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Secondary invariants for string bordism and tmf

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Secondary Invariants for String Bordism and tmf

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Abstract

Using spectral invariants of Dirac operators we construct a secondary version of the Witten genus, namely a bordism invariant of string manifolds in dimensions $4m-1$. We prove a secondary index theorem which relates this global-analytic construction with its homotopy-theoretic analog. The latter will be calculated through its factorization over topological modular forms.

Contents

1 Introduction **2**
- 1.1 Review of the contents . . . 2
- 1.2 Thom spectra . . . 4
- 1.3 The Witten genus, T_{2m}, and b^{geom} . . . 4
- 1.4 Calculations . . . 7
- 1.5 Open problems . . . 8

2 Three-manifolds **8**
- 2.1 A string bordism invariant in dimension 3 . . . 8
- 2.2 A generator for $MString_3$ . . . 11
- 2.3 Atiyah's canonical 2-framing . . . 12
- 2.4 Examples . . . 13

3 Invariants from the Witten genus **15**
- 3.1 Introduction . . . 15
- 3.2 Tamings and η-invariants . . . 16
- 3.3 The Witten genus . . . 16
- 3.4 A string bordism invariant in dimension $4m-1$ . . . 19

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3.5 A geometric expression . . . 21
3.6 A topological expression . . . 22
3.7 Calculation in the case $m = 1$ . . . 23

4 The secondary index theorem **24**
4.1 Construction of b^{top} . . . 24
4.2 The index theorem . . . 28
4.3 Factorization over tmf . . . 29

5 Some details about Adams spectral sequences **31**
5.1 The E_1-term and the cobar complex . . . 31
5.2 Application to tmf . . . 32
5.3 $\mathbb{Q}/\mathbb{Z}$-theory . . . 32
5.4 Application to tmf . . . 34
5.5 Application to ko . . . 34

6 Calculations at 2 **37**
6.1 The result . . . 37
6.2 Transition to $\mathbb{Q}/\mathbb{Z}$-theory . . . 39
6.3 The map to $ko[[q]]$ . . . 39

7 Calculations at 3 **41**
7.1 The result . . . 41
7.2 Transition to $\mathbb{Q}/\mathbb{Z}$-theory . . . 41
7.3 The map to $ko[[q]]$ . . . 42
7.4 How to detect $\nu\Delta$? . . . 43

8 Geometric and homotopy theoretic string bordism **43**

9 Modular forms and the structure of T_{2m} **47**

10 Table of $MSpin_*$ **51**

1 Introduction

1.1 Review of the contents

A well-established principle in the border area between algebraic topology and global analysis predicts that local geometric representations of primary topological invariants lead to interesting secondary constructions. Instances of this principle are the construction of secondary characteristic classes [CS85], the discussion of Adams's e-invariant in [APS75] (see also [BN08, Introduction]), and the construction of Kreck-Stolz invariants [KS].

In the present paper, based on this principle, we construct a secondary invariant for string bordism classes and elements of the homotopy of the spectrum of topological modular forms. We will employ the notion of geometric string structures which was developed recently by Waldorf [Wal].
Here is a short summary of our main results. For all $m \geq 1$ we define an abelian group T_{2m} (Definition 1.1) and construct a homomorphism (Definition 3.2)

$$b^{an} : MString_{4m-1} \to T_{2m} \tag{1}$$

using spectral invariants of Dirac operators and differential geometry. It is a secondary version of the Witten genus. Because of the appearance of spectral invariants, a direct evaluation of b^{an} is complicated but we do have additional tools to analyze b^{an} restricted to the subgroup

$$A_{4m-1} := \ker(MString_{4m-1} \xrightarrow{j} MSpin_{4m-1}) \subseteq MString_{4m-1} \ . \tag{2}$$

This restriction will be shown to coincide with homomorphisms

$$b^{geom}, b^{top} : A_{4m-1} \to T_{2m} \tag{3}$$

defined using differential geometry (28) and homotopy theory (Definition 4.1), respectively. The constructions of the homomorphisms b^{an} and b^{geom} are very similar to the analytic and geometric description of the Adams e-invariant given in [APS75]. Our first main result asserts that

$$b^{an}_{|A_{4m-1}} = b^{geom} = b^{top} \ . \tag{4}$$

The first equality $b^{an}_{|A_{4m-1}} = b^{geom}$ is essentially a consequence of the Atiyah-Patodi-Singer index theorem [APS75] and will be verified during the construction of b^{geom} in Subsection 3.5. The second equality $b^{geom} = b^{top}$, shown in Theorem 4.2, is deeper, and the resulting equality $b^{an}_{|A_{4m-1}} = b^{top}$ can be considered as a secondary index theorem. In the proof we use ideas we learned from [Lau00]. The analogy with the secondary index theorem for the e-invariant of Adams [Ada63] as formulated in [BN08, Introduction] will be explained in Subsection 4.2.
We will give a complete and explicit homotopy theoretic calculation of b^{top}. It is based on the factorization (Proposition 4.3)

$$\begin{array}{ccc} MString_{4m-1} & \xrightarrow{\sigma} & tmf_{4m-1} \\ \uparrow & & \downarrow b^{tmf} \\ A_{4m-1} & \xrightarrow{b^{top}} & T_{2m} \end{array}$$

and the explicit calculation of b^{tmf} (Propositions 6.1 and 7.1) the construction of which is similar to the one of b^{top}. Here, tmf denotes the connective spectrum of topological modular forms of Goerss-Hopkins-Miller and Lurie [Goe09], and σ is induced by the tmf-valued Witten genus which has been constructed as an E_∞-ring map $\sigma : MString \to tmf$ in [AHR].

Since the homotopy groups of string bordism and tmf are interesting and complicated, it is useful to relate these objects non-trivially to geometry and to global analysis.

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The charts of the spectral sequences have been compiled using the TEX-package of T. Bauer. The homotopy groups of MSpin have been calculated with the help of MAPLE.

1.2 Thom spectra

In the remainder of this introduction we give a more detailed overview. We first recall the definition of the Thom spectra appearing in the present paper. Let

$$\cdots \to BO\langle 8\rangle \to BO\langle 4\rangle \to BO\langle 2\rangle \to BO \tag{5}$$

be the first stages of the Postnikov tower of the H-space BO. The classical names of these spaces are

$$BSO = BO\langle 2\rangle\ , \quad BSpin = BO\langle 4\rangle\ , \quad BString = BO\langle 8\rangle\ .$$

There is a functor from the category of H-spaces over BO to ring spectra which associates to a map $\xi : X \to BO$ the Thom spectrum X^ξ. The Thom spectrum for the map $BO\langle i\rangle \to BO$ will be denoted with $MO\langle i\rangle$. In particular we will write

$$MSO = MO\langle 2\rangle\ , \quad MSpin = MO\langle 4\rangle\ , \quad MString = MO\langle 8\rangle\ .$$

These spectra are related by morphisms of ring spectra

$$MString \xrightarrow{j} MSpin \to MSO \to MO\ .$$

1.3 The Witten genus, T_{2m}, and b^{geom}

For the purpose of this introduction, we explain the construction of b^{geom} which is the simplest of the three maps in (4). In particular we want to motivate the Definition 1.1 of the group T_{2m}. The map b^{geom} is a secondary version of the Witten genus. We let KO denote the real K-theory spectrum. For a (ring) spectrum X we let $X[[q]]$ denote the (ring) spectrum which represents the homology theory $X[[q]]_*(B) := X_*B[[q]]$ for all finite CW-complexes B. The Witten genus is a morphism of ring spectra

$$R : MSpin \to KO[[q]]\ .$$

We refer to Subsection 3.3 for a description of this map as a transformation of homology theories.

For a spectrum X we let $X_* := \pi_*(X)$ denote the homotopy groups of X. Via the Thom-Pontrjagin construction, elements in the homotopy of Thom spectra like $MSpin$ or $MString$ correspond to bordism classes of manifolds with corresponding structures on the tangent bundles (see Subsection 8). Thus homotopy classes of $MString$ correspond to bordism classes of closed string manifolds $[(M,\alpha^{top})]$, where a string manifold (M,α^{top}) consists of a spin manifold M together with the choice of a topological string structure α^{top}, see Subsection 8 for details. The morphism $j : MString_* \to MSpin_*$ just forgets the topological string structure, i.e. it is given geometrically by $j([(M,\alpha^{top})]) = [M]$.
We now give a description of the Witten genus following the exposition [HBJ92]. Let $[M] \in MSpin_{4m}$ be a homotopy class represented by a closed $4m$-dimensional spin manifold M. We choose a connection ∇^{TM} on the tangent bundle TM of M. In general, if ∇^V is a connection on a real vector bundle $V \to B$ over some manifold B, then by $p_i(\nabla^V) \in \Omega^{4i}(B)$ we denote the Chern-Weil representative of the i'th Pontrjagin class $p_i(V) \in H^{4i}(B;\mathbb{Z})$. We set $\kappa_m := 1$ for even m, and $\kappa_m := \frac{1}{2}$ in the case of odd m. In (17) we define a power series

$$\Phi \in \mathbb{Q}[[q]][[p_1, p_2, \dots]] \ .$$

With this notation we have the following local expression for the Witten genus

$$R([M]) = \kappa_m \int_M \Phi(p_1(\nabla^{TM}), p_2(\nabla^{TM}), \dots) \ , \tag{6}$$

which a priori is an element of $\mathbb{Q}[[q]]$. By the Atiyah-Singer index theorem, the coefficients of this formal power series can be identified with indices of twisted Dirac operators. More precisely, we have

$$R([M]) = \kappa_m \sum_{n \geq 0} q^n \mathtt{index}(D_M \otimes R_n(TM)) \ ,$$

where $R_n(TM)$ is a certain virtual bundle derived from the tangent bundle (see (19)) for all $n \geq 0$, and $D_M \otimes R_n(TM)$ denotes the spin Dirac operator of M twisted by $R_n(TM)$. Since the indices are integers and even in the case of odd m (because of an additional real symmetry) we see that

$$R([M]) \in \mathbb{Z}[[q]] \ .$$

For a subring $R \subseteq \mathbb{C}$ we denote by $\mathcal{M}^R_{2m}$ the R-module of modular forms for $SL(2,\mathbb{Z})$ of weight $2m$ with a q-expansion in the subring $R[[q]] \subseteq \mathbb{C}[[q]]$. Using the q-expansion we will identify $\mathcal{M}^R_{2m}$ with a sub-R-module of $R[[q]]$. For more information on modular forms we refer to Subsection 9.
It is a crucial observation for our constructions that the Witten genus has the following factorization

$$\begin{array}{ccccccc} & & & & & & \mathcal{M}^{\mathbb{Z}}_{2m} \\ & & & & & \nearrow^{\cdots} & \big\downarrow \\ MString_{4m} & \xrightarrow[j]{} & MSpin_{4m} & \xrightarrow[R]{} & KO[[q]]_{4m} & \xrightarrow[\cong]{} & \mathbb{Z}[[q]] \end{array} \ .$$

The construction of the secondary invariant b^{geom} starts with the local formula (an integral over characteristic forms) for the Witten genus given by the right-hand side of (6). It will be applied to a spin zero-bordism Z of a string manifold (M, α^{top}) and gives, combined with a contribution from a geometric string structure α, a formal power series in $\mathbb{R}[[q]]$. In order to ensure independence of the choices of the geometry and the zero bordism we have to calculate modulo everything that comes from closed spin manifolds and string zero bordisms. This leads us to the definition of the following quotient group.

Definition 1.1 *The group T_{2m} is defined by*

$$T_{2m} := \frac{\mathbb{R}[[q]]}{\mathbb{Z}[[q]] + \mathcal{M}^{\mathbb{R}}_{2m}} .$$

We now turn to the construction of the secondary invariant $b^{geom}([(M, \alpha^{top})])$. It is based on the existence of a spin zero bordism of the string manifold (M, α^{top}). We therefore have to assume that the class $[(M, \alpha^{top})] \in MString_{4m-1}$ is in the subgroup $A_{4m-1} \subseteq MString_{4m-1}$ defined in (2).
Let (M, α^{top}) be a $4m-1$-dimensional string manifold such that $[(M, \alpha^{top})] \in A_{4m-1}$. Then we can choose a spin zero bordism Z of M. Furthermore, we choose a connection ∇^{TZ} extending a connection ∇^{TM} on the tangent bundle TM of M.
A topological string structure α^{top} is by definition a "trivialization" of the spin characteristic class $\frac{p_1}{2}(TM) \in H^4(M; \mathbb{Z})$. A geometric refinement α of α^{top} trivializes this class on the form level, i.e. α gives rise to a form $H_\alpha \in \Omega^3(M)$ such that $dH_\alpha = \frac{1}{2} p_1(\nabla^{TM})$ (see [Wal]). This can be used to define a refinement $\tilde{p}_1(\nabla^{TZ}, \alpha) \in \Omega^4(Z)$ of $p_1(\nabla^{TZ})$ which vanishes on the boundary M of Z (see Subsection 3.6). The value of b^{geom} on the class $[(M, \alpha^{top})] \in MString_{4m-1}$ is then given (compare Lemma 3.5) as the class

$$b^{geom}([(M, \alpha^{top})]) := \left[\int_Z \Phi(\tilde{p}_1(\nabla^{TZ}, \alpha), p_2(\nabla^{TZ}), \dots) \right] \in T_{2m} .$$

In contrast to b^{geom}, the analytic variant $b^{an}([(M, \alpha^{top})]) \in T_{2m}$ in Definition 3.1 provides an intrinsic formula which does not depend on the choice of a spin zero bordism and is therefore defined on all of $MString_{4m-1}$ rather than just on the subgroup $A_{4m-1} \subseteq MString_{4m-1}$. It involves spectral invariants of the twisted Dirac operators $D_M \otimes R_n(TM)$ for all $n \geq 0$. In Lemma 3.4 we establish the equality

$$b^{geom} = b^{an}_{|A_{4m-1}} .$$

The construction of all variants of b is based on the interplay between the facts that $MString_{4m-1}$ is torsion and $KO[[q]]_{4m} \otimes \mathbb{Q}$ is non-trivial. This explains the restriction to dimensions of the form $4m-1$. In order to detect elements of $MString_*$ in dimensions $0, 1, 2, 4 \bmod 8$ one can use the Witten genus $R \circ j : MString \to KO[[q]]$ directly.

1.4 Calculations

In Section 2 we consider the case of three-manifolds, i.e. the case $m = 1$. In this case it suffices to consider the constant term of the formal power series representing $b^{geom}([(M, \alpha^{top})])$. This simplifies matters considerably and justifies a separate discussion. We will see that

$$b^{geom} : MString_3 \cong \mathbb{Z}/24\mathbb{Z} \hookrightarrow T_2 = \mathbb{R}[[q]]/\mathbb{Z}[[q]]$$

is injective. As a side result we get the following analog for spin manifolds of Atiyah's canonical 2-framings of oriented three-manifolds [Ati90]: *A 3-dimensional connected closed spin manifold has a canonical topological string structure (Definition 2.6).*
In higher dimensions the homotopy groups $MString_*$ are not fully understood. The following facts are known [Hov97].

1. $MString_k$ is a finite group for $k \equiv 1, 2, 3 \mod 4$.
2. The p-torsion of $MString_*$ is trivial for every prime $p \geq 5$.
3. The 3-torsion of $MString_*$ is annihilated by multiplication with 3.

The spin bordism groups $MSpin_*$ are calculated additively in [ABP66] (see Section 10 for a table of $MSpin_*$). We will use the following facts:

1. $MSpin_k$ is a finite group for $k \equiv 1, 2, 3 \mod 4$.
2. For all $k \geq 0$ the torsion $MSpin_{k,tors} \subseteq MSpin_k$ is a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$.
3. $MSpin_{4m-1} = 0$ for $m \leq 9$.

We get the following consequences for the subgroup $A_{4m-1} \subseteq MString_{4m-1}$:

1. For all $m \geq 1$, the group $A_{4m-1} \subseteq MString_{4m-1}$ contains all 3-torsion elements.
2. For all $m \geq 1$ we have $2 \cdot MString_{4m-1} \subseteq A_{4m-1}$.
3. We have $A_{4m-1} = MString_{4m-1}$ for $m \leq 9$.

Because of our lack of knowledge of A_{4m-1}, for an explicit description of b^{top} Proposition 4.3 is a crucial observation. It asserts that b^{top} has a factorization

$$\begin{array}{ccc} MString_{4m-1} & \xrightarrow{\sigma} & tmf_{4m-1} \\ \uparrow & & \downarrow b^{tmf} \\ A_{4m-1} & \xrightarrow{b^{top}} & T_{2m} \end{array} \quad .$$

In contrast to $MString$, the homotopy groups of tmf are known [Bau08]. In Propositions 6.1 and 7.1 we obtain a complete calculation of b^{tmf}. In particular, the 3-torsion of tmf_{4m-1} is completely detected by b^{tmf}.
This calculation implies that b^{top} is non-trivial in arbitrarily high dimensions. Let us explain some examples.

- There exists a 3-torsion element $x \in tmf_{27}$ which goes to the element with the name $\nu\Delta \in tmf_{(3),27}$ under localization at 3. By Proposition 7.1 we know that $b^{tmf}(x) \neq 0$.

  By a result of Hopkins-Mahowald [Hop02, Theorem 6.25] the tmf-valued Witten genus
$$\sigma : MString_* \to tmf_*$$
  is surjective. We can choose an element $y \in A_{27} = MString_{27}$ such that $\sigma(y) = x$. We then have
$$b^{top}(y) \stackrel{4.3}{=} b^{tmf}(x) \neq 0 \ .$$

  If (M, α) is a closed 27-dimensional string manifold one can try to calculate the element $\sigma([(M, \alpha)]) \in tmf_{(3),27} = (\mathbb{Z}/3\mathbb{Z})\,\nu\Delta$. An answer in terms of characteristic numbers is given in Subsection 7.4. Similar statements can be produced in higher dimensions using the 72-periodicity of $tmf_{(3),*}$ given by multiplication with Δ^3.

- Let $x \in tmf_{192}$ be an element which goes to $\Delta^8 \in tmf_{(2),192}$ under localization at 2. Then by Hopkins-Mahowald [Hop02, Theorem 6.25] there exists a class $y \in MString_{192}$ such that $\sigma(y) = x$. Let $g \in A_3 = MString_3$ be the generator (13). Note that $yg \in A_{195}$. The element $\sigma(yg) \in tmf_{195}$ goes to $\nu\Delta^8 \in tmf_{(2),195}$ under localization at 2. By Proposition 6.1, the order of $b^{top}(yg) = b^{tmf}(\nu\Delta^8)$ is 8.

1.5 Open problems

We close with stating some open problems:

1. What is $A_{4m-1} \subseteq MString_{4m-1}$? For which m do we have an equality?
2. What is the image $\sigma(A_{4m-1}) \subseteq tmf_{4m-1}$? For which m do we have an equality?
3. What is $b^{an}(x) \in T_{2m}$ for $x \in MString_{4m-1} \setminus A_{4m-1}$? One could conjecture that
$$b^{an} = b^{tmf} \circ \sigma \ .$$

2 Three-manifolds

2.1 A string bordism invariant in dimension 3

In the present subsection we give a geometric construction of a homomorphism
$$d : MString_3 \to \mathbb{Z}/24\mathbb{Z} \ .$$

We will see in Corollary 2.4 that d is an isomorphism. It is known that $MSpin_3 = 0$. Let M be a closed spin three-manifold M. Then we can find a spin zero bordism Z. We choose a connection ∇^{TM} on TM. It naturally induces a connection on the $Spin(3)$-principal

bundle given by the spin structure. We furthermore choose an extension of ∇^{TM} to a connection ∇^{TZ}.
Let us fix a topological string structure α^{top} on M. As a principal bundle can be equipped with a connection, a topological string structure α^{top} on a spin bundle with connection can be refined to a geometric string structure α. For details we refer to [Wal]. A geometric string structure α on M gives rise to a 3-form H_α which satisfies $dH_\alpha = \frac{1}{2}p_1(\nabla^{TM})$ (this condition is non-vacuous in higher dimensions). We can form the difference[1]

$$d_Z(M,\alpha) := \frac{1}{2}\int_Z p_1(\nabla^{TZ}) - \int_M H_\alpha \in \mathbb{R} \ .$$

Lemma 2.1 *The real number $d_Z(M,\alpha)$ is independent of the choice of connections and the geometric data of the string structure.*

Proof. First observe that the difference does not depend on the extension of the connection to the bordism Z. Indeed, given two extensions $\nabla^{TZ}, \nabla^{TZ,\prime}$ we form the closed manifold $W := Z \cup_M -Z$ with the induced connection ∇^{TW} which restricts to ∇^{TZ} and $\nabla^{TZ,\prime}$ on the two obvious copies of Z in W. We then have

$$\int_Z p_1(\nabla^{TZ}) - \int_Z p_1(\nabla^{TZ,\prime}) = \int_W p_1(\nabla^{TW}) = 3\ \mathtt{sign}(W) \ .$$

But $\mathtt{sign}(W) = 0$ because of the obvious orientation-reversing $\mathbb{Z}/2\mathbb{Z}$-symmetry of W.
Two connections on TM can be joined by a connection $\nabla^{T(I\times M)}$ on the cylinder $I \times M$. Similarly, two geometric string structures α and α' refining the same underlying topological string structure can be connected by a geometric string structure $\tilde{\alpha}$ on $I \times M$. Then we have by Stokes' theorem

$$\frac{1}{2}\int_{I\times M} p_1(\nabla^{T(I\times M)}) = \int_{I\times M} dH_{\tilde{\alpha}} = \int_M H_\alpha - \int_M H_{\alpha'} \ . \tag{7}$$

We choose $Z' := Z \cup_M (I \times M)$ as the zero bordism for the primed choices. Then

$$\frac{1}{2}\int_{Z'} p_1(\nabla^{TZ,\prime}) - \frac{1}{2}\int_Z p_1(\nabla^{TZ}) = \frac{1}{2}\int_{I\times M} p_1(\nabla^{T(I\times M)}) \ . \tag{8}$$

The assertion now follows from the combination of (7) and (8). Alternatively one could conclude the result from Lemma 2.2 below and a continuity argument. □

As a consequence of Lemma 2.1, the integral $\int_M H_\alpha$ does not depend on the geometry of α so that we can also write it as $\int_M H_{\alpha^{top}}$. Moreover we can write $d_Z(M,\alpha) = d_Z(M,\alpha^{top})$.

