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Comodule categories and the geometry of the stack of formal groups

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Abstract

We generalise recent results of M. Hovey and N. Strickland on comodule categories for Landweber exact algebras using the formalism of algebraic stacks.

1. Introduction

Extensions of comodules over flat Hopf algebroids play an important role in algebraic topology as the E_2-term of the Adams-Novikov spectral sequence based on a sufficiently well-behaved ring theory. It is "well-known" that the category of comodules is equivalent to the category of quasi-coherent sheaves of modules on an algebraic stack associated to the flat Hopf algebroid. The purpose of this note is to make this precise and to demonstrate that the switch of perspective from flat Hopf algebroids to algebraic stacks is not purely formal. To this end, we generalise recent results of M. Hovey and N. Strickland ([HS]) using only a minimum of the theory of formal groups and general facts about algebraic stacks. We hope to make clear that all these results are rather immediate consequences of the simple geometric structure of the stack of formal groups.

We review the individual sections in more detail. In section 2 we give the relation between flat Hopf algebroids and algebraic stacks, following essentially [P]. In section 3 we collect a number of technical results on algebraic stacks. The analogues for flat Hopf algebroids of some of these results are known. In section 4 we apply this theory to the stack $\mathfrak{X}_{FG}$ of formal groups over $\mathbb{Z}_{(p)}$ and isolate the relation of Landweber exactness (as considered in [HS]) with the geometry of $\mathfrak{X}_{FG}$ (theorem 18). We then deduce the equivalences of comodule categories and change of rings theorems generalising those of [HS].

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2. Algebraic stacks and flat Hopf algebroids

In this section we review parts of [P] giving the relation between flat Hopf algebroids and their categories of comodules and a certain class of stacks and their catagories of quasi-coherent sheaves of modules.

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2.1 The 2-category of flat Hopf algebroids

We refer to [R1], Appendix A for the notion of (flat) Hopf algebroid. To give a Hopf algebroid (A,Γ) is equivalent to giving $(X_0 := \mathrm{Spec}\,(A), X_1 := \mathrm{Spec}\,(\Gamma))$ as a groupoid in affine schemes [LM-B], 2.4.3 and we will formulate most results involving Hopf algebroids this way.
Recall that this means that X_0 and X_1 are affine schemes and we are given morphisms $s,t : X_1 \longrightarrow X_0$ (source and target), $\epsilon : X_0 \longrightarrow X_1$ (identity), $\delta : X_1 \underset{s,X_0,t}{\times} X_1 \longrightarrow X_1$ (composition) and $i : X_1 \longrightarrow X_1$ (inverse) verifying suitable identities. The corresponding maps of rings are denoted η_L, η_R (left- and right unit), ϵ (augmentation), Δ (comultiplication) and c (antipode).
The 2-category of flat Hopf algebroids $\mathcal{H}$ is defined as follows. Objects are Hopf algebroids (X_0, X_1) such that s and t are flat (and thus faithfully flat because they allow ϵ as a right inverse). A 1-morphism of flat Hopf algebroids from (X_0, X_1) to (Y_0, Y_1) is a pair of morphisms of affine schemes $f_i : X_i \longrightarrow Y_i$ $(i = 0, 1)$ commuting with all the structure. The composition of 1-morphisms is component wise. Given two 1-morphisms $(f_0, f_1), (g_0, g_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$, a 2-morphism $c : (f_0, f_1) \longrightarrow (g_0, g_1)$ is a morphism of affine schemes $c : X_0 \longrightarrow Y_1$ such that $sc = f_0, tc = g_0$ and the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{(g_1, cs)} & Y_1 \underset{s,Y_0,t}{\times} Y_1 \\ {\scriptstyle (ct,f_1)}\downarrow & & \downarrow{\scriptstyle \delta} \\ Y_1 \underset{s,Y_0,t}{\times} Y_1 & \xrightarrow{\delta} & Y_1 \end{array}$$

commutes. For $(f_0, f_1) = (g_0, g_1)$ the identity 2-morphism is given by $c := \epsilon f_0$. Given two 2-morphisms $(f_0, f_1) \xrightarrow{c} (g_0, g_1) \xrightarrow{c'} (h_0, h_1)$ their composition is defined as

$$c' \circ c : \; X_0 \xrightarrow{(c',c)} Y_1 \underset{s,Y_0,t}{\times} Y_1 \xrightarrow{\delta} Y_1 \, .$$

One checks that the above definitions make $\mathcal{H}$ a 2-category which is in fact clear because (except for the flatness of s and t) they are merely a functorial way of stating the axioms of a groupoid, a functor and a natural transformation. For technical reasons we will sometimes consider Hopf algebroids for which s and t are not flat.

2.2 The 2-category of rigidified algebraic stacks

Let S be an affine scheme. We denote by Aff_S the category of affine S-schemes (with some cardinality bound as to make it small). We generally drop S from the notation. We endow Aff with with the *fpqc* topology, i.e. a cover of $X \in \mathrm{Aff}$ is a finite family of flat morphisms $X_i \longrightarrow X$ in Aff such that $\coprod X_i \longrightarrow X$ is faithfully flat. We denote by Aff also the site thus defined. We will consider stacks over Aff and all notations and conventions concerning stacks will be those of [LM-B] except that we work with the *fpqc* rather than the étale topology, c.f. [LM-B], §9.

Definition 1. *A stack $\mathfrak{X}$ (over the site* Aff*) is* algebraic *if its diagonal $\Delta_{\mathfrak{X}}$ is representable and affine and there is an affine scheme X_0 and a faithfully flat morphism $P : X_0 \longrightarrow \mathfrak{X}$.*

Any 1-morphism of algebraic stacks from an algebraic space to an algebraic stack is representable and affine, see the proof of [LM-B], 3.13. In particular, it makes sense to say that P is faithfully flat. By definition, every algebraic stack is quasi-compact, hence so is any 1-morphism between algebraic stacks ([LM-B], 4.16, 4.17). One can check that finite limits and colimits of algebraic stacks (taken

in the 2-category of *fpqc*-stacks, [LM-B] 3.3) are again algebraic stacks.
A morphism P as in definition 1 is called a presentation of $\mathfrak{X}$. As far as we are aware, this definition of "algebraic" is due to P. Goerss [G] and is certainly motivated by the equivalence given in subsection 2.3 below. We point out that the notion of "algebraic stack" well-establish in algebraic geometry ([LM-B],4.1) is different from the above, for example the stack associated to (BP_*, BP_*BP) (c.f. section 4) is algebraic in the above sense but not in the sense of algebraic geometry because its diagonal is not of finite type, [LM-B] 4.2. Of course, in the following we will use the term "algebraic stack" in the sense defined above.
The 2-category $\mathcal{S}$ of rigidified algebraic stacks is defined as follows. Objects are presentations $P : X_0 \longrightarrow \mathfrak{X}$ as in definition 1. A 1-morphism from $P : X_0 \longrightarrow \mathfrak{X}$ to $Q : Y_0 \longrightarrow \mathfrak{Y}$ is a pair consisting of $f_0 : X_0 \longrightarrow Y_0$ (a morphism in Aff) and $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ (a 1-morphism of stacks) such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

is 2-commutative. Composition of 1-morphisms is component wise. Given 1-morphisms $(f_0, f), (g_0, g) : (X_0 \longrightarrow \mathfrak{X}) \longrightarrow (Y_0 \longrightarrow \mathfrak{Y})$ a 2-morphism in $\mathcal{S}$ from (f_0, f) to (g_0, g) is by definition a 2-morphism from f to g in the 2-category of stacks, [LM-B], 3.

2.3 The equivalence of $\mathcal{H}$ and $\mathcal{S}$

We now establish an equivalence of 2-categories between $\mathcal{H}$ and $\mathcal{S}$. We define a functor $K : \mathcal{S} \longrightarrow \mathcal{H}$ as follows.

