{4.24}$$

According to the second inequality in (4.21), the function $\sigma^2 U$ is strictly increasing and thus the expression on the right of (4.24) is positive for sufficiently small ε. Hence (4.19) is satisfied. Letting $\varepsilon \to 0$, we obtain that the circle $\overline{B}_R(m)$ is invariant.

In order to prove (4.23) we note that in the case $T < 1$ the inequality is obvious because even $\mathrm{Re}\, y \leq \alpha - \sqrt{U}$ (see Figure 2). Thus we can assume $T \geq 1$, and the estimate

$$\mathrm{Re}\, y \;\geq\; \alpha - R \;\geq\; \alpha - \frac{\sqrt{U}}{2}\,(T+2)$$

together with (4.22) gives the claim. ■

If the potential V is monotone increasing, by choosing $\alpha \equiv 0$ we obtain the following simple estimate.

Corollary 4.3 *Assume that the potential V is monotone increasing on $[0, u_{max}]$. For a constant $T_0 > 0$ with $T_0^2 \geq -V(0)$ we introduce the functions*

$$\beta \;=\; \frac{1}{2}\left(T_0 - \frac{V}{T_0}\right), \qquad R \;=\; \frac{1}{2}\left(T_0 + \frac{V}{T_0}\right). \tag{4.25}$$

If a solution of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$y \;\in\; \Big\{ z \,|\, |z - i\beta| \leq R,\ \mathrm{Re}\, z, \mathrm{Im}\, z \geq 0 \Big\} \cup \left\{ z \,\middle|\, \left| z - \frac{iT_0}{2} \right| \leq \frac{T_0}{2},\ \mathrm{Re}\, z \leq 0 \right\},$$

then this condition holds for all $u \in [0, u_{max}]$ (see Figure 3).

Proof. Choosing $\alpha \equiv 0$ and β, T according to (4.25), we know from Lemma 4.1 and Lemma 4.2 that the circles $|y - m| \leq R$ are invariant. Furthermore, we note that the arc Λ in Figure 3 is the flow line of the equation $y' + y^2 = 0$, and thus it cannot be crossed from the right to the left when V is positive. This gives the result in the case that V has no zeros. If V has a zero, the invariant disks in the regions $V \leq 0$ and $V \geq 0$ coincide at

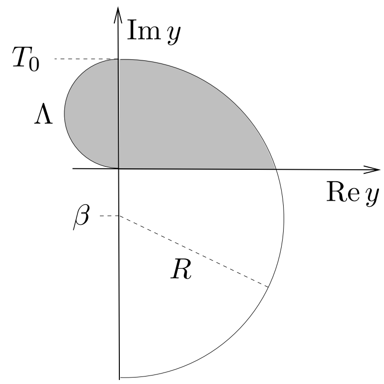


Figure 3: Invariant region estimate for monotone V.

the zero of V. ■

The invariant disk estimates of Lemma 4.1 and Lemma 4.2 can also be used if the functions α and U have a discontinuity at some $v \in [0, u_{\max}]$, i.e.

$$\alpha_l \;:=\; \lim_{u \nearrow v} \alpha(u) \;\neq\; \lim_{u \searrow v} \alpha(u) \;=:\; \alpha_r\,, \qquad U_l \;:=\; \lim_{u \nearrow v} U(u) \;\neq\; \lim_{u \searrow v} U(u) \;=:\; U_r\,.$$

In this case we choose the function T also to be discontinuous at v,

$$T_l \;:=\; \lim_{u \nearrow v} T(u) \;\neq\; \lim_{u \searrow v} T(u) \;=:\; T_r\,,$$

in such a way that the circle corresponding to (α_r, U_r, T_r) contains that corresponding to (α_l, U_l, T_l) (see Figure 4). In the next lemma we give sufficient "jump conditions" for this "matching."

Lemma 4.4 (matching of invariant disks) *Suppose that $U_l < 0$. Depending on the sign of U_r, we set*

$$T_r \;=\; T_l\, \frac{(\alpha_r - \alpha_l)^2 + |U_l + U_r|}{\sqrt{|U_l|\,|U_r|}} \qquad \text{if } U_r < 0 \tag{4.26}$$

$$T_r \;=\; T_l\, \frac{(\alpha_r - \alpha_l)^2 + |U_l + U_r| + \sqrt{|U_l|\,U_r}}{\sqrt{|U_l|\,U_r}} \qquad \text{if } U_r > 0 \tag{4.27}$$

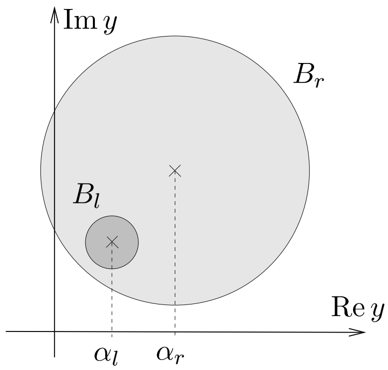
Let $B_{l/r}$ be the disks with centers $m_{l/r} = \alpha_{l/r} + i\beta_{l/r}$ and radii $R_{l/r}$ as given by (4.13) or (4.14). Then $B_l \subset B_r$.

Proof. We must satisfy the condition $R_r \geq |m_r - m_l| + R_l$. Taking squares, we obtain the equivalent conditions $R_r \geq R_l$ and

$$(R_r - R_l)^2 \;\geq\; (\alpha_r - \alpha_l)^2 + (\beta_r - \beta_l)^2\,.$$

This last condition can also be written as

$$(\alpha_r - \alpha_l)^2 + (\beta_l^2 - R_l^2) + (\beta_r^2 - R_r^2) \;\leq\; 2\,(\beta_l \beta_r - R_l R_r)\,. \tag{4.28}$$



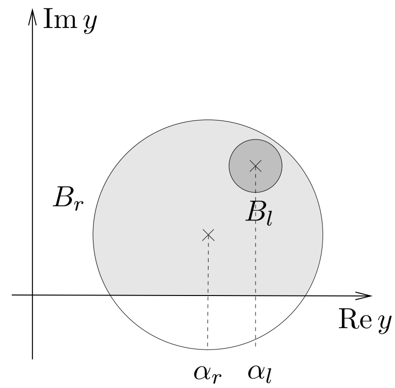


Figure 4: Matching of invariant disks in the cases $U_r < 0$ (left) and $U_r > 0$ (right).

In the case $U_r < 0$, we can substitute the ansatz (4.13) into (4.28) to obtain the equivalent inequality

$$(\alpha_r - \alpha_l)^2 + |U_l| + |U_r| \;\leq\; \sqrt{|U_l|\,|U_r|} \left(\frac{T_r}{T_l} + \frac{T_l}{T_r} \right).$$

Dropping the last summand on the right and solving for T_r, we obtain (4.26), which is thus a sufficient condition.

In the case $U_r > 0$, we substitute (4.13, 4.14) into (4.28) to obtain the equivalent condition

$$(\alpha_r - \alpha_l)^2 + |U_l| - U_r \;\leq\; \sqrt{|U_l|\,U_r} \left(\frac{T_r}{T_l} - \frac{T_l}{T_r} \right).$$

Using the inequality $|U_l| - U_r \leq |U_l + U_r|$, replacing the factor T_l/T_r on the right by one and solving for T_r, we obtain the sufficient condition (4.27). ■

4.3 Bounds for the Wronskian and the Fundamental Solutions

We now consider the solutions $\acute{\phi}$ and $\grave{\phi}$ as defined in Section 3.1 for ω on the real axis and set

$$\acute{y} \;=\; \frac{\acute{\phi}'}{\acute{\phi}}\,, \qquad \grave{y} \;=\; \frac{\grave{\phi}'}{\grave{\phi}}\,.$$

We keep k fixed. Since taking the complex conjugate of the separated wave equation flips the sign of k, we may assume that $k \geq 0$. Then ω_0 as defined by (2.12) is negative.

Proposition 4.5 *If $\omega \not\in [\omega_0, 0]$, the Wronskian $w(\acute{\phi}, \grave{\phi})$ is non-zero.*

Proof. According to (2.13), ω and Ω have the same sign. From (4.10) we know that the functions $\acute{y}$ and $\grave{y}$ both stay either in the upper or lower half plane. In view of the asymptotics (3.1, 3.2), we know that they must be in be in opposite half planes. Thus

$$w(\acute{\phi}, \grave{\phi}) \;=\; \acute{\phi}\,\grave{\phi}\,(\acute{y} - \grave{y}) \;\neq\; 0\,.$$ ■

In the case $\omega \in (\omega_0, 0)$, we need the following global estimate for large λ.

Proposition 4.6 *For any $u_1 \in \mathbb{R}$ there are constants $c, \lambda_0 > 0$ such that*

$$\left| \frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})} \right| \;\leq\; \frac{c}{\lambda} \qquad \text{for all } \lambda > \lambda_0,\ \omega \in (\omega_0, 0),\ u < u_1\,.$$

The remainder of this section is devoted to the proof of this proposition. Let $u_1 \in \mathbb{R}$ and $\omega \in (\omega_0, 0)$. Possibly by increasing u_1 and λ_0 we can clearly arrange that V is monotone decreasing on $[u_1, \infty)$. Then we have the following estimate.

Lemma 4.7 *The functions $\grave{\phi}$ and $\grave{y}$ satisfy the inequalities*

$$|\grave{\phi}(u)| \;\geq\; 1\,, \quad \mathrm{Re}\,\grave{y}(u) \;\leq\; |\omega| \qquad \text{on } [u_1, \infty)\,.$$

Proof. From the asymptotics (3.2) we know that $\lim_{u\to\infty} \grave{y}(u) = -i\omega$. Thus for v sufficiently large, $|\grave{y}(v) - i|\omega|| < \varepsilon$, and we can apply Corollary 4.3 on the interval $[u_1, v]$ backwards in u with $T_0 = |\omega| + 2\varepsilon$. Since ε can be chosen arbitrarily small, we conclude that Corollary 4.3 applies even on $[u_1, \infty)$ with $T_0 = |\omega|$. This means that

$$0 \;\leq\; \mathrm{Im}\,\grave{y} \;\leq\; |\omega|\,, \quad \mathrm{Re}\,\grave{y} \;\leq\; |\omega| \qquad \text{on } [u_1, \infty)\,.$$

Finally, we use (4.9) with $w = i|\omega|$. ∎

We now come to the estimates for $\acute{\phi}$, which are more difficult because we need a stronger result. The next lemma specifies the behavior of the potential on $(-\infty, 2u_1]$.

Lemma 4.8 *For any $u_1 \in \mathbb{R}$ there are constants c, λ_0 such that the potential V has for all $\omega \in (\omega_0, 0)$ and all $\lambda > \lambda_0$ the following properties. There are unique points $u_- < u_0 < u_+ < u_1$ such that*

$$V(u_-) \;=\; -\frac{\Omega^2}{2}\,, \qquad V(u_0) \;=\; 0\,, \qquad V(u_+) \;=\; \Omega^2\,.$$

V is monotone increasing on $(-\infty, u_+]$. Furthermore,

$$u_+ - u_- \;\leq\; c \tag{4.29}$$

$$\gamma\, u_+ \;\geq\; \log \Omega^2 - \log \lambda - c \tag{4.30}$$

$$|V'|^{\frac{2}{3}} + |V''|^{\frac{1}{2}} \;\leq\; \frac{1}{4}\, |V| \qquad \text{on } [u_+, 2u_1], \tag{4.31}$$

with γ as in (3.9).