[1] This difference has also been considered in the recent (and independent) paper [Red], where it is put in relation with the e-invariant of Adams.

The set of topological string structures on M is a torsor under $H^3(M;\mathbb{Z}) \cong \mathbb{Z}$. We will write the action additively, see (46). Note that (see [Wal])

$$\int_M H_{\alpha^{top}+x} = \int_M H_{\alpha^{top}} + \langle x, [M]\rangle \ , \quad \forall x \in H^3(M,\mathbb{Z}) \ . \tag{9}$$

Therefore

$$d_Z(M, \alpha^{top} + x) = d_Z(M, \alpha^{top}) - \langle x, [M]\rangle \ . \tag{10}$$

Lemma 2.2 *We have $d_Z(M, \alpha) \in \mathbb{Z}$.*

Proof. We let $\widehat{H\mathbb{Z}}^*$ denote the differential integral cohomology functor. It has first been introduced by Cheeger-Simons [CS85]. For an axiomatic picture see [BS09]. Differential integral cohomology is the home for geometric refinements of Chern and Pontrjagin classes. Let

$$\frac{\hat{p}_1}{2}(\nabla^{TM}) \in \widehat{H\mathbb{Z}}^4(M)$$

denote the lift introduced in [CS85] of the spin characteristic class $\frac{p_1}{2}(TM)$ (see also [Bun09b, Sec. 4.2]). Its integral over M is an element of $\mathbb{R}/\mathbb{Z}$. By the bordism formula for the evaluation of differential cohomology classes we have the equality

$$[\frac{1}{2}\int_Z p_1(\nabla^{TZ})] = \int_M \frac{\hat{p}_1}{2}(\nabla^{TM})$$

in $\mathbb{R}/\mathbb{Z}$. We also know (see e.g. [Bun09b, Sec. 4.2]) that

$$a(H_\alpha) = \frac{\hat{p}_1}{2}(\nabla^{TM}) \ ,$$

where $a : \Omega^3(M)/\mathtt{im}(d) \to \widehat{H\mathbb{Z}}^4(M)$ is one of the structure maps of differential cohomology. It follows that

$$[d_Z(M, \alpha^{top})] = [\frac{1}{2}\int_Z p_1(\nabla^{TZ}) - \int_M H_\alpha] = 0 \in \mathbb{R}/\mathbb{Z} \ .$$

□

By the specialization of Hirzebruch's signature theorem to the four-dimensional case, the signature of a closed oriented 4-dimensional spin manifold W is given by

$$\mathtt{sign}(W) = \frac{1}{3}\int_W p_1(\nabla^{TW}) \ . \tag{11}$$

Similarly, by the specialization of the Atiyah-Singer index theorem, the index of the spin Dirac operator of a four-dimensional spin manifold W is given by

$$\mathtt{index}(D_W) = -\frac{1}{24}\int_W p_1(\nabla^{TW}) \ . \tag{12}$$

In dimensions $8m+4$ the spin Dirac operator has an additional real symmetry which forces its index to be even. It follows that

$$\frac{1}{2}\int_W p_1(\nabla^W) \equiv 0 \mod 24 .$$

¿From the additivity of the signature we conclude that the class

$$d(M,\alpha^{top}) := [d_Z(M,\alpha^{top})] \in \mathbb{Z}/24\mathbb{Z}$$

is independent of the choice of the zero bordism Z.

Lemma 2.3 *$d(M,\alpha^{top}) \in \mathbb{Z}/24\mathbb{Z}$ is a string bordism invariant.*

Proof. If α has an extension to a geometric string structure $\tilde{\alpha}$ over Z, then by Stokes' theorem

$$\frac{1}{2}\int_Z p_1(\nabla^{TZ}) = \int_Z dH_{\tilde{\alpha}} = \int_M H_\alpha .$$

□

It is now clear that

$$d : MString_3 \to \mathbb{Z}/24\mathbb{Z} , \quad [(M,\alpha^{top})] \mapsto d(M,\alpha^{top})$$

is well-defined and a homomorphism.

2.2 A generator for $MString_3$

In this subsection we construct a specific generator $g \in MString_3$ such that $d(g) = [1] \in \mathbb{Z}/24\mathbb{Z}$. It will be used in later calculations. Let S denote the sphere spectrum, and let $\epsilon : S \to MString$ be the unit of the ring spectrum $MString$. It is known [Hov97, Thm 2.2.1] that $\epsilon : S_3 \to MString_3$ is an isomorphism so that $MString_3 \cong S_3 \cong \mathbb{Z}/24\mathbb{Z}$.
We consider the sphere $S^3 \subset \mathbb{R}^4$. As the boundary of the disc $D^4 \subset \mathbb{R}^4$ it has a preferred orientation, spin structure and string structure α^{top}. We define

$$g := [S^3, \alpha^{top} - \mathtt{or}_{S^3}] \in MString_3 , \tag{13}$$

where $\mathtt{or}_{S^3} \in H^3(S^3;\mathbb{Z})$ is the orientation class of S^3. Since (S^3,α^{top}) bounds a string manifold we have $d_Z(S^3,\alpha^{top}) = 0$. It follows immediately from (10) that $d(g) = [1] \in \mathbb{Z}/24\mathbb{Z}$. Since the order of $d(g)$ is 24 we see that $g \in MString_3$ is a generator.

Corollary 2.4 *The homomorphism*

$$d : MString_3 \to \mathbb{Z}/24\mathbb{Z} , \quad [(M,\alpha^{top})] \mapsto d(M,\alpha^{top})$$

is an isomorphism

We will systematically extend this construction to higher dimensions in Section 3.

2.3 Atiyah's canonical 2-framing

In this subsection we discuss, as a side aspect of the main topic of the present paper, an analogue for spin manifolds of an observation by Atiyah [Ati90] saying that oriented three-manifolds have canonical 2-framings. We will show that every closed connected spin three-manifold has a canonical string structure.

Let us first explain the result of Atiyah [Ati90]. For all $n \geq 2$ the horizontal composition in the following diagram of Lie groups

$$\begin{array}{ccccc} & & & & Spin(2n) \\ & & & \nearrow & \downarrow \\ SO(n) & \xrightarrow[\mathtt{diag}]{} & SO(n) \times SO(n) & \longrightarrow & SO(2n) \end{array}$$

has a unique lift indicated by the dotted arrow. This implies that the double $2V := V \oplus V$ of a n-dimensional real oriented vector bundle V has a canonical spin structure.

A 2-framing of a closed oriented three-manifold M is by definition a spin-trivialization of the double $2TM$ of its tangent bundle. Atiyah now considers an oriented zero bordism Z of M. The spin bundle $2TZ$ is trivialized by the 2-framing α at the boundary $\partial Z \cong M$. This trivialization refines the spin characteristic class $\frac{p_1}{2}(2TZ) \in H^4(Z;\mathbb{Z})$ to a relative cohomology class

$$\frac{p_1}{2}(2TZ, \alpha) \in H^4_c(Z, M; \mathbb{Z}) \ .$$

Atiyah then observes that

$$\sigma(\alpha) := 3\mathtt{sign}(Z) - \langle \frac{p_1}{2}(2TZ, \alpha), [Z, M] \rangle \in \mathbb{Z}$$

does not depend on the oriented zero bordism Z. Furthermore, by changing α, we can alter the right-hand side by any integer. It follows that there is a unique 2-framing α_0 such that $\sigma(\alpha_0) = 0$. This is the canonical 2-framing of M.

We now define the canonical topological string structure of a three-dimensional closed connected spin manifold M. Let α^{top} be any topological string structure on M. The combination

$$\sigma(M, \alpha^{top}) := 3\mathtt{sign}(Z) - 2d_Z(M, \alpha^{top}) \in \mathbb{Z}$$

is independent of the choice of the spin zero bordism Z of M (by the same argument as in [Ati90] which employs additivity of the signature and the formula (11)). Its class

$$\sigma(M) := [\sigma(M, \alpha^{top})] = [3\mathtt{sign}(Z) - 2d_Z(M, \alpha^{top})] = [\mathtt{sign}(Z)] \in \mathbb{Z}/2\mathbb{Z}$$

is independent of the choice of the string structure α^{top} as well. We will see from the examples below that both possible values do occur.

Proposition 2.5 *A closed connected spin three-manifold M has a unique topological string structure α_0^{top} characterized by $\sigma(M, \alpha_0^{top}) \in \{0, 1\}$.*

Definition 2.6 *The topological string structure uniquely characterized in Proposition 2.5 will be called the canonical string structure.*

2.4 Examples

In this subsection we calculate two examples which show that both possible values in of the invariant $\sigma(M) \in \mathbb{Z}/2\mathbb{Z}$ and thus of $\sigma(M, \alpha_0^{top}) \in \{0, 1\}$ of a connected three-dimensional spin manifold do really occur.

1. The three-dimensional compact group $SO(3)$ is diffeomorphic to $\mathbb{R}??\mathbb{P}^3$. We thus have

$$H^*(SO(3); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^4)\ , \quad |x| = 1\ .$$

The manifold $SO(3)$ fits into a principal bundle $SO(2) \to SO(3) \xrightarrow{\pi} S^2$. The circle $SO(2)$ acts by the adjoint representation on the quotient of Lie algebras $so(3)/so(2)$. We fix a basis element of the Lie algebra $so(2)$. It determines a complex structure on $so(3)/so(2)$ by requiring that it rotates counterclockwise. It furthermore induces a fundamental vector field on $SO(3)$ which trivializes the vertical bundle $T^v\pi$. The vertical bundle therefore gets a spin and a topological string structure $\alpha^{top}_{T^v\pi}$. This spin structure induces the bounding spin structure on all fibers of π.

Furthermore, using the identification $TS^2 \cong SO(3) \times_{SO(2)} so(3)/so(2)$, we fix a complex structure on TS^2. This complex structure determines the orientation of S^2. Since $H^1(S^2; \mathbb{Z}/2\mathbb{Z}) = 0$ and $w_2(TS^2) = 0$ it can be refined to a spin structure in a unique way. Furthermore, there exists a unique topological string structure $\alpha^{top}_{S^2}$ on the spin manifold S^2. Up to homotopy there is a unique splitting

$$TSO(3) \cong T^v\pi \oplus \pi^*TS^2. \tag{14}$$

The spin structures of the summands induce a spin structure s_0 on $SO(3)$. Since $H^1(SO(3); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ there is actually a second spin structure s_1. Since the restriction of $H^1(SO(3); \mathbb{Z}/2\mathbb{Z}) \to H^1(\pi^{-1}(x); \mathbb{Z}/2\mathbb{Z})$ is an isomorphism for all $x \in S^2$ the spin structure s_1 induces the non-bounding spin structure on every fiber of π. We continue with the spin structure s_0 and choose the topological string structure α^{top} such that (14) becomes an isomorphism of string bundles.

The trivialization of $T^v\pi$ induces a geometric string structure $\alpha_{T^v\pi}$ refining $\alpha^{top}_{T^v\pi}$ with $H_{\alpha_{T^v\pi}} = 0$. We further choose a geometric string structure α_{S^2} which refines $\alpha^{top}_{S^2}$. We have $H_{\alpha_{S^2}} = 0$ since S^2 is two-dimensional. We choose the geometric string structure α refining α^{top} as the sum of $\alpha_{T^v\pi}$ and $\pi^*\alpha_{S^2}$. Then we have $H_\alpha = 0$.

The action of $SO(2)$ on $so(3)/so(2)$ fixes a metric and induces an action on the unit disc D^2. If we fix the point $1 \in D^2$, then we can identify $SO(2)$ with the boundary of D^2, the orbit of 1. In this way we get an identification $SO(3) \cong \partial Z$ with the boundary of the four-dimensional manifold

$$Z := SO(3) \times_{SO(2)} D^2\ .$$

The D^2-bundle $q : Z \to S^2$ exhibits this manifold as a fiberwise zero bordism of $SO(3)$. It can be identified with the unit-disc bundle in the tangent bundle TS^2.

The vertical bundle of q is therefore given by $T^v q \cong q^* TS^2$. Up to homotopy we have a unique decomposition

$$TZ \cong T^v q \oplus q^* TS^2 \cong q^*(TS^2 \oplus TS^2) \ .$$

It gives TZ a complex structure and therefore an orientation. Since the projection $q : Z \to S^2$ is a homotopy equivalence and $H^1(S^2; \mathbb{Z}/2\mathbb{Z}) = 0$ we see that the spin structure on TZ induced by the spin structure on TS^2 and this decomposition is the unique one. The restriction of the spin structure on $T^v q$ to each fiber of π is obviously the bounding one. It follows that the restriction of the spin structure on Z to its boundary is the spin structure s_0 fixed above. Hence Z is a spin zero bordism of $SO(3)$.

We have identified $SO(3)$ with the unit sphere bundle in TS^2. This unit sphere bundle is the same as the orthonormal complex frame bundle. Therefore the complex bundle $\pi^* TS^2 \cong T^v q_{|SO(3)}$ is canonically trivialized.

We can choose a complex connection $\nabla^{T^v q}$ which is compatible with this trivialization, and we define ∇^{TZ} as the sum of $\nabla^{T^v q}$ and $q^* \nabla^{TS^2}$, where ∇^{TS^2} is any complex connection of TS^2. Then we have

$$p_1(\nabla^{TZ}) = -c_2(\nabla^{TZ}) = -c_1(\nabla^{T^v q}) \wedge q^* c_1(\nabla^{TS^2}) \ .$$

By a standard calculation[2] we get $\int_{Z/S^2} c_1(\nabla^{T^v q}) = 1$. It follows that

$$\int_Z p_1(\nabla^{TZ}) = -\int_{S^2} c_1(\nabla^{TS^2}) = -2 \ .$$

This gives $d_Z(SO(3), \alpha^{top}) = -1$.

Using again that $q : Z \to S^2$ is a homotopy equivalence we see that $H_2(Z; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the orientation of S^2. The self intersection of this class, which geometrically can be considered as the zero section of TS^2, is the Euler characteristic $\chi(S^2) = 2$. It is positive so that $\mathtt{sign}(Z) = 1$. We conclude that

$$\sigma(SO(3), \alpha^{top}) = 3 + 2 = 5 \ .$$

It follows that

$$\sigma(SO(3)) = 1 \ .$$

[2]Without any explicit calculation of integrals this can be seen as follows. We consider the complex line bundle $D^2 \times \mathbb{C} \to D^2$ with the trivialization over S^1 given by the tautological section $s : S^1 \to \mathbb{C}$ on the boundary. Using this trivialization we glue it with the trivial bundle $\bar{D}^2 \times \mathbb{C}$, where $\bar{D}$ has the opposite complex structure, in order to get a holomorphic bundle $L \to S^2 \cong \mathbb{CP}^1$. We choose a connection on $D^2 \times \mathbb{C}$ which is compatible with the trivialization at the boundary. This connection can then be extended to ∇^L as a trivial connection across $\bar{D}$. The section $z : D^2 \to D^2 \times \mathbb{C}$ extends by 1 to a holomorphic section of L. Furthermore, the constant section $1 : D^2 \to D^2 \times \mathbb{C}$ can be extended by the holomorphic function $\bar{z}$ to $\bar{D}$. It is easy to see that these two generate the two-dimensional space of holomorphic sections of $L \to \mathbb{CP}^2$. By the Riemann-Roch theorem $2 = 1 + \int_{\mathbb{CP}^1} c_1(L) = 1 + \int_{D^2} c_1(\nabla^L)$ and therefore $\int_{D^2} c_1(\nabla^L) = 1$.

The canonical string structure of $SO(3)$ with the spin structure s_0 fixed above is given by
$$\alpha_0^{top} = \alpha^{top} - 2\ \mathtt{or}_{SO(3)}\ .$$

2. We now consider the torus $\mathbb{T}^3 \cong \mathbb{R}^3/\mathbb{Z}^3$. This representation as a quotient fixes a trivialization of $T\mathbb{T}^3$ and therefore a spin and geometric string structure α with $H_\alpha = 0$. The induced spin structure on each circle subgroup of $\mathbb{T}^3$ is bounding. We write $\mathbb{T}^3 \cong S^1 \times \mathbb{T}^2$ and can consider S^1 as the spin boundary of the disc D^2. Therefore $Z := D^2 \times \mathbb{T}^2$ is a spin zero bordism of $\mathbb{T}^3$. We choose ∇^{TZ} such that respects this decomposition and the factor $\mathbb{T}^2$ is flat. Then $p_1(\nabla^{TZ}) = 0$. It follows that $d_Z(\mathbb{T}^3, \alpha^{top}) = 0$. The intersection form on $H_2(Z;\mathbb{Z})$ vanishes so that $\mathtt{sign}(Z) = 0$. We conclude that
$$\sigma(\mathbb{T}^3) = 0$$
for the spin structure chosen above, and that the canonical string structure of $\mathbb{T}^3$ with this spin structure is given by
$$\alpha_0^{top} = \alpha^{top}\ .$$

3 Invariants from the Witten genus

3.1 Introduction

In Section 2, using geometric string structures, we have defined a homomorphism
$$d : MString_3 \to \mathbb{Z}/24\mathbb{Z}$$
which turned out to be an isomorphism. In the present section, with the construction of the homomorphisms $b^{geom} : A_{4m-1} \to T_{2m}$, we generalize this to higher dimensions.
It is interesting to observe that using the index theorem of Atiyah-Patodi-Singer [APS75] we can give an alternative expression for d which does not involve the zero bordism Z. This parallels the treatment of the e-invariant of Adams given in [APS75]. We will actually work in the slightly different setting of index theory for manifolds with boundary which has been developed in [Bun09a], and which we will briefly review in Subsection 3.2.
We consider a closed string three-manifold (M, α^{top}) and a spin zero bordism Z. Let us assume that ∇^{TM} is the Levi-Civita connection associated to a Riemannian metric on M, and that ∇^{TZ} is the Levi-Civita connection for an extension of that metric to Z with a product structure. These geometric structures turn the manifolds M and Z into geometric manifolds $\mathcal{M}$ and $\mathcal{Z}$. If we choose a (real) taming $\mathcal{M}_t$, then we get a boundary tamed manifold $\mathcal{Z}_{bt}$. The index $\mathtt{index}(\mathcal{Z}_{bt})$ of the Fredholm operator associated to a boundary tamed manifold can be calculated by the formula [Bun09a, Thm. 2.2.18]. It follows from the presence of a real structure that $\mathtt{index}(\mathcal{Z}_{bt})$ is even. In the case at hand we have
$$\mathtt{index}(\mathcal{Z}_{bt}) = -\frac{1}{24}\int_Z p_1(\nabla^{TZ}) + \eta(\mathcal{M}_t)\ ,$$

where $\eta(\mathcal{M}_t)$ is the eta-invariant of the tamed manifold $\mathcal{M}_t$, see Subsection 3.2. Therefore we have the following equality in $\mathbb{R}/\mathbb{Z}$:

$$[\frac{1}{2}\int_Z p_1(\nabla^{TZ})] = [12\eta(\mathcal{M}_t)] \ .$$

We furthermore choose a geometric refinement α of the topological string structure α^{top} based on the spin connection induced by ∇^{TM}. Then we get the expression

$$d(M,\alpha) = [12\eta(\mathcal{M}_t) - \int_M H_\alpha] \in \mathbb{Z}/24\mathbb{Z}$$

which is now intrinsic to M. With the construction of $b^{an} : MString_{4m-1} \to T_{2m}$ (see Definition 3.1) we generalize this analytic formula to higher dimensions.

