Universität Regensburg
Mathematik

A Spectral Sequence for Iwasawa Adjoints

Uwe Jannsen

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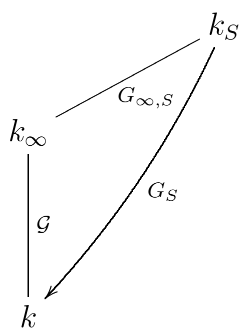
Let k be a number field, fix a prime p, and let k_∞ be some Galois extension of k such that $\mathcal{G} = \mathrm{Gal}\,(k_\infty/k)$ is a p-adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geqslant 1$). Let S be a finite set of primes containing all primes above p and ∞, and all primes ramified in k_∞/k, and let k_S be the maximal S-ramified extension of k; by assumption, $k_\infty \subseteq k_S$. Let $G_S = \mathrm{Gal}\,(k_S/k)$ and $G_{\infty,S} = \mathrm{Gal}\,(k_S/k_\infty)$.



Let A be a discrete G_S-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geqslant 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^\infty]$, the group of p-power torsion points of an elliptic curve E/k with good reduction outside S). We are **not** assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated Λ-module M we put

$$E^i(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)\ .$$

Hence $E^0(M) = \mathrm{Hom}_\Lambda(M, \Lambda) =: M^+$ is just the Λ-dual of M. This has a natural structure of a Λ-module, by letting $\sigma \in \mathcal{G}$ act via

$$\sigma f(m) \;=\; \sigma f(\sigma^{-1})m)$$

for $f \in M^+$, $m \in M$. It is known that Λ is a noetherian ring (here we use that $\mathcal{G}$ is a p-adic Lie group), by results of Lazard [La]. Hence M^+ is a finitely generated Λ-module again (choose a projection $\Lambda^r \twoheadrightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated Λ-modules for all $i \geqslant 0$ which we call the (generalzed) Iwasawa adjoints of M.

Examples

a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a Λ-torsion module M, $E^1(M)$ is isomorphic to the Iwasawa adjoint, which can be defined as

$$\mathrm{ad}\,(M) = \varprojlim_n\,(M/\alpha_n M)^\vee$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of elements in Λ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and (α_n) is prime to the support of M for every $n \geqslant 1$, and where

$$N^\vee = \mathrm{Hom}\,(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact $\mathbb{Z}_p$-module N. For any finitely generated Λ-module M, $E^1(M)$is quasi-isomorphic to $\mathrm{Tor}_\Lambda(M)^\sim$, where $\mathrm{Tor}_\Lambda(M)$ is the Λ-torsion submodule of M, and $M^\sim$ is the "Iwasawa twist" of a Λ-module M: the action of $\gamma \in \mathcal{G}$ is changed to the action of γ^{-1}.

b) If $\mathcal{G} = \mathbb{Z}_p^r$, $r \geqslant 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

Theorem 1 *There is a spectral sequence of finitely generated Λ-modules*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S},\, A)^\vee) \;\Rightarrow\; \varprojlim_{k',\, m} H^{p+q}(G_S(k'),\, A[p^m]) \;=\; \varprojlim_{k'} H^{p+q}(G_S(k'),\, T_pA)\,.$$

Here the limit runs through the natural numbers m and the finite extensions k'/k contained in k_∞, respectively, via the natural maps $H^n(G_S(k'), A[p^{m+1}]) \to H^n(G_S(k'), A[p^m])$ and the corestrictions. The rightmost group is the continuous cohomology of the Tate module $T_pA = \varprojlim_m A[p^m]$.

Before we give the proof of a slightly more general result (cf. Theorem 11 below), we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$\begin{aligned} 0 \;\to\; & E^1(H^0(G_{\infty,S},\, A)^\vee) \xrightarrow{inf^1} \varprojlim_{k'} H^1(G_S(k'),\, T_pA) \\ \to\; & (H^1(G_{\infty,S},\, A)^\vee)^+ \longrightarrow E^2(H^0(G_{\infty,S},\, A)^\vee) \xrightarrow{inf^2} \varprojlim_{k'} H^2(G_S(k'),\, T_pA)\,. \end{aligned}$$

To say more, we consider some assumptions.

