Possibility of generalized monogamy relations for multipartite entanglement beyond three qubits

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We discuss the possibility to interpret the residual entanglement for more than three qubits in terms of distributed multipartite entanglement or, in other words, possible extensions of the Coffman-Kundu-Wootters monogamy equality to higher qubit numbers. Existing knowledge on entanglement in multipartite systems puts narrow constraints on the form of such extensions. We study various examples for families of pure four-qubit states for which the characterization of three-qubit and four-qubit entanglement in terms of polynomial invariants is known. These examples indicate that, although families with such extensions do exist, a generalized monogamy equality cannot be found along those lines.

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I. INTRODUCTION

Getting insight into multipartite entanglement is one of the challenges in quantum information theory. A seminal step toward this goal was the discovery of the analytic expression for pairwise qubit entanglement—the concurrence of arbitrary two-qubit states [1,2]. Interestingly, this measure very soon led to a further breakthrough as there is rather restricted freedom to distribute pairwise entanglement in a three-qubit pure state. This constraint can be cast into the so-called *monogamy relation* [3]: the total amount of entanglement for a given qubit (quantified by the *tangle* or linear entropy) bounds the sum of two-qubit entanglement (measured by the two-tangle) of all pairs with the qubit under consideration.

As for an arbitrary pure three-qubit state, the discrepancy between tangle and the sum of two-tangles is nonzero it was attributed to three-partite entanglement, the three-tangle [3]. Interestingly it turned out that the three-tangle fulfills all requirements for an entanglement measure [4–6], and therefore it indeed quantifies the genuine three-party entanglement [5]. Later, Osborne and Verstraete presented a proof that also for arbitrary pure N-qubit states the tangle is a bound for the total amount of shared pairwise entanglement [7]. However, even to date it is not clear whether also in the general case $N>3$ the difference between tangle and the sum of two-tangles can be expressed in some way in terms of quantities that quantify the distributed multipartite entanglement.

The first studies in this direction have been performed recently [8,9] where specific pure four-qubit states have been analyzed with respect to their tangle and concurrence. As a working hypothesis, the authors assumed monogamylike relations for certain multipartite quantum correlations, with a single four-party correlation for all four monogamy equalities. On that basis, they derived the three-partite and four-partite correlation terms as solutions of the resulting set of linear equations. The conclusion from their analysis was that these three-partite correlations cannot, in general, be identified with the mixed-state three-tangles.

In this work, we choose an alternative approach that is based on polynomial $SL(2,\mathbb{C})$ invariants as multipartite entanglement measures for three qubits [3] and four qubits [10–15]. These invariants are entanglement monotones with respect to stochastic local operations and classical communication (SLOCC) [6]. The relevant ingredient is the analytical solution for the convex roof of the three-tangle for rank-two mixed three-qubit states. A recent analysis provided solutions for various families of such states [16,17] and even for rank-three states [18]. We mention that there are different approaches to describe monogamy properties of multipartite entanglement, e.g., in terms of different entanglement measures [19–23] and also for continuous-variable systems [24,25].

It is important to note that monogamy relations emerge from the concept of distributing entanglement in various ways among many parties (quantified by the corresponding measures) and thus implicitly generate also a classification of multipartite entangled states. On the other hand, it is not *a priori* clear if a complete generalization of monogamy is possible and which one among the many existing approaches to classify multipartite entanglement (e.g., Refs. [5,11,26–28]) allows for such an extension.

In this paper, we first explain in detail which type of generalized monogamy relation we would like to consider (Sec. II). In Secs. III and IV we present various examples for states that do obey the specified type of monogamy, as well as counterexamples. It turns out that there is a family of pure four-qubit states (which we call "telescope states") whose monogamy relation relies on a straightforward extension of the three-qubit Coffman-Kundu-Wootters equality. Conclusions are presented in Sec. V.

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II. STRUCTURE OF GENERALIZED MONOGAMY RELATIONS

The fundamental quantities entering the Coffman-Kundu-Wootters monogamy inequality for multipartite qubit systems are the tangle $\tau_1^{(j)}$ (or "one-tangle") of qubit number j and the two-tangle $\tau_2^{(jk)}=C_{jk}^2$ of qubits number j and k, where C_{jk} is the concurrence of qubits j and k. They are defined from the single and two-qubit reduced density matrices, $\rho_j^{(1)}$ and $\rho_{jk}^{(2)}$, of the N-qubit pure state $\rho=|\psi\rangle\langle\psi|$ as

$$\tau_1^{(j)} := 4 \det \rho_j^{(1)} \tag{1}$$

and

$$C_{jk} := \text{máx}\{0, 2\lambda_{\max} - \text{tr}\sqrt{R_{jk}}\}, \tag{2}$$

where $\lambda_{\max}^2$ is the largest eigenvalue of the positive Hermitian operator

$$R_{jk} := \sqrt{\rho_{jk}^{(2)}}(\sigma_2 \otimes \sigma_2)\rho_{jk}^{*}(\sigma_2 \otimes \sigma_2)\sqrt{\rho_{jk}^{(2)}}, \tag{3}$$

where σ_μ, $\mu=1,2,3$ denote the Pauli matrices and $\sigma_0 \equiv \mathbb{1}$. In terms of these quantities the monogamy relation is expressed as [3,7]

$$\mathcal{R}^{(j)} := \tau_1^{(j)} - \sum_{k\neq j} \tau_2^{(jk)} \geq 0. \tag{4}$$

