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On the étale homotopy type of Morel-Voevodsky spaces

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In the 1960s Artin and Mazur [AM] constructed a functor which associates to each locally noetherian scheme X its *étale homotopy type* X_{et}, an object of *pro-*$\mathcal{H}$, the pro-category of the homotopy category $\mathcal{H}$ of simplicial sets. For any geometric point x of X, the (pro)groups $\pi_i((X,x)_{et})$, $i \geq 1$, give a natural definition of homotopy groups in algebraic geometry. Artin and Mazur proved that the étale homotopy type X_{et} of a smooth complex variety X is isomorphic in *pro-*$\mathcal{H}$ to the profinite completion of the topological space $X(\mathbb{C})$. This strongly refined previously known comparison theorems between étale and singular cohomology and led to interesting applications to both algebraic geometry and topology, most prominent, the proofs of the Adam's conjecture given by Friedlander/Quillen [Fr1] and by Sullivan [Su].

In the 1990s Morel and Voevodsky [MV] defined a natural categorical framework for the use of topological methods in algebraic geometry. They embedded the category of smooth schemes of finite type over a field k into a larger category of 'k-spaces', which carries the structure of a closed model category, namely the $\mathbb{A}^1$-model structure. The associated homotopy category is the celebrated $\mathbb{A}^1$-homotopy category of smooth schemes over k.

The aim of this paper is to show how both concepts interact. More precisely, we show that over any field k, the functor 'étale homotopy type' has a natural extension to k-spaces. Furthermore, if k has characteristic zero and is of finite virtual cohomological dimension, then a weak $\mathbb{A}^1$-homotopy equivalence $X \to Y$ of k-spaces induces an isomorphism $X_{et} \to Y_{et}$, i.e. we obtain an induced functor on the $\mathbb{A}^1$-homotopy category of smooth schemes over k. In particular, we obtain the notion of étale homotopy groups of k-spaces. In contrast, if k has positive characteristic, then the affine line has a highly non-trivial fundamental group [Ra] and the étale homotopy type does not factor through the $\mathbb{A}^1$-homotopy category. However, factorization holds after completion away from the characteristics.

I want to thank Eric Friedlander and Jens Hornbostel for their comments on a preliminary version of this article. After completing this work I learned that D. Isaksen [Is] has obtained related results by using a completely different method.

1 Hypercoverings

Slightly changing the usual notation we call a k-scheme X smooth if it is the disjoint union of smooth schemes of finite type over k and we say that a morphism in $Sm(k)$ is étale if it is the disjoint union of étale morphisms in the usual sense. Note that this change in terminology does not affect the notion of étale sheaves on the category $Sm(k)$. Let $\mathcal{H}$ be the homotopy category of simplicial sets and let *pro*-$\mathcal{H}$ be its pro-category. Following [AM] have a natural functor

$$et : Sm(k) \longrightarrow pro\text{-}\mathcal{H},$$

which associates to each smooth scheme its 'etale homotopy type'. The construction works for arbitrary locally noetherian schemes and we recall it briefly. Let C be any site.

Definition 1.1. *A hypercovering $X.$ of C is a simplicial object with values in C such that the following conditions hold*

(i) *the natural morphism $X_0 \to e$ to the final object of C is a covering*

(ii) *for all n the natural morphism*

$$X_{n+1} \longrightarrow (\mathrm{cosk}_n X.)_{n+1}$$

is a covering.

Recall that the functor cosk_n is the right adjoint to the functor 'truncation at level n' and that $(\mathrm{cosk}_n X.)_k = X_k$ for $k \leq n$.

Example: Let C be the site $(Sets)$ where coverings are surjective families of maps. A simplicial set is a hypercovering if and only if it satisfies the Kan-condition and is contractible (cf. [AM], 8.5.a).

Let $X.$ be a simplicial object of C and let $T.$ be a simplicial set. We form the simplicial object of C

$$(X \otimes T).$$

which is given in degree n as the coproduct of copies of X_n indexed by T_n. For a non-decreasing map $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ the operator α^* acts on the copy of X_n indexed by $t \in T_n$ via $\alpha^* : (X_n)_t \to (X_m)_{\alpha^*(t)}$. Of course, this construction requires the existence of infinite coproducts in C unless T_n is finite for all n.