$$K(\ X_0 \xrightarrow{P} \mathfrak{X}\ ) := (X_0, X_1 :=\ X_0 \underset{P,\mathfrak{X},P}{\times} X_0\ )$$

has a canonical structure of groupoid ([LM-B], 3.8), X_1 is affine because X_0 is affine and P is representable and affine and the projections $s, t :\ X_1 \rightrightarrows X_0$ are flat because P is; so (X_0, X_1) is a flat Hopf algebroid. If $(f_0, f) :\ (X_0 \xrightarrow{P} \mathfrak{X}) \ \longrightarrow\ (Y_0 \xrightarrow{Q} \mathfrak{Y})$ is a 1-morphism in $\mathcal{S}$ we define $K((f_0, f)) := (f_0, f_0 \times f_0)$. If we have 1-morphisms $(f_0, f), (g_0, g) :\ (X_0 \xrightarrow{P} \mathfrak{X}) \ \longrightarrow\ (Y_0 \xrightarrow{Q} \mathfrak{Y})$ in $\mathcal{S}$ and a 2-morphism $(f_0, f) \longrightarrow (g_0, g)$ then we have by definition a 2-morphism $f \xrightarrow{\Theta} g$: $\mathfrak{X} \longrightarrow \mathfrak{Y}$. In particular, we have $\Theta_{X_0} : \mathrm{Ob}(\mathfrak{X}_{X_0}) \longrightarrow \mathrm{Mor}\ (\mathfrak{Y}_{X_0}) = \mathrm{Hom}_{\mathrm{Aff}}(X_0, Y_1)$ and we define $K(\Theta) := \Theta_{X_0}(\mathrm{id}_{X_0})$. One checks that $K : \mathcal{S} \longrightarrow \mathcal{H}$ is a 2-functor.
We define a 2-functor $G : \mathcal{H} \longrightarrow \mathcal{S}$ as follows. On objects we put $G((X_0, X_1)) := (X_0 \xrightarrow{can} [\ X_1 \rightrightarrows X_0\ ])$, the stack associated with the groupoid (X_0, X_1) together with its canonical presentation ([LM-B], 3.4.3; identify the X_i with the flat sheaves they represent to consider them as "S-espaces", see also subsection 3.1). This is a rigidified algebraic stack: Saying that the diagonal of $\mathfrak{X}$ is representable and affine means that for any algebraic space X and morphisms $x_1, x_2 : X \longrightarrow \mathfrak{X}$ the sheaf $\underline{\mathrm{Isom}}_X(x_1, x_2)$ on X is representable by an affine X-scheme. This problem is local in the *fpqc* topology on X because affine morphisms satisfy effective descent in the *fpqc* topology [SGA1], exposé VIII, theorem 2.1. So we can assume that the x_i lift to X_0 and the assertion follows because $(s, t) : X_1 \longrightarrow\ X_0 \underset{S}{\times} X_0$ is affine. A similar argument shows that $P : X_0 \longrightarrow \mathfrak{X}$ is (representable

and) faithfully flat because s and t are faithfully flat.
Given a 1-morphism $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ in $\mathcal{H}$ there is a unique 1-morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ making

$$\begin{array}{ccccc} X_1 & \rightrightarrows & X_0 & \xrightarrow{P} & \mathfrak{X} \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \vdots\,{\scriptstyle f} \\ Y_1 & \rightrightarrows & Y_0 & \xrightarrow{Q} & \mathfrak{Y} \end{array}$$

2-commutative ([LM-B], proof of 4.18) and we define $G((f_0, f_1)) := f$.
Given a 2-morphism $c : X_0 \longrightarrow Y_1$ from the 1-morphism $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ to the 1-morphism $(g_0, g_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ in $\mathcal{H}$ we have a diagram

$$\begin{array}{ccccc} X_1 & \rightrightarrows & X_0 & \xrightarrow{P} & \mathfrak{X} \\ {\scriptstyle f_1}\downarrow\downarrow{\scriptstyle g_1} & & {\scriptstyle f_0}\downarrow\downarrow{\scriptstyle g_0} & & {\scriptstyle f}\big(\;\big){\scriptstyle g} \\ Y_1 & \rightrightarrows & Y_0 & \xrightarrow{Q} & \mathfrak{Y} \end{array}$$

and need to construct a 2-morphism $\Theta = G(c) : f \longrightarrow g$ (in the 2-category of stacks). We will do this in some detail because we omit numerous similar arguments.
Fix $U \in \mathrm{Aff}$, $x \in \mathrm{Ob}(\mathfrak{X}_U)$ and a representation of x (c.f. [LM-B], proof of 3.2)

$$(U' \longrightarrow U,\, x' : U' \longrightarrow X_0,\, U'' := U' \underset{U}{\times} U' \xrightarrow{\sigma} X_1),$$

i.e. $U' \longrightarrow U$ is a cover in Aff, $x' \in X_0(U') = \mathrm{Hom}_{\mathrm{Aff}}(U', X_0)$ and σ is a descent datum for x' with respect to the cover $U' \longrightarrow U$. Hence, denoting by $\pi_1, \pi_2 : U'' \longrightarrow U'$ and $\pi : U' \longrightarrow U$ the projections we have $\sigma : \pi_1^* x' \xrightarrow{\sim} \pi_2^* x'$ in $\mathfrak{X}_{U''}$, i.e. $x'\pi_1 = s\sigma$ and $x'\pi_2 = t\sigma$. Furthermore, σ satisfies a cocycle condition which we do not spell out.
We have to construct a morphism

$$\Theta_x : f(x) \longrightarrow g(x) \text{ in } \mathfrak{Y}_U$$

which we do by descent from U' as follows. We have a morphism

$$\pi^*(f(x)) = f(\pi^*(x) = x') = f_0 x' \xrightarrow{\phi'} \pi^*(g(x)) = g_0 x' \text{ in } \mathfrak{Y}_{U'}$$

given by $\phi' := cx' : U' \longrightarrow Y_1$. We also have a diagram

$$\begin{array}{ccc} \pi_1^*(\pi^*(f(x))) = f_0 x'\pi_1 & \xrightarrow{\pi_1^*(\phi')} & \pi_1^*(\pi^*(g(x))) = g_0 x'\pi_1 \\ {\scriptstyle \sigma_f}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ \pi_2^*(\pi^*(f(x))) = f_0 x'\pi_2 & \xrightarrow{\pi_2^*(\phi')} & \pi_2^*(\pi^*(g(x))) = g_0 x'\pi_2 \end{array}$$

in $\mathfrak{Y}_{U''}$ where σ_f and σ_g are descent isomorphisms for $f(x')$ and $g(x')$ given by $\sigma_f = f_1\sigma$ and $\sigma_g = g_1\sigma$. We check that this diagram commutes by computing in Mor $(\mathfrak{Y}_{U''})$:

$$\sigma_g \circ \pi_1^*(\phi') = \delta_Y(g_1\sigma, cx'\pi_1) = \delta_Y(g_1\sigma, cs\sigma) = \delta_Y(g_1, cs)\sigma \overset{(*)}{=}$$

$$= \delta_Y(ct, f_1)\sigma = \delta_Y(ct\sigma, f_1\sigma) = \delta_Y(cx'\pi_2, f_1\sigma) = \pi_2^*(\phi') \circ \sigma_f.$$

Here δ_Y is the composition of (Y_0, Y_1) and in $(*)$ we used the commutative square in the definition of 2-morphisms in $\mathcal{H}$.
So ϕ' is compatible with descent data and thus descents to the desired $\Theta_x : f(x) \longrightarrow g(x)$. We omit

the verification that Θ_x is independent of the chosen representation of x and natural in x and U. We now check that K and G are inverse equivalences.

We have $G \circ K(X_0 \xrightarrow{P} \mathfrak{X}) = (\ X_0 \xrightarrow{can} [X_0 \underset{\mathfrak{X}}{\times} X_0 \rightrightarrows X_0]\ )$ and there is a unique 1-isomorphism $\nu_P :\ [X_0 \underset{\mathfrak{X}}{\times} X_0 \rightrightarrows X_0] \ \longrightarrow \mathfrak{X}$ with $\nu_p \circ can = P$ ([LM-B], 3.8). One checks that this defines an isomorphism of 2-functors $GK \xrightarrow{\simeq} \mathrm{id}_{\mathcal{S}}$.

Next we have $K \circ G(X_0, X_1) = (X_0,\ X_0 \underset{P,\mathfrak{X},P}{\times} X_0\ )$, where $(X_0 \xrightarrow{P} \mathfrak{X}) = G(X_0, X_1)$, and $X_1 \simeq X_0 \underset{P,\mathfrak{X},P}{\times} X_0$ ([LM-B], 3.4.3) and one checks that this defines an isomorphism of 2-functors $\mathrm{id}_{\mathcal{H}} \xrightarrow{\simeq} KG$.

The forgetful functor from rigidified algebraic stacks to algebraic stacks is not full but we have the following.

Proposition 2. *If (X_0, X_1) and (Y_0, Y_1) are flat Hopf algebroids with associated rigidified algebraic stacks $P : X_0 \longrightarrow \mathfrak{X}$ and $Q : X_0 \longrightarrow \mathfrak{Y}$ and $\mathfrak{X}$ and $\mathfrak{Y}$ are 1-isomorphic as stacks then there is a chain of 1-morphisms of flat Hopf algebroids from (X_0, X_1) to (Y_0, Y_1) such that every morphism in this chain induces a 1-isomorphism on the associated algebraic stacks.*

This result implies theorem 6.5 of [HS]: As we will see in section 4, the assumptions of *loc. cit.* imply that the flat Hopf algebroids (B, Γ_B) and $(B', \Gamma_{B'})$ considered there have the same open substack of the stack of formal groups as their associated stack. So they are connected by a chain of weak equivalences by proposition 2 (see also remark 6 for the notion of weak equivalence).