Proof. We expand V in a Taylor series around the event horizon,

$$V \;=\; -\Omega^2 + (\lambda + c_0)(r - r_1) + \lambda\, \mathcal{O}((r - r_1)^2)\,.$$

Hence for sufficiently large λ_0 there are near the event horizon unique points u_-, u_0, u_+ where the potential has the required value. Integrating (2.1) we get near the event horizon the asymptotic formula

$$u \;\sim\; \frac{1}{\gamma}\, \log(r - r_1)\,.$$

Getting asymptotic expansions for $u_\pm$ we immediately obtain (4.29, 4.30). Furthermore, using (2.1) to transform r-derivatives into u-derivatives, we obtain in the region $(r_1, r_1 + \varepsilon) \cap (u_+, \infty)$ the estimates

$$\begin{aligned} \frac{\lambda}{c}\, e^{\gamma u} \;\leq\; V(u) \;&\leq\; \lambda\, c\, e^{\gamma u} \\ |V'(u)| + |V''(u)| \;&\leq\; \lambda\, c\, e^{\gamma u}\,, \end{aligned}$$

uniformly in λ and ω. Hence for sufficiently large λ_0, (4.31) will be satisified near the event horizon.

In the region $r > r_1 + \varepsilon$ away from the event horizon, V is strictly positive, $V > \lambda/c$, and since the derivatives of V can clearly be bounded by $|V'| + |V''| < c\lambda$, it follows that (4.31) is again satisfied. ■

First we apply Corollary 4.3 on the interval $(-\infty, u_-)$ to obtain the following result.

Corollary 4.9 *There is a constant $c > 0$ such that for all $\omega \in (\omega_0, 0)$ and $\lambda > \lambda_0$,*

$$\frac{\Omega}{2} \;\leq\; \mathrm{Im}\, y \;\leq\; \Omega\,, \quad |\mathrm{Re}\, y| \;\leq\; \frac{\Omega}{2} \qquad \text{on } (-\infty, u_-]\,.$$

Also, at $u = u_-$ we have an invariant disk with

$$\alpha_l \;=\; 0\,, \qquad U_l \;=\; -\frac{\Omega^2}{2}\,, \qquad T_l \;=\; \sqrt{2}\,. \tag{4.32}$$

On the interval $[u_-, u_+]$ we use the method described in the next lemma.

Lemma 4.10 *Assume that the potential V is monotone increasing on $[0, u_{max}]$. We set*

$$\alpha \;=\; \sqrt{\max(2\, V(u_{max}), 0)}$$

and introduce for a given constant $T_0 > 1$ the functions U, σ, β, R, and T by (4.12, 4.14) and

$$T(u) \;=\; T_0\, e^{2\alpha u}\, \frac{\sqrt{|U(0)|}}{\sqrt{|U(u)|}}\,. \tag{4.33}$$

If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,,$$

then this condition holds for all $u \in [0, u_{max}]$.

Proof. By definition of α, the function $U = V - \alpha^2$ is negative and monotone increasing. Using furthermore that $\sigma = e^{2\alpha u}$, we can estimate the total variation in (4.16) as follows,

$$\mathrm{TV}_{[0,u)} \log|\sigma^2 U| \;=\; \int_0^u \left(4\alpha - \frac{|U|'}{|U|}\right) \;=\; 4\alpha u + \log|U(0)| - \log|U(u)|.$$

This gives (4.33). ■

Thus we match the invariant disk (4.32) to a disk with $U_r = V(u_-) - \alpha_r^2$ and $\alpha_r = \sqrt{2}\,\Omega$. From (4.29) we see that $(u_+ - u_-)\,\alpha$ is uniformly bounded, and thus we obtain the following estimate.

Corollary 4.11 *There is a constant $c > 0$ such that for all $\omega \in (\omega_0, 0)$ and $\lambda > \lambda_0$,*

$$\frac{\Omega}{c} \;\leq\; \mathrm{Im}\, y \;\leq\; c\,\Omega\,, \quad |\mathrm{Re}\, y| \;\leq\; c\,\Omega \qquad \text{on } [u_-, u_+]\,.$$

At $u = u_+$ we get an invariant disk with

$$0 \;\leq\; \alpha_l \;\leq\; \Omega\,, \quad -c\,U_l \;=\; -\Omega^2\,, \qquad T_l \;\leq\; c\,. \tag{4.34}$$

In the remaining interval $[u_+, 2u_1]$ an approximate solution of the Schrödinger equation (2.14) is available from semi-classical analysis: the WKB wave function

$$\phi(u) \;=\; V^{-\frac{1}{4}} \exp\left(\int^u \sqrt{V}\right).$$

The corresponding function y is given by

$$y(u) \;=\; \sqrt{V} - \frac{V'}{4V}\,.$$

In order to get an invariant disk estimate which quantifies the exponential increase of φ, we choose α such that it also becomes large as $V \gg 0$. For technical simplicity, we choose

$$\alpha(u) \;=\; \frac{7}{8}\,\sqrt{V(u)}\,, \tag{4.35}$$

giving rise to the following general result.

Lemma 4.12 *Assume that the potential V is positive on $[0, u_{max}]$ and that*

$$|V'(u)| \;\leq\; \frac{1}{2}\,V(u)^{\frac{3}{2}}\,, \qquad |V''(u)| \;\leq\; \frac{1}{4}\,V(u)^2\,. \tag{4.36}$$

We introduce for a given constant $T_0 > 0$ the functions α, U, σ, β, R, and T by (4.35, 4.12, 4.14, 4.22). If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,,$$

then this condition holds for all $u \in [0, u_{max}]$. Furthermore,

$$\mathrm{Re}\, y \;\geq\; \frac{\sqrt{V}}{8} \;-\; \frac{T_0}{2}\,|\Omega|\,. \tag{4.37}$$

Proof. A short calculation yields

$$\begin{aligned} U \;&=\; V - \alpha^2 - \alpha' \;=\; \frac{15}{64}\,V - \frac{7}{16}\,\frac{V'}{\sqrt{V}} \\ U' + 4\alpha U \;&=\; \frac{105}{128}\,V^{\frac{3}{2}} - \frac{83}{64}\,V' + \frac{7}{32}\,\frac{V'^2}{V^{\frac{3}{2}}} - \frac{7}{16}\,\frac{V''}{\sqrt{V}}\,. \end{aligned}$$

Using (4.36) we obtain the estimates

$$\frac{V}{64} \;\leq\; U \;\leq\; \frac{V}{2} \qquad \text{and} \qquad U' + 4\alpha U \;\geq\; \frac{V^{\frac{3}{2}}}{16}\,.$$

Hence the conditions (4.21) are satisified, and Lemma 4.2 applies. The inequality (4.37) follows from (4.23), the just-derived upper bound for U and the fact that $\sigma \geq 1$. ■

Matching the invariant disk (4.34) to the invariant disk with $\alpha_r = \alpha(u_r)$ and $U_r = V(u_r) - \alpha^2(u_r) - \alpha'(u_r)$ with α according to (4.35), we obtain

$$U_r \;\leq\; \Omega^2\,, \qquad\qquad T_r \;\leq\; c\,. \tag{4.38}$$

We can then apply the last lemma on the interval $[u_+, 2u_1]$.

Proof of Proposition 4.6. Suppose that $u < u_1$. Using the definition of $\acute{y}$ and $\grave{y}$, we can rewrite the Wronskian as

$$w(\acute{\phi}, \grave{\phi}) \;=\; \acute{\phi}\,\grave{\phi}\,(\acute{y} - \grave{y})\,.$$

Applying Lemma 4.7 at $u = 2u_1$ gives

$$\left|\frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})}\right| \;\leq\; \left|\frac{\acute{\phi}(u)}{\acute{\phi}(2u_1)}\right| \, \frac{1}{\mathrm{Re}\,\acute{y}(2u_1) - |\omega|}\,. \tag{4.39}$$

We combine (4.37) with $T_0 = T_r$ satisfying (4.38) to get

$$\mathrm{Re}\,\acute{y} \;\geq\; \frac{\sqrt{V}}{8} \;-\; c\Omega\,. \tag{4.40}$$

Since the potential V is strictly positive on the interval $[u_1, 2u_1]$, we can, possibly by increasing λ_0 and c, arrange that

$$\sqrt{V} \;\geq\; \frac{\sqrt{\lambda}}{c} \qquad\qquad \text{on } [u_1, 2u_1] \tag{4.41}$$

and thus also that

$$\mathrm{Re}\,\acute{y} \;\geq\; \frac{\sqrt{\lambda}}{16\,c} \qquad\qquad \text{on } [u_1, 2u_1]\,. \tag{4.42}$$

This inequality allows us to bound the fraction in (4.39),

$$\left|\frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})}\right| \;\leq\; \left|\frac{\acute{\phi}(u)}{\acute{\phi}(2u_1)}\right|\,. \tag{4.43}$$

Thus it remains to control the last quotient. We omit the accent and use the notation $\rho = |\phi|$. In the case $u < u_+$, we can use (4.9),

$$\frac{\rho(u)^2}{\rho(u_+)^2} \;=\; \frac{\mathrm{Im}\,y(u_+)}{\mathrm{Im}\,y(u)}\,,$$

and the last quotient is controlled from above and below by Corollary 4.9 and Corollary 4.11. Hence, rewriting the quotient on the right of (4.43) as

$$\frac{\rho(u)}{\rho(2u_1)} \;=\; \frac{\rho(u)}{\rho(u_+)}\,\frac{\rho(u_+)}{\rho(2u_1)}\,,$$

it remains to consider the case $u \geq u_+$. Applying (4.7) and (4.40), we obtain

$$A \;:=\; \log\left|\frac{\phi(u)}{\phi(2u_1)}\right| \;=\; -\int_u^{2u_1} \mathrm{Re}\,y(u) \;\leq\; c\,\Omega\,(2u_1 - u_+) \;-\; \frac{1}{8}\int_u^{2u_1} \sqrt{V}\,.$$

Now we use (4.30) and the fact that the function $\Omega \log \Omega$ is bounded,

$$A \;\leq\; c \,\log \lambda \;-\; \frac{1}{8} \int_u^{2u_1} \sqrt{V} \,.$$

Estimating the last summand with (4.41),

$$\int_u^{2u_1} \sqrt{V} \;\geq\; \int_{u_1}^{2u_1} \sqrt{V} \;\geq\; \frac{\sqrt{\lambda}}{c}\, u_1 \,,$$

we conclude that for large λ this summand dominates the term $c \log \lambda$, and thus (4.43) decays in λ even like $\exp(-\sqrt{\lambda}/c)$. ∎