Definition 1.2. *Two maps $f_0, f_1 : X. \to Y.$ of simplicial objects in C are homotopic if there exists a map*

$$H : X. \otimes \Delta(1) \longrightarrow Y.$$

with $H \circ i_\nu = f_\nu$ for $\nu = 0, 1$. Here $i_\nu : X. \to X. \otimes \Delta(1)$, $\nu = 0, 1$ are the obvious inclusions.

Lemma 1.3. *For a site C, let $\mathrm{HR}(C)$ denote the category whose objects are hypercoverings and whose maps are homotopy classes of morphisms. Then $\mathrm{HR}(C)$ is left filtering.*

Proof. See [AM], 8.13. □

Definition 1.4. *An object $X \in C$ is connected if it has no non-trivial coproduct decomposition. C is called locally connected if every object has a coproduct decomposition into connected objects. C is connected if it is locally connected and its final object e is connected.*

If C is locally connected, then the expression of an object X as a coproduct of connected objects is essentially unique. Moreover, the rule associating to an object its set of connected components is a functor. We denote this functor by

$$\Pi : C \longrightarrow (Sets)$$

and call it the connected component functor. Applying Π to each X_n separately, we obtain a simplicial set $\Pi(X.)$ associated to each hypercovering $X.$ of C. This assignment induces a functor $\Pi : \mathrm{HR}(C) \to \mathcal{H}$ and therefore defines a pro-object

$$\Pi C \in pro\text{-}\mathcal{H}.$$

Now, if X is locally noetherian scheme, we obtain its étale homotopy type $X_{et} \in pro\text{-}\mathcal{H}$ by applying the above construction to the small étale site over X.

2 Local fibrations versus hypercoverings

Next we recall several definitions and facts on simplicial sheaves from [MV], only that we work with the étale site. We work in the category $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ of simplicial étale sheaves (of sets) on $Sm(k)$. By a point we will always mean a geometric point. A map of simplicial sheaves $f : F \to G$ is called a simplicial weak equivalence if for every point x the map $F_x \to G_x$ is a weak equivalence of simplicial sets. f is called a (trivial) cofibration if it is injective (injective and a weak equivalence). Fibrations are maps satisfying the right lifting property with respect to trivial cofibrations. The category of simplicial sheaves together with these three classes of morphisms is a simplicial closed model category and we denote the associated homotopy category by $\mathcal{H}_{s,et}(Sm(k))$.

A map of simplicial sheaves $F \to G$ is called a (trivial) local fibration if for every point x the map $F_x \to G_x$ is a fibration (fibration and weak equivalence). A local fibration has the right lifting property after an étale refinement. Kan-simplicial sets considered as constant simplicial sheaves are locally fibrant.

For simplicial sheaves $\mathcal{X}$, $\mathcal{Y}$ denote by $\pi(\mathcal{X}, \mathcal{Y})$ the quotient of $Hom(\mathcal{X}, \mathcal{Y}) = S_0(\mathcal{X}, \mathcal{Y})$ with respect to the equivalence relation generated by simplicial homotopies, i.e. the set of connected components of the simplicial function object $S(\mathcal{X}, \mathcal{Y})$, and call it the set of *simplicial homotopy classes of morphisms* from $\mathcal{X}$ to $\mathcal{Y}$. One easily checks that the simplicial homotopy relation is compatible with composition and thus one gets a category $\pi\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ with objects the simplicial sheaves and morphisms the simplicial homotopy classes of morphisms. For any simplicial sheaf $\mathcal{X}$ denote by $\pi\, Triv/\mathcal{X}$ the category whose objects are the trivial local fibrations to $\mathcal{X}$ and whose morphisms are the obvious commutative triangles in $\pi\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$. This category is filtering and essentially small ([MV], Lemma 2.1.12). We will need the following

Proposition 2.1 ([MV],2.1.13). *For any simplicial sheaves $\mathcal{X}$, $\mathcal{Y}$, with $\mathcal{Y}$ locally fibrant, the canonical map*

$$colim_{p:\mathcal{X}'\to\mathcal{X}\in\pi Triv/\mathcal{X}}\ \pi(\mathcal{X}',\mathcal{Y}) \longrightarrow Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{X},\mathcal{Y})$$

is a bijection.