For pure three qubit states, the residue $\mathcal{R}^{(j)}$ in Eq. (4) turns out to be an entanglement monotone, namely, the three-tangle (or residual tangle):

$$\tau_1^{(j)} - \sum_{k\neq j} \tau_2^{(jk)} = \tau_3. \tag{5}$$

This is the celebrated Coffman-Kundu-Wootters monogamy equality [3]. The three-tangle is most conveniently expressed as

$$\begin{aligned}\tau_3(\psi) &= |\langle\psi^*|\sigma_\mu \otimes \sigma_2 \otimes \sigma_2|\psi\rangle\langle\psi^*|\sigma^\mu \otimes \sigma_2 \otimes \sigma_2|\psi\rangle| \\ &\equiv |\langle\psi^*| \bullet \langle\psi^*|(\sigma_\mu\sigma_2\sigma_2) \bullet (\sigma^\mu\sigma_2\sigma_2)|\psi\rangle \bullet |\psi\rangle| \\ &\equiv |(\sigma_\mu\sigma_2\sigma_2) \bullet (\sigma^\mu\sigma_2\sigma_2)|,\end{aligned} \tag{6}$$

where $\sigma_\mu \bullet \sigma^\mu = \Sigma_\mu G_\mu \sigma_\mu \bullet \sigma_\mu$ with $(G_0, G_1, G_2, G_3) = (-1, 1, 0, 1)$. That is, the three-tangle can be written as an expectation value of an antilinear operator with respect to a twofold copy of the state $|\psi\rangle$. The $\bullet$ in the second and third line of Eq. (6) represents a tensor product and emphasizes the action of the operator on multiple copies (see [11,12,15]).

The main question addressed in this article is whether, for arbitrary number of qubits N, the residue in the monogamy relation (4) can be expressed as a sum of higher tangles, i.e., polynomial SL$(2,\mathbb{C})$ invariants such as the three, four,..., N tangles. This question arises from the intuition of multipartite entanglement as a resource that can be distributed in different ways among the parties [3].

Let us first discuss the possible structure of such an extension in more detail. The Coffman-Kundu-Wootters monogamy relation Eq. (5) as well as the Osborne-Verstraete inequality [Eq. (4)] suggest that entanglement might be an additive resource, i.e., entanglement of a given qubit j with the others is distributed in two- and three-tangle (and possibly higher) which mathematically have to be summed up in some way to give the tangle $\tau_1^{(j)}$.

An important restriction on the structure of a monogamy equality arises from the fact that, for an arbitrary qubit number, inequality (4) saturates for W states [3], i.e., $\mathcal{R}_W^{(j)}=\tau_1^{(j)} - \Sigma_{k\neq j}\tau_2^{(jk)}=0$ (i.e., the entanglement of these globally entangled multiqubit states is distributed in genuine two-qubit entanglement). Consequently, any generalized monogamy relation must be an additive extension to the original monogamy equality, in which the one- and two-tangle must appear exactly in the combination as given in $\mathcal{R}^{(j)}$. For example, one could not have just a power of the sum of two-tangles—the only way to maintain the validity of the relation for W states would be to take a power of $\mathcal{R}^{(j)}$ (however, we note that this would lead to multiply counting pairwise correlations and, in a sense, neglected the interpretation as an additive resource). Thus, a generalized monogamy relation could be of the form

$$\mathcal{R}^{(j)} = h(\tau_3^{(jkl)}, \tau_4^{(jklm)}, \ldots), \tag{7}$$

where h is a positive function of the three, four, and higher tangles involving the jth qubit.

As we would like to discuss monogamy relations containing polynomial invariants, the homogeneity degree (i.e., the number of wave function component factors that occur in the invariants) becomes relevant. In the original monogamy equality (5), we observe that the homogeneity degree is 4 on both sides of the equation. From this we infer that the homogeneity degree on the right-hand side (rhs) of Eq. (7) has to be 4 as well, as we will argue in the following.

Imagine the situation of a pure N-qubit state with *only* N-tangle [no $(N-1)$-, $(N-2)$-tangle, and so on]. The Greenberger-Horne-Zeilinger (GHZ) states are examples for such states. Now, without loss of generality and for better highlighting the central argument, we consider real state coefficients only. Then, the left-hand side (lhs) is a polynomial of degree 4 of the wave function coefficients, whereas for the rhs we are looking for a polynomial invariant in the same wave function coefficients. Clearly, one expects identical expressions on both sides. This means that (at least with the restriction to real states with only N-tangle) the N-tangle on the rhs must be functionally dependent on the one-tangle on the lhs of the relation (as is indeed the case for pure states of two qubits as well as for three qubits).

When focusing on homogeneous polynomial invariants, every invariant of degree larger than 4 satisfying the monogamy relation (or maybe an integer power of it) must be a certain integer power of the one-tangle. The corresponding root of that N-tangle then coincides with the one-tangle, and therefore is a homogeneous function of degree 4.

We can find a further good reason to assume homogeneity degree 4 also for the rhs by considering the invariance of the monogamy relation under SU$(2)^{\otimes N}$ operations *for general states*. The fact that each polynomial SU invariant can be expressed uniquely as a sum of some given generating set of homogeneous polynomials provides another indication that all terms on the rhs must have homogeneity degree 4 as well.