5 Contour Deformations to the Real Axis

In this section we fix the angular momentum number k throughout and omit the angular variable φ. We can again assume without loss of generality that $k \geq 0$. Also, since here we are interested in the situation only locally in u, we evaluate weakly. Thus we write the integral representation (2.16) for compactly supported initial data Ψ_0 and a test function $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$ as

$$<\eta,\, \Psi(t)> \;=\; -\frac{1}{2\pi i} \sum_{n \in \mathbb{N}} \lim_{\varepsilon \searrow 0} \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}} \right) d\omega \; e^{-i\omega t} <\eta, Q_n(\omega)\, S_\infty(\omega)\, \Psi_0> . \qquad (5.1)$$

The integration contour in (5.1) can be moved to the real axis provided that the integrand is continuous. In the next lemma we specify when this is the case and simplify the integrand. For ω real, the complex conjugates of $\acute{\phi}$ and $\grave{\phi}$ are again solutions of the ODE. Thus, apart from the exceptional cases $\omega \in \{0, \omega_0\}$, we can express $\grave{\phi}$ as a linear combination of $\acute{\phi}$ and $\overline{\acute{\phi}}$,

$$\grave{\phi} \;=\; \alpha\, \acute{\phi} + \beta\, \overline{\acute{\phi}} \qquad\qquad (\omega \in \mathbb{R} \setminus \{0, \omega_0\}) \,. \qquad (5.2)$$

The complex coefficients α and β are called *transmission coefficients*. The Wronskian of $\acute{\phi}$ and $\grave{\phi}$ can then be expressed by

$$w(\acute{\phi}, \grave{\phi}) \;=\; \beta\, w(\acute{\phi}, \overline{\acute{\phi}}) \;=\; 2i\Omega\, \beta \,, \qquad (5.3)$$

where in the last step we used the asymptotics (3.1). Furthermore, it is convenient to introduce the real fundamental solutions

$$\phi_1 \;=\; \mathrm{Re}\, \acute{\phi} \,, \qquad\qquad \phi_2 \;=\; \mathrm{Im}\, \acute{\phi} \,,$$

and to denote the corresponding solutions of the wave equation in Hamiltonian form by $\Psi^{\omega n}_{1/2}$.

Lemma 5.1 *If the Wronskian $w(\acute{\phi}, \grave{\phi})$ is non-zero at $\omega \in \mathbb{R} \setminus \{0, \omega_0\}$, then the integrand in (5.1) is continuous at ω and*

$$(\lim_{\varepsilon \nearrow 0} - \lim_{\varepsilon \searrow 0})(Q_n(\omega + i\varepsilon)\, S_\infty(\omega + i\varepsilon)\, \Psi)(r, \vartheta) \;=\; -\frac{i}{\omega \Omega} \sum_{a,b=1}^{2} t^{\omega n}_{ab}\; \Psi^{\omega n}_a <\Psi^{\omega n}_b, \Psi> , \qquad (5.4)$$

where the coefficients t_{ab} are given by

$$t_{11} \;=\; 1+\mathrm{Re}\,\frac{\alpha}{\beta}\,, \qquad t_{12} \;=\; t_{21} \;=\; -\mathrm{Im}\,\frac{\alpha}{\beta}\,, \qquad t_{22} \;=\; 1-\mathrm{Re}\,\frac{\alpha}{\beta}\,. \tag{5.5}$$

Proof. We start from the explicit formula for the operator product $Q_n S_\infty$ given in [5, Proposition 5.4]. Since the angular operator $Q_n(\omega+i\varepsilon)$ can be diagonalized for ε sufficiently small, the kernel $g(u,u')$ is simply the Green's function of the radial ODE, i.e. for ω in the lower half plane,

$$g(u,u') \;:=\; \frac{1}{w(\acute{\phi},\grave{\phi})} \times \left\{ \begin{array}{ll} \acute{\phi}(u)\,\grave{\phi}(u') & \text{if } u \leq u' \\ \grave{\phi}(u)\,\acute{\phi}(u') & \text{if } u > u'\,. \end{array} \right. , \tag{5.6}$$

whereas the formula in the upper half plane is obtained by complex conjugation. Using that $\lim_{\varepsilon\nearrow 0}\acute{\phi} = \lim_{\varepsilon\searrow 0}\overline{\acute{\phi}}$ and $\lim_{\varepsilon\nearrow 0}\grave{\phi} = \lim_{\varepsilon\searrow 0}\overline{\grave{\phi}}$, we find that

$$(\lim_{\varepsilon\nearrow 0} - \lim_{\varepsilon\searrow 0})\; g(u,u') \;=\; 2i\,\mathrm{Im}\,g(u,u')\,,$$

and a short calculation using (5.2, 5.3) gives

$$(\lim_{\varepsilon\nearrow 0} - \lim_{\varepsilon\searrow 0})\; g(u,u') \;=\; -\frac{i}{\Omega}\sum_{a,b=1}^{2} t_{ab}\,\phi^a(u)\,\phi^b(u')$$

with t_{ab} according to (5.5).

Except for the function $g(u,u')$, all the functions appearing in the formula for $Q_n S_\infty$ in [5, Proposition 5.4] are continuous on the real axis. A direct calculation shows that

$$\begin{aligned} &(\lim_{\varepsilon\nearrow 0} - \lim_{\varepsilon\searrow 0})(Q_{k,n}(\omega+i\varepsilon)\,S_\infty(\omega+i\varepsilon)\,\Psi) \\ &= \;-\frac{i}{\omega\Omega}\sum_{a,b=1}^{2} t_{ab}\,\Psi_a <\Psi_b, \begin{pmatrix} (\omega-\beta)\omega & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \end{aligned}$$

with $d\mu$ given by (2.7). Since the Ψ_b are eigenfunctions of the Hamiltonian, we know according to (2.3) that $A\Psi_b = (\omega-\beta)\omega\Psi_b$. Using furthermore that the operator A is symmetric on $L^2(d\mu)$, we conclude that

$$<\Psi_b, \begin{pmatrix} (\omega-\beta)\omega & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \;=\; <\Psi_b, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \;=\; <\Psi_b,\,\Psi>\,,$$

where in the last step we used (2.8). ∎

Let us now consider for which values of ω and n the contour can be moved to the real axis. According to Proposition 4.5, the Wronskian $w(\acute{\phi},\grave{\phi})$ is non-zero unless $\omega \in [\omega_0, 0]$. We now analyze carefully the exceptional cases $\omega = 0, \omega_0$. From Theorem 3.1, Theorem 3.2 and Theorem 3.5 we know that the functions $\acute{\phi}$ and $\phi_\omega = \omega^\mu \grave{\phi}$ are continuous for all $\omega \in \mathbb{R}$. If $ak = 0$ and $\omega = 0$, the functions $\acute{\phi}$ and ϕ_ω degenerate to real solutions with the asymptotics

$$\lim_{u\to-\infty}\acute{\phi}(u) \;=\; 1\,, \qquad \lim_{u\to\infty} u^{\mu-\frac{1}{2}}\,\phi_0(u) \;=\; \frac{\Gamma(\mu)}{\sqrt{\pi}}\,.$$

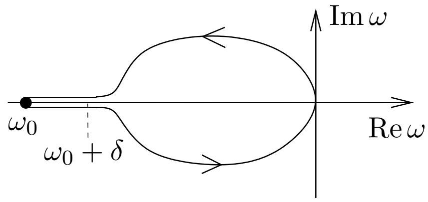


Figure 5: The integration contour D_δ.

Noting that the function

$$\partial_u \sqrt{r^2+a^2} \;=\; \frac{r\,\Delta}{(r^2+a^2)^{\frac{3}{2}}} \;=\; \frac{r}{\sqrt{r^2+a^2}} \left(1 - \frac{2Mr}{r^2+a^2}\right)$$

is monotone increasing, the potential V, (2.15), is everywhere positive. Hence solutions of the Schrödinger equation (2.14) are convex. This implies that the functions $\acute{\phi}$ and $\grave{\phi}$ do not coincide, and thus their Wronskian is non-zero. As a consequence, the Green's function (5.6), and thus the whole integrand in (5.1), is bounded and continuous near $\omega = 0$ (note that (5.6) is invariant under rescalings of $\grave{\phi}$, and thus we can in this formula replace $\grave{\phi}$ by ϕ_ω). In the case $ak \neq 0$ and $\omega = 0$, the function ϕ_0 is real, whereas $\acute{\phi}$ is complex, and thus $w(\acute{\phi}, \phi_0) \neq 0$. If on the other hand $ak \neq 0$ and $\omega = \omega_0$, $\acute{\phi}$ is real and $\grave{\phi}$ is complex, and again $w(\acute{\phi}, \phi_0) \neq 0$. Hence the integrand in (5.1) is continuous and bounded at the points $\omega = 0, \omega_0$. We conclude that for every $n \in \mathbb{N}$, the integrand in (5.1) is continuous on an open neighborhood of on $\omega \in \mathbb{R} \setminus (\omega_0, 0)$. Furthermore, according to Proposition 4.6, $w(\acute{\phi}, \grave{\phi}) \neq 0$ if $\omega \in (\omega_0, 0)$ and λ is sufficiently large. We have thus proved the following result.

Proposition 5.2 *There is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $\Psi_0 \in C^\infty_0(\mathbb{R} \times S^2)^2$, the completeness relation*

$$\begin{aligned} \Psi_0 \;&=\; \frac{1}{2\pi} \left(\sum_{n=0}^{\infty} \int_{\mathbb{R}\setminus[\omega_0,0]} + \sum_{n>n_0} \int_{\omega_0}^{0} \right) \frac{d\omega}{\omega\Omega} \sum_{a,b=1}^{2} t^{\omega n}_{ab}\; \Psi^{\omega n}_a \,<\!\Psi^{\omega n}_b, \Psi_0\!> \\ &\quad + \sum_{n \leq n_0} \oint_{D_\delta} (Q_n\, S_\infty\, \Psi_0)\, d\omega \end{aligned}$$

holds, with the contour D_δ as in Figure 5.

We point out that the contour D_δ passes along the line segment $[\omega_0, \omega_0 + \delta)$ twice, once as the limit of the contour in the lower half plane, and once as limit of the contour in the upper half plane. These two integrals can be combined to one integral over $[\omega_0, \omega_0 + \delta)$ with the integrand given by (5.4).

Let us now consider how the remaining contour integrals over C_ε can be moved to the real line. According to Theorems 3.1 and 3.2, the functions $\acute{\phi}$ and $\grave{\phi}$ have for every $n \leq n_0$ and for every $\omega \in (\omega_0, 0)$ a holomorphic extension to a neighborhood of ω. Thus their Wronskian is also holomorphic in this neighborhood, and consequently they can have only isolated zeros of finite order. Since $w(\acute{\phi}, \grave{\phi}) \neq 0$ for ω near 0 and ω_0, we conclude that the numbers of zeros must be finite. Since we only need to consider a finite number of angular momentum modes, there is at most a finite number of points $\omega_1, \ldots, \omega_K \in (\omega_0, 0)$, $K \geq 0$, where any of the Wronskians $w(\acute{\phi}_n, \grave{\phi}_n)$ has a zero. We denote the maximum of the orders of these zeros at ω_i by $l_i \in \mathbb{N}$.