As usual, we consider a sheaf F as a simplicial sheaf with F in each degree and all face and degeneracy morphisms the identity of F. We will also make no difference in notation between a smooth scheme X and the sheaf that it represents. A hypercovering $U.$ of X is a hypercovering in the small étale site over X and will be also considered as an object in $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$.

Lemma 2.2. *Let X be a smooth scheme and let $U.$ be a hypercovering of X. Then, considered as a map in $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$, the projection*

$$U. \longrightarrow X$$

is a trivial local fibration.

Proof. For every point x, the associated map $U._x \to X_x$ is a hypercovering in $(Sets)$, hence a Kan-fibration and a weak equivalence. □

Lemma 2.3. *Let X be a smooth scheme and let $\mathcal{X} \to X$ be a trivial local fibration of simplicial sheaves. Then there exists a hypercovering $U. \to X$ of X and a map of simplicial sheaves $U. \to \mathcal{X}$ that commutes with the respective projections to X.*

Proof. For each geometric point $x : \mathrm{Spec}(K) \to X$, the stalk $\mathcal{X}_x$ is a contractible Kan-simplicial set. In particular, these stalks are nonempty and, for all n, the restriction of the map

$$\mathcal{X}_{n+1} \longrightarrow (\mathrm{cosk}_n\mathcal{X})_{n+1}$$

to the small etale site over X is an epimorphism of sheaves.

Now we construct the required hypercovering by induction. First of all, there exists an etale covering $U_0 \to X$ such that $\mathcal{X}_0(U_0) \neq \varnothing$. This gives a map in degree zero. Assume we have already constructed the hypercovering $U.$ up to level n together with a map from $U.$ to the level n truncation of $\mathcal{X}$. This gives a map of sheaves $\alpha : (\mathrm{cosk}_n U.)_{n+1} \to (\mathrm{cosk}_n\mathcal{X})_{n+1}$ or, equivalently, a section $\alpha \in (\mathrm{cosk}_n\mathcal{X})_{n+1}\big((\mathrm{cosk}_n U.)_{n+1}\big)$. Since the map of sheaves $\mathcal{X}_{n+1} \longrightarrow (\mathrm{cosk}_n\mathcal{X})_{n+1}$ is an epimorphism in the small etale site over X, we find an etale covering $W \to (\mathrm{cosk}_n U.)_{n+1}$ and a lift of α to $\mathcal{X}_{n+1}$ over W. Then we put $U_{n+1} = W$. □

Summarizing, we have proven the following

Proposition 2.4. *Let X be a smooth scheme. Every hypercovering $U. \to X$ defines an object of $\pi\,Triv/X$. The category* $\mathrm{HR}(X)$ *is a full and cofinal subcategory of $\pi\,Triv/X$.*

In order not to overload notation, given a simplicial set, we will denote several associated objects by the same letter: the associated constant pro-simplicial set, the associated constant object in *pro-*$\mathcal{H}$, the associated constant simplicial sheaf in $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$, its image in $\mathcal{H}_{s,et}(Sm(k))$, and so on.

Corollary 2.5. *Let M be a simplicial set and let X be a smooth scheme. Then we have a natural isomorphism*

$$Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(X, M) = Hom_{pro\text{-}\mathcal{H}}(X_{et}, M).$$

Proof. First we may replace M by a weakly equivalent Kan-simplicial set. Then M is locally fibrant as an object in $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$. By propositions 2.1, 2.4, we obtain

$$\begin{aligned} Hom_{\mathcal{H}_s(\mathrm{Sm}(k)_{et})}(X, M) &= colim_{p:\mathcal{X}\to X\in\pi Triv/X}\ \pi(\mathcal{X}, M) \\ &= colim_{p:U.\to X \text{ hypercovering}}\ \pi(U., M) \\ &= colim_{p:U.\to X \text{ hypercovering}}\ \pi(\Pi(U.), M) \\ &= Hom_{pro\text{-}\mathcal{H}}(X_{et}, M), \end{aligned}$$

where, in the third line, we considered $\Pi(U.)$ as a constant simplicial sheaf. This proves the corollary. □

Proposition 2.4 suggests to define the étale homotopy type of a simplicial sheaf $\mathcal{X}$ as the functor of connected components

$$\Pi : \pi Triv/\mathcal{X} \longrightarrow \mathcal{H}$$

In order to do this, one has to show that the category of sheaves is locally connected. We will do this in the next section.

