

# Universität Regensburg Mathematik



## On determinant functors and *K*-theory

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# ON DETERMINANT FUNCTORS AND $K$ -THEORY

FERNANDO MURO, ANDREW TONKS, AND MALTE WITTE

ABSTRACT. In this paper we introduce a new approach to determinant functors which allows us to extend Deligne's determinant functors for exact categories to Waldhausen categories, (strongly) triangulated categories, and derivators. We construct universal determinant functors in all cases by original methods which are interesting even for the known cases. Moreover, we show that the target of each universal determinant functor computes the corresponding  $K$ -theory in dimensions 0 and 1. As applications, we answer open questions by Maltziniotis and Neeman on the  $K$ -theory of (strongly) triangulated categories and a question of Grothendieck to Knudsen on determinant functors. We also prove additivity and localization theorems for low-dimensional  $K$ -theory and obtain generators and (some) relations for various  $K_1$ -groups.

## CONTENTS

Introduction	2
1. Determinant functors	4
1.1. For Waldhausen categories	5
1.2. Derived determinant functors	8
1.3. For triangulated categories	8
1.4. Virtual determinant functors	10
1.5. For strongly triangulated categories	11
1.6. Graded determinant functors on abelian categories	11
1.7. A unified approach to determinant functors	14
2. Strict Picard groupoids	19
2.1. Crossed modules and categorical groups	19
2.2. Strictifying tensor functors	22
2.3. Stable quadratic modules	25
2.4. Presentations	26
3. Universal determinant functors	27
3.1. The category of determinant functors	27
3.2. The existence of universal determinant functors	28
3.3. Non-commutative determinant functors	30
3.4. Examples	32
3.5. The connection to $K$ -theory	33
4. Applications	35

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4.1. Generators and (some) relations for $K_1$	35
4.2. Graded and ungraded determinant functors on an abelian category	48
4.3. Low-dimensional $K$ -theory of a triangulated category with a $t$ -structure	55
4.4. On additivity and localization for low-dimensional $K$ -theory of triangulated categories	58
4.5. Derived and non-derived determinant functors on a Waldhausen category	63
4.6. A counterexample to two conjectures by Maltasiniotis	63
4.7. The $K$ -theory of some unusual triangulated categories	65
References	70

## INTRODUCTION

Determinant functors, considered first by Knudsen and Mumford [KM76], categorify the usual notion of determinant of invertible matrices. The most elementary instance of such a functor sends a vector space  $V$  to a pair

$$\det V = (\wedge^{\dim V} V, \dim V).$$

This makes sense since the highest exterior power of an automorphism  $f: V \cong V$ , with matrix  $A$ , is multiplication by the determinant  $\wedge^{\dim V} f = \det A$ .

Deligne [Del87] axiomatised the properties of these functors, defining the notion of determinant functor on any exact category  $\mathcal{E}$  taking values in a Picard groupoid  $\mathcal{P}$ . He sketched the construction of a Picard groupoid of ‘virtual objects’  $V(\mathcal{E})$  which is the target of a *universal* determinant functor, in the sense that any other determinant functor factors through it in an essentially unique way.

The set of isomorphism classes of objects of the Picard groupoid  $V(\mathcal{E})$  is in natural bijection with Quillen’s  $K$ -theory group  $K_0(\mathcal{E})$  and the automorphism group of any object is isomorphic to  $K_1(\mathcal{E})$ . This shows that any interesting exact category has highly non-trivial determinant functors.

Knudsen [Knu02a, Knu02b] showed by elementary methods that determinant functors on an exact category  $\mathcal{E}$  extend to the category of bounded complexes  $C^b(\mathcal{E})$  in an essentially unique way, generalising results with Mumford [KM76].

More recently Breuning defined a notion of determinant functor for triangulated categories and showed that any triangulated category  $\mathcal{T}$  possesses a universal determinant functor [Bre06]. Moreover, if  $\mathcal{T}$  has a bounded non-degenerate  $t$ -structure with heart  $\mathcal{A}$  he proved that determinant functors on  $\mathcal{T}$  coincide with those on the abelian category  $\mathcal{A}$  [Bre06, Theorem 5.2]. Breuning also defined  $K$ -theory groups, that we denote  $K_0({}^b\mathcal{T})$  and  $K_1({}^b\mathcal{T})$ , from the target of the universal determinant functor, by analogy with the exact case considered by Deligne.

The  $K$ -theory of triangulated categories has been an object of discussion for several years. Schlichting showed that there cannot exist any higher  $K$ -theory of triangulated categories satisfying desirable properties such as functoriality, additivity, localisation, and agreement with Quillen’s  $K$ -theory [Sch02]. Nevertheless, Neeman defined several  $K$ -theories for a given triangulated category  $\mathcal{T}$ , given by spectra equipped with comparison maps [Nee05],

$$K({}^w\mathcal{T}) \longrightarrow K({}^d\mathcal{T}) \longrightarrow K({}^v\mathcal{T}).$$

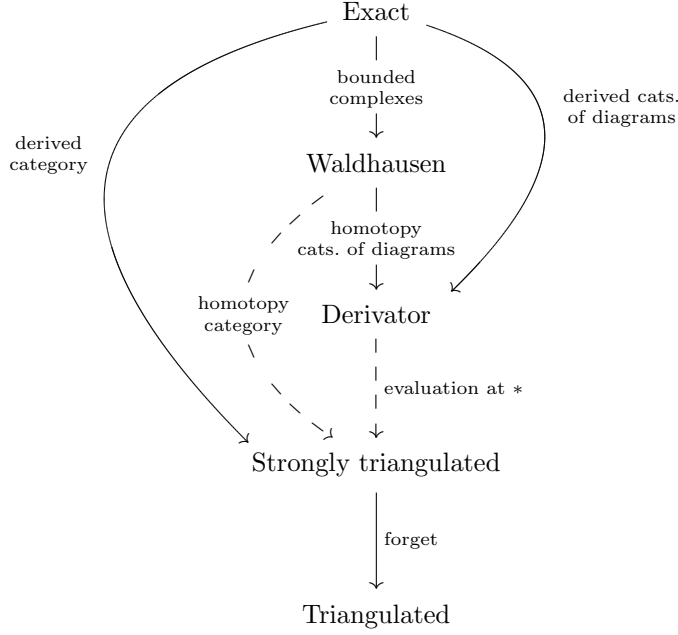


FIGURE 1. The hierarchy between exact and triangulated categories. The dashed arrows indicate that well-known stability properties are required.

Here  $K^{(d)\mathcal{T}}$  and  $K^{(v)\mathcal{T}}$  are functorial with respect to exact functors between triangulated categories, while  $K^{(w)\mathcal{T}}$  is not. The definition of  $K^{(d)\mathcal{T}}$  is based on the classical notion of distinguished triangles, octahedra, etc., while  $K^{(v)\mathcal{T}}$  uses Vaknin’s notion of virtual triangle [Vak01c]. If  $\mathcal{T}$  has a bounded non-degenerate  $t$ -structure with heart  $\mathcal{A}$ , Neeman constructs a comparison map

$$K(\mathcal{A}) \longrightarrow K^{(d)\mathcal{T}}.$$

All comparison maps are easily seen to induce isomorphisms in  $K_0$ . Neeman explicitly poses the open question of what happens in  $K_1$  [Nee05, Problem 1]. The spectrum  $K^{(w)\mathcal{T}}$  is only defined when  $\mathcal{T}$  has models in the sense of Thomason [TT90]. In this case, Neeman showed in a series of papers that the last comparison map factors through a weak equivalence,

$$K(\mathcal{A}) \xrightarrow{\sim} K^{(w)\mathcal{T}}.$$

Restricted to  $K_0$  and  $K_1$  this result is in some sense parallel to the aforementioned result of Breuning.

In this paper we are able to generalise the notions above to the different levels of the hierarchy interpolating between exact and triangulated categories (Figure 1). We define suitable notions of determinant functor, and show they correspond to  $K_0$  and  $K_1$  via categories of ‘virtual objects’. In consequence we obtain several new results.

More precisely, our aims in this paper are the following.

- Introduce a unified approach to determinant functors (Section 1.7) and construct categories of virtual objects and universal determinant functors (Section 3). For this purpose we use original methods which are interesting even for the known cases, since they are more explicit than [Del87] and less technical than [Bre06].
- Use this approach to define determinant functors for Waldhausen categories, (strongly) triangulated categories, graded abelian categories (Sections 1.1–1.6), and Grothendieck derivators (Example 1.7.4), and prove in each case that the category of virtual objects encodes  $K_0$  and  $K_1$  in the sense of Waldhausen [Wal85], Neeman [Nee05] and Maltsiniotis [Mal06, Mal07], respectively (Section 3.5).
- Obtain generators and (some) relations for  $K_1$  in all these cases (Section 4.1), along the lines of [Nen98, Vak01b, MT08].
- Answer Neeman’s question positively: we show that if  $\mathcal{T}$  is a triangulated category with a bounded non-degenerate  $t$ -structure with heart  $\mathcal{A}$ , then the natural comparison homomorphisms,

$$K_1(\mathcal{A}) \longrightarrow K_1({}^d\mathcal{T}) \longrightarrow K_1({}^v\mathcal{T}),$$

are isomorphisms. We do not assume the existence of any kind of models (Section 4.3).

- Prove new additivity and localization theorems for low-dimensional  $K$ -theories of triangulated categories (Section 4.4). We know by Schlichting’s results that these theorems cannot be extended to higher dimensions.
- Prove that determinant functors on an exact category  $\mathcal{E}$  and its bounded derived category  $D^b(\mathcal{E})$  coincide if we regard the latter as a triangulated category with a ‘category of true triangles’, and extend the result to Waldhausen categories (Section 4.5). This was posed as a question in a letter of Grothendieck to Knudsen [Knu02a, Appendix B].
- Disprove the conjecture of Maltsiniotis that the  $K$ -theories of  $\mathcal{E}$  and  $D^b(\mathcal{E})$  regarded as a strongly triangulated category agree, and also the conjecture that the  $K$ -theory of a triangulated derivator  $\mathbb{D}$  coincides with the  $K$ -theory of the strongly triangulated category  $\mathbb{D}(\ast)$  (Section 4.6), see [Mal06, Conjectures 1 and 2].
- Give examples where the comparison homomorphism  $K_1({}^d\mathcal{T}) \rightarrow K_1({}^v\mathcal{T})$  is not an isomorphism (Section 4.7).

Note that determinant functors on Waldhausen categories have already been successfully applied in non-commutative Iwasawa theory [Wit08, Wit10], and in  $\mathbb{A}^1$ -homotopy theory [Eri09]. They have also been discussed in the Geometric Langlands Seminar of the University of Chicago [Boy], see Remark 1.1.5. Fukaya and Kato give in [FK06] an alternative construction of the category of virtual objects for  $\mathcal{E}$  the exact category of projective modules of finite type over a ring  $R$ .

## 1. DETERMINANT FUNCTORS

Recall that a *Picard groupoid*  $\mathcal{P}$  is a symmetric monoidal category [Mac71, VII.1, 7] such that all morphisms are invertible and tensoring with any object  $x$  in  $\mathcal{P}$  yields an equivalence of categories

$$x \otimes - : \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Some examples are:

- The category  $\mathbf{Pic}(X)$  of line bundles over a scheme or manifold  $X$  with the tensor product over the structure sheaf  $\otimes_{\mathcal{O}_X}$ . If  $X = \mathrm{Spec} R$  is the spectrum of a commutative ring  $R$  then  $\mathbf{Pic}(R) = \mathbf{Pic}(X)$  is the groupoid of invertible  $R$ -modules with respect to the tensor product  $\otimes_R$ .
- The category  $\mathbf{Pic}^{\mathbb{Z}}(X)$  of graded line bundles over a scheme or manifold  $X$ . Objects are pairs  $(L, n)$  with  $L$  a line bundle over  $X$  and  $n: X \rightarrow \mathbb{Z}$  a locally constant map. There are only morphisms between objects with the same degree  $(L, n) \rightarrow (L', n)$ , given by isomorphisms  $L \rightarrow L'$ . The symmetric monoidal structure is  $(L, n) \otimes (L', m) = (L \otimes_{\mathcal{O}_X} L', n + m)$  with the usual associativity and unit constraints, and the graded symmetry constraint,

$$(L, n) \otimes (L', m) \rightarrow (L', m) \otimes (L, n): a \otimes b \mapsto (-1)^{n+m} b \otimes a.$$

**1.1. For Waldhausen categories.** A *Waldhausen category*  $\mathscr{W}$  is a category together with a distinguished zero object  $0$  and two subcategories  $\mathrm{cof}(\mathscr{W})$  and  $\mathrm{we}(\mathscr{W})$  containing  $\mathrm{iso}(\mathscr{W})$ , whose morphisms are called *cofibrations*  $\rightarrow$  and *weak equivalences*  $\xrightarrow{\sim}$ , respectively. The following axioms must hold:

- The morphism  $0 \rightarrow A$  is always a cofibration.
- The pushout of any map and a cofibration  $B \leftarrow A \rightarrow C$  exists in  $\mathscr{W}$ , and is denoted  $B \cup_A C$ .
- Given a commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

the induced morphism  $B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$  is a weak equivalence.

These categories were introduced by Waldhausen under the name of *categories with cofibrations and weak equivalences* as a general setting where a reasonable  $K$ -theory can be defined extending Quillen's [Wal85, Section 1.2].

*Example 1.1.1.* The following are three simple examples of Waldhausen categories:

- An *exact category*  $\mathcal{E}$  is a full additive subcategory of an abelian category closed under extensions. A short exact sequence in  $\mathcal{E}$  is a short exact sequence in the ambient abelian category between objects in  $\mathcal{E}$ . The first arrow of a short exact sequence in  $\mathcal{E}$  is called an *admissible monomorphism*. Admissible monomorphisms are the cofibrations of a Waldhausen category structure on  $\mathcal{E}$  with weak equivalences given by isomorphisms  $\mathrm{we}(\mathcal{E}) = \mathrm{iso}(\mathcal{E})$ . One must also choose a zero object  $0$  in  $\mathcal{E}$ . Examples of exact categories are abelian categories, the category  $\mathbf{Proj}(R)$  of finitely generated projective modules over a ring  $R$ , and the category  $\mathbf{Vect}(X)$  of vector bundles over a scheme or a manifold  $X$ .
- The category  $C^b(\mathcal{E})$  of bounded complexes in an exact category  $\mathcal{E}$ . Cofibrations are levelwise split monomorphisms and weak equivalences are quasi-isomorphisms, i.e. chain morphisms inducing isomorphism in homology computed in the ambient abelian category. The distinguished zero object is the complex with  $0$  everywhere.
- The category  $C^b(\mathcal{E})$  with the same weak equivalences and distinguished zero object as above, but levelwise admissible monomorphisms as cofibrations.

This Waldhausen category has the same  $K$ -theory as the previous one. We will always assume that  $C^b(\mathcal{E})$  is endowed with this Waldhausen category structure so that the inclusion  $\mathcal{E} \rightarrow C^b(\mathcal{E})$  of complexes concentrated in degree 0 preserves cofibrations, weak equivalences and distinguished zero objects.

Coproducts  $A \sqcup B = A \cup_0 B$  exist in  $\mathcal{W}$ . Also for any cofibration  $A \rightarrow B$  we have a *cofiber sequence*

$$A \rightarrow B \rightarrow B/A = 0 \cup_A B.$$

The *cofiber*  $B/A$  is only well defined up to canonical isomorphism under  $B$ , however this notation is standard in the literature. Coiber sequences in exact categories are short exact sequences.

**Definition 1.1.2.** A *determinant functor* from a Waldhausen category  $\mathcal{W}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor from the subcategory of weak equivalences,

$$\det: \text{we}(\mathcal{W}) \longrightarrow \mathcal{P},$$

together with *additivity data*: for any cofiber sequence  $\Delta: A \rightarrow B \rightarrow B/A$  in  $\mathcal{W}$ , a morphism in  $\mathcal{P}$ ,

$$\det(\Delta): \det(B/A) \otimes \det(A) \longrightarrow \det(B),$$

natural with respect to weak equivalences of cofiber sequences, given by commutative diagrams in  $\mathcal{W}$ ,

$$\begin{array}{ccccc} A & \longrightarrow & B & \twoheadrightarrow & B/A \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ A' & \longrightarrow & B' & \twoheadrightarrow & B'/A' \end{array}.$$

The following two axioms must be satisfied.

- (1) *Associativity*: let  $A \xrightarrow{f} B \xrightarrow{g} C$  be two cofibrations so that there are four cofiber sequences in  $\mathcal{W}$ ,

$$\begin{aligned} \Delta_f: A \xrightarrow{f} B \rightarrow B/A, & & \Delta_g: B \xrightarrow{g} C \rightarrow C/B, \\ \Delta_{gf}: A \xrightarrow{gf} C \rightarrow C/A, & & \tilde{\Delta}: B/A \rightarrow C/A \rightarrow C/B, \end{aligned}$$

fitting into a commutative diagram,

$$(1.1.3) \quad \begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow \\ & & & & B/A \longrightarrow C/A \\ & & \uparrow & & \uparrow \\ A \longrightarrow B & \longrightarrow & C & & \end{array}$$

Then the following diagram in  $\mathcal{P}$  commutes,

$$\begin{array}{ccc}
 & \det(C) & \\
 \det(\Delta_g) \nearrow & & \nwarrow \det(\Delta_{gf}) \\
 \det(C/B) \otimes \det(B) & & \det(C/A) \otimes \det(A) \\
 \uparrow 1 \otimes \det(\Delta_f) & & \uparrow \det(\tilde{\Delta}) \otimes 1 \\
 \det(C/B) \otimes (\det(B/A) \otimes \det(A)) & \xrightarrow{\text{associativity}} & (\det(C/B) \otimes \det(B/A)) \otimes \det(A).
 \end{array}$$

(2) *Commutativity*: let  $A, B$  be two objects so that there are two cofiber sequences associated to the inclusions and projections of a coproduct,

$$\Delta_1: A \rightarrow A \sqcup B \rightarrow B, \quad \Delta_2: B \rightarrow A \sqcup B \rightarrow A.$$

Then the following triangle commutes,

$$\begin{array}{ccc}
 & \det(A \sqcup B) & \\
 \det(\Delta_1) \nearrow & & \nwarrow \det(\Delta_2) \\
 \det(B) \otimes \det(A) & \xrightarrow{\text{symmetry}} & \det(A) \otimes \det(B).
 \end{array}$$

This definition of determinant functor generalizes Deligne's definition for the special case of exact categories [Del87, 4.2].