A.1 Assume that $p > 2$ or that k_∞ is totally imaginary. Then

$$H^r(G_{\infty,S},\, A) = 0 = \varprojlim_{k'} H^r(G_S(k'),\, T_pA)$$

for all $r > 2$.

Corollary 2 *Assume that $H^2(G_{\infty,S},\, A) = 0$ (This is the so-called "weak Leopoldt conjecture" for A. It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$, and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja 2]).*

Then the cokernel of inf^2 *is*

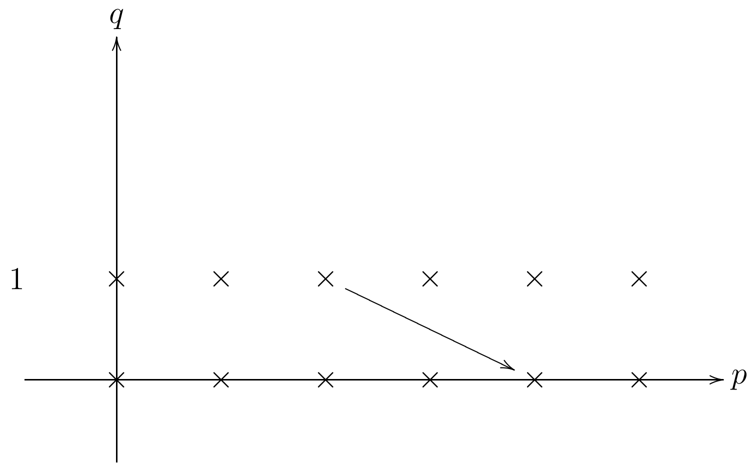
$$\ker\,(E^1(H^1(G_{\infty,S},\, A)^\vee) \;\twoheadrightarrow\; E^3(H^0(G_{\infty,S},\, A)^\vee)),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty,S},\, A)^\vee) \;\xrightarrow{\sim}\; E^{i+2}(H^0(G_{\infty,S},\, A)^\vee)$$

for $i \geqslant 2$.

Proof This comes from A.1 and the following picture of the spectral sequence



Corollary 3 *Assume that* $H^0(G_{\infty,S},\, A) = 0$*. Then*

(a)

$$\varprojlim_{k'} H^1(G_S(k'),\, T_pA) \xrightarrow{\sim} H^1(G_{\infty,S},\, A)^\vee)^+$$

(b) *There is an exact sequence*

$$0 \to E^1(H^1(G_{\infty,S},\, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\, T_pA)$$

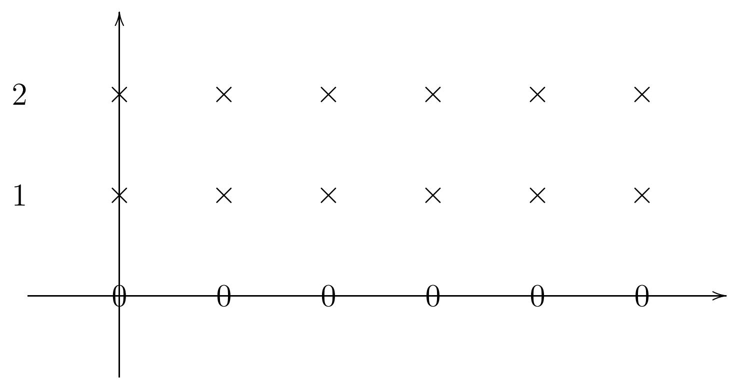
$$\to (H^2(G_{\infty,S},\, A)^\vee)^+ \to E^2(H^1(G_{\infty,S},\, A)^\vee) \to 0$$

(c) *There are isomorphisms*

$$E^i(H^2(G_{\infty,S},\, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty,S},\, A)^\vee)$$

for $i \geqslant 1$.

