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# Mathematik

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**Abstract**

We formulate a variational principle for a collection of projectors in an indefinite inner product space. The existence of minimizers is proved in various situations.

In a recent book it was proposed to formulate physics with a new variational principle in space-time [2]. In the present paper we construct minimizers of this variational principle. In order to make the presentation self-contained and easily accessible, we introduce the mathematical framework from the basics (see Sections 1 and 2). Thus this paper can be used as an introduction to the mathematical setting of the principle of the fermionic projector. However, the reader who wants to get a physical understanding is referred to [2].

Our variational principle is set up in finite dimension, and thus the continuity of the action is not an issue. The difficulties are the lack of compactness and the fact that there is no notion of convexity. Therefore, we need to derive suitable estimates (Sections 4 and 5) before we can use the direct method of the calculus of variations (Sections 7 and 8). Our main results are stated in Section 2, whereas in Section 3 we explain our variational principle and illustrate it with a few simple examples.

## 1 Discrete Space-Time and the Fermionic Projector

Let $H$ be a finite-dimensional complex vector space, endowed with a sesquilinear form $<.|.> \,:\, H \times H \to \mathbb{C}$, i.e. for all $u, v, w \in H$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} <u \,|\, \alpha v + \beta w> &= \alpha <u \,|\, v> + \beta <u \,|\, w> \\ <\alpha u + \beta v \,|\, w> &= \overline{\alpha} <u \,|\, w> + \overline{\beta} <v \,|\, w> . \end{aligned}$$

We assume that $<.|.>$ is symmetric,

$$\overline{<u \,|\, v>} \;=\; <v \,|\, u>\,,$$

and non-degenerate,

$$<u \,|\, v> \;=\; 0 \;\;\; \forall\, v \in H \quad \Longrightarrow \quad u \;=\; 0\,.$$

Note that $<.|.>$ is in general not positive, and it is therefore not a scalar product. We also refer to $(H, <.|.>)$ as an *indefinite inner product space*. To a non-degenerate subspace of $H$ we can associate its *signature* $(p, q)$, where $p$ and $q$ are the maximal dimensions of positive and negative definite subspaces, respectively (for more details see [1, 3] and the examples in Section 3).

Many constructions familiar from scalar product spaces can be carried over to indefinite inner product spaces. In particular, we define the *adjoint* of a linear operator $A : H \to H$ by the relation

$$<\!u \,|\, Av\!> \;=\; <\!A^* u \,|\, v\!> \qquad \forall\, u, v \in H\,.$$

A linear operator $A$ is said to be *unitary* if $A^* = A^{-1}$ and *symmetric* if $A^* = A$. It is called a *projector* if it is symmetric and idempotent,

$$A^* \;=\; A \;=\; A^2\,.$$

Let $M$ be a finite set. To every point $x \in M$ we associate a projector $E_x$. We assume that these projectors are orthogonal and complete in the sense that

$$E_x\, E_y \;=\; \delta_{xy}\, E_x \qquad \text{and} \qquad \sum_{x \in M} E_x \;=\; \mathbb{1}\,. \tag{1}$$

Equivalently, we can say that the images of the projectors $E_x$ give a decomposition of $H$ into orthogonal subspaces,

$$H \;=\; \bigoplus_{x \in M} E_x(H)\,. \tag{2}$$

Furthermore, we assume that the images $E_x(H) \subset H$ of these projectors are non-degenerate and all have the same signature $(n, n)$. We refer to $(n, n)$ as the *spin dimension*. Relation (2) shows that the dimension of $H$ must be equal to $m \cdot 2n$, where $m = \#M$ denotes the number of points of $M$. The points $x \in M$ are called *discrete space-time points*, and the corresponding projectors $E_x$ are the *space-time projectors*. The structure $(H, <\!.|.\!>, (E_x)_{x \in M})$ is called *discrete space-time*.

We now introduce one more projector $P$ on $H$, the so-called *fermionic projector*, which has the additional property that its image $P(H)$ is *negative* definite. In other words, $P(H)$ has signature $(0, f)$ with $f \in \mathbb{N}$. The vectors in the image of $P$ have the interpretation as the quantum mechanical states of the particles of our system, and we call $f = \dim P(H)$ the *number of particles*. We remark that in physical applications [2] these particles are Dirac particles, which are fermions, giving rise to the name "fermionic projector".

A space-time projector $E_x$ can be used to restrict an operator to the subspace $E_x(H) \subset H$. Using a more graphic notion, we also refer to this restriction as the *localization* at the space-time point $x$. For example, using the completeness of the space-time projectors (1), we readily see that

$$f \;=\; \mathrm{Tr}\, P \;=\; \sum_{x \in M} \mathrm{Tr}(E_x P)\,. \tag{3}$$

The expression $\mathrm{Tr}(E_x P)$ can be understood as the localization of the trace at the space-time point $x$, and summing over all space-time points gives the total trace. We call $\mathrm{Tr}(E_x P)$ the *local trace* of $P$. When forming more complicated composite expressions in the projectors $P$ and $(E_x)_{x \in M}$, it is convenient to use the short notations

$$P(x,y) \;=\; E_x\, P\, E_y \qquad \text{and} \qquad u(x) \;=\; E_x\, u\,.$$

Referring to the orthogonal decomposition (2), $P(x,y)$ maps $E_y(H)$ to $E_x(H)$ and vanishes otherwise. It is often useful to regard $P(x,y)$ as a mapping only between these subspaces,

$$P(x,y) \;:\; E_y(H) \;\to\; E_x(H)\,.$$

Using (1), we can write the product $Pu$ as follows,

$$(Pu)(x) \;=\; E_x\, Pu \;=\; \sum_{y \in M} E_x\, P\, E_y\, u \;=\; \sum_{y \in M} (E_x\, P\, E_y)\, (E_y\, u)\,,$$

and thus

$$(Pu)(x) \;=\; \sum_{y \in M} P(x,y)\, u(y)\,.$$

This relation resembles the representation of an operator with an integral kernel. Therefore, we call $P(x, y)$ the *discrete kernel* of the fermionic projector. The discrete kernel can be used for expressing general operator products; for example,

$$P\, E_x\, P\, E_y \;=\; \sum_{z \in M} P(z,x)\, P(x,y)\,.$$

# 2 A Variational Principle, Statement of the Main Results

We want to form a positive quantity which depends on the form of the fermionic projector relative to the space-time projectors. Since scalar invariants (like the trace or the determinant) can be introduced only for operators which map a vector space into itself, we first define the *closed chain* $A_{xy}$ by

$$A_{xy} \;=\; P(x,y)\, P(y,x) \;=\; E_x\, P\, E_y\, P\, E_x \;:\; E_x(H) \;\to\; E_x(H)\,. \tag{4}$$

We shall often omit the subscripts 'xy'. Let $\lambda_1, \ldots, \lambda_{2n}$ be the zeros of the characteristic polynomial of $A$, counted with multiplicities. We define the *spectral weight* $|A|$ by

$$|A| \;=\; \sum_{j=1}^{2n} |\lambda_j|\,.$$

More generally, one can take the spectral weight of powers of $A_{xy}$, and by summing over the space-time points we get positive numbers depending only on the projectors $P$ and $(E_x)_{x \in M}$.

For a given parameter $\kappa > 0$ we consider the family of fermionic projectors $\mathcal{P}(\kappa)$ defined by

$$\mathcal{P}(\kappa) \;=\; \Big\{ P \text{ with } \sum_{x,y \in M} |A_{xy}|^2 = \kappa \Big\}\,. \tag{5}$$

Our variational principle as introduced in [2, §3.5] is to

$$\text{minimize} \quad \sum_{x,y \in M} |A_{xy}^2| \quad \text{by varying } P \text{ in } \mathcal{P}(\kappa)\,, \tag{6}$$

keeping the number of particles $f$ as well as discrete space-time $(H, <.|.>, (E_x)_{x \in M})$ fixed. The next theorem completely settles the existence problem.

**Theorem 2.1** *For every $\kappa$ for which the family $\mathcal{P}(\kappa)$ is not empty, the variational principle (6) attains its minimum.*

This theorem makes no statement on uniqueness, and indeed we do not see a reason why the minimizers should be unique. Using the method of Lagrangian multipliers, for every minimizer $P$ there is a real parameter $\mu$ such that $P$ is a stationary point of the *action*

$$\mathcal{S}_\mu[P] \;=\; \sum_{x,y \in M} \mathcal{L}_\mu[A_{xy}] \tag{7}$$

with the *Lagrangian*

$$\mathcal{L}_\mu[A] \;=\; |A^2| - \mu\, |A|^2 \,. \tag{8}$$

Unfortunately, the above theorem does not give information on the value of the Lagrangian multiplier $\mu$. Knowing $\mu$ is important because in physical applications the Lagrangian multiplier is determined by the model (more precisely, for the fermionic projector of the standard model [2, Chapter 5] one takes $n = 16$ and $\mu = 1/28$, whereas for modeling the simpler system of one sector one takes $n = 2$ and $\mu = 1/4$). Thus we would like to construct stationary points of the action (7) for a given value of $\mu$. The simplest method to achieve this is to minimize directly the action (7) in the class of all fermionic projectors. This is our motivation for considering also the variational principle

$$\text{minimize } \mathcal{S}_\mu[P] \text{ by varying } P \,, \tag{9}$$

again keeping the number of particles as well as discrete space-time fixed. We point out that every minimizer $P$ of the variational principle (9) is also a minimizer of the variational principle with constraint (6, 5) for the corresponding value of $\kappa = \sum_{x,y} |P(x,y)P(y,x)|^2$. Since we regard the variational principle (9) merely as a technical tool for constructing minimizers of (6), we refer to (9) as the *auxiliary variational principle*.