Keeping in mind the conjectured character of multipartite entanglement as an additive resource, we restrict the rhs in Eq. (7) to sums of the form

$$h(\tau_3^{(jkl)},\tau_4^{(jklm)},\ldots)=\sum_{kl\neq j} f_3(\tau_3^{(jkl)})+\sum_{klm\neq j} f_4(\tau_4^{(jklm)})+\cdots.$$

A further restriction comes from the fact that the three-tangle enters the monogamy equality for pure three qubit states. This limits the tripartite entanglement monotone to coincide with the three-tangle on pure states. A remaining freedom is to choose the tripartite measure as the convex roof $\widehat{f(\tau_3)}$ of $f(\tau_3)$, where $f:[0,1]\to[0,1]$ is a strictly monotonous function, and then to consider $f^{-1}(\widehat{f(\tau_3)})$ to obtain a homogeneous function of degree 4. In the remainder of this article we consider monogamy relations for pure states of at most four qubits, i.e., $N\leq 4$. Therefore, the only quantities involved in the residue $\mathcal{R}$ are pure-state four-tangles and mixed-state three tangles. Although physically unmotivated, it is not clear *a priori*, whether a single four-tangle might fix all four monogamy relations. Hence, we analyze possible extended monogamy relations for four qubits of the form

$$\mathcal{R}^{(j)}=\tau_1^{(j)}-\sum_{k\neq j}\tau_2^{(jk)}=\sum_{kl\neq j} f^{-1}(f(\widehat{\tau_3^{(jkl)}}))+\tau_{4;j}. \tag{8}$$

To this end, we will investigate various families of interesting pure four-qubit states for which we are able to compute the mixed-state three-tangle and for which we can make statements about their genuine four-qubit entanglement.

It is worth mentioning that the residual tangle $\mathcal{R}^{(j)}$ vanishes not only for W states but also for product states. This implies that $\tau_4=0$ for all product states, which is a further justification to give major importance to multipartite entanglement measures with this property. The notion of genuine multipartite entanglement measures as introduced in Refs. [11,12] include the requirement for the measure to vanish on arbitrary product states. Such measures form an ideal in the algebra of polynomial $\mathrm{SL}(2,\mathbb{C})$ invariants [15].

III. EXAMPLE

In order to test the possibility of a generalized monogamy relation in a simple but nontrivial case, we may consider four-qubit states for which, however, the three-tangle of the reduced density matrix has to be known. Recently, the three-tangle of a whole family of mixed three-qubit states has been found—namely, for rank-2 mixtures of GHZ states and W states [16,17]. Therefore, we consider four-qubit states that are purifications of those rank-2 states

$$|\Psi_p\rangle=\sqrt{\frac{p}{2}}(|1111\rangle+|1000\rangle)+\sqrt{\frac{1-p}{3}}(|0100\rangle+|0010\rangle+|0001\rangle). \tag{9}$$

In Refs. [11,12], SLOCC invariants for genuine four-partite entanglement in four-qubit states have been studied. The four-tangle of the states [Eq. (9)] is measured only by the quantity

$$\mathcal{F}_1^{(4)}=|(\sigma_\mu\sigma_\nu\sigma_2\sigma_2)\bullet(\sigma^\mu\sigma_2\sigma_\lambda\sigma_2)\bullet(\sigma_2\sigma^\nu\sigma^\lambda\sigma_2)|. \tag{10}$$

The correct homogeneous degree 4 is obtained via $\tau_{4;j}:=s_j(\mathcal{F}_1^{(4)})^{2/3}$. Note that the normalization of τ_4 is not *a priori* clear. We account for it with a scaling factor s_j and find

$$\tau_{4;j}(\Psi_p)=s_j 4\sqrt[3]{\frac{2}{3}}p(1-p). \tag{11}$$

All other four tangles are zero for this state, and therefore the index j can only occur in the scaling factor. Due to the permutation symmetry on the last three qubits, there are two different values for the three-tangle: $\tau_3^{(234)}$ has been determined in Ref. [16] and is zero for $p\leq p_0=4\sqrt[3]{2}/(3+4\sqrt[3]{2})\sim 0.62$, whereas from Ref. [17] a direct calculation leads to $\tau_3^{(123)}=\tau_3^{(124)}=\tau_3^{(134)}=0$ for all p. Furthermore do all two tangles including qubit number 1 vanish and all remaining two-tangles are equal and vanish for $p\geq p_c:=7-\sqrt{45}\sim 0.2918$ [16]. The one-tangles are $\tau_1^{(1)}=4p(1-p)$ and $\tau_1^{(j)}=(2+p)(4-p)/9$ for $j\neq 1$. The validity of a monogamy relation such as Eq. (8) in the interval $0\leq p\leq p_0$ would then imply

$$0=4p(1-p)-4s_1p(1-p) \tag{12}$$

and hence $s_1=1$ for the first qubit, and for the other qubits $s_2=s_3=s_4$ and

$$0=\frac{(2+p)(4-p)}{9}-4s_2p(1-p),\quad p_c\leq p\leq p_0, \tag{13}$$

$$0=\frac{3p(2-5p)}{9}+8(1-p)\sqrt{\frac{p(2+p)}{27}}-4s_2p(1-p),$$

for

$$0\leq p\leq p_c. \tag{14}$$

No scaling factor s_2 can be found to adjust the monogamy relation in all cases. We mention that the monogamy relations cannot even be satisfied on average (that is, for the equally weighted sum of all one-tangles [8]) with a p-independent s_2. We conclude that no extended monogamy relation of form (8) can exist that includes the three-tangle and/or four-tangles, and is valid for arbitrary pure four-qubit states. An analogous analysis can be carried out for other families of states discussed in Ref. [8] and leads to the same conclusion (see the Appendix).