Proof In this case, the spectral sequence looks like



Corollary 4 *Assume that $\mathcal{G}$ is a p-adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geqslant 3$. Let*

$$B = \text{im}\ (\inf{}^2 : E^2(H^0(G_{\infty,S},\, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\, T_pA))$$

Then B is finite, and there is an exact sequence

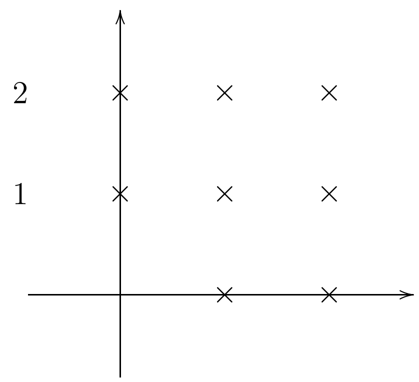
$$0 \;\to\; E^1(H^1(G_{\infty,S},\, A)^\vee) \;\to\; \varprojlim_{k'} H^2(G_S(k'),\, T_pA)/B \;\to\; (H^2(G_{\infty,S},\, A)^\vee)^+$$
$$\to E^2(H^1(G_{\infty,S},\, A)^\vee) \to 0\,,$$

and

$$E^1(H^2(G_{\infty,S},\, A)^\vee) = 0 = E^2(H^2(G_{\infty,S},\, A)^\vee)\,,$$

i.e., $(H^2(G_{\infty,S},\, A)^\vee$ is a projective Λ-module.

Proof Quite generally, for a p-adic Lie group $\mathcal{G}$ of dimension n one has $vcd_p(\mathcal{G}) = n$ for the virtual cohomological p-dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n+1$, cf. [Ja 3]. The finiteness of $E^2(M)$ (for a finitely generated Λ-module M) in our case is well-known, cf. [Ja 3]. The remaining claims follow from the following shape of the spectral sequence:



Lemma 5 *Assume that $\mathcal{G}$ is a p-adic Lie group of dimension n (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$). Then $E^i(H^0(G_{\infty,S},\, A)^\vee) = 0$ for $i \neq n,\, n+1$.*

(a) *If $H^0(G_{\infty,S},\, A)$ is divisible (e.g., if $G_{\infty,S}$ acts trivially on A), then*

$$E^i(H^0(G_{\infty,S},\, A)^\vee) = \begin{cases} 0 & \text{for } \ i \neq n \\ \text{Hom }(D,\, H^0(G_{\infty,S},\, A)), & \text{for } \ i = n\,, \end{cases}$$

where D is the dualising module for $\mathcal{G}(D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n)$.

(b) *If $H^0(G_{\infty,S},\, A)$ is finite, then*

$$E^i(H^0(G_{\infty,S},\, A)^\vee) = \begin{cases} 0 & \text{for } \ i \neq n+1 \\ \text{Hom }(H^0(G_{\infty,S},\, A),\, D)^\vee & \text{for } \ i = n+1 \end{cases}$$

Proof This is well-known, cf. [Ja 3].

Corollary 6 *Let $\mathcal{G}$ is a p-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on A, then there are exact sequences*

$$0 \to \varprojlim_{k'} H^1(G_S(k'), T_pA) \to (H^1(G_{\infty,S}, A)^\vee)^+$$

$$\to T_pA \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_pA)$$

and

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'), T_pA)\big/_{\operatorname{im}\inf^2}$$

$$\to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0\,,$$

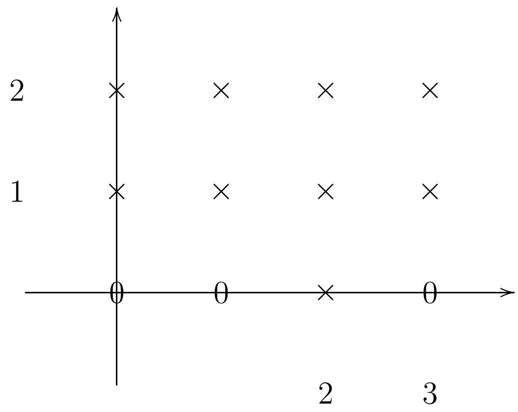
an isomorphism

$$E^1(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty,S}, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S}, A)^\vee)\,.$$