3.2 Tamings and η-invariants

In this subsection we recall the necessary language of local index theory for manifolds with boundary. The main reference for the set up is [Bun09a] which specialized to manifolds with boundary can be considered as a variant of [APS75]. If M is a $4m-1$-dimensional closed Riemannian spin manifold and $V \to M$ is a real vector bundle with metric h^V and connection ∇^V, then we can form the twisted Dirac operator $D_M \otimes V$ which acts on sections of the bundle $S(M) \otimes_{\mathbb{R}} V$, where $S(M) \to M$ is the spinor bundle. A taming of $D_M \otimes V$ is by definition a selfadjoint operator Q acting on sections of $S(M) \otimes V$ and given by a smooth integral kernel such that $D_M \otimes V + Q$ is invertible. If m is odd, then the spinor bundle has a quaternionic structure. If the taming respects this quaternionic structure, then we call it a real taming. If m is even, then $S(M)$ has a real structure, and a real taming should respect this structure. The obstruction against the existence of a real taming is $\mathtt{index}(D_M \otimes V) \in KO_{4m-1}$. Since $KO_{4m-1} = 0$ for all $m \geq 1$ such real tamings always exist. Following the notation introduced in [Bun09a] we will call a Riemannian spin manifold a geometric manifold $\mathcal{M}$, and we denote such a manifold with an additional choice of a geometric bundle $\mathbf{V} := (V, h^V, \nabla^V)$ by $\mathcal{M} \otimes \mathbf{V}$. Finally, this data together with a choice of a real taming will be denoted by $(\mathcal{M} \otimes \mathbf{V})_t$ and called a tamed manifold. The η-invariant $\eta((\mathcal{M} \otimes \mathbf{V})_t) \in \mathbb{R}$ of a tamed manifold is defined by

$$\eta((\mathcal{M} \otimes \mathbf{V})_t) := \frac{-1}{\sqrt{\pi}} \int_0^\infty \mathtt{Tr}(D_M + Q)e^{-t^2(D_M+Q)^2} dt \ .$$

These definitions can be extended to graded or virtual bundles V in a natural way.

3.3 The Witten genus

In this subsection we recall the Witten genus $R : MSpin \to KO[[q]]$ and introduce several related formal power serieses. We use the correspondence between an even formal power series $\phi \in A[[x^2]]$ and a power series $K_\phi \in A[[p_1, p_2, \dots]]$ given by

$$\prod_{i=1}^{\infty} \phi(x_i) = K_\phi(p_1, p_2, \dots) \ , \quad \sum_{i \geq 0} p_i = \prod_{i=1}^{\infty} (1 + x_i^2) \ ,$$

where A is some commutative $\mathbb{Q}$-algebra, e.g. $A = \mathbb{Q}[[q]]$ in the example which follows. We let (see [HBJ92, Ch. 6.3])

$$\phi_W(x, q) \in \mathbb{Q}[[q]][[x]]$$

be given by

$$\begin{aligned} \phi_W(x, q) \ &:= \ \frac{\frac{x}{2}}{\sinh(\frac{x}{2})} \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^x)(1 - q^n e^{-x})} \\ &= \ \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} x^{2k}\right] e^{G_2(q)x^2} \ . \end{aligned} \tag{15}$$

Here

$$G_{2k} := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \in \mathbb{Q}[[q]] \tag{16}$$

with the Bernoulli numbers B_{2k} and the sum $\sigma_{2k-1}(n)$ of the $(2k-1)$'th-powers of the positive divisors of n. For $k \geq 2$ the power series G_{2k} is the q-expansion of a modular form of weight $2k$. We have $G_{2k} \in \mathcal{M}_{2k}^{\mathbb{Q}}$, since the constant term of the formal power series G_{2k} is rational, while all higher terms are integral.
We will also need the series

$$\theta_W(x, q) := \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} x^{2k}\right] \in \mathbb{Q}[[q]][[x]] \ .$$

We define

$$\Phi := K_{\phi_W} \in \mathbb{Q}[[q]][[p_1, p_2, \dots]] \ , \quad \Theta := K_{\theta_W} \in \mathbb{Q}[[q]][[p_1, p_2, \dots]] \ . \tag{17}$$

For $k \geq 1$ we let

$$N_{2k}(p_1, \dots) = \sum_{j=1}^{\infty} x_j^{2k} \in \mathbb{Q}[[p_1, \dots]]$$

be the Newton polynomials. Then we can write

$$\Phi(p_1, \dots) = \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} N_{2k}(p_1, \dots)\right] e^{G_2 p_1} \in \mathbb{Q}[[q]][[p_1, \dots]] \ .$$

We define

$$\tilde{\Phi}(p_1, \dots,) := \exp\left[\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k} N_{2k}(p_1, \dots)\right] \sum_{j=1}^{\infty} \frac{G_2^j p_1^{j-1}}{j!} \in \mathbb{Q}[[q]][[p_1, \dots]] \ .$$

More systematically

$$\tilde{\Phi} = \Theta \frac{e^{G_2 p_1} - 1}{p_1} \ . \tag{18}$$

We now turn to the definition of the Witten genus. For a real l-dimensional vector bundle $V \to B$ we define the formal power series of virtual bundles

$$R(V) := \sum_{n \geq 0} q^n R_n(V) = \prod_{k \geq 1} (1 - q^k)^{2l} \bigotimes_{k \geq 1} Sym_{q^k}(V \otimes_{\mathbb{R}} \mathbb{C}) \ , \tag{19}$$

where

$$Sym_q(V) := \bigoplus_{n \geq 0} q^n Sym^n(V)$$

is the generating power series of the symmetric powers $Sym^n(V)$ of V. For a manifold Z we then have

$$\hat{\mathbf{A}}(TZ) \cup \mathbf{ch}(R(TZ)) = \Phi(p_1(TZ), \dots) \ . \tag{20}$$

If ∇^{TZ} is a connection on TZ, then we will write

$$\Phi(\nabla^{TZ}) := \Phi(p_1(\nabla^{TZ}), \dots) \in \Omega^{4*}(Z)[[q]] \ ,$$

and we will use a similar convention for $\tilde{\Phi}$ and Θ.
The homomorphism

$$R : MSpin_{4m} \to \mathbb{Z}[[q]] \cong KO[[q]]_{4m}$$

is given by

$$R([M]) = \kappa_m \mathtt{index}(D_M \otimes R(TM)) = \kappa_m \langle \Phi(p_1(TM), p_2(TM), \dots), [M] \rangle . \tag{21}$$

In the case of odd m the factor $\kappa_m = \frac{1}{2}$ takes care of the factor 2 on the lower horizontal arrow of the commutative diagram

$$\begin{array}{ccc} KO_{4m} & \longrightarrow & K_{4m} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{\ 2\ } & \mathbb{Z} \end{array} \ ,$$

since we take the index of the twisted Dirac operator as a complex operator.
The above construction can be refined to a multiplicative natural transformation of homology theories

$$R : MSpin \to KO[[q]] \ .$$

Let $f : M \to X$ be a map from a closed l-dimensional spin manifold M to a space X representing the bordism class $[M, f] \in MSpin_l(X)$. If we choose a Riemannian metric on M, and $V \to M$ is a real vector bundle with connection, then we can define the real K-homology class $[D_M \otimes V] \in KO_l(M)$ of the Dirac operator of M twisted by V. It is independent of the choice of the geometric structures. The value of

$$R : MSpin_l(X) \to KO[[q]]_l(X)$$

on $[M,f]\in MSpin_l(X)$ is then given by

$$R([M,f]) := \sum_{n\geq 0} q^n f_*[D_M\otimes R_n(TM)] \in KO_l(X)[[q]] = KO[[q]]_l(X)\ .$$

The formal power series defined in (19) is multiplicative in the sense that $R(V\oplus W) = R(V)\otimes R(W)$. This easily implies that the transformation R is multiplicative.

3.4 A string bordism invariant in dimension $4m-1$

Let $m\geq 1$ and consider a $4m-1$-dimensional closed string manifold (M,α^{top}). We choose a Riemannian metric on M and get a geometric manifold $\mathcal{M}$. The Riemannian metric gives a Levi-Civita connection ∇^{TM} which together with the trivial connection on the trivial bundle $\Theta_{\mathbb{R}} = M\times\mathbb{R}$ induces a connection on the virtual bundles $R_n(TM\oplus\Theta_{\mathbb{R}})$ for all $n\geq 0$. We choose real tamings $(\mathcal{M}\otimes R_n(TM\oplus\Theta_{\mathbb{R}}))_t$ for all $n\geq 0$. Finally we choose a geometric refinement α of the topological string structure α^{top}. It gives rise to a form $H_\alpha\in\Omega^3(M)$ satisfying $dH_\alpha = \frac{1}{2}p_1(\nabla^{TM})$.

Definition 3.1 *We define the formal power series*

$$\tilde{b}^{an}(\mathcal{M},\alpha,t) := 2\kappa_m\int_M H_\alpha\wedge\tilde{\Phi}(\nabla^{TM}) + \kappa_m\sum_{n\geq 0} q^n\eta((\mathcal{M}\otimes R_n(TM\oplus\Theta_{\mathbb{R}}))_t) \in \mathbb{R}[[q]]\ , \quad (22)$$

where

$$\kappa_m = \begin{cases} 1 & m\equiv 0(2) \\ \frac{1}{2} & m\equiv 1(2)\end{cases}\ ,$$

and we choose tamings which are compatible with the real structure.

The last entry in the list of variables of $\tilde{b}^{an}$ indicates the dependence of the formal power series on the choice of the tamings.
Recall the Definition 1.1 of the group T_{2m}.

Definition 3.2 *We let*

$$b^{an}(M,\alpha^{top}) := [\tilde{b}^{an}(\mathcal{M},\alpha,t)] \in T_{2m}$$

denote the class in T_{2m} represented by the formal power series $\tilde{b}^{an}(\mathcal{M},\alpha,t)$.

This notation is justified by the following Lemma.

Lemma 3.3 *The class $b^{an}(M,\alpha^{top})\in T_{2m}$ is an invariant of the string bordism class of the $4m-1$-dimensional string manifold (M,α^{top}). In particular, it is independent of the choice of geometric structures and the taming involved in the definition of $\tilde{b}^{an}(\mathcal{M},\alpha,t)$.*

Proof. Let $(Z,\tilde{\alpha})$ be a string bordism between (M,α) and (M',α'). Then we choose a Riemannian metric on Z which extends the given metrics on M and M' with product structures. We have decompositions of geometric bundles $TZ_{|M}\cong TM\oplus\Theta_{\mathbb{R}}$ and $TZ_{|M'}\cong TM'\oplus\Theta_{\mathbb{R}}$, where the trivial summands correspond to the normal bundle. Therefore, for all $n\geq 0$ the geometric bundles $R_n(TZ)$ extend the geometric bundles $R_n(TM\oplus\Theta_{\mathbb{R}})$ and $R_n(TM'\oplus\Theta_{\mathbb{R}})$ The tamings $(\mathcal{M}\otimes R_n(TM\oplus\Theta_{\mathbb{R}}))_t$ and $(\mathcal{M}'\otimes R_n(TM'\oplus\Theta_{\mathbb{R}}))_{t'}$ induce a boundary taming $(\mathcal{Z}\otimes R_n(TZ))_{bt}$. The index theorem [Bun09a, Thm. 2.2.18] for boundary tamed manifolds gives

$$\begin{aligned}&\mathtt{index}(\mathcal{Z}_{bt}\otimes R_n(TZ))\\ &\quad=\int_Z \hat{\mathbf{A}}(\nabla^{TZ})\wedge\mathbf{ch}(\nabla^{R_n(TZ)})+\eta((\mathcal{M}'\otimes R_n(TM'\oplus\Theta_{\mathbb{R}}))_{t'})-\eta((\mathcal{M}\otimes R_n(TM\oplus\Theta_{\mathbb{R}}))_t)\ .\end{aligned}$$

If m is odd, then $\mathtt{index}(\mathcal{Z}_{bt})$ is even. Using that by (20) we have

$$\sum_{n\geq 0} q^n\hat{\mathbf{A}}(\nabla^{TZ})\wedge\mathbf{ch}(\nabla^{R_n(TZ)})=\Phi(\nabla^{TZ})\ ,\tag{23}$$

we get the following equality in T_{2m}.

$$\begin{aligned}&\left[\kappa_m\sum_{n\geq 0}q^n\big(\eta((\mathcal{M}'\otimes R_n(TM'\oplus\mathbb{R}))_{t'})-\eta((\mathcal{M}\otimes R_n(TM\oplus\mathbb{R}))_t)\big)\right]\\ &\qquad\qquad = \ [-\kappa_m\int_Z\Phi(\nabla^{TZ})]\end{aligned}\tag{24}$$

Using Stoke's theorem, (18), and $2dH_{\tilde{\alpha}}=p_1(\nabla^{TZ})$ we calculate

$$\begin{aligned}2\int_{M'}H_{\alpha'}\wedge\tilde{\Phi}(\nabla^{TM'})-2\int_M H_\alpha\wedge\tilde{\Phi}(\nabla^{TM}) &= \int_Z 2dH_{\tilde{\alpha}}\wedge\tilde{\Phi}(\nabla^{TZ})\\ &= \int_Z\Theta(\nabla^{TZ})e^{G_2p_1(\nabla^{TZ})}-\int_Z\Theta(\nabla^{TZ})\\ &= \int_Z\Phi(\nabla^{TZ})-\int_Z\Theta(\nabla^{TZ})\ .\end{aligned}\tag{25}$$

Note that $\int_Z\Theta(\nabla^{TZ})\in\mathcal{M}^{\mathbb{R}}_{2m}$ so that

$$\left[2\kappa_m\int_{M'}H_{\alpha'}\wedge\tilde{\Phi}(\nabla^{TM'})-2\kappa_m\int_M H_\alpha\wedge\tilde{\Phi}(\nabla^{TM})\right]=\left[\kappa_m\int_Z\Phi(\nabla^{TZ})\right]\tag{26}$$

in T_{2m}. If we combine (26) and (18), then we get $b^{an}(M,\alpha)=b^{an}(M',\alpha')$.
The independence of $b^{an}(M,\alpha^{top})$ from the geometric structures and tamings follows from the bordism invariance since we can connect two choices of geometric structures and tamings by corresponding structures on a cylinder $Z=[0,1]\times M$ which provides a bordism. □

The invariant $b^{an}(M,\alpha^{top})$ is clearly additive under disjoint unions of string manifolds. We have therefore defined a homomorphism

$$b^{an} : MString_{4m-1} \to T_{2m} \ , \quad [(M,\alpha^{top})] \mapsto b^{an}(M,\alpha^{top}) \ . \tag{27}$$

The main goal of the present paper is to understand the homotopy theoretic meaning of this global analytic construction.

3.5 A geometric expression

Since the construction of b^{an} involves spectral invariants of Dirac operators, a direct evaluation of b^{an} is complicated. Our further analysis of b^{an} relies on the comparison with a homotopy theoretic version $b^{top} : A_{4m-1} \to T_{2m}$. The comparison between $b^{an}_{|A_{4m-1}}$ and b^{top} is achieved via an intermediate construction

$$b^{geom} : A_{4m-1} \to T_{2m}$$

using differential geometry which we will present in the present subsection.
We consider a closed string manifold (M,α^{top}) of dimension $4m-1$ which represents an element in $[M,\alpha^{top}] \in A_{4m-1} \subseteq MString_{4m-1}$. Then we can choose a spin zero bordism Z of M. We choose a connection ∇^{TZ} on TZ and let ∇^{TM} be its restriction to $M \cong \partial Z$. Furthermore, we let α be a geometric string structure which refines α^{top} based on the spin connection on TM induced by ∇^{TM}. Then we can define the formal power series

$$\tilde{b}^{geom}(M,\alpha,\nabla^{TZ}) := 2\kappa_m \int_M H_\alpha \wedge \tilde{\Phi}(\nabla^{TM}) - \kappa_m \int_Z \Phi(\nabla^{TZ}) \in \mathbb{R}[[q]] \ . \tag{28}$$

Lemma 3.4 *In T_{2m} we have the equality*

$$[\tilde{b}^{geom}(M,\alpha,\nabla^{TZ})] = b^{an}(M,\alpha^{top}) \ .$$

Proof. We first observe that $[\tilde{b}^{geom}(M,\alpha,\nabla^{TZ})]$ does not depend on the choice of the connection ∇^{TZ} and the corresponding geometric refinement α of α^{top}. Let ∇^{TZ}_i and α_i, $i=0,1$, be two choices of geometric structures. Then we can find a connection ∇^{TW} on $W := [0,1]\times Z$ which restricts to ∇^{TZ}_i on $\{i\}\times Z$. Furthermore we can find a string structure $\tilde{\alpha}$ on $V := [0,1]\times M$ which connects the string structures α_i, $i=0,1$. From (25) we get

$$2\int_M H_{\alpha_1} \wedge \tilde{\Phi}(\nabla^{TM}_1) - 2\int_M H_{\alpha_0} \wedge \tilde{\Phi}(\nabla^{TM}_0) = \int_V \Phi(\nabla^{TV}) - \int_V \Theta(\nabla^{TV}) \ .$$

The last term belongs to $\mathcal{M}^{\mathbb{R}}_{2m}$ so that

$$\left[2\kappa_m \int_M H_{\alpha_1} \wedge \tilde{\Phi}(\nabla^{TM}_1) - 2\kappa_m \int_M H_{\alpha_0} \wedge \tilde{\Phi}(\nabla^{TM}_0)\right] = \left[\kappa_m \int_V \Phi(\nabla^{TV})\right] \tag{29}$$

in T_{2m}. Again by Stoke's theorem we have

$$\begin{aligned} 0 &= \int_W d\Phi(\nabla^{TW}) = \int_{\partial W} \Phi(\nabla^{TW}) \\ &= \int_Z \Phi(\nabla_1^{TZ}) - \int_Z \Phi(\nabla_0^{TZ}) - \int_V \Phi(\nabla_0^{TV}) \ . \end{aligned} \tag{30}$$

If we combine (30) and (29), then we get $[\tilde{b}^{geom}(M, \alpha_1, \nabla_1^{TZ})] = [\tilde{b}^{geom}(M, \alpha_0, \nabla_0^{TZ})]$.
We now choose a Riemannian metric on Z with product structure near the boundary which extends the metric of M. Then we get the geometric manifold $\mathcal{Z}$. The taming $(\mathcal{M} \otimes R_n(TM \oplus \Theta_{\mathbb{R}}))_t$ induces a boundary taming $(\mathcal{Z} \otimes R_n(TZ))_{bt}$, and by the index formula

$$\kappa_m \int_Z \hat{\mathbf{A}}(\nabla^{TZ}) \wedge \mathbf{ch}(\nabla^{R_n(TZ)}) + \kappa_m \eta((\mathcal{M} \otimes R_n(TM \oplus \mathbb{R}))_t) = \kappa_m \mathtt{index}((\mathcal{Z} \otimes R_n(TZ))_{bt}) \in \mathbb{Z}$$

for all $n \geq 0$. Using (23) we get the equality of classes

$$\left[-\kappa_m \int_Z \Phi(\nabla^{TZ})\right] = \left[\kappa_m \sum_{n\geq 0} q^n \eta((\mathcal{M} \otimes R_n(TM \oplus \mathbb{R}))_t)\right]$$

in T_{2m}. Combining this with (22) and (28) we conclude that

$$[\tilde{b}^{geom}(M, \alpha, \nabla^{TZ})] = [\tilde{b}^{an}(\mathcal{M}, \alpha, t)] = b^{an}(M, \alpha^{top}) \ .$$

□

As a consequence of Lemma 3.4 we have constructed a homomorphism

$$b^{geom} : A_{4m-1} \to T_{2m}$$

which maps $[M, \alpha^{top}] \in A_{4m-1}$ to the class in T_{2m} represented by the formal power series $b^{geom}(M, \alpha, \nabla^{TZ})$ for some choices of the spin zero bordism Z and the geometric structures. Of course, we have the equality

$$b^{an}_{|A_{4m-1}} = b^{geom} \ .$$

3.6 A topological expression

In this subsection we retain the assumptions of Subsection 3.5 and transform the geometric expression $[\tilde{b}(M, \alpha, \nabla^{TZ})] = b^{geom}(M, \alpha^{top})$ into a purely topological formula. Let $[0, 1) \times M \hookrightarrow Z$ be a collar and consider a function $\chi \in C_c^\infty[0, 1)$ with $\chi \equiv 1$ near 0. Using the normal variable of the collar this function can be transported to the collar. Its extension by zero gives a smooth function on Z which we will also denote by $\chi \in C^\infty(Z)$.
Let $p : [0, 1) \times M \to M$ denote the projection. The form $d(\chi p^* H_\alpha)$ can be considered as a closed form on Z supported near M. We now define the closed form

$$\tilde{p}_1(\nabla^{TZ}, \alpha) := p_1(\nabla^{TZ}) - 2d(\chi p^* H_\alpha) \in \Omega^4_c(Z) \ .$$

It represents a relative cohomology class

$$\tilde{p}_1(TZ,\alpha^{top}) := [\tilde{p}_1(\nabla^{TZ},\alpha)] \in H^4(Z,M;\mathbb{R}) \ . \tag{31}$$

We define

$$\hat{b}^{geom}(M,\alpha^{top},Z) := \langle -\kappa_m \tilde{\Phi}(TZ) \cup \tilde{p}_1(TZ,\alpha^{top}), [Z,M]\rangle \in \mathbb{R}[[q]] \ .$$

Lemma 3.5 *In T_{2m} we have the equality*

$$b^{geom}(M,\alpha^{top}) = [\hat{b}^{geom}(M,\alpha^{top},Z)] \ .$$

Proof. Using (18) we write

$$\begin{aligned}
\Theta(\nabla^{TZ}) &= \Phi_W(\nabla^{TZ}) - \tilde{\Phi}_W(\nabla^{TZ}) \wedge p_1(\nabla^{TZ}) \\
&= \Phi_W(\nabla^{TZ}) - \tilde{\Phi}_W(\nabla^{TZ}) \wedge \tilde{p}_1 - 2\tilde{\Phi}_W(\nabla^{TZ}) \wedge d(\chi H_\alpha) \\
&= \Phi_W(\nabla^{TZ}) - \tilde{\Phi}_W(\nabla^{TZ}) \wedge \tilde{p}_1 - 2d(\tilde{\Phi}_W(\nabla^{TZ}) \wedge \chi H_\alpha) \ .
\end{aligned}$$

Since $\int_Z \Theta(\nabla^{TZ}) \in \mathcal{M}^{\mathbb{R}}_{2m}$ and

$$\int_Z 2d(\tilde{\Phi}(\nabla^{TZ}) \wedge \chi H_\alpha) = 2\int_M \tilde{\Phi}(\nabla^{TZ}) \wedge H_\alpha$$

we conclude that

$$[b^{geom}(M,\alpha,\nabla^{TZ})] = [-\kappa_m \int_Z \tilde{\Phi}(\nabla^{TZ}) \wedge \tilde{p}_1] = [\hat{b}^{geom}(M,\alpha,Z)]$$

in T_{2m}.