3 Geometry of sheaves

The goal of the section is to show that $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is a connected site. The results of this section are rather formal and extend to sheaves for any subcanonical topology.

The category $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ has fibre products, in contrast to *Sm*(k). If $X \to Z$ and $Y \to Z$ are morphisms in *Sm*(k) and if at least one of them is smooth, then the scheme-theoretical fibre product $X \times_Z Y$ is smooth over k and represents the sheaf-theoretical fibre product.

Definition 3.1. *A morphism $F \to G$ in $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is étale, if for any $X \in$ Sm(k) and any morphism $X \to G$ the fibre product*

$$F \times_G X \longrightarrow X$$

is represented by an étale morphism in Sm(k). An étale covering of a sheaf is a surjective family of étale morphisms.

Using the identity morphism of a smooth scheme X, we see that an étale morphism to X in $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is nothing else but an étale morphism to X in *Sm*(k). Therefore the notion of étale morphisms in $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ extends that of *Sm*(k).

Definition 3.2. *An open subsheaf $F \subset G$ is an injective sheaf morphism which is étale.*

This notion agrees with the scheme-theoretical one when applied to a sheaf which is represented by a smooth scheme.

Lemma 3.3. *Any morphism of sheaves is continuous, i.e. the inverse image of an open subsheaf is open. Etale morphisms of sheaves are stable under base change.*

Proof. The first statement follows from the second one. Let $F' \to F$ be etale, $G \to F$ any map and $G' = F' \times_F G$. In order to show that $G' \to G$ is etale we have to show that for any map $U \to G$ with U a smooth scheme, the pullback U' of G' to U is represented by an etale scheme over U. But, with respect to the composite map $U \to G \to F$, we have $U' = U \times_F F'$. □

Corollary 3.4. *If F is the disjoint union of subsheaves F_1, F_2, then F_1 and F_2 are open in F.*

Proof. Let $pt = \mathrm{Spec}(k)$ and put $B = pt \,\dot\cup\, pt$, which is obviously a decomposition into open subsheaves. The canonical projections $F_i \to pt$, $i = 1, 2$, give a map $F \to B$ such that the F_i are the preimages of open subsheaves. Therefore the statement follows from lemma 3.3. □

A sheaf F is connected if it cannot be written as a nontrivial disjoint union of subsheaves.

Lemma 3.5. *A smooth connected scheme represents a connected sheaf.*

Proof. By corollary 3.4, a disjoint union decomposition of a representable sheaf corresponds to a disjoint union decomposition of the representing scheme. □

Lemma 3.6. *Let $F \in \mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ be connected. Then every map from F to a constant sheaf is constant, i.e. factors through $F \to pt$.*

Proof. The constant sheaf over a set M is the disjoint union over the final sheaf pt indexed by M. If $f : F \to M$ is a map of sheaves, then F is the disjoint union $F = \dot\cup_{m \in M} f^{-1}(pt_m)$ and therefore only one of these components can be nontrivial, if F is connected. □

The following lemma (obvious for schemes) will be essential for the construction of the étale homotopy type.

Proposition 3.7. *Each $G \in \mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ can be written in a unique way as the disjoint union of connected subsheaves. The connected components are open.*

Proof. If $f : F \to G$ is a morphism with F connected then the image sheaf $\mathrm{im}(F) \subset G$ is connected. We consider the natural surjection

$$\dot\bigcup_{(X,\alpha)} X \quad \twoheadrightarrow G,$$

where (X, α) runs through the pairs $X \in \mathrm{S}m(k)$ connected, $\alpha \in G(X)$. Let I be the set of equivalence classes of such pairs with respect to the smallest equivalence relation containing the relations '$(X_1, \alpha_1) \sim (X_2, \alpha_2)$ if $\mathrm{im}(\alpha_1) \cap \mathrm{im}(\alpha_2) \neq \varnothing$'. Then G is the disjoint union of its connected subsheaves

$$G_i := \bigcup_{(X,\alpha) \in i} \mathrm{im}(\alpha), \quad i \in I.$$

The uniqueness of the decomposition is obvious. That the G_i are open follows from corollary 3.4 (or can be easily seen directly). □