The above zeros of the Wronskian lead to poles in the integrand of (5.1) and correspond to radiant modes. We will prove in Section 7 by contradiction that these radiant modes are actually absent. Therefore, we now make the assumption that there are radiant modes, i.e. that the Wronskians $w(\acute{\phi}_n, \grave{\phi}_n)$ have at least one zero on the real axis. As a preparation for the analysis of Section 7, we now choose a special configuration where radiant modes appear, but in the simplest possible way. We choose new initial data

$$\Phi_0 \;=\; \mathcal{P}(H)\, \Psi_0\,, \tag{5.7}$$

where $\mathcal{P}$ is the polynomial

$$\mathcal{P}(x) \;=\; \omega\,(\omega - \omega_0)\; (x - \omega_1)^{l_1 - 1} \prod_{i=2}^{K} (x - \omega_i)^{l_i}\,.$$

Then Φ_0 again has compact support, and using the spectral calculus, the corresponding solution $\Phi(t)$ of the Cauchy problem is obtained from (5.1) by multiplying the integrand by $\mathcal{P}(\omega)$. Then the poles of the integrand at $\omega_2, \ldots, \omega_K$ disappear, and at ω_1 a simple pole remains. Subtracting this pole, the integrand becomes analytic, whereas for the pole itself we get a contour integral which can be computed with residues.

Let us summarize the result of the above construction with a compact notation. For a test function $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$ we introduce the vectors $\eta^{\omega n}$ by

$$\eta^{\omega n} \;=\; (\eta^{\omega n}_1, \eta^{\omega n}_2) \qquad \text{where} \qquad \eta^{\omega n}_a \;=\; <\Psi^{\omega n}_a, \eta>\,.$$

Proposition 5.3 *Assume that there are radiant modes, $K \geq 1$. Then the Cauchy development $\Phi(t)$ of the initial data (5.7) satisfies the relation*

$$<\eta, \Phi(t)> \;=\; \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\infty}^{\infty} e^{-i\omega t}\, \langle \eta^{\omega n}, T^{\omega n} \Phi^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \;+\; e^{-i\omega_1 t} \sum_{n \leq n_0} \langle \eta^{\omega_1 n}, \sigma^n \rangle_{\mathbb{C}^2}\,.$$

Here $\omega_1 \in (\omega_0, 0)$. The $(\sigma^n)_{n=1,\ldots,n_0}$ are vectors in $\mathbb{C}^2$, at least one of which is non-zero. The matrices $T^{\omega n}$ have the following properties,

(1) *If $\omega \notin [\omega_0, 0]$ or $n > n_0$,*

$$(T^{\omega n})_{ab} \;=\; t^{\omega n}_{ab} \tag{5.8}$$

with t_{ab} according to (5.5).

(2) *For each n, the function $T^{\omega n}$ is continuous in $\omega \in \mathbb{R}$ and analytic in $(\omega_0, 0)$.*

6 Energy Splitting Estimates

In this section, we consider the family of test functions

$$\eta_L(u) \;=\; \eta(u + L)$$

for a fixed $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$. Our goal is to control the inner product $<\eta_L, \Phi(t)>$ in the limit $L \to \infty$ when the support of η_L moves towards the event horizon. Our method is to split up the inner product into a positive and an indefinite part. Once the indefinite part is bounded using the ODE estimates of Section 4, we can use the Schwarz inequality and energy conservation to also control the positive part.

We choose $u_1 \in \mathbb{R}$ and general test functions $\eta, \zeta \in C_0^\infty((-\infty, u_1) \times S^2)^2$ which are supported to the left of u_1 (for later use we often work more generally with ζ instead of Φ_0). Since for each fixed n, the $T^{\omega n}$ are continuous and the eigensolutions $\Psi_a^{\omega n}(u)$ are, according to Theorem 3.1, also continuous in ω, uniformly for $u \in (-\infty, u_1)$, we have no difficulty controlling the expressions $\langle \eta^{\omega n}, T^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}$ for $n \leq n_0$ and $\omega \in [\omega_0, 0]$. Hence we only need to consider the case when the matrix $T^{\omega n}$ is given by (5.8). Using (5.5), the eigenvalues $\lambda_\pm$ of this matrix are

$$\lambda_\pm \;=\; 1 \pm \left| \frac{\alpha}{\beta} \right| . \tag{6.1}$$

In order to determine the sign of these eigenvalues, we first use the asymptotics (3.1, 3.2) to compute the Wronskians $w(\grave{\phi}, \overline{\grave{\phi}}) = -2i\omega$ and $w(\acute{\phi}, \overline{\acute{\phi}}) = 2i\Omega$. Furthermore, we obtain from (5.2) and its complex conjugate that

$$w(\grave{\phi}, \overline{\grave{\phi}}) \;=\; (|\alpha|^2 - |\beta|^2)\, w(\acute{\phi}, \overline{\acute{\phi}}) \,.$$

Combining these identities, we find that

$$|\alpha|^2 - |\beta|^2 \;=\; -\frac{\omega}{\Omega} \tag{6.2}$$

From (6.1, 6.2) we see that in the case $\omega \notin [\omega_0, 0]$, where ω and Ω have the same sign, the eigenvalues $\lambda_\pm$ are both positive. However, if $\omega \in (\omega_0, 0)$, one of the eigenvalues is negative. This result is not surprising, because the lack of positivity corresponds to the fact that for $\omega \in [\omega_0, 0]$ the energy density can be negative inside the ergosphere. In the case when $T^{\omega n}$ is not positive, we decompose it into the difference of two positive matrices,

$$T^{\omega n} \;=\; T_+^{\omega n} - T_-^{\omega n} \qquad \text{for } \omega \in (\omega_0, 0),\; n > n_0 \,,$$

where

$$T_-^{\omega n} \;=\; -\lambda_- \, \mathbb{1} \,.$$

In the next lemma we bound the integral over $T_-^{\omega n}$ using ODE techniques.

Lemma 6.1 *For any $\varepsilon > 0$ we can, possibly by increasing n_0, arrange that for all $L \leq 0$,*

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_-^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| \, d\omega \;\leq\; \varepsilon \,.$$

Proof. Using (6.2, 5.3) we can estimate the norm of T_- by

$$\|T_-\| \;=\; |\lambda_-| \;=\; \frac{|\alpha|^2 - |\beta|^2}{|\beta|\,(|\alpha| + |\beta|)} \;\leq\; \left| \frac{\omega}{\Omega} \right| \frac{1}{2\,|\beta|^2} \;=\; \frac{2|\omega\Omega|}{|w(\acute{\phi}, \grave{\phi})|^2} \,.$$

Hence

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_-^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| \, d\omega \;\leq\; 2 \sum_{n > n_0} \int_{\omega_0}^0 \frac{|\eta_L^{\omega n}|}{|w(\acute{\phi}, \grave{\phi})|} \, \frac{|\zeta^{\omega n}|}{|w(\acute{\phi}, \grave{\phi})|} \, |\omega\Omega| \, d\omega \,.$$

Writing out the energy scalar product using [5, eq. (2.14)] and expressing the fundamental solutions $\Psi_a^{\omega n}$ in terms of the radial solution $\acute{\phi}$, one sees that

$$|\eta_L^{\omega n}| \;\leq\; c \,\sup_{\mathbb{R}} |\eta_L \, \acute{\phi}| \,, \qquad\qquad |\zeta^{\omega n}| \;\leq\; c \,\sup_{\mathbb{R}} |\zeta \, \acute{\phi}| \,,$$

where the constant $c = c(\omega)$ is independent of λ. Now we apply Proposition 4.6 and use that the eigenvalues λ_n grow quadratically in n, (2.11). ∎

Lemma 6.2 *There is a constant $C > 0$ such that for all $L \geq 0$,*

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} |\langle \eta_L^{\omega n}, T^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}| \; d\omega \;\leq\; C \,.$$

Proof. First of all, using the positivity of the matrix T,

$$\begin{aligned} &\sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} |\langle \eta_L^{\omega n}, T^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}| \; d\omega \\ &\quad\leq\; \frac{1}{2} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} \left(\langle \eta_L^{\omega n}, T^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2} \;+\; \langle \zeta^{\omega n}, T^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right) \, d\omega. \end{aligned}$$

The two summands can be treated in exactly the same way; we treat the summand involving $\eta_L^{\omega n}$ because of the additional L-dependence. Applying Proposition 5.2 and dropping all negative terms, we get

$$\begin{aligned} \int_{\mathbb{R}\setminus[\omega_0,0]} \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;&\leq\; <\!\eta_L, H\,(H-\omega_0)\, \eta_L\!> \\ &+ \sum_{n>n_0} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_-^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;+\; \sum_{n\leq n_0} \oint_{D_\delta} |\!<\!\eta_L, \, (Q_n \, S_\infty \, H\,(H-\omega_0)\, \eta_L)\!>\!| \; d\omega \,. \end{aligned}$$

Using the asymptotic form of the energy scalar product and the Hamiltonian near the event horizon, it is obvious that the first term stays bounded as $L \to \infty$. The second term is bounded according to Lemma 6.1. For the contour integrals we can use the formula (5.4) on the real interval $[\omega_0, \omega_0 + \delta)$. Since Theorem 3.1 gives us control of the asymptotics of fundamental solution $\acute{\phi}$ uniformly as $u \to -\infty$, it is clear that the integral over $[\omega_0, \omega_0 + \delta)$ is bounded uniformly in L. For the contour in the complex plane, we cannot work with (5.4), but we must instead consider the formula for the operator product $Q_n \, S_\infty$ give in [5, Proposition 5.4] together with the estimate for the Green's function given in Lemma 6.3 below. ∎

Lemma 6.3 *For every $\tilde{\omega} \in D_\delta$ with $\omega \neq \omega_0$, there are constants $C, \epsilon > 0$ and $u_0 \in \mathbb{R}$ such the Green's function satisfies for all $\omega \in D_\delta \cap B_\epsilon(\tilde{\omega})$ the inequality*

$$|g(u,v)| \;\leq\; C \qquad\quad \text{for all } u, v \leq u_0 \,.$$

Proof. It suffices to consider the case $\text{Im}\,\omega \leq 0$, because the Green's function in the upper half plane is obtained simply by complex conjugation. By symmetry, we can furthermore assume that $u \leq v$. Thus, according to (5.6), we must prove the inequality

$$\left| \frac{\acute{\phi}(u)\, \grave{\phi}(v)}{w(\acute{\phi}, \grave{\phi})} \right| \;\leq\; C \qquad \text{for all } u \leq v \leq u_0\,.$$

According to Whiting's mode stability [11], the Wronskian $w(\acute{\phi}, \grave{\phi})$ has no zeros away from the real line, and thus by choosing δ so small that $B_\delta(\tilde{\omega})$ lies entirely in the lower half plane, we can arrange that $|w(\acute{\phi}, \grave{\phi})|$ is bounded away from zero on $B_\delta(\tilde{\omega})$. Hence our task is to bound the factor $|\acute{\phi}(u)\, \grave{\phi}(v)|$. Solving the defining equation for $w(\acute{\phi}, \grave{\phi})$ for $\grave{\phi}$ and integrating, we obtain

$$\frac{\grave{\phi}}{\acute{\phi}}\Big|_v^{u_0} \;=\; -w(\acute{\phi}, \grave{\phi}) \int_v^{u_0} \frac{du}{\acute{\phi}(u)^2}\,.$$