*Example 1.1.4.* The prototypical example of determinant functor on an exact category is the following. Suppose  $X$  is a scheme or manifold. Then the rank of a vector bundle  $E$  over  $X$  is a locally constant function  $\text{rk } E: X \rightarrow \mathbb{Z}$ , and we can define a determinant functor from  $\mathbf{Vect}(X)$  to  $\mathbf{Pic}^{\mathbb{Z}}(X)$  as follows

$$\det(E) = (\wedge_{\mathcal{O}_X}^{\text{rk } E} E, \text{rk } E).$$

As a particular case, when  $X = \text{Spec } R$  we get a determinant functor from  $\mathbf{Proj}(R)$  to  $\mathbf{Pic}^{\mathbb{Z}}(R)$ .

Knudsen–Mumford showed in [KM76] that this example can be extended to a determinant functor from  $C^b(\mathbf{Vect}(X))$  to  $\mathbf{Pic}^{\mathbb{Z}}(X)$  in an essentially unique way. Knudsen [Knu02a, Knu02b] generalized this result to arbitrary determinant functors on an exact category. These results are proved by a lengthy direct computation. We here derive this result from the existence of universal determinant functors with values in a Picard groupoid computing the first two  $K$ -theory groups (Corollary 4.5.1) and the Gillet–Waldhausen theorem.

*Remark 1.1.5.* In the seminar notes [Boy] a tentative definition of determinant functor is given. Drinfeld wonders whether this notion is such that a universal determinant functor exists and whether the target is associated to Waldhausen's  $K$ -theory [Boy, Endnote 7]). Our results on non-commutative determinant functors (Section 3.3) show that the answer is yes provided we introduce a slight correction in [Boy, (ii) in Section 2], we must require the induced map  $A \cup_{A'} B' \rightarrow B$  to be a cofibration, compare [MT08, Proposition 1.6]. The same correction must be made in [Eri09, Definition 2.2.1 (c)].



**1.2. Derived determinant functors.** Any Waldhausen category  $\mathcal{W}$  has an associated homotopy category  $\mathrm{Ho}(\mathcal{W})$  obtained by formally inverting weak equivalences in  $\mathcal{W}$ . We can also consider the Waldhausen category  $S_2\mathcal{W}$  of cofiber sequences in  $\mathcal{W}$ .

**Definition 1.2.1.** A *derived determinant functor* from a Waldhausen category  $\mathcal{W}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor from the category of isomorphisms in the homotopy category,

$$\det: \mathrm{iso}(\mathrm{Ho}(\mathcal{W})) \longrightarrow \mathcal{P},$$

together with *additivity data*: for any cofiber sequence  $\Delta: A \twoheadrightarrow B \rightarrow B/A$  in  $\mathcal{W}$ , a morphism in  $\mathcal{P}$ ,

$$\det(\Delta): \det(B/A) \otimes \det(A) \longrightarrow \det(B),$$

natural in  $\mathrm{Ho}(S_2\mathcal{W})$ . Axioms (1) and (2) in Definition 1.1.2 must be satisfied.

Derived determinant functors are related to Grothendieck's question to Knudsen that we answer positively in Section 4.5.

**1.3. For triangulated categories.** A *triangulated category*  $\mathcal{T}$  is an additive category together with an equivalence  $\Sigma: \mathcal{T} \xrightarrow{\sim} \mathcal{T}$  and a class of diagrams called *distinguished triangles*,

$$X \xrightarrow{f} Y \xrightarrow{i^f} C^f \xrightarrow{q^f} \Sigma X,$$

also depicted as

$$(1.3.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \swarrow^{-1} & \searrow^{i^f} \\ & C^f & \end{array}$$

where  $X \xrightarrow{-1} Y$  denotes a morphism  $X \rightarrow \Sigma Y$  (we use  $-1$  instead of the usual  $+1$  since we later use homological grading). Any diagram like (1.3.1) where two consecutive morphisms compose to 0 will be called a *triangle*. We say that  $f$  is the *base* of the triangle. The class of distinguished triangles is contained in the class of all triangles.

Distinguished triangles must satisfy a set of well-known axioms, see [Nee01]. Verdier's octahedral axiom says that given composable morphisms,

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

and three distinguished triangles  $\Delta_f$ ,  $\Delta_g$  and  $\Delta_{gf}$  with bases  $f$ ,  $g$  and  $gf$ , respectively, then there exists a diagram with the shape of an octahedron

$$(1.3.2) \quad \begin{array}{ccccc} & & Z & \xrightarrow{\quad} & C^g \\ & & \uparrow & \nearrow & \uparrow \\ & & C^{gf} & \xrightarrow{\bar{f}} & C^g \\ & & \uparrow & \nwarrow & \downarrow \\ gf & & X & \xrightarrow{\quad} & C^f \\ & & \downarrow & \searrow & \downarrow \\ & & Y & \xrightarrow{\quad} & C^f \end{array}$$

(Note: The diagram above is a schematic representation of the octahedron diagram (1.3.2). The vertices are  $Z$  (top),  $C^g$  (top-right),  $C^{gf}$  (center),  $X$  (bottom-left),  $C^f$  (bottom-right), and  $Y$  (bottom-center). Solid arrows represent the main morphisms:  $Z \rightarrow C^g$ ,  $Z \rightarrow C^{gf}$  (labeled  $g$ ),  $C^g \rightarrow C^{gf}$  (labeled  $\bar{f}$ ),  $C^{gf} \rightarrow X$  (labeled  $-1$ ),  $X \rightarrow C^f$  (labeled  $f$ ),  $C^f \rightarrow C^{gf}$  (labeled  $\bar{g}$ ), and  $C^g \rightarrow C^f$  (labeled  $-1$ ). Dotted arrows represent the other faces:  $Z \rightarrow Y$ ,  $C^g \rightarrow Y$ ,  $X \rightarrow Y$ , and  $C^f \rightarrow Y$ . The triangles  $\Delta_f$ ,  $\Delta_g$ , and  $\Delta_{gf}$  are distinguished triangles with bases  $f$ ,  $g$ , and  $gf$  respectively.)

in which three faces are  $\Delta_f$ ,  $\Delta_g$  and  $\Delta_{gf}$ , four faces are commutative triangles, and the remaining face,

$$\begin{array}{ccc} C^f & \xrightarrow{\bar{g}} & C^{gf} \\ & \nwarrow & \downarrow \\ & & C^g \end{array}$$

is also a distinguished triangle  $\tilde{\Delta}$ . Moreover, three planes divide the octahedron into two square pyramids. The squares perpendicular to the page must be commutative.

Verdier's axiom is about the existence of  $\bar{f}$  and  $\bar{g}$ ; the rest is given. Any diagram with the properties of (1.3.2) will be called an *octahedron*.

A *special octahedron* is an octahedron (1.3.2) such that the two commutative squares are homotopy push-outs, i.e. the following triangles are distinguished

$$(1.3.3) \quad \begin{array}{ccc} Y & \xrightarrow{\begin{pmatrix} g \\ -if \end{pmatrix}} & Z \oplus C^f \xrightarrow{(i^{gf}, \bar{g})} C^{gf} \xrightarrow{q^g \bar{f}} \Sigma Y, \\ C^{gf} & \xrightarrow{\begin{pmatrix} q^{gf} \\ -\bar{f} \end{pmatrix}} & \Sigma X \oplus C^g \xrightarrow{(\Sigma f, q^g)} \Sigma Y \xrightarrow{\Sigma(\bar{g}q^f)} \Sigma C^{gf}. \end{array}$$

Special octahedra were first introduced by Neeman [Nee05]. If  $\mathcal{T}$  is a derived category, or more generally a stable homotopy category, then it is well known that the standard octahedral completion of two composable morphisms  $X \rightarrow Y \rightarrow Z$  is special in this sense. In general, the octahedral axiom completion can be chosen so that one of the two triangles in (1.3.3) is distinguished, see [Nee01, Proposition 1.4.6].

**Definition 1.3.4.** A *Breuning determinant functor* from a triangulated category  $\mathcal{T}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor,

$$\det: \text{iso}(\mathcal{T}) \longrightarrow \mathcal{P},$$

together with *additivity data*: for any distinguished triangle  $\Delta: X \xrightarrow{f} Y \rightarrow C^f \rightarrow \Sigma X$ , a morphism in  $\mathcal{P}$ ,

$$\det(\Delta): \det(C^f) \otimes \det(X) \longrightarrow \det(Y),$$

natural with respect to distinguished triangle isomorphisms. The following two axioms must be satisfied, see [Bre06, Definition 3.1].

- (1) *Associativity*: for any octahedron as in (1.3.2) the following diagram in  $\mathcal{P}$  commutes,

$$\begin{array}{ccc}
 & \det(Z) & \\
 \det(\Delta_g) \nearrow & & \nwarrow \det(\Delta_{gf}) \\
 \det(C^g) \otimes \det(Y) & & \det(C^{gf}) \otimes \det(X) \\
 \uparrow 1 \otimes \det(\Delta_f) & & \uparrow \det(\tilde{\Delta}) \otimes 1 \\
 \det(C^g) \otimes (\det(C^f) \otimes \det(X)) & \xrightarrow{\text{associativity}} & (\det(C^g) \otimes \det(C^f)) \otimes \det(X)
 \end{array}$$

- (2) *Commutativity*: given two objects  $X, Y$  in  $\mathcal{T}$ , if we consider the two distinguished triangles associated to the inclusions and projections of a direct sum,

$$\Delta_1: X \rightarrow X \oplus Y \rightarrow Y \xrightarrow{0} \Sigma X, \quad \Delta_2: Y \rightarrow X \oplus Y \rightarrow X \xrightarrow{0} \Sigma Y,$$

then the following diagram commutes,

$$\begin{array}{ccc}
 & \det(X \oplus Y) & \\
 \det(\Delta_1) \nearrow & & \nwarrow \det(\Delta_2) \\
 \det(Y) \otimes \det(X) & \xrightarrow{\text{symmetry}} & \det(X) \otimes \det(Y)
 \end{array}$$

A *special determinant functor* is defined in the same way, but we only require associativity with respect to special octahedra.

**1.4. Virtual determinant functors.** The following notion of determinant functor is based on Vaknin's notion of virtual triangle [Vak01c]. Let  $\mathcal{T}$  be a triangulated category. A *contractible triangle* is a direct sum of triangles of the form,

$$A \xrightarrow{1} A \rightarrow 0 \rightarrow \Sigma A, \quad 0 \rightarrow B \xrightarrow{1} B \rightarrow 0, \quad C \rightarrow 0 \rightarrow \Sigma C \xrightarrow{1} \Sigma C,$$

i.e.

$$A \oplus C \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} B \oplus A \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \Sigma C \oplus B \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \Sigma A \oplus \Sigma C.$$

Contractible triangles are always distinguished.

A *virtual triangle*  $X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X$  is a direct summand with contractible complement of a triangle,

$$\begin{array}{ccccccc}
 X' & \xrightarrow{f'} & Y' & \xrightarrow{i'} & C^{f'} & \xrightarrow{q'} & \Sigma X' \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 X \oplus A \oplus C & \xrightarrow{f \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & Y \oplus B \oplus A & \xrightarrow{i \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & C^f \oplus \Sigma C \oplus B & \xrightarrow{q \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & \Sigma X \oplus \Sigma A \oplus \Sigma C.
 \end{array}$$

such that there exist distinguished triangles as follows,

$$X' \xrightarrow{f''} Y' \xrightarrow{i'} C^{f'} \xrightarrow{q'} \Sigma X', \quad X' \xrightarrow{f'} Y' \xrightarrow{i''} C^{f'} \xrightarrow{q'} \Sigma X', \quad X' \xrightarrow{f'} Y' \xrightarrow{i'} C^{f'} \xrightarrow{q''} \Sigma X',$$

i.e. each morphism in  $X' \xrightarrow{f'} Y' \xrightarrow{i'} C^{f'} \xrightarrow{q'} \Sigma X'$  can be replaced to obtain a distinguished triangle.

A *virtual octahedron* is a diagram like (1.3.2) where four faces  $\Delta_f, \Delta_g, \Delta_{gf}, \tilde{\Delta}$ , are virtual triangles, the remaining four faces are commutative triangles, and we have two commutative squares as in classical octahedra.

*Remark 1.4.1.* In a virtual octahedron, the triangles (1.3.3) are always virtual triangles by Vaknin’s two-out-of-three property [Vak01c, Section 1.3] applied to,

$$\begin{array}{ccc}
 Z & \xrightarrow{i^{gf}} & C^{gf} \\
 \uparrow g & & \uparrow \bar{g} \\
 Y & \xrightarrow{i^f} & C^f \\
 \uparrow f & & \uparrow \\
 X & \longrightarrow & 0
 \end{array}
 \qquad
 \begin{array}{ccc}
 C^g & \xrightarrow{q^g} & \Sigma Y \\
 \uparrow \bar{f} & & \uparrow \Sigma f \\
 C^{gf} & \xrightarrow{q^{gf}} & \Sigma X \\
 \uparrow i^{gf} & & \uparrow \\
 Z & \longrightarrow & 0
 \end{array}$$

**Definition 1.4.2.** A *virtual determinant functor* from a triangulated category  $\mathcal{T}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor

$$\det: \text{iso}(\mathcal{T}) \longrightarrow \mathcal{P}$$

together with *additivity data*: for any virtual triangle  $\Delta: X \xrightarrow{f} Y \rightarrow C^f \rightarrow \Sigma X$ , a morphism in  $\mathcal{P}$ ,

$$\det(\Delta): \det(C^f) \otimes \det(X) \longrightarrow \det(Y),$$

natural with respect to virtual triangle isomorphisms. In addition we require associativity for virtual octahedra and commutativity as in Definition 1.3.4.

**1.5. For strongly triangulated categories.** Following a remark of Beilinson–Bernstein–Deligne [BBD82, 1.1.14], Maltiniotis defined the notion of *strongly triangulated category*, also termed  *$\infty$ -triangulated category* [Mal06]. He showed that the bounded derived category  $D^b(\mathcal{E})$  can be endowed with such a structure. He also defined the truncated version, called  *$n$ -pretriangulated category*. A *3-pretriangulated category*  $\mathcal{T}_3$  is a triangulated category together with a family of *distinguished octahedra* (*3-triangles* in Maltiniotis’s terminology), which must satisfy some axioms generalizing the axioms for distinguished triangles in a triangulated category, see [Mal06, 1.3 and 1.4].

**Definition 1.5.1.** A *determinant functor* from a 3-pretriangulated category to a Picard groupoid is the same as a determinant functor on the underlying triangulated category, except that we only require the associativity axiom (1) to hold for distinguished octahedra. A determinant functor from a strongly triangulated category is a determinant functor on the underlying 3-pretriangulated category.

**1.6. Graded determinant functors on abelian categories.** In this section we define determinant functors on abelian categories with additivity data associated to long exact sequences, rather than to short exact sequences.

**Definition 1.6.1.** A bounded graded object  $X = \{X_n\}_{n \in \mathbb{Z}}$  in  $\mathcal{A}$  is a collection of objects  $X_n$  in  $\mathcal{A}$  such that  $X_n = 0$  for  $|n| \gg 0$ . The category of bounded graded objects in  $\mathcal{A}$  will be denoted by  $\text{Gr}^b \mathcal{A}$ .

A *graded determinant functor* from  $\mathcal{A}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor from the subcategory of isomorphisms,

$$\det: \text{iso}(\text{Gr}^b \mathcal{A}) \longrightarrow \mathcal{P},$$

together with *additivity data*: for any three bounded graded objects  $X$ ,  $Y$  and  $C^f$ , and any long exact sequence,

$$(1.6.2) \quad \cdots \rightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{i_n} C_n^f \xrightarrow{q_n} X_{n-1} \rightarrow \cdots,$$

a morphism in  $\mathcal{P}$ ,

$$\det(1.6.2): \det(C^f) \otimes \det(X) \longrightarrow \det(Y),$$

natural with respect to isomorphisms of long exact sequences. The following two axioms must be satisfied.