Proof The spectral sequence looks like



Corollary 7 *Let $\mathcal{G}$ be a p-adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geqslant 4$). If $H^0(G_{\infty,S}, A)$ is finite, then*

$$\varprojlim_{k'} H^1(G_S(k'), T_pA) \cong (H^1(G_{\infty,S}, A)^\vee)^+\ .$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty,S}, A)^\vee) \;\to\; E^3(H^0(G_{\infty,S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$0 \;\to\; \ker d_2^{1,1} \;\to\; \varprojlim_{k'} H^2(G_S(k'), T_pA) \;\to$$

$$\to\; \ker(d_2^{0,2}\colon (H^2(G_{\infty,S}, A)^\vee)^+ \twoheadrightarrow E^2(H^1(G_{\infty,S}, A)^\vee)) \to\; \operatorname{coker} d_2^{1,1} \to 0\,,$$

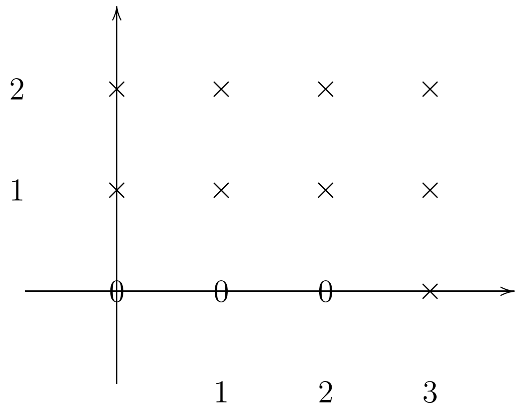
an isomorphism

$$E^1(H^2(G_{\infty,S},\, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S},\, A)^\vee)\,,$$

and the vanishing

$$E^2(H^2(G_{\infty,S},\, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S},\, A)^\vee)\;.$$

Proof The spectral sequence looks like



Remark In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

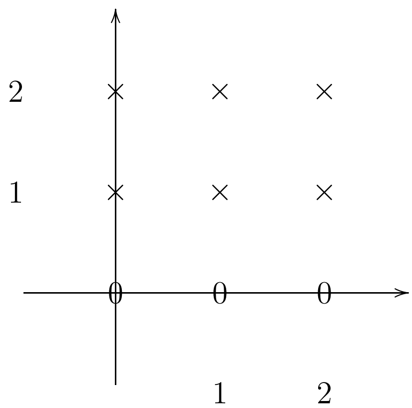
$$0 \to E^1(H^1(G_{\infty,S},\, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),\, T_pA)$$

$$\to (H^2(G_{\infty,S},\, A)^\vee)^+ \to E^2(H^1(G_{\infty,S},\, A)^\vee) \to 0\,.$$

Corollary 8 *Let $\mathcal{G}$ be a p-adic Lie group of dimension > 2. Then*

$$(H^1(G_{\infty,S},\, A)^\vee)^+ \;\cong\; \varprojlim_{k'} H^1(G_S(k'),\, T_pA)$$

Proof The first three columns of the spectral sequence look like



We will now prove Theorem 1, by proving a somewhat more general result. For any profinite group G, let $\Lambda(G) = \mathbb{Z}_p[[G]]$ be the completed group ring over $\mathbb{Z}_p$, and let $M_G = M_{G,p}$ be the category of discrete (left) $\Lambda(G)$-modules. These are the discrete G-modules A which are p-primary torsion abelian groups. For such a module A, its Pontrjagin dual $A^\vee = Hom(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is a compact $\Lambda(G)$-module. In fact, Pontrjagin duality gives an anti-equivalence between M_G and the category $C_G = C_{G,p}$ of compact (right) $\Lambda(G)$-modules.

Let $M_G^{\mathbb{N}}$ be the category of inverse systems

$$(A_n): \quad \ldots \to A_3 \to A_2 \to A_1$$

in M_G as in [Ja 1]. Denote by $H^i_{cont}(G, (A_n))$ the continuous cohomology of such a system and recall that one has an exact sequence for each i

$$0 \to \varprojlim_n{}^1 \; H^{i-1}(G, A_n) \to H^i_{cont}(G, (A_n)) \to \varprojlim_n H^i(G, A_n) \to 0\,,$$

in which the $\lim^1$-term vanishes if the groups $H^{i-1}(G, A_n)$ are finite for all n (cf. loc. cit.).