Proof. Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a 1-isomorphism of stacks and form the cartesian diagram

$$\begin{array}{ccc}
X_1' & \xrightarrow{f_1} & Y_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
X_0' & \xrightarrow{f_0} & Y_0 \\
{\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}.
\end{array}$$

To be precise, the upper square is cartesian for either both source or both target morphisms. Then (f_0, f_1) is a 1-isomorphism of flat Hopf algebroids. Next, $Z := X_0' \underset{P',\mathfrak{X},P}{\times} X_0$ is an affine scheme because X_0' is and P is representable and affine. The obvious 1-morphism $Z \longrightarrow \mathfrak{X}$ is representable, affine and faithfully flat because P and P' are. Writing $W := Z \underset{\mathfrak{X}}{\times} Z \simeq X_1' \underset{\mathfrak{X}}{\times} X_1$ we have that $\mathfrak{X} \simeq [\ W \rightrightarrows Z\ ]$ by (the flat version of) [LM-B], 4.3.2. There are obvious 1-morphisms of flat Hopf algebroids $(Z, W) \longrightarrow (X_0', X_1')$ and $(Z, W) \longrightarrow (X_0, X_1)$ covering $\mathrm{id}_{\mathfrak{X}}$ (in particular inducing an isomorphism on stacks) and we get the sought for chain as $(Y_0, Y_1) \longleftarrow (X_0', X_1') \longleftarrow (Z, W) \longrightarrow (X_0, X_1)$. □

2.4 Comodules and quasi-coherent sheaves of modules

For basic results concerning the category $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ of quasi-coherent sheaves of modules on an algebraic stack $\mathfrak{X}$ we refer the reader to [LM-B], 13.

Fix a rigidified algebraic stack $X_0 \xrightarrow{P} \mathfrak{X}$ corresponding as in subsection 2.3 to the flat Hopf algebroid $(X_0 = \mathrm{Spec}\,(A), X_1 = \mathrm{Spec}\,(\Gamma))$ with structure morphisms s and t. As P is affine it is in particular quasi-compact, hence *fpqc*, and thus of effective cohomological descent for quasi-coherent modules, [LM-B],13.5.5,i). In particular, P^* induces an equivalence

$$P^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \xrightarrow{\simeq} \{F \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0}) + \text{ descent data}\},$$

c.f. [BLR], Chapter 6 for similar examples of descent. A descent datum on $F \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0})$ is an isomorphism $\alpha : s^*F \longrightarrow t^*F$ in $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_1})$ satisfying a cocycle condition. Giving α is equivalent to giving either its adjoint $\psi_l : F \longrightarrow s_*t^*F$ or the adjoint of α^{-1}, $\psi_r : F \longrightarrow t_*s^*F$. Writing M for the A-module corresponding to F, α corresponds to an isomorphism $\Gamma \underset{\eta_L, A}{\otimes} M \longrightarrow \Gamma \underset{\eta_R, A}{\otimes} M$ of Γ-modules and ψ_r and ψ_l correspond respectively to morphisms $M \longrightarrow \Gamma \otimes_A M$ and $M \longrightarrow M \otimes_A \Gamma$ of A-modules. One checks that this is a 1-1 correspondence between descent data on F and left- (respectively right-)Γ-comodule structures on M. For example, the cocycle condition of α corresponds to the coassociativity of the coaction. In the following we will work with left-Γ-comodules only and call them simply Γ-comodules. The above construction provides an identification of the category of Γ-comodules and $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ which can also be proved using the Baer-Beck theorem, [P], 3.22.

The identification of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ with Γ-comodules allows to (re)understand a number of results on Γ-comodules from the "geometric" point of view and we now give a short list of such applications which we will use later.

The adjunction $(P^*, P_*) : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{X_0})$ corresponds to the forgetful functor from Γ-comodules to A-modules, respectively to the functor "induced/extended comodule". The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ corresponds to the trivial Γ-comodule A, hence taking the primitives of a Γ-comodule (i.e. the functor $\mathrm{Hom}_\Gamma(A, \cdot)$ from Γ-comodules to abelian groups) corresponds to $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X}}, \cdot) = H^0(\mathfrak{X}, \cdot)$ and $\mathrm{Ext}^*_\Gamma(A, \cdot)$ corresponds to quasi-coherent cohomology $H^*(\mathfrak{X}, \cdot)$.

By [LM-B], 14.2.7 there is a 1-1 correspondence between closed substacks $\mathfrak{Z} \subseteq \mathfrak{X}$ and quasi-coherent ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ such that $\mathcal{O}_{\mathfrak{Z}} \simeq \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$ and these $\mathcal{I}$ correspond to Γ-subcomodules $I \subseteq A$, i.e. invariant ideals. In this situation, the diagram

$$\begin{array}{ccc}
\mathrm{Spec}\,(\Gamma/I\Gamma) & \longrightarrow & \mathrm{Spec}\,(\Gamma) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathrm{Spec}\,(A/I) & \longrightarrow & \mathrm{Spec}\,(A) \\
\downarrow & & \downarrow \\
\mathfrak{Z} & \longrightarrow & \mathfrak{X}
\end{array}$$

is cartesian. Note that the Hopf algebroid $(A/I, \Gamma/I\Gamma)$ is induced from (A, Γ) by the map $A \longrightarrow A/I$ because $A/I \otimes_A \Gamma \otimes_A A/I \simeq \Gamma/(\eta_L I + \eta_R I)\Gamma = \Gamma/I\Gamma$ because I is invariant.

If $\mathfrak{U} \overset{i}{\hookrightarrow} \mathfrak{X}$ is a quasi-compact open immersion of stacks then the stack $\mathfrak{U}$ is algebraic as one easily checks. In general, an open substack of an algebraic stack need not be algebraic (c.f. the introduction of section 4).

We conclude this subsection by giving a finiteness result for quasi-coherent sheaves of modules. Let

$\mathfrak{X}$ be an algebraic stack. We say that $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ if *finitely generated* if there is a presentation $P : X_0 = \mathrm{Spec}\,(A) \longrightarrow \mathfrak{X}$ such that the A-module corresponding to $P^*\mathcal{F}$ is finitely generated. If $\mathcal{F}$ is finitely generated then for any presentation $P : X_0^{'} = \mathrm{Spec}\,(A^{'}) \longrightarrow \mathfrak{X}$ the $A^{'}$-module corresponding to $P^{'*}\mathcal{F}$ is finitely generated as one sees using [Bou], I, §3, proposition 11.

Proposition 3. *Let (A, Γ) be a flat Hopf algebroid, M a Γ-comodule and $M^{'} \subseteq M$ a finitely generated A-submodule. Then $M^{'}$ is contained in a Γ-subcomodule of M which is finitely generated as an A-module.*

Proof. [W], proposition 5.7. □

Note that in this result, "finitely generated" cannot be strengthened to "coherent" as is shown by the example of the simple BP_*-comodule $BP_*/(v_0, \ldots)$ which is not coherent as a BP_*-module.

Proposition 4. *Let $\mathfrak{X}$ be an algebraic stack. Then any $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ is the filtering union of its finitely generated quasi-coherent subsheaves.*

Proof. Choose a presentation of $\mathfrak{X}$ and apply proposition 3 to the resulting flat Hopf algebroid. □

Compare to [LM-B], 15.4.

3. Properties of morphisms

In this section we relate properties of 1-morphisms (f_0, f_1) of flat Hopf algebroids to properties of the induced morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of algebraic stacks and the adjoint pair $(f^*, f_*) : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ of functors.

3.1 The epi/monic factorisation

By a *flat sheaf* we will mean a set valued sheaf on the site Aff. The topology of Aff is subcanonical, i.e. every representable presheaf is a sheaf. We can thus identify the category underlying Aff with a full subcategory of the category of flat sheaves.
Every 1-morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of stacks factors canonically $\mathfrak{X} \longrightarrow \mathfrak{X}^{'} \longrightarrow \mathfrak{Y}$ into an epimorphism followed by a monomorphism, [LM-B], 3.7. The stack $\mathfrak{X}^{'}$ is determined up to unique 1-isomorphism and is called the image of f.
For a 1-morphism $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ of flat Hopf algebroids we denote

(1) $$\alpha := t\pi_2 : \; X_0 \underset{f_0, Y_0, s}{\times} Y_1 \longrightarrow Y_0 \text{ and}$$

$$\beta := (s, f_1, t) : X_1 \longrightarrow \; X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0 \, .$$

The 1-morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ induced by (f_0, f_1) on algebraic stacks is an epimorphism if and only if α is an epimorphism of flat sheaves; this is clear from the definition of epimorphism of stacks, [LM-B], 3.6; f is a monomorphism if and only if β is an isomorphism, as is easily checked.

Writing $X_1' := X_0 \underset{f_0,Y_0,s}{\times} Y_1 \underset{t,Y_0,f_0}{\times} X_0$, (f_0, f_1) factors as

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1' := \beta} & X_1' & \xrightarrow{\pi_2} & Y_1 \\ \downdownarrows & & \pi_1 \downdownarrows \pi_3 & & \downdownarrows \\ X_0 & \xrightarrow{f_0' := \mathrm{id}_{X_0}} & X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

and the factorisation of f induced by this is the epi/monic factorisation. Note that even when (X_0, X_1) and (Y_0, Y_1) are flat Hopf algebroids, (X_0, X_1') does not have to be flat.

3.2 Flatness and isomorphisms

The proof of the next result will be given at the end of this subsection. The equivalence of $ii)$ and $iii)$ is equivalent to theorem 6.2 of [HS] but we will obtain refinements of it below (proposition 11 and proposition 12).