Let us discuss the behavior of the auxiliary variational principle for different values of $\mu$. Clearly, a necessary condition for the existence of minimizers is that the action is bounded from below. According to the Schwarz inequality,

$$|A| \;=\; \sum_{j=1}^{2n} |\lambda_j| \;\leq\; \left( \sum_{j=1}^{2n} 1 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{2n} |\lambda_j|^2 \right)^{\frac{1}{2}} \;=\; \sqrt{2n}\, |A^2|^{\frac{1}{2}} \,,$$

and squaring both sides we find that

$$\mathcal{L}_\mu \;\geq\; 0 \qquad \text{if} \quad \mu \leq \frac{1}{2n} \,.$$

If the last inequality is strict, we get existence:

**Theorem 2.2** *If $\mu < \frac{1}{2n}$, the auxiliary variational principle (9) attains its minimum.*

If conversely $\mu > \frac{1}{2n}$, one sees in the example of the matrices $A_k = k\mathbb{1}$ that $|A_k^2| = 2nk^2$, $|A_k|^2 = 4n^2k^2$ and thus $\mathcal{L}_\mu[A_k] \to -\infty$ as $k \to \infty$. Hence the action is not bounded below, and we cannot expect the existence of minimizers.

The remaining *critical case* $\mu = \frac{1}{2n}$ is the most interesting but also the most difficult case. For notational convenience, we set $\mathcal{L} \equiv \mathcal{L}_{1/2n}$ and $\mathcal{S} \equiv \mathcal{S}_{1/2n}$. Then our Lagrangian can also be written in the form

$$\mathcal{L}[A] \;=\; |A^2| - \frac{1}{2n}\, |A|^2 \;=\; \frac{1}{4n} \sum_{i,j=1}^{2n} \left( |\lambda_i| - |\lambda_j| \right)^2 \,, \tag{10}$$

as is easily verified by multiplying out the last square in (10). This shows that the Lagrangian vanishes only if the $|\lambda_j|$ are all equal. Thus one can say qualitatively that the critical variational principle tries to achieve that the zeros of the characteristic polynomial of $A$ all have the same absolute value.

In the critical case we prove the following existence theorem.

**Theorem 2.3** *Suppose that $(P_k)_{k \in \mathrm{I\!N}}$ is a minimal sequence of the auxiliary variational principle (9) in the critical case $\mu = \frac{1}{2n}$. Assume that the local trace is bounded away from zero in the sense that for suitable $\delta > 0$,*

$$|\mathrm{Tr}(E_x P_k)| \;\geq\; \delta \qquad \forall\, k \in \mathbb{N},\; x \in M\,.$$

*Then there exists a minimizer $P$.*

Here we need the additional condition that in a minimal sequence the local trace must not go to zero at any space-time point. It is an open problem whether this condition is only a technicality needed in our proof, or whether it is really necessary for the theorem to hold.

We will prove a general existence theorem (see Theorem 6.1 below), which is useful for constructing minimizers under various constraints. As an example, we here consider homogeneous operators.

**Def. 2.4** *A fermionic projector $P$ is called* **homogeneous** *if for any $x_0, x_1 \in M$ there is a permutation $\sigma : M \to M$ with $\sigma(x_0) = x_1$ and a gauge transformation $U \in \mathcal{G}$ such that*

$$P(\sigma(x), \sigma(y)) \;=\; U\, P(x,y)\, U^{-1} \qquad \forall\, x,y \in M\,.$$

We remark that this definition generalizes the usual notion of "homogeneity" as defined via a symmetry group $K$ acting transitively on space-time. Namely, in this case we take for any $x_0, x_1 \in M$ a group element $g \in K$ with $g(x_0) = x_1$ and set $\sigma(x) = g(x)$, together with unitary maps $U_x : E_x(H) \to E_{g(x)}(H)$ which identify the corresponding spinor spaces. Homogeneous operators seem of physical interest because the vacuum should be described by a homogeneous fermionic projector.

**Theorem 2.5** *Consider the auxiliary variational principle (9) in the critical case $\mu = \frac{1}{2n}$. Varying $P$ in the class of homogeneous fermionic projectors, the action (7) attains its minimum.*

In the course of our analysis, it will be convenient to consider our variational principles more generally on operators $P$ which are not necessarily projectors. These generalizations are of interest if one considers the above variational principles on a subspace of $H$, disregarding the overall normalization of the fermionic states (for example, one may consider a system corresponding to a subset of space-time points or modeling only one sector). In this situation, the restriction of $P$ to the subspace is no longer a projector. In order to specify which class of operators $P$ we want to consider, we need the following notion.

**Def. 2.6** *A symmetric operator $A$ on an indefinite inner product space of signature $(p,q)$ is said to be* **positive** *if*

$$<\!u \,|\, A\, u\!> \;\geq\; 0 \qquad \forall\, u \in H\,.$$

If $P$ is a projector on a negative definite subspace, then the operator $(-P)$ is positive because

$$<\!u \,|\, (-P)\, u\!> \;=\; -<\!u \,|\, P^2\, u\!> \;=\; -<\!Pu \,|\, Pu\!> \;\geq\; 0\,. \tag{11}$$

Therefore, the next definition really extends the class of fermionic projectors of rank $f$.

**Def. 2.7** *An operator $P$ on an inner product space $(H, <.|.>)$ is said to be* **of class** $\mathcal{P}^f$ *if*

**(i)** *The operator $(-P)$ is positive.*

**(ii)** *The operator $P$ has trace $f$ and rank at most $f$.*

**Theorem 2.8** *The variational principle (6, 5) considered for $P \in \mathcal{P}^f$ attains its minimum.*

**Theorem 2.9** *For every $\mu \leq \frac{1}{2n}$, the auxiliary variational principle (7) attains its minimum in $\mathcal{P}^f$.*

We point out that the last theorem also applies in the critical case $\mu = \frac{1}{2n}$.

In Def. 2.7 **(ii)**, the operator $P$ was normalized by prescribing its trace. Such a normalization is essential for the auxiliary variational principle in order to rule out the trivial minimizer $P = 0$. However, for the variational principle (6), the constraint (5) prevents trivial solutions, and thus it makes sense to drop the normalization.

**Theorem 2.10** *Consider for any parameters $\kappa > 0$ and $f \in \mathbb{N}$ the variational principle (6, 5), where $P$ now is a general operator such that $(-P)$ is positive and has rank at most $f$. Then there exists a minimizer $P$. It is a stationary point of the action (7, 8) with the Lagrangian multiplier chosen such that*

$$\mathcal{S}_\mu[P] \;=\; 0\,. \tag{12}$$

This theorem is interesting because of the following argument, which explains why the condition for $P$ being a projector is fundamental: One may wonder why the physical fermionic projector was introduced in [2] as a projector. The only reason was to ensure the correct normalization of the fermionic states, in accordance with physical observations. But since physical observations are limited to the low-energy region, it is a-priori not clear if the physical $P$ really is a projector, or whether only its low-energy states give the impression that $P$ is a projector. Theorem 2.10 gives us information on what would happen if the normalization condition for $P$ were dropped. Then the action of the minimizer would vanish (12), in contradiction to the fact that for a Dirac sea configuration, the Lagrangian $\mathcal{L}_\mu[A_{xy}]$ is strictly positive if the vector $y - x$ is timelike (see [2, §5.6]).

For the proof of the above theorems we will use the direct method of the calculus of variations. By starting with different minimal sequences, our method allows to construct all minimizers.