- (1) *Associativity*: given six bounded graded objects  $X$ ,  $Y$ ,  $Z$ ,  $C^f$ ,  $C^g$  and  $C^{gf}$ , and a commutative diagram,

(1.6.3)

formed by four long exact sequences,

$$(1.6.4) \quad \cdots \rightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{i_n^f} C_n^f \xrightarrow{q_n^f} X_{n-1} \rightarrow \cdots,$$

$$(1.6.5) \quad \cdots \rightarrow X_n \xrightarrow{g_n f_n} Z_n \xrightarrow{i_n^{gf}} C_n^{gf} \xrightarrow{q_n^{gf}} X_{n-1} \rightarrow \cdots,$$

$$(1.6.6) \quad \cdots \rightarrow Y_n \xrightarrow{g_n} Z_n \xrightarrow{i_n^g} C_n^g \xrightarrow{q_n^g} Y_{n-1} \rightarrow \cdots,$$

$$(1.6.7) \quad \cdots \rightarrow C_n^f \xrightarrow{\bar{g}_n} C_n^{gf} \xrightarrow{\bar{f}_n} C_n^g \xrightarrow{i_{n-1}^f q_n^g} C_{n-1}^f \rightarrow \cdots,$$

the following diagram in  $\mathcal{P}$  commutes,

$$\begin{array}{ccc} & \det(Z) & \\ \det(1.6.6) \nearrow & & \nwarrow \det(1.6.5) \\ \det(C^g) \otimes \det(Y) & & \det(C^{gf}) \otimes \det(X) \\ \uparrow 1 \otimes \det(1.6.4) & & \uparrow \det(1.6.7) \otimes 1 \\ \det(C^g) \otimes (\det(C^f) \otimes \det(X)) & \xrightarrow{\text{associativity}} & (\det(C^g) \otimes \det(C^f)) \otimes \det(X). \end{array}$$

(2) *Commutativity*: given two bounded graded objects  $X, Y$ , if we consider the following two long exact sequences,

$$(1.6.8) \quad \cdots \rightarrow X_n \xrightarrow{\binom{1}{0}} X_n \oplus Y_n \xrightarrow{(0,1)} Y_n \xrightarrow{0} X_{n-1} \rightarrow \cdots,$$

$$(1.6.9) \quad \cdots \rightarrow Y_n \xrightarrow{\binom{0}{1}} X_n \oplus Y_n \xrightarrow{(1,0)} X_n \xrightarrow{0} Y_{n-1} \rightarrow \cdots,$$

the following triangle commutes,

$$\begin{array}{ccc} & \det(X \oplus Y) & \\ \det(1.6.8) \nearrow & & \nwarrow \det(1.6.9) \\ \det(Y) \otimes \det(X) & \xrightarrow{\text{symmetry}} & \det(X) \otimes \det(Y). \end{array}$$

*Remark 1.6.10.* A long exact sequence (1.6.2) can also be depicted as a triangular diagram of bounded graded objects,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \swarrow \scriptstyle q^{-1} & \searrow \scriptstyle i \\ & C^f & \end{array}$$

such that  $f$  and  $i$  are degree 0 morphisms and  $q$  is a morphism of degree  $-1$ . If we denote  $\Sigma X$  the graded object with  $(\Sigma X)_n = X_{n-1}$  we can also denote the long exact sequence as follows,

$$X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X.$$

In the same fashion, the diagram of four long exact sequences (1.6.3) can be depicted as an octahedron,

$$\begin{array}{ccccc} & & Z & \xrightarrow{i^g} & C^g \\ & & \nearrow \scriptstyle i^{gf} & & \nwarrow \scriptstyle \bar{f} \\ & & C^{gf} & & \\ & & \nearrow \scriptstyle g & & \nwarrow \scriptstyle q^g \\ & & Y & & \\ & & \nearrow \scriptstyle q^{gf} & & \nwarrow \scriptstyle \bar{g} \\ & & X & \xrightarrow{q^f} & C^f \\ & & \nwarrow \scriptstyle f & & \nearrow \scriptstyle i^f \\ & & & & \end{array}$$

**Definition 1.6.11.** Given integers  $n \leq m$ , a bounded graded object  $X$  in  $\mathcal{A}$  is said to be *concentrated in the interval  $[n, m]$*  if  $X_k = 0$  provided  $k \notin [n, m]$ . It also makes sense to take  $n = -\infty$  and  $m = +\infty$ , in this case we use round brackets instead of square brackets, as usual. The full subcategory of graded objects in  $\mathcal{A}$  concentrated in  $[n, m]$  will be denoted by  $\text{Gr}^{[n, m]} \mathcal{A}$ . A long exact sequence (1.6.2) in  $\mathcal{A}$  is concentrated in  $[n, m]$  if  $X, Y$  and  $C^f$  are concentrated in  $[n, m]$ .

A *graded determinant functor concentrated in  $[n, m]$*  from  $\mathcal{A}$  to a Picard groupoid  $\mathcal{P}$  consists of a functor from the subcategory of isomorphisms,

$$\det: \text{iso}(\text{Gr}^{[n, m]} \mathcal{A}) \longrightarrow \mathcal{P}$$

together with additivity data associated to long exact sequences concentrated in  $[n, m]$  satisfying associativity and commutativity properties as above.

**1.7. A unified approach to determinant functors.** The following notion of determinant functor for simplicial categories generalizes the definitions above.

**Definition 1.7.1.** Let  $\mathcal{C}_\bullet$  be a reduced simplicial category,

$$\cdots \mathcal{C}_3 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xrightarrow{d_3} \end{array} \mathcal{C}_2 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \mathcal{C}_1 \xrightarrow{d_0} *,$$

$\begin{array}{c} \xrightarrow{s_0} \\ \xrightarrow{s_1} \end{array}$ 
 $\begin{array}{c} \xrightarrow{s_0} \end{array}$

i.e.  $*$  is the terminal category, with only one object  $*$  and one morphism (the identity). We assume that  $\mathcal{C}_n$  has coproducts for all  $n \geq 0$ , and faces and degeneracies preserve coproducts. Moreover,  $\mathcal{C}_\bullet$  is endowed with a simplicial subcategory  $\text{we}\mathcal{C}_\bullet$ , containing all isomorphisms  $\text{iso}\mathcal{C}_\bullet \subset \text{we}\mathcal{C}_\bullet$ , whose morphisms are termed *weak equivalences*. Finite coproducts of weak equivalences are required to be weak equivalences. We refer to such a  $\mathcal{C}_\bullet$  as a *simplicial category with weak equivalences*.

A *determinant functor* from  $\mathcal{C}_\bullet$  to a Picard groupoid  $\mathcal{P}$  consists of a functor,

$$\det: \text{we}\mathcal{C}_1 \longrightarrow \mathcal{P},$$

together with *additivity data*: for any object  $\Delta$  in  $\mathcal{C}_2$ , a morphism in  $\mathcal{P}$ ,

$$\det(\Delta): \det(d_0\Delta) \otimes \det(d_2\Delta) \longrightarrow \det(d_1\Delta),$$

natural with respect to morphisms in  $\text{we}\mathcal{C}_2$ . The following two axioms must be satisfied.

- (1) *Associativity*: Let  $\Theta$  be an object in  $\mathcal{C}_3$ . The following diagram in  $\mathcal{P}$  commutes,

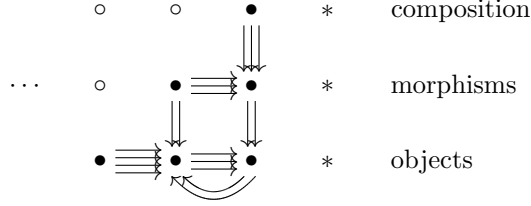
$$\begin{array}{ccc} & \det(d_1 d_2 \Theta) & \\ \det(d_1 \Theta) \nearrow & & \nwarrow \det(d_2 \Theta) \\ \det(d_0 d_1 \Theta) \otimes \det(d_1 d_3 \Theta) & & \det(d_0 d_2 \Theta) \otimes \det(d_2 d_3 \Theta) \\ \uparrow 1 \otimes \det(d_3 \Theta) & & \uparrow \det(d_0 \Theta) \otimes 1 \\ \det(d_0 d_1 \Theta) \otimes (\det(d_0 d_3 \Theta) \otimes \det(d_2 d_3 \Theta)) & \xrightarrow{\text{assoc.}} & (\det(d_0 d_3 \Theta) \otimes \det(d_0 d_3 \Theta)) \otimes \det(d_2 d_3 \Theta) \end{array}$$

- (2) *Commutativity*: given two objects  $X, Y$  in  $\mathcal{C}_1$  the following triangle commutes,

$$\begin{array}{ccc} & \det(X \sqcup Y) & \\ \det(s_1 X \sqcup s_0 Y) \nearrow & & \nwarrow \det(s_0 X \sqcup s_1 Y) \\ \det(Y) \otimes \det(X) & \xrightarrow{\text{symmetry}} & \det(X) \otimes \det(Y) \end{array}$$

*Remark 1.7.2.* Notice that in the previous definition we do not use all the structure of the reduced simplicial category  $\mathcal{C}_\bullet$  but only the piece of  $\mathcal{C}_\bullet$  depicted in the diagram above. Moreover we do not use all the structure of that diagram, but just the coproduct operation in  $\text{we}\mathcal{C}_1$  and  $\text{we}\mathcal{C}_2$ , the category structure of  $\text{we}\mathcal{C}_1$ , the

underlying graph of we  $\mathcal{C}_2$ , and the set of objects of  $\mathcal{C}_3$ . This can be illustrated by the following diagram,



*Example 1.7.3.* We now see how the determinant functors presented in Section 1 are covered by our unified approach. Weak equivalences in  $\mathcal{C}_\bullet$  are isomorphisms in all examples below, except from the first one. We need a distinguished zero object for the definition of degeneracies. This is not a real problem because all zero objects are canonically isomorphic.

- (1) Determinant functors on a Waldhausen category  $\mathcal{W}$  coincide with determinant functors on Waldhausen's  $S_\bullet(\mathcal{W})$  [Wal85]. This follows from the fact that  $S_1(\mathcal{W})$  is just  $\mathcal{W}$ ,  $S_2(\mathcal{W})$  is the category of cofiber sequences in  $\mathcal{W}$ ,  $S_3(\mathcal{W})$  is the category of diagrams in  $\mathcal{W}$  with shape (1.1.3), and the non-trivial faces and degeneracies in low dimensions are,

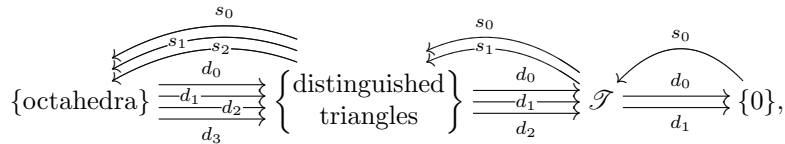
$$d_i(A \rightrightarrows B \rightrightarrows B/A) = \begin{cases} B/A, & i = 0; \\ B, & i = 1; \\ A, & i = 2; \end{cases}$$

$$s_i(A) = \begin{cases} 0 \rightrightarrows A \xrightarrow{1} A, & i = 0; \\ A \xrightarrow{1} A \rightrightarrows 0, & i = 1; \end{cases}$$

$$d_i(1.1.3) = \begin{cases} B/A \rightrightarrows C/A \rightrightarrows C/B, & i = 0; \\ B \rightrightarrows C \rightrightarrows C/B, & i = 1; \\ A \rightrightarrows C \rightrightarrows C/A, & i = 2; \\ A \rightrightarrows B \rightrightarrows B/A, & i = 3. \end{cases}$$

Waldhausen's  $S_\bullet$  construction is 2-functorial with respect to exact functors and natural weak equivalences between them.

- (2) From this description of the low-dimensional part of  $S_\bullet(\mathcal{W})$  it also follows that derived determinant functors on a Waldhausen category  $\mathcal{W}$  coincide with determinant functors on  $\text{Ho } S_\bullet(\mathcal{W})$ .
- (3) Given a triangulated category  $\mathcal{T}$  we can consider the reduced 3-truncated simplicial category  $\bar{S}_{\leq 3}({}^b\mathcal{T})$ ,



with faces and degeneracies

$$d_i(X \xrightarrow{f} Y \rightarrow C^f \rightarrow \Sigma X) = \begin{cases} C^f, & i = 0; \\ Y, & i = 1; \\ X, & i = 2; \end{cases}$$



$$s_i(X) = \begin{cases} 0 \rightarrow X \xrightarrow{1} X \rightarrow 0, & i = 0; \\ X \xrightarrow{1} X \rightarrow 0 \rightarrow \Sigma X, & i = 1; \end{cases}$$

$$d_i(1.3.2) = \begin{cases} C^f \rightarrow C^{gf} \rightarrow C^g \rightarrow \Sigma C^f, & i = 0; \\ Y \xrightarrow{g} Z \rightarrow C^g \rightarrow \Sigma Y, & i = 1; \\ X \xrightarrow{gf} Z \rightarrow C^{gf} \rightarrow \Sigma X, & i = 2; \\ X \xrightarrow{f} Y \rightarrow C^f \rightarrow \Sigma X, & i = 3. \end{cases}$$

The degeneracies  $s_i(X \xrightarrow{f} Y \rightarrow C^f \rightarrow \Sigma X)$ ,  $i = 0, 1, 2$ , are defined as the unique octahedra with the required faces.

The 3-truncated simplicial category  $\bar{S}_{\leq 3}({}^b\mathcal{T})$  can be extended to a simplicial category  $\bar{S}_\bullet({}^b\mathcal{T})$  by applying the 3-coskeleton functor, i.e. the right adjoint to the 3-truncation functor. Determinant functors on  $\mathcal{T}$  and  $\bar{S}_\bullet({}^b\mathcal{T})$  coincide.

The simplicial category  $\bar{S}_\bullet({}^b\mathcal{T})$  is 2-functorial with respect to exact functors between triangulated categories and natural isomorphisms between them.

- (4) We can also restrict ourselves to special octahedra,

$$\left\{ \begin{array}{l} \text{special} \\ \text{octahedra} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xleftarrow{s_2} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xrightarrow{d_3} \end{array} \left\{ \begin{array}{l} \text{distinguished} \\ \text{triangles} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \{0\}.$$

Then we essentially obtain the 3-skeleton of Neeman's simplicial set  $S_\bullet({}^d\mathcal{T})$  [Nee05]. More precisely,  $S_\bullet({}^d\mathcal{T})$  is the simplicial set of objects of a simplicial category  $\bar{S}_\bullet({}^d\mathcal{T})$  whose 3-skeleton is as defined above, and the inclusion  $S_\bullet({}^d\mathcal{T}) \subset \text{iso}(\bar{S}_\bullet({}^d\mathcal{T}))$  induces a homotopy equivalence on geometric realizations, compare [Wal85, Lemma 1.4.1]. Determinant functors on  $\bar{S}_\bullet({}^d\mathcal{T})$  are essentially special determinant functors in  $\mathcal{T}$ .

The simplicial category  $\bar{S}_\bullet({}^d\mathcal{T})$  is also 2-functorial with respect to exact functors between triangulated categories and natural isomorphisms.

- (5) We can also consider a 3-skeleton defined as above,

$$\left\{ \begin{array}{l} \text{virtual} \\ \text{octahedra} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xleftarrow{s_2} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xrightarrow{d_3} \end{array} \left\{ \begin{array}{l} \text{virtual} \\ \text{triangles} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \{0\}.$$

This is essentially the 3-skeleton of Neeman's simplicial set  $S_\bullet({}^v\mathcal{T})$  [Nee05]. In fact, as in the previous case,  $S_\bullet({}^v\mathcal{T})$  is the simplicial sets of objects of a simplicial category  $\bar{S}_\bullet({}^v\mathcal{T})$  whose 3-skeleton is as defined above, and such that the inclusion  $S_\bullet({}^v\mathcal{T}) \subset \text{iso}(\bar{S}_\bullet({}^v\mathcal{T}))$  induces a homotopy equivalence on geometric realizations. Determinant functors on  $\text{iso}(\bar{S}_\bullet({}^v\mathcal{T}))$  are essentially virtual determinant functors in  $\mathcal{T}$ .

Again, the simplicial category  $\bar{S}_\bullet({}^v\mathcal{T})$  turns out to be 2-functorial with respect to exact functors between triangulated categories and natural isomorphisms.

(6) Given a strongly triangulated category  $\mathcal{T}$  we consider,

$$\left\{ \begin{array}{l} \text{distinguished} \\ \text{octahedra} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \xleftarrow{s_2} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \xrightarrow{d_2} \\ \xrightarrow{d_3} \end{array} \left\{ \begin{array}{l} \text{distinguished} \\ \text{triangles} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \{0\},$$

This is essentially the 3-skeleton of Maltiniotis's simplicial set  $Q_\bullet(\mathcal{T})$  [Mal06]. Again,  $Q_\bullet(\mathcal{T})$  is the simplicial set of objects of the simplicial category  $\bar{Q}_\bullet(\mathcal{T})$  whose 3-skeleton is as defined above, and the inclusion  $Q_\bullet(\mathcal{T}) \subset \text{iso}(\bar{Q}_\bullet(\mathcal{T}))$  induces a homotopy equivalence on geometric realizations. Therefore, determinant functors on  $\bar{Q}_\bullet(\mathcal{T})$  essentially coincide with determinant functors in  $\mathcal{T}$ .

The simplicial category  $\bar{Q}_\bullet(\mathcal{T})$  is 2-functorial with respect to exact functors between strongly triangulated categories and natural isomorphisms between them.