Definition 9 *For a closed subgroup $H \leq G$ and a discrete G-module A in M_G define the relative cohomology $H^m(G, H; A)$ as the value at A of the m-th derived functor of the left exact functor (with Ab being the category of abelian groups)*

$$\begin{array}{rcl} H^0(G,\, H;\, -) : M_G & \to & Ab \\ A & \mapsto & \varprojlim_U H^0(U,\, A)\,, \end{array}$$

where U runs through all open subgroups $U \subset G$ containing H, and the transition maps are the corestriction maps. For an inverse system (A_n) of modules in M_G define the continuous relative cohomology $H^m_{cont}(G, H; (A_n))$ as the value at (A_n) of the m-th right derivative of the functor

$$\begin{array}{rcl} H^0_{cont}(G,\, H;\, -) : M_G^{\mathbb{N}} & \to & Ab \\ (A_n) & \mapsto & \varprojlim_n \varprojlim_U H^0(U,\, A_n)\,, \end{array}$$

where the limit over U is as before, and the limit over n is induced by the transition maps $A_{n+1} \to A_n$.

Lemma 10 *If G/H has a countable basis of neighbourhoods of identity, i.e., if there is a countable family U_ν of open subgroup, $H \leq U_\nu \leq G$, with $\bigcap_\nu U_\nu = H$, and if, in addition, $H^i(U, A_n)$ is finite for all these U and all n, then*

$$H^n_{cont}(G,\, H;\, (A_n)) = \varprojlim_n \varprojlim_U H^n(U,\, A_n)\,.$$

Proof In general, by deriving the inverse limit $\varprojlim_{n,U}$, one gets a Grothendieck spectral sequence

$$E_2^{p,q} = R^p \varprojlim_{n,U} H^q(U,\, A_n) \Rightarrow H^{p+q}_{\mathrm{cont}}(G,\, H;\, (A_n))\,.$$

If the limit is over a countable family, then $R^p \varprojlim_{n,U} = 0$ for $p > 1$, and $R^1 \varprojlim_{n,U}$ has the usual description ([Ja 1]). If, in addition, all $H^q(U, A_n)$ are finite, then $R^1 \varprojlim_{n,U} H^q(U, A_n) = 0$, and we get the claimed isomorphisms.

Now we come to the spectral sequence in theorem 1. Any module A in M_G gives rise to two inverse systems, viz., the system $(A[p^n])$, where the transition maps $A[p^{n+1}] \to A[p^n]$ are induced by multiplication with p in A, and the system (A/p^n), where the transition maps are induced by the identity of A. For reasons explained later, denote by $H^m_{\mathrm{cont}}(G, H; R\underline{T}_p A)$ the value at A of the m-th derived functor of the left exact functor

$$F : A \mapsto \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where U runs through all open subgroups $U \subset G$ containing H, and the transition maps are the corestriction maps and those coming from $A[p^{n+1}] \to A[p^n]$, respectively. If H is a normal subgroup, then we may restrict to normal open subgroups $U \leq G$ containing H in the above inverse limit, and the limit is a (left) $\Lambda(G/H)$-module in a natural way.

Theorem 11 *Let H be a closed subgroup of a profinite group G, and let A be a discrete $\Lambda(G)$-module.*

(a) There are short exact sequences

$$0 \to H^n_{cont}(G, H; (A[p^n])) \to H^n_{cont}(G, H; R\underline{T}_p A) \to H^{n-1}_{cont}(G, H; (A/p^n)) \to 0\,.$$

If H is a normal subgroup, then these are exact sequences of $\Lambda(G/H)$-modules.

(b) Let H' be a normal subgroup of G, with $H' \subset H$. There is a spectral sequence

$$E_2^{p,q} = H^p_{cont}(G/H', H/H'; R\underline{T}_p H^q(H', A)) \;\Rightarrow\; H^{p+q}_{cont}(G, H; R\underline{T}_p A)\,.$$

If H is a normal subgroup, too, this is a spectral sequence of $\Lambda(G/H)$-modules.