IV. TELESCOPE STATES

The findings in the previous section raise the question: are there any families of states for which monogamy persists? A simple example is

$$|\Psi_{\mathrm{tel}}\rangle:=\alpha|1111\rangle+\beta|1000\rangle+\gamma|0110\rangle. \tag{15}$$

It is straightforward algebra to check that this state contains only two-tangle and three-tangle and that it satisfies the monogamy relations of form (8) with $f\equiv\mathbb{1}$ for all four qubits. This specific state is an example for a pure quantum state in which one (or more) single qubits have a one-to-one correspondence to one (or more) single qubits of a pure quantum state with a reduced number of qubits. Such an $(N+m)$-qubit state emerges from a given pure N-qubit reference state by doubling one (or more) selected qubits by what we will call *telescoping*. This concept has been useful already in Ref. [12] for the creation of maximally entangled states for q qubits from those known for $q-1$ qubits. To give a specific example, from the three-qubit reference state $|\mathcal{M}\rangle = \Sigma_{k=0}^{1} m_k |\mathcal{M}^{(k)}\rangle_{12} \otimes |k\rangle_3$ the four-qubit telescoped state,

$$|\mathcal{T}_{\mathcal{M}}\rangle = \sum_{k=0}^{1} m_k |\mathcal{M}^{(k)}\rangle_{12} \otimes |kk\rangle_{34}, \tag{16}$$

is obtained by simply doubling the third qubit. It is worth mentioning that the concept of telescoping is not reduced to this specific form of extension. It is clear that instead of simple qubit doubling

$$|\psi\rangle \otimes |1\rangle \to |\psi\rangle \otimes |1\rangle \otimes |1\rangle,$$

$$|\psi\rangle \otimes |0\rangle \to |\psi\rangle \otimes |0\rangle \otimes |0\rangle,$$

an arbitrary pair of orthonormal single qubit states, $|\uparrow\rangle_{\vec{n}}$ and $|\downarrow\rangle_{\vec{n}}$, can be used for the extension as

$$|\psi\rangle \otimes |1\rangle \to |\psi\rangle \otimes |1\rangle \otimes |\uparrow\rangle_{\vec{n}},$$

$$|\psi\rangle \otimes |0\rangle \to |\psi\rangle \otimes |0\rangle \otimes |\downarrow\rangle_{\vec{n}}.$$

This amounts to a local unitary transformation on the added qubit *after* telescoping. Note that one can also apply a local unitary transformation on the original state *before* telescoping or even combine both. It is interesting that telescoped product states are product states on the partition induced by the telescoping procedure. Furthermore, telescoping and qubit permutation do not commute.

In the following we analyze the entanglement pattern of the telescoped states. After tracing out one of the telescoped qubits, a biseparable density matrix is obtained. For state (16) this implies

$$\tau_3^{(123)}(\mathcal{T}_{\mathcal{M}}) = \tau_3^{(124)}(\mathcal{T}_{\mathcal{M}}) = 0, \tag{17}$$

$$\tau_2^{(13)}(\mathcal{T}_{\mathcal{M}}) = \tau_2^{(14)}(\mathcal{T}_{\mathcal{M}}) = \tau_2^{(23)}(\mathcal{T}_{\mathcal{M}}) = \tau_2^{(24)}(\mathcal{T}_{\mathcal{M}}) = 0. \tag{18}$$

Furthermore we have $\mathrm{tr}_{3,4}|\mathcal{T}_{\mathcal{M}}\rangle\langle\mathcal{T}_{\mathcal{M}}| = \mathrm{tr}_3|\mathcal{M}\rangle\langle\mathcal{M}|$ and therefore $\tau_2^{(12)}(\mathcal{T}_{\mathcal{M}}) = \tau_2^{(12)}(\mathcal{M})$. Consequently, all single-qubit reduced density matrices and hence all one-tangles coincide for both states. Invoking the three-qubit monogamy relation for the reference state $|\mathcal{M}\rangle$ fixes the values for the four-tangles entering the monogamy relations for the four-qubit telescoped state

$$\tau_{4;1} = \tau_3(\mathcal{M}) + \tau_2^{(13)}(\mathcal{M}) - \tau_3^{(134)}(\mathcal{T}_{\mathcal{M}}), \tag{19}$$

$$\tau_{4;2} = \tau_3(\mathcal{M}) + \tau_2^{(23)}(\mathcal{M}) - \tau_3^{(234)}(\mathcal{T}_{\mathcal{M}}). \tag{20}$$

By using the notation $\tau_{4;j}$ we allow for the possibility that the monogamy relations on different qubits might be satisfied mathematically with different four-tangles—although, from a physical point of view, this would be questionable.