# 3 Discussion and Simple Examples

We begin with a few general remarks on the mathematical structure of the variational principle introduced in the previous section. First, we point out that the Lagrangian $\mathcal{L}_\mu[A_{xy}]$ is symmetric in its two arguments $x$ and $y$, as the following consideration shows. For any two quadratic matrices $B$ and $C$, we choose $\varepsilon$ not in the spectrum of $C$ and set $C^\varepsilon = C - \varepsilon \mathbb{1}$. Taking the determinant of the relation $C^\varepsilon(BC^\varepsilon - \lambda) = (C^\varepsilon B - \lambda)C^\varepsilon$, we can use that the determinant is multiplicative and that $\det C^\varepsilon \neq 0$ to obtain the equation $\det(BC^\varepsilon - \lambda) = \det(C^\varepsilon B - \lambda)$. Since both determinants are continuous in $\varepsilon$, this equation holds even for all $\varepsilon \in \mathbb{R}$, proving the elementary identity

$$\det(BC - \lambda \mathbb{1}) \;=\; \det(CB - \lambda \mathbb{1})\,.$$

Applying this identity to the closed chain,

$$\begin{aligned}\det(A_{xy} - \lambda \mathbb{1}) &= \det(P(x,y)\,P(y,x) - \lambda \mathbb{1}) \\ &= \det(P(y,x)\,P(x,y) - \lambda \mathbb{1}) \;=\; \det(A_{yx} - \lambda \mathbb{1})\,,\end{aligned}$$

we conclude that the operators $A_{xy}$ and $A_{yx}$ have the same characteristic polynomial, and thus

$$\mathcal{L}_\mu[A_{xy}] \;=\; \mathcal{L}_\mu[A_{yx}] \qquad \forall\, x, y \in M\,. \tag{13}$$

It is a simple but important observation that a joint unitary transformation of all projectors,

$$E_x \;\to\; U E_x U^{-1}\,, \qquad P \;\to\; U P U^{-1} \qquad \text{with } U \text{ unitary} \tag{14}$$

keeps the action unchanged, because

$$\begin{aligned} P(x,y) &\;\to\; U\,P(x,y)\,U^{-1}\,, \qquad A_{xy} \;\to\; U A_{xy} U^{-1} \\ \det(A_{xy} - \lambda \mathbb{1}) &\;\to\; \det\big(U (A_{xy} - \lambda \mathbb{1})\,U^{-1}\big) \;=\; \det(A_{xy} - \lambda \mathbb{1})\,,\end{aligned}$$

and so the $\lambda_j$ stay the same. Such unitary transformations can also be used to vary the fermionic projector. However, since we want to keep discrete space-time fixed, we are only allowed to consider unitary transformations which do not change the space-time projectors,

$$E_x \;=\; U E_x U^{-1} \qquad \forall\, x \in M\,. \tag{15}$$

Then (14) reduces to the transformation of the fermionic projector

$$P \;\to\; U P U^{-1}\,. \tag{16}$$

Unitary transformations of the form (15, 16) are called *gauge transformations*. The conditions (15) mean that $U$ maps every subspace $E_x(H)$ into itself. Hence $U$ splits into a direct sum of unitary transformations

$$U(x) \;:=\; U E_x \;:\; E_x(H) \;\to\; E_x(H)\,, \tag{17}$$

which act "locally" on the subspaces associated to the individual space-time points. Obviously, the gauge transformations form a group, referred to as the *gauge group* $\mathcal{G}$. Localizing the gauge transformations according to (17), we obtain at any space-time point $x$ the so-called *local gauge group*. The local gauge group is the group of isometries of $E_x(H)$ and can thus be identified with the group $U(n,n)$.

One may ask why the space-time projectors are to be kept fixed in our variational principles. More generally, one could vary both $P$ and the $(E_x)_{x\in M}$, fixing only the integer parameters $f$ and $n$. Recall that the space-time projectors are equivalently described by the orthogonal decomposition (2) together with the condition that the subspaces $E_x(H)$ should all have signature $(n,n)$. For two different sets of space-time projectors, we can find a unitary transformation which maps the corresponding subspaces $E_x(H)$ onto each other. Then the transition from one set of space-time projectors to the other is described by the unitary transformation $E_x \to U E_x U^{-1}$. Since such unitary transformations leave the action unchanged if also the fermionic projector is transformed according to (14), it is no loss in generality to fix the space-time projectors throughout.

It is instructive to consider our framework in a concrete basis of $H$. Then our inner product can be represented in the form

$$<u \,|\, v> \;=\; (u \,|\, S v)\,,$$

where $(.|.)$ is the canonical scalar product on $\mathbb{C}^{2mn}$. Here $S$ is a Hermitian matrix (meaning that $(u \mid Sv) = (Su \mid v) \;\forall\, u, v \in H$), referred to as the *signature matrix*. By choosing the basis of $H$ appropriately, we can arrange that $S$ is diagonal with eigenvalues equal to $\pm 1$. In particular, $S$ is unitary and $S^2 = \mathbb{1}$. The signature matrix is useful for calculations. For example,

$$<u \mid Av> \;=\; (u \mid SAv) \;=\; (A^\dagger S u \mid v) \;=\; (SA^\dagger S u \mid Sv) \;=\; <SA^\dagger S u \mid v>\,,$$

where the dagger denotes transposition and complex conjugation. Thus the adjoint can be expressed by

$$A^* \;=\; SA^\dagger S\,.$$

In particular, a matrix is symmetric if and only if the matrix $SA$ is Hermitian. As one already sees in the two-dimensional example

$$S \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \tag{18}$$

a symmetric matrix in an indefinite inner product space need not be diagonalizable. This explains why after (4) we had to speak of "zeros of the characteristic polynomial" and not of "eigenvalues." Note that the matrix $A$ in (18) is nilpotent and thus $|A| = 0$. This shows that the spectral weight is not a matrix norm, not even on symmetric operators. We remark that it seems impossible to introduce any other basis independent matrix norm; in particular, the analogue of the Hilbert-Schmidt norm $(\mathrm{Tr}(A^* A))^{\frac{1}{2}}$ vanishes in the example (18). Even if a symmetric matrix *is* diagonalizable, its eigenvalues are in general not real, as can be seen in the example

$$S \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

At least, the calculation

$$\overline{\det(A - \lambda \mathbb{1})} \;=\; \det\Big(A^\dagger - \overline{\lambda}\Big) \;=\; \det\Big(S(A^\dagger - \overline{\lambda})S\Big) \;=\; \det\big(A^* - \overline{\lambda}\big) \;=\; \det\big(A - \overline{\lambda}\mathbb{1}\big)$$

shows that the characteristic polynomial of a symmetric matrix $A$ has real coefficients. In other words, the non-real $\lambda_j$ always appear in complex conjugate pairs.

Using the above matrix representations, we can now consider a few simple examples. We restrict attention to the auxiliary variational principle (9) in the critical case $\mu = \frac{1}{2n}$ because this case seems most interesting. The examples are generalized in a straightforward way to the other cases and to the variational principle (6). We begin with the case $m = 1$ of one space-time point. In this case, the only space-time projector $E$ is the identity, and the sum over the space-time points in (7) drops out. Thus

$$\mathcal{S} \;=\; |A^2| - \frac{1}{2n}\,|A|^2$$

with $A = P^2$. Using that $P$ is idempotent and that its only non-vanishing eigenvalue is one with multiplicity $f$, we find that

$$\mathcal{S} \;=\; |P| - \frac{1}{2n}\,|P|^2 \;=\; f - \frac{f^2}{2n}\,.$$

Hence the action is unchanged if the fermionic projector is varied. This can also be understood from the fact that with only one space-time point, the condition (15) is trivial, and therefore any variation of $P$ can be realized as a gauge transformation (16). The situation becomes more interesting with two space-time points, as the next example shows.

**Example 3.1** Choose $M = \{1, 2\}$ with spin dimension $(1, 1)$ and $f = 1$. Then $H$ is 4-dimensional, and by choosing a suitable basis we can arrange that

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{20}$$

where for $E_{1/2}$ we used a block matrix notation (thus every matrix entry stands for a $2 \times 2$-matrix). Again in this block matrix notation, the gauge transformations (15) are of the form

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{21}$$

where $(U_x)_{x \in M}$ are two independent "local" unitary transformations on $E_x(H)$ of the form

$$U_x = e^{i\alpha} \begin{pmatrix} e^{i\beta} \cosh \vartheta & e^{i\gamma} \sinh \vartheta \\ e^{-i\gamma} \sinh \vartheta & e^{-i\beta} \cosh \vartheta \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma, \vartheta \in \mathbb{R}.$$

Thus the local gauge group is $U(1, 1)$, and the gauge transformations (21) are elements of the gauge group $\mathcal{G} = U(1, 1) \otimes U(1, 1)$.

Since we consider a system of one particle ($f = 1$), the fermionic projector $P$ must be a projector on a one-dimensional, negative definite subspace. It is convenient to write $P$ using bra/ket-notation as

$$P = -|u\rangle\langle u| \qquad \text{with} \qquad \langle u \,|\, u\rangle = -1\,. \tag{22}$$

A possible choice is

$$u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and thus} \qquad P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

A short calculation yields that $|A_{11}| = |A_{11}^2| = 1$, and all other $A_{ij}$ vanish. Thus

$$\mathcal{S} = \mathcal{L}(A_{11}) = |A_{11}^2| - \frac{1}{2}\, |A_{11}|^2 = \frac{1}{2}\,.$$

It turns out that the above $P$ is not a minimizer. Namely, choosing

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad P = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \tag{24}$$

we get a smaller value for the action,

$$\begin{aligned} |A_{ij}|^2 &= |A_{ij}^2| = \frac{1}{16} \quad \text{for all } i, j \in M \\ \mathcal{S} &= 4\,\mathcal{L}(A_{11}) = 4\left(\frac{1}{16} - \frac{1}{2 \cdot 16}\right) = \frac{1}{8}\,. \end{aligned}$$