(c) If H is a normal subgroup of G, then for every discrete $\Lambda(G)$-module A one has canonical isomorphisms of $\Lambda(G/H)$-modules

$$H^m(G, H; R\underline{T}_p A) \;\cong\; Ext^m_{\Lambda(G)}(A^\vee, \Lambda(G/H))$$

for all $m \geq 0$, where $\Lambda(G/H)$ is regarded as a $\Lambda(G)$-module via the ring homomorphism $\Lambda(G) \to \Lambda(G/H)$. More precisely, the δ-functor

$$M_G \;\to\; Mod_{\Lambda(G/H)} \quad , \quad A \;\rightsquigarrow\; (H^m(G, H; R\underline{T}_p A) \mid m \geq 0)$$

is canonically isomorphic to the δ-functor

$$M_G \;\to\; Mod_{\Lambda(G/H)} \quad , \quad A \;\rightsquigarrow\; (Ext^m_{\Lambda(G)}(A^\vee, \Lambda(G/H)) \mid m \geq 0)\,.$$

Here and in the following, the Ext-groups $Ext_{\Lambda(G)}(-,-)$ are taken in the category C_G of compact $\Lambda(G)$-modules. We note that these Ext-groups are $\Lambda(G)$-modules, but not necessarily compact.

(d) *In particular, let H be a normal subgroup of G, and let $\mathcal{G} = G/H$. If A is a discrete $\Lambda(G)$-module, then one has a spectral sequence of $\Lambda(\mathcal{G})$-modules*

$$E_2^{p,q} = Ext^p_{\Lambda(\mathcal{G})}(H^q(H,\, A)^\vee, \Lambda(\mathcal{G})) \;\Rightarrow\; H^{p+q}_{cont}(G,\, H;\, R\underline{T}_pA) = Ext^{p+q}_{\Lambda(G)}(A^\vee, \Lambda(\mathcal{G}))\,.$$

Before we give the proof of Theorem 11, we note that it implies Theorem 1. In fact, we apply Theorem 11 to $G = G_S$ and $H = G_{\infty,S}$. If A is a G_S-module of cofinite type as in Theorem 1, then $A/p^n = 0$ and $A[p^n]$ is finite, for all n. Moreover, $H^i(U, B)$ is known to be finite for all open subgroups $U \le G_S$ and all finite U-modules B. By (a) and Lemma 10 we deduce

$$H^m_{cont}(G_S,\, G_{\infty,S};\, RT_pA) = \varprojlim_{n,U} H^m(U, A[p^n]) = \varprojlim_{n,k'} H^m(G_S(k'), A[p^n])\,,$$

where k' runs through all finite subextensions of k_∞/k. Moreover, one has canonical isomorphisms

$$\varprojlim_{n} H^m(U, A[p^n]) \;\cong\; H^m(U, T_pA)$$

where the latter group is continuous cochain group cohomology, cf. [Ja 1]. By applying Theorem 11 (d) we thus get the desired spectral sequence. Finally, $H^m(H, A)^\vee$ is a finitely generated $\Lambda(\mathcal{G})$-module for all $m \ge 0$, so that the initial terms of the spectral sequence are finitely generated $\Lambda(\mathcal{G})$-modules as well, and so are the limit terms. In fact, let N be the kernel of the homomorphism $G_S \to Aut(A)$ given by the action of G_S on A, and let $H' = H \cap N$. Then G/H' is a p-adic analytic Lie group, since G/H and G/N are. It is well-known that $H^m(H', \mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated discrete $\Lambda(G/H')$-module for all $m \ge 0$; hence the same is true for $H^m(H', A) \cong H^m(H', \mathbb{Q}_p/\mathbb{Z}_p) \otimes T_pA$. The claim then follows from the Hochschild-Serre spectral sequence $H^p(H/H', H^q(H', A)) \Rightarrow H^{p+q}(H, A)$.

Proof of Theorem 11 (a): We can write F as the composition of the two left exact functors

$$\begin{aligned} \underline{T}_p : M_G &\;\to\; M_G^{\mathbb{N}} \\ A &\;\mapsto\; (A[p^n]) \end{aligned}$$

and

$$\begin{aligned} H^0_{\text{cont}}(G,\, H;\, -) : M_G^{\mathbb{N}} &\;\to\; Ab \\ (A_n) &\;\rightsquigarrow\; \varprojlim_n \varprojlim_U H^0(U,\, A_n)\,, \end{aligned}$$

where U runs through all open (normal) subgroups of G containing H. Because $\underline{T}_p$ maps injectives to $H^0_{\text{cont}}(G,\, H;\, -)$-acyclics we get a spectral sequence

$$E_2^{p,q} = H^p_{\text{cont}}(G,\, H;\, R^q\underline{T}_pA) \Rightarrow H^{p+q}_{\text{cont}}(G,\, H;\, R\underline{T}_pA)\;.$$