3.7 Calculation in the case $m = 1$

In the present subsection we explain the relation between the three-dimensional constructions in Section 2 and the higher-dimensional theory of the present section. We have $\kappa_1 = \frac{1}{2}$. In (13) we have fixed a generator $g \in MString_3$ in the form

$$g = [S^3, \alpha^{top} - \mathtt{or}_{S^3}] = [S^3, \alpha^{top} - \mathtt{or}_{S^3}] - [S^3, \alpha^{top}] \ ,$$

where α^{top} is the topological string structure coming from the representation of S^3 as the boundary of the disc $D^4 \subset \mathbb{R}^4$ with its natural topological string structure.
We use formula (28) in order to calculate $b^{geom}(g) \in T_2$. Because of the structure of the representative of g as a difference, the contribution of the zero bordism Z drops out. In the three-dimensional case only the cohomological-degree 0 part $\tilde{\Phi}_{[0]} = G_2$ of $\tilde{\Phi}$ contributes. Using (9) we obtain

$$b^{geom}(g) = b^{geom}([S^3,\alpha^{top} - \mathtt{or}_{S^3}]) - b^{geom}([S^3,\alpha^{top}]) = \left[-2\kappa_m G_2 \langle \mathtt{or}_{S^3}, [S^3]\rangle\right] = [-G_2] \ .$$

Note that

$$G_2 = -\frac{1}{24} + q + \dots \ .$$

The higher terms of the q-expansion are all integral. This implies

$$b^{geom}(g) = [\frac{1}{24}] \in T_2 = \mathbb{R}/\mathbb{Z}[[q]] \ . \tag{32}$$

Therefore $b^{geom}(g)$ has order 24, and we see that $b^{geom} : MString_3 \to T_2$ is injective. This shows in particular that the homomorphism $b^{geom} = b^{an}$ is non-trivial at least in some examples, an observation which will be generalized substantially in Sections 6.1 and 7.1. Let $j : \mathbb{Z}/24\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the inclusion such that $j([1]) = [\frac{1}{24}]$. Then the relation between d in Corollary 2.4 and b^{geom} can be subsumed by saying that the following diagram commutes:

$$\begin{array}{ccc} MString_3 & \xrightarrow{d} & \mathbb{Z}/24\mathbb{Z} \\ \downarrow{\scriptstyle b^{geom}} & & \downarrow{\scriptstyle j} \\ T_2 & \xrightarrow{p_0} & \mathbb{R}/\mathbb{Z} \end{array} \ .$$

Here, $p_0 : T_2 = \mathbb{R}/\mathbb{Z}[[q]] \to \mathbb{R}/\mathbb{Z}$ takes the constant coefficient.

4 The secondary index theorem

4.1 Construction of b^{top}

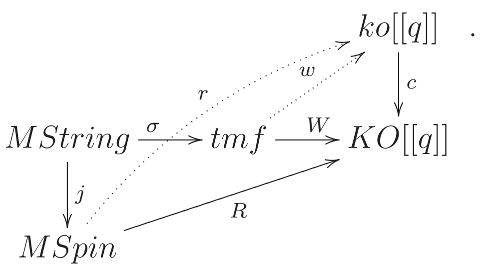
In this subsection for all $m \geq 1$ we give a homotopy theoretic construction of a homomorphism

$$b^{top} : A_{4m-1} \to T_{2m} \ .$$

We let $ko \to KO$ denote a connective cover of the real K-theory spectrum. Then we get an induced connective cover

$$c : ko[[q]] \to KO[[q]] \ .$$

We let tmf denote the connective spectrum of topological modular forms constructed by Goerss, Hopkins, Miller and Lurie [Hop02], [GH04],[Goe09]. It fits into the following commutative diagram of multiplicative transformations



The map σ will be called the tmf-valued Witten genus. The tmf-valued Witten genus and the factorization $W \circ \sigma = R \circ j : MString \to KO[[q]]$ (for R see Subsection 3.3) have

been constructed by Ando, Hopkins and Rezk [AHR]. Since tmf is connective we have a unique factorization of W over a ring map $w : tmf \to ko[[q]]$. Similarly, since $MSpin$ is connective, we have a unique ring map r which lifts R.
For an abelian group G we let MG denote the associated Moore spectrum. For a spectrum X we write $XG := X \wedge MG$. The inclusion $\mathbb{Z} \to \mathbb{Q}$ induces a map $M\mathbb{Z} \to M\mathbb{Q}$, and we get an exact triangle

$$M\mathbb{Z} \to M\mathbb{Q} \to M\mathbb{Q}/\mathbb{Z} \to \Sigma M\mathbb{Z}$$

in the stable homotopy category. By smashing with X it induces an exact triangle

$$X \to X\mathbb{Q} \to X\mathbb{Q}/\mathbb{Z} \to \Sigma X$$

which is functorial in the spectrum X. For a spectrum X we write $X^l := \Sigma^l X$ for all $l \in \mathbb{Z}$. We define the spectra F and G to fit into exact triangles

$$G^{-1} \to tmf \xrightarrow{w} ko[[q]] \to G$$

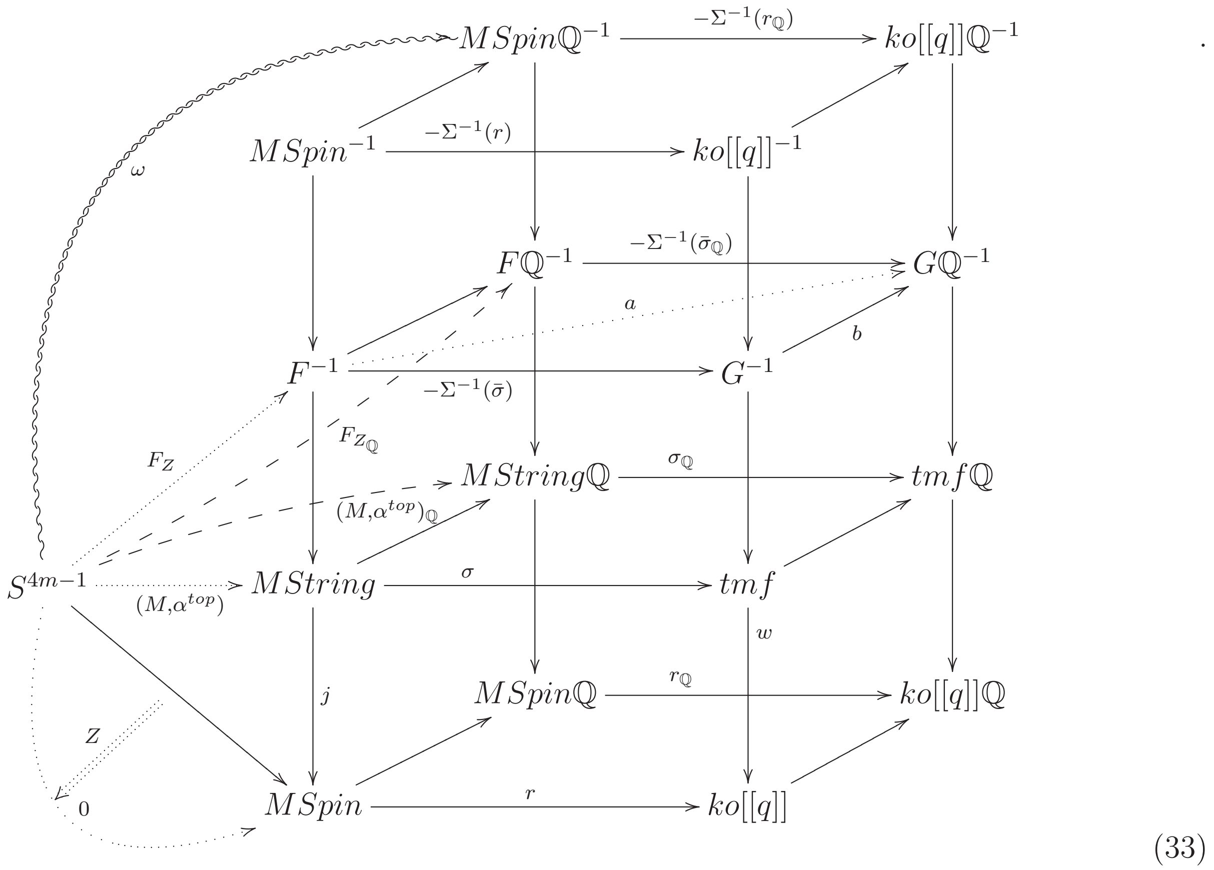
and

$$F^{-1} \xrightarrow{\nu} MString \xrightarrow{j} MSpin \to F \ .$$

We choose a map $\bar{\sigma} : F \to G$ such that the following diagram becomes a morphism of exact triangles

$$\begin{array}{ccccccc} F^{-1} & \xrightarrow{\nu} & MString & \xrightarrow{j} & MSpin & \longrightarrow & F \\ \downarrow{\scriptstyle -\Sigma^{-1}(\bar{\sigma})} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \bar{\sigma}} \\ G^{-1} & \longrightarrow & tmf & \xrightarrow{w} & ko[[q]] & \longrightarrow & G \end{array} \ .$$

We now consider the following commutative diagram:



(33)

The maps $b, \sigma_{\mathbb{Q}}$ and $\bar\sigma_{\mathbb{Q}}$ are the smash products of $\mathtt{id}_{G^{-1}}, \sigma$ and $\bar\sigma$ with the canonical map $S = M\mathbb{Z} \to M\mathbb{Q}$, respectively. The map a is defined as the composition $a := -b \circ \Sigma^{-1}(\bar\sigma)$. Note that a does not depend on the choice of $\bar\sigma$ because by construction

$$\begin{array}{ccccccc} F^{-1} & \xrightarrow{\nu} & MString & \xrightarrow{j} & MSpin & \longrightarrow & F \\ \downarrow{\scriptstyle a} & & \downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle -\Sigma(a)} \\ G\mathbb{Q}^{-1} & \longrightarrow & tmf\mathbb{Q} & \xrightarrow{w_{\mathbb{Q}}} & ko[[q]]\mathbb{Q} & \longrightarrow & G\mathbb{Q} \end{array}$$

is a morphism of triangles. This fact determines a up to elements coming from $[MString, ko[[q]]\mathbb{Q}^{-1}]$. Since $MString$ is rationally even we have $[MString, ko[[q]]\mathbb{Q}^{-1}] = 0$.
We calculate the homotopy groups of $G\mathbb{Q}$ from the exact sequence

$$\dots \to tmf\mathbb{Q}_j \to ko[[q]]\mathbb{Q}_j \to G\mathbb{Q}_j \to tmf\mathbb{Q}_{j+1} \to ko[[q]]\mathbb{Q}_{j+1} \to \dots\ .$$

In particular, since tmf is rationally even, we get the short exact sequences

$$0 \to tmf\mathbb{Q}_{4m} \to ko[[q]]\mathbb{Q}_{4m} \to G\mathbb{Q}_{4m} \to 0\ .$$

For $m \geq 0$ we have $ko[[q]]_{4m} \cong \mathbb{Z}[[q]]$, so that $ko[[q]]\mathbb{Q}_{4m} \cong \mathbb{Z}[[q]] \otimes \mathbb{Q} \subset \mathbb{Q}[[q]]$. Note that

$$\mathcal{M}^{\mathbb{Q}}_{2m} \subset \mathbb{Z}[[q]] \otimes \mathbb{Q} \subset \mathbb{Q}[[q]]\ ,$$

and by [Hop02] the subgroup $\mathcal{M}^{\mathbb{Q}}_{2m}$ coincides with the image of

$$\sigma_{\mathbb{Q}} : tmf\mathbb{Q}_{4m} \to ko[[q]]\mathbb{Q}_{4m} \ .$$

We therefore get an identification

$$G\mathbb{Q}_{4m} \cong \frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathcal{M}^{\mathbb{Q}}_{2m}} \ .$$

Let (M, α^{top}) be a closed string manifold of dimension $4m - 1$. The associated string bordism class $[M, \alpha^{top}] \in MString_{4m-1}$ is represented by a map $(M, \alpha^{top}) : S^{4m-1} \to MString$ as indicated in diagram (33). We assume that

$$[(M, \alpha)] \in A_{4m-1} \subseteq MString_{4m-1} \ .$$

Then we can choose a spin zero bordism Z of M. This zero bordism can be interpreted as a zero homotopy Z of the composition

$$S^{4m-1} \stackrel{(M,\alpha^{top})}{\longrightarrow} MString \stackrel{j}{\to} MSpin$$

indicated in (33). It thus gives rise to a map $F_Z : S^{4m-1} \to F^{-1}$ representing the element $[F_Z] \in F_{4m}$. We have

$$a([F_Z]) \in G\mathbb{Q}_{4m} \cong \frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathcal{M}^{\mathbb{Q}}_{2m}} \ .$$

Note that $[F_Z]$ is determined by $[M, \alpha^{top}] \in MString_{4m-1}$ up to elements in the image of $MSpin_{4m} \to F_{4m}$. By a diagram chase we see that these go into the image of $ko[[q]]_{4m} \to G\mathbb{Q}_{4m}$, i.e. the image of

$$\mathbb{Z}[[q]] \to \frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathcal{M}^{\mathbb{Q}}_{2m}} \ .$$

The natural chain of inclusions

$$\mathbb{Z}[[q]] \otimes \mathbb{Q} \to \mathbb{Q}[[q]] \to \mathbb{R}[[q]]$$

induces an inclusion

$$\frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathbb{Z}[[q]] + \mathcal{M}^{\mathbb{Q}}_{2m}} \to \frac{\mathbb{R}[[q]]}{\mathbb{Z}[[q]] + \mathcal{M}^{\mathbb{R}}_{2m}} = T_{2m} \ .$$

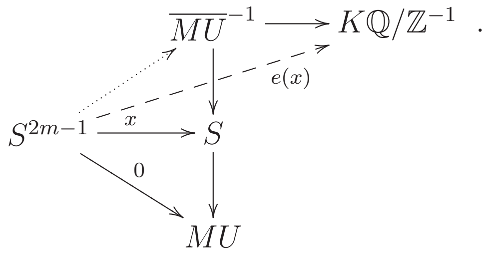
In the following we will use these inclusions implicitly. By construction, the class $[a([F_Z])] \in T_{2m}$ only depends on the class $[M, \alpha^{top}] \in A_{4m-1}$. Moreover, it depends on this bordism class additively thus justifying the following definition.

Definition 4.1 *We define the homomorphism*

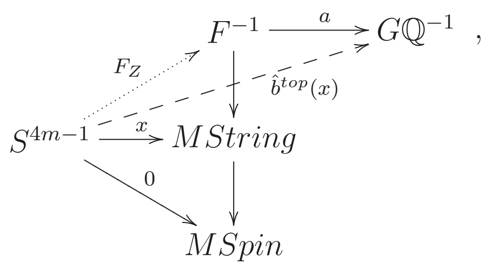
$$b^{top} : A_{4m-1} \to T_{2m} \ , \quad [(M, \alpha^{top})] \mapsto [a([F_Z])] \ . \tag{34}$$

4.2 The index theorem

In this subsection we show the secondary index theorem $b^{top} = b^{geom}$. It is very similar to the secondary index theorem $e^{an} = e^{top}$ for the e-invariant stated in [BN08, Introduction]. Let us explain the analogy in greater detail. The primary invariant for a map $S^k \to S$ in the case of the e-invariant was induced by the unit $x : S \to MU$. The e-invariant was obtained from a zero homotopy of the composition $S^{2m-1} \to S \to MU$, seen through the eyes of K-theory. The construction can be visualized in the diagram



Here $\overline{MU}$ is the cofiber of the unit, the dotted arrow is induced by the zero homotopy, and the map $\overline{MU} \to K\mathbb{Q}/\mathbb{Z}$ is constructed from the familiar orientation $MU \to K$ as explained in [BN08]. In the present case the primary invariant for a map $x : S^k \to MString$ is induced by the map $j : MString \to MSpin$. The secondary invariant b^{top} measures a zero homotopy of the composition $S^{4m-1} \to MString \to MSpin$, which exists since we assume that $x \in A_{4m-1}$, by means of the Witten genus r with values in $ko[[q]]$. The visualization is



where the class

$$\hat{b}^{top}(x) = a \circ F_Z \in G\mathbb{Q}_{4m} \cong \frac{\mathbb{Z}[[q]] \otimes \mathbb{Q}}{\mathcal{M}^{\mathbb{Q}}_{2m}}$$

represents $b^{top}(x) \in T_{2m}$. The reason that we cannot go directly to $ko[[q]]\mathbb{Q}/\mathbb{Z}$ is that in contrast to $S_{2m} \otimes \mathbb{Q} = 0$ for $m \geq 1$ in the case of the e-invariant, in our situation we have $MString_{4m} \otimes \mathbb{Q} \neq 0$.