Substituting the identity

$$\frac{1}{\acute{\phi}(u)^2} \;=\; \frac{e^{-2i\Omega u}}{2i\Omega}\, \frac{d}{du} \left(\frac{e^{2i\Omega u}}{\acute{\phi}(u)^2} \right) \;-\; \frac{1}{2i\Omega}\, \frac{d}{du} \left(\frac{1}{\acute{\phi}(u)^2} \right),$$

the integral over the last term gives a boundary term,

$$\int_v^{u_0} \frac{1}{2i\Omega}\, \frac{d}{du} \left(\frac{1}{\acute{\phi}(u)^2} \right) \;=\; \frac{1}{2i\Omega}\, \frac{1}{\acute{\phi}(u)^2} \Big|_v^{u_0}\,.$$

The integral over the other term can be estimated by

$$\frac{1}{2\Omega} \int_v^{u_0} \left| e^{-2i\Omega u} \left(\frac{e^{2i\Omega u}}{\acute{\phi}(u)^2} \right)' \right| dv \;\leq\; \frac{e^{-2\text{Im}\,\Omega\, v}}{\Omega} \int_v^{u_0} \left| \frac{(e^{-i\Omega u}\, \acute{\phi}(u))'}{(e^{-i\Omega u}\, \acute{\phi}(u))^3} \right| dv\,.$$

Using the asymptotics (3.1) one sees that the last integrand vanishes at the event horizon. From the series expansion for $\acute{\phi}$, (3.7, 3.8), we see that this integrand decays even exponentially fast. Therefore, the last integral is finite, uniformly in v and locally uniformly in ω. Collecting all the obtained terms and using the known asymptotics (3.1) of $\acute{\phi}$, the result follows. ∎

Lemma 6.4 *For any $\varepsilon > 0$ we can, possibly by increasing n_0, arrange that for all $L \geq 0$,*

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_+^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| d\omega \;\leq\; \varepsilon\,.$$

Proof. Again using positivity, it suffices to bound the terms

$$\sum_{n > n_0} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \quad \text{and} \quad \sum_{n > n_0} \int_{\omega_0}^0 \langle \zeta^{\omega n}, T_+^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega\,.$$

They can be treated similarly, consider for example the first term. For any $n_1 > n_0$,

$$\inf_{(\omega_0,0)} \lambda_{n_1}^2 \sum_{n \geq n_1} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;\leq\; \sum_{n > n_0} \int_{\omega_0}^0 \langle (\mathcal{A}\eta_L)^{\omega n}, T_+^{\omega n} (\mathcal{A}\eta_L)^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega$$

$$\leq \; <\mathcal{A}\eta_L, \, H\,(H-\omega_0)\,\mathcal{A}\eta_L> + \sum_{n > n_0} \int_{\omega_0}^0 \langle (\mathcal{A}\eta_L)^{\omega n}, T_-^{\omega n} (\mathcal{A}\eta_L)^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega$$

$$+ \sum_{n \leq n_0} \oint_{D_\delta} |<\mathcal{A}\eta_L, \, (Q_n \, S_\infty \, H\,(H-\omega_0)\,\mathcal{A}\eta_L)>| \; d\omega \,.$$

Here $\mathcal{A}$ is the angular operator. When it acts on a test function, we always get rid of the time-derivatives with the replacement $i\partial_t \to H$. We now argue as in the proof of Lemma 6.2 (with η_L replaced by $\mathcal{A}\eta_L$) and choose n_1 sufficiently large. ∎

7 An Integral Representation on the Real Axis

We now use a causality argument together with the estimates of the previous section to show that the radiant modes in Proposition 5.3 must be absent. This will be a contradiction to the assumption that there are radiant modes, ruling out the possibility that there are radiant modes at all. This will lead us to an integral representation of the propagator on the real axis.

Let us return to the setting of Proposition 5.3. Choosing the ϑ-dependence of η such that it is orthogonal to the angular wave functions $(\Psi_a^{\omega_1 n})_{n \leq n_0}$ except for one n, and choosing the u-dependence of η such that it is orthogonal only to one of the plane waves $e^{\pm i(\omega_1 - \omega_0)u}$, we can clearly arrange that

$$\limsup_{L \to \infty} |\sigma(L)| \;=:\; \kappa \;>\; 0 \qquad \text{where} \qquad \sigma(L) \;:=\; \sum_{n \leq n_0} <\eta_L^{\omega_1 n}, \sigma^n>_{\mathbb{C}^2} \,. \tag{7.1}$$

Furthermore, we choose η_L such that its support lies to the left of $\text{supp}\, \Phi_0$, i.e.

$$\text{dist}(\text{supp}\, \eta_L, \, \text{supp}\, \Phi_0) \;>\; L \qquad \text{for all } L \geq 0.$$

Due to the finite propagation speed (which in the (t, u)-coordinates is equal to one),

$$\text{supp}\, \eta_L \,\cap\, \text{supp}\, \Phi(t) \;=\; \emptyset \qquad \text{if } |t| \leq L.$$

Hence for all $L > 0$,

$$\begin{aligned} 0 \;&=\; \frac{1}{2L} \int_{-L}^{L} e^{i\omega_1 t} <\eta_L, \, \Phi(t)> \\ &=\; \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\infty}^{\infty} \frac{\sin((\omega - \omega_1)L)}{(\omega - \omega_1)L} \, \langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;+\; \sigma(L) \,. \end{aligned}$$

We apply Lemma 6.1 and Lemma 6.4 with $\varepsilon = \kappa/(8\pi)$ to obtain

$$\frac{1}{2\pi} \sum_{n > n_0} \int_{\omega_0}^0 |\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}| \; d\omega \;\leq\; \frac{\kappa}{2} \,.$$

Furthermore, Lemma 6.2 gives rise to the estimate

$$\left| \int_{\mathbb{R}\setminus[\omega_0,0]} \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \,\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \right| \;\leq\; C \sup_{\mathbb{R}\setminus[\omega_0,0]} \left| \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \right| ,$$

and since this supremum tends to zero as $L \to \infty$, we conclude that the expression on the left vanishes in the limit $L \to \infty$. Combining these estimates with (7.1), we obtain

$$\limsup_{L\to\infty} \left| \sum_{n \leq n_0} \int_{\omega_0}^{0} \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \,\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \right| \;\geq\; \pi\kappa \,. \tag{7.2}$$

Since the matrices $T^{\omega n}$ are continuous in ω and the fundamental solutions $\Psi_a^{\omega n}(u)$ are according to Theorem 3.1 uniformly bounded as $u \to -\infty$, there is a constant C such that

$$|\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}| \;\leq\; C \qquad \text{for all } L \geq 0 \text{ and } \omega \in (\omega_0, 0),\; n \leq n_0 \,.$$

Hence we can apply Lebesgue's dominated convergence theorem on the left of (7.2) and take the limit $L \to \infty$ inside the integral, giving zero. This is a contradiction.

Since radiant modes have been ruled out, we know that the Wronskian $w(\acute{\phi}, \grave{\phi})$ has no zeros on the real axis. Thus we can move all contours up to the real axis. This gives the following integral representation for the propagator.

Theorem 7.1 *For any initial data $\Psi_0 \in C_0^\infty(\mathbb{R}\times S^2)^2$, the solution of the Cauchy problem has the integral representation*

$$\Psi(t,r,\vartheta,\varphi) \;=\; \frac{1}{2\pi} \sum_{k\in\mathbf{Z}} e^{-ik\varphi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n}\, \Psi^a_{k\omega n}(r,\vartheta) < \Psi^b_{k\omega n}, \Psi_0 >$$

with the coefficients t_{ab} as given by (5.5, 5.2). Here the sums and the integrals converge in L^2_{loc}.

8 Proof of Decay

As was recently pointed out to us by Thierry Daudé, there is an error on the last page of the printed version of this paper (which is identical with the previous online version). In order to keep the best possible agreement between the printed and online versions, in this section we simply give the erratum which corrects the printed version. The error was that the inequality (8.3) in the printed version of this paper cannot be applied to the function $\Psi(t)$ because it does not satisfy the correct boundary conditions. This invalidates the last two inequalities of the printed version, and thus the proof of decay is incomplete. We here fill the gap using a different method. At the same time, we will clarify in which sense the sum over the angular momentum modes converges in [5, Theorem 1.1] and Theorem 7.1, an issue which so far was not treated in sufficient detail. The arguments so far certainly yield *weak* convergence in L^2_{loc}; here we will prove strong convergence.

Our method here is to split the wave function into the high and low energy components. For the high energy component, we show that the L^2-norm of the wave function can be bounded by the energy integral, even though the energy density need not be everywhere positive (Section 8.1). For the low energy component we refine our ODE techniques

(Section 8.2). Combining these arguments with a Sobolev estimate and the Riemann-Lebesgue lemma completes the proof (Section 8.3).

We begin by considering the integral representation of Theorem 7.1, for fixed k and a finite number n_0 of angular momentum modes,

$$\Phi^{n_0}(t,r,\vartheta) \;:=\; \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t}\,\hat{\Phi}^{n_0}(\omega,r,\vartheta)\;, \tag{8.1}$$

where for notational convenience we have omitted the φ-dependence (i.e. the factor $e^{-ik\varphi}$), and $\hat{\Phi}^{n_0}(\omega)$ is defined by

$$\hat{\Phi}^{n_0}(\omega,r,\vartheta) \;=\; \frac{1}{2\pi}\,\frac{1}{\omega\Omega}\sum_{n=1}^{n_0}\sum_{a,b=1}^{2} t_{ab}^{\omega n}\;\Phi^a_{\omega n}(r,\vartheta)\;<\Psi^b_{\omega n},\Psi_0>$$

(as in [5], we always denote the scalar wave function by Φ, whereas $\Psi = (\Phi, \partial_t\Phi)$ is a two-component vector). We recall that for large ω, the WKB-estimates of [5, Section 6] ensure that the fundamental solutions $\Phi^b_{k\omega n}$ go over to plane waves, and thus, since the initial data Ψ_0 is smooth and compactly supported, the function $\hat{\Phi}^{n_0}(\omega,r,\vartheta)$ decays rapidly in ω (for details on this method see [8, proof of Theorem 6.5]). As a consequence, Φ^{n_0} and its derivatives are, for r and ϑ in any compact set, uniformly bounded in time. Our goal is to obtain estimates uniform in n_0, justifying that, as $n_0 \to \infty$, Φ^{n_0} converges in L^2_{loc} to the solution of the wave equation.