4 The étale homotopy type

The rule associating to a sheaf its set of connected components defines the *connected component functor*

$$\Pi : \mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k)) \longrightarrow (Sets)\ .$$

This functor naturally extends to simplicial sheaves (taking values in simplicial sets). Furthermore, simplicial homotopies between simplicial sheaves carry over to homotopies between simplicial sets.

Definition 4.1. *The étale homotopy type $\mathcal{X}_{et}$ of an $\mathcal{X} \in \Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is the induced functor*

$$\Pi : \pi Triv/\mathcal{X} \longrightarrow \mathcal{H}$$

By proposition 2.4, the etale homotopy type of a smooth scheme is naturally isomorphic to the etale homotopy type of the sheaf that it represents.

Corollary 4.2. *Let M be a simplicial set and let $\mathcal{X} \in \Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$. Then we have a natural isomorphism*

$$Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{X}, M) = Hom_{pro\text{-}\mathcal{H}}(\mathcal{X}_{et}, M).$$

Proof. We may replace M by a weakly equivalent Kan-simplicial set. Then the statement follows from proposition 2.1 and lemma 3.6. □

Corollary 4.3. *If a morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is a simplicial weak equivalence, then the induced map $f_{et} : \mathcal{X}_{et} \to \mathcal{Y}_{et}$ is an isomorphism in pro-$\mathcal{H}$.*

Proof. This follows from corollary 4.2 and from the fact that pro-objects in a category are uniquely determined by their morphisms to constant objects. □

Summarizing, we have proven the

Theorem 4.4. *There exists a natural functor*

$$et : \mathcal{H}_{s,et}(Sm(k)) \longrightarrow pro\text{-}\mathcal{H}$$

such that the composite $Sm(k) \to \mathcal{H}_{s,et}(Sm(k)) \to pro\text{-}\mathcal{H}$ is the functor 'étale homotopy type' of Artin-Mazur.

Remark 4.5. Having the notion of an etale covering of a sheaf F, we can define the left filtering category of étale hypercoverings of F. Then the analog of proposition 2.4 holds and we can use hypercoverings to define F_{et}.

In order to extend the above constructions to a pointed setting, we have to define the notion of a geometric point on a sheaf. Let us fix a separable closure $\bar{k}$ of k. We define a (geometrically) *pointed sheaf* (F, f) as a sheaf F together with an element $f \in F(\mathrm{Spec}(\bar{k}))$. As is well-known, the set of geometric points of the form $x : \mathrm{Spec}(\bar{k}) \to X$, $X \in Sm(k)$, is a conservative set of points for $Sm(k)_{et}$. Therefore every non-empty sheaf can be pointed. A pointed simplicial sheaf is a simplicial object in the category of pointed sheaves. Calling a map of pointed simplicial sheaves a fibration, cofibration or simplicial weak equivalence if and only if the underlying unpointed map has this property, we obtain the pointed simplicial homotopy category. We say that a simplicial sheaf $\mathcal{X}$ is connected if the simplicial set ΠX is. We then can form the category $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))_0$ of pointed connected simplicial sheaves on $Sm(k)_{et}$ and we denote its simplicial homotopy category by $\mathcal{H}_{s,et,0}(Sm(k))$. Similarly, denote by $\mathcal{H}_0$ the category of pointed connected simplicial sets. For each $(T,t) \in$ *pro-*$\mathcal{H}_0$, we have the pro-sets $\pi_i(T,t)$, which are pro-groups for $i \geq 1$ (abelian, if $i \geq 2$).

The proof of the following lemma is straightforward.