Theorem 4.2 *For all $m \geq 1$ we have the following equality of homomorphisms*

$$b^{geom} = b^{top} : A_{4m-1} \to T_{2m} \ .$$

Proof. We let $(M, \alpha^{top}) : S^{4m-1} \to MString$ be a map which represents a class $[(M, \alpha^{top})] \in A_{4m-1}$. It produces a closed $(4m-1)$-dimensional string manifold, also denoted by (M, α^{top}), via the Thom-Pontrjagin construction and Lemma 8.5. Since $MString$

is rationally even the element $[F_{Z_{\mathbb{Q}}}] \in F\mathbb{Q}_{4m}$ maps to zero in $MString\mathbb{Q}_{4m}$. This implies the existence of a lift $\omega : S^{4m-1} \to MSpin\mathbb{Q}^{-1}$ as indicated in (33). By a diagram chase we see that the formal power series $-\Sigma^{-1}(r_{\mathbb{Q}})(\omega) \in ko[[q]]\mathbb{Q}_{4m} \cong \mathbb{Z}[[q]] \otimes \mathbb{Q}$ represents $b^{top}([M, \alpha^{top}])$.
Let us explicate the geometric interpretation of ω. Since $MString\mathbb{Q}_{4m-1} = 0$ there exists a $4m$-dimensional string manifold (V, β^{top}) and a natural number $L \in \mathbb{N}$ such that $L(M, \alpha^{top}) = \partial(V, \beta^{top})$, where by $L(M, \alpha^{top})$ we denote the disjoint union of L copies of (M, α^{top}). We define a spin manifold $LZ \cup_M -V$, where $-V$ denotes the spin manifold V with the opposite orientation and spin structure. Then $\omega = \frac{1}{L}[LZ \cup_M -V]$.
We choose Riemannian metrics on Z and V with product structures which glue nicely. Then we have associated Levi-Civita connections on the tangent bundles of the manifolds. By (6) or (21) we have

$$-\Sigma^{-1}(r_{\mathbb{Q}})(\omega) = \frac{\kappa_m}{L} \int_{LZ \cup_M -V} \Phi(\nabla^{T(LZ \cup_M -V)}) \in \mathbb{Z}[[q]] \otimes \mathbb{Q} \cong k[[q]]\mathbb{Q}_{4m} \ .$$

Using a geometric refinement β of the string structure β^{top} with restriction α to M we can write by (18)

$$\Phi(\nabla^{TV}) = 2dH_\beta \wedge \tilde{\Phi}(\nabla^{TV}) + \Theta(\nabla^{TV}) \ .$$

By Stokes' theorem we get

$$\begin{aligned} -\Sigma^{-1}(r_{\mathbb{Q}})(\omega) &= \frac{\kappa_m}{L} \int_{LZ \cup_M -V} \Phi(\nabla^{T(LZ \cup_M -V)}) \\ &= \kappa_m \int_Z \Phi(\nabla^{TZ}) - 2\kappa_m \int_M H_\alpha \wedge \tilde{\Phi}(\nabla^{TM}) - \frac{\kappa_m}{L} \int_V \Theta(\nabla^{TV}) \ . \end{aligned}$$

The last integral belongs to $\mathcal{M}^{\mathbb{R}}_{2m}$, and the first two terms together equal $\tilde{b}^{geom}(M, \alpha, \nabla^{TZ})$, compare with (28). This shows that $b^{top}([M, \alpha]) = b^{geom}([M, \alpha])$. □

4.3 Factorization over tmf

In this subsection we show that

$$b^{top} : A_{4m-1} \to T_{2m}$$

defined in Definition 4.1 admits a factorization over a homomorphism

$$b^{tmf} : tmf_{4m-1} \to T_{2m} \ .$$

For the construction of b^{tmf} we consider the following diagram of vertical and horizontal fiber sequences

$$\begin{array}{ccccccc}
 & & & & & & tmf\mathbb{Q}^{-1} \\
 & & & & & & \downarrow \\
 & & & & ko[[q]]^{-1} & \longrightarrow & {}^{\hat{x}_{\mathbb{Q}}}\ ko[[q]]\mathbb{Q}^{-1} \\
 & & & & \downarrow & & \downarrow \\
 & & G\mathbb{Q}/\mathbb{Z}^{-2} & \longrightarrow & {}^{\tilde{x}}\ G^{-1} & \longrightarrow & {}^{\tilde{x}_{\mathbb{Q}}}\ G\mathbb{Q}^{-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & {}^{x_{\mathbb{Q}/\mathbb{Z}}}\ tmf\mathbb{Q}/\mathbb{Z}^{-1} & \longrightarrow & {}^{x}\ tmf & \longrightarrow & {}^{x_{\mathbb{Q}}}\ tmf\mathbb{Q} \\
 & & \downarrow{\scriptstyle \bar{w}} & & \downarrow{\scriptstyle w} & & \downarrow \\
{}^{z}\ ko[[q]]\mathbb{Q}^{-1} & \longrightarrow & {}^{\bar{w}(x_{\mathbb{Q}/\mathbb{Z}})}\ ko[[q]]\mathbb{Q}/\mathbb{Z}^{-1} & \longrightarrow & ko[[q]] & \longrightarrow & ko[[q]]\mathbb{Q}
\end{array} \quad . \qquad (35)$$

The script-size symbols denote elements which will be chased through the diagram during the following discussion. Let $x \in tmf_{4m-1}$. We are going to construct $b^{tmf}(x) \in T_{2m}$ by a diagram chase. Since $ko[[q]]_{4m-1} = 0$ we can choose a lift $\tilde{x} \in G_{4m}$. Its image $\tilde{x}_{\mathbb{Q}} \in G\mathbb{Q}_{4m}$ maps to $x_{\mathbb{Q}} = 0$ in $tmf\mathbb{Q}_{4m-1} = 0$. Therefore we can further choose a lift $\hat{x}_{\mathbb{Q}} \in ko[[q]]\mathbb{Q}_{4m} \cong \mathbb{Z}[[q]] \otimes \mathbb{Q}$ which we define to represent $b^{tmf}(x) \in T_{2m}$.
Let us check that $b^{tmf}(x)$ is well-defined. The lift $\tilde{x}$ is defined uniquely up to elements in the image $ko[[q]]_{4m} \to G_{4m}$. These are mapped to integral power serieses in $ko[[q]]\mathbb{Q}_{4m} \cong \mathbb{Z}[[q]]\otimes\mathbb{Q}$. The lift $\hat{x}_{\mathbb{Q}}$ is well-defined up to elements in the image of $tmf\mathbb{Q}_{4m} \to ko[[q]]\mathbb{Q}_{4m}$, i.e. the space of power series $\mathcal{M}^{\mathbb{Q}}_{2m}$. Since in T_{2m} integral and modular power serieses are factored out, it follows that $b^{tmf}(x)$ is independent of the choices and that b^{tmf} is a homomorphism.

Proposition 4.3 *For every $m \geq 1$ we have a factorization*

$$\begin{array}{ccc}
MString_{4m-1} & \xrightarrow{\sigma} & tmf_{4m-1} \\
\uparrow & & \downarrow{\scriptstyle b^{tmf}} \\
A_{4m-1} & \xrightarrow{b^{top}} & T_{2m}
\end{array} \quad .$$

Proof. We chase in the diagrams (33) and (35). Let $y \in A_{4m-1} \subseteq MString_{4m-1}$ and $x = \sigma(y) \in tmf_{4m-1}$. Then we choose a lift $y_F \in F_{4m}$. The element $\tilde{x} := -\Sigma^{-1}(\bar{\sigma})(y_F) \in G_{4m}$ can serve as a lift of x in the construction of b^{tmf}. The image $\tilde{y}_{\mathbb{Q}} \in F\mathbb{Q}_{4m}$ can further be lifted to $\hat{y}_{\mathbb{Q}} \in MSpin\mathbb{Q}^{-1}_{4m}$. On the one hand, the element $-\Sigma^{-1}(r_{\mathbb{Q}})(\hat{y}_{\mathbb{Q}}) \in ko[[q]]\mathbb{Q}_{4m} \cong \mathbb{Z}[[q]] \otimes \mathbb{Q}$ represents $b^{top}(y)$ by construction. On the other hand, it can serve as a lift $\hat{x}_{\mathbb{Q}}$ in the construction of $b^{tmf}(x)$ and thus represents $b^{tmf}(x)$. □

Using diagram (35), we can give the following alternative description of $b^{tmf}(x)$ which will be employed in the Adams spectral sequence calculations in Subsections 6.3 and 7.3. Since $x \in tmf_{4m-1}$ maps to $x_{\mathbb{Q}} = 0$ we can choose a lift $x_{\mathbb{Q}/\mathbb{Z}} \in tmf\mathbb{Q}/\mathbb{Z}_{4m}$. We let $\bar{w}(x_{\mathbb{Q}/\mathbb{Z}}) \in ko[[q]]\mathbb{Q}/\mathbb{Z}_{4m}$ be its image under the map $\bar{w} := w \wedge \mathtt{id}_{M\mathbb{Q}/\mathbb{Z}} : tmf\mathbb{Q}/\mathbb{Z} \to ko[[q]]\mathbb{Q}/\mathbb{Z}$ induced by w. Note that there is a natural homomorphism $p : ko[[q]]\mathbb{Q}/\mathbb{Z}_{4m} \cong \mathbb{Z}[[q]] \otimes \mathbb{Q}/\mathbb{Z} \to T_{2m}$.

Lemma 4.4 *We have*

$$-p(\bar{w}(x_{\mathbb{Q}/\mathbb{Z}})) = b^{tmf}(x)$$

in T_{2m}.

Proof. We can lift $\bar{w}(x_{\mathbb{Q}/\mathbb{Z}})$ to an element $z \in ko[[q]]\mathbb{Q}_{4m}$. By a diagram chase we see that

$$z + \hat{x}_{\mathbb{Q}} \in \mathtt{im}(tmf\mathbb{Q}_{4m} \to ko[[q]]\mathbb{Q}_{4m}) + \mathtt{im}(ko[[q]]_{4m} \to ko[[q]]\mathbb{Q}_{4k})$$

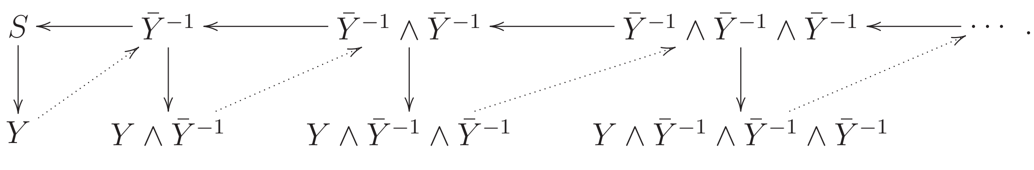
so that $\hat{x}_{\mathbb{Q}}$ and $-z$ represent the same element in T_{2m}. □

5 Some details about Adams spectral sequences

In this section we collect some facts and small results about the generalized Adams spectral sequence in connection with $\mathbb{Q}/\mathbb{Z}$-versions of spectra. This material will be used in the analysis of the invariant $b^{tmf} : tmf_{4m-1} \to T_{2m}$ in later sections.

5.1 The E_1-term and the cobar complex

Let us recall some details of the construction of the generalized Adams spectral sequence ${}^{Y}E_{*}^{*,*}(X)$ for a spectrum X and a commutative ring spectrum Y such that Y_*Y is a flat Y_*-module. As a first step we form the Adams resolution of the sphere spectrum S [Rav86, Def. 2.2.10]



Then we obtain the standard Y-based Adams resolution of X by smashing this resolution of S with X.

Note that Y_*X is a comodule over the Hopf algebroid (Y_*, Y_*Y). It gives rise to the reduced cobar complex $\bar{C}^*(Y_*X)$, see [Rav86, A.1.2.11]. There is a decomposition $\overline{Y_*Y} \oplus Y_* \cong Y_*Y$ of Y_*-modules (actually there are two such decompositions induced by the left or right unit of the Hopf algebroid), where $\overline{Y_*Y} \subset Y_*Y$ is the kernel of the counit. Hence the reduced cobar complex is a subcomplex of the cobar complex $C^*(Y_*X) \cong Y_*Y^{\otimes_{Y_*}*} \otimes_{Y_*} Y_*X$.

According to [Rav86, Prop. 2.11] the E_1-term of the generalized Adams spectral sequence is given by

$$({}^Y E_1^{*,*}(X), d_1) \cong (\bar{C}^*(Y_* X), d^{cobar}) \ .$$

5.2 Application to tmf

In this subsection we recall the relevant example of a certain ring spectrum Y introduced by Ravenel [Rav84]. The K-theory functor K^0 is represented by the H-space $\mathbb{Z} \times BU$. We have an equivalence $\Omega(\mathbb{Z} \times BU) \cong \Omega BU \cong U$. This H-space represents the functor K^{-1}. Hence K^{-2} is represented by the H-space ΩU. The Bott periodicity transformation $K^{-2} \to K^0$ is represented by a map of H-spaces $\Omega U \to \mathbb{Z} \times BU$. By restriction we get a map of H-spaces

$$\Omega U(4) \to \mathbb{Z} \times BU \tag{36}$$

To a map $\xi : X \to \mathbb{Z} \times BU$ we can associate a Thom spectrum X^ξ. If ξ is a map of H-spaces, then X^ξ is a ring spectrum which is commutative if ξ deloops twice.
It is known that the map (36) is a two-fold loop map and we let Y denote the associated Thom spectrum which is a commutative ring spectrum.
The spectrum tmf has the characterizing property that the cobar complex $C^*(Y_* tmf)$ is the cobar complex of the Weierstrass Hopf algebroid [Rez07, 14.5]. In particular, every group in the complex $C^*(Y_* tmf)$ is torsion-free. The same is true for the reduced cobar complex, and thus for the E_1-terms ${}^Y E_1^{*,*}(tmf_{(p)})$ of the Y-based generalized Adams spectral sequence for the p-localization $tmf_{(p)}$ of tmf for all primes p.

5.3 $\mathbb{Q}/\mathbb{Z}$-theory

For abelian groups A, B we denote $A * B := Tor_1^{\mathbb{Z}}(A, B)$. One can identify

$$A * \mathbb{Q}/\mathbb{Z} \cong A_{tors} \subseteq A \tag{37}$$

in a canonical way, where $A_{tors} \subseteq A$ denotes the torsion subgroup of A. For a spectrum X we have a functorial short exact sequence

$$0 \to X_* \otimes \mathbb{Q}/\mathbb{Z} \to X\mathbb{Q}/\mathbb{Z}_* \to X_{*-1} * \mathbb{Q}/\mathbb{Z} \to 0 \ . \tag{38}$$

The Y-based Adams resolution of $X\mathbb{Q}/\mathbb{Z}$ is obtained from the Y-based Adams resolution of X by smashing with the Moore spectrum $M\mathbb{Q}/\mathbb{Z}$. Therefore we get short exact sequences

$$0 \to {}^Y E_1^{s,t}(X) \otimes \mathbb{Q}/\mathbb{Z} \to {}^Y E_1^{s,t}(X\mathbb{Q}/\mathbb{Z}) \to {}^Y E_1^{s,t-1}(X) * \mathbb{Q}/\mathbb{Z} \to 0 \ .$$

The differentials of the E_1-term of the Y-based generalized Adams spectral sequence are induced by maps of spectra defined before smashing with $M\mathbb{Q}/\mathbb{Z}$. Hence, from the functoriality of (38) in X, we get a short exact sequences of complexes whose cohomology groups form long exact sequences involving constituents of the E_2-terms of the generalized Adams spectral sequences of X and $X\mathbb{Q}/\mathbb{Z}$.

Now we assume that ${}^Y E_1^{*,*}(X)$ is torsion-free. Then we have an isomorphism of complexes

$$ {}^Y E_1^{*,*}(X) \otimes \mathbb{Q}/\mathbb{Z} \cong {}^Y E_1^{*,*}(X\mathbb{Q}/\mathbb{Z}) \ . $$

¿From the universal coefficient formula we deduce exact sequences

$$ 0 \to {}^Y E_2^{s,t}(X) \otimes \mathbb{Q}/\mathbb{Z} \to {}^Y E_2^{s,t}(X\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^Y E_2^{s+1,t}(X) * \mathbb{Q}/\mathbb{Z} \to 0 \ . $$

Now we consider the distinguished triangle in the stable homotopy category

$$ X \to X\mathbb{Q} \to X\mathbb{Q}/\mathbb{Z} \xrightarrow{h} \Sigma X \ . $$

If $Y_*(h) = 0$, then by the geometric boundary theorem [Rav86, Thm. 2.3.4] for $r \geq 2$ there exist connecting maps

$$ \delta_r : {}^Y E_r^{s,t}(X\mathbb{Q}/\mathbb{Z}) \to {}^Y E_r^{s+1,t}(X) $$

which are compatible with the differentials of the spectral sequences. Furthermore they are filtered versions of

$$ h_* : X\mathbb{Q}/\mathbb{Z}_{*+1} \to X_* \ . $$

Assume again that ${}^Y E_1^{s,t}(X)$ is torsion free for all s, t. The case $s = 0$ yields $Y_*(h) = 0$ so that the geometric boundary theorem applies. In this way, using (37), we can compare the two maps

$$ {}^Y E_2^{s,t}(X\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^Y E_2^{s+1,t}(X)_{tors} \xrightarrow{i} {}^Y E_2^{s+1,t}(X) \ , \qquad \delta_2 : {}^Y E_2^{s,t}(X\mathbb{Q}/\mathbb{Z}) \to {}^Y E_2^{s+1,t}(X) $$

where i denotes the canonical embedding.

Lemma 5.1 *We have $i \circ \partial = \delta_2$.*

Proof. Let (A, Γ) be a Hopf algebroid such that Γ is a flat A-module, and let

$$ 0 \to M \to N \to Q \to 0 \tag{39} $$

be an exact sequence of (A, Γ)-comodules. Then the associated complex of reduced cobar complexes

$$ 0 \to \bar{C}^*(M) \to \bar{C}^*(N) \to \bar{C}^*(Q) \to 0 $$

is exact, too. The boundary operator

$$ \mathtt{Ext}^s_{(A,\Gamma)}(A, Q) \to \mathtt{Ext}^{s+1}_{(A,\Gamma)}(A, M) $$

in the associated long exact sequence is the $\mathtt{Ext}_{(A,\Gamma)}$-multiplication with the extension class of the sequence (39) in $\mathtt{Ext}^1_{(A,\Gamma)}(Q, M)$. We now apply this to the exact complex

$$ 0 \to Y_*X \to Y_*X\mathbb{Q} \to Y_*X\mathbb{Q}/\mathbb{Z} \to 0 \tag{40} $$

of Y_*Y-comodules. The Adams E_1-terms give the exact sequences of reduced cobar complexes. The map $i \circ \partial$ is the boundary operator of this sequence. The map δ_2 is by [Rav86, Thm 2.3.4 (a)] the $\mathtt{Ext}^1_{(A,\Gamma)}(Q, M)$-multiplication with the extension class of (40). □

5.4 Application to tmf

In this short subsection we specialize the facts obtained above to the case of tmf. We let $Y := \Omega U(4)^{\xi}$ be as in Subsection 5.2. Since ${}^{Y}E_1^{s,t}(tmf_{(p)})$ is torsion-free for all s,t we get a short exact sequence

$$0 \to {}^{Y}E_2^{s,t}(tmf_{(p)}) \otimes \mathbb{Q}/\mathbb{Z} \to {}^{Y}E_2^{s,t}(tmf_{(p)}\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^{Y}E_2^{s+1,t}(tmf_{(p)})_{tors} \to 0 \ .$$

Since ∂ is induced by δ_2 it is compatible with the boundary operators of the spectral sequences and a filtered version of

$$tmf_{(p)}\mathbb{Q}/\mathbb{Z}_{t-s} \to tmf_{(p),t-s-1} \ .$$

5.5 Application to ko

As before we let ko denote the connective real K-theory spectrum. Replacing $U(4)$ by $U(2)$ in the construction of the spectrum Y in Subsection 5.2 one gets a commutative ring spectrum Y'. The Hopf algebroid (Y'_*, Y'_*Y') is flat. It can be used to calculate ko_*. The Hopf algebroid

$$(Y'_*ko, Y'_*Y' \otimes_{Y'_*} Y'_*ko) \cong (\mathbb{Z}[b,c],\ \mathbb{Z}[b,c,r]) \tag{41}$$

with right unit

$$\begin{pmatrix} b \\ c \end{pmatrix} \mapsto \begin{pmatrix} b+2r \\ c+br+r^2 \end{pmatrix}$$

has been identified in [Hi07]. Furthermore, in this reference it has been shown that the Hopf algebroid (41) is equivalent to the Hopf algebroid

$$(\mathbb{Z}[b],\ \mathbb{Z}[b,r]/(r^2+2r))$$

with the right unit $b \mapsto b+2r$. It can be used to calculate the E_2-term of the generalized Adams spectral sequence

$${}^{Y'}E_r^{s,t}(ko) \Rightarrow ko_{t-s} \ .$$

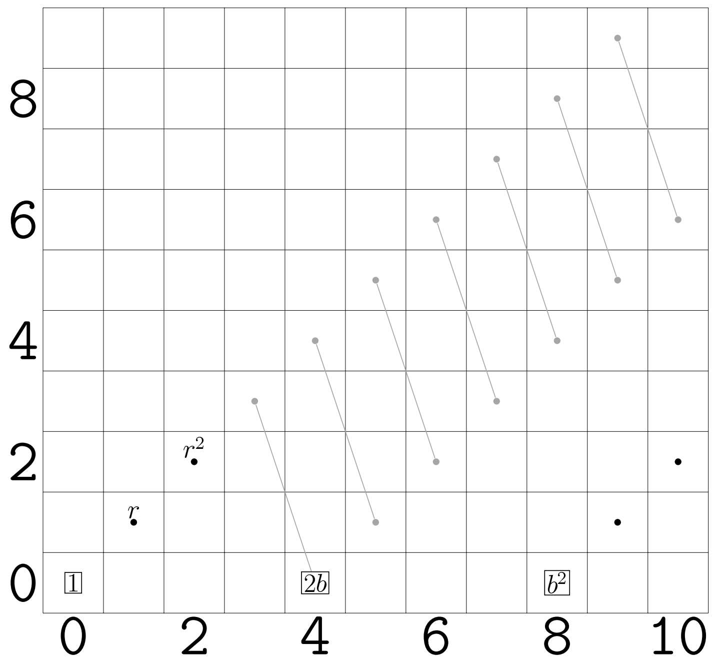
The inclusion $U(2) \to U(4)$ induces a map of ring spectra $Y' \to Y$, and therefore a morphism of spectral sequences ${}^{Y'}E_*^{*,*}(ko) \to {}^{Y}E_*^{*,*}(ko)$. One can check that this map is an isomorphism from the E_2-term on. The E_1-term is the reduced cobar complex of the Y'_*Y'-comodule Y'_*ko. It consists of torsion-free abelian groups. The same is true after base-change to Y.
Using the chain of invariant ideals

$$(2) \subseteq (2,b) \subseteq \mathbb{Z}[b,c]/(r^2+2r)$$

and a routine calculation with Bockstein spectral sequences we obtain

$${}^{Y}E_2^{*,*}(ko) \cong \mathbb{Z}[b,r]/(2r) \ , \quad |b| = (0,4) \ , \ |r| = (1,2) \ .$$

The equality $ko_3 = 0$ forces the differential $d_3b = r^3$ which determines d_3 algebraically. By sparseness, the spectral sequence degenerates at E_4. Its $E_4 = E_\infty$-term reproduces the familiar computation of ko_*. The result is visualized in the following chart.