The most surprising feature is the connection between a certain two-tangle of the reference state and a three-tangle of the telescope state. To see this, consider the two decomposition states $|\mathcal{M}^{(k)}\rangle_{12} =: \alpha_{ij}^{(k)}|ij\rangle$ $(k=0,1)$ of $\rho_{23}^{(2)}(\mathcal{M}) = \mathrm{tr}_1|\mathcal{M}\rangle\langle\mathcal{M}|$ and $|\mathcal{T}_{\mathcal{M}}^{(k)}\rangle_{123} =: \alpha_{ij}^{(k)}|ijj\rangle$ of $\rho_{234}^{(3)}(\mathcal{T}_{\mathcal{M}}) = \mathrm{tr}_1|\mathcal{T}_{\mathcal{M}}\rangle\langle\mathcal{T}_{\mathcal{M}}|$ where $i,j=0,1$ represent the computational basis for the respective qubit (we drop the symbol Σ_{ij} for brevity). It is clear that any decomposition of $\rho_{23}^{(2)}(\mathcal{M})$ is telescoped into a decomposition of $\rho_{234}^{(3)}(\mathcal{T}_{\mathcal{M}})$ and vice versa. We now use the expression of the two- and three-tangle in terms of antilinear expectation values [11] and obtain

$$\tau_2(\alpha_{ij}^{(k)}|ij\rangle) = |\alpha_{ij}^{(k)}\alpha_{lm}^{(k)}\alpha_{np}^{(k)}\alpha_{qr}^{(k)}\langle ij|\sigma_2\sigma_2|lm\rangle\langle np|\sigma_2\sigma_2|qr\rangle|, \tag{21}$$

$$\begin{aligned}\tau_3(\alpha_{ij}^{(k)}|ijj\rangle) &= |\alpha_{ij}^{(k)}\alpha_{lm}^{(k)}\alpha_{np}^{(k)}\alpha_{qr}^{(k)}\langle ijj|\sigma_2\sigma_2\sigma_\mu|lmm\rangle \\ &\quad\times\langle npp|\sigma_2\sigma_2\sigma^\mu|qrr\rangle| \\ &= |\alpha_{ij}^{(k)}\alpha_{lm}^{(k)}\alpha_{np}^{(k)}\alpha_{qr}^{(k)}\langle i|\bullet\langle n|\sigma_2\bullet\sigma_2|l\rangle\bullet|q\rangle \\ &\quad\times\langle j|\bullet\langle p|\sigma_2\bullet\sigma_2|m\rangle\bullet|r\rangle \\ &\quad\times\langle j|\bullet\langle p|\sigma_\mu\bullet\sigma^\mu|m\rangle\bullet|r\rangle|. \end{aligned} \tag{22}$$

Now it is sufficient to observe that for states of the computational basis

$$\begin{aligned}\langle j|\bullet\langle p|\sigma_2\bullet\sigma_2|m\rangle\bullet|r\rangle\langle j|\bullet\langle p|\sigma_\mu\bullet\sigma^\mu|m\rangle\bullet|r\rangle \\ = \langle j|\bullet\langle p|\sigma_2\bullet\sigma_2|m\rangle\bullet|r\rangle,\end{aligned} \tag{23}$$

in order to establish that indeed

$$\tau_2^{(23)}(\mathcal{M}) = \tau_3^{(234)}(\mathcal{T}_{\mathcal{M}}), \quad \tau_2^{(13)}(\mathcal{M}) = \tau_3^{(134)}(\mathcal{T}_{\mathcal{M}}). \tag{24}$$

Inserting these results into Eqs. (19) and (20) leads to

$$\tau_{4;1}(\mathcal{T}_{\mathcal{M}}) = \tau_3(\mathcal{M}), \quad \tau_{4;2}(\mathcal{T}_{\mathcal{M}}) = \tau_3(\mathcal{M}). \tag{25}$$

A simple calculation shows that all four-qubit SL-invariant tangles evaluated on telescope states [Eq. (16)] contain the three-tangle of the reference state as a common factor. Hence, if the reference state has no three-tangle, the telescope four-qubit state has no four-tangle. Then the monogamy equality for four qubits is readily satisfied on qubits 1 and 2. Otherwise both four-tangles must coincide with the three-tangle of the reference state.

In order to analyze the general case where the four-tangle is nonzero, we continue by verifying the monogamy relations for qubits 3 and 4. We consider two cases: (i) $\langle\mathcal{M}^{(2)}|\mathcal{M}^{(1)}\rangle_{12}=0$ and (ii) $\langle\mathcal{M}^{(2)}|\mathcal{M}^{(1)}\rangle_{12}\neq 0$. In case (i) $\rho_{34}^{(2)}$ is separable and consequently $\tau_2^{(34)}(\mathcal{T}_{\mathcal{M}})=0$. In addition, the single qubit density matrices on sites 3 and 4 of the telescope state is identical to that on site 3 of the reference state. This implies $\tau_1^{(3)}(\mathcal{T}_{\mathcal{M}}) = \tau_1^{(4)}(\mathcal{T}_{\mathcal{M}}) = \tau_1^{(3)}(\mathcal{M})$, and we are ready to calculate the value of the four-tangle that appears in the monogamy relation

$$\tau_{4;3}(\mathcal{T}_{\mathcal{M}}) = \tau_3(\mathcal{M}); \quad \tau_{4;4}(\mathcal{T}_{\mathcal{M}}) = \tau_3(\mathcal{M}). \tag{26}$$