(7) Given an abelian category  $\mathcal{A}$  we consider,

$$\left\{ \begin{array}{l} \text{diagrams} \\ \text{like (1.6.3)} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \xleftarrow{s_2} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \xrightarrow{d_2} \\ \xrightarrow{d_3} \end{array} \left\{ \begin{array}{l} \text{long exact} \\ \text{sequences} \end{array} \right\} \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{s_1} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \text{Gr}^b \mathcal{A} \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \{0\},$$

with faces and degeneracies,

$$d_i(1.6.2) = \begin{cases} C^f, & i = 0; \\ Y, & i = 1; \\ X, & i = 2. \end{cases}$$

$$s_i(X) = \begin{cases} \cdots \rightarrow 0 \rightarrow X_n \xrightarrow{1} X_n \rightarrow 0 \rightarrow \cdots, & i = 0; \\ \cdots \rightarrow X_n \xrightarrow{1} X_n \rightarrow 0 \rightarrow X_{n-1} \rightarrow \cdots, & i = 1; \end{cases}$$

$$d_i(1.6.3) = \begin{cases} (1.6.7), & i = 0; \\ (1.6.6), & i = 1; \\ (1.6.5), & i = 2; \\ (1.6.4), & i = 3. \end{cases}$$

This is essentially the 3-skeleton of the simplicial set  $S_\bullet(\text{Gr}^b \mathcal{A})$  defined by Neeman in [Nee05]. Once again, this is the simplicial set of objects of the simplicial category  $\bar{S}_\bullet(\text{Gr}^b \mathcal{A})$  whose 3-skeleton is as defined above, and the inclusion  $S_\bullet(\text{Gr}^b \mathcal{A}) \subset \text{iso}(\bar{S}_\bullet(\text{Gr}^b \mathcal{A}))$  induces a homotopy equivalence on geometric realizations. Determinant functors on  $\bar{S}_\bullet(\text{Gr}^b \mathcal{A})$  essentially coincide with graded determinant functors on  $\mathcal{A}$ .

The simplicial category  $\bar{S}_\bullet(\text{Gr}^b \mathcal{A})$  is 2-functorial with respect to exact functors between abelian categories and natural isomorphisms between them.

The full simplicial subcategory  $\bar{S}_\bullet(\text{Gr}^{[n,m]} \mathcal{A})$  spanned by graded objects concentrated in an interval  $[n, m]$  satisfies the same formal properties as  $\bar{S}_\bullet(\text{Gr}^b \mathcal{A})$ . Notice that  $\bar{S}_\bullet(\text{Gr}^{[0,0]} \mathcal{A})$  coincides with Waldhausen's  $S_\bullet(\mathcal{A})$ , and

$$\bar{S}_\bullet(\text{Gr}^b \mathcal{A}) = \text{colim}_{n \rightarrow +\infty} \bar{S}_\bullet(\text{Gr}^{[-n,n]} \mathcal{A}).$$

*Example 1.7.4.* The unified approach to determinant functors given by Definition 1.7.1 allows to define determinant functors for triangulated derivators, and more generally for *right* pointed derivators, using the terminology of [Cis08]. Notice that these are called *left* pointed derivators in [Gar06, Gar05].

Let  $\mathbf{Cat}$  be the 2-category of small categories and  $\mathbf{Dir}_f \subset \mathbf{Cat}$  the full sub-2-category of directed finite categories, i.e. those categories whose nerve have a finite number of non-degenerate simplices, e.g. finite posets. The canonical example of derivator is defined from a Waldhausen category  $\mathscr{W}$  with cylinders whose weak equivalences satisfy the two-out-of-three axiom, as for instance  $\mathscr{W} = C^b(\mathcal{E})$ . It is the contravariant 2-functor on the category of small categories,

$$\begin{aligned} \mathbb{D}_{\mathscr{W}} : \mathbf{Dir}_f^{\text{op}} &\longrightarrow \mathbf{Cat}, \\ J &\longmapsto \text{Ho}(\mathscr{W}^J), \end{aligned}$$

which takes a small category  $J$  to the homotopy category of  $J$ -indexed diagrams in  $\mathscr{W}$ . For  $\mathscr{W} = C^b(\mathcal{E})$ ,  $\mathbb{D}^b(\mathcal{E}) = \mathbb{D}_{C^b(\mathcal{E})}$  is

$$\begin{aligned} \mathbb{D}^b(\mathcal{E}) : \mathbf{Dir}_f^{\text{op}} &\longrightarrow \mathbf{Cat}, \\ J &\longmapsto D^b(\mathcal{E}^J). \end{aligned}$$

In general, a right pointed derivator is a 2-functor  $\mathbb{D} : \mathbf{Dir}_f^{\text{op}} \rightarrow \mathbf{Cat}$  satisfying the formal properties of  $\mathbb{D}_{\mathscr{W}}$ . Garkusha defined in [Gar06] a connected simplicial category  $S_{\bullet}\mathbb{D}$ , 2-functorial with respect to right exact pseudo-natural transformations between right pointed derivators and invertible modifications between them, in the same manner as Waldhausen's  $S_{\bullet}$ .

We define a determinant functor on  $\mathbb{D}$  to be a determinant functor on the simplicial category  $S_{\bullet}(\mathbb{D})$  with isomorphisms as weak equivalences. The interested reader may also work out the explicit definition of determinant functors for right pointed derivators along the lines of the previous section. Nevertheless, we warn that the outcome is not simple at all.

Given a simplicial functor  $f_{\bullet} : \mathcal{C}_{\bullet} \rightarrow \mathcal{C}'_{\bullet}$  between simplicial categories with weak equivalences preserving weak equivalences and coproducts, and a determinant functor  $\det' : \mathcal{C}'_1 \rightarrow \mathcal{P}$  on  $\mathcal{C}'_{\bullet}$ , the composite  $\det = \det' \circ f_1 : \mathcal{C}_1 \rightarrow \mathcal{P}$  is a determinant functor on  $\mathcal{C}_{\bullet}$  with  $\det(\Delta) = \det'(f_2(\Delta))$  for any object  $\Delta$  in  $\mathcal{C}_2$ . Actually, it is enough to have such a simplicial functor defined on the 3-skeletons  $f_{\leq 3} : \mathcal{C}_{\leq 3} \rightarrow \mathcal{C}'_{\leq 3}$ , and even less, compare Remark 1.7.2.

*Example 1.7.5.* We will consider the following particular instances.

- (1) Weak equivalences in a Waldhausen  $\mathscr{W}$  category project to isomorphisms in the homotopy category, so we have a simplicial functor as above given by the projection to the homotopy category,

$$S_{\bullet}(\mathscr{W}) \longrightarrow \text{Ho} S_{\bullet}(\mathscr{W}).$$

In particular any derived determinant functor on  $\mathscr{W}$  yields an honest determinant functor.

- (2) In a triangulated category  $\mathcal{T}$  any distinguished triangle is virtual, and any special octahedron is virtual, therefore there is an obvious simplicial faithful functor,

$$\bar{S}_{\bullet}(d\mathcal{T}) \longrightarrow \bar{S}_{\bullet}(v\mathcal{T}).$$

Moreover, there is also a simplicial faithful functor between the 3-skeletons,

$$\bar{S}_{\leq 3}(^d\mathcal{T}) \longrightarrow \bar{S}_{\leq 3}(^b\mathcal{T}).$$

This functor extends uniquely to  $\bar{S}_{\bullet}(^d\mathcal{T}) \rightarrow \bar{S}_{\bullet}(^b\mathcal{T})$  since the coskeleton construction is a right adjoint.

We deduce that Breuning and virtual determinant functors yield special determinant functors, which is actually obvious from the definitions.

- (3) Any strongly triangulated category  $\mathcal{T}$  has an underlying triangulated structure, therefore we have a 3-truncated simplicial functor,

$$\bar{Q}_{\leq 3}(\mathcal{T}) \longrightarrow \bar{S}_{\leq 3}(^b\mathcal{T}),$$

which has an adjoint  $\bar{Q}_{\bullet}(\mathcal{T}) \longrightarrow \bar{S}_{\bullet}(^b\mathcal{T})$ .

- (4) Maltsiniotis showed that a triangulated derivator  $\mathbb{D}$  induces a strongly triangulated category structure on  $\mathbb{D}(\ast)$ , and there is a comparison map,

$$S_{\bullet}(\mathbb{D}) \longrightarrow \bar{Q}(\mathbb{D}(\ast)),$$

defined by using the canonical evaluation functors from  $\mathbb{D}(J)$  to the category of functors  $J \rightarrow \mathbb{D}(\ast)$ .

- (5) If  $\mathcal{T}$  is a triangulated category with a  $t$ -structure with heart  $\mathcal{A}$ , any short exact sequence in  $\mathcal{A}$ ,

$$A \xrightarrow{j} B \xrightarrow{r} C,$$

extends uniquely to a distinguished triangle in  $\mathcal{T}$ ,

$$A \xrightarrow{j} B \xrightarrow{r} C \longrightarrow \Sigma A.$$

In this way the inclusion  $\mathcal{A} \subset \mathcal{T}$  induces a simplicial fully faithful functor [Nee05],

$$S_{\bullet}(\mathcal{A}) \longrightarrow \bar{S}_{\bullet}(^d\mathcal{T}).$$

We also have the following (truncated) simplicial functors defined by taking homology,

$$\begin{array}{ccc} & \bar{S}_{\bullet}(^d\mathcal{T}) & \\ & \swarrow & \downarrow H_* \\ \bar{S}_{\bullet}(^v\mathcal{T}) & \xrightarrow{H_*} & \bar{S}_{\bullet}(\mathrm{Gr}^b\mathcal{A}) \end{array} \quad \begin{array}{ccc} & \bar{S}_{\leq 3}(^d\mathcal{T}) & \\ & \swarrow & \downarrow H_* \\ \bar{S}_{\leq 3}(\mathrm{Gr}^b\mathcal{A}) & \xleftarrow{H_*} & \bar{S}_{\leq 3}(^v\mathcal{T}) \end{array}$$

## 2. STRICT PICARD GROUPOIDS

We will show that, without loss of generality, we may work entirely with strict Picard groupoids and strict determinant functors. This simplifies considerably definitions and proofs in later sections.

**2.1. Crossed modules and categorical groups.** Recall that a *crossed module* is a group homomorphism  $\partial: C_1 \rightarrow C_0$  together with a right action of  $C_0$  on  $C_1$  such that

- (1)  $\partial(c_1^{c_0}) = -c_0 + \partial(c_1) + c_0$ ,
- (2)  $c_1^{\partial(c'_1)} = -c'_1 + c_1 + c'_1$ .

It follows that the image of  $\partial$  is always a normal subgroup, and the kernel is always central. The *homotopy groups* of a crossed module  $C_*$  are

$$\pi_0(C_*) = C_0/\partial C_1, \quad \pi_1(C_*) = \text{Ker } \partial.$$

The action of  $C_0$  on  $C_1$  induces an action of  $\pi_0(C_*)$  on  $\pi_1(C_*)$ .

The *commutator* of two elements in a group  $x, y \in G$  is,

$$[x, y] = -x - y + x + y.$$

A *reduced 2-module* is a crossed module together with a map,

$$\langle \cdot, \cdot \rangle: C_0 \times C_0 \longrightarrow C_1,$$

which controls commutators. It must satisfy:

- (3)  $\partial \langle c_0, c'_0 \rangle = [c'_0, c_0]$ ,
- (4)  $c_1^{c'_0} = c_1 + \langle c_0, \partial(c_1) \rangle$ ,
- (5)  $\langle c_0, \partial(c_1) \rangle + \langle \partial(c_1), c_0 \rangle = 0$ ,
- (6)  $\langle c_0, c'_0 + c''_0 \rangle = \langle c_0, c'_0 \rangle^{c''_0} + \langle c_0, c''_0 \rangle$ ,
- (7)  $\langle c_0 + c'_0, c''_0 \rangle = \langle c'_0, c''_0 \rangle + \langle c_0, c''_0 \rangle^{c'_0}$ .

The crossed module  $\partial$  and the bracket  $\langle \cdot, \cdot \rangle$  form a *stable 2-module* if (3), (4), (6) and

$$(8) \quad \langle c_0, c'_0 \rangle + \langle c'_0, c_0 \rangle = 0$$

are satisfied. In a reduced or stable 2-module the action of  $C_0$  on  $C_1$  is completely determined by the bracket  $\langle \cdot, \cdot \rangle$ , by (4), so (1) is redundant and (2) becomes

$$(9) \quad \langle \partial(c_1), \partial(c'_1) \rangle = [c'_1, c_1].$$

The *k-invariant* of a reduced 2-module  $C_*$  is the natural quadratic map,

$$\begin{aligned} \eta: \pi_0(C_*) &\longrightarrow \pi_1(C_*), \\ [c_0] &\longmapsto \langle c_0, c_0 \rangle. \end{aligned}$$

In fact  $C_*$  is stable if and only if the *k-invariant* factors through a homomorphism,

$$\eta: \pi_0(C_*) \otimes \mathbb{Z}/2 \longrightarrow \pi_1(C_*).$$

A *crossed module morphism*  $f: C_* \rightarrow D_*$  is a pair of group homomorphisms  $f_i: C_i \rightarrow D_i$ ,  $i = 0, 1$ , which respect the actions and satisfy  $\partial f_1 = f_0 \partial$ . A *reduced* or *stable 2-module morphism* is a morphism  $f$  between the underlying crossed modules which preserves the bracket,  $\langle f_0, f_0 \rangle = f_1 \langle \cdot, \cdot \rangle$ .

A *homotopy*  $\alpha: f \Rightarrow g$  between two such morphisms is a function  $\alpha: C_0 \rightarrow D_1$  such that

$$\begin{aligned} \alpha(c_0 + c'_0) &= \alpha(c_0)^{g_0(c'_0)} + \alpha(c'_0), \\ \partial' \alpha(c_0) &= -g_0(c_0) + f_0(c_0), \\ \alpha \partial(c_1) &= -g_1(c_1) + f_1(c_1). \end{aligned}$$

Here we follow the conventions in [Wit08], which are opposite to [MT07, MT08].

Thus we obtain 2-categories of crossed modules and of reduced and stable 2-modules, together with their morphisms and homotopies of morphisms. Horizontal composition is given by composition of maps and the vertical composition of two homotopies

$$f \xRightarrow{\alpha} g \xRightarrow{\beta} h$$

is given by the map  $\beta + \alpha$ , compare [BM08, Proposition 7.2].

A *strong deformation retraction* is special kind of homotopy equivalence given by a diagram,

$$\begin{array}{ccc} & \alpha \curvearrowright & \\ & C_* & \xleftarrow[p]{j} D_* \\ & & \end{array},$$

where  $p$  and  $j$  are morphisms such that  $pj = 1_{D_*}$  and  $\alpha: jp \Rightarrow 1_{C_*}$  is a homotopy satisfying  $\alpha j = 0$  and  $p\alpha = 0$ .

A monoidal groupoid  $(\mathcal{G}, \otimes, I)$  with unit object  $I$  is a *categorical group* if for each object  $x$  of  $\mathcal{G}$  there is an object  $x^*$  and a map  $j_x: x^* \otimes x \cong I$ . Equivalently, there is a contravariant functor  $*$  on  $\mathcal{G}$  such that the endofunctors  $_ \otimes x$  and  $x \otimes _$  are equivalences of categories with inverses  $_ \otimes x^*$  and  $x^* \otimes _$ , respectively [Lap83].

A categorical group is *braided* or *symmetric* if the underlying monoidal category is. Recall that a *braiding* is a natural isomorphism,

$$\text{sym}_{x,y}: x \otimes y \longrightarrow y \otimes x,$$

satisfying certain coherence laws [JS93], and is a *symmetry* if  $\text{sym}_{y,x} \circ \text{sym}_{x,y} = 1_{x \otimes y}$  is the identity. A Picard groupoid is just a symmetric categorical group.

A *tensor functor* between categorical groups is a functor  $F: \mathcal{G} \rightarrow \mathcal{H}$  together with comparison maps for multiplication,

$$\text{mult.}: F(x) \otimes F(y) \longrightarrow F(x \otimes y),$$

which are natural and compatible with the associativity isomorphisms [Eps66]. A tensor functor between braided (or symmetric) categorical groups is *symmetric* if it is also compatible with the braiding isomorphisms. A *tensor natural transformation*  $\alpha: F \rightarrow G$  is one which commutes with the comparison maps for multiplication. (Braided, symmetric) categorical groups, (symmetric) tensor functors and tensor natural transformations also form 2-categories.

The *homotopy groups* of a (braided, symmetric) categorical group  $\mathcal{G}$  are,

$$\begin{aligned} \pi_0(\mathcal{G}) &= \text{isomorphism classes of objects, with } + \text{ induced by } \otimes, \\ \pi_1(\mathcal{G}) &= \text{Aut}_{\mathcal{G}}(I). \end{aligned}$$

Homotopy groups detect equivalences. The group  $\pi_0(\mathcal{G})$  acts on  $\pi_1(\mathcal{G})$  by

$$(I \xrightarrow{f} I)^{[x]} = x^* \otimes (I \xrightarrow{f} I) \otimes x,$$

and the action is trivial in the braided case. One can also define the *k-invariant* in the braided case as the natural quadratic map

$$\eta: \pi_0(\mathcal{G}) \longrightarrow \pi_1(\mathcal{G}),$$

such that  $x \otimes x \otimes \eta([x]) = \text{sym}_{x,x}$ , and  $\mathcal{G}$  is stable if and only if the *k-invariant* factors through a homomorphism

$$\eta: \pi_0(\mathcal{G}) \otimes \mathbb{Z}/2 \longrightarrow \pi_1(\mathcal{G}).$$

A (braided, symmetric) categorical group is *strict* if the associativity and unit isomorphisms are identities and the isomorphisms  $j_x$  can be chosen to be identities. Thus the underlying monoidal category is strict and the functors  $_ \otimes x$  and  $x \otimes _$  are isomorphisms of categories. If  $\mathcal{G}$  and  $\mathcal{H}$  are strict then  $F: \mathcal{G} \rightarrow \mathcal{H}$  is a *strict tensor functor* if the comparison maps for multiplication are all identities.