${}^Y E_r^{s,t}(ko) \Rightarrow ko_*$

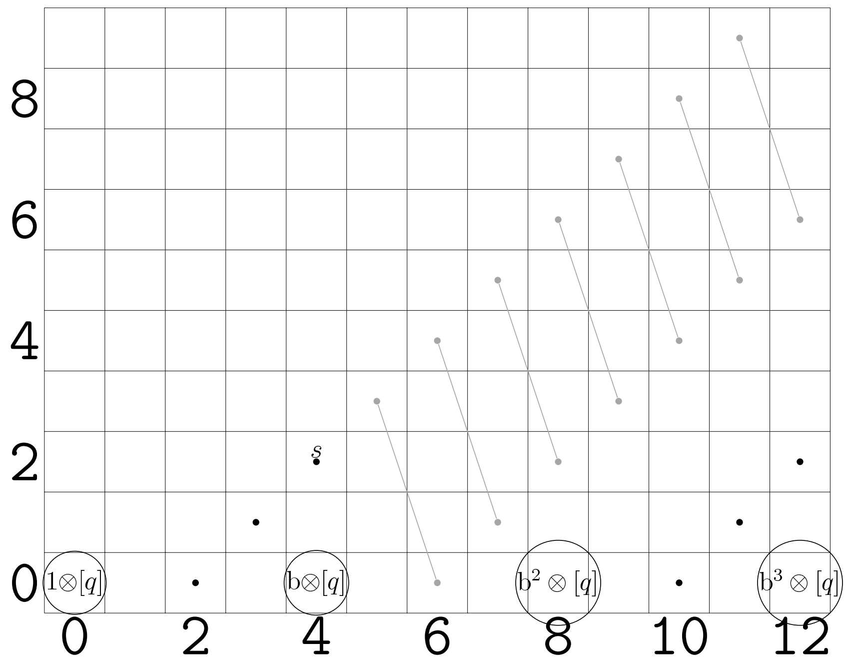
The horizontal (resp. vertical) axis represents $t - s$ (rep. s). Boxes indicate copies of $\mathbb{Z}$ and dots copies of $\mathbb{Z}/2\mathbb{Z}$. The E_∞-term is black. The E_2-term and the differentials are gray. The element $b^2 \in {}^Y E_2^{0,8}(ko)$ is a periodicity generator of dimension 8.
Now we turn to $ko\mathbb{Q}/\mathbb{Z}$. We have a distinguished triangle

$$ko \to ko\mathbb{Q} \to ko\mathbb{Q}/\mathbb{Z} \xrightarrow{l} \Sigma ko \tag{42}$$

in ko-modules. Since $Y_* ko$ is torsion-free the map $Y_*(l)$ vanishes. The geometric boundary theorem applies and shows that the second map in the exact sequence

$$0 \to {}^Y E_2^{s,t}(ko) \otimes \mathbb{Q}/\mathbb{Z} \to {}^Y E_2^{s,t}(ko\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^Y E_2^{s+1,t}(ko)_{tors} \to 0$$

is compatible with the differential d_3. We get the following chart of ${}^Y E_*^{*,*}(ko\mathbb{Q}/\mathbb{Z})$.



$^Y E_r^{s,t}(ko\mathbb{Q}/\mathbb{Z}) \Rightarrow ko\mathbb{Q}/\mathbb{Z}_*$

The circles indicate copies of $\mathbb{Q}/\mathbb{Z}$ whose elements can be written as $b^k \otimes [q]$ with $q \in \mathbb{Q}$ and $[q] \in \mathbb{Q}/\mathbb{Z}$. The dots indicate copies of $\mathbb{Z}/2\mathbb{Z}$. All indicated differentials are forced by the compatibility of ∂ with d_3. The spectral sequence degenerates at $E_4 = E_\infty$. Starting in dimension 2, the E_∞-term is 8-periodic with periodicity generator b^2 acting via the module structure under $^Y E_\infty^{*,*}(ko)$.
For our applications we are especially interested in $ko\mathbb{Q}/\mathbb{Z}_{4m}$. For all $m \geq 0$ we have

$$^Y E_2^{0,4m}(ko\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \ , \quad ^Y E_2^{2,8m+6}(ko\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \ .$$

We let $s \in {}^Y E_2^{2,6}(ko\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ be the non-trivial element. It is a permanent cycle. There is an extension problem in dimensions $8m-4$. By the 8-periodicity it suffices to solve it in dimension 4. The long exact sequence in homotopy groups associated with the triangle (42) gives isomorphisms

$$ko\mathbb{Q}/\mathbb{Z}_4 \cong ko_4 \otimes \mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \ .$$

This solves the extension problem for $ko\mathbb{Q}/\mathbb{Z}_4$ as follows: The Adams filtration of $ko\mathbb{Q}/\mathbb{Z}_4 \cong \mathbb{Q}/\mathbb{Z}$ is given by $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$, where the subgroup is in filtration ≥ 2. We let $\vartheta \in \mathtt{Filt}^2 ko\mathbb{Q}/\mathbb{Z}_4$ denote the non-trivial element. It is detected by $[\vartheta]_2 = s$. We now observe from the chart of the spectral sequence $^Y E_2^{*,*}(ko\mathbb{Q}/\mathbb{Z})$ that

$$\mathtt{Filt}^3 ko\mathbb{Q}/\mathbb{Z}_* = 0 \ .$$

In the following we introduce some notation to write elements in $ko\mathbb{Q}/\mathbb{Z}_*$ and their relatives in $^Y E_\infty^{*,*}(ko\mathbb{Q}/\mathbb{Z})$. Let $a_m \in ko_{8m-4} \cong \mathbb{Z}$ be the generator detected by the

uniquely determined element $[a_m]_2 = 2b^{2m-1}$. For $q \in \mathbb{Q}$ we can then consider the class $[qa_m] \in ko\mathbb{Q}/\mathbb{Z}_{8m-4} \cong ko\mathbb{Q}_{8m-4}/ko_{8m-4}$. If $2q \notin \mathbb{Z}$, then this class is detected by $2b^{2m-1} \otimes [q] \in {}^Y E_2^{0,8m-4}(ko\mathbb{Q}/\mathbb{Z})$. If $q = \frac{1}{2}$, then $[qa_m]$ is detected by the unique element $[qa_m]_2 = b^{2m-2}s \in {}^Y E_2^{2,8m-2}(ko\mathbb{Q}/\mathbb{Z})$.
If $a \in ko_{8m} \cong \mathbb{Z}$ is detected by $[a]_2 = b^{2m} \in {}^Y E_2^{0,8m}(ko)$, then for $q \in \mathbb{Q}$ the class $[qa] \in ko\mathbb{Q}/\mathbb{Z}_{8m}$ is detected by $b^{2m} \otimes [q] \in {}^Y E_2^{0,8m}(ko\mathbb{Q}/\mathbb{Z})$.
It is easy to see that for a spectrum $X \in \{ko,\ ko\mathbb{Q}/\mathbb{Z}\}$ and any prime p we have

$${}^Y E_*^{*,*}(X_{(p)}) \cong {}^Y E_*^{*,*}(X) \otimes \mathbb{Z}_{(p)} ,$$

where $X_{(p)}$ denotes the p-localization of the spectrum X. The above spectral sequences can be localized at $p = 2$ without any essential changes. If localized at a prime $p \neq 2$ they simplify considerably. In this case the spectral sequences degenerate at the E_2-terms, and

$${}^Y E_2^{*,*}(ko_{(p)}) \cong \mathbb{Z}_{(p)}[b] ,$$

$${}^Y E_2^{0,4m}(ko_{(p)}\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}_{(p)}$$

and vanishes elsewhere.

6 Calculations at 2

6.1 The result

In this section we analyze the localization at 2

$$b^{tmf} : tmf_{(2),4m-1} \to T_{(2),2m} , \quad m \geq 1 ,$$

of the maps $b^{tmf} : tmf_{4m-1} \to T_{2m}$ constructed in Subsection 4.3, where

$$T_{(2),2m} := (T_{2m})_{(2)} := T_{2m} \otimes \mathbb{Z}_{(2)}$$

is the localization of T_{2m} at 2. In view of the surjectivity of the Witten genus $\sigma : MString_* \to tmf_*$ ([Hop02, Theorem 6.25]) and Proposition 4.3, the result implies a non-triviality statement for the homomorphism $b^{top} : A_{4m-1} \to T_{2m}$ for an infinite number of $m \geq 1$.
In order to state the result we must name some elements of $tmf_{(2),*}$. To this end we first recall some parts of the calculations of [Bau08] of the spectral sequence ${}^Y E_*^{*,*}(tmf_{(2)})$ and of $tmf_{(2),*}$. Let $\Delta \in \mathcal{M}_{12}^{\mathbb{Z}}$ be the unique cusp form which is normalized such that $\Delta = q + \dots$. There exists an unique element $\Delta_2 \in {}^Y E_2^{0,24}(tmf_{(2)}) \cong \mathbb{Z}_{(2)}^2$ which corresponds to Δ under the identification (50). We know that $8\Delta_2$ is a permanent cycle. The element $\Delta_2^8 \in {}^Y E_2^{0,192}(tmf_{(2)})$ is a permanent cycle, too. It detects a unique element $\Delta^8 \in tmf_{(2),192}$ which is a periodicity generator of $tmf_{(2),*}$.
The map $\sigma : MString \to tmf$ induces an isomorphism $\sigma : MString_3 \to tmf_3$. The generator $g \in MString_3$ defined in (13) maps to a generator $\nu := \sigma(g)$ of $tmf_{(2),3}$, where

we omit the localization map $tmf \to tmf_{(2)}$ from notation. The order of ν is 8, it belongs to filtration 1, and it is detected by a unique element $[\nu]_2 \in {}^Y E_2^{1,4}(tmf_{(2)}) \cong \mathbb{Z}/2\mathbb{Z}$. Let $\eta \in tmf_1$ be the image of the Hopf map in S_1 under the unit $S \to tmf$. It is detected by $[\eta]_2 \in {}^Y E_2^{1,2}(tmf_{(2)}) \cong \mathbb{Z}/2\mathbb{Z}$. For $i \in \{14, 38, 74, 110, 134\}$ we let $[a_i]_2 \in {}^Y E_2^{2,i+2}(tmf_{(2)}) \cong \mathbb{Z}/2\mathbb{Z}$. denote generators of the corresponding groups.
The upper half of the fourth column of the following table displays all additive generators of ${}^Y E_\infty^{1,4m}(tmf_{(2)}) \subseteq E_2^{1,4m}(tmf_{(2)})$ and dimension $4m-1 < 195$. We fix elements of $tmf_{(2),4m-1}$ detected by these generators and which are named in the second column. These elements are unique up to elements of filtration ≥ 3 (with the exception of ν which we have fixed above). The third column lists the order of the elements in the first column. If we multiply the elements of order 8 (or 4, respectively) by 4 (or 2, respectively), then we obtain elements in filtration ≥ 3. These are detected by elements in ${}^Y E_\infty^{3,4m+2}(tmf_{(2)})$ which do not appear separately in the table. The lower half of the fourth column lists the remaining permanent cycles of ${}^Y E_2^{3,4m+2}(tmf_{(2)})$ in dimension $4m-1 < 192$.
The second column is a complete list of additive generators of

$$tmf_{(2),4m-1}/\mathtt{Filt}^4 tmf_{(2),4m-1}$$

in this range of dimension.

m	*name*	`ord`	${}^Y E_2^{*,*}(tmf_{(2)})$	$b^{tmf}(\dots)$	$c \in \{\dots\}$
1	ν	8	$[\nu]_2$	$[\frac{3}{8}]$	
7	$2\nu\Delta$	4	$2[\nu]_2\Delta_2$	$[\frac{c}{4}\Delta]$	$1, 3$
13	$\nu\Delta^2$	8	$[\nu]_2\Delta_2^2$	$[\frac{c}{8}\Delta^2]$	$1, 5$
25	$\nu\Delta^4$	8	$[\nu]_2\Delta_2^4$	$[\frac{c}{8}\Delta^4]$	$1, 5$
31	$2\nu\Delta^5$	4	$2[\nu]_2\Delta_2^5$	$[\frac{c}{4}\Delta^5]$	$1, 3$
37	$\nu\Delta^6$	8	$[\nu]_2\Delta_2^6$	$[\frac{c}{8}\Delta^6]$	$1, 5$
4	ηa_{14}	2	$[\eta]_2[a_{14}]_2$	0	
10	ηa_{38}	2	$[\eta]_2[a_{38}]_2$	0	
19	ηa_{74}	2	$[\eta]_2[a_{74}]_2$	0	
29	ηa_{110}	2	$[\eta]_2[a_{110}]_2$	0	
34	ηa_{134}	2	$[\eta]_2[a_{134}]_2$	0	

.

The table can be continued in the obvious 192-periodic way using the multiplication by Δ^8. There are more elements in dimensions $4m-1$ which are products of the listed elements with other elements in dimensions divisible by 4. All these products are of filtration ≥ 4. The main result of the present section is the following proposition.

Proposition 6.1 *1. The values of b^{tmf} on the elements listed in the second column are represented by the formal power series listed in the fifth column of the table.*

2. The map b^{tmf} annihilates all elements in filtration ≥ 4.

3. For all $m \geq 0$ and $[m]_{mod\ 48} \in \{1, 7, 13, 25, 31, 37\}$ the map b^{tmf} induces an injective map

$$\bar{b}^{tmf} : tmf_{(2),4m-1}/\mathtt{Filt}^4 tmf_{(2),4m-1} \to T_{(2),2m}\ .$$

4. For all $m \geq 0$ not listed in 3 the map $\bar{b}^{tmf} : tmf_{(2),4m-1} \to T_{(2),2m}$ is trivial.

The remainder of the present section is devoted to the proof of this proposition.

6.2 Transition to $\mathbb{Q}/\mathbb{Z}$-theory

From Subsection 5.4 we have an exact sequence

$$0 \to {}^Y E_2^{s,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z} \to {}^Y E_2^{s,t}(tmf_{(2)}\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^Y E_2^{s+1,t}(tmf_{(2)})_{tors} \to 0 \tag{43}$$

for all $t, s \in \mathbb{Z}$. The group ${}^Y E_2^{s,t}(tmf_{(2)})$ is finite for all $s \geq 1$ and $t \in \mathbb{Z}$. This implies that ${}^Y E_2^{s,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z} = 0$ for $s \geq 1$. Note that ${}^Y E_2^{0,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z}$ is a divisible group. There are no non-trivial maps from divisible groups to finite groups. Hence the elements in the subgroup ${}^Y E_2^{0,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z} \subseteq {}^Y E_2^{0,t}(tmf_{(2)}\mathbb{Q}/\mathbb{Z})$ are annihilated by all differentials, i.e. they are permanent cycles. In particular we get an embedding

$$0 \to {}^Y E_2^{0,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z} \to {}^Y E_\infty^{0,t}(tmf_{(2)}\mathbb{Q}/\mathbb{Z}) \ .$$

By an inspection of the calculation [Bau08] we see that the maps

$$\delta_r : {}^Y E_r^{s,4m}(tmf_{(2)}\mathbb{Q}/\mathbb{Z}) \to {}^Y E_r^{s+1,4m}(tmf_{(2)})$$

are isomorphisms for all $r \geq 2$ and $s \geq 1$. Indeed, they map cycles to cycles. Since the differentials of ${}^Y E_r^{*,*}(tmf_{(2)}\mathbb{Q}/\mathbb{Z})$ annihilate the subgroups ${}^Y E_2^{0,t}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z}$ and otherwise are induced by the differentials of ${}^Y E_r^{*,*}(tmf_{(2)})$ we further conclude that if $\delta_r(x)$ is a boundary, then so is x. We therefore have isomorphisms

$$\delta_r : \frac{{}^Y E_r^{0,4m}(tmf_{(2)}\mathbb{Q}/\mathbb{Z})}{{}^Y E_2^{0,4m}(tmf_{(2)}) \otimes \mathbb{Q}/\mathbb{Z}} \xrightarrow{\sim} E_r^{1,4m}(tmf_{(2)})$$

for all $r \geq 2$ and $r = \infty$.

6.3 The map to $ko[[q]]$

The morphism of ring spectra $w : tmf \to ko[[q]]$ induces a morphism $E_r(w)$ of Y-based generalized Adams spectral sequences. Note that ${}^Y E_2^{*,*}(ko[[q]]) \cong {}^Y E_2^{*,*}(ko)[[q]]$. The spectra $ko[[q]]_{(2)}$, $tmf_{(2)}\mathbb{Q}/\mathbb{Z}$ and $ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}$ are modules over $tmf_{(2)}$. We get induced module structures on the spectral sequences.
Recall the element $\Delta_2 \in {}^Y E_2^{0,24}(tmf_{(2)})$ introduced in Subsection 6.1. We have

$$E_2(w)(\Delta_2) = b^6\Delta \in {}^Y E_2^{0,24}(ko[[q]]_{(2)}) \cong {}^Y E_2^{0,24}(ko)[[q]]_{(2)} \ .$$

We let $\bar{w} = w \wedge \mathtt{id}_{M\mathbb{Q}/\mathbb{Z}} : tmf_{(2)}\mathbb{Q}/\mathbb{Z} \to ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}$ denote the map induced by w. Since

$$\mathtt{Filt}^3 ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}_* = 0$$

it is clear that $\bar{w}$ annihilates $\mathtt{Filt}^3 tmf_{(2)}\mathbb{Q}/\mathbb{Z}_*$. It now follows from Lemma 4.4 that b^{tmf} annihilates $\mathtt{Filt}^4(tmf_{(2)})$. This is Assertion 2. of Proposition 6.1. Consequently b^{tmf} annihilates all elements except for those listed in the table in Subsection 6.1 in dimensions $4m-1$.
We now explain how we get the listed values of b^{tmf}.
Recall the generator $\nu \in tmf_{(2),3}$ of order 8. We choose a lift $\nu_{\mathbb{Q}/\mathbb{Z}} \in tmf_{(2)}\mathbb{Q}/\mathbb{Z}_4$ as in (35). By (32) know that $b^{top}(g) = [\frac{1}{24}] \in T_2 = \mathbb{R}[[q]]/\mathbb{Z}[[q]]$. Therefore, using that $[\frac{1}{24}] = [\frac{3}{8}]$ in $\mathbb{Q}/\mathbb{Z}_{(2)}$, we get by Lemma 4.4

$$\bar{w}(\nu_{\mathbb{Q}/\mathbb{Z}}) = [-\frac{3}{8}] \in ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}_4 \cong \mathbb{Z}_{(2)}[[q]] \otimes \mathbb{Q}/\mathbb{Z} \ .$$

¿From the discussion at the end of Subsection 5.5 it follows that

$$E_2(\bar{w})([\nu_{\mathbb{Q}/\mathbb{Z}}]_2) = b \otimes [-\frac{3}{4}] \in E_2^{0,4}(ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}[[q]]_{(2)} \otimes \mathbb{Q}/\mathbb{Z} \ .$$