In case (ii) we can write uniquely $|\mathcal{M}^{(2)}\rangle_{12} = \alpha|\mathcal{M}^{(1)}\rangle_{12} + \beta|\perp\rangle_{12}$ with $\langle\perp|\mathcal{M}^{(1)}\rangle_{12}=0$, and a straightforward calculation shows that the difference between the one-tangles for reference and telescope state compensate precisely with the resulting nonzero two-tangle $\tau_2^{(34)}(\mathcal{T}_{\mathcal{M}})=|\alpha m_0 m_1|^2$. Therefore, Eq. (26) remains unaltered.

Summarizing the above discussion, we conclude that the monogamy relation can be adjusted for telescope states with a single choice for the *value* of the hypothetic four-tangle.

We will now use the trick involved in the equality of the reference state two-tangle with the telescope state three-tangle in order to construct this unknown four-tangle. It can be derived from our finding that the monogamy equality holds if and only if the four-tangle of the telescope state coincides with the three-tangle of the reference state. Using the identity

$$\begin{aligned}\tau_3(\psi) &= |\langle\psi^*|\sigma_\mu\sigma_2\sigma_2|\psi\rangle\langle\psi^*|\sigma^\mu\sigma_2\sigma_2|\psi\rangle| \\ &= |\langle\psi^*|\sigma_2\sigma_\mu\sigma_2|\psi\rangle\langle\psi^*|\sigma_2\sigma^\mu\sigma_2|\psi\rangle| \\ &= |\langle\psi^*|\sigma_2\sigma_2\sigma_\mu|\psi\rangle\langle\psi^*|\sigma_2\sigma_2\sigma^\mu|\psi\rangle|,\end{aligned}$$

we derive the relevant four-qubit polynomial $SL(2,\mathbb{C})$ invariants (notations from Ref. [15]) as

$$\mathcal{C}^{(4)}_{4;(1,4)} \coloneqq (\sigma_\mu\sigma_2\sigma_2\sigma_\nu)\bullet(\sigma^\mu\sigma_2\sigma_2\sigma^\nu), \tag{27}$$

$$\mathcal{C}^{(4)}_{4;(2,4)} \coloneqq (\sigma_2\sigma_\mu\sigma_2\sigma_\nu)\bullet(\sigma_2\sigma^\mu\sigma_2\sigma^\nu). \tag{28}$$

Their absolute values give the corresponding four-tangles that fix all four monogamy relations simultaneously. Due to the relations[15] $\mathcal{C}^{(4)}_{4;(1,4)}=\mathcal{C}^{(4)}_{4;(2,3)}$, $\mathcal{C}^{(4)}_{4;(1,3)}=\mathcal{C}^{(4)}_{4;(2,4)}$, $\mathcal{C}^{(4)}_{4;(1,2)}=\mathcal{C}^{(4)}_{4;(3,4)}$, and $\mathcal{C}^{(4)}_{4;(1,4)}+\mathcal{C}^{(4)}_{4;(2,4)}+\mathcal{C}^{(4)}_{4;(3,4)}=12H^2$, where $H=(\sigma_2^{\otimes 4})/2$ is the four-concurrence from Ref. [29], we can also use $6H^2-\frac{1}{2}\mathcal{C}^{(4)}_{4;(3,4)}$ as the four-tangle. This implies that the three possible four-tangles $\mathcal{C}^{(4)}_{4;(1,4)}$, $\mathcal{C}^{(4)}_{4;(2,4)}$, and $6H^2-\frac{1}{2}\mathcal{C}^{(4)}_{4;(3,4)}$ have the same value on telescope states generated by doubling qubit number 3. It is clear that doubling qubits 1 or 2 leads to analogous expressions. Interestingly, the algebra of polynomial invariants of four-qubit telescope states is generated by two independent elements only. When the third qubit is doubled, then $\mathcal{C}^{(4)}_{4;(1,2)}$ and $\mathcal{C}^{(4)}_{4;(1,3)}$ can be chosen as independent generators. Consequently, all other four-tangles can be expressed uniquely as a polynomial function of them.

However, we stress that there is no *unique* genuine four-qubit entanglement measure that satisfies the four-qubit monogamy equalities for all four-qubit telescope states.

The above-mentioned correspondence of q-tangles of some q-qubit reference state to a set of $(q+m)$-tangles for telescope states generated from the reference state by m-fold qubit doubling is a generic feature and appears for general q and m. Monogamy relations for $3+m$ qubits emerge directly from the Coffman-Kundu-Wootters monogamy relation for pure three-qubit states. The $(q+m)$-tangles satisfying the monogamy relations are found to depend on the specific qubit-doubling procedure that creates the $(q+m)$-qubit state from its reference q-qubit state. We conclude that no general extension to the monogamy relation [Eq. (4)] exist that includes q-tangles with $q>3$ not even for telescope states.