Theorem 5. *Let $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ be a 1-morphism of flat Hopf algebroids with associated morphisms α and β as in (1) and inducing $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ on algebraic stacks. Then the following are equivalent:*
i) f is a 1-isomorphism of stacks.
ii) $f^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ is an equivalence.
iii) α is faithfully flat and β is an isomorphism.

Remark 6. *This result shows that the weak equivalences of [H], 1.1.4 are exactly those 1-morphisms of flat Hopf algebroids which induce 1-isomorphisms on the associated algebraic stacks.*
It is possible for $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ and $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ to be equivalent without $\mathfrak{X}$ and $\mathfrak{Y}$ being isomorphic which answers conjecture 6.3 of [HS] to the negative. In [R], 1.4. one finds two non-isomorphic finite groups G_1 and G_2 with identical character table. This implies that their categories of representations over $\mathbb{C}$ are equivalent as abelian *categories. The constant groups schemes over $\mathbb{C}$ defined by G_1 and G_2 define algebraic stacks $\mathfrak{X}_1$ and $\mathfrak{X}_2$ (their classifying stacks, [LM-B] 2.4.2). The category of representations over $\mathbb{C}$ of G_i is equivalent to $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_i})$, hence one gets the desired example because $\mathfrak{X}_1 \not\simeq \mathfrak{X}_2$ by Tannaka theory ([D]).*
Even though this invalidates the above mentioned conjecture as it is stated, it leaves room for some speculation: The example of [R] was meant to illustrate that in Tannakian theory one cannot ignore the additional structure on the representation categories furnished by the tensor product. Taking this structure into account, Tannakian theory may be subsumed by saying that there is an equivalence of 2-categories between a category of (rather special) algebraic stacks (namely gerbes bound by affine group schemes over fields of characteristic zero) and the category of Tannakian categories. Even though this has been generalised lately ([W]) there is no such result for algebraic stacks as general as those considered here. It is still conceivable that partial results may be interesting:
Using the notation from section 4, the classification of hereditary torsion theories of BP_*-comodules ([HS], theorem B) amounts to saying that the hereditary torsion theories inside $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ are exactly given as $\ker(j_n^*)$ for the open immersions $j_n : \mathfrak{U}^n \hookrightarrow \mathfrak{X}_{FG}$ $(0 \leqslant n < \infty)$. It is easy to see that the $\mathfrak{U}^n$ exhaust all *quasi-compact open substacks of $\mathfrak{X}_{FG}$ and a suitable "Tannakian" correspondence (between quasi-compact open substacks and hereditary torsion theories) would allow to recover [HS], theorem B from this simple geometric fact. See [Lu], theorem 5.11 for a result in this direction.*

We next give two results about the flatness of morphisms.

PROPOSITION 7. *Let* $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ *be a* 1*-morphism of flat Hopf algebroids,* $P : X_0 \longrightarrow \mathfrak{X}$ *and* $Q : Y_0 \longrightarrow \mathfrak{Y}$ *the associated rigidified algebraic stacks and* $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ *the induced* 1*-morphism of algebraic stacks. Then the following are equivalent:*
i) f *is (faithfully) flat.*
ii) $f^* : \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}}) \longrightarrow \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ *is exact (and faithful).*
iii) $\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \longrightarrow Y_0$ *is (faithfully) flat.*
iv) the composition $X_0 \xrightarrow{P} \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ *is (faithfully) flat.*

Proof. The equivalence of i) and ii) is by definition, the one of i) and iv) is because P is $fpqc$ and being (faithfully) flat is a local property for the *fpqc* topology. Abbreviating $Z := X_0 \underset{f_0, Y_0, s}{\times} Y_1$ we have a cartesian diagram

$$\begin{array}{ccccc} Z & & \xrightarrow{\quad\alpha\quad} & & Y_0 \\ \downarrow{\scriptstyle \pi_1} & & \overset{f_0}{\nearrow} & & \downarrow{\scriptstyle Q} \\ X_0 & \xrightarrow{P} & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

which, as Q is $fpqc$, shows that iv) and iii) are equivalent. We check that this diagram is in fact cartesian by computing:

$$X_0 \underset{fP, \mathfrak{Y}, Q}{\times} Y_0 = X_0 \underset{Qf_0, \mathfrak{Y}, Q}{\times} Y_0 \simeq$$

$$\simeq X_0 \underset{f_0, Y_0, \mathrm{id}}{\times} Y_0 \underset{Q, \mathfrak{Y}, Q}{\times} Y_0 \simeq X_0 \underset{f_0, Y_0, s}{\times} Y_1 = Z,$$

and under this isomorphism the projection onto the second factor corresponds to α. □

PROPOSITION 8. *Let* (Y_0, Y_1) *be a flat Hopf algebroid,* $f_0 : X_0 \longrightarrow Y_0$ *a morphism in* Aff *and* $(f_0, f_1) : (X_0, X_1 := X_0 \underset{f_0, Y_0, s}{\times} Y_1 \underset{t, Y_0, f_0}{\times} X_0) \longrightarrow (Y_0, Y_1)$ *the* 1*-morphism of Hopf algebroids from the induced Hopf algebroid and* $Q : Y_0 \longrightarrow \mathfrak{Y}$ *the rigidified algebraic stack associated to* (Y_0, Y_1)*. Then the following are equivalent:*
i) the composition $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{Q} \mathfrak{Y}$ *is (faithfully) flat.*
ii) $\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \longrightarrow Y_0$ *is (faithfully) flat.*
If either of this maps is flat, then (X_0, X_1) *is a* flat *Hopf algebroid.*

The last assertion of this proposition does not admit a converse: For $(Y_0, Y_1) = (\mathrm{Spec}\,(BP_*), \mathrm{Spec}\,(BP_*BP))$ and $X_0 := \mathrm{Spec}\,(BP_*/I_n) \longrightarrow Y_0$, the induced Hopf algebroid is flat but $X_0 \longrightarrow \mathfrak{Y}$ is not (c.f. section 4).

Proof. The proof of the equivalence of i) and ii) is the same as in proposition 7, using that Q is $fpqc$. Again denoting $Z := X_0 \underset{f_0, Y_0, s}{\times} Y_1$ one checks that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Y_0 \\ \uparrow & & \uparrow{\scriptstyle f_0} \\ X_1 & \xrightarrow{t} & X_0 \end{array}$$

is cartesian, hence the final assertion of the proposition follows because flatness is stable under base change. □

We will need the next result in section 4.

PROPOSITION 9. *Let* (Y_0, Y_1) *be a flat Hopf algebroid,* $f_0 : X_0 \longrightarrow Y_0$ *a morphism in* Aff *such that the composition* $X_0 \xrightarrow{f_0} Y_0 \xrightarrow{Q} \mathfrak{Y}$ *is faithfully flat, where* $Q : Y_0 \longrightarrow \mathfrak{Y}$ *is the rigidified algebraic stack associated to* (Y_0, Y_1)*. Let* $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ *be the canonical* 1*-morphism with* (X_0, X_1) *the Hopf algebroid induced from* (Y_0, Y_1) *by* f_0*. Then* (X_0, X_1) *is a flat Hopf algebroid and* (f_0, f_1) *induces a* 1*-isomorphism on the associated algebraic stacks.*

Proof. The 1-morphism f induced on the associated algebraic stacks is a monomorphism by construction. Proposition 8 shows that (X_0, X_1) is a flat Hopf algebroid and that f is an epimorphism, hence a 1-isomorphism by [LM-B], 3.7.1. □

We now start to take the module categories into consideration.
Given $f : X \longrightarrow Y$ in Aff we have an adjunction $\psi_f : \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_Y)} \longrightarrow f_* f^*$.
We recognise the epimorphisms of representable flat sheaves as follows.

PROPOSITION 10. *Let* $f : X \longrightarrow Y$ *be a morphism in* Aff*. Then the following are equivalent:*
i) f *is an epimorphism of flat sheaves.*
ii) There is some $\phi : Z \longrightarrow X$ *in* Aff *such that* $f\phi$ *is faithfully flat.*
If i) and ii) hold, then ψ_f *is injective.*
If f *is flat, the conditions i) and ii) are equivalent to* f *being faithfully flat.*

For an example of such an f which is not flat, c.f. [Bou], I, §3, ex. 5.

Proof. That i) implies ii) is seen by lifting $\mathrm{id}_Y \in Y(Y)$ after a suitable faithfully flat cover $Z \longrightarrow Y$ to some $\phi \in X(Z)$.
To see that ii) implies i), fix some $U \in \mathrm{Aff}$ and $u \in Y(U)$ and form the cartesian diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\phi} & X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle v} & & & & \uparrow{\scriptstyle u} \\ W & & \longrightarrow & & U. \end{array}$$

Then $W \longrightarrow U$ is faithfully flat and u lifts to $v \in Z(W)$ and hence to $\phi v \in X(W)$.
To see the assertion about flat f, note first that a faithfully flat map is trivially an epimorphism of flat sheaves. Secondly, if f is flat and an epimorphism of flat sheaves, then there is some $\phi : Z \longrightarrow X$ as in ii) and the composition $f\phi$ is surjective (on the topological spaces underlying these affine schemes), hence so is f, i.e. f is faithfully flat. The injectivity of ψ_f is a special case of [Bou], I, §3, proposition 8 i). □

We have a similar result for epimorphisms of algebraic stacks.