Strict (braided, symmetric) categorical groups, strict (symmetric) tensor functors and tensor natural transformations again form a 2-category.

**Proposition 2.1.1.** *There are equivalences between the 2-categories of*

- *strict categorical groups and crossed modules,*
- *braided strict categorical groups and reduced 2-modules,*
- *symmetric strict categorical groups (i.e. strict Picard groupoids), and stable 2-modules.*

*Proof.* The result is essentially due to Verdier, see [BS76] for some history. For any crossed module  $C_*$  the corresponding strict categorical group has object group  $C_0$  and morphism group  $C_0 \times C_1$ . Multiplication and inverses in these groups define the tensor and  $*$  operations. The morphisms have source and target as follows,

$$(c_0, c_1): c_0 + \partial(c_1) \longrightarrow c_0,$$

and the composite  $(c_0, c_1) \circ (c_0 + \partial(c_1), c'_1)$  is given by  $(c_0, c_1 + c'_1)$ . If  $C_*$  is a reduced or stable 2-module then the bracket defines a braiding or, symmetry respectively,

$$(c'_0 + c_0, \langle c'_0, c_0 \rangle): c_0 + c'_0 \longrightarrow c'_0 + c_0.$$

If  $f: C_* \rightarrow D_*$  is a morphism of crossed modules or (reduced, stable) 2-modules the associated functor is defined as  $f_0$  on objects and  $(c_0, c_1) \mapsto (f_0(c_0), f_1(c_1))$  on morphisms. Moreover, if  $\alpha: f \Rightarrow g$  is a homotopy between morphisms  $f, g: C_* \rightarrow D_*$  then the associated natural transformation is given by

$$(g_0(c_0), \alpha(c_0)): f_0(c_0) \longrightarrow g_0(c_0).$$

To recover a crossed module from a strict categorical group  $\mathcal{G}$  is straightforward:  $C_0$  is the object group,  $C_1$  is the kernel of target homomorphism, and

$$\partial(x \xrightarrow{a} I) = x, \quad (x \xrightarrow{a} I)^y = y^* \otimes (x \xrightarrow{a} I) \otimes y.$$

A braiding or symmetry also defines a bracket on this crossed module,

$$\langle x, y \rangle = y^* \otimes x^* \otimes (y \otimes x \xrightarrow{\text{sym.}} x \otimes y).$$

The morphism defined by a strict functor is the obvious one, and a tensor natural transformation  $\alpha: f \Rightarrow g$  between strict (symmetric) tensor functors  $f, g: \mathcal{G} \rightarrow \mathcal{H}$  yields a homotopy defined by the map  $x \mapsto g(x)^* \otimes (\alpha(x): f(x) \rightarrow g(x))$ .  $\square$

**2.2. Strictifying tensor functors.** A (braided, symmetric) strict categorical group is called *0-free* if the group of objects is free, and a 2-module or crossed module  $C_*$  is 0-free if  $C_0$  is a free group. In this section we shall prove the following result.

**Proposition 2.2.1.** *There is a weak equivalence between the 2-categories of:*

- *(braided, symmetric) categorical groups, (symmetric) tensor functors and tensor natural transformations,*
- *0-free (braided, symmetric) strict categorical groups, strict (symmetric) tensor functors and tensor natural transformations.*

Obviously the latter is a sub-2-category of the former. We give some details of the (folklore) results that (braided, symmetric) categorical groups can be strictified, and one can replace a strict categorical group by a 0-free one.

**Lemma 2.2.2.** *Any (braided, symmetric) categorical group is (symmetric) tensor equivalent to a 0-free strict one.*

*Proof.* We know that tensor equivalence classes of categorical groups  $\mathcal{G}$  with fixed isomorphisms  $\pi_0(\mathcal{G}) \cong G$ ,  $\pi_1(\mathcal{G}) \cong M$  of groups and  $G$ -modules, respectively, are in bijection with cohomology classes  $H^3(G, M)$  [Sin75, Chapitre 1 §1, Proposition 10]. We also know that any such class can be represented by a crossed module [Mac49],

therefore any categorical group is equivalent to a strict one. In addition a crossed module  $C_*$  can be replaced by a 0-free one  $D_*$  via the pull-back construction

$$\begin{array}{ccc} D_1 & \xrightarrow{\partial} & \langle E \rangle \\ \downarrow & \text{pull} & \downarrow \\ C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

Here  $E \subset C_0$  is a set of generators of  $\pi_0(C_*)$ ,  $\langle E \rangle$  is the free group with basis  $E$ , and  $D_0 = \langle E \rangle \rightarrow C_0$  is induced by the inclusion. This commutative square is a morphism of crossed modules which induces isomorphisms on homotopy groups, compare [BM08, Proposition 4.15], and therefore an equivalence between the corresponding categorical groups. Notice however that the inverse equivalence need not be strict.

The braided and symmetric case go along the same lines. If  $\mathcal{G}$  is braided or symmetric, we can strictify the underlying categorical group and then transfer the symmetry constraint along the equivalence. In this way we obtain an equivalent (braided, symmetric) strict categorical group. The pull-back construction allows us again to replace any reduced or stable 2-module by a 0-free one, compare [BM08, Proposition 4.15].  $\square$

When the source is 0-free, (symmetric) tensor functors can also be strictified. We have not found any reference for the following lemma in the literature.

**Lemma 2.2.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be strict categorical groups where  $\mathcal{G}$  is 0-free. Then for any tensor functor  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  there exists a strict tensor functor  $\phi^s : \mathcal{G} \rightarrow \mathcal{H}$  together with a tensor natural transformation  $\alpha : \phi^s \Rightarrow \phi$ .*

*Moreover, if  $\phi$  is a symmetric tensor functor between braided or symmetric categorical groups  $\mathcal{G}$  and  $\mathcal{H}$ , then  $\phi^s$  can be taken to be symmetric.*

*Proof.* Suppose  $\text{Ob}(\mathcal{G})$  is free on a set  $B$ , and define  $\phi^s : \text{Ob}(\mathcal{G}) \rightarrow \text{Ob}(\mathcal{H})$  to be the unique group homomorphism with  $\phi^s(b) = \phi(b)$  for  $b \in B$ . The transformation  $\alpha : \phi^s \Rightarrow \phi$  is defined on the neutral element  $\mathbb{1}_{\mathcal{G}}$ , elements  $b \in B$  and products  $b \otimes b'$ , for  $b, b' \in B$ , as follows:

$$\begin{aligned} \alpha(\mathbb{1}_{\mathcal{G}}) &= \mathbb{1}_{\mathcal{H}} \xrightarrow{\text{unit.}} \phi(\mathbb{1}_{\mathcal{G}}), \\ \alpha(b) &= \phi(b) \xrightarrow{1} \phi(b), \\ \alpha(b \otimes b') &= \phi(b) \otimes \phi(b') \xrightarrow{\text{mult}_{b,b'}} \phi(b \otimes b'). \end{aligned}$$

In general  $\alpha$  is defined on objects by induction on the reduced word length in the free group, by the following commutative diagram

$$\begin{array}{ccc} \phi^s(x) \otimes \phi^s(b) & \xrightarrow{=} & \phi^s(x \otimes b) \\ \alpha(x) \otimes 1 \downarrow & & \downarrow \alpha(x \otimes b) \\ \phi(x) \otimes \phi(b) & \xrightarrow{\text{mult}_{x,b}} & \phi(x \otimes b) \end{array}$$

This diagram defines  $\alpha(x \otimes b)$  from  $\alpha(x)$  provided the last letter in the reduced word  $x$  is not  $b^{-1}$ . At the same, if  $x = y \otimes b^{-1}$  is a reduced word, it defines  $\alpha(y \otimes b^{-1})$



from  $\alpha(y) = \alpha(x \otimes b)$ . Something similar happens later for the definition of  $\phi^s$  on morphisms. Notice also that this diagram is one case of the condition that  $\alpha$  is a tensor natural transformation. The condition is verified in general using induction (on word length of  $y$ ) and the following commutative diagram

$$\begin{array}{ccc}
 \phi^s(x) \otimes \phi^s(y \otimes b) & \xrightarrow{=} & \phi^s(x \otimes y \otimes b) \\
 \downarrow \alpha(x) \otimes \alpha(y \otimes b) & \swarrow \phi^s(x) \otimes \phi^s(y) \otimes \phi^s(b) \xrightarrow{=} \phi^s(x \otimes y) \otimes \phi^s(b) & \searrow \alpha(x \otimes y \otimes b) \\
 & \downarrow \alpha(x) \otimes \alpha(y) \otimes 1 & \downarrow \alpha(x \otimes y) \otimes 1 \\
 & \phi(x) \otimes \phi(y) \otimes \phi(b) \xrightarrow{\text{mult}_{x,y} \otimes 1} \phi(x \otimes y) \otimes \phi(b) & \\
 & \swarrow 1 \otimes \text{mult}_{y,b} & \searrow \text{mult}_{x \otimes y, b} \\
 \phi(x) \otimes \phi(y \otimes b) & \xrightarrow{\text{mult}_{x,y \otimes b}} & \phi(x \otimes y \otimes b) \\
 & & \downarrow \alpha(x \otimes y \otimes b)
 \end{array}$$

Now  $\phi^s$  is defined on morphisms  $f: x \rightarrow y$  by the following commutative diagram:

$$\begin{array}{ccc}
 \phi^s(x) & \xrightarrow{\phi^s(f)} & \phi^s(y) \\
 \alpha(x) \downarrow & & \downarrow \alpha(y) \\
 \phi(x) & \xrightarrow{\phi(f)} & \phi(y)
 \end{array}$$

This is just the naturality condition for  $\alpha$ .

The following diagram shows the functor  $\phi^s$  so defined is a tensor functor:

$$\begin{array}{ccc}
 \phi^s(x) \otimes \phi^s(x') & \xrightarrow{=} & \phi^s(x \otimes x') \\
 \downarrow \alpha(x) \otimes \alpha(x') & \swarrow \phi(x) \otimes \phi(x') \xrightarrow{\text{mult}_{x,x'}} \phi(x \otimes x') & \searrow \alpha(x \otimes x') \\
 & \downarrow \phi(f) \otimes \phi(f') & \downarrow \phi(f \otimes f') \\
 & \phi(y) \otimes \phi(y') \xrightarrow{\text{mult}_{y,y'}} \phi(y \otimes y') & \\
 \downarrow \alpha(y) \otimes \alpha(y') & & \downarrow \alpha(y \otimes y') \\
 \phi^s(y) \otimes \phi^s(y') & \xrightarrow{=} & \phi^s(y \otimes y') \\
 \downarrow \phi^s(f) \otimes \phi^s(f') & & \downarrow \phi^s(f \otimes f')
 \end{array}$$

Finally, we note that if  $\phi$  is symmetric then so is  $\phi^s$ , by the following commutative diagram:

$$\begin{array}{ccc}
\phi^s(y) \otimes \phi^s(x) & \xrightarrow{\text{sym.}} & \phi^s(x) \otimes \phi^s(y) \\
\downarrow \alpha(y) \otimes \alpha(x) & \searrow & \swarrow \alpha(x) \otimes \alpha(y) \\
\phi(y) \otimes \phi(x) & \xrightarrow{\text{sym.}} & \phi(x) \otimes \phi(y) \\
\downarrow \text{mult}_{y,x} & & \downarrow \text{mult}_{x,y} \\
\phi(y \otimes x) & \xrightarrow{\phi(\text{sym.})} & \phi(x \otimes y) \\
\downarrow \alpha(y \otimes x) & \swarrow & \searrow \alpha(x \otimes y) \\
\phi^s(y \otimes x) & \xrightarrow{\phi^s(\text{sym.})} & \phi^s(x \otimes y)
\end{array}$$

□

Now Proposition 2.2.1 follows from Lemmas 2.2.2 and 2.2.3.

**2.3. Stable quadratic modules.** The category of *stable quadratic modules* is the full reflective subcategory of the category of stable 2-modules given by those objects  $C_*$  for which the bracket vanishes whenever one argument lies in the commutator subgroup of  $C_0$ ,

$$(2.3.1) \quad \langle c_0, [c'_0, c''_0] \rangle = 0.$$

This holds if and only if the bracket factors through the tensor square of  $C_0^{ab}$ . Hence a stable quadratic module consists just of group homomorphisms,

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle \cdot, \cdot \rangle} C_1 \xrightarrow{\partial} C_0,$$

satisfying equations (3), (8) and (9) in Section 2.1, if we use the same notation for elements of  $C_0$  as for their images in  $C_0^{ab}$ .

The laws of a stable quadratic module imply that  $C_0$  and  $C_1$  are groups of nilpotence class 2, and that the image of  $\langle \cdot, \cdot \rangle$  is central.

We call a stable quadratic module  $C_*$  *0-free* if  $C_0$  is free as an object in the category of groups of nilpotence class 2, i.e. it is the quotient of a free group by triple commutators.

**Proposition 2.3.2.** *There is a weak equivalence between the 2-categories of 0-free stable 2-modules and 0-free stable quadratic modules.*

The equivalence is realized by the inclusion and the reflection functors, this follows from [MT07, Remark 4.21]. The following result is now a combination of this last proposition and Propositions 2.1.1 and 2.2.1.

**Corollary 2.3.3.** *There is a weak equivalence between the 2-categories of Picard groupoids and 0-free stable quadratic modules.*

We finish this section by stating a useful lemma about homotopies, see [Wit08, Lemmas 2.1.13 and 2.1.14].

**Lemma 2.3.4.** *Let  $g: C_* \rightarrow D_*$  be a morphism of stable quadratic modules with  $C_0$  free of nilpotency class 2 with basis  $E$ . Any map  $E \rightarrow D_1$  extends to a map,*

$$\alpha: C_0 \longrightarrow D_1,$$

satisfying,

$$\alpha(c_0 + c'_0) = \alpha(c_0)^{g_0(c'_0)} + \alpha(c'_0), \quad c_0, c'_0 \in C_0.$$

Moreover, there is a unique morphism  $f = g + \alpha: C_* \rightarrow D_*$ , defined as,

$$f(c_0) = g(c_0) + \partial\alpha(c_0), \quad f(c_1) = g(c_1) + \alpha\partial(c_1), \quad c_i \in C_i, i = 0, 1,$$

such that  $\alpha$  is a homotopy  $\alpha: f \Rightarrow g$ .

**2.4. Presentations.** We consider the adjoints of the functors sending a crossed module or a stable quadratic module  $C_*$  to the pair of sets  $(C_0, C_1)$ . Objects in the image of this left adjoint are said to be *free*.

Let  $\langle E \rangle$  denote the free group on a set  $E$ . The free crossed module  $F_*^c(E_0, E_1)$  on a pair of sets  $(E_0, E_1)$  is defined as follows:  $F_0^c(E_0, E_1) = \langle E_0 \sqcup E_1 \rangle$  is a free group,  $F_1^c(E_0, E_1) = \text{Ker } p$  is the kernel of the homomorphism,

$$\langle E_0 \sqcup E_1 \rangle \xrightarrow{p} \langle E_0 \rangle, \quad E_0 \ni e_0 \mapsto e_0, \quad E_1 \ni e_1 \mapsto 0,$$

the homomorphism  $\partial: F_1^c(E_0, E_1) \hookrightarrow F_0^c(E_0, E_1)$  is the inclusion, and  $F_0^c(E_0, E_1)$  acts on  $F_1^c(E_0, E_1)$  by conjugation. The universal property of a free crossed module holds since  $F_1^c(E_0, E_1)$  is freely generated as a group by the conjugates,

$$e_1^{c_0} = -c_0 + e_1 + c_0, \quad e_1 \in E_1, c_0 \in \langle E_0 \rangle.$$

Given two sets of relations  $R_i \subset F_i^c(E_0, E_1)$ ,  $i = 0, 1$ , the crossed module  $C_*$  with generators  $(E_0, E_1)$  and relations  $(R_0, R_1)$  is defined as follows:  $C_0$  is the quotient of  $F_0^c(E_0, E_1)$  by the normal subgroup  $N_0$  generated by  $R_0 \cup \partial R_1$ , and  $C_1$  is the quotient of  $F_1^c(E_0, E_1)$  by the normal subgroup generated by,

$$r_1^{c_0}, \quad r_1 \in R_1, c_0 \in C_0; \quad -c_1 + c_1^{n_0}, \quad c_1 \in C_1, n_0 \in N_0.$$

The action of  $C_0$  on  $C_1$  and the homomorphism  $\partial: C_1 \rightarrow C_0$  are defined so that the natural projection  $F_*^c(E_0, E_1) \twoheadrightarrow C_*$  is a morphism of crossed modules.