An interesting representative for telescope states is the four-qubit cluster state [30]

$$a|0000\rangle - b|0111\rangle - c|1100\rangle + d|1011\rangle, \tag{29}$$

which has been considered in Ref. [9] (up to a permutation of qubits 2 and 3). The working hypothesis is the same as in Ref. [8]. We confirm the nonzero three-tangles to be $\tau_3^{(134)}=4|ad-bc|^2$ and $\tau_3^{(234)}=4|ab-cd|^2$. With the remaining one and two-tangles the four-tangle that adjusts all four monogamy relations must take the value

$$\tau_{4;j} = \tau_{4,\mathrm{av}} = 4|abcd|. \tag{30}$$

The four-qubit cluster state [Eq. (29)] is detected only by two independent four-qubit SL invariants that vanish on product states. Using the notation from Ref. [11], these are

$$\begin{aligned}\mathcal{F}_2^{(4)} = |\mathfrak{S}\{&(\sigma_\mu\sigma_\nu\sigma_y\sigma_y)\bullet(\sigma^\mu\sigma_y\sigma_\lambda\sigma_y)\bullet(\sigma_y\sigma^\nu\sigma_y\sigma_\tau) \\ &\bullet(\sigma_y\sigma_y\sigma^\lambda\sigma^\tau)\}|,\end{aligned} \tag{31}$$

$$\begin{aligned}\mathcal{F}_3^{(4)} = \Big|\frac{1}{2}&(\sigma_\mu\sigma_\nu\sigma_y\sigma_y)\bullet(\sigma^\mu\sigma^\nu\sigma_y\sigma_y)\bullet(\sigma_\rho\sigma_y\sigma_\tau\sigma_y) \\ &\bullet(\sigma^\rho\sigma_y\sigma^\tau\sigma_y)\bullet(\sigma_y\sigma_\rho\sigma_\tau\sigma_y)\bullet(\sigma_y\sigma^\rho\sigma^\tau\sigma_y)\Big|,\end{aligned} \tag{32}$$

where $\mathfrak{S}$ indicates the symmetrization under four-qubit permutations. It is interesting to note that the value of those measures exponentiated to homogeneous degree 4 is $16|abcd|/\sqrt{3}$, respectively, $16|abcd|$. When we restrict ourselves to the family of telescope states from the third qubit, we find

$$\mathcal{F}_2^{(4)} = \mathcal{C}^{(4)}_{4;(1,3)}\left(\frac{7}{9}\mathcal{C}^{(4)}_{4;(1,3)} + \frac{2}{9}\mathcal{C}^{(4)}_{4;(1,2)}\right),$$

$$\mathcal{F}_3^{(4)} = \frac{1}{2}[\mathcal{C}^{(4)}_{4;(1,3)}]^2\mathcal{C}^{(4)}_{4;(1,2)}.$$

V. CONCLUSIONS

We have analyzed possible extensions of the Coffman-Kundu-Wootters monogamy equality to pure four-qubit states. The known monogamy relations impose tight constraints on such extensions: the tripartite entanglement measure must coincide with the three-tangle on pure states, and the bipartite entanglement has to be measured by the two-tangle in order to respect the inequality due to Osborne and Verstraete.

We have presented a detailed analysis of specific families of pure four-qubit states. The example of the family [Eq. (9)] (as well as the state $|\chi_1\rangle$ in the Appendix) basically rules out that a monogamy relation of form (8) can exist. In particular, there are states that contain only permutation-invariant four-tangle (vanishing two-tangle and three-tangle) while the one-tangles are different. Since any reasonable four-tangle—as a global measure for entanglement—should be permutation invariant, this indicates clearly that a meaningful (i.e., state-independent) extension of the Coffman-Kundu-Wootters monogamy relation to multipartite tangles does not exist. Even averaging over the one-tangles does not eliminate this problem. This also points out that *a priori* assumption of a single four-qubit correlation to fix all four monogamy relations is problematic.

Nevertheless, there are interesting exceptions, that is, families of states which systematically do obey monogamy equalities. We have called these states telescope states as their monogamy properties can be retraced to those of the corresponding three-qubit states from which they can be generated by a qubit-doubling procedure. Interestingly, the four-tangles in *these* states do coincide with the values one can obtain from the known four-qubit polynomial invariants [11,15] which justifies us to name them four-tangles. Note that their homogeneity degree is equal to 4, in analogy with the two- and the three-tangle. We emphasize that the four-tangle in general needs to be chosen according to the qubit-doubling procedure applied to the three-qubit reference state. Consequently, even for the four-qubit telescope states there is no unique extended monogamy relation of the form in Eq. (8).

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APPENDIX: MORE EXAMPLES

Here we reconsider some pure four-qubit states previously analyzed in Ref. [8]. We begin our analysis with

$$|\chi_1\rangle \coloneqq \frac{1}{2}(|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle), \quad \text{(A1)}$$

$$|\chi_2\rangle \coloneqq a|0000\rangle + b|0101\rangle + c|1000\rangle + d|1110\rangle. \quad \text{(A2)}$$