PROPOSITION 11. *Let* $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ *be a* 1*-morphism of flat Hopf algebroids inducing* $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ *on associated algebraic stacks and write* $\alpha := t\pi_2 : X_0 \underset{f_0, Y_0, s}{\times} Y_1 \longrightarrow Y_0$*. Then the following are equivalent:*
i) f *is an epimorphism.*
ii) α *is an epimorphism of flat sheaves.*
iii) There is some $\phi : Z \longrightarrow X_0 \underset{f_0, Y_0, s}{\times} Y_1$ *in* Aff *such that* $\alpha\phi$ *is faithfully flat.*
If these conditions hold then $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \longrightarrow f_* f^*$ *is injective.*

Proof. The equivalence of i) and ii) is "mise pour memoire", the one of ii) and iii) has been proved in proposition 10. Assume that these conditions hold and let $g : \mathfrak{X}' \longrightarrow \mathfrak{X}$ be any morphism of algebraic stacks. Assume that $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \hookrightarrow (fg)_*(fg)^*$. Then we have that the composition $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \longrightarrow f_*f^* \longrightarrow f_*g_*g^*f^* = (fg)_*(fg)^*$ is injective and hence so is $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \longrightarrow f_*f^*$. Taking $g := P : X_0 \longrightarrow \mathfrak{X}$ the canonical presentation we see that we can assume that $\mathfrak{X} = X_0$, in particular $f : X_0 \longrightarrow \mathfrak{Y}$ is representable and affine (and an epimorphism). Now let $Q : Y_0 \longrightarrow \mathfrak{Y}$ be the canonical presentation and form the cartesian diagram

(2)
$$\begin{array}{ccc} Z_0 & \xrightarrow{g_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ X_0 & \xrightarrow{f} & \mathfrak{Y}. \end{array}$$

As Q is $fpqc$ we have $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})} \hookrightarrow f_*f^*$ if and only if $Q^* \hookrightarrow Q^*f_*f^* \simeq g_{0,*}P^*f^* \simeq g_{0,*}g_0^*Q^*$ (we used flat base change, [LM-B] 13.1.9) and this will follow from $\mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Y_0})} \hookrightarrow g_{0,*}g_0^*$ because Q is flat.
As f is representable and affine, Z_0 is an affine scheme hence, by proposition 10, we are done because g_0 is an epimorphism of flat sheaves, [LM-B], 3.8.1. □

There is an analogous result for monomorphisms of algebraic stacks.

Proposition 12. *Let $(f_0, f_1) : (X_0, X_1) \longrightarrow (Y_0, Y_1)$ be a 1-morphism of flat Hopf algebroids, $P : X_0 \longrightarrow \mathfrak{X}$ the rigidified algebraic stack associated to (X_0, X_1), $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ the associated 1-morphism of algebraic stacks, $\Theta : f^*f_* \longrightarrow \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})}$ the adjunction and $\beta = (s, f_1, t) : X_1 \longrightarrow X_0 \underset{f_0,Y_0,s}{\times} Y_1 \underset{t,Y_0,f_0}{\times} X_0$. Then the following are equivalent:*
i) f is a monomorphism.
ii) β is an isomorphism.
iii) $\Theta_{P_*\mathcal{O}_{X_0}}$ is an isomorphism.
If f is representable then these conditions are equivalent to:
iiia) Θ is an isomorphism.
iiib) f_* is fully faithful.

Remark 13. *This result may be compared to the first assertion of theorem 2.5 of [HS]. There it is proved that Θ is an isomorphism if f is a* flat *monomorphism.*
We will determine the essential image of f_* below.
I do not know whether every monomorphism of algebraic stacks is representable, c.f. [LM-B], 8.1.3.

Proof. We already know that *i)* and *ii)* are equivalent. Consider the diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{P} & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ {\scriptstyle \Delta'}\downarrow\uparrow{\scriptstyle \pi_1'} & & {\scriptstyle \Delta_f}\downarrow\uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle f} \\ \pi : \mathfrak{Z} & \xrightarrow[P']{} & \mathfrak{X} \underset{f,\mathfrak{Y},f}{\times} \mathfrak{X} & \xrightarrow[\pi_2]{} & \mathfrak{X} \end{array}$$

in which the squares made of straight arrows are cartesian. As fP is representable and affine, we have $fP = \underline{\mathrm{Spec}}(f_*P_*\mathcal{O}_{X_0})$ (c.f. [LM-B], 14.2) and $\pi = \underline{\mathrm{Spec}}(f^*f_*P_*\mathcal{O}_{X_0})$. We know that *i)* is equivalent to the diagonal of f, Δ_f, being an isomorphism [LM-B], 2.3.1. As Δ_f is a section of π_1 this is equivalent to π_1 being an isomorphism. As P is an epimorphism, this is equivalent to π_1' being an isomorphism by [LM-B], 3.8.1. Of course, π_1' admits $\Delta' := (\mathrm{id}_{X_0}, \Delta_f P)$ as a section so,

finally, i) is equivalent to Δ' being an isomorphism. One checks that $\Delta' = \underline{\mathrm{Spec}}\,(\Theta_{P_*\mathcal{O}_{X_0}})$ and this proves the equivalence of i) and iii).
Now assume that f is representable and a monomorphism. We will show that $iiia$) holds. Consider the cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array}$$

We have

$$P^* f^* f_* \simeq f'^* Q^* f_* \simeq f'^* f'_* P^*.$$

As P^* reflects isomorphism, $iiia$) will hold if the adjunction $f'^* f'_* \longrightarrow \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_Z)}$ is an isomorphism. As f is representable, this can be checked at the stalks of $z \in Z$, and we can replace f' by the induced morphism $\mathrm{Spec}\,(\mathcal{O}_{Z,z}) \longrightarrow \mathrm{Spec}\,(\mathcal{O}_{Y_0,y})$ $(y := f'(z))$ which is a monomorphism. In particular, we have reduced the proof of $iiia$) to the case of affine schemes, i.e. the following assertion: If $\phi : A \longrightarrow B$ is a ring homomorphism such that $\mathrm{Spec}\,(\phi)$ is a monomorphism (i.e. the ring homomorphism corresponding to the diagonal $B \otimes_A B \longrightarrow B$, $b_1 \otimes b_2 \mapsto b_1 b_2$ is an isomorphism) then, for any B-module M, the canonical homomorphism of B-modules $M \otimes_A B \longrightarrow M$ is an isomorphism. This is however easy:

$$M \otimes_A B \simeq (M \otimes_B B) \otimes_A B \simeq M \otimes_B (B \otimes_A B) \simeq M \otimes_B B \simeq M,$$

and we leave it to the reader to check that the composition of these isomorphisms is the natural map $M \otimes_A B \longrightarrow M$.
Finally, the proof that $iiia$) and $iiib$) are equivalent is a formal manipulation with adjunctions which we leave to the reader, and trivially $iiia$) implies iii). □

We promised to identify the essential image of f_*.

Proposition 14. *In the situation of proposition 12 assume that f is representable and a monomorphism, let $Q : Y_0 \longrightarrow \mathfrak{Y}$ be the rigidified algebraic stack associated to (Y_0, Y_1) and form the cartesian diagram*

(3)
$$\begin{array}{ccc} Z_0 & \xrightarrow{g_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array}$$

Then Z_0 is an algebraic space and a given $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ is in the essential image of f_* if and only if $Q^*\mathcal{F}$ is in the essential image of $g_{0,*}$. Consequently, f_* induces an equivalence between $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ and the full subcategory of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ consisting of such $\mathcal{F}$.