The free stable quadratic module  $F_*^s(E_0, E_1)$  on a pair of sets  $(E_0, E_1)$  is defined as follows:  $F_0^s(E_0, E_1) = \langle E_0 \sqcup E_1 \rangle^{nil}$  is the nilpotent group of class two freely generated by the set  $E_0 \sqcup E_1$ , i.e. the quotient of  $\langle E_0 \sqcup E_1 \rangle$  by triple commutators. For any abelian group  $A$  we define  $\hat{\otimes}^2 A$  as the quotient of the tensor square  $A \otimes A$  by  $a \otimes b + b \otimes a$ ,  $a, b \in A$ . Moreover, if we denote  $\langle E \rangle^{ab}$  the free abelian group on a set  $E$ , then,

$$F_1^s(E_0, E_1) = \hat{\otimes}^2 \langle E_0 \rangle^{ab} \times \langle E_0 \times E_1 \rangle^{ab} \times \langle E_1 \rangle^{nil}.$$

The homomorphism  $\partial$  and the bracket  $\langle \cdot, \cdot \rangle$  in  $F_*^s(E_0, E_1)$  are defined by the following formulas:

$$\begin{aligned} \partial(e_0 \otimes e'_0, (e_0'', e_1), e'_1) &= [e'_0, e_0] + [e_1, e_0''] + e'_1; & \langle e_0, e'_0 \rangle &= (e_0 \otimes e'_0, 0, 0); \\ \langle e_0, e_1 \rangle &= (0, (e_0, e_1), 0); & \langle e_1, e'_1 \rangle &= (0, 0, [e'_1, e_1]). \end{aligned}$$

Given two sets of relations  $R_i \subset F_i^s(E_0, E_1)$ ,  $i = 0, 1$ , the stable quadratic module  $C_*$  with generators  $(E_0, E_1)$  and relations  $(R_0, R_1)$  is defined as follows:  $C_0$  is the quotient of  $F_0^s(E_0, E_1)$  by the normal subgroup  $N_0$  generated by  $R_0 \cup \partial R_1$ , and  $C_1$  is the quotient of  $F_1^s(E_0, E_1)$  by the normal subgroup generated by  $R_1$  and  $\langle F_0^s(E_0, E_1), N_0 \rangle$ . The homomorphism  $\partial$  and the bracket  $\langle \cdot, \cdot \rangle$  are defined so that the natural projection  $F_*^s(E_0, E_1) \twoheadrightarrow C_*$  is a morphism of stable quadratic modules.

Crossed modules and stable quadratic modules defined by a presentation satisfy the obvious universal property.

## 3. UNIVERSAL DETERMINANT FUNCTORS

**3.1. The category of determinant functors.** Let  $\mathcal{C}_\bullet$  be a simplicial category with weak equivalences.

**Definition 3.1.1.** Let  $\mathcal{P}$  be a Picard groupoid. A *morphism of determinant functors*  $f: \det \rightarrow \det'$  on  $\mathcal{C}_\bullet$  is a natural transformation between the functors  $\det, \det': \text{we } \mathcal{C}_1 \rightarrow \mathcal{P}$  compatible with the additivity data, i.e. given an object  $\Delta$  in  $\mathcal{C}_2$ , the following diagram commutes:

$$\begin{array}{ccc} \det(d_2\Delta) \otimes \det(d_0\Delta) & \xrightarrow{\det(\Delta)} & \det(d_1\Delta) \\ f(d_2\Delta) \otimes f(d_0\Delta) \downarrow & & \downarrow f(d_1\Delta) \\ \det'(d_2\Delta) \otimes \det'(d_0\Delta) & \xrightarrow[\det'(\Delta)]{} & \det'(d_1\Delta) \end{array}$$

The resulting *category of determinant functors* is denoted by  $\text{Det}(\mathcal{C}_\bullet, \mathcal{P})$ .

*Remark 3.1.2.* The category  $\text{Det}(\mathcal{C}_\bullet, \mathcal{P})$  is itself a Picard groupoid. The tensor structure is given as follows. For any determinants  $\det, \det', \det''$ , any object  $X$  and any morphism  $\alpha$  in  $\text{we } \mathcal{C}_1$  as well as any object  $\Delta$  in  $\mathcal{C}_2$ , we define:

$$\begin{aligned} (\det \otimes \det')(X) &= \det(X) \otimes \det'(X), \\ (\det \otimes \det')(\alpha) &= \det(\alpha) \otimes \det'(\alpha), \\ (\det \otimes \det')(\Delta) &= \det(\Delta) \otimes \det'(\Delta) \circ 1 \otimes \text{comm} \otimes 1, \\ \text{ass}(\det, \det', \det'')(X) &= \text{ass}(\det(X), \det'(X), \det''(X)), \\ \text{comm}(\det, \det')(X) &= \text{comm}(\det(X), \det'(X)). \end{aligned}$$

This structure has already been considered in [Knu02a, Proposition 1.13] for determinant functors of exact categories.

For determinant functors with values in strict Picard groupoids it is convenient to introduce also the following notion.

**Definition 3.1.3.** A determinant functor  $\det: \mathcal{C}_\bullet \rightarrow \mathcal{P}$  with values in a strict Picard groupoid  $\mathcal{P}$  is *strict* if it satisfies

$$\det(s_0^2(*)) = \text{id}_I$$

for the initial object  $s_0^2(*)$  of  $\mathcal{C}_2$ .

We denote by  $\text{Det}_s(\mathcal{C}_\bullet, \mathcal{P})$  the full subcategory of  $\text{Det}(\mathcal{C}_\bullet, \mathcal{P})$  whose objects are the strict determinant functors.

**Lemma 3.1.4.** *The inclusion  $\text{Det}_s(\mathcal{C}_\bullet, \mathcal{P}) \subset \text{Det}(\mathcal{C}_\bullet, \mathcal{P})$  is an equivalence, natural in the strict Picard groupoid  $\mathcal{P}$ .*

*Proof.* Let  $D: \mathcal{C}_\bullet \rightarrow \mathcal{P}$  be a determinant functor. Then we can define a strict determinant functor  $D'$  by

$$D' = D \otimes D(s_0(*))^{-1}: \text{we } \mathcal{C}_1 \rightarrow \mathcal{P}$$

and by

$$D'(\Delta) = (D(\Delta) \otimes \text{id}_{D(s_0(*))^{-1}}) \circ (D(s_1 d_2 \Delta)^{-1} \otimes \text{id}_{D(s_0(*))^{-1} \otimes D'(d_0 \Delta)})$$

for objects  $\Delta$  in  $\mathcal{C}_2$ . Moreover,  $X \mapsto D(s_1 X) \otimes \text{id}_{D(s_0(*))^{-1}}$  is a morphism of determinant functors from  $D$  to  $D'$ .  $\square$

**Definition 3.1.5.** Let  $\mathcal{C}_\bullet$  be a simplicial category as in Definition 1.7.1. A determinant functor  $\det: \mathcal{C}_\bullet \rightarrow V(\mathcal{C}_\bullet)$  is *universal* if any determinant functor  $\det': \mathcal{C}_\bullet \rightarrow \mathcal{P}$  factors through  $\det$  in an essentially unique way, i.e. there exists a factorization

$$(3.1.6) \quad \begin{array}{ccc} \mathcal{C}_\bullet & \xrightarrow{\det} & V(\mathcal{C}_\bullet) \\ & \searrow \alpha & \downarrow f \\ & \det' & \mathcal{P} \end{array}$$

where  $f$  is a symmetric tensor functor and  $\alpha$  is a natural transformation of determinant functors, and moreover, if

$$\begin{array}{ccc} \mathcal{C}_\bullet & \xrightarrow{\det} & V(\mathcal{C}_\bullet) \\ & \searrow \alpha' & \downarrow f' \\ & \det' & \mathcal{P} \end{array}$$

is another such factorization, then there exists a unique tensor natural transformation  $\beta: f \Rightarrow f'$  such that (3.1.6) coincides with the pasting of

$$\begin{array}{ccc} \mathcal{C}_\bullet & \xrightarrow{\det} & V(\mathcal{C}_\bullet) \\ & \searrow \alpha' & \downarrow f' \\ & \det' & \mathcal{P} \end{array} \quad \begin{array}{c} \xleftarrow{\beta} \\ \xleftarrow{\beta} \\ \xleftarrow{\beta} \end{array} \quad \begin{array}{c} f \\ \downarrow \\ \mathcal{P} \end{array}$$

*Remark 3.1.7.* Let  $\text{Hom}_c^\otimes(\mathcal{P}, \mathcal{P}')$  denote the category of symmetric tensor functors  $F: \mathcal{P} \rightarrow \mathcal{P}'$ . Then the above definition is equivalent to saying that the natural transformation induced by  $\det$ ,

$$\text{Hom}_c^\otimes(V(\mathcal{C}_\bullet), -) \longrightarrow \text{Det}(\mathcal{C}_\bullet, -),$$

is a natural equivalence.

**3.2. The existence of universal determinant functors.** In this section we will show that universal determinant functors always exist. We will actually construct universal determinant functors by using presentations of stable quadratic modules.

**Definition 3.2.1.** Let  $\mathcal{C}_\bullet$  be a simplicial category with weak equivalences. We define the stable quadratic module  $\mathcal{D}_*(\mathcal{C}_\bullet)$  by generators,

- (G1)  $[X]$  for any object in  $\mathcal{C}_1$ , in dimension 0,
- (G2)  $[X \rightarrow X']$  for any weak equivalence in  $\mathcal{C}_1$ , in dimension 1,
- (G3)  $[\Delta]$  for any object in  $\mathcal{C}_2$ , in dimension 1,

and relations,

- (R1)  $\partial[X \xrightarrow{\sim} X'] = -[X'] + [X]$ ,
- (R2)  $\partial[\Delta] = -[d_1\Delta] + [d_0\Delta] + [d_2\Delta]$ ,
- (R3)  $[s_0(*)] = 0$  for the degenerate object of  $\mathcal{C}_1$ ,
- (R4)  $[X \xrightarrow{1} X] = 0$ , for all identity morphisms in  $\mathcal{C}_1$ ,
- (R5)  $[s_0X] = 0 = [s_1X]$  for any object  $X$  in  $\mathcal{C}_1$ ,
- (R6) for any pair of composable weak equivalences  $X \xrightarrow{\sim} Y \xrightarrow{\sim} Z$  in  $\mathcal{C}_1$ ,

$$[X \xrightarrow{\sim} Z] = [Y \xrightarrow{\sim} Z] + [X \xrightarrow{\sim} Y],$$

(R7) for any weak equivalence  $\Phi: \Delta \xrightarrow{\sim} \Delta'$  in  $\mathcal{C}_2$ ,

$$[d_2\Phi] + [d_0\Phi]^{[d_2\Delta]} = -[\Delta'] + [d_1\Phi] + [\Delta],$$

(R8) for any object  $\Theta$  in  $\mathcal{C}_3$ ,

$$[d_1\Theta] + [d_3\Theta] = [d_2\Theta] + [d_0\Theta]^{[d_2d_3\Theta]},$$

(R9) for any two objects  $A$  and  $B$  in  $\mathcal{C}_1$ ,

$$\langle [X], [Y] \rangle = -[s_0X \sqcup s_1Y] + [s_1X \sqcup s_0Y].$$

*Remark 3.2.2.* This is not a minimal presentation, compare [MT08, Remark 1.4], but it is the most intuitive. Relation (R4) follows from (R6) applied to

$$X \xrightarrow{1} X \xrightarrow{1} X, \quad [X \xrightarrow{1} X] = [X \xrightarrow{1} X] + [X \xrightarrow{1} X].$$

Relation (R3) follows from (R5),

$$0 = \partial[s_1X] = -[X] + [X] + [s_0(*)] = [s_0(*)].$$

Relation (R5) is equivalent to imposing  $[s_0^2(*)] = 0$  for the degenerate object of  $\mathcal{C}_2$ . Indeed, applying (R8) to  $s_0^2X$  and  $s_1^2X$ , respectively, we obtain,

$$\begin{aligned} [s_0X] + [s_0^2(*)] &= [s_0X] + [s_0X]^{[X]}, \\ [s_1X] + [s_1^2(*)] &= [s_1X] + [s_0^2(*)]^{[s_0^2(*)]}. \end{aligned}$$

*Remark 3.2.3.* The stable quadratic module  $\mathcal{D}_*(\mathcal{C}_\bullet)$  is functorial with respect to simplicial functors  $f_\bullet: \mathcal{C}_\bullet \rightarrow \mathcal{C}'_\bullet$  preserving weak equivalences and coproducts,

$$\begin{aligned} \mathcal{D}_0(f_\bullet): \mathcal{D}_0(\mathcal{C}_\bullet) &\longrightarrow \mathcal{D}_0(\mathcal{C}'_\bullet), & \mathcal{D}_1(f_\bullet): \mathcal{D}_1(\mathcal{C}_\bullet) &\longrightarrow \mathcal{D}_1(\mathcal{C}'_\bullet), \\ [X] &\mapsto [f_1(X)], & [\phi: X \xrightarrow{\sim} X'] &\mapsto [f(\phi): f_1(X) \xrightarrow{\sim} f_1(X')], \\ & & [\Delta] &\mapsto [f_2(\Delta)]. \end{aligned}$$

Moreover, it is 2-functorial with respect to simplicial natural weak equivalences  $\alpha_\bullet: f_\bullet \Rightarrow g_\bullet$  between simplicial functors  $f_\bullet, g_\bullet: \mathcal{C}_\bullet \rightarrow \mathcal{C}'_\bullet$  as above,

$$\begin{aligned} \mathcal{D}_*(\alpha_\bullet): \mathcal{D}_0(\mathcal{C}_\bullet) &\longrightarrow \mathcal{D}_1(\mathcal{C}'_\bullet), \\ [X] &\mapsto [\alpha_1(X): f_1(X) \xrightarrow{\sim} g_1(X)]. \end{aligned}$$

**Theorem 3.2.4.** *There is defined a determinant functor on  $\mathcal{C}_\bullet$  with*

$$\begin{aligned} \det: \text{we } \mathcal{C}_1 &\longrightarrow \mathcal{D}_*(\mathcal{C}_\bullet), \\ X &\mapsto [X], \\ (X \rightarrow X') &\mapsto [X \rightarrow X'], \end{aligned}$$

and for any object  $\Delta$  in  $\mathcal{C}_2$ ,

$$\det(\Delta) = [\Delta]: [d_0\Delta] + [d_2\Delta] \longrightarrow [d_1\Delta].$$

Moreover, let  $\text{Hom}_{\text{cs}}^\otimes(\mathcal{P}, \mathcal{P}')$  denote the category of strict tensor functors between strict Picard groupoids  $\mathcal{P}$  and  $\mathcal{P}'$ . Then  $\det$  induces an isomorphism

$$\text{Hom}_{\text{cs}}^\otimes(\mathcal{D}_*(\mathcal{C}_\bullet), \mathcal{P}) \xrightarrow{\cong} \text{Det}_s(\mathcal{C}_\bullet, \mathcal{P})$$

for any strict Picard groupoid  $\mathcal{P}$  corresponding to a stable quadratic module.

*Proof.* This follows immediately from the presentation of  $\mathcal{D}_*(\mathcal{C}_\bullet)$  given in Definition 3.2.1.  $\square$

**Corollary 3.2.5.** *The determinant functor  $\det: \mathcal{C}_1 \rightarrow \mathcal{D}_*(\mathcal{C}_\bullet)$  in Theorem 3.2.4 is universal.*

*Proof.* Let  $\mathcal{P}$  be any Picard groupoid and let  $\mathcal{P}^s$  be the stable quadratic module associated to the corresponding 0-free strict Picard groupoid constructed in Lemma 2.2.2. By Corollary 2.3.3, Lemma 3.1.4 and Theorem 3.2.4 we have a chain of natural equivalences,

$$\begin{aligned} \mathrm{Hom}_c^\otimes(\mathcal{D}_*(\mathcal{C}_\bullet), \mathcal{P}) &\simeq \mathrm{Hom}_{cs}^\otimes(\mathcal{D}_*(\mathcal{C}_\bullet), \mathcal{P}^s) \cong \\ &\mathrm{Det}_s(\mathcal{C}_\bullet, \mathcal{P}^s) \simeq \mathrm{Det}(\mathcal{C}_\bullet, \mathcal{P}^s) \simeq \mathrm{Det}(\mathcal{C}_\bullet, \mathcal{P}). \end{aligned}$$

□

**3.3. Non-commutative determinant functors.** In [Del87], Deligne also considers determinant functors into categorical groups that are not necessarily symmetric. Of course, one has to omit the commutativity axiom in Definition 1.7.1 if one chooses to work in this context. We will call those determinant functors *non-commutative determinant functors*. However, as Deligne already noticed, it turns out that this notion is not essentially more general than the theory of commutative determinant functors considered above.

**Definition 3.3.1.** Given a simplicial category with weak equivalences  $\mathcal{C}_\bullet$  we define  $\mathcal{D}'_*(\mathcal{C}_\bullet)$  as the crossed module presented by generators (G1–3) and relations (R1–8) as in Definition 3.2.1.

**Proposition 3.3.2.** *There exists a unique map*

$$\langle \cdot, \cdot \rangle: \mathcal{D}'_*(\mathcal{C}_\bullet) \times \mathcal{D}'_*(\mathcal{C}_\bullet) \rightarrow \mathcal{D}'_*(\mathcal{C}_\bullet)$$

such that

- (1)  $\langle [X], [Y] \rangle = -[s_0X \sqcup s_1Y] + [s_1X \sqcup s_0Y]$  for any two objects  $X, Y$  in  $\mathcal{C}_1$ .
- (2)  $(\mathcal{D}'_*(\mathcal{C}_\bullet), \langle \cdot, \cdot \rangle)$  is a reduced 2-module.

Moreover, this map satisfies

$$\langle a, b \rangle + \langle b, a \rangle = 0$$

for any  $a, b \in \mathcal{D}'_0(\mathcal{C}_\bullet)$ , i. e.  $(\mathcal{D}'_*(\mathcal{C}_\bullet), \langle \cdot, \cdot \rangle)$  is a stable 2-module.