Let us verify that this is indeed the minimum. We represent a general $P$ in the form (22). Since at least one of the inner products $<u \,|\, E_1\, u>$ or $<u \,|\, E_2\, u>$ must be negative, we must distinguish the two cases where these two inner products either have the opposite sign or are both non-positive. In the first case, we can assume that $<u|E_1\, u> > 0$ and $<u|E_2\, u> < 0$. Using the gauge freedom, we can arrange that $u$ is of the form $u = (\sinh\varphi, 0, 0, \cosh\varphi)$ with $\varphi \in \mathbb{R}$. A short calculation yields that

$$\begin{aligned} |A_{11}|^2 &= |A_{11}^2| \;=\; \sinh^8\varphi\,, \qquad |A_{22}|^2 \;=\; |A_{22}^2| \;=\; \cosh^8\varphi \\ |A_{12}|^2 &= |A_{12}^2| \;=\; |A_{21}|^2 \;=\; |A_{21}^2| \;=\; \sinh^4\varphi\, \cosh^4\varphi \\ \mathcal{S} &= \sum_{i,j\in M} |A_{ij}^2| - \frac{1}{2}\,|A_{ij}|^2 \;=\; \frac{1}{2}\left(\cosh^4\varphi + \sinh^4\varphi\right)^2 \;\geq\; \frac{1}{2} \;>\; \frac{1}{8}\,. \end{aligned}$$

In the remaining case when the inner products $<u \,|\, E_1\, u>$ and $<u \,|\, E_2\, u>$ are both non-positive, we can use the gauge freedom (21) to arrange that $u$ is of the form $u = (0, \cos\varphi, 0, \sin\varphi)$ with $\varphi \in [0, 2\pi)$. It follows that

$$\begin{aligned} |A_{11}|^2 &= |A_{11}^2| \;=\; \cos^8\varphi\,, \qquad |A_{22}|^2 \;=\; |A_{22}^2| \;=\; \sin^8\varphi \\ |A_{12}|^2 &= |A_{12}^2| \;=\; |A_{21}|^2 \;=\; |A_{21}^2| \;=\; \sin^4\varphi\, \cos^4\varphi \\ \mathcal{S} &= \sum_{i,j\in M} |A_{ij}^2| - \frac{1}{2}\,|A_{ij}|^2 \;=\; \frac{1}{2}\left(\cos^4\varphi + \sin^4\varphi\right)^2 \;=\; \frac{1}{2}\left(2\sin^4\varphi - 2\sin^2\varphi + 1\right)^2\,, \end{aligned}$$

and the last function really attains its minimum when $\sin^2\varphi = 1/2$. ♦

In the above example, our variational principle has, up to gauge transformations, a unique minimum. The fact that the configuration (23), where the particle is localized at the first space-time point, is not optimal can be understood qualitatively by saying that our variational principle "tends to spread out particles in space-time." We will quantify this observation later (see Lemma 5.1); it will be important in our analysis. We also point out that the local gauge group is *non-compact*, and that the set of gauge-equivalent minima $UPU^{-1}$ with $P$ and $U$ according to (24, 21) form an *unbounded* family of matrices. This explains why minimizers cannot be constructed with simple compactness arguments.

We next consider a system of two particles.

**Example 3.2** Choose $M = \{1, 2\}$ with spin dimension $(1, 1)$ and $f = 2$. Thus the discrete space-time is the same as in Example 3.1; it is again described by (20). However, $P$ now maps onto a two-dimensional negative subspace of $H$. An example for $P$ is obtained by localizing one particle at each space-time point,

$$P \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

This is indeed the minimizer, as the following consideration shows. Let $P$ be a general femionic projector of rank two. We first consider the case that both operators $E_1 P E_1$ and $E_2 P E_2$ have a non-trivial kernel. We set $\alpha_x = \mathrm{Tr}(E_x P)$. Then $\alpha_1 + \alpha_2 = 2$ and $\mathrm{Tr}((E_x P)^2) = \mathrm{Tr}(E_x P)^2 = \alpha_x^2$. Furthermore,

$$\alpha_1 \;=\; \mathrm{Tr}(E_1 P^2) \;=\; \mathrm{Tr}(E_1\, P\, E_1\, P) + \mathrm{Tr}(E_1\, P\, E_2\, P) \;=\; \alpha_x^2 + \mathrm{Tr}(E_1\, P\, E_2\, P)$$

and similarly for $\alpha_2$. It follows that

$$\alpha_1 - \alpha_2 \;=\; \alpha_1^2 - \alpha_2^2 \;=\; (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \;=\; 2\,(\alpha_1 - \alpha_2)\,.$$

The only solution to the above equations is $\alpha_1 = \alpha_2 = 1$. Using that $P$ is a projector on a two-dimensional negative definite subspace, it is easily verified that $P$ is gauge equivalent to the fermionic projector (25).

It remains to consider the case that for example $E_1 P E_1$ has a trivial kernel. Then its characteristic polynomial has two non-vanishing roots. Anticipating results of Section 4, the operator $(-E_1 P E_1)$ is positive (Lemma 4.1 **(i)**), and the two non-zero roots have opposite signs (Lemma 4.2). Thus $E_1 P E_1$ can be diagonalized. More precisely, we can choose a new pseudo-orthonormal basis in $E_1(H)$ such that the operator $P$ takes the form

$$P \;=\; \begin{pmatrix} a & 0 & * & * \\ 0 & b & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \qquad \text{with } a<0 \text{ and } b>0,$$

where the stars denote arbitrary entries. The first column of this matrix is a vector in the negative definite subspace $P(H)$. Using a gauge transformation in $E_2(H)$, we can arrange that this vector is of the form $u = (a,0,0,c)$ with $|c| > |a|$. Using that $Pu = u$ and $P^* = P$, we can arrange by a suitable gauge transformation in $E_2(H)$ that $P$ is of the form

$$P \;=\; \begin{pmatrix} -\sinh^2\alpha & 0 & 0 & \sinh\alpha\cosh\alpha \\ 0 & b & * & 0 \\ 0 & * & * & 0 \\ -\sinh\alpha\cosh\alpha & 0 & 0 & \cosh^2\alpha \end{pmatrix} \qquad \text{with } \alpha \neq 0.$$

Again using that $P$ is a projector on a negative definite subspace, we can arrange that

$$P \;=\; \begin{pmatrix} -\sinh^2\alpha & 0 & 0 & \sinh\alpha\cosh\alpha \\ 0 & \cosh^2\beta & -\sinh\beta\cosh\beta & 0 \\ 0 & \sinh\beta\cosh\beta & -\sinh^2\beta & 0 \\ -\sinh\alpha\cosh\alpha & 0 & 0 & \cosh^2\alpha \end{pmatrix}.$$

In the limiting case $\alpha = \beta = 0$, this formula also includes (25). The Lagrangian corresponding to this fermionic projector is computed to be

$$\begin{aligned} \mathcal{L}[A_{11}] \;&=\; \frac{1}{2}\left(\sinh^4\alpha - \cosh^4\beta\right)^2 \\ \mathcal{L}[A_{22}] \;&=\; \frac{1}{2}\left(\sinh^4\beta - \cosh^4\alpha\right)^2 \\ \mathcal{L}[A_{12}] \;=\; \mathcal{L}[A_{21}] \;&=\; \frac{1}{2}\left(\cosh^2\alpha\sinh^2\alpha - \cosh^2\beta\sinh^2\beta\right)^2. \end{aligned}$$

Adding these terms and using trigonometric identities for the hyperbolic functions, we obtain for the action the expression

$$\mathcal{S}[P] \;=\; \frac{1}{8}\Big(2 + (\cosh 2\alpha - \cosh 2\beta)^2\Big)\,(\cosh 2\alpha + \cosh 2\beta)^2\;,$$

and this function clearly attains its unique absolute minimum at $\alpha = 0 = \beta$. ♦

The most interesting case is when the number of particles is large, but still much smaller than the number of space-time points.

**Example 3.3** Choose $1 \ll f \ll m$ with spin dimension $(1,1)$. In this case, we can represent discrete space-time by the following matrices,

$$S = \begin{pmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & 0 & \\ & & 0 & -1 & \\ & & & & \ddots \end{pmatrix}$$

and

$$E_1 = \begin{pmatrix} \mathbb{1} & & \\ & 0 & \\ & & \ddots \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & & \\ & \mathbb{1} & \\ & & \ddots \end{pmatrix}, \ldots, E_m .$$

One possibility to choose $P$ is to localize each of the $f$ particles similar to (25) at one of the space-time points. However, the resulting value for the action

$$\mathcal{S} = \frac{f}{2}$$

is certainly not minimal. It is better if, in analogy to (24), each particle is evenly spread over $m/f$ space-time points (we here assume for simplicity that $m/f$ is an integer). A short calculation yields

$$\mathcal{S} = \frac{f}{2} \left( \frac{f}{m} \right)^2 .$$

There is no reason why this configuration should be optimal. It is completely unknown how the minimizer looks like in general. ♦

The case of physical interest is spin dimension $(2,2)$ (or more generally $(2N,2N)$ with $N \geq 1$), because in this case the vectors of $H$ can be identified with the Dirac wave functions of relativistic quantum mechanics. We expect the general structure of the minima to be very complicated. Our qualitative picture is that the minimizers should induce relations between the discrete space-time points which for large $m$ and $f$ should correspond to specific geometric configurations of the space-time points. As explained in [2, §5.6], such relations should, in a suitable limit in which discrete space-time goes over to a continuum space-time, give the causal structure of a Lorentzian manifold.