Proof. Z_0 is an algebraic space because f is representable. We know that f_* is fully faithful by proposition 12, $iiib$) and need to show that the above description of its essential image is correct. If $\mathcal{F} \simeq f_*\mathcal{G}$ then $Q^*\mathcal{F} \simeq Q^* f_* \mathcal{G} \simeq g_{0,*} P^* \mathcal{G}$ so $Q^*\mathcal{F}$ lies in the essential image of $g_{0,*}$. To see the converse, extend (3) to a cartesian diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{g_1} & Y_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ Z_0 & \xrightarrow{g_0} & Y_0 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array}$$

Note that $\mathfrak{X} \simeq [\, Z_1 \rightrightarrows Z_0 \,]$, hence (Z_0, Z_1) is a flat groupoid (in algebraic spaces) representing $\mathfrak{X}$. Now let there be given $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Y}})$ and $G \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Z_0})$ with $Q^*\mathcal{F} \simeq g_{0,*}G$. We define σ to make the following diagram commutative:

$$\begin{array}{ccc} s^*Q^*\mathcal{F} & \xrightarrow[\sim]{can} & t^*Q^*\mathcal{F} \\ \sim\downarrow & & \sim\downarrow \\ s^*g_{0,*}G & & t^*g_{0,*}G \\ \sim\downarrow & & \sim\downarrow \\ g_{1,*}s^*G & \xrightarrow[\sigma]{\sim} & g_{1,*}t^*G. \end{array}$$

As f is representable and a monomorphism, so is g_1 and thus $g_1^* g_{1,*} \xrightarrow{\sim} \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{Z_1})}$ and $g_{1,*}$ is fully faithful by proposition 12,*iiia*), *iiib*). We define τ to make the following diagram commutative:

$$\begin{array}{ccc} g_1^*g_{1,*}s^*G & \xrightarrow[\sim]{g_1^*(\sigma)} & g_1^*g_{1,*}t^*G \\ \sim\downarrow & & \sim\downarrow \\ s^*G & \xrightarrow{\tau} & t^*G. \end{array}$$

Then τ satisfies the cocycle condition because it does so after applying the faithful functor $g_{1,*}$. So τ is a descent datum on G, and G descents to $\mathcal{G} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})$ with $P^*\mathcal{G} \simeq G$ and we have $Q^*f_*\mathcal{G} \simeq g_{0,*}P^*\mathcal{G} \simeq Q^*\mathcal{F}$, hence $f_*\mathcal{G} \simeq \mathcal{F}$, i.e. $\mathcal{F}$ lies in the essential image of f_* as was to be shown. □

To conclude this subsection we give the proof of theorem 5 the notations and assumptions of which we now resume.

Proof. If iii) holds then f is an epimorphism and a monomorphism (by proposition 11, *iii*) $\Rightarrow$ *i*) and proposition 12, *ii*) $\Rightarrow$ *i*)) hence i) holds by [LM-B], 3.7.1. The proof that i) implies ii) is left to the reader and we assume that ii) holds. Since (f^*, f_*) is an adjoint pair of functors, f_* is a quasi-inverse for f^* and $\Theta : f^*f_* \longrightarrow \mathrm{id}_{\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}})}$ is an isomorphism so β is an isomorphism by 12, *iii*) $\Rightarrow$ *ii*). As f^* is in particular exact and faithful, α is faithfully flat by proposition 7, *ii*) $\Rightarrow$ *iii*) and iii) holds. □

4. Landweber exactness and change of rings

Let p be a prime number. In this section we will consider the algebraic stack associated to the flat Hopf algebroid (BP_*, BP_*BP) where BP denotes Brown-Peterson homology at p.
We will work over $S := \mathrm{Spec}\,(\mathbb{Z}_{(p)})$, i.e. Aff will be the category of $\mathbb{Z}_{(p)}$-algebras with its *fpqc* topology. We refer the reader to [R1], Chapter 4 for basic facts about BP, e.g. $BP_* = \mathbb{Z}_{(p)}[v_1, \ldots]$ where the v_i denote either the Hazewinkel- or the Araki-generators, it does not matter but the reader is free to make a definite choice at this point if she feels like doing so.
$(V := \mathrm{Spec}\,(BP_*), W := \mathrm{Spec}\,(BP_*BP))$ is a flat Hopf algebroid and we denote by $P : V \longrightarrow \mathfrak{X}_{FG}$ the corresponding rigidified algebraic stack. We point out that it is not a priori clear what $\mathfrak{X}_{FG}$ is: For $U = \mathrm{Spec}\,(R) \in$ Aff, $\mathfrak{X}_U$ should of course be the groupoid of one dimensional, commutative formal groups (not group laws) over the $\mathbb{Z}_{(p)}$-algebra R and checking this amounts to understanding *fpqc* descent for formal groups. More than enough material to do this should be contained in [S] but we do not claim to have checked the details. Of course, we will not use the above description of $\mathfrak{X}_{FG}$ but always consider it as the stack associated to (V, W).
For $n \geqslant 1$ the ideal $I_n := (v_0, \ldots, v_{n-1}) \subseteq BP_*$ is an invariant prime ideal where we agree that $v_0 := p$, $I_0 := (0)$ and $I_\infty := (v_0, v_1, \ldots)$.
As explained in subsection 2.4, corresponding to these invariant ideals there is a sequence of closed substacks

$$\mathfrak{X}_{FG} = \mathfrak{Z}^0 \supseteq \mathfrak{Z}^1 \supseteq \ldots \supseteq \mathfrak{Z}^\infty.$$

The stack $\mathfrak{Z}^n$ should be the stack of formal groups all of whose geometric fibres have height at least n. We denote by $\mathfrak{U}^n := \mathfrak{X}_{FG} - \mathfrak{Z}^n$ $(0 \leqslant n \leqslant \infty)$ the open substack complementary to $\mathfrak{Z}^n$ and have an ascending chain

$$\emptyset = \mathfrak{U}^0 \subseteq \mathfrak{U}^1 \subseteq \ldots \subseteq \mathfrak{U}^\infty \subseteq \mathfrak{X}_{FG}.$$

For $0 \leqslant n < \infty$, I_n if finitely generated, hence the open immersion $\mathfrak{U}^n \subseteq \mathfrak{X}_{FG}$ is quasi-compact and $\mathfrak{U}^n$ is an algebraic stack. However, $\mathfrak{U}^\infty$ is not algebraic: If it was, it could be covered by an affine (hence quasi-compact) scheme and the open covering $\mathfrak{U}^\infty = \cup_{n \geqslant 0, n \neq \infty} \mathfrak{U}^n$ would allow a finite subcover, which it does not.

4.1 Flatness and the Landweber condition

Fix some $0 \leqslant n < \infty$. The stack $\mathfrak{Z}^n$ is associated to the flat Hopf algebroid (V_n, W_n) where $V_n := \mathrm{Spec}\,(BP_*/I_n)$ and $W_n := \mathrm{Spec}\,(BP_*BP/I_nBP_*BP)$ (the flatness of this Hopf algebroid is established by direct inspection) and we have a cartesian diagram

(4)
$$\begin{array}{ccc}
W_n & \hookrightarrow & W = W_0 \\
\downarrow\downarrow & & \downarrow\downarrow \\
V_n & \overset{i_n}{\hookrightarrow} & V = V_0 \\
{\scriptstyle Q_n}\downarrow & & \downarrow{\scriptstyle Q} \\
\mathfrak{Z}^n & \hookrightarrow & \mathfrak{X}_{FG}
\end{array}$$

in which the horizontal arrows are closed immersions.
We have an ascending chain of open substacks

$$\emptyset = \mathfrak{Z}^n \cap \mathfrak{U}^n \subseteq \mathfrak{Z}^n \cap \mathfrak{U}^{n+1} \subseteq \ldots \subseteq \mathfrak{Z}^n \cap \mathfrak{U}^\infty \subseteq \mathfrak{Z}^n$$

and $\mathfrak{Z}^n \cap \mathfrak{U}^{n+1}$ should be the stack of formal groups all of whose geometric fibres have height exactly n.
Let $X_0 \xrightarrow{\phi} V_n$ be a morphism in Aff corresponding to $BP_*/I_n \longrightarrow R := \Gamma(X_0, \mathcal{O}_{X_0})$. Slightly generalising definition 4.1 of [HS] we define the height of ϕ as

$$\mathrm{ht}(\phi) := \max\{N \geqslant 0 | R/I_N R \neq 0\}$$

which may be ∞ and we agree to put $\mathrm{ht}(\phi) := -1$ in case $R = 0$, i.e. $X_0 = \emptyset$. Recall that a geometric point of X_0 is a morphism $\Omega \xrightarrow{\alpha} X_0$ in Aff where $\Omega = \mathrm{Spec}\,(K)$ is the spectrum of an algebraically closed field K. The composition $\Omega \xrightarrow{\alpha} X_0 \xrightarrow{\phi} V_n \overset{i_n}{\hookrightarrow} V$ specifies a p-typical formal group law over K and $\mathrm{ht}(i_n \phi \alpha)$ is the height of this formal group law. The relation between $\mathrm{ht}(\phi)$ and the height of formal group laws is the following.

Proposition 15. *In the above situation we have*

$$\mathrm{ht}(\phi) = \max\{\mathrm{ht}(i_n \alpha \phi) | \alpha : \Omega \longrightarrow X_0 \textit{ a geometric point}\},$$

with the convention that $\max \emptyset = -1$.

This proposition means that $\mathrm{ht}(\phi)$ is the maximum height in a geometric fibre of the formal group law over X_0 parametrised by $i_n \phi$.

Proof. Clearly, $\mathrm{ht}(\psi\phi) \leqslant \mathrm{ht}(\phi)$ for any morphism $\psi : Y \longrightarrow X_0$ in Aff. For any $0 \leqslant N' \leqslant \mathrm{ht}(\phi)$ we have that $I_{N'} R \neq R$ so there is a maximal ideal of R containing $I_{N'} R$ and a geometric point α of X_0 supported at this maximal ideal will satisfy $\mathrm{ht}(i_n \alpha \phi) \geqslant N'$. □

Another geometric interpretation of $\mathrm{ht}(\phi)$ is given by considering the composition $f : X_0 \xrightarrow{\phi} V_n \xrightarrow{Q_n} \mathfrak{Z}^n$.