*Proof.* We use the same argument as in [Wit08, Lemma 2.2.3] The relations for the objects  $s_i s_j(X)$  in  $\mathcal{C}_3$  imply  $\langle [X], [s_0(*)] \rangle = \langle [s_0(*)], [X] \rangle = 0$  for any object in  $\mathcal{C}_1$ . Recall that  $[s_0(*)] = 0$ . Since the group  $\mathcal{D}'_0(\mathcal{C}_\bullet)$  is the free group over the set  $E$  of objects of  $\mathcal{C}_1$  minus the degenerate object  $s_0(*)$ , an induction over the reduced word length of the two arguments shows that the map

$$E \times E \rightarrow \mathcal{D}'_1(\mathcal{C}_\bullet), \quad (X, Y) \mapsto -[s_0X \sqcup s_1Y] + [s_1X \sqcup s_0Y]$$

extends in a unique way to a map

$$\langle \cdot, \cdot \rangle: \mathcal{D}'_*(\mathcal{C}_\bullet) \times \mathcal{D}'_*(\mathcal{C}_\bullet) \rightarrow \mathcal{D}'_*(\mathcal{C}_\bullet)$$

satisfying

- (1)  $\langle c, c' + c'' \rangle = \langle c, c' \rangle^{c''} + \langle c, c'' \rangle$ ,
- (2)  $\langle c + c', c'' \rangle = \langle c', c'' \rangle + \langle c, c'' \rangle^{c'}$ .

It remains to show that  $(\mathcal{D}'_*(\mathcal{C}_\bullet), \langle \cdot, \cdot \rangle)$  is a stable 2-module. For this, it suffices to check the axioms (3), (4), (6), and (8) in Section 2.1. Axioms (3) and (6) are immediate from the definition of  $\langle \cdot, \cdot \rangle$ .

We verify axiom (8). Let  $X$  and  $Y$  be objects of  $\mathcal{C}_1$ . Given two coproducts  $s_1X \sqcup s_0Y$  and  $s_0Y \sqcup s_1X$  their universal property yields a unique isomorphism fitting into the following commutative diagram,

$$\begin{array}{ccccc} s_1X & \longrightarrow & s_1X \sqcup s_0Y & \longleftarrow & s_0Y \\ \parallel & & \downarrow \sim & & \parallel \\ s_1X & \longrightarrow & s_0Y \sqcup s_1X & \longleftarrow & s_0Y \end{array}$$

where the horizontal arrows are the inclusions of the factors. This isomorphism and the corresponding one after exchanging  $X$  and  $Y$  yield the following relations,

$$\begin{aligned} & [s_0Y \sqcup s_1X] + [d_0(s_1X \sqcup s_0Y \xrightarrow{\sim} s_0Y \sqcup s_1X)] + [d_2(s_1X \sqcup s_0Y \xrightarrow{\sim} s_0Y \sqcup s_1X)]^{[X]} \\ &= [d_1(s_1X \sqcup s_0Y \xrightarrow{\sim} s_0Y \sqcup s_1X)] + [s_1X \sqcup s_0Y], \\ & [s_0X \sqcup s_1Y] + [d_0(s_1Y \sqcup s_0X \xrightarrow{\sim} s_0X \sqcup s_1Y)] + [d_2(s_1Y \sqcup s_0X \xrightarrow{\sim} s_0X \sqcup s_1Y)]^{[Y]} \\ &= [d_1(s_1Y \sqcup s_0X \xrightarrow{\sim} s_0X \sqcup s_1Y)] + [s_1Y \sqcup s_0X], \end{aligned}$$

Moreover, we have

$$d_i(s_1X \sqcup s_0Y \xrightarrow{\sim} s_0Y \sqcup s_1X) = \begin{cases} 1_X, & i = 0; \\ X \sqcup Y \xrightarrow{\sim} Y \sqcup X, & i = 1; \\ 1_Y, & i = 2; \end{cases}$$

$$d_i(s_1Y \sqcup s_0X \xrightarrow{\sim} s_0X \sqcup s_1Y) = \begin{cases} 1_Y, & i = 0; \\ Y \sqcup X \xrightarrow{\sim} X \sqcup Y, & i = 1; \\ 1_X, & i = 2; \end{cases}$$

and hence,

$$\begin{aligned} \langle [X], [Y] \rangle + \langle [Y], [X] \rangle &= -([X \sqcup Y \xrightarrow{\sim} Y \sqcup X] + [Y \sqcup X \xrightarrow{\sim} X \sqcup Y])^{\partial[s_0X \sqcup s_1Y]} \\ &= -[1_{Y \sqcup X}]^{\partial[s_0X \sqcup s_1Y]} = 0. \end{aligned}$$

By induction it follows that  $\langle c, c' \rangle + \langle c', c \rangle = 0$  for any pair of elements  $c, c'$  in  $\mathcal{D}'_0(\mathcal{C}_\bullet)$ .

Finally, we verify axiom (4). Since both sides of the axiom define operations of  $\mathcal{D}'_0(\mathcal{C})$  on  $\mathcal{D}'_1(\mathcal{C})$ , it suffices to check the relation for the action of an object  $U$  of  $\mathcal{C}_1$  on a weak equivalence  $\alpha: X \rightarrow X'$  in  $\mathcal{C}_1$  and on an object  $\Delta$  in  $\mathcal{C}_2$ , respectively. The weak equivalences  $s_1\alpha \sqcup s_01_U$  and  $s_0\alpha \sqcup s_11_U$  in  $\mathcal{C}_2$  imply

$$\begin{aligned} [s_1X' \sqcup s_0U] + [\alpha] &= [\alpha \sqcup 1_U] + [s_1X \sqcup s_0U], \\ [s_0X' \sqcup s_1U] + [\alpha]^{[U]} &= [\alpha \sqcup 1_U] + [s_0X \sqcup s_1U], \end{aligned}$$

and hence,

$$\begin{aligned} [\alpha]^{[U]} &= \langle [X'], [U] \rangle + [\alpha] + \langle [U], [X] \rangle \\ &= [\alpha] + \langle [U], -[X'] \rangle^{[X]} + \langle [U], [X] \rangle = [\alpha] + \langle [U], \partial[\alpha] \rangle. \end{aligned}$$

The objects  $s_0(\Delta) \sqcup s_1s_1(U)$ ,  $s_1(\Delta) \sqcup s_0s_1(U)$ ,  $s_2(\Delta) \sqcup s_0s_1(U)$ , and  $s_2(\Delta) \sqcup s_1s_0(U)$  in  $\mathcal{C}_3$  imply the relations

$$\begin{aligned} [\Delta \sqcup s_1U] + [s_0d_0\Delta \sqcup s_1U] &= [s_0d_1\Delta \sqcup s_1U] + [\Delta]^{[U]}, \\ [\Delta \sqcup s_1U] + [s_1d_0\Delta] \sqcup s_0U &= [\Delta \sqcup s_0U] + [s_0d_2\Delta \sqcup s_1U]^{[d_0\Delta]}, \\ [s_1d_1\Delta \sqcup s_0U] + [\Delta] &= [\Delta \sqcup s_0U] + [s_1d_2\Delta \sqcup s_0U]^{[d_0\Delta]}. \end{aligned}$$



From these, one deduces easily the relation

$$[\Delta]^{[U]} = [\Delta] + \langle [U], \partial[\Delta] \rangle.$$

□

*Remark 3.3.3.* The stable 2-module  $\mathcal{D}'_*(\mathcal{C}_\bullet)$  admits generators (G1–3) and relations (R1–9) as a stable 2-module, i.e. it satisfies the obvious universal property. In particular,  $\mathcal{D}_*(\mathcal{C}_\bullet)$  is the stable quadratic module associated to  $\mathcal{D}'_*(\mathcal{C}_\bullet)$ . Moreover, replacing  $\mathcal{D}_*(\mathcal{C}_\bullet)$  by  $\mathcal{D}'_*(\mathcal{C}_\bullet)$  in Theorem 3.2.4 we have a natural isomorphism,

$$\mathrm{Hom}_{\mathrm{cs}}^{\otimes}(\mathcal{D}'_*(\mathcal{C}_\bullet), \mathcal{P}) \xrightarrow{\cong} \mathrm{Det}_s(\mathcal{C}_\bullet, \mathcal{P})$$

for *any* strict Picard groupoid  $\mathcal{P}$ .

Indeed, if  $C_*$  is a stable 2-module, then any map from the generators (G0) to  $C_0$ , and (G1–2) to  $C_1$ , compatible with relations (R1–9), can be extended to a morphism of crossed modules  $\mathcal{D}'_*(\mathcal{C}) \rightarrow C_*$ , by the very definition of the crossed module  $\mathcal{D}'_*(\mathcal{C}_\bullet)$ . Now we just have to check compatibility with the bracket. It is enough to check it on generators, and this follows from the previous proposition.

**Corollary 3.3.4.** *The universal determinant functor is also universal for non-commutative determinant functors.*

**3.4. Examples.** We can consider  $\mathcal{D}_*(\mathcal{C}_\bullet)$  for the simplicial categories in Examples 1.7.3 and 1.7.4. Given a Waldhausen category  $\mathcal{W}$ , a right pointed derivator  $\mathbb{D}$ , a (strongly) triangulated category  $\mathcal{T}$ , and an abelian category  $\mathcal{A}$ , we have,

$$\begin{aligned} \mathcal{D}_*(\mathcal{W}) &= \mathcal{D}_*(S_\bullet(\mathcal{W})); & \mathcal{D}_*^{\mathrm{der}}(\mathcal{W}) &= \mathcal{D}_*(\mathrm{Ho} S_\bullet(\mathcal{W})); \\ \mathcal{D}_*(\mathbb{D}) &= \mathcal{D}_*(S_\bullet(\mathbb{D})); & \mathcal{D}_*({}^s\mathcal{T}) &= \mathcal{D}_*(\bar{Q}_\bullet(\mathcal{T})); \\ \mathcal{D}_*(\blacklozenge\mathcal{T}) &= \mathcal{D}_*(\bar{S}_\bullet(\blacklozenge\mathcal{T})), \quad \blacklozenge = b, d, v; & \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}) &= \mathcal{D}_*(\bar{S}_\bullet(\mathrm{Gr}^b\mathcal{A})); \\ \mathcal{D}_*(\mathrm{Gr}^{[n,m]}\mathcal{A}) &= \mathcal{D}_*(\bar{S}_\bullet(\mathrm{Gr}^{[n,m]}\mathcal{A})), \quad n \leq m. \end{aligned}$$

In this way we obtain universal determinant functors for Waldhausen categories, strongly triangulated categories, and right pointed derivators, as well a universal derived determinant functor on any Waldhausen category and a universal graded determinant functor on any abelian category (concentrated in an interval). For any triangulated category we also obtain universal Breuning, special and virtual determinant functors. These eight stable quadratic modules are 2-functorial with respect to exact functors between Waldhausen categories, (strongly) triangulated categories, and abelian categories, and natural transformations between them, see Remark 3.2.3; and with respect to right exact pseudo-natural transformations between right pointed derivators and invertible modifications between them, see Examples 1.7.3 and 1.7.4.

There are natural comparison morphisms of stable quadratic modules,

$$\begin{aligned} \mathcal{D}_*({}^v\mathcal{T}) &\longleftarrow \mathcal{D}_*({}^d\mathcal{T}) \longrightarrow \mathcal{D}_*({}^b\mathcal{T}) \longleftarrow \mathcal{D}_*({}^s\mathcal{T}); \\ \mathcal{D}_*(\mathbb{D}) &\longrightarrow \mathcal{D}_*({}^s\mathbb{D}(*)); & \mathcal{D}_*(\mathcal{W}) &\longrightarrow \mathcal{D}_*^{\mathrm{der}}(\mathcal{W}); \\ \mathcal{D}_*(\mathrm{Gr}^{[n,m]}\mathcal{A}) &\longrightarrow \mathcal{D}_*(\mathrm{Gr}^{[n',m']}\mathcal{A}), \quad n' < n, m < m'; & \mathcal{D}_*(\mathcal{A}) &\longrightarrow \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}); \end{aligned}$$

see Example 1.7.5. The last one is a special case of the previous one since

$$\mathcal{D}_*(\mathrm{Gr}^{[0,0]}\mathcal{A}) = \mathcal{D}_*(\mathcal{A}), \quad \mathcal{D}_*(\mathrm{Gr}^{(-\infty, +\infty)}\mathcal{A}) = \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}).$$

Moreover,

$$\begin{aligned}\mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}) &= \operatorname{colim}_{n \rightarrow +\infty} \mathcal{D}_*(\mathrm{Gr}^{[-n,n]}\mathcal{A}), \\ \mathcal{D}_*(\mathrm{Gr}^{[n,+\infty]}\mathcal{A}) &= \operatorname{colim}_{m \rightarrow +\infty} \mathcal{D}_*(\mathrm{Gr}^{[n,m]}\mathcal{A}), \\ \mathcal{D}_*(\mathrm{Gr}^{(-\infty,m]}\mathcal{A}) &= \operatorname{colim}_{n \rightarrow -\infty} \mathcal{D}_*(\mathrm{Gr}^{[n,m]}\mathcal{A}).\end{aligned}$$

In addition, if  $\mathcal{T}$  has a  $t$ -structure with heart  $\mathcal{A}$  we have,

$$(3.4.1) \quad \begin{array}{ccccc} & & \mathcal{D}_*(\mathcal{T}) & & \\ & \nearrow & & \searrow & \\ \mathcal{D}_*(\mathcal{A}) & \longrightarrow & \mathcal{D}_*({}^d\mathcal{T}) & & \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}). \\ & & \searrow & \nearrow & \\ & & \mathcal{D}_*({}^v\mathcal{T}) & & \end{array}$$

This diagram is commutative and the composite  $\mathcal{D}_*(\mathcal{A}) \rightarrow \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A})$  is the direct comparison morphism above, that we show later to be always a weak equivalence.

It is easy to check that all these morphisms induce an isomorphism on  $\pi_0$ , which is always a certain  $K_0$  group. In [Mur08] it is proven that the comparison morphism  $\mathcal{D}_*(\mathcal{W}) \rightarrow \mathcal{D}_*^{\mathrm{der}}(\mathcal{W})$  is an isomorphism provided  $\mathcal{W}$  has cylinders and a saturated class of weak equivalences. We will give an example that  $\mathcal{D}_*(\mathbb{D}) \rightarrow \mathcal{D}_*(\mathbb{D}(*))$  may fail to induce an isomorphism on  $\pi_1$ , disproving a conjecture of Maltiniotis. We will also show that if  $\mathcal{T}$  is a triangulated category with a bounded, non-degenerate  $t$ -structure with heart  $\mathcal{A}$  then the comparison morphisms in (3.4.1) are weak equivalences. We show with examples that  $\mathcal{D}_*({}^d\mathcal{T}) \rightarrow \mathcal{D}_*({}^v\mathcal{T})$  need not be surjective.

It remains open whether  $\mathcal{D}_*({}^d\mathcal{T}) \rightarrow \mathcal{D}_*({}^b\mathcal{T})$  and  $\mathcal{D}_*({}^s\mathcal{T}) \rightarrow \mathcal{D}_*({}^b\mathcal{T})$  are weak equivalences under some weaker assumptions. The only completely obvious property is that these morphism are identities in dimension 0, and moreover,  $\mathcal{D}_*({}^d\mathcal{T}) \rightarrow \mathcal{D}_*({}^b\mathcal{T})$ ,  $\mathcal{D}_*({}^s\mathcal{T}) \rightarrow \mathcal{D}_*({}^b\mathcal{T})$  and  $\mathcal{D}_*(\mathbb{D}) \rightarrow \mathcal{D}_*({}^s\mathbb{D}(*))$  are surjective in dimension 1, so they also induce a surjection on  $\pi_1$ .

**3.5. The connection to  $K$ -theory.** Let  $\mathrm{Ho}\mathbf{Spec}_0$  be the full coreflective subcategory of the stable homotopy category spanned by *connective spectra*, i.e. spectra with trivial homotopy groups in negative dimensions. Let  $\mathrm{Ho}\mathbf{Spec}_0^1$  be the full reflective subcategory of spectra with homotopy groups concentrated in dimensions 0 and 1. The reflection functor  $\mathrm{Ho}\mathbf{Spec}_0 \rightarrow \mathrm{Ho}\mathbf{Spec}_0^1$  takes a connective spectrum to its 1-*type*. It is well known that the homotopy category of Picard groupoids is equivalent to  $\mathrm{Ho}\mathbf{Spec}_0^1$ , and the equivalence is compatible with the corresponding notions of homotopy groups and  $k$ -invariant. There are several ways of realizing this equivalence. The equivalence in [MT07], between  $\mathrm{Ho}\mathbf{Spec}_0^1$  and the homotopy category of stable quadratic modules  $\mathrm{Ho}\mathbf{squad}$ , is particularly adapted to the goal of this paper.

**Lemma 3.5.1** ([MT07, Lemma 4.22]). *There is a functor*

$$\lambda_0: \mathrm{Ho}\mathbf{Spec}_0 \longrightarrow \mathrm{Ho}\mathbf{squad}$$

*together with natural isomorphisms*

$$\pi_i \lambda_0 X \cong \pi_i X, \quad i = 0, 1,$$

compatible with the  $k$ -invariants, which induces an equivalence of categories

$$\lambda_0: \mathrm{Ho} \mathbf{Spec}_0^1 \xrightarrow{\sim} \mathrm{Ho} \mathbf{squad}.$$

Therefore the functor  $\lambda_0$  can be regarded as an algebraic model for the 1-type of a connective spectrum.

**Theorem 3.5.2.** *Let  $\mathcal{C}_\bullet$  be a simplicial category with weak equivalences and let  $X$  be a spectrum of simplicial sets, such that the geometric realizations of the simplicial subcategory of weak equivalences  $\mathrm{we}(\mathcal{C}_\bullet)$  and the 1-stage of  $X$  are homotopy equivalent as  $H$ -spaces. Then  $\lambda_0 X \cong \mathcal{D}_*(\mathcal{C}_\bullet)$  in  $\mathrm{Ho} \mathbf{squad}$ .*

This theorem is a straightforward generalization of [MT07, Theorem 1.7]. Exactly the same proof works with the appropriate changes in notation.