To arrange the energy splitting we choose for a given parameter $J > 0$, a positive smooth function $\chi_{\text{H}+}$ supported on (J,∞) with $\chi_{\text{H}+}|_{[2J,\infty)} \equiv 1$. We define $\chi_{\text{H}-}$ by $\chi_{\text{H}-}(\omega) = \chi_{\text{H}+}(-\omega)$ and set $\chi_{\text{L}} = 1 - \chi_{\text{H}+} - \chi_{\text{H}-}$. We introduce the high-energy contributions $\hat{\Phi}^{n_0}_{\text{H}\pm}$ and the low-energy contribution $\hat{\Phi}^{n_0}_{\text{L}}$ by

$$\hat{\Phi}^{n_0}_{\text{H}\pm}(\omega,r,\vartheta) \;=\; \chi_{\text{H}\pm}(\omega)\,\hat{\Phi}^{n_0}(\omega,r,\vartheta)\;, \qquad \hat{\Phi}^{n_0}_{\text{L}}(\omega,r,\vartheta) \;=\; \chi_{\text{L}}(\omega)\,\hat{\Phi}^{n_0}(\omega,r,\vartheta)\;.$$

8.1 L^2-Estimates of the High-Energy Contribution

We recall from [5, (2.5)] that the energy density of a wave function Φ in the Kerr geometry is given by

$$\begin{aligned}\mathcal{E}(\Phi) \;=\;& \left(\frac{(r^2+a^2)^2}{\Delta} - a^2\,\sin^2\vartheta\right)|\partial_t\Phi|^2 + \Delta\,|\partial_r\Phi|^2 \\ &+ \sin^2\vartheta\,|\partial_{\cos\vartheta}\Phi|^2 + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta}\right)k^2\;|\Phi|^2\,.\end{aligned} \tag{8.2}$$

Note that the energy density need not be positive due to the last term. However, the next theorem shows that the energy integral

$$E(\Phi) \;:=\; \int_{r_1}^{\infty} dr \int_{-1}^{1} d\cos\vartheta\;\mathcal{E}(\Phi(t))$$

(which is independent of time due to energy conservation), in the high-energy region is both positive and can be bounded from below by the L^2-norm. In what follows, we only consider $\Phi^{n_0}_{\text{H}+}$ because $\Phi^{n_0}_{\text{H}-}$ can be treated similarly.

Theorem 8.1 *There exists a positive constant J_0 (depending only on k, but independent of n_0 and Ψ_0), such that for all $J \geq J_0$ the following inequality holds for every t:*

$$E(\Phi^{n_0}_{H+}) \;\geq\; \frac{J^2}{2} \int_{r_1}^{\infty} \frac{(r^2+a^2)^2}{\Delta}\, dr \int_{-1}^{1} d\cos\vartheta\, |\Phi^{n_0}_{H+}(t)|^2 \,.$$

The remainder of this section is devoted to the proof of Theorem 8.1. We begin with the following lemma.

Lemma 8.2 *Let g, Φ be measurable functions, g real and Φ complex, such that Φ and $g\Phi$ are in $L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} d\omega \int_{\mathbb{R}} d\omega' \;\min\Big(g(\omega), g(\omega')\Big)\; \Phi(\omega)\, \overline{\Phi(\omega')} \;\geq\; \inf g \;\left| \int_{\mathbb{R}} \Phi \right|^2 .$$

Proof. Using a standard approximation argument, it suffices to consider the case that g and Φ are simple functions of the form

$$g(\omega) \;=\; \sum_{a=1}^{A} g_a\, \chi(K_a)\,, \qquad \Phi(\omega) \;=\; \sum_{a=1}^{A} \Phi_a\, \chi(K_a)\,,$$

where $\chi(K_a)$ is the characteristic function of the set K_a, and $(K_a)_{a=1,\ldots,A}$ forms a partition of $\mathbb{R}$. Then the above inequality reduces to

$$\sum_{a,b=1}^{A} \min(g_a, g_b)\; \Phi_a\, |K_a|\; \overline{\Phi_b}\, |K_b| \;\geq\; \min g \sum_{a,b=1}^{A}\; \Phi_a\, |K_a|\; \overline{\Phi_b}\, |K_b|\,.$$

In the case $A = 2$ and $g_1 \leq g_2$, this inequality follows immediately from the calculation

$$g_1\, |c_1|^2 + g_1\, (c_1\, \overline{c_2} + \overline{c_1}\, c_2) + g_2\, |c_2|^2 \;\geq\; g_1\, |c_1|^2 + g_1\, (c_1\, \overline{c_2} + \overline{c_1}\, c_2) + g_1\, |c_2|^2 \;=\; g_1\, |c_1 + c_2|^2\,,$$

where $c_a := \Phi_a\, |K_a|$. In the case $A = 3$ and $g_1 \leq g_2 \leq g_3$, we get

$$\begin{aligned} & g_1\left(|c_1|^2 + 2\mathrm{Re}(c_1\overline{c_2} + c_1\overline{c_3})\right) + g_2\left(|c_2|^2 + 2\mathrm{Re}(c_2\overline{c_3})\right) + g_3\, |c_3|^2 \\ &\quad \geq\; g_1\left(|c_1|^2 + 2\mathrm{Re}(c_1\overline{c_2} + c_1\overline{c_3})\right) + g_2\, |c_2 + c_3|^2 \;\geq\; g_1\, |c_1 + c_2 + c_3|^2 \,. \end{aligned}$$

The general case is similar. ■

The next lemma bounds the L^2-norm of $\Phi^{n_0}_{\mathrm{H}+}$ and its partial derivatives by a constant depending on n_0 and Ψ_0.

Lemma 8.3 *There is a constant $C = C(n_0, \Psi_0)$ such that for every t,*

$$\int_{r_1}^{\infty} \frac{(r^2+a^2)^2}{\Delta}\, dr \int_{-1}^{1} d\cos\vartheta\, \left(|\Phi^{n_0}_{H+}(t)|^2 + |\partial_r \Phi^{n_0}_{H+}(t)|^2 + \sum_{k=1}^{3} |\partial_t^k \Phi^{n_0}_{H+}(t)|^2 \right) \;\leq\; C\,.$$

Proof. It suffices to consider one angular momentum mode. For notational simplicity we omit the angular dependence. Since $\Phi^{n_0}_{\mathrm{H}+}$ and its derivatives are locally pointwise bounded uniformly in time, it follows that their L^2-norms on any compact set are bounded in time.

Near the event horizon, we work with the fundamental solution $\acute{\phi}$ in the Regge-Wheeler variable u (see [5, (2.18, 5.2)] and Section 3.1. Then for any sufficiently small u_0, the integral of $|\Phi|^2$ over the region $u < u_0$ can be written as

$$\int_{-\infty}^{u_0} du\, |\phi|^2 \,,$$

where it is now convenient to write our integral representation (8.1) in the form

$$\phi(t,u) \;=\; \int_{-\infty}^{\infty} d\omega \left(h_+(\omega)\, \acute{\phi}_\omega(u) + h_-(\omega)\, \overline{\acute{\phi}_\omega(u)} \right) e^{-i\omega t} \,.$$

Here the functions $h_\pm$ have rapid decay and, as they are supported away from the set $\{0, \omega_0\}$, they are also smooth (see Section 3.1). Using the Jost representation (3.7, 3.8), the function $\phi(t,u)$ can be decomposed as

$$\phi(t,u) \;=\; \phi_+(t,u) + \phi_-(t,u) + \rho(t,u)$$

where

$$\begin{aligned} \phi_\pm(t,u) &:= \int_{-\infty}^{\infty} d\omega\, h_\pm(\omega)\, e^{\pm i(\omega-\omega_0)u - i\omega t} \\ |\rho(t,u)| &\leq C\, e^{\gamma u} \quad \text{for all } t, \text{ where } C, \gamma > 0. \end{aligned}$$

Note that the smoothness of $h_\pm$ implies that $\phi_\pm$ decay rapidly in u. From the exponential decay of the factor $e^{\gamma u}$ it is obvious that the L^2-norm of ρ is bounded uniformly in t. The L^2-norms of $\phi_\pm$ can be estimated as

$$\int_{-\infty}^{u_0} |\phi_\pm(t,u)|^2 du \;\leq\; \int_{-\infty}^{\infty} |\phi_\pm(t,u)|^2 du \;=\; \int_{-\infty}^{\infty} |\phi_\pm(0,u')|^2 du' \;=:\; c \,,$$

where $u' = u \mp t$.

Near infinity, we work similarly with the fundamental solutions $\grave{\phi}$ (3.2). Again using the Jost representation (see (pe2) and Lemma 3.3), we get terms depending only on $t \pm u$ as well as error terms which decay like $1/u$ and are thus in L^2.

The time derivatives can be treated in the same way, since a time derivative merely gives a factor of ω which can be absorbed into $h_\pm$. For the spatial derivatives we use similarly the estimates for the first derivatives of the Jost functions. ∎

Our next step is to decompose the energy integral into a convenient form. To do this, we introduce a positive mollifier $\alpha \in C^\infty_0([-1,1])$ with the properties $\alpha(-\omega) = \alpha(\omega)$ and $\int \alpha(\omega) d\omega = 1$. We define the function $\Gamma(\omega - \omega')$ by mollifying the Heaviside function Θ,

$$\Gamma(\omega - \omega') \;=\; (\Theta * \alpha)(\omega - \omega') \,. \tag{8.3}$$

We now substitute the Fourier representation of $\Phi^{n_0}_{\mathrm{H}+}$ into the formula for the energy density (8.2). For simplicity we omit the indices n_0 and $H+$ in what follows. Omitting the first positive summand in (8.2), we get the inequality

$$\mathcal{E}(\Phi)(t,r,\vartheta) \;\geq\; \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, e^{-i(\omega-\omega')t}$$

$$\times\Bigg\{ \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta}\right) k^2\, \hat{\Phi}(\omega)\, \overline{\hat{\Phi}(\omega')} \tag{8.4}$$

$$+\Gamma(\omega-\omega') \Big(\Delta\, \partial_r \hat{\Phi}(\omega)\, \overline{\partial_r \hat{\Phi}(\omega')} + \sin^2\vartheta\, \partial_{\cos\vartheta} \hat{\Phi}(\omega)\, \overline{\partial_{\cos\vartheta} \hat{\Phi}(\omega')}\Big) \tag{8.5}$$

$$+(1-\Gamma)(\omega-\omega') \Big(\Delta\, \partial_r \hat{\Phi}(\omega)\, \overline{\partial_r \hat{\Phi}(\omega')} + \sin^2\vartheta\, \partial_{\cos\vartheta} \hat{\Phi}(\omega)\, \overline{\partial_{\cos\vartheta} \hat{\Phi}(\omega')}\Big) \Bigg\}. \tag{8.6}$$