Proposition 16. *In this situation we have*

$$\mathrm{ht}(\phi) + 1 = \min\{N \geqslant 0 | f \textit{ factors through } \mathfrak{Z}^n \cap \mathfrak{U}^N \hookrightarrow \mathfrak{Z}^n\}$$

with the convention that $\min \emptyset = \infty$ *and* $\infty + 1 = \infty$.

Proof. For any $\infty > N \geqslant n$ we have a cartesian square

(5)
$$\begin{array}{ccc} V_n^N & \xrightarrow{j} & V_n \\ \downarrow & & \downarrow Q_n \\ \mathfrak{Z}^n \cap \mathfrak{U}^N & \xrightarrow{i} & \mathfrak{Z}^n \end{array}$$

where $V_n^N = V_n - \mathrm{Spec}\,(BP_*/I_N) = \bigcup_{i=n}^{N-1} \mathrm{Spec}\,((BP_*/I_n)[v_i^{-1}])$ hence f factors through i if and only if $\phi : X_0 \longrightarrow V_n$ factors through j. As j is an open immersion, this is equivalent to $|\phi|(|X_0|) \subseteq |V_n^N| \subseteq |V_n|$ where $|\cdot|$ denotes the topological space underlying a scheme. But this condition can be checked using geometric points and the rest is easy, using proposition 15. □

Recall from [HS], 2.1 that, if (A, Γ) is a flat Hopf algebroid, an A-algebra $f : A \longrightarrow B$ is said to be *Landweber exact* over (A, Γ) if the functor $M \mapsto M \otimes_A B$ from Γ-comodules to B-modules is exact. For $(X_0 := \mathrm{Spec}\,(A), X_1 := \mathrm{Spec}\,(\Gamma))$, $\phi := \mathrm{Spec}\,(f) : Y_0 := \mathrm{Spec}\,(B) \longrightarrow X_0$ and $P : X_0 \longrightarrow \mathfrak{X}$ the rigidified algebraic stack associated to (X_0, X_1) this exactness is equivalent to the flatness of the composition $Y_0 \xrightarrow{\phi} X_0 \xrightarrow{P} \mathfrak{X}$ (see also proposition 8). In case $\mathfrak{X} = \mathfrak{Z}^n$ this flatness has the following decisive consequence which paraphrases the fact that the image of a flat morphism is stable under generalisation.

PROPOSITION 17. *Assume that* $n \geqslant 0$ *and that* $\phi : \emptyset \neq X_0 \longrightarrow V_n$ *is Landweber exact of height* $N := \mathrm{ht}(\phi)$ *(hence* $n \leqslant N \leqslant \infty$*). Then for any* $n \leqslant j \leqslant N$ *there is a geometric point* $\alpha : \Omega \longrightarrow X_0$ *such that* $\mathrm{ht}(i_n \phi \alpha) = j$*.*

Proof. Let ϕ correspond to $BP_*/I_n \longrightarrow R$. We first note that $v_n, v_{n+1}, \ldots \in R$ is a regular sequence: Assume to the contrary that there is some $i \geqslant n$ such that $K := \ker(R/I_{i-1}R \xrightarrow{\cdot v_i} R/I_{i-1}R) \neq 0$. We have an injective homomorphism $BP_*/I_{i-1} \overset{\cdot v_i}{\hookrightarrow} BP_*/I_{i-1}$ of $(BP_*/I_n, BP_*BP/I_n)$-comodules which by flatness (i.e. Landweber exactness) pulls back to give the contradiction $K = 0$.
Now fix $n \leqslant j \leqslant N$. Then $v_j \in R/I_{j-1}R \neq 0$ is not a zero divisor and thus there is a minimal prime ideal of $R/I_{j-1}R$ not containing v_j. A geometric point supported at this prime ideal solves the problem. □

The main result of this subsection is the following.

THEOREM 18. *Assume that* $n \geqslant 0$ *and that* $\emptyset \neq X_0 \longrightarrow V_n$ *is Landweber exact of height* N *(hence* $n \leqslant N \leqslant \infty$*). Let* (X_0, X_1) *be the Hopf algebroid induced from* (V, W) *along the composition* $X_0 \xrightarrow{\phi} V_n \overset{i_n}{\hookrightarrow} V$*. Then* (X_0, X_1) *is a flat Hopf algebroid and its associated algebraic stack is given as*

$$[\, X_1 \rightrightarrows X_0 \,] \simeq \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \text{ if } N \neq \infty \text{ and}$$

$$[\, X_1 \rightrightarrows X_0 \,] \simeq \mathfrak{Z}^n \text{ if } N = \infty.$$

Proof. Note that (X_0, X_1) is also induced from the flat Hopf algebroid (V_n, W_n) along ϕ and thus is flat using the final statement of proposition 8 and the Landweber exactness of ϕ. We first assume that $N \neq \infty$. Then by proposition 16 the composition $X_0 \xrightarrow{\phi} V_n \longrightarrow \mathfrak{Z}^n$ factors as $X_0 \xrightarrow{\psi} \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \xrightarrow{i} \mathfrak{Z}^n$ and ψ is flat because i is an open immersion and $X_0 \longrightarrow \mathfrak{Z}^n$ is flat by assumption. By proposition 9 we will be done if we can show that ψ is in fact faithfully flat. For this we consider the presentation $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \simeq [\, W_n^{N+1} \rightrightarrows V_n^{N+1} \,]$ given by the cartesian diagram

$$\begin{array}{ccc} W_n^{N+1} & \longrightarrow & W_n \\ \downdownarrows & & \downdownarrows \\ V_n^{N+1} & \longrightarrow & V_n \\ \downarrow & & \downarrow {\scriptstyle Q_n} \\ \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} & \longrightarrow & \mathfrak{Z}^n \end{array}$$

and note that ψ lifts to $\rho : X_0 \longrightarrow V_n^{N+1}$ and induces $\alpha := t\pi_2 : \; X_0 \underset{\rho, V_n^{N+1}, s}{\times} W_n^{N+1} \longrightarrow V_n^{N+1}$ which is flat and we need it to be faithfully flat (to apply proposition 7, *iii*) $\Rightarrow$ *iv*) and conclude that ψ is faithfully flat), i.e. surjective (on the topological spaces underlying the schemes involved).
This surjectivity can be checked on geometric points and for any such geometric point $\Omega \xrightarrow{\mu} V_n^{N+1}$ we have that $j := \mathrm{ht}(\Omega \xrightarrow{\mu} V_n^{N+1} \longrightarrow V_n)$ satisfies $n \leqslant j \leqslant N$. By proposition 17 there is a geometric point $\Omega' \xrightarrow{\nu} X_0$ with $\mathrm{ht}(\Omega' \xrightarrow{\nu} X_0 \longrightarrow V_n) = j$ and we can assume that $\Omega = \Omega'$ because the corresponding fields have the same characteristic (namely 0 if $j = 0$ and p otherwise). As any

two formal group laws over an algebraically closed field having the same height are isomorphic we find some $\sigma : \Omega \longrightarrow W_n^{N+1}$ fitting into a commutative diagram

$$\begin{array}{ccc} X_0 \times_{\rho, V_n^{N+1}, s} W_n^{N+1} & \xrightarrow{\alpha} & V_n^{N+1} \\ \uparrow{\scriptstyle (\nu,\sigma)} & \nearrow{\scriptstyle \mu} & \\ \Omega. & & \end{array}$$

As μ was arbitrary this shows that α is surjective. We leave the obvious modifications for the case $N = \infty$ to the reader. □

Remark 19. *We will see in the next subsection that this result implies many of the recent results of M. Hovey and N. Strickland ([HS]) so it may be worthwhile to point out that we have not used any of the fundamental results of P. Landweber ([L]) except for the invariance of the $I_n \subseteq BP_*$. Besides the language of stacks we only used basic facts about (p-typical) formal group laws, and one may wonder if the results of [L] may be recovered from this point of view.*
We can get the classification of finitely generated radical ideals $I \subseteq BP_*$ as follows: I corresponds to some closed substack $\mathfrak{Z} \subseteq \mathfrak{X}_{FG}$ with complement $\mathfrak{U} \subseteq \mathfrak{X}_{FG}$ which is algebraic because I is finitely generated. Composing a presentation $X_0 \longrightarrow \mathfrak{U}$ with the inclusion $\mathfrak{U} \subseteq \mathfrak{X}_{FG}$ we have a flat 1-morphism $f : X_0 \longrightarrow \mathfrak{X}_{FG}$ hence by theorem 18 (with $n = 0$) the image of f equals some $\mathfrak{U}^n$, so $\mathfrak{U} = \mathfrak{U}^n$. As $\mathfrak{Z}$ is reduced (because I is radical) we conclude $\mathfrak{Z} = \mathfrak{Z}^n$, i.e. $I = I_n$.
The fact that any non-zero BP_*-comodule has a non-zero primitive means that any non-zero quasi-coherent $\mathcal{O}_{\mathfrak{X}_{FG}}$-module $\mathcal{F}$ has $H^0(\mathfrak{X}_{FG}, \mathcal{F}) \neq 0$, certainly a striking result. We believe that this is due to the faithful $\mathbb{G}_m$-action on $\mathfrak{X}_{FG}$ (corresponding to the grading) and it might be interesting to generalise this result from the example $\mathfrak{X}_{FG}$ to general algebraic stacks with $\mathbb{G}_m$-action.
Finally, the result that any BP_*-comodule is the union of its finitely generated subcomodules generalises, c.f. proposition 4.