From the snake lemma one immediately gets

$$R^q\underline{T}_pA = \begin{cases} (A/p^mA) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences

$$0 \to H^n_{\text{cont}}(G,\, H;\, \underline{T}_pA) \to H^n_{\text{cont}}(G,\, H;\, R\underline{T}_pA) \to H^{n-1}_{\text{cont}}(G,\, H;\, R^1\underline{T}_pA) \to 0\,.$$

This shows (a) and also explains the notation for R^nF. In fact, $H^n_{\mathrm{cont}}(G,\, H;\, R\underline{T}_pA)$ is the hypercohomology with respect to $H^0_{\mathrm{cont}}(G,\, H;\, -)$ of a complex $R\underline{T}_pA$ in $M_G^{\mathbb{N}}$ computing the $R^i\underline{T}_pA$.

(b): If H is a normal subgroup, we can regard the functor F as a functor from M_G to the category $Mod_{\Lambda(G/H)}$ of $\Lambda(G/H)$-modules. On the other hand, we can also write F as the composition of the left exact functors

$$H^0(H,\, -):\; M_G \to M_{G/H} \quad , \quad A \rightsquigarrow A^H$$

and

$$\tilde{F}:\; M_{G/H} \to Mod_{\Lambda(G/H)} \quad , \quad B \mapsto \varprojlim_n \varprojlim_{U/H} H^0(U/H,\, B[p^n]) = H^0(G/H, \{1\}; RT_pB)\,.$$

(Note that U/H runs through all open (normal) subgroups of G/H.) This immediately gives the spectral sequence in (b).

(c): We claim that the functor F is isomorphic to the functor

$$\begin{array}{rcl} M_G & \to & Mod_{\Lambda(G/H)} \\ B & \mapsto & \mathrm{Hom}_{\Lambda(G)}(B^\vee,\, \Lambda(G/H)). \end{array}$$

In fact, writing $\mathrm{Hom}_{\Lambda(G)}(-,-)$ for the homomorphism groups of compact $\Lambda(G)$-modules, we have (cf. [Ja 3] p. 179)

$$\begin{aligned} \mathrm{Hom}_{\Lambda(G)}(B^\vee,\, \Lambda(G/H)) &= \varprojlim_U \mathrm{Hom}_{\Lambda(G)}(B^\vee,\, \mathbb{Z}_p[G/U]) \\ &= \varprojlim_n \varprojlim_U \mathrm{Hom}_{cont}(H^0(U,B)^\vee,\, \mathbb{Z}/p^n\mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U \mathrm{Hom}_{cont}(H^0(U,B[p^n])^\vee,\, \mathbb{Z}/p^n\mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U H^0(U,B[p^n])\,, \end{aligned}$$

where U runs through all open subgroups of G containing H, and hence

$$\mathrm{Hom}_{\Lambda(G)}(B^\vee,\, \Lambda(G/H)) = H^0(G,H;RT_pB).$$

Since taking Pontrjagin duals is an exact functor $M_G \to C_G$ taking injectives to projectives, the derived functors of the functor $B \rightsquigarrow \mathrm{Hom}_{\Lambda(G)}(B^\vee,\, \Lambda(G/H))$ are the functors $B \rightsquigarrow \mathrm{Ext}^i_{\Lambda(G)}(B^\vee,\, \Lambda(G/H))$, and we get (c). Finally, by applying (b) for $H' = H$ and (c) for $H = \{1\}$ we get (d).

Let us note that the proof of theorem 11 gives the following $\mathbb{Z}/p^n$-analogue (by 'omitting the inverse limits over n'). For a profinite group G let $\Lambda_n(G) = \Lambda(G)/p^n = \mathbb{Z}/p^n[[G]]$ be the completed group ring over $\mathbb{Z}/p^n$.