We multiply by a positive test function $\eta(u) \in C^\infty_0(\mathbb{R})$ and integrate over r and $\cos\vartheta$. Integrating by parts in (8.5) and (8.6) to the right and left, respectively, we can use the wave equation

$$\Bigg[-\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\Delta} \left((r^2+a^2)\omega + ak\right)^2 \\ - \frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial \cos\vartheta} + \frac{1}{\sin^2\vartheta} (a\omega \sin^2\vartheta + k)^2 \Bigg] \hat{\Phi}(\omega) \;=\; 0$$

to obtain

$$\int_{r_1}^\infty dr \int_{-1}^1 d\cos\vartheta\; \eta(u) \Big((8.5)+(8.6)\Big) \;=\; \int_{r_1}^\infty dr \int_{-1}^1 d\cos\vartheta \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty d\omega'\; e^{-i(\omega-\omega')t}$$

$$\times\Bigg\{ -\eta'(u)\, (r^2+a^2) \Big(\Gamma\, \hat{\Phi}(\omega)\, \overline{\partial_r \hat{\Phi}(\omega')} + (1-\Gamma)\, \partial_r \hat{\Phi}(\omega)\, \overline{\hat{\Phi}(\omega')}\Big) \tag{8.7}$$

$$+\eta(u) \Big(\Gamma\, g(\omega') + (1-\Gamma)\, g(\omega)\Big)\, \hat{\Phi}(\omega)\, \overline{\hat{\Phi}(\omega')} \Bigg\}, \tag{8.8}$$

where

$$g(\omega, r, \vartheta) \;=\; \frac{1}{\Delta} \left((r^2+a^2)\omega + ak\right)^2 - \frac{1}{\sin^2\vartheta} (a\omega \sin^2\vartheta + k)^2\,,$$

and we used that

$$\frac{d}{dr} \eta(u) \;=\; \eta'(u)\, \frac{r^2+a^2}{\Delta}\,.$$

We interchange the orders of integration of the spatial and frequency integrals and let η tend to the constant function one. In the term corresponding to (8.4), Lemma 8.3 allows us to pass to the limit. In (8.7, 8.8) the situation is a bit more involved due to the factors of Γ. However, since multiplication by $\Gamma(\omega-\omega')$ corresponds to convolution with its Fourier transform $\check{\Gamma}(t)$, we can again apply Lemma 8.3 and pass to the limit using Lebesgue's dominated convergence theorem. To make this method more precise, let us show in detail that the expression

$$\int_{r_1}^\infty dr \int_{-1}^1 d\cos\vartheta \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty d\omega'\; e^{-i(\omega-\omega')t} \Big(1-\eta(u)\Big)\, \Gamma(\omega-\omega')\, g(\omega')\, \hat{\Phi}(\omega)\, \overline{\hat{\phi}(\omega')} \tag{8.9}$$

tends to zero as η converges to the constant function one. Rewriting the factor Γ with a time convolution, we obtain the expression

$$\int_{-\infty}^\infty d\tau\; \check{\Gamma}(\tau)\, F(\tau)$$

where F and $\check{\Gamma}$ are defined by

$$F(\tau) \;=\; \int_{r_1}^\infty dr \int_{-1}^1 d\cos\vartheta\; (1-\eta(u))\, \Phi(t-\tau, r, \vartheta)\, (\check{g} * \overline{\Phi})(t-\tau, r, \vartheta)$$

$$\check{\Gamma}(\tau) \;=\; \frac{1}{2\pi} \int_{-\infty}^\infty \Gamma(b)\, e^{-ib\tau}\, db\,.$$

Writing g as a polynomial in ω,

$$g(\omega) \;=\; g_0 + g_1\,\omega + g_2\,\omega^2\,, \tag{8.10}$$

where

$$g_0 \;=\; \frac{a^2k^2}{\Delta} - \frac{k^2}{\sin^2\vartheta}\,, \quad g_1 \;=\; 2ak\left(\frac{r^2+a^2}{\Delta}-1\right), \quad g_2 \;=\; \frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\vartheta\,, \tag{8.11}$$

the function $\check{g} * \overline{\Phi}$ can be expressed explicitly in terms of $\overline{\Phi}$ and its time derivatives of order at most two. In order to compute $\check{\Gamma}$, we first note that the Fourier transform of the Heaviside function Θ is

$$\check{\Theta}(\tau) \;=\; \frac{1}{2\pi}\left(-i\,\frac{\mathrm{PP}}{\tau} + \pi\delta(\tau)\right), \tag{8.12}$$

where "PP" denotes the principal part. Using (8.3) together with the fact that convolution in momentum space corresponds to multiplication in position space, we find that

$$\check{\Gamma}(\tau) \;=\; \left(-i\,\frac{\mathrm{PP}}{\tau} + \pi\delta(\tau)\right)\check{\alpha}(\tau)\,, \tag{8.13}$$

where $\check{\alpha}$ is a Schwartz function with $\check{\alpha}(0) = (2\pi)^{-1}$. According to Lemma 8.3, the function F is uniformly bounded,

$$|F(\tau)| \;\leq\; \sup|\eta| \quad \text{for all } \tau \in \mathbb{R}.$$

Using the rapid decay of $\check{\Gamma}$, we can for any given ε choose a parameter $L > 0$ such that

$$\left|\int_{\mathbb{R}\setminus[-L,L]} \check{\Gamma}(\tau)\,F(\tau)\right| \;\leq\; \varepsilon\,\sup|\eta|\,.$$

On the interval $[-L, L]$, on the other hand, the singularity of $\hat{\Gamma}$ at $\tau = 0$ can be controlled by at most first derivatives of F, and thus for a suitable constant $C = C(L)$,

$$\left|\int_{-L}^{L} \check{\Gamma}(\tau)\,F(\tau)\right| \;\leq\; 2C \sup_{[-L,L]}\left(|F| + |F'|\right).$$

Using the rapid decay of Φ and its time derivatives in u, locally uniformly in τ, we can make $\sup_{[-L,L]}(|F| + |F'|)$ as small as we like. This shows that (8.9) really tends to zero as η goes to the constant function one.

We conclude that

$$E(\Phi^{n_0}_{\mathrm{H}+}) \;\geq\; \int_{r_1}^{\infty} dr \int_{-1}^{1} d\cos\vartheta \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\; e^{-i(\omega-\omega')t}$$

$$\times\left\{\left(\min(g(\omega), g(\omega')) + \frac{k^2}{\sin^2\vartheta} - \frac{a^2k^2}{\Delta}\right)\hat{\Phi}(\omega)\,\overline{\hat{\Phi}(\omega')}\right. \tag{8.14}$$

$$\left. + \Big(\Gamma\,g(\omega') + (1-\Gamma)\,g(\omega) - \min(g(\omega), g(\omega'))\Big)\hat{\Phi}(\omega)\,\overline{\hat{\Phi}(\omega')}\right\}. \tag{8.15}$$

We apply Lemma 8.2 to obtain

$$(8.14) \;\geq\; \int_{r_1}^{\infty} dr \int_{-1}^{1} d\cos\vartheta\, \left(\inf_{\omega\geq J} g(\omega) + \frac{k^2}{\sin^2\vartheta} - \frac{a^2k^2}{\Delta}\right)\, |\Phi(t,r,\vartheta)|^2\,.$$

Using the explicit form of g, (8.10), we find that for sufficiently large J,

$$(8.14) \;\geq\; \frac{J^2}{2} \int_{r_1}^\infty \frac{(r^2+a^2)^2}{\Delta}\, dr \int_{-1}^1 d\cos\vartheta\, |\Phi(t,r,\vartheta)|^2\,. \tag{8.16}$$

It remains to control the term (8.15). We write the ω, ω'-integral of (8.15) in the form

$$B \;:=\; \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty d\omega'\; e^{-i(\omega-\omega')t} h(\omega,\omega')\, \hat{\Phi}(\omega)\, \overline{\Phi(\omega')}\,,$$

where

$$h(\omega,\omega') \;=\; \Gamma\, g(\omega') + (1-\Gamma)\, g(\omega) - \min(g(\omega), g(\omega'))\,.$$

Introducing the variables $a = \frac{1}{2}(\omega+\omega')$ and $b = \frac{1}{2}(\omega-\omega')$, and using that $g(\omega)$ is a polynomial in ω, a short calculation yields

$$h(a+b, a-b) \;=\; (g_1 + 2g_2\, a)\;\; S(2b) \qquad \text{where} \qquad S(b) \;:=\; b\,\Big(\Theta(b) - \Gamma(b)\Big)\,.$$

Using (8.12, 8.13) together with the fact that the factor b corresponds to a derivative in position space, we obtain

$$\check{S}(\tau) \;=\; \frac{1}{2\pi} \frac{d}{d\tau} \left(\frac{1 - 2\pi \check{\alpha}(\tau)}{\tau} \right) \;=\; -\frac{d}{d\tau} \int_0^1 \alpha'(s\tau)\, ds\,.$$

This is a smooth function which decays quadratically at infinity; in particular, it is integrable.

We thus obtain for the Fourier transform of h the explicit formula

$$\begin{aligned} \check{h}(\tau,\tau') \;&=\; \frac{1}{(4\pi)^2} \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty d\omega'\; h(\omega,\omega')\, e^{-i(\omega t - \omega' t')} \\ &=\; \Big(g_1 \delta(\tau-\tau') + 2ig_2\, \delta'(\tau-\tau')\Big)\, \check{S}\Big(\frac{\tau+\tau'}{2}\Big)\,. \end{aligned}$$

Using Plancherel for distributions, we obtain

$$\begin{aligned} B \;&=\; \int_{-\infty}^\infty d\tau \int_{-\infty}^\infty d\tau'\; \check{h}(\tau,\tau')\, \Phi(t-\tau)\, \overline{\Phi(t-\tau')} \\ &=\; \int_{-\infty}^\infty d\tau\, \check{S}(\tau) \Big[g_1\, \Phi(t-\tau)\, \overline{\Phi(t-\tau)} \\ &\qquad\qquad + ig_2 \Big(\partial_t \Phi(t-\tau)\, \overline{\Phi(t+\tau)} - \Phi(t-\tau)\, \overline{\partial_t \Phi(t+\tau)} \Big) \Big]\,. \end{aligned}$$

Integrating over space, we can use the explicit formulas for g_1 and g_2 and apply Lemma 8.3 to obtain

$$(8.15) \;=\; \int_{r_1}^\infty dr \int_{-1}^1 d\cos\vartheta\; B \;\geq\; -C(n_0, \Psi_0) \int_{-\infty}^\infty |\check{S}(\tau)|\, d\tau\,.$$

We now let α tend to the Dirac delta; then $\check{\alpha}$ tends to the constant function $(2\pi)^{-1}$. As a consequence, the L^1-norm of $\check{S}$ tends to zero, and thus (8.15) becomes positive in this limit. Hence the energy is bounded from below by (8.16). This concludes the proof of Theorem 8.1.

8.2 Pointwise Estimates for the Low-Energy Contribution

The low-energy contribution can be written as

$$\Phi_{\rm L}^{n_0}(t,r,\vartheta) \;=\; \frac{1}{2\pi} \sum_{n=1}^{n_0} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t}\, \chi_{\rm L}(\omega) \sum_{a,b=1}^{2} t_{ab}^{\omega n}\, \Phi_{\omega n}^a(r,\vartheta)\, <\Psi_{\omega n}^b, \Psi_0> .$$

We now derive pointwise estimates for the large angular momentum modes.