Let $\lambda = 2\nu\Delta \in \mathtt{Filt}^1 tmf_{(2),27}$ be detected by $[\lambda]_2 := 2[\nu]_2\Delta_2$. Then we can choose a lift $\lambda_{\mathbb{Q}/\mathbb{Z}} \in tmf_{(2)}\mathbb{Q}/\mathbb{Z}_{27}$ which is detected by $[\lambda_{\mathbb{Q}/\mathbb{Z}}]_2 = 2[\nu_{\mathbb{Q}/\mathbb{Z}}]_2\Delta_2$ (note that the lift $\lambda_{\mathbb{Q}/\mathbb{Z}}$ is unique up to elements coming from $tmf_{(2)}\mathbb{Q}_{28} \cong \mathbb{Q}$). This follows from the condition $\partial[\lambda_{\mathbb{Q}/\mathbb{Z}}]_2 = [\lambda]_2$, where ∂ is as in (43), and $\partial(2[\nu_{\mathbb{Q}/\mathbb{Z}}]_2\Delta_2) = 2\Delta_2\partial([\nu_{\mathbb{Q}/\mathbb{Z}}]_2)$.
We conclude that

$$E_2(\bar{w})(\lambda_{\mathbb{Q}/\mathbb{Z}}) = E_2(\bar{w})([\nu_{\mathbb{Q}/\mathbb{Z}}]_2)2\Delta_2 = b^7 \otimes [\frac{1}{2}\Delta] \in E_2^{0,28}(ko[[q]]_{(2)}\mathbb{Q}/\mathbb{Z}) \ .$$

Lemma 4.4 implies that

$$b^{tmf}(2\nu\Delta) = [\frac{c}{4}\Delta] \in T_{(2),28} \ ,$$

where $c = 1$ or 3. This indeterminacy is unavoidable since $2\nu\Delta$ is only determined up to elements of higher filtration.
By a similar argument

$$b^{tmf}(\nu\Delta^2) = [\frac{c}{8}\Delta^2] \in T_{(2),52}$$

with $c \in \{3, 7\}$. In a similar manner we get the value of b^{tmf} on the remaining entries of the upper part of the second column.
Turning to the elements listed in the lower part of the second column one shows $b^{tmf}(\eta a_i) = 0$ as follows. Note that $\eta a_i \in \mathtt{Filt}^3 tmf_{(2),i+1}$. Therefore $b^{tmf}(\eta a_i)$ is represented by an element in $\mathtt{Filt}^2 ko[[q]]\mathbb{Q}/\mathbb{Z}_{i+2}$. In all cases we have $i + 2 \equiv 0 \bmod 8$. But then $\mathtt{Filt}^2 ko[[q]]\mathbb{Q}/\mathbb{Z}_{i+2} = 0$. We thus have shown Assertions 1 and 4 of Proposition 6.1.
It remains to show Assertion 3. It suffices to show that $\mathtt{ord}(b^{tmf}(x)) = \mathtt{ord}(x)$ for the generators of the cyclic groups $tmf_{(2),4m-1}/\mathtt{Filt}^4 tmf_{(2),4m-1}$ listed in the first column of the table and their multiples by Δ^8. The assertion immediately follows from Corollary 9.4. $\square$

7 Calculations at 3

7.1 The result

Compared with the localization at 2, the structure of the spectral sequence ${}^YE_*^{*,*}(tmf_{(3)})$ and of $tmf_{(3),*}$ is much simpler. We again start by recalling the calculations of [Bau08]. There is an unique element $\Delta_2 \in {}^YE_2^{0,24}(tmf_{(3)})$ which corresponds to Δ under the identification (50). We know that $3\Delta_2$ is a permanent cycle. The element $\Delta_2^3 \in {}^YE_2^{0,72}(tmf_{(3)})$ is a permanent cycle and detects a unique element denoted by $\Delta^3 \in tmf_{(3),72}$ which is a periodicity generator.
The generator $g \in MString_3$ defined in (13) maps to a generator $\nu := \sigma(g)$ of $tmf_{(3),3}$ of order 3 in filtration 1 detected by $[\nu]_2 \in {}^YE_2^{1,4}(tmf_{(3)})$. The following table gives the complete list of additive generators of $tmf_{(3),4m-1}$ for $4k-1 < 75$. All these elements live in filtration one and are uniquely determined. The table can be continued in the obvious 72-periodic way using multiplication by Δ^3.

m	*name*	`ord`	${}^YE_2^{*,*}(tmf_{(3)})$	$b^{tmf}(\dots)$
1	ν	3	$[\nu]_2$	$[\frac{2}{3}]$
7	$\nu\Delta$	3	$[\nu]_2\Delta_2$	$[\frac{2}{3}\Delta]$

.

The third column displays the order of the element. The last two columns display the elements in ${}^YE_2^{*,*}(tmf_{(3)})$ which detect the elements in the second column, and the value of the invariant b^{tmf} in $T_{(3),2k}$, expressed as a class of a rational formal power series.

Proposition 7.1 *1. The value of b^{tmf} on the generator listed in the second column of the table is as indicated in the last column of the table.*

2. For all $m \geq 1$ we have injections

$$b^{tmf} : tmf_{(3),4m-1} \to T_{(3),2m} \ .$$

The following two subsections are devoted to the proof of this proposition.

7.2 Transition to $\mathbb{Q}/\mathbb{Z}$-theory

By Subsection 5.4 we have an exact sequence

$$0 \to {}^YE_2^{s,t}(tmf_{(3)}) \otimes \mathbb{Q}/\mathbb{Z} \to {}^YE_2^{s,t}(tmf_{(3)}\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} {}^YE_2^{s+1,t}(tmf_{(3)})_{tors} \to 0 \ . \quad (44)$$

As in Subsection 6.2 we see that the elements in the subgroup ${}^YE_2^{0,t}(tmf_{(3)}) \otimes \mathbb{Q}/\mathbb{Z} \subseteq {}^YE_2^{0,t}(tmf_{(3)}\mathbb{Q}/\mathbb{Z})$ are permanent cycles and we get an embedding

$$0 \to {}^YE_2^{0,t}(tmf_{(3)}) \otimes \mathbb{Q}/\mathbb{Z} \to {}^YE_\infty^{0,t}(tmf_{(3)}\mathbb{Q}/\mathbb{Z}) \ .$$

By an inspection of the calculation [Bau08] we see again that the maps

$$\delta_r : {}^YE_r^{s,4m}(tmf_{(3)}\mathbb{Q}/\mathbb{Z}) \to E_r^{s+1,4m}(tmf_{(3)})$$

are isomorphisms for all $r \geq 2$ and $s \geq 1$, so that the maps

$$\delta_r : \frac{{}^Y E_r^{0,4m}(tmf_{(3)}\mathbb{Q}/\mathbb{Z})}{{}^Y E_2^{0,4m}(tmf_{(3)}) \otimes \mathbb{Q}/\mathbb{Z}} \xrightarrow{\sim} E_r^{1,4m}(tmf_{(3)})$$

are isomorphisms for all $r \geq 2$ and $r = \infty$.

7.3 The map to $ko[[q]]$

The element $\Delta_2 \in {}^Y E_2^{0,24}(tmf_{(3)})$ satisfies

$$E_2(w)(\Delta_2) = b^6\Delta \in {}^Y E_2^{0,24}(ko[[q]]_{(3)}) \cong {}^Y E_2^{0,24}(ko)[[q]]_{(3)} \ .$$

Recall the generator $\nu \in tmf_{(3),3}$ of order 3 and the lift $\nu_{\mathbb{Q}/\mathbb{Z}} \in tmf_{(3)}\mathbb{Q}/\mathbb{Z}_4$ as in (35).
By (32) know that $b^{top}(g) = [\frac{1}{24}] \in T_2 \cong \mathbb{Q}/\mathbb{Z}[[q]]$. Therefore, using that $[\frac{1}{24}] = [-\frac{1}{3}]$ in $\mathbb{Q}/\mathbb{Z}_{(3)}$, we get

$$\bar{w}(\nu_{\mathbb{Q}/\mathbb{Z}}) = [\frac{1}{3}] \in ko[[q]]_{(3)}\mathbb{Q}/\mathbb{Z}_4 \cong \mathbb{Z}[[q]]_{(3)} \otimes \mathbb{Q}/\mathbb{Z} \ .$$

With Lemma 4.4 this gives

$$b^{tmf}(\nu) = \left[\frac{2}{3}\right] \in T_{(3),2} \ .$$

Furthermore, we have

$$E_2(\bar{w})([\tilde{\nu}]_2) = b \otimes [\frac{2}{3}] \in E_2^{0,4}(ko[[q]]_{(3)}\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[[q]]_{(3)} \ .$$

Let $\lambda = \nu\Delta \in \mathtt{Filt}^1 tmf_{(3),27}$ be detected by $[\lambda]_2 := [\nu]_2\Delta_2$. Then we can choose a lift $\lambda_{\mathbb{Q}/\mathbb{Z}} \in tmf_{(3)}\mathbb{Q}/\mathbb{Z}_{27}$ which is detected by $[\lambda_{\mathbb{Q}/\mathbb{Z}}]_2 = [\tilde{\nu}]_2\Delta_2$ (note that the lift $\lambda_{\mathbb{Q}/\mathbb{Z}}$ is unique up to elements coming from $tmf_{(3)}\mathbb{Q}_{28} \cong \mathbb{Q}$).
We conclude that

$$E_2(\bar{w})(\lambda_{\mathbb{Q}/\mathbb{Z}}) = E_2(\bar{w})(\nu_{\mathbb{Q}/\mathbb{Z}}]_2)\Delta_2 = b^7 \otimes [\frac{2}{3}\Delta] \in E_2^{0,28}(ko[[q]]_{(3)}\mathbb{Q}/\mathbb{Z}) \ .$$

This gives

$$b^{tmf}(\nu\Delta) = \left[\frac{2}{3}\Delta\right] \in T_{(3),14} \ .$$

This finishes the proof of Assertion 1 of Proposition 7.1. .
For Assertion 2. it suffices to show that $\mathtt{ord}(b^{tmf}(x)) = \mathtt{ord}(x)$ for the generators of the cyclic groups $tmf_{(3),4m-1}$ listed in the first column of the table and their multiples by Δ^3.
The assertion immediately follows from Corollary 9.4. □

7.4 How to detect $\nu\Delta$?

The equality $b^{top} = b^{geom}$ can be used to do some explicit calculations of the tmf-valued Witten genus. In the following we discuss an example. Let (M, α^{top}) be a 27-dimensional closed string manifold. It represents a string bordism class $[(M, \alpha^{top})] \in MString_{27}$. Moreover, $\sigma([(M, \alpha^{top})]) = c\Delta\nu \in tmf_{(3),27}$ for a uniquely determined $c \in \mathbb{Z}/3\mathbb{Z}$. The problem consists in the explicit calculation of the value c.
First observe that by the calculations in Subsection 7.1 we have an equality of classes in $T_{(3),27}$

$$[b^{top}([(M, \alpha^{top})])] = [\frac{2c}{3}\Delta] \ .$$

Expand $b^{top}(M, \alpha^{top}) = b_0 + qb_1 + \dots$. Then we form

$$b^{top}(M, \alpha^{top}) - b_0 E_{14} = (b_1 - 24b_0)q + \dots \ ,$$

where $E_{14} := -24G_{14}$ (with G_{14} given by (16)) denotes the normalized Eisenstein series. Since $\mathcal{M}_{14}^{\mathbb{Q}} = \mathbb{Q} \cdot E_{14}$, we have $3(b^{top}(M, \alpha^{top}) - b_0 E_{14}) \in \mathbb{Z}[[q]]_{(3)}$. Furthermore,

$$\frac{2}{3}\Delta = \frac{2}{3}q + \dots \quad .$$

Therefore we get

$$c = -3(b_1 - 24b_0) \ (\mathtt{mod}\, 3) \ .$$

Let Z be a spin zero bordism of M. Then specializing Lemma 3.5 we get

$$\begin{aligned} c \quad = \quad & -\langle \tilde{p}_1 \cup \left(\frac{1967}{729} {N_4}^3 + \frac{356}{243} N_8 {p_1}^2 + \frac{2575}{2187} N_6 {p_1}^3 \right. \\ & + \frac{152}{81} N_8 N_4 + \frac{941}{729} N_4 p_1 N_6 + \frac{6232}{2187} {p_1}^6 \\ & + \frac{898}{729} N_4 {p_1}^4 + \frac{541}{243} N_{10} p_1 + \frac{623}{729} {N_4}^2 {p_1}^2 \\ & \left. + \frac{457}{729} N_{12} + \frac{2398}{2187} {N_6}^2 \right), [Z, M] \rangle \quad \mathrm{mod}\ 3 \end{aligned}$$

Here $N_{2i} := N_{2i}(p_1, p_2, ...)$ denotes the i'th Newton polynomial and we use the abbreviations $p_i := p_i(TZ)$ and $\tilde{p}_1 := \tilde{p}_1(TZ, \alpha^{top})$ for the class defined in (31). It is a non-trivial integrality statement that the right-hand side is an integer (before taking its value modulo 3).

8 Geometric and homotopy theoretic string bordism

In the present paper we use a geometric picture of string bordism. The goal of the present subsection is to give the geometric definition of string bordism and to relate it with the homotopy theoretic picture. This material is familiar to geometers and we include it mainly as a reference.

We first recall some facts about string structures. The following diagram gives the unstable variation of the Postnikov tower (5). For $n \geq 3$ we consider

$$\begin{array}{ccccc} & & BString(n) & & \\ & \overset{\alpha^{top}}{\nearrow} & \downarrow & & \\ X & \xrightarrow{\xi} & BSpin(n) & \xrightarrow{\frac{p_1}{2}} & K(\mathbb{Z},4) \\ & {\scriptstyle V}\searrow & \downarrow & & \\ & & BSO(n) & \xrightarrow{w_2} & K(\mathbb{Z}/2\mathbb{Z},2) \\ & & \downarrow & & \\ & & BO(n) & \xrightarrow{w_1} & K(\mathbb{Z}/2\mathbb{Z},1) \end{array} \quad . \tag{45}$$

The space $BSO(n)$ is the homotopy fiber of the first Stiefel-Whitney class w_1. Similarly, $BSpin(n)$ and $BString(n)$ are the homotopy fibers of the second Stiefel-Whitney class w_2 and the spin refinement $\frac{p_1}{2}$ of the first Pontrjagin class, respectively.
Let $V \to X$ be a n-dimensional real vector bundle over a CW-complex X. We use same symbol in order to denote the classifying map $V : X \to BO(n)$. An orientation and spin structure on V is given by the choice of a lift $\xi : X \to BSpin(n)$. By definition, a spin bundle is a pair (V, ξ). We will usually drop ξ from the notation and refer to V as a spin bundle.

Definition 8.1 *A topological string structure on a spin bundle V is a homotopy class of lifts α^{top}. The pair (V, α^{top}) is called a string bundle.*

The set of topological string structures on the spin bundle V is a torsor under $H^3(X; \mathbb{Z})$. We will write the action of $x \in H^3(X; \mathbb{Z})$ as

$$(x, \alpha^{top}) \mapsto \alpha^{top} + x \ . \tag{46}$$

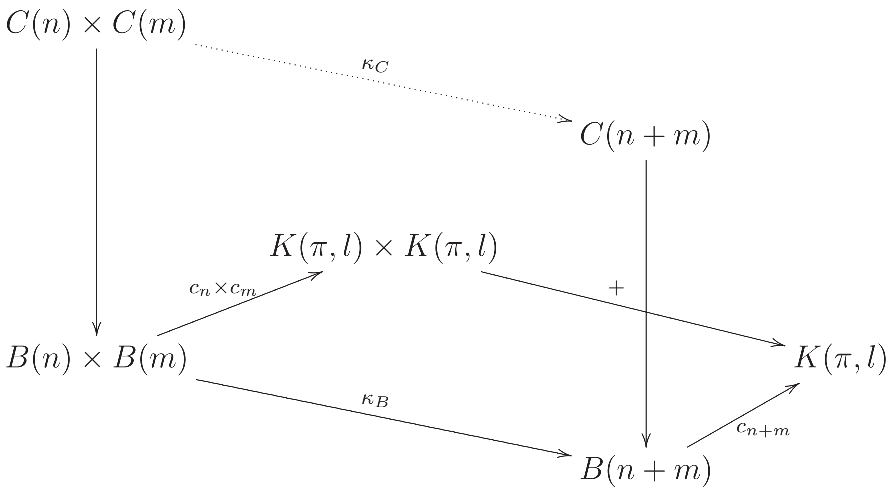
In differential geometry one works with orientations, spin, or string structures on the (unstabilized) tangent bundle, while in homotopy theory one considers those structures on the stable normal bundle. In order to transfer structures back and forth between the tangent and stable normal bundles, the two-out-of three principle Lemma 8.3 below is crucial.
In the following we describe an abstract setting for this principle which can inductively be applied to the layers of the tower (45). Assume that we have a sequence of maps $(B(n) \to BO(n))_{n \geq n_0}$ together with homotopy commutative diagrams

$$\begin{array}{ccc} B(n) \times B(m) & \xrightarrow{\kappa_B} & B(n+m) \\ \downarrow & & \downarrow \\ BO(n) \times BO(m) & \xrightarrow{\oplus} & BO(n+m) \end{array}$$

satisfying obvious associativity constraints. We further assume that the spaces $B(n)$ are $(l-1)$-connected for some integer l, and that we have a collection of maps $(c_n : B(n) \to K(\pi, l))_{n \geq n_0}$ inducing isomorphisms in $\pi_l(B(n)) \xrightarrow{\sim} \pi$ for all $n \geq n_0$ and some fixed abelian group π. We define the l-connected spaces $C(n)$ as homotopy pull-backs

$$\begin{array}{ccc} C(n) & \longrightarrow & * \\ \downarrow & & \downarrow \\ B(n) & \xrightarrow{c_n} & K(l,\pi) \end{array} \quad .$$

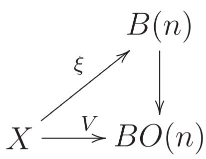
Finally we assume that the collection of maps $(c_n)_{n \geq n_0}$ is additive in the sense that the lower squares of the following diagrams are homotopy commutative:



.

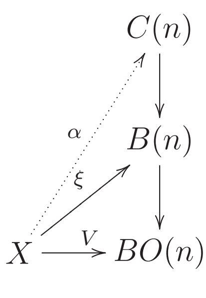
Then we get unique (up to homotopy) lifts κ_C of κ_B indicated by the dotted arrow which again satisfy an associativity constraint.

Let X be a CW-complex. A real n-dimensional vector bundle $V \to X$ with a B-structure is a homotopy commutative diagram



,

where V is the classifying map of the vector bundle and the lift ξ defines the $B(n)$-structure. The pair (V, ξ) will be called a B-bundle. A C-structure on a B-bundle (V, ξ) is given by a further lift α as indicated in



.

The set of isomorphism classes of C-structures on the B-bundle (V, ξ) is in one-to-one correspondence with the set of homotopy classes of lifts η of ξ. By obstruction theory this set is a torsor under $H^l(X; \pi)$. If $x \in H^l(X; \pi)$, then we let $\alpha + x$ denote the result of the action of x on α. The maps κ_B allow to define the sum of B-bundles

$$(V, \xi) \oplus (W, \zeta) := (V \oplus W, \kappa_B(\xi, \zeta)) \ .$$

Furthermore, using the maps κ_C we define the sum of B-bundles with C-structures by

$$((V, \xi), \alpha) \oplus ((W, \zeta), \beta) := ((V \oplus W, \kappa_B(\xi, \zeta)), \kappa_C(\alpha, \beta)) \ .$$

One can check that κ_C is bilinear with respect to the action of $H^l(X; \pi)$, i.e. we have

$$\kappa_C(\alpha + x, \beta) = \kappa_C(\alpha, \beta + x) = \kappa_C(\alpha, \beta) + x \ , \quad \forall x \in H^l(X; \pi) \ .$$

We can now formulate the following two-out-of-three principle for C-structures on B-bundles (V, ξ) and (W, ζ). The verification is left to the reader.

Lemma 8.2 *The choice of two C-structures out of α, β, γ on the B-bundles*

$$(V, \xi) \ , \quad (W, \zeta) \ , \quad (V \oplus W, \kappa_B(\xi, \zeta))$$

uniquely fixes the third such that there is an isomorphism of C-bundles

$$((V, \xi), \alpha) \oplus ((W, \zeta), \beta) \cong ((V \oplus W, \kappa_B(\xi, \zeta)), \gamma) \ .$$

The characteristic classes which define the layers of the Postnikov tower (45) are additive. Hence the theory described above can be applied. Recalling the stability statements $\pi_0(O(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 1$, $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_3(Spin(n)) = \mathbb{Z}$ for $n \geq 3$ we thus obtain the following.