The state $|\chi_1\rangle$ is symmetric under permutation of the last three qubits. In contrast to the nonzero four-partite correlations in Ref. [8], this state has zero four-tangle since every polynomial SL invariant gives zero for that state. This can be easily checked by explicit evaluation of the generating set of SL invariants for four qubits [10,14]. As observed in Ref. [8] $\tau_3^{(234)}=0$, since the reduced three-qubit density matrix is a mixture of a W state with a product state. For the other mixed three-tangles the reduced density matrix is a rank-2 mixture of a GHZ state with a (biseparable) product state such that the three-tangle can be computed by using the methods of Ref. [17]. We obtain $\tau_3^{(123)}=\tau_3^{(124)}=\tau_3^{(134)}=1/4$. Together with the one-tangles $\tau_1^{(1)}=3/4$, $\tau_1^{(2)}=\tau_1^{(3)}=\tau_1^{(4)}=1$ (the two-tangles vanish), this leads to a valid monogamy relation for the first qubit only, whereas for qubits 2,3,4 a mismatch of 1/2 occurs. It must be stressed at this point that no alternative convex-roof extended function of the three-tangle can fix this discrepancy. This is because the reduced density matrices in this case are mixtures of GHZ states and orthogonal product states, and the corresponding characteristic curve is the convex function p^2 (here $p=1/2$) where p is the weight of the GHZ state in the mixture. In this particular case $\widehat{f(\tau_3)} \leq f(\widehat{\tau_3})$ [31] and then $f^{-1}(\widehat{f(\tau_3)}) \leq \hat{\tau}_3$. This is a further proof that no monogamy relation of form (8) including the three-tangle (in some form) can exist for pure states of more than three qubits. This example clearly indicates (in analogy to $|\Psi_p\rangle$ in Sec. III) that—although appealing from a physical point of view—it must not be assumed that the four party residue $\mathcal{R}$ in the monogamy relation be independent of the number of the distinguished qubit.

Finally we analyze $|\chi_2\rangle$ (cf. Ref. [8]). This state has no four-tangle; the three-tangles are calculated as [16,32] $\tau_3^{(123)}=4|ad|^2$, $\tau_3^{(124)}=4|bc|^2$, $\tau_3^{(134)}=4|bd|^2$, and $\tau_3^{(234)}=0$; the two-tangles are $\tau_2^{12}=\tau_2^{13}=\tau_2^{14}=\tau_2^{34}=0$, $\tau_2^{23}=4|dc|^2$, and $\tau_2^{24}=4|ab|^2$; and the one-tangles are obtained as $\tau_1^{(1)}=4(|bc|^2+|d|^2(|a|^2+|b|^2)$, $\tau_1^{(2)}=4(|a|^2+|c|^2)(|b|^2+|d|^2)$, $\tau_1^{(3)}=4|d|^2(1-|d|^2)$, and $\tau_1^{(4)}=4|b|^2(1-|b|^2)$. In this case, the monogamy relations are indeed fulfilled. Since this state is at least not obviously a telescope state, this might be a hint that also nontelescope states can satisfy an extended monogamy relation.

[1] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[2] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[3] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[4] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
[5] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[6] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A **68**, 012103 (2003).
[7] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
[8] Y.-K. Bai, D. Yang, and Z. D. Wang, Phys. Rev. A **76**, 022336 (2007).
[9] Y.-K. Bai and Z. D. Wang, Phys. Rev. A **77**, 032313 (2008).
[10] J.-G. Luque and J.-Y. Thibon, Phys. Rev. A **67**, 042303 (2003).
[11] A. Osterloh and J. Siewert, Phys. Rev. A **72**, 012337 (2005).
[12] A. Osterloh and J. Siewert, Int. J. Quantum Inf. **4**, 531 (2006).
[13] P. Lévay, J. Phys. A **39**, 9533 (2006).
[14] O. Chterental and D. Ž. Đoković, *Linear Algebra Research Advances* (Nova Science, Hauppauge, NY, 2007), Chap. 4, p. 133.
[15] D. Ž. Đoković and A. Osterloh, J. Math. Phys. **50**, 033509 (2009).
[16] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. **97**, 260502 (2006).
[17] C. Eltschka, A. Osterloh, J. Siewert, and A. Uhlmann, New J. Phys. **10**, 043014 (2008).
[18] E. Jung, M.-R. Hwang, D. K. Park, and J.-W. Son, Phys. Rev. A **79**, 024306 (2009).
[19] Y.-C. Ou and H. Fan, Phys. Rev. A **75**, 062308 (2007).
[20] G. Gour, S. Bandyopadhyay, and B. C. Sanders, J. Math. Phys.

48, 012108 (2007).
[21] C. S. Yu and H. S. Song, Phys. Rev. A **77**, 032329 (2008).
[22] S. S. Sharma and N. K. Sharma, Phys. Rev. A **78**, 012113 (2008).
[23] J. S. Kim, A. Das, and B. C. Sanders, Phys. Rev. A **79**, 012329 (2009).
[24] G. Adesso and F. Illuminati, J. Phys. A **40**, 7821 (2007).
[25] G. Adesso and F. Illuminati, Phys. Rev. Lett. **99**, 150501 (2007).
[26] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
[27] A. Mandilara, V. M. Akulin, A. V. Smilga, and L. Viola, Phys. Rev. A **74**, 022331 (2006).
[28] L. Lamata, J. Leon, D. Salgado, and E. Solano, Phys. Rev. A **75**, 022318 (2007).
[29] A. Wong and N. Christensen, Phys. Rev. A **63**, 044301 (2001).
[30] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
[31] In the case of nonconvex characteristic curve, this inequality would apply to concave functions f.
[32] A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. A **77**, 032310 (2008).