Theorem 12 *Let H and H' be normal subgroups of a profinite group G, with $H' \subset H$, and let A be a discrete $\Lambda_n(G)$-module.*

(a) There is a spectral sequence of $\Lambda_n(G/H)$-modules

$$E_2^{p,q} = H^p(G/H', H/H';\, H^q(H',\, A)) \;\Rightarrow\; H^{p+q}(G,\, H;\, A)\,.$$

(b) *On the category of discrete $\Lambda_n(G)$-modules the δ-functor $A \rightsquigarrow (H^m(G,H;A) \mid m \geq 0)$ with values in the category of $\Lambda_n(G/H)$-modules is canonically isomorphic to the δ-functor $A \rightsquigarrow (Ext^m_{\Lambda_n(G)}(A^\vee, \Lambda_n(G/H)) \mid m \geq 0)$, where the Ext-groups are taken in the category of compact $\Lambda_n(G)$-modules.*

(c) *In particular, if $\mathcal{G} = G/H$, and A is a discrete $\Lambda_n(G)$-module, then one has a spectral sequence of $\Lambda_n(\mathcal{G})$-modules*

$$E_2^{p,q} = Ext^p_{\Lambda_n(\mathcal{G})}(H^q(H,\, A)^\vee, \Lambda_n(\mathcal{G})) \;\Rightarrow\; H^{p+q}(G,\, H;\, A) = Ext^{p+q}_{\Lambda_n(G)}(A^\vee, \Lambda(\mathcal{G}))\,.$$

Corollary 13 *With the notations as for Theorem 1, let A be a finite $\Lambda_n(G_S)$-module, and $\Lambda_n = \Lambda(\mathcal{G})$. Then there is a spectral sequence of finitely generated Λ_n-modules*

$$E_2^{p,q} = Ext^p_{\Lambda_n}(H^q(G_{\infty,S}, A)^\vee, \Lambda_n) \quad\Rightarrow\quad \varprojlim_{k'} \; H^{p+q}(G_S(k'),\, A) = Ext^{p+q}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n)\,,$$

where k' runs through the finite subextensions k'/k of k_∞/k.

On the other hand, Theorem 1 also has the following counterpart for finite modules.

Theorem 14 *With notations as for Theorem 1, let A be a finite p-primary G_S-module, of exponent p^n. Then there is a spectral sequence*

$$E_2^{p,q} = Ext^p_{\Lambda}(H^q(G_{\infty,S}, A)^\vee, \Lambda) \quad\Rightarrow\quad \varprojlim_{k'} \; H^{p+q-1}(G_S(k'),\, A) = Ext^{p+q-1}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n)\,,$$

where, in the inverse limit, k' runs through the finite extension k' of k inside k_∞ and the transition maps are the corestrictions.

Proof As in the proof of Theorem 1, Theorem 11 (d) applies to $G = G_S$ and $H = G_{\infty,S}$. But now the inverse system $(A[p^n])$ is Mittag-Leffler-zero in the sense of [Ja 1]: if the exponent of A is p^d, then the transition maps $A[p^{n+d}] \to A[p^n]$ are zero. This implies that $H^m_{cont}(G_S, (A[p^n])) = 0$ for all $m \geq 0$, cf. [Ja 1]. On the other hand it is clear that the system (A/p^n) is essentially constant ($A/p^n = A$ for $n \geq d$). From Theorem 11 (a) and Lemma 10 we immediately get

$$H^m_{cont}(G_S, G_{\infty,S}; RT_pA) \cong H^{m-1}_{cont}(G_S, G_{\infty,S}; (A/p^n)) \cong \varprojlim_{k'} H^{p+q-1}(G_S(k'),\, A)\,,$$

and hence the claim, by applying Theorem 12 (b) in addition.

References

[Ja1] Jannsen, U.: *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245

[Ja2] Jannsen, U.: *On the l-adic cohomology of varieties over number fields and its Galois cohomology*, Galois groups over Q (Berkeley, CA, 1987), 315–360, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989

[Ja3] Jannsen, U.: *Iwasawa modules up to isomorphism*, Algebraic Number theory, 171–207, Adv. Stud. Pure Math. **17**, Academic Press Boston 1989

[La] M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. IHES **26** (1965), 389–603.