Lemma 4.6. *Let $(\mathcal{X}, x)$ be a pointed simplicial sheaf and denote by $\pi_0(\mathcal{X}, x)$ the pointed set of connected components of the pointed simplicial set $\Pi(\mathcal{X}, x)$. Then the pointed pro-set $\pi_0((\mathcal{X}, x)_{et})$ is isomorphic in pro-$(Sets_*)$ to the pointed set $\pi_0(\mathcal{X}, x)$. Furthermore, $\mathcal{X}$ is a disjoint union of non-trivial connected simplicial sheaves $\mathcal{X}^\alpha$ corresponding to elements $\alpha \in \pi_0(\mathcal{X}, x)$.*

We obtain a natural functor

$$et : \mathcal{H}_{s,et,0}(Sm(k)) \longrightarrow pro\text{-}\mathcal{H}_0.$$

Definition 4.7. *Let $(\mathcal{X}, x)$ be a pointed connected simplicial sheaf. We call the pro-groups*

$$\pi_i(\mathcal{X}, x) := \pi_i((\mathcal{X}, x)_{et}),\ i \geq 1,$$

the étale homotopy groups of $(\mathcal{X}, x)$.

5 $\mathbb{A}^1$-factorization

We say that a field k has finite virtual cohomological dimension if there exists a finite extension $K|k$ and a number d such that $H^i(K, A) = 0$ for all $i > d$ and every discrete $G(\bar{K}|K)$-module A. All fields of arithmetic interest, as absolutely finitely generated fields and their various completions and henselizations have this property.

The $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1,et}(Sm(k))$ is obtained from $\mathcal{H}_{s,et}(Sm(k))$ by a process, which essentially inverts the morphism $\mathbb{A}^1_k \to \mathrm{Spec}(k)$, see [MV]. The aim of this section is to prove the

Theorem 5.1. *Let k be a field of characteristic zero having finite virtual cohomological dimension. Then the functor*

$$et : \mathcal{H}_{s,et}(Sm(k)) \longrightarrow pro\text{-}\mathcal{H}$$

factors through the étale $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1,et}(Sm(k))$.

We recall the notion of weak equivalences in *pro-*$\mathcal{H}$ from [AM]. Let $X = \{X_i\}$ be in *pro-*$\mathcal{H}$. The various coskeletons $\mathrm{cosk}_n X_i$ form a pro-object $X^\natural$ indexed by pairs (i,n). We have a natural map $X \to X^\natural$ and $X^\natural \to X^{\natural\natural}$ is an isomorphism. We call a map $f : X \to Y$ in *pro-*$\mathcal{H}$ a weak equivalence ($\natural$-isomorphism in [AM]) if $f^\natural : X^\natural \to Y^\natural$ is an isomorphism. Let

$$(pro\text{-}\mathcal{H})_w$$

be the full subcategory in *pro-*$\mathcal{H}$ consisting of objects isomorphic to $X^\natural$ for some X. Then $(pro\text{-}\mathcal{H})_w$ is the localization of *pro-*$\mathcal{H}$ with respect to the class of weak equivalences.

In *pro-*$\mathcal{H}_0$, weak equivalences can be detected on homotopy groups. For a proof of the next theorem see [AM], Theorem 4.4.

Theorem 5.2. *A map in pro-*$\mathcal{H}_0$ *is a weak equivalence, if and only if it induces isomorphisms on the homotopy groups.*

We say that an object X of a site C has dimension $\leq d$ if for every locally constant sheaf A of abelian groups on C we have $H^q(X, A) = 0$ for $q > d$. The site C is said to have local dimension $\leq d$ if for every $X \in C$, there is a covering $X' \to X$ such that X' has dimension $\leq d$. As is well known, if the field k has virtual finite cohomological dimension and if $X \in Sm(k)$, then the small étale site over X has finite local dimension.

If C is a connected pointed site, then ΠC is naturally pointed and we consider it as an object in *pro-*$\mathcal{H}_0$. The following theorem is a special case of [AM], Theorem 12.5.

Theorem 5.3. *Let $f : C \to D$ be a morphism of pointed connected sites. Suppose that C, D have finite local dimension and that $\Pi f : \Pi C \to \Pi D$ is a weak equivalence in pro-*$\mathcal{H}_0$*. Then Πf is an isomorphism in pro-*$\mathcal{H}_0$*.*

The above results enable us to show $\mathbb{A}^1$-factorization.