# 4 Positive Operators, Lower Bounds for the Lagrangian

We return to the concept of a positive operator as introduced in Def. 2.6. Expressed with the signature matrix, we can say that a self-adjoint operator $A$ on an inner product space $H$ of signature $(p,q)$ is positive if and only if the matrix $SA$ is positive semi-definite on $\mathbb{C}^{p+q}$ endowed with the standard Euclidean scalar product. To avoid confusion, we point out that the statements "$A$ is positive" and "the image of $A$ is positive" are completely different. In the two-dimensional examples

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (26)$$

the operator $A_1$ is positive, although its image has signature $(1,1)$. The operator $A_2$ is also positive, but its image is negative. The last example also shows that the trace of a positive operator can be negative. At least, the argument (11) shows that a *projector* on a positive subspace is a positive operator.

We now collect a few elementary but useful properties of positive operators.

**Lemma 4.1** *Suppose that $A$ is a positive operator on $H$. Then*

**(i)** *If $Q$ is a projector in $H$, the operator $QAQ$ is again positive.*

**(ii)** *For all $u, v \in H$,*

$$|{<}u \,|\, A\,v{>}| \;\leq\; \sqrt{{<}u \,|\, A\,u{>}}\;\sqrt{{<}v \,|\, A\,v{>}}\,. \tag{27}$$

*Proof.* Part **(i)** is obvious because ${<}u\,|\,QAQ\,u{>} = {<}Qu\,|\,A\,Qu{>} \geq 0$. Part **(ii)** can be regarded as the Schwarz inequality for the positive semi-definite inner product $(.|.)_A := {<}.|A.{>}$. The proof is almost as simple as in scalar product spaces: First note that for all $a, b \in H$,

$$0 \;\leq\; (a-b \,|\, a-b)_A \;=\; (a \,|\, a)_A \;+\; (b \,|\, b)_A \;-\; 2\,\mathrm{Re}\,(a \,|\, b)_A\,.$$

By changing the phase of the vector $a$, we can arrange that $(a\,|\,b)_A \geq 0$. Thus

$$2\,|(a \,|\, b)_A| \;\leq\; (a \,|\, a)_A \;+\; (b \,|\, b)_A \qquad \forall\, a, b \in H\,. \tag{28}$$

Suppose that $(u\,|\,u)_A = 0$. Then applying (28) for $a = u/\varepsilon$ and $b = \epsilon u$ with a parameter $\epsilon > 0$ gives

$$2\,|(u \,|\, v)_A| \;\leq\; \varepsilon^2\,(v \,|\, v)_A\,,$$

and letting $\varepsilon \to 0$, we see that (27) is trivially satisfied. The same argument applies if $(v\,|\,v)_A = 0$. In the remaining case $(u\,|\,u)_A \neq 0$ and $(v\,|\,v)_A \neq 0$, we apply (28) with

$$a \;=\; \left(\frac{(v \,|\, v)_A}{(u \,|\, u)_A}\right)^{\frac{1}{4}} u\,, \qquad b \;=\; \left(\frac{(u \,|\, u)_A}{(v \,|\, v)_A}\right)^{\frac{1}{4}} v\,. \qquad \blacksquare$$

Compared to the situation for general symmetric operators as explained after (18), positive operators have nice spectral properties, as the following approximation argument shows.

**Lemma 4.2** *A positive operator $A$ on an indefinite inner product space of signature $(p,q)$ has a purely real spectrum. The zeros $(\lambda_j)_{j=1,\ldots,p+q}$ of its characteristic polynomial (again counted with multiplicities) can be ordered as follows,*

$$\lambda_1 \leq \cdots \leq \lambda_q \;\leq\; 0 \;\leq\; \lambda_{q+1} \leq \cdots \leq \lambda_{p+q}\,.$$

*Proof.* We choose a matrix representation with signature matrix $S$ and set $A^\varepsilon = A + \varepsilon S$. Clearly, the matrices $A^\varepsilon$ converge to $A$ as $\varepsilon \to 0$. Since the spectrum is continuous in $\varepsilon$, it suffices to prove the lemma for the matrix $A^\varepsilon$ and any $\varepsilon > 0$.

The matrix $A^\varepsilon$ is symmetric and strictly positive in the sense that for all $u \neq 0$,

$${<}u \,|\, A^\varepsilon u{>} \;=\; {<}u \,|\, A\,u{>} \;+\; \varepsilon\,{<}u \,|\, S\,u{>} \;\geq\; \varepsilon\,{<}u \,|\, S\,u{>} \;=\; \varepsilon\,(u \,|\, u) \;>\; 0\,.$$

Hence we can introduce a scalar product by

$$(u \,|\, v)_{A^\varepsilon} \;:=\; <u \,|\, A^\varepsilon v> .$$

Since the operator $A^\varepsilon$ is symmetric and commutes with itself, it is clearly self-adjoint in the Hilbert space $(H, (.|.)_{A^\varepsilon})$. Thus we can choose an eigenvector basis $(u_j)_{j=1,\ldots,p+q}$. The corresponding eigenvalues $\lambda_j$ satisfy the identity

$$\lambda_j <u_j \,|\, u_j> \;=\; <u_j \,|\, A^\varepsilon u_j> \;=\; (u_j \,|\, u_j)_{A^\varepsilon} \;>\; 0 .$$

Thus $p$ of the eigenvalues are positive, whereas the other $q$ eigenvalues are negative. ■

Note that a positive operator is in general not diagonalizable, as the example (18) shows.

The above lemmas can be used to get lower estimates of our Lagrangian (8) and the corresponding action, which shed some light on the mathematical behavior of our variational principle. We consider the case $\mu \leq \frac{1}{2n}$. Using the elementary inequality

$$\mathcal{L}_\mu[A] \;=\; \mathcal{L}[A] + \left( \frac{1}{2n} - \mu \right) |A|^2 \;\geq\; \mathcal{L}[A] \qquad \text{if } \mu \leq \frac{1}{2n} , \tag{29}$$

we may restrict attention to the critical case $\mu = \frac{1}{2n}$. It is obvious from (3) that the local trace is non-zero at least at some $x \in M$. The next lemma shows that the Lagrangian of $A_{xx}$ can be bounded below by expressions involving the local trace at $x$.

**Proposition 4.3** *Let $P$ be a symmetric operator on $(H, <.|.>)$ such that $(-P)$ is positive. Then, using again the notation (4, 8),*

$$\mathcal{L}[A_{xx}] \;\geq\; \frac{|\mathrm{Tr}(E_x P)|^4}{256\, n^3} \tag{30}$$

$$\mathcal{L}[A_{xx}] \;\geq\; \frac{1}{4n}\, |\mathrm{Tr}(E_x P)|^2 \;\inf \sigma\Big( A_{xx}|_{E_x(H)} \Big) . \tag{31}$$

*Proof.* According to Lemma 4.1 **(i)**, the operator $(-P(x,x)) \;:\; E_x(H) \to E_x(H)$ is positive. Lemma 4.2 tells us that the zeros of the characteristic polynomial of $P(x,x)$, which we denote by $(\nu_j)_{j=1,\ldots,2n}$, are all real and have the ordering

$$\nu_1 \leq \cdots \leq \nu_n \;\leq\; 0 \;\leq\; \nu_{n+1} \leq \cdots \leq \nu_{2n} . \tag{32}$$

This allows us to write the local trace as follows,

$$\mathrm{Tr}(E_x P) \;=\; \sum_{j=1}^{2n} \nu_j \;=\; \sum_{i=n+1}^{2n} |\nu_i| - \sum_{j=1}^{n} |\nu_j| \;=\; \frac{1}{n} \sum_{i=n+1}^{2n} \sum_{j=1}^{n} (|\nu_i| - |\nu_j|) , \tag{33}$$

where the last equality is obvious if one notices that when for example adding up the $|\nu_i|$, the sum over $j$ can be carried out giving a factor $n$. We now take absolute values and increase the right side by taking more summands,

$$|\mathrm{Tr}(E_x P)| \;\leq\; \frac{1}{n} \sum_{i,j=1}^{2n} \Big| |\nu_i| - |\nu_j| \Big| . \tag{34}$$

Now we can proceed with Hölder's inequality to obtain

$$|\mathrm{Tr}(E_x P)| \;\leq\; \frac{1}{n}\left(4n^2\right)^{\frac{1}{2}} \left( \sum_{i,j=1}^{2n} \Big| |\nu_i| - |\nu_j| \Big|^2 \right)^{\frac{1}{2}} \;=\; 2 \left( \sum_{i,j=1}^{2n} \Big| |\nu_i| - |\nu_j| \Big|^2 \right)^{\frac{1}{2}} \tag{35}$$

$$|\mathrm{Tr}(E_x P)| \;\leq\; \frac{1}{n}\left(4n^2\right)^{\frac{3}{4}} \left( \sum_{i,j=1}^{2n} \Big| |\nu_i| - |\nu_j| \Big|^4 \right)^{\frac{1}{4}} \;=\; \sqrt{8n} \left( \sum_{i,j=1}^{2n} \Big| |\nu_i| - |\nu_j| \Big|^4 \right)^{\frac{1}{4}}. \tag{36}$$