Theorem 8.4 *For any $u_0 < u_1$ there is a constant $C > 0$ such that for all $\omega \in (-2J, 2J) \setminus \{\omega_0, 0\}$ and for all $u, u' \in (u_0, u_1)$,*

$$\sum_{n=1}^{\infty} \left| \frac{1}{\Omega} \sum_{a,b=1}^{2} t_{ab}\, \phi^a(u)\, \phi^b(u') \right| \;<\; C\, . \tag{8.17}$$

Proof. From (5.5) the coefficients t_{ab} have the explicit form

$$T \;:=\; (t_{ab}) \;=\; \begin{pmatrix} 1+{\rm Re}\,\dfrac{\alpha}{\beta} & -{\rm Im}\,\dfrac{\alpha}{\beta} \\ -{\rm Im}\,\dfrac{\alpha}{\beta} & 1-{\rm Re}\,\dfrac{\alpha}{\beta} \end{pmatrix},$$

where the transmission coefficients α and β are defined by

$$\grave{\phi} \;=\; \alpha\, \acute{\phi} + \beta\, \overline{\acute{\phi}}\,,$$

and $\phi_1 = {\rm Re}\,\acute{\phi}$, $\phi_2 = {\rm Im}\,\acute{\phi}$. The estimates of Section 4.3 are obviously valid for ω in any bounded set; in particular for $\omega \in (-2J, 2J) \setminus \{\omega_0, 0\}$. We use these estimates in what follows, also using the same notation. We choose n_1 so large that $u_+ < u_0$, and thus on the whole interval $(u_0, 2u_1)$ the invariant disk estimates of Lemmas 4.2 and 4.8 hold.

Rewriting the expression $t_{ab}\phi^a\phi^b$ with the Green's function (see the proof of Lemma 5.1), this expression is clearly invariant under the phase transformation $\acute{\phi} \to e^{i\vartheta}\acute{\phi}$. Thus we can arrange that $\acute{\phi}(2u_1)$ is real. Then the transmission coefficients are computed at $u = 2u_1$ by

$$\begin{pmatrix} \grave{\phi} \\ \grave{\phi}' \end{pmatrix} \;=\; \begin{pmatrix} \acute{\phi} & \overline{\acute{\phi}} \\ \acute{\phi}' & \overline{\acute{\phi}'} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \;=\; \acute{\phi} \begin{pmatrix} 1 & 1 \\ \acute{y} & \overline{\acute{y}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

We thus obtain

$$\alpha \;=\; -\frac{\grave{\phi}}{2\,\acute{\phi}\,{\rm Im}\,\acute{y}}\, (\overline{\acute{y}} - \grave{y})\Big|_{u=2u_1} \qquad \beta \;=\; \frac{\grave{\phi}}{2\,\acute{\phi}\,{\rm Im}\,\acute{y}}\, (\acute{y} - \grave{y})\Big|_{u=2u_1}.$$

Hence

$$\left| 1 + \frac{\alpha}{\beta} \right| \;=\; 2\, \frac{|{\rm Im}\,\acute{y}|}{|\acute{y} - \grave{y}|} \;\leq\; \frac{4\,|\Omega|}{\acute{\rho}^2\, {\rm Re}\,(\acute{y} - \grave{y})}\,,$$

where all functions are evaluated at $u = 2u_1$, and where we used the relation

$$\acute{\rho}^2 \;=\; \frac{|\Omega|}{{\rm Im}\,\acute{y}}\,, \tag{8.18}$$

which is an immediate consequence of (4.9) and $w(\acute{\phi}, \overline{\acute{\phi}}) = 2i\Omega$. From Lemma 4.7 and (4.42) we know that for λ sufficiently large,

$$\mathrm{Re}\,(\acute{y} - \grave{y}) \;\geq\; \frac{1}{C}\,.$$

Using the above formulas for t_{ab}, we conclude that

$$\left| \sum_{(a,b) \neq (2,2)} \frac{1}{\Omega}\, t_{ab}\, \phi^a(u)\, \phi^b(u') \right| \;\leq\; 12\, C\, \frac{\acute{\rho}(u)}{\acute{\rho}(2u_1)}\, \frac{\acute{\rho}(u')}{\acute{\rho}(2u_1)}.$$

The argument after (4.43) shows that the two factors on the right decay like $\exp(-\sqrt{\lambda}/c)$.

It remains to consider the case $a = b = 2$. Taking the imaginary part of the identity

$$\acute{\phi}(u) \;=\; \acute{\phi}(2u_1)\, \exp\left(\int_{2u_1}^u \acute{y} \right)$$

and using that $\acute{\phi}(2u_1)$ is real, we find that

$$\acute{\rho}(u) \;=\; \acute{\rho}(2u_1)\, \exp\left(\int_{2u_1}^u \mathrm{Re}\, \acute{y} \right)$$

and thus

$$|\phi_2(u)| \;=\; \acute{\rho}(u) \left| \sin\left(\int_{2u_1}^u \mathrm{Im}\, \acute{y} \right) \right| \;\leq\; \acute{\rho}(u) \left| \int_u^{2u_1} \mathrm{Im}\, \acute{y} \right|\,.$$

From (4.40) we see that $\mathrm{Re}\, y$ is positive on $[u, 2u_1]$. Using the relation $\rho'/\rho = \mathrm{Re}\, y$, we conclude that $\acute{\rho}$ is monotone increasing, and (8.18) yields that $\mathrm{Im}\, \acute{y}$ is decreasing. Hence, again using (8.18),

$$|\phi_2(u)| \;\leq\; \acute{\rho}(u)\, \mathrm{Im}\, \acute{y}(u)\, (2u_1 - u_0) \;\leq\; \sqrt{|\Omega|\, \mathrm{Im}\, \acute{y}(u_0)}\, (2u_1 - u_0)\,.$$

Using the above estimates for α/β, we conclude that

$$\left| \frac{1}{\Omega}\, t_{22}\, \phi^2(u)\, \phi^2(u') \right| \;\leq\; 3\, \mathrm{Im}\, \acute{y}(u_0)\, (2u_1 - u_0)^2\,. \tag{8.19}$$

The invariant region estimate of Lemmas 4.2 and 4.8 yield that

$$\mathrm{Im}\, \acute{y}(u_0) \;\leq\; c|\Omega|\, \exp\left(-\frac{7}{8} \int_{u_+}^{u_0} \sqrt{V} \right)\,,$$

and we conclude that (8.19) again decays like $\exp(-\sqrt{\lambda}/c)$.

Since the eigenvalues λ_n scale quadratically in n (2.11), the summands in (8.17) decay exponentially in n. ∎

This theorem gives us pointwise control of the low-energy contribution, uniformly in time and in n_0, locally uniformly in space. To see this, we estimate the integral representation (8.1) by

$$|\Phi^{n_0}_{\mathrm{L}}(t,r,\vartheta)| \;\leq\; \frac{1}{2\pi} \sum_{n=1}^{n_0} \int_{-\infty}^{\infty} d\omega\, |\chi_L(\omega)| \sum_{a,b=1}^2 \left| \frac{1}{\omega\Omega}\, t^{\omega n}_{ab}\, \Phi^a_{\omega n}(r,\vartheta) < \Psi^b_{\omega n}, \Psi_0 > \right|.$$

In order to control the factor ω^{-1}, we write the energy scalar product on the right in the form [5, (2.15)], which involves an overall factor ω. Now we can apply Theorem 8.4.

8.3 Decay in L^∞_{loc}

In this section we complete the proof of Theorem 1.1. Let $K \subset (r_1, \infty) \times S^2$ be a compact set. The L^2-norm of Φ^{n_0} can be estimated by

$$\|\Phi^{n_0}(t)\|_{L^2(K)} \;\leq\; \|\Phi^{n_0}_{\text{H}+}(t)\|_{L^2(K)} + \|\Phi^{n_0}_{\text{H}-}(t)\|_{L^2(K)} + \|\Phi^{n_0}_{\text{L}}(t)\|_{L^2(K)}\,.$$

According to Theorems 8.1 and 8.4, these norms are bounded uniformly in n_0 and t. Furthermore, our estimates imply that the sequence $\Phi^{n_0}(t)$ forms a Cauchy sequence in $L^2(K)$. To see this, we note that for any n_1, n_2,

$$\|\Phi^{n_1} - \Phi^{n_2}\|_{L^2(K)} \;\leq\; E\Big(\Phi^{n_1}_{\text{H}+} - \Phi^{n_2}_{\text{H}+}\Big) + E\Big(\Phi^{n_1}_{\text{H}-} - \Phi^{n_2}_{\text{H}-}\Big) + \|\Phi^{n_1}_{\text{L}} - \Phi^{n_2}_{\text{L}}\|_{L^2(K)}\,,$$

uniformly in t. The energy terms on the right are the sums of the energies of the individual angular momentum modes. All the summands are positive due to Theorem 8.1, and thus the energy terms become small as $n_1, n_2 \to \infty$. The same is true for the last summand due to our OD estimates of Theorem 8.4. We conclude that $\Phi^{n_0}(t)$ converges in L^2_{loc} as $n_0 \to \infty$, and the limit coincides with the weak limit, which from [5] we know to be the solution $\Phi(t)$ of the Cauchy problem.

To prove decay, given any $\varepsilon > 0$ we choose n_0 such that $\|\Phi(t) - \Phi^{n_0}(t)\|_{L^2(K)} < \varepsilon$ for all t. Since $\hat{\Phi}^{n_0}$ is continuous in ω and has rapid decay, uniformly on K, the Riemann-Lebesgue lemma yields that $\Phi^{n_0}(t)$ decays in $L^\infty(K) \subset L^2(K)$. Since ε is arbitrary, we conclude that $\Phi(t)$ decays in $L^2(K)$.

Applying the same argument to the initial data $H^n \Psi_0$, we conclude that the partial derivatives of $\Phi(t)$ also decay in $L^2(K)$. The Sobolev embedding $H^{2,2}(K) \hookrightarrow L^\infty(K)$ completes the proof.

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References

[1] V. Cardoso and S. Yoshida, "Superradiant instabilities of rotating black branes and strings," hep-th/0502206.

[2] S. Chandrasekhar, "The mathematical theory of black holes," *Oxford University Press* (1983)

[3] De Alfaro, Regge, "Potential Scattering," *North-Holland Publishing Company*, Amsterdam (1965)

[4] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, "The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry," *Adv. Theor. Math. Phys.* **7** (2003) 25–52

[5] F. Finster, N. Kamran, J. Smoller, S.-T. Yau, "An integral spectral representation of the propagator for the wave equation in the Kerr geometry," gr-qc/0310024, *Commun. Math. Phys.* **260** (2005) 257-298

[6] F. Finster, H. Schmid, "Spectral estimates and non-selfadjoint perturbations of spheroidal wave operators," math-ph/0405010, *J. Reine Angew. Math.* **601** (2006) 71-107 (2006)

[7] B. Kay, R. Wald, "Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere," *Classical Quantum Gravity* **4** (1987) 893–898.

[8] J. Kronthaler, "The Cauchy problem for the wave equation in the Schwarzschild geometry," gr-qc/0601131, *J. Math. Phys.* **47** (2006) 042501

[9] W.H. Press, S.A. Teukolsky, "Perturbations of a rotating black hole. II. Dynamical stability of the Kerr metric," Astrophys. J. **185** (1973) 649

[10] R.H. Price, "Nonspherical perturbations of relativistic gravitational collapse I, scalar and gravitational perturbations," *Phys. Rev.* D (3) **5** (1972) 2419–2438

[11] B. Whiting, "Mode stability of the Kerr black hole," *J. Math. Phys.* **30** (1989) 1301–1305

[12] T. Regge, J.A. Wheeler, "Stability of the Schwarzschild singularity," Phys. Rev. (2) **108** (1957) 1063–1069

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