Proposition 5.4. *Let k be a field of characteristic zero. Then for any smooth scheme U over k the projection $\mathbb{A}^1_k \times U \to U$ induces a weak equivalence*

$$(\mathbb{A}^1_k \times U)_{et} \to U_{et}$$

in pro-$\mathcal{H}$*. If k has virtual finite cohomological dimension, then this map is an isomorphism.*

Proof. First of all, we may assume that U is connected. Choosing a (geometric) point u of U and the point $u' = (0, u)$ of $\mathbb{A}^1_k \times U$, we obtain a morphism of smooth connected pointed schemes. By theorem 5.2, it suffices to show that the projection induces an isomorphism on homotopy groups. By [AM], Theorem 11.1, these groups are profinite. By [Fr2], Theorem 11.5, we have a long exact sequence

$$\cdots \to \pi_i(\mathbb{A}^1_k, 0) \to \pi_i(\mathbb{A}^1 \times U, u') \to \pi_i(U, u) \to \cdots .$$

It therefore remains to show that $\mathbb{A}^1_k$ is weakly contractible if k has characteristic zero. It is well known, that the fundamental group is trivial (use Hurwitz' genera formula). The statement on the higher homotopy groups follows via the Hurewicz-homomorphism ([AM], 4.5, 9.3) from the triviality of the cohomology of $\mathbb{A}^1_k$ with values in constant sheaves of abelian groups. The last statement follows from theorem 5.3. □

Now let B be the class of morphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{H}_{s,et}(Sm(k))$ such that for every simplicial set M the induced map

$$Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{Y}, M) \longrightarrow Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{X}, M)$$

is an isomorphism. Slightly abusing language, we say that a morphism of simplicial sheaves is in B, if its class in $\mathcal{H}_{s,et}(Sm(k))$ is in B. The proof of the next lemma is strictly parallel to the proof of [MV] 2.2.12 (where the class of $\mathbb{A}^1$-weak equivalences is considered), and therefore we omit it.

Lemma 5.5. *Let I be a small category, $\mathcal{X}, \mathcal{Y}$ functors from I to $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ and $f : \mathcal{X} \to \mathcal{Y}$ a natural transformation such that all the morphisms f_i, $i \in I$, are in B. Then the morphism*

$$hocolim_I \mathcal{X} \longrightarrow hocolim_I \mathcal{Y}$$

is in B.

Lemma 5.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial sheaves such that for each $n \geq 0$ the morphism of sheaves $f_n : \mathcal{X}_n \to \mathcal{Y}_n$ is in B. Then f is in B.*

Proof. (cf. the proof of [MV] 2.1.14) Consider $\mathcal{X}$ and $\mathcal{Y}$ as diagrams of simplicial sheaves of simplicial dimension zero indexed by Δ^{op}. By [BK], XII, 3.4, the obvious morphisms $hocolim_{\Delta^{op}} \mathcal{X} \to \mathcal{X}$ and $hocolim_{\Delta^{op}} \mathcal{Y} \to \mathcal{Y}$ are simplicial weak equivalences. Therefore the statement follows from lemma 5.5. □

Proposition 5.7. *Suppose that k has characteristic zero and is of finite virtual cohomological dimension. Then*

(i) *the class B contains the class of $\mathbb{A}^1$-weak equivalences.*

(ii) *a simplicial set, considered as an element in $\mathcal{H}_{s,et}(Sm(k))$ is $\mathbb{A}^1$-local.*

Proof. Let M be simplicial set. By definition, M is $\mathbb{A}^1$-local if for every $\mathcal{X} \in \mathcal{H}_{s,et}(Sm(k))$ the projection $\mathcal{X} \times \mathbb{A}^1_k \to \mathcal{X}$ induces an isomorphism

$$Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{X}, M) \longrightarrow Hom_{\mathcal{H}_{s,et}(\mathrm{Sm}(k))}(\mathcal{X} \times \mathbb{A}^1_k, M).$$

In other words, we have to show that the projections $\mathcal{X} \times \mathbb{A}^1_k \to \mathcal{X}$ are in B. If $\mathcal{X}$ is a smooth scheme, then this follows from proposition 5.4 and corollary 4.2. The case of a smooth simplicial scheme follows from lemma 5.6. Finally, by [MV] Lemma 2.1.16, applied to the class of representable sheaves, each object in $\Delta^{op}\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}(k))$ is simplicially weakly equivalent to a smooth simplicial scheme. This shows (ii), and (i) follows easily. □

Now, theorem 5.1 follows from proposition 5.7, corollary 4.2 and the fact that an object in the pro-category is detected by its morphisms to constant objects.