Since the matrix $A_{xx}$ is the square of $P(x,x)$, the zeros of its characteristic polynomial, again denoted by $(\lambda_j)_{j=1,\ldots,2n}$, satisfy the relations

$$0 \;\leq\; \lambda_j \;=\; |\nu_j|^2 \qquad \forall\, j = 1, \ldots, 2n\,. \tag{37}$$

Using the formula (10) for the critical Lagrangian, it follows that

$$4n\,\mathcal{L}(A_{xx}) \;=\; \sum_{i,j=1}^{2n} (\lambda_i - \lambda_j)^2 \;=\; \sum_{i,j=1}^{2n} (|\nu_i| - |\nu_j|)^2\,(|\nu_i| + |\nu_j|)^2\,. \tag{38}$$

The last expression can be bounded from below in two ways. Either we use the inequality

$$(|\nu_i| + |\nu_j|)^2 \;\geq\; (|\nu_i| - |\nu_j|)^2$$

and apply (36) to obtain (30). Or we use the estimate

$$\sum_{i,j=1}^{2n} (|\nu_i| - |\nu_j|)^2\,(|\nu_i| + |\nu_j|)^2 \;\geq\; 4\,\min_j |\nu_j|^2 \sum_{i,j=1}^{2n} (|\nu_i| - |\nu_j|)^2$$

together with (37) and (35), giving (31). ■

The inequality (30) immediately gives a positive lower bound for the action.

**Corollary 4.4** *For every $P \in \mathcal{P}^f$, the critical action satisfies the inequality*

$$\mathcal{S}[P] \;\geq\; \frac{f^4}{256\,n^3\,m^3}\,.$$

*Proof.* We first apply Hölder's inequality in (3),

$$f \;\leq\; m^{\frac{3}{4}} \left( \sum_{x \in M} |\mathrm{Tr}(E_x P)|^4 \right)^{\frac{1}{4}}\,. \tag{39}$$

Dropping the contributions for $x \neq y$ in (7), we obtain the lower bound

$$\mathcal{S}[P] \;=\; \sum_{x,y \in M} \mathcal{L}[A_{xy}] \;\geq\; \sum_{x \in M} \mathcal{L}[A_{xx}]\,, \tag{40}$$

and using (30) and (39) gives the claim. ■

We point out that for the estimates of Proposition 4.3 it is crucial that the maximal dimensions of the positive and negative definite subspaces of $E_x(H)$ coincide. If we considered more general discrete space-times with spin dimension $(p,q)$, then in the case $p \neq q$ the last transformation in (33) would no longer be valid, and the statements of Lemma 4.3 and Corollary 4.4 would break down. This can be seen most easily in the extreme example of spin dimension $(0,q)$, where by "localizing" $q$ particles similar to (23) at the space-time point $x$ we could arrange that $P(x,x) = \mathbb{1}_{|E_x(H)}$. Then $A_{xx}$ would be the identity, and $\mathcal{L}[A_{xx}]$ would vanish, although the local trace $\mathrm{Tr}(E_x P)$ would be equal to $q$. By localizing all particles in this way at individual space-time points, we could construct minimizers of the action which are not particularly interesting. This consideration is the reason why in this paper we only consider systems of spin dimension $(n,n)$. We feel that, apart from their physical significance, these systems are the ones for which the minimizers of our variational principle should have the most interesting mathematical structure.

## 5 A Lower Bound for the Local Trace

In this section we shall analyze how the infimum of our action depends on the number of space-time points. This will lead us to an estimate for the local trace of $P$ (Proposition 5.2), which is needed for the proof of Theorem 2.9 in the critical case (the reader not interested in Theorem 2.9 may skip this section).

For fixed spin dimension $(n,n)$ and a fixed number of particles $f$, we consider for any $m \in \mathbb{N}$ a discrete space-time $(H, <.|.>, (E_x)_{x \in M})$ with $m = \#M$ (note that this discrete space-time is unique up to isomorphisms). We define for any fixed $\mu \leq \frac{1}{2n}$ the quantities

$$\left. \begin{array}{rcl} I(f,m) &=& \inf\{\mathcal{S}_\mu[P] \,|\, P \in \mathcal{P}^f\} \\ J(f,m) &=& \inf\{\mathcal{S}_\mu[P] \,|\, P \text{ fermionic projector}\} \end{array} \right\} . \tag{41}$$

In the case $f > mn$, when the set of fermionic projectors is empty, we set $J(f,m) = \infty$. The functions $I$ and $J$ are strictly positive by Corollary 4.4. Also, it is obvious that $I(f,m) \leq J(f,m)$. Apart from simple examples as considered in Section 3, nothing is known about the values of $I(f,m)$ and $J(f,m)$. In particular, it would be interesting to know whether $I(f,m)$ is always strictly smaller than $J(f,m)$.

Our next lemma shows that the functions $I(f,m)$ and $J(f,m)$ are strictly decreasing in the parameter $m$. This can be understood from the fact that if $m$ is increased, the particles can spread out over more space-time points, making the infimum of the action smaller.

**Lemma 5.1** *The functions $I$ and $J$ defined by (41) satisfy the inequalities*

$$I(f,m+1) \;\leq\; \left(1 - \frac{3}{4m}\right) I(f,m)\,, \qquad J(f,m+1) \;\leq\; \left(1 - \frac{3}{4m}\right) J(f,m)\,. \tag{42}$$

*Proof.* Let $P$ be an operator of class $\mathcal{P}^f$ in a discrete space-time $(H, <.|.>, (E_x)_{x \in M})$ with $M = \{1, \ldots, m\}$. Introducing a discrete space-time $(\hat{H}, <.|.>, \hat{M})$ where $\hat{M} = \{0, \ldots, m\}$ consists of one more space-time point, there is a unitary transformation $U$ from $H$ to the subspace $K = \oplus_{x=1}^m \hat{E}_x(\hat{H})$ of $\hat{H}$ which maps the space-time projectors $E_x$ to the $\hat{E}_x$ in the sense that $E_x = U^{-1} \hat{E}_x U$ for all $x = 1, \ldots, m$. In other words, we can identify $(H, <.|.>, (E_x)_{x \in M})$ with the discrete space-time $(K, <.|.>, (\hat{E}_x)_{x \in M})$. Using this identification, the operator $P$ maps $K$ to itself, and extending it by zero to $\hat{E}_0(\hat{H})$, we obtain an operator

$$P \,:\, \hat{H} \to \hat{H} \qquad \text{with} \qquad E_0\, P \;=\; 0 \;=\; P\, E_0\,.$$

Since $P(x,y)$ vanishes when $x=0$ or $y=0$, the action of $P$ is given by

$$\mathcal{S}_\mu[P] \;=\; \sum_{x,y \in M} \mathcal{L}_\mu[A_{xy}]\;,$$

and this also shows that our reinterpretation of $P$ did not change its action.

Our method is to construct a unitary transformation $V \,:\, \hat{H} \to \hat{H}$ such that the action of the operator

$$\hat{P} \;:=\; V\, P\, V^{-1} \tag{43}$$

is strictly smaller than that of $P$. First, in

$$\mathcal{S}_\mu[P] \;=\; \sum_{x \in M} \left( \sum_{y \in M} \mathcal{L}_\mu[A_{xy}] \right)$$

we choose a point $x \in M$ for which the inner sum is maximal. Then

$$\sum_{y \in M} \mathcal{L}_\mu[A_{xy}] \;\geq\; \frac{\mathcal{S}_\mu[P]}{m}\,. \tag{44}$$

We choose $V$ such that it is the identity on the subspaces $\hat{E}_y(\hat{H})$ for $y \notin \{0,x\}$, whereas on the subspace $\hat{E}_0(\hat{H}) \oplus \hat{E}_x(\hat{H})$ it has in block matrix notation the form

$$V \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}, \qquad V^{-1} \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}.$$

A short calculation shows that the discrete kernels of $P$ and $\hat{P}$ are related by

$$\left\{ \begin{array}{lllll} \hat{P}(y,z) & = & P(y,z) & & \text{if } y,z \notin \{0,x\} \\ \hat{P}(y,z) & = & \dfrac{1}{\sqrt{2}}\, P(x,z)\,,\;\; \hat{P}(z,y) & = \;\; \dfrac{1}{\sqrt{2}}\, P(z,x) & \text{if } y \in \{0,x\} \text{ and } z \notin \{0,x\} \\ \hat{P}(z,y) & = & \dfrac{1}{2}\, P(x,x) & & \text{if } y,z \in \{0,x\}\,. \end{array} \right.$$