Corollary 8.3 *We have a two-out-of-three principle for*

- *orientations on real vector bundles*
- *spin-structures on oriented vector bundles of rank ≥ 3 and*
- *string structures on spin bundles of rank ≥ 3.*

In contrast, this principle does not apply to (homotopy classes of) framings as is obvious from $TS^2 \oplus \mathbb{R} \simeq S^2 \times \mathbb{R}^3$.
We now describe the geometric picture of string bordism. A spin manifold M is a manifold with a chosen orientation and spin structure on its tangent bundle TM.

Definition 8.4 *A string manifold (M, α^{top}) is a spin manifold M with a chosen topological string structure α^{top} on the spin bundle TM.*

If (Z, β^{top}) is a string manifold with boundary M, then M has an induced string structure α^{top} defined as follows. The inner normal field induces a decomposition

$$TZ_{|M} \cong TM \oplus \Theta_{\mathbb{R}} \ , \tag{47}$$

where $\Theta_{\mathbb{R}} := M \times \mathbb{R}$ denotes the trivial bundle. A trivial bundle has a canonical orientation, as well as a canonical spin structure, and a canonical string structure *can*. We define the orientation, spin structure, and string structure on TM by the two-out-of-three principle such that (47) becomes a decomposition of structured bundles.
We can now define the n'th geometric string bordism homology group $\Omega_n^{String}(X)$ of a space X in the usual manner by cycles and relations. A cycle for $\Omega_n^{String}(X)$ is a pair $((M, \alpha^{top}), f)$ of an n-dimensional closed string manifold (M, α^{top}) and a map $f : M \to X$. The set of isomorphism classes of cycles forms a semigroup with respect to disjoint union. The group $\Omega_n^{String}(X)$ is the defined as the quotient of the group completion of this semigroup by the group generated by string manifolds with maps to X which are boundaries of $n+1$-dimensional compact string manifolds with maps to X. The remaining structures for the homology theory (e.g. the suspension isomorphism) are defined in the standard way and will be neglected in the discussion below, too.
We have the following consequence of the Pontrjagin-Thom construction and the two-out-of-three principle Lemma 8.3.

Lemma 8.5 *There is a canonical isomorphism*

$$\Omega_*^{String}(X) \cong MString_*(X)$$

for all $* \geq 3$.

Proof. The standard Thom-Pontrjagin construction produces an isomorphism between the homotopy theoretic string bordism and a version of the geometric string bordism with topological string structures on stable normal bundles [S, Theorem, Chapter II]. Using the two-out-of-three principle Lemma 8.3 we can go back and forth between topological string structures on stable normal bundles and on the (unstable) tangent bundle. □

In the main body of the text, we will use the notation $MString_*(X)$ in order to denote both, the geometric and the homotopy theoretic string bordism groups of X.

9 Modular forms and the structure of T_{2m}

The present subsection has two goals. One is to make the structure of the group T_{2m} more explicit. In applications the important problem is to decide whether a formal power series $f \in \mathbb{R}[[q]]$ represents a non-trivial element in T_{2m}. This can be solved using Lemma 9.2. The other goal is to fix the isomorphism between the E_2-term ${}^{Y}E_2^{0,4m}(tmf)$ of the Y-based generalized Adams spectral sequence for tmf and the free $\mathbb{Z}$-module of modular forms $\mathcal{M}_{2m}^{\mathbb{Z}}$. We have used this relation in order to name elements of tmf_*.

We start by recalling some material from [Rez07]. Let A be a commutative ring. A generalized Weierstrass equation over A is an equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \tag{48}$$

with $a_i \in A$. We let

$$W_0 := \mathtt{spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$$

be the affine scheme over $\mathbb{Z}$ representing Weierstrass equations. We furthermore let

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & W_0 \times \mathbb{P}^2 \\ & \searrow & \downarrow{\scriptstyle pr_{W_0}} \\ & & W_0 \end{array}$$

denote the universal Weierstrass curve defined by equation (48).
By $G \subseteq PGL_{3,\mathbb{Z}}$ we denote the subgroup scheme of automorphisms of $\mathbb{P}^2_{\mathbb{Z}}$ which respect generalized Weierstrass equations. The group scheme G is affine and acts on the universal Weierstrass curve $\mathcal{C} \to W_0$.
The scheme $W_1 := G \times W_0$ is the scheme of morphisms of the action groupoid $W_1 \rightrightarrows W_0$ in the category of schemes. It will be called the Weierstrass groupoid and is the spectrum of the so-called Weierstrass Hopf algebroid $(\mathcal{O}_{W_0}(W_0), \mathcal{O}_{W_1}(W_1))$.
The stack represented by the Weierstrass groupoid will be denoted by $\mathfrak{W}$. The action of G on the universal Weierstrass curve $\mathcal{C}$ gives an action groupoid $G \times \mathcal{C} \rightrightarrows \mathcal{C}$, and by $\mathfrak{C}$ we denote the corresponding stack. The G-equivariant projection $\mathcal{C} \to W_0$ induces a morphism of stacks $\pi : \mathfrak{C} \to \mathfrak{W}$, the universal Weierstrass curve.
For every ring A, the point $[0 : 1 : 0] \in \mathbb{P}^2_A$ belongs to each Weierstrass curve and is fixed by every automorphism in $G(A)$. We therefore get a section

$$\begin{array}{c} \mathfrak{C} \\ s \uparrow \downarrow \pi \\ \mathfrak{W} \end{array} .$$

The section s lies in the smooth locus of π. The pull-back of the relative cotangent bundle $s^*T^*(\mathfrak{C}/\mathfrak{W}) \to \mathfrak{W}$ is thus a line bundle over $\mathfrak{W}$. Its sheaf of sections will be denoted by ω.
We define an evenly graded Hopf algebroid (A_*, Γ_*) whose underlying graded rings are given by

$$A_* := \bigoplus_{n \geq 0} \omega^{2n}(W_0) \ , \quad \Gamma_* := \bigoplus_{n \geq 0} \omega^{2n}(W_1) \ ,$$

using the map $W_1 \xrightarrow{s} W_0 \to \mathfrak{W}$ in order to view W_1 as a scheme over $\mathfrak{W}$. The structure maps of this Hopf algebroid are induced in the canonical way. The spectrum tmf is characterized by the property that the complex $({}^YE_1^{*,2*}(tmf), d_1)$ has a preferred bigraded isomorphism to the cobar complex $(C^*(A_*), d^{cobar})$ of the Hopf-algebroid (A_*, Γ_*). This isomorphism will be used to name elements of ${}^YE_r^{*,*}(tmf_{(p)})$.

Let $\bar{\mathfrak{M}}$ denote the Deligne-Rapoport compactification of the moduli stack of elliptic curves over $\mathbb{Z}$, i.e. the stack of generalized elliptic curves with irreducible geometric fibers, denoted M_1 in [DR73]. Since generalized elliptic curves admit Weierstrass equations (Zariski locally), there is a unique (1-)morphism $i : \bar{\mathfrak{M}} \to \mathfrak{W}$ such that $i^*(\mathfrak{C})$ is isomorphic to the universal curve over $\bar{\mathfrak{M}}$. One can check that i is an open immersion.
The algebro-geometric definition of the $\mathbb{Z}$-module of integral modular forms of weight m is

$$^{alg}\mathcal{M}_m^{\mathbb{Z}} := H^0(\bar{\mathfrak{M}}, i^*\omega^m) \ .$$

Since $\mathfrak{W}$ is normal and the codimension of $\mathfrak{W} \setminus i(\bar{\mathfrak{M}})$ is ≥ 2 the inclusion $i : \bar{\mathfrak{M}} \to \mathfrak{W}$ induces an isomorphism

$$i^* : H^0(\mathfrak{W}, \omega^m) \to H^0(\bar{\mathfrak{M}}, i^*\omega^m) \ .$$

The following isomorphism will be used to identify elements in ${}^Y E_2^{0,*}(tmf)$ with modular forms.

$$^{alg}\mathcal{M}_{2m}^{\mathbb{Z}} \stackrel{(i^*)^{-1}}{\to} H^0(\mathfrak{W}, \omega^{2m}) \cong H^0(C^*(A_*, d^{cobar}))_{2m} \cong {}^Y E_2^{0,4m}(tmf) \ , \qquad (49)$$

where

$$H^0(C^*(A_*, d^{cobar})) = \bigoplus_{m\geq 0} H^0(C^*(A_*, d^{cobar}))_{2m}$$

is the decomposition of the cohomology by degree.
We now relate the algebraists' version $^{alg}\mathcal{M}_{2m}^{\mathbb{Z}}$ of modular forms with our formal power series version $\mathcal{M}_{2m}^{\mathbb{Z}} \subset \mathbb{Z}[[q]]$. In inhomogeneous coordinates $x := X/Z$, $y := Y/Z$ we can consider the 1-form

$$\eta \in \Gamma(\mathcal{C}, T^*(\mathcal{C}/W_0)) \ , \quad \eta := \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y} \ .$$

For all $n \geq 0$ the section $s^*\eta^n \in \omega^n(W_0)$ is nowhere vanishing and induces isomorphisms of groups

$$\eta^n : \mathcal{O}_{W_0}(W_0) \stackrel{\sim}{\to} \omega^n(W_0) \ .$$

The homomorphism $Tate : \mathtt{spec}\mathbb{Z}[[q]] \to W_0$ defined by

$$a_1 \mapsto 1 \ , a_2 \mapsto 0 \ , a_3 \mapsto 0 \ , a_4 \mapsto B \ , a_6 \mapsto C$$

with $B, C \in \mathbb{Z}[[q]]$ given by

$$B := -5 \sum_{n\geq 1} n^3 \cdot \frac{q^n}{1-q^n}, C := -\frac{1}{12} \sum_{n\geq 1} (7n^5 + 5n^3) \cdot \frac{q^n}{1-q^n}$$

defines the Tate curve over $\mathbb{Z}[[q]]$. It induces an injective map

$$r : {}^{alg}\mathcal{M}_{2m}^{\mathbb{Z}} \stackrel{(i^*)^{-1}}{\to} H^0(\mathfrak{W}, \omega^{2m}) \hookrightarrow \omega^{2m}(W_0) \stackrel{(\eta^{2m})^{-2}}{\to} \mathcal{O}_{W_0}(W_0) \stackrel{Tate^*}{\to} \mathbb{Z}[[q]] \ .$$

The following Lemma is a consequence of the q-expansion principle, [K, Theorem 1.6.1].

Lemma 9.1 *The map r defined above induces an isomorphism $r : {}^{alg}\mathcal{M}_{2m}^{\mathbb{Z}} \xrightarrow{\sim} \mathcal{M}_{2m}^{\mathbb{Z}}$.*

If we combine the q-expansion principle with (49) we get the identification

$$ {}^{Y}E_2^{0,4m}(tmf) \cong \mathcal{M}_{2m}^{\mathbb{Z}} \tag{50} $$

used in the present paper.
We now turn to the structure of the groups T_{2m}. For $\nu \geq 0$ we denote by $p_\nu : \mathbb{R}[[q]] \to \mathbb{R}$ the projection onto the ν-th coefficient. Let $m \geq 2$ and $k_m := \dim_{\mathbb{C}} \mathcal{M}_{2m}^{\mathbb{C}}$ be the dimension of the space of modular forms of weight $2m$.

Lemma 9.2 *i) There exists a $\mathbb{Z}$-basis $f_0, \ldots, f_{k_m-1} \in \mathcal{M}_{2m}^{\mathbb{Z}}$ such that $p_i(f_j) = \delta_{i,j}$ for $0 \leq i, j \leq k_m - 1$.*

ii) For a basis as in part i), the map

$$ \alpha : T_{2m} = \frac{\mathbb{R}[[q]]}{\mathbb{Z}[[q]] + \mathcal{M}_{2m}^{\mathbb{R}}} \longrightarrow \prod_{\nu \geq k_m} (\mathbb{R}/\mathbb{Z}) \quad , \quad f \mapsto \left(\left[p_\nu(f - \sum_{i=0}^{k-1} p_i(f) f_i) \right] \right)_{\nu \geq k_m} $$

is well-defined and an isomorphism of abelian groups.

Proof. It is easy to see that *i*) implies *ii*). We now show *i*).
According to [L, Part IV, chapter X, Theorem 4.4], there are $f_1, \ldots, f_{k_m-1} \in \mathcal{M}_{2m}^{\mathbb{Z}}$ such that

$$ p_i(f_j) = \delta_{ij} \text{ and } p_0(f_j) = 0 \ \forall 1 \leq i, j \leq k_m - 1. $$

Since $m \geq 2$, there are $\alpha, \beta \geq 0$ such that $4\alpha + 6\beta = 2m$. Denoting (cf. (16))

$$ E_4 := -\frac{8}{B_4} G_4 = 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{1 \leq d | n} d^3 \right) q^n \text{ and} $$

$$ E_6 := -\frac{12}{B_6} G_6 = 1 - 504 \sum_{n=1}^{\infty} \left(\sum_{1 \leq d | n} d^5 \right) q^n $$

the normalized Eisenstein series of the indicated weight, it is clear that

$$ f_0 := E_4^\alpha E_6^\beta - \sum_{i=1}^{k_m - 1} p_i(E_4^\alpha E_6^\beta) f_i \in \mathcal{M}_{2m}^{\mathbb{Z}} $$

is as desired. □

Remark 9.3 *One may be inclined to take the normalized Eisenstein series of weight $2m$ in place of f_0 in the above proof but this fails interestingly, e.g. in weight $2m = 12$ as follows:*
One has $k_{12} = 2$,

$$E_{12} = 1 + \frac{65220}{691} \sum_{n=1}^{\infty} \left(\sum_{1 \le d | n} d^{11} \right) q^n \in \mathcal{M}_{12}^{\mathbb{Q}} \setminus \mathcal{M}_{12}^{\mathbb{Z}}$$

and $f := E_{12} - \frac{65220}{691}\Delta \in \mathcal{M}_{12}^{\mathbb{Q}}$ satisfies $q^0(f), q^1(f) \in \mathbb{Z}$ which by the above implies that $q^n(f) \in \mathbb{Z}$ for all $n \geq 0$. Unwinding definitions, this is equivalent to saying that for all n

$$\tau(n) \equiv \sum_{1 \le d | n} d^{11} \ \ mod\ 691,$$

which is a classical congruence for Ramanujan's tau-function defined in terms of the q-expansion

$$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n.$$

The following corollary will be used later. Let $\Delta \in \mathcal{M}_{12}^{\mathbb{Z}}$ be as above the unique cusp form normalized such that $\Delta = q + \ldots$.

Corollary 9.4 *For every $k \geq 0$ and integer $a \in \mathbb{N}$ the element $\left[\frac{1}{a}\Delta^k\right] \in T_{12k+2}$ has order a.*

Proof. The case $k = 0$ is clear since $\mathcal{M}_2 = 0$. Let us now assume that $k \geq 1$. We have $\dim \mathcal{M}_{12k+2} = k$. Since $p_\nu(\frac{1}{a}\Delta^k) = 0$ for $\nu = 0, \ldots, k-1$ we have

$$\alpha(\left[\frac{1}{a}\Delta^k\right]) = \left(p_\nu(\frac{1}{a}\Delta^k)\right)_{\nu \geq k} = \left(\left[\frac{1}{a}\right], \left[\frac{a_1}{a}\right], \ldots\right) \in (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$$

for suitable $a_1, \ldots \in \mathbb{Z}$. Therefore $\alpha(\left[\frac{1}{a}\Delta^k\right])$ has order a. □

10 Table of $MSpin_*$

Anderson-Brown-Peterson [ABP66] have obtained the following additive decomposition

$$MSpin\hat{_2} \cong \prod_{n(J)\ \text{even}} ko\hat{_2}\langle n(J)\rangle \vee \prod_{1 \neq n(J)\ \text{odd}} ko\hat{_2}\langle n(J) - 2\rangle \vee \prod_{i=0}^{\infty} \prod_{j=1}^{\dim(Z_i)} K(\mathbb{Z}/2\mathbb{Z}, i)$$

of the 2-completion of $MSpin$. In the first two products the index J runs over all unordered tuples $J = (j_1, \ldots, j_k)$ for $k \geq 0$ and $j_i \geq 2$, $n(J) := \sum_{i=1}^{k} j_i$, and $Z := \bigoplus_{i=0}^{\infty} Z_i$

is some $\mathbb{N}$-graded $\mathbb{F}_2$-vector space. Furthermore, $\hat{ko_2}\langle k\rangle$ denotes the 2-completed and k-connective cover of ko. The Poincaré series given in [ABP66, Thm.1.11] can be combined to determine the dimensions $\dim(Z_i)$ for all $i \geq 0$. Using MAPLE and these formulas, we have compiled the following table. Recalling that there is no odd torsion in $MSpin_*$, this table determines $MSpin_*$ additively for $* \leq 127$. The final column separates off the torsion which is accounted for by the $K(\mathbb{Z}/2\mathbb{Z}, i)$'s, i.e. those Spin-manifolds which are detected by Stiefel-Whitney classes. For the purpose of the present paper, note that the familiar structure of ko_* implies that $MSpin_{4m-1}$ is entirely accounted for by the $K(\mathbb{Z}/2\mathbb{Z}, i)$'s. In the smallest dimension congruent to 3 modulo 4 in which $0 \neq MSpin_{39} \simeq \mathbb{Z}/2\mathbb{Z}$, one can use computations of Mahowald/Gorbunov [MG] to show that $MString_{39} \to MSpin_{39}$ is zero, i.e. $A_{39} = MSpin_{39}$, but for all $m \geq 11$ such that $MSpin_{4m-1} \neq 0$ (the table strongly suggests this is in fact true for *all* $m \geq 11$) we cannot decide whether or not $A_{4m-1} \subseteq MString_{4m-1}$ is an equality.

i	$\dim_{\mathbb{Q}}(MSpin\mathbb{Q}_i)$	$\dim_{\mathbb{F}_2}(MSpin_{i,tors})$	$\dim_{\mathbb{F}_2} Z_i$
0	1	0	0
1	0	1	0
2	0	1	0
3	0	0	0
4	1	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	2	0	0
9	0	2	0
10	0	3	0
11	0	0	0
12	3	0	0
13	0	0	0
14	0	0	0
15	0	0	0
16	5	0	0
17	0	5	0
18	0	7	0
19	0	0	0
20	7	1	1
21	0	0	0
22	0	1	1
23	0	0	0
24	11	0	0
25	0	11	0
26	0	15	0
27	0	0	0
28	15	2	2
29	0	1	1
30	0	3	3
31	0	0	0
32	22	1	1
33	0	23	1
34	0	31	1
35	0	0	0
36	30	6	6
37	0	2	2
38	0	7	7
39	0	1	1
40	42	4	4
41	0	45	3
42	0	60	4
43	0	2	2

i	$\dim_{\mathbb{Q}}(MSpin\mathbb{Q}_i)$	$\dim_{\mathbb{F}_2}(MSpin_{i,tors})$	$\dim_{\mathbb{F}_2} Z_i$
44	56	14	14
45	0	6	6
46	0	17	17
47	0	4	4
48	77	11	11
49	0	86	9
50	0	114	13
51	0	7	7
52	101	31	31
53	0	15	15
54	0	38	38
55	0	13	13
56	135	29	29
57	0	159	24
58	0	210	34
59	0	22	22
60	176	67	67
61	0	38	38
62	0	80	80
63	0	36	36
64	231	70	70
65	0	290	59
66	0	379	82
67	0	58	58
68	297	142	142
69	0	90	90
70	0	169	169
71	0	92	92
72	385	158	158
73	0	521	136
74	0	676	186
75	0	143	143
76	490	291	291
77	0	205	205
78	0	347	347
79	0	219	219
80	627	343	343
81	0	931	304
82	0	1196	404
83	0	330	330
84	792	589	589
85	0	448	448
86	0	698	698
87	0	494	494

i	$\dim_{\mathbb{Q}}(MSpin\mathbb{Q}_i)$	$\dim_{\mathbb{F}_2}(MSpin_{i,tors})$	$\dim_{\mathbb{F}_2} Z_i$
88	1002	721	721
89	0	1658	656
90	0	2103	848
91	0	729	729
92	1255	1171	1171
93	0	952	952
94	0	1385	1385
95	0	1068	1068
96	1575	1472	1472
97	0	2948	1373
98	0	3689	1731
99	0	1550	1550
100	1958	2296	2296
101	0	1967	1967
102	0	2706	2706
103	0	2233	2233
104	2436	2941	2941
105	0	5239	2803
106	0	6461	3451
107	0	3194	3194
108	3010	4438	4438
109	0	3969	3969
110	0	5215	5215
111	0	4539	4539
112	3718	5760	5760
113	0	9312	5594
114	0	11311	6746
115	0	6411	6411
116	4565	8470	8470
117	0	7839	7839
118	0	9925	9925
119	0	9005	9005
120	5604	11086	11086
121	0	16544	10940
122	0	19796	12954
123	0	12582	12582
124	6842	15963	15963
125	0	15193	15193
126	0	18656	18656
127	0	17493	17493

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