We finally mention a variant which does not make use of the assumption on the virtual cohomological dimension of the ground field k. Consider the functor

$$ht = \natural \circ et : Sm(k) \longrightarrow (pro\text{-}\mathcal{H})_w.$$

Proceeding as above, but using only those simplicial sets M as test objects which have non-trivial homotopy only in finitely many dimensions, we obtain:

Theorem 5.8. (i) *There exists a natural functor*

$$ht : \mathcal{H}_{s,et}(Sm(k)) \longrightarrow (pro\text{-}\mathcal{H})_w$$

such that the composite $Sm(k) \to \mathcal{H}_{s,et}(Sm(k)) \to (pro\text{-}\mathcal{H})_w$ *is the functor* ht *defined above.*

(ii) *If* k *has characteristic zero, then* ht *factors through the* $\mathbb{A}^1$*-homotopy category, inducing a functor*

$$ht : \mathcal{H}_{\mathbb{A}^1,et}(Sm(k)) \longrightarrow (pro\text{-}\mathcal{H})_w$$

6 Closing remarks

1. Composing et with the natural functor $\mathcal{H}_{\mathbb{A}^1,Nis}(Sm(k)) \longrightarrow \mathcal{H}_{\mathbb{A}^1,et}(Sm(k))$, we obtain an étale homotopy type functor on the usual Morel-Voevodsky category. Alternatively, we could have worked with Nisnevich sheaves all through this paper, without any difference.

2. Our construction of the étale homotopy type for simplicial sheaves particularly applies to smooth simplicial schemes (via the represented simplicial sheaf). In this case the étale homotopy type was defined previously by Friedlander [Fr2] using bisimplicial hypercoverings. Both constructions coincide up to weak equivalence. This can be best seen by using Isaksen's "hypercover descent theorem" [Is], theorems 11 and 12.

3. If k is a field of positive characteristic, then the functor ht factors through $\mathbb{A}^1$-equivalence after completion away from the characteristics. More precisely, let C be the class of finite groups of order prime to the characteristic of k and let $C^\wedge : pro\text{-}\mathcal{H}_0 \to pro\text{-}\mathcal{H}_0$ be the C-completion functor of [AM], Thm.3.4. Then the composite

$$\mathcal{H}_{s,et,0}(Sm(k)) \xrightarrow{ht} (pro\text{-}\mathcal{H}_0)_w \xrightarrow{C^\wedge} (pro\text{-}\mathcal{H}_0)_w$$

factors through $\mathcal{H}_{\mathbb{A}^1,et,0}(Sm(k))$. If k has finite virtual C-cohomological dimension, then, as above, we may work with $(pro\text{-}\mathcal{H}_0)$ instead of $(pro\text{-}\mathcal{H}_0)_w$.

Finally, with the obvious modifications, everything works over an arbitrary noetherian base scheme S instead of Spec(k).

4. One can stabilize the construction of the étale homotopy type with respect to the simplicial sphere S^1_s of [MV] (use [Fr2], prop. 4.7).

5. As already mentioned in the introduction, there exists another construction of an étale realization functor due to Isaksen [Is], who, however, does not construct a functor with values in *pro*-$\mathcal{H}$. His approach is the following: he fixes a prime number p different to the characteristic of k and constructs a functors which takes values in the localization $\mathcal{L}_p$ of the category (*pro*-Δ^{op} *Sets*) with respect to a model structure, in which the weak equivalences are those morphisms which induce isomorphisms in mod-p cohomology. One advantage of our construction is the rather useful adjunction formula 4.2. Furthermore, we do not have to fix one particular prime number p. A problem in comparing both approaches is to find an appropriate common target category.

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