Using that the Lagrangian is homogeneous in $P$ of degree four, we obtain with the obvious notation $\hat{A}_{xy} = \hat{P}(x,y)\, \hat{P}(y,x)$ that

$$\begin{aligned} \mathcal{S}_\mu[\hat{P}] \;&=\; \sum_{y,z \notin \{0,x\}} \mathcal{L}_\mu[\hat{A}_{y,z}] \;+\; 4 \sum_{y \notin \{0,x\}} \mathcal{L}_\mu[\hat{A}_{xy}] \;+\; 4\, \mathcal{L}_\mu[\hat{A}_{xx}] \\ &=\; \sum_{y,z \notin \{0,x\}} \mathcal{L}_\mu[A_{y,z}] \;+\; \sum_{y \notin \{0,x\}} \mathcal{L}_\mu[A_{xy}] \;+\; \frac{1}{4}\, \mathcal{L}_\mu[A_{xx}] \end{aligned}$$

and thus

$$\mathcal{S}_\mu[P] - \mathcal{S}_\mu[\hat{P}] \;=\; \sum_{y \notin \{0,x\}} \mathcal{L}_\mu[A_{xy}] \;+\; \frac{3}{4}\, \mathcal{L}_\mu[A_{xx}] \;\geq\; \frac{3}{4} \sum_{y \in M} \mathcal{L}_\mu[A_{xy}]\,.$$

Now we can put in (44) to obtain the inequality

$$\mathcal{S}_\mu[\hat{P}] \;\leq\; \left(1 - \frac{3}{4m}\right) \mathcal{S}_\mu[P]\,.$$

Consider a minimal sequence $P_k \in \mathcal{P}^f(H)$. Then

$$I(m+1, f) \;\leq\; \mathcal{S}_\mu[\hat{P}_k] \;\leq\; \left(1 - \frac{3}{4m}\right) \mathcal{S}_\mu[P_k] \;\stackrel{k\to\infty}{\longrightarrow}\; \left(1 - \frac{3}{4m}\right) I(m, f)\;,$$

proving the left inequality in (42). Similarly, if we let $P_k$ be a minimal sequence of projectors, then the corresponding operators $\hat{P}_k$ are also projectors (because (43) is a unitary transformation), and we obtain the right inequality in (42). ∎

**Proposition 5.2** *Let $P_k \in \mathcal{P}^f$ be a minimal sequence for the action (7), i.e.*

$$\lim_{k\to\infty} \mathcal{S}_\mu[P_k] \;=\; I(f, m)\,. \tag{45}$$

*Then there is $\delta > 0$ such that*

$$\mathrm{Tr}(E_x P_k) \;\geq\; \delta \qquad \forall\, k \in \mathbb{N},\; x \in M\,.$$

*Proof.* We argue by contradiction. Assume that there is $x \in M$ and a subsequence of $(P_k)$ (again denoted by $(P_k)_{k\in\mathbb{N}}$) such that $\lim_{k\to\infty} \mathrm{Tr}(E_x P_k) \leq 0$. Then we must clearly have more than one space-time point, because otherwise $\mathrm{Tr}(E_x P) = \mathrm{Tr}(P) = f > 0$. We introduce the projector $F = \mathbb{1} - E_x$ and define for large $k$ the series of operators $Q_k$ by

$$Q_k \;=\; c_k\, F P_k F \qquad \text{with} \qquad c_k \;:=\; \frac{f}{\mathrm{Tr}(F P_k)}\,. \tag{46}$$

Since $\mathrm{Tr}(FP_k) = \mathrm{Tr}(P_k) - \mathrm{Tr}(E_x P_k) \to f$, we know that

$$\lim_{k\to\infty} c_k \;\leq\; 1\,. \tag{47}$$

According to Lemma 4.1 **(i)**, the operators $(-Q_k)$ are positive, and we normalized them such that $\mathrm{Tr}\, Q_k = f$. Therefore, the operators $Q_k$ are again of class $\mathcal{P}^f$. Since they vanish identically on $E_x(H)$, we can regard them as operators in a discrete space-time consisting of $m-1$ space-time points, and thus

$$\mathcal{S}_\mu[Q_k] \;\geq\; I(m-1, f)\,. \tag{48}$$

Using that the Lagrangian is homogeneous of degree four, we obtain furthermore

$$\mathcal{S}_\mu[Q_k] \;=\; c_k^4\, \mathcal{S}_\mu[F P_k F] \;\leq\; c_k^4\, \mathcal{S}_\mu[P_k]\,, \tag{49}$$

where in the last step we used that the Lagrangians of $P_k$ and $FP_kF$ coincide away of the space-time point $x$; more precisely,

$$\mathcal{S}_\mu[P_k] - \mathcal{S}_\mu[FP_kF] \;=\; \mathcal{L}_\mu[A_{xx}] + 2\sum_{y\neq x} \mathcal{L}_\mu[A_{xy}] \;\geq\; 0\,.$$

Taking in (49) the limit $k \to \infty$ and using (47, 45), we obtain in view of (48) that

$$I(m-1, f) \;\leq\; \lim_{k\to\infty} \mathcal{S}_\mu[Q_k] \;\leq\; \lim_{k\to\infty} \mathcal{S}_\mu[P_k] \;=\; I(m,f)\;,$$

in contradiction to Lemma 5.1. ∎

We point out that, unfortunately, the above argument does not apply to a minimal sequence of projectors, because the property to be idempotent gets lost when the operators are restricted similar to (46) to a subspace of $H$.

# 6 A General Existence Theorem

In this section we will show that all the results stated in Section 2 are a consequence of the following general existence theorem.

**Theorem 6.1** *Suppose that $(P_k)_{k\in\mathbb{N}}$ is a sequence of operators in $H$ such that the operators $(-P_k)$ are all positive. Assume furthermore that the corresponding sequence of critical actions $\mathcal{S}[P_k]$ is bounded and that one of the following two conditions is satisfied:*

**(C1)** *The local trace is bounded away from zero in the sense that for suitable $\delta > 0$,*

$$|\mathrm{Tr}(E_x P_k)| \;\geq\; \delta \qquad \forall\, k \in \mathbb{N},\; x \in M\;.$$

**(C2)** *The spectral weights $|(A_k)_{xx}|$ are bounded from above in the sense that for suitable $C > 0$,*

$$|(A_k)_{xx}| \;\leq\; C \qquad \forall\, k \in \mathbb{N},\; x \in M\;.$$

*Then there is a subsequence $(P_{k_l})$ and a sequence of gauge transformations $U_l \in \mathcal{G}$ such that the gauge-transformed operators have a limit*

$$P \;:=\; \lim_{l\to\infty} U_l P_{k_l} U_l^{-1}\;.$$

The proof of this theorem will be given in Sections 7 and 8. Here we simply assume that Theorem 6.1 holds and deduce the theorems in Section 2:

We let $(P_k)_{k\in\mathbb{N}}$ be a minimal sequence. Since all the matrix functionals considered here are continuous, the limit $P$ constructed with the above theorem will certainly be a minimizer. Furthermore, the limit of projectors of rank $f$ is again a projector of rank $f$, whereas for general operators the rank only decreases in the limit. For this reason, it is obvious that by taking limits we do not leave class of operators under consideration.

When considering the variational principle (6, 5), the inequality $|A_{xx}|^2 \leq \kappa$ shows that condition **(C2)** holds. Furthermore, $\mathcal{L}[P_k] \leq \sum_{x,y} |A^2_{xy}|$. Hence Theorem 6.1 applies and gives the desired minimizer $P$. This proves Theorem 2.1, Theorem 2.8 and the existence part of Theorem 2.10. In order to derive the relation (12), we consider the variation $P(\tau) = (1+\tau)\,P$. Using that the action is homogeneous in $P$ of degree 4, we find that

$$0 \;=\; \frac{d}{d\tau}\mathcal{S}_\mu(P(\tau))\Big|_{\tau=0} \;=\; 4\,\mathcal{S}_\mu(P)\;,$$

and so the action vanishes.

To prove Theorem 2.2, we decompose the Lagrangian as in (29) into a sum of two positive terms. This shows that condition **(C2)** is satisfied, and Theorem 6.1 applies. In the setting of Theorem 2.3, the assumption on the local trace ensures that condition **(C1)** holds, and again Theorem 6.1 applies. To prove Theorem 2.5 we let $P$ be a homogeneous fermionic projector. Then, with $\sigma$ and $U$ as in Definition 2.4,

$$\mathrm{Tr}(E_{x_1} P) \;=\; \mathrm{Tr}(P(x_1, x_1)) \;=\; \mathrm{Tr}(U\, P(x_0, x_0)\, U^{-1}) \;=\; \mathrm{Tr}(P(x_0, x_0))\,.$$

Thus the local trace is the same at all space-time points, and from (3) we conclude that

$$\mathrm{Tr}(E_x P) \;=\; \frac{f}{m} \qquad \forall\, x \in M\,.$$

Hence the condition **(C1)** is satisfied, and we can again apply Theorem 6.1.

Finally, to prove Theorem 2.9, we can in the case $\mu < \frac{1}{2n}$ again use the decomposition (29), whereas in the critical case $\mu = \frac{1}{2n}$ Proposition 5.2 ensures that condition **(C1)** holds. This concludes the proof of all the theorems in Section 2, provided that Theorem 6.1 is true.

# 7 Gauge Fixing, Rescaling

We now enter the proof of Theorem 6.1. Thus let $(P_k)_{k \in \mathbb{N}}$ be a sequence of operators satisfying the assumptions of Theorem 6.1. We again choose a basis in $H$ and let $(.|.)$ be the canonical scalar product on $\mathbb{C}^{2nm}$. We let $\|.\|$ be the corresponding Hilbert-Schmidt norm, $\|A\| := (\mathrm{Tr}(A^\dagger A))^{\frac{1}{2}}$.