We conclude this subsection by proving the expected characterisation of Landweber exactness.

Proposition 20. *Let $n \geqslant 0$ and $\phi : X_0 \longrightarrow V_n$ be a morphism in* Aff *corresponding to $BP_*/I_n \longrightarrow R$. Then the following are equivalent:*
i) ϕ is Landweber exact.
ii) The composition $X_0 \xrightarrow{\phi} V_n \longrightarrow \mathfrak{Z}^n$ is flat.
iii) The sequence $v_n, v_{n+1}, \ldots \in R$ is regular.

Proof. The equivalence of i) and ii) was explained in the paragraph preceding proposition 17 and the implication $i) \Rightarrow iii)$ has been established during the proof of proposition 17. The proof that iii) implies ii) is an immediate generalisation of the proof of the exact functor theorem [L]. To show that $\mathrm{Tor}_1^{\mathcal{O}_{\mathfrak{Z}^n}}(\mathcal{O}_{X_0}, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ we can assume that $\mathcal{F}$ is finitely generated by proposition 4. By proposition 21 below, then, $\mathcal{F}$ corresponds to a finitely generated BP_*-comodule M such that $I_n M = 0$. Hence every subquotient of the Landweber filtration of M is (isomorphic to a shift of) BP_*/I_m for some $m \geqslant n$ and the result follows. □

4.2 Equivalence of comodule categories and change of rings

In this subsection we will spell out some consequences of the above results in the language of comodules but need some elementary preliminaries first.
Let A be a ring, $I = (f_1, \ldots, f_n) \subseteq A$ $(n \geqslant 1)$ a finitely generated ideal and M an A-module. We

have a canonical map

$$\bigoplus_i M_{f_i} \longrightarrow \bigoplus_{i<j} M_{f_i f_j},\ (x_i)_i \mapsto \Big(\frac{x_i}{1} - \frac{x_j}{1}\Big)_{i,j}$$

and a canonical map

$$\alpha_M : M \longrightarrow \ker(\bigoplus_i M_{f_i} \longrightarrow \bigoplus_{i<j} M_{f_i f_j}).$$

For $X := \mathrm{Spec}\,(A)$, $Z := \mathrm{Spec}\,(A/I)$, $j : U := X - Z \hookrightarrow X$ the open immersion and $\mathcal{F}$ the quasi-coherent $\mathcal{O}_X$-module corresponding to M, α_M corresponds to the adjunction $\mathcal{F} \longrightarrow j_* j^* \mathcal{F}$. Note that $\ker(\alpha_M)$ is the I-torsion submodule of M. The cokernel of α_M corresponds to the local cohomology $H^1_Z(X, \mathcal{F})$, c.f. [Ha]. We say that M is I-local if α_M is an isomorphism. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is in the essential image of j_* if and only if $\mathcal{F} \longrightarrow j_* j^* \mathcal{F}$ is an isomorphism if and only if the A-module corresponding to $\mathcal{F}$ is I-local. If $n = 1$ then M is $I = (f_1)$-local if and only if f_1 acts invertibly on M.

We now formulate a special case of proposition 14 in terms of comodules.

Proposition 21. *For any $n \geqslant 0$ the category $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ is equivalent to the full subcategory of BP_*-comodules M such that $I_n M = 0$.*
For any $0 \leqslant n \leqslant N < \infty$ the category $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ is equivalent to the full subcategory of BP_*-comodules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a BP_*/I_n-module.

Proof. Fix $0 \leqslant n < \infty$. The 1-morphism $\mathfrak{Z}^n \hookrightarrow \mathfrak{X}_{FG}$ is representable and a closed immersion (in particular a monomorphism) because its base change along $V \longrightarrow \mathfrak{X}_{FG}$ is a closed immersion and being a closed immersion is $fpqc$-local on the base, [EGA IV$_2$], 2.7.1, xii). Proposition 14 identifies $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$ with the full subcategory of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ consisting of those $\mathcal{F}$ such that $Q^*\mathcal{F} \simeq i_{n,*}G$ for some $G \in \mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{V_n})$ (with notations as in (4)). Identifying, as in subsection 2.4, $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ with the category of BP_*-comodules, $\mathcal{F}$ corresponds to some BP_*-comodule M and $Q^*\mathcal{F}$ corresponds to the BP_*-module underlying M. So the condition of proposition 14 is that the BP_*-module M is in the essential image of $i_{n,*}$, i.e. M is an BP_*/I_n-module, i.e. $I_n M = 0$.
Now fix $0 \leqslant n \leqslant N < \infty$. We apply proposition 14 to $i : \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \longrightarrow \mathfrak{X}_{FG}$ which is representable and a quasi-compact immersion (in particular a monomorphism) because it sits in a cartesian diagram

$$\begin{array}{ccc} V_n^{N+1} & \xrightarrow{j} & V \\ \downarrow & & \downarrow Q \\ \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} & \xrightarrow{i} & \mathfrak{X}_{FG}, \end{array}$$

c.f. (5), in which j is a quasi-compact immersion and one uses [EGA IV$_2$], 2.7.1, xi) as above. Arguing as above, we are left with identifying the essential image of j_* which, as explained at the beginning of this subsection, corresponds to the BP_*-modules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a BP_*/I_n-module. □

Corollary 22. *Let $n \geqslant 0$ and let $BP_*/I_n \longrightarrow R \neq 0$ be Landweber exact of height N (hence $n \leqslant N \leqslant \infty$). Then $(R, \Gamma) := (R, R \otimes_{BP_*} BP_*BP \otimes_{BP_*} R)$ is a flat Hopf algebroid and its category of comodules is equivalent to the full subcategory of BP_*-comodules M such that $I_n M = 0$ and M is I_{N+1}/I_n-local as a BP_*/I_n-module. (The last condition is to be ignored in case $N = \infty$)*

Proof. By theorem 18, (R, Γ) is a flat Hopf algebroid with associated algebraic stack $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ (resp. $\mathfrak{Z}^n$ if $N = \infty$). So the category of (R, Γ)-comodules is equivalent to $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ (resp. $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n})$). Now use proposition 21. □

The case $n = 0$ corresponds to the situation treated in [HS] where (translated into the present terminology) $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{U}^{N+1}})$ is identified as a *localisation* of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$. This can be done because $f : \mathfrak{U}^{N+1} \longrightarrow \mathfrak{X}_{FG}$ is flat, hence f^* exact. To relate more generally $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ to $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ it seems more appropriate to identify the former as a full subcategory of the latter as we did above. However, using proposition 1.4 of *loc. cit.* and proposition 12 one sees that $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}})$ is equivalent to the localisation of $\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{O}_{\mathfrak{X}_{FG}})$ with respect to all morphisms α such that $f^*(\alpha)$ is an isomorphism where $f : \mathfrak{Z}^n \cap \mathfrak{U}^{N+1} \longrightarrow \mathfrak{X}_{FG}$ is the immersion. As f is not flat for $n \geqslant 1$ this condition seems less tractable than the one in corollary 22.

Of course, equivalences of comodule categories give rise to change of rings theorems and we refer to [HS] for numerous examples (in the case $n = 0$) and only point out the following (c.f. [R2], theorem B.8.8 for the notation and a special case): If $n \geqslant 1$ and M is a BP_*-comodule such that $I_n M = 0$ and v_n acts invertibly on M then

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, M) \simeq \mathrm{Ext}^*_{\Sigma(n)}(\mathbb{F}_p[v_n, v_n^{-1}], M \otimes_{BP_*} \mathbb{F}_p[v_n, v_n^{-1}]).$$

In fact, this is clear from the case $n = N$ of corollary 22 applied to the obvious map $BP_*/I_n \longrightarrow \mathbb{F}_p[v_n, v_n^{-1}]$ which is Landweber exact of height n by proposition 20.

To make a final point, in [HS] we also find many of the fundamental results of [L] generalised to Landweber exact algebras (whose induced Hopf algebroids are presentations of our $\mathfrak{U}^{N+1}$). One may generalise these further to the present case (i.e. to $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ for $n \geqslant 1$) but again we leave the fun of doing this to the reader and only point out an example: In the situation of corollary 22 every non-zero (R, Γ)-comodule has a non-zero primitive.

To prove this, consider the comodule as a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}$ and use that the primitives we are looking at are $H^0(\mathfrak{Z}^n \cap \mathfrak{U}^{N+1}, \mathcal{F}) \simeq H^0(\mathfrak{X}_{FG}, f_*\mathcal{F}) \neq 0$ because f_* is faithful and using the result of P. Landweber about $\mathfrak{X}_{FG}$ recalled in remark 19.

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