Our first task is to treat the non-compact gauge group $\mathcal{G}$ (as defined after (17)). We denote the equivalence class of gauge-equivalent operators by $\langle . \rangle_{\mathcal{G}}$, i.e.

$$\langle P \rangle_{\mathcal{G}} \;=\; \{ U P U^{-1} \,|\, U \in \mathcal{G} \}\,.$$

We consider for any fixed $k \in \mathbb{N}$ the variational principle

$$\text{minimize} \quad \{ \|Q\| \mid Q \in \langle P_k \rangle_{\mathcal{G}} \}\,. \tag{50}$$

If $(Q_l)_{l \in \mathbb{N}}$ is a minimal sequence of this variational principle, the Hilbert-Schmidt norms of the $Q_l$ are uniformly bounded. Thus we can use a compactness argument to select a convergent subsequence. We conclude that the variational principle (50) attains its minimum. We choose for each $k$ a minimizer and denote it by $\hat{P}_k$. We point out that the above construction of the $\hat{P}_k$ involves the norm $\|.\|$ and thus depends on the choice of our basis of $H$. This will be no problem in what follows because the minimizers obtained by choosing different norms will be gauge equivalent. We refer to our method of arbitrarily choosing one representative of each gauge equivalence class as *gauge fixing*; it can be understood in analogy to the gauge fixing used in electrodynamics or in general relativity.

In the case that the sequence of operators $(\hat{P}_k)$ has a subsequence of bounded Hilbert-Schmidt norm, we can by compactness choose a subsequence which converges to an operator $P$. Thus it remains to consider the case when the Hilbert-Schmidt norm is unbounded for any subsequence of $(\hat{P}_k)$; in other words, that

$$\|\hat{P}_k\| \;\to\; \infty\,. \tag{51}$$

We introduce new operators $R_k$ by rescaling the $\hat{P}_k$,

$$R_k \;=\; \alpha_k \, \hat{P}_k \qquad \text{with} \qquad \alpha_k \;:=\; \frac{1}{\|\hat{P}_k\|} \;\stackrel{k\to\infty}{\longrightarrow}\; 0 \,.$$

Then obviously $\|R_k\| \equiv 1$, and thus we can, again after choosing a subsequence, assume that the $R_k$ converge,

$$R_k \;\to\; R \,.$$

It is clear from their construction that the operators $R_k$ and $R$ have the following properties: The operators $(-R_k)$ and $(-R)$ are positive and normalized by

$$\|R_k\| \;=\; 1 \;=\; \|R\| \,. \tag{52}$$

Their action is computed to be

$$\mathcal{S}[R_k] \;=\; \alpha_k^4 \, \mathcal{S}[P_k] \,, \qquad\qquad \mathcal{S}[R] \;=\; 0 \,. \tag{53}$$

Finally, the conditions **(C1)** and **(C2)** give

$$\left\{ \begin{array}{lll} |R(x,x)^2| \;=\; 0 \quad \forall\, x \in M & & \text{in case } \mathbf{(C1)} \\ |\mathrm{Tr}(E_x R_k)| \;\geq\; \delta\, \alpha_k \quad \forall\, k \in \mathbb{N},\; x \in M & & \text{in case } \mathbf{(C2)} \,. \end{array} \right. \tag{54}$$

## 8 Existence of Minimizers

Our goal is to show that the properties (52–54) contradict the fact that the $\hat{P}_k$ are minimizers of (50) (this then implies that the case (51) cannot occur, completing the proof of Theorem 6.1). For any $x \in M$, the operator $T := -R(x,x)$ is positive according to Lemma 4.1 **(i)**. From Lemma 4.2 we conclude that the zeros $(\nu_j)_{j=1,\ldots,2n}$ of its characteristic polynomial are all real and ordered as in (32). Since $\mathcal{L}[T^2] = 0$, the absolute values of the $\nu_j$ must all be equal, and thus there is a parameter $\nu \geq 0$ such that

$$\nu_1 = \ldots = \nu_n = -\nu \qquad \text{and} \qquad \nu_{n+1} = \ldots = \nu_{2n} = \nu \,.$$

Let us rule out the case $\nu > 0$. If the condition **(C1)** is satisfied, we obtain from (54) that $0 = |T^2| = 2n\nu^2$, a contradiction. If on the other hand the condition **(C2)** holds, we know by the continuity of the spectrum that for large $k$,

$$\inf \sigma\Big( (T_k)^2|_{E_x(H)} \Big) \;\geq\; \frac{\nu^2}{2}$$

(with $T_k := -R_k(x,x)$). Combining the lower bound (31) with (54) and (53), we obtain

$$\frac{1}{4n}\, \frac{\nu^2}{2}\, \delta^2\, \alpha_k^2 \;\leq\; \mathcal{L}[T_k^2] \;\leq\; \mathcal{S}[R_k] \;=\; \alpha_k^4\, \mathcal{S}[P_k] \,.$$

Dividing by $\alpha_k^2$ and taking the limit $k \to \infty$, we obtain a contradiction to the boundedness of the sequence $\mathcal{S}[P_k]$.

It remains to consider the case $\nu = 0$ where the operator $T$ is nilpotent. As in the proof of Lemma 4.2, we approximate $T$ by the strictly positive operators $T_\varepsilon = T + \varepsilon S$. Diagonalizing the $T_\varepsilon$ by unitary transformations $U_\varepsilon$ on $E_x(H)$, the diagonal matrices $U_\varepsilon T_\varepsilon U_\varepsilon^{-1}$ converge to zero as $\varepsilon \to 0$. Hence for any $\Psi \in H$,

$$<\!\Psi \,|\, U_\varepsilon\, T\, U_\varepsilon^{-1}\, \Psi\!> \;+\; \varepsilon <\!\Psi \,|\, U_\varepsilon\, S\, U_\varepsilon^{-1}\, \Psi\!> \;=\; <\!\Psi \,|\, U_\varepsilon\, T_\varepsilon\, U_\varepsilon^{-1}\, \Psi\!> \;\stackrel{\varepsilon\to 0}{\longrightarrow}\; 0 \,.$$

Since the summands on the left are both positive, we conclude that $<\Psi \,|\, U_\varepsilon T U_\varepsilon^{-1}\, \Psi> \to 0$ for all $\Psi \in H$ and thus

$$\lim_{\varepsilon \to 0} U_\varepsilon T U_\varepsilon^{-1} \;=\; 0\,.$$

For given $\kappa > 0$ we choose $\varepsilon$ such that $\|U_\varepsilon T U_\varepsilon^{-1}\| < \kappa/2$ and subsequently $k$ so large that $\|T_k - T\| < \kappa/(2\,\|U_\varepsilon\|\,\|U_\varepsilon^{-1}\|)$. Then

$$\|U_\varepsilon\, T_k\, U_\varepsilon^{-1}\| \;\leq\; \|U_\varepsilon\|\,\|T_k - T\|\,\|U_\varepsilon^{-1}\| \;+\; \|U_\varepsilon\, T\, U_\varepsilon^{-1}\| \;\leq\; \kappa\,.$$

Since $\kappa$ can be chosen arbitrarily small, we conclude that there is a subsequence of the $T_k$ (which we denote again by $(T_k)_{k \in \mathbb{N}}$) together with unitary transformations $U_k$ such that

$$\lim_{k \to \infty} U_k\, T_k\, U_k^{-1} \;=\; 0\,.$$

Extending the $U_k$ by the identity to the subspaces $E_y(H)$, $y \neq x$, we obtain a sequence of gauge transformations such that

$$U_k\, R_k\, U_k^{-1} \;\to\; \tilde{R}\,.$$

Since these gauge transformations act only on $E_x(H)$, it is clear that $R(y,z) = \tilde{R}(y,z)$ if $y, z \neq x$. By construction, $\tilde{R}(x,x) = 0$. The Schwarz inequality, Lemma 4.1 **(ii)**, tells us that also the entries $\tilde{R}(x,y)$ and $\tilde{R}(y,x)$ for $y \neq x$ vanish. Since we chose the operators $\hat{P}_k$ such that their Hilbert-Schmidt norm was minimal among all gauge-equivalent operators, the Hilbert-Schmidt norm of the operators $R_k$ (which were obtained from the $\hat{P}_k$ only by rescaling) cannot be decreased by a subsequent gauge transformation, and thus $\|U_k\, R_k\, U_k^{-1}\| \geq \|R_k\|$. Taking the limit $k \to \infty$, we find that $\|\tilde{R}\| \geq \|R\|$. Since these operators coincide up to matrix elements where $\tilde{R}$ vanishes, the operators $\tilde{R}$ and $R$ must coincide. In particular, $R(x,x) = 0$.

We conclude that the diagonal entries $R(x,x)$ of $R$ all vanish. Again applying the Schwarz inequality, Lemma 4.1 **(ii)**, we see that the off-diagonal entries of $R$ are also zero. Thus $R = 0$, in contradiction to (52).

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