Examples of connective spectra are Quillen's  $K$ -theory of an exact category  $K(\mathcal{E})$  [Qui73], Waldhausen's  $K$ -theory of a category with cofibrations and weak equivalences  $K(\mathcal{W})$  [Wal85], Garkusha's derived  $K$ -theory  $DK(\mathcal{W})$  [Gar05], Maltiniotis's  $K$ -theory of a triangulated derivator [Mal07], and more generally Garkusha's  $K$ -theory of a right pointed derivator  $K(\mathbb{D})$  [Gar06], Maltiniotis's  $K$ -theory of a strongly triangulated category  $K({}^s\mathcal{T})$  [Mal06], two of Neeman's  $K$ -theories of a triangulated category  $K({}^d\mathcal{T})$ ,  $K({}^v\mathcal{T})$  and Neeman's  $K$ -theory of a graded abelian category  $K(\mathrm{Gr}^b \mathcal{A})$  [Nee05]. The 1-stage of each of these spectra is the geometric realization of the subcategory of weak equivalences of the corresponding simplicial category with weak equivalences in Example 1.7.3.

**Corollary 3.5.3.** *Let  $\mathcal{W}$  be a Waldhausen category,  $\mathcal{T}$  a (strongly) triangulated category, and  $\mathbb{D}$  a right pointed derivator. Then there are natural isomorphisms in  $\mathrm{Ho} \mathbf{squad}$ ,*

$$\begin{aligned} \mathcal{D}_*(\mathcal{W}) &\cong \lambda_0 K(\mathcal{W}), & \mathcal{D}_*^{\mathrm{der}}(\mathcal{W}) &\cong \lambda_0 DK(\mathcal{W}), \\ \mathcal{D}_*(\mathbb{D}) &\cong \lambda_0 K(\mathbb{D}), & \mathcal{D}_*({}^s\mathcal{T}) &\cong \lambda_0 K({}^s\mathcal{T}), \\ \mathcal{D}_*({}^d\mathcal{T}) &\cong \lambda_0 K({}^d\mathcal{T}), & \mathcal{D}_*({}^v\mathcal{T}) &\cong \lambda_0 K({}^v\mathcal{T}), \\ \mathcal{D}_*(\mathrm{Gr}^b \mathcal{A}) &\cong \lambda_0 K(\mathrm{Gr}^b \mathcal{A}). \end{aligned}$$

Breuning defines the  $K$ -theory of a triangulated category in dimensions  $i = 0, 1$  via universal determinant functors, so by definition  $K_i({}^b\mathcal{T}) = \pi_i \mathcal{D}_*({}^b\mathcal{T})$ . It is then reasonable to define  $K({}^b\mathcal{T})$  as the classifying spectrum of  $\mathcal{D}_*({}^b\mathcal{T})$ , i.e. the spectrum satisfying  $\lambda_0 K({}^b\mathcal{T}) \cong \mathcal{D}_*({}^b\mathcal{T})$ .

These spectra come equipped with natural comparison morphisms,

$$\begin{aligned} K({}^v\mathcal{T}) &\longleftarrow K({}^d\mathcal{T}) \longrightarrow K({}^b\mathcal{T}) \longleftarrow K({}^s\mathcal{T}); \\ K(\mathbb{D}) &\longrightarrow K({}^s\mathbb{D}(*)); & K(\mathcal{W}) &\longrightarrow K^{\mathrm{der}}(\mathcal{W}); \\ K(\mathrm{Gr}^{[n,m]} \mathcal{A}) &\longrightarrow K(\mathrm{Gr}^{[n',m']} \mathcal{A}), \quad n' < n, \quad m < m'; & K(\mathcal{A}) &\longrightarrow K(\mathrm{Gr}^b \mathcal{A}). \end{aligned}$$

In addition, if  $\mathcal{T}$  is a triangulated category with a  $t$ -structure with heart  $\mathcal{A}$  we have,

$$\begin{array}{ccccc}
 & & K(\mathcal{T}) & & \\
 & \nearrow & & \searrow & \\
 K(\mathcal{A}) & \longrightarrow & K({}^d\mathcal{T}) & & K(\mathrm{Gr}^b\mathcal{A}). \\
 & \searrow & & \nearrow & \\
 & & K({}^v\mathcal{T}) & & 
 \end{array}$$

This diagram is commutative and the composite  $K(\mathcal{A}) \rightarrow K(\mathrm{Gr}^b\mathcal{A})$  is the direct comparison morphism above.

The 1-stage of these comparison morphisms is given by the geometric realizations of the simplicial functors in Example 1.7.5 between the subcategories of weak equivalences, therefore these morphisms of spectra are compatible with the morphisms of stable quadratic modules in Section 3.4 via the isomorphisms in Corollary 3.5.3. This has obvious implications in  $K_0$  and  $K_1$ .

#### 4. APPLICATIONS

**4.1. Generators and (some) relations for  $K_1$ .** Nenashev [Nen98] considered pairs of short exact sequences over the same objects in an exact category  $\mathcal{E}$ ,

$$(4.1.1) \quad A \begin{array}{c} \xrightarrow{j} \\ \xrightarrow{j'} \end{array} B \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{r'} \end{array} C.$$

Such a pair yields an element in  $K_1(\mathcal{E})$ . Nenashev proved that any element in  $K_1(\mathcal{E})$  is of this kind and computed a set of relations among them, associated to  $3 \times 3$  diagrams, yielding a presentation of  $K_1(\mathcal{E})$ .

Vaknin [Vak01b] considered pairs of distinguished triangles over the same objects in a triangulated category  $\mathcal{T}$ ,

$$(4.1.2) \quad X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{f'} \end{array} Y \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{i'} \end{array} Z \begin{array}{c} \xrightarrow{q} \\ \xrightarrow{q'} \end{array} \Sigma X.$$

Using similar techniques, Vaknin proved that any element in Neeman's  $K_1({}^d\mathcal{T})$  is of this kind and computed a set of relations among them, extending Nenashev's, yielding a presentation of  $K_1({}^d\mathcal{T})$ , as in the exact case.

Muro and Tonks considered in [MT08] diagrams in a Waldhausen category  $\mathcal{W}$ ,

$$(4.1.3) \quad A \begin{array}{c} \xrightarrow{j} \\ \xrightarrow{j'} \end{array} B \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} C_1 \\ C_2 \end{array} \begin{array}{c} \nwarrow \\ \swarrow \end{array} C.$$

consisting of two cofiber sequences and two weak equivalences. They extended Nenashev's results, showing that any element in  $K_1(\mathcal{W})$  is of this kind and computing a set of relations among them generalizing Nenashev's. Some evidence was given for the conjecture that these relations define a presentation of  $K_1(\mathcal{W})$ .

In this section we extend the results in [MT08] to the unified context introduced in this paper. This yields new results for Breuning's  $K_1({}^b\mathcal{T})$ , Neeman's  $K_1({}^v\mathcal{T})$ ,

and Maltiniotis's  $K_1({}^s\mathcal{T})$ . We need our simplicial category  $\mathcal{C}_\bullet$  with weak equivalences to satisfy an additional property, which will remain throughout this section:

- The functor sending an  $(n+1)$ -simplex to its  $n+2$  faces,

$$\phi: \mathcal{C}_{n+1} \longrightarrow \mathcal{C}_n \times_{\mathcal{C}_{n-1}} \cdots \times_{\mathcal{C}_{n-1}} \mathcal{C}_n, \quad n \geq 0,$$

is a fibration of categories, i.e. it satisfies the isomorphism lifting property: any isomorphism  $\phi(x) \rightarrow y$  in the target is the image by  $\phi$  of an isomorphism in the source, in particular the object  $y$  is in the image of  $\phi$ .

This property is satisfied by all examples in Example 1.7.3, but it needs not be satisfied by Example 1.7.4 on derivators.

In the language of this paper, the elements (4.1.1), (4.1.2), (4.1.3) in  $K_1$  are, respectively,

$$\begin{aligned} & -[A \xrightarrow{j} B \xrightarrow{r} C] + [A \xrightarrow{j'} B \xrightarrow{r'} C] \in \mathcal{D}_1(\mathcal{E}), \\ & -[X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X] + [X \xrightarrow{f'} Y \xrightarrow{i'} Z \xrightarrow{q'} \Sigma X] \in \mathcal{D}_1({}^d\mathcal{T}), \\ & -[C \xrightarrow{\sim} C_1]^{[A]} - [A \xrightarrow{j} B \rightarrow C_1] + [A \xrightarrow{j'} B \rightarrow C_2] + [C \xrightarrow{\sim} C_2]^{[A]} \in \mathcal{D}_1(\mathcal{W}). \end{aligned}$$

Actually (4.1.2) also defines elements in Breuning's  $K_1({}^b\mathcal{T})$ , in Maltiniotis's  $K_1({}^s\mathcal{T})$ , and also in Neeman's  $K_1({}^v\mathcal{T})$  if (4.1.2) is a pair of virtual triangles. For the sake of simplicity we will write in all (strongly) triangulated cases,

$$[X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{f'} \end{array} Y \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{i'} \end{array} Z \begin{array}{c} \xrightarrow{q} \\ \xrightarrow{q'} \end{array} \Sigma X] = -[X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X] + [X \xrightarrow{f'} Y \xrightarrow{i'} Z \xrightarrow{q'} \Sigma X].$$

We now generalize these elements to our unified framework.

**Definition 4.1.4.** A *triangle*  $\Delta$  in  $\mathcal{C}_\bullet$  is just an object of  $\mathcal{C}_2$ . A *weak triangle*  $(\Delta, f)$  in  $\mathcal{C}_\bullet$  consists of a triangle  $\Delta$  and a morphism  $f: C \xrightarrow{\sim} d_0\Delta$  in  $\text{we}(\mathcal{C}_1)$ . We denote,

$$[\Delta, f] = [\Delta] + [f]^{[d_2\Delta]} \in \mathcal{D}_1(\mathcal{C}_\bullet).$$

A *pair of triangles*  $(\Delta_1; \Delta_2)$  consists of two triangles,  $\Delta_1$  and  $\Delta_2$ , with the same edges  $d_i\Delta_1 = d_i\Delta_2$ ,  $i = 0, 1, 2$ . A pair of triangles yields an element,

$$[\Delta_1; \Delta_2] = -[\Delta_1] + [\Delta_2] \in \pi_1\mathcal{D}_*(\mathcal{C}_\bullet).$$

A *pair of weak triangles*  $(\Delta_1, f_1; \Delta_2, f_2)$  consists of two weak triangles,  $(\Delta_1, f_1)$  and  $(\Delta_2, f_2)$ , such that  $\Delta_1$  and  $\Delta_2$  have the same first and second edge,

$$d_1\Delta_1 = d_1\Delta_2, \quad d_2\Delta_1 = d_2\Delta_2,$$

and  $f_1$  and  $f_2$  have the same source,

$$d_0\Delta_1 \xleftarrow{f_1} C \xrightarrow{f_2} d_0\Delta_2.$$

Any pair of weak triangles yields an element,

$$[\Delta_1, f_1; \Delta_2, f_2] = -[\Delta_1, f_1] + [\Delta_2, f_2] \in \pi_1\mathcal{D}_*(\mathcal{C}_\bullet).$$

*Remark 4.1.5.* Notice that a trivial pair of weak triangles is trivial, i.e. given a weak triangle  $(\Delta, f)$ ,

$$(S2) \quad [\Delta, f; \Delta, f] = 0 \in \pi_1\mathcal{D}_*(\mathcal{C}_\bullet).$$

**Theorem 4.1.6.** *Any element in  $\pi_1\mathcal{D}_*(\mathcal{C}_\bullet)$  is represented by a pair of weak triangles.*

Now we extend to the unified framework in this paper all results in [MT08] needed so that the proof of [MT08, Theorem 2.1] works for Theorem 4.1.6.

The following corollary follows from Theorem 4.1.6 and from Proposition 4.1.24 below.

**Corollary 4.1.7.** *Given a (strongly) triangulated category  $\mathcal{T}$  and a strongly triangulated category  $\mathcal{S}$ , any element in Breuning's  $K_1({}^b\mathcal{T})$ , Neeman's  $K_1({}^d\mathcal{T})$  and Maltziniotis's  $K_1({}^s\mathcal{T})$  is represented by a pair of distinguished triangles, and any element in Neeman's  $K_1({}^v\mathcal{T})$  is represented by a pair of virtual triangles.*

A  $3 \times 3$  diagram in an exact category  $\mathcal{E}$  is a commutative diagram of short exact sequences,

$$(4.1.8) \quad \begin{array}{ccccc} A' & \longrightarrow & A & \twoheadrightarrow & A'' \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & B & \twoheadrightarrow & B'' \\ \downarrow & & \downarrow & & \downarrow \\ C' & \longrightarrow & C & \twoheadrightarrow & C'' \end{array}$$

Such a diagram yields four objects in  $S_3\mathcal{E}$ ,

$$(4.1.9) \quad \begin{array}{c} \begin{array}{c} \text{Diagram 1: } \begin{array}{c} \text{Top face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \\ \text{Bottom face: } A' \xrightarrow{f'} A'' \\ \text{Left face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \\ \text{Right face: } A \xrightarrow{g} A'' \\ \text{Back face: } A' \xrightarrow{f'} A'' \\ \text{Front face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \end{array} \\ \text{Diagram 2: } \begin{array}{c} \text{Top face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \\ \text{Bottom face: } A' \xrightarrow{f'} A'' \\ \text{Left face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \\ \text{Right face: } A \xrightarrow{g} A'' \\ \text{Back face: } A' \xrightarrow{f'} A'' \\ \text{Front face: } A' \xrightarrow{f} A \xrightarrow{g} A'' \end{array} \\ \text{Diagram 3: } \begin{array}{c} \text{Top face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \\ \text{Bottom face: } B' \xrightarrow{f'} B'' \\ \text{Left face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \\ \text{Right face: } B \xrightarrow{g} B'' \\ \text{Back face: } B' \xrightarrow{f'} B'' \\ \text{Front face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \end{array} \\ \text{Diagram 4: } \begin{array}{c} \text{Top face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \\ \text{Bottom face: } B' \xrightarrow{f'} B'' \\ \text{Left face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \\ \text{Right face: } B \xrightarrow{g} B'' \\ \text{Back face: } B' \xrightarrow{f'} B'' \\ \text{Front face: } B' \xrightarrow{f} B \xrightarrow{g} B'' \end{array} \end{array}$$

Based on this fact, we make the following definition of a  $3 \times 3$  diagram in a simplicial category  $\mathcal{C}_\bullet$  with weak equivalences.

**Definition 4.1.10.** A  $3 \times 3$  diagram in  $\mathcal{C}_\bullet$  consists of four objects  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$  in  $\mathcal{C}_3$  such that,

$$\begin{array}{ll} d_2\Theta_1 = d_2\Theta_2, & d_1\Theta_3 = d_1\Theta_4, \\ d_1\Theta_1 = d_3\Theta_3, & d_1\Theta_2 = d_3\Theta_4, \\ d_0d_3\Theta_1 = d_0d_1\Theta_2, & d_0d_1\Theta_1 = d_0d_3\Theta_2, \\ d_0\Theta_1 = s_1d_0d_3\Theta_1 \sqcup s_0d_0d_1\Theta_1, & d_0\Theta_2 = s_0d_0d_1\Theta_2 \sqcup s_1d_0d_3\Theta_2. \end{array}$$

*Remark 4.1.11.* Given a Waldhausen category  $\mathscr{W}$ , a  $3 \times 3$  diagram in  $S_\bullet \mathscr{W}$  yields a commutative diagram (4.1.8) of cofiber sequences in  $\mathscr{W}$ , but not all such commutative diagrams come from a  $3 \times 3$  diagram. The required condition is that the natural map  $W = A \cup_{A'} B' \rightarrow B$  be a cofibration, see the proof of [MT08, Proposition 1.6].

The triangulated case is not so simple. Given a (strongly) triangulated category  $\mathscr{T}$ , a  $3 \times 3$  diagram in  $\bar{S}_\bullet(b\mathscr{T})$ ,  $\bar{S}_\bullet(d\mathscr{T})$  or  $\bar{Q}_\bullet(\mathscr{T})$  yields a diagram of distinguished triangles,

$$(4.1.12) \quad \begin{array}{ccccccc} X' & \xrightarrow{f^X} & X & \xrightarrow{i^X} & X'' & \xrightarrow{q^X} & \Sigma X' \\ f' \downarrow & & \downarrow f & & \downarrow f'' & & \downarrow \Sigma f' \\ Y' & \xrightarrow{f^Y} & Y & \xrightarrow{i^Y} & Y'' & \xrightarrow{q^Y} & \Sigma Y' \\ i' \downarrow & & \downarrow i & & \downarrow i'' & & \downarrow \Sigma i' \\ Z' & \xrightarrow{f^Z} & Z & \xrightarrow{i^Z} & Z'' & \xrightarrow{q^Z} & \Sigma Z' \\ q' \downarrow & & \downarrow q & & \downarrow q'' & \ominus & \downarrow -\Sigma q' \\ \Sigma X' & \xrightarrow{\Sigma f^X} & \Sigma X & \xrightarrow{\Sigma i^X} & \Sigma X'' & \xrightarrow{-\Sigma q^X} & \Sigma^2 X' \end{array}$$

which is commutative except from the bottom right square, which is anticommutative,  $(\Sigma q')q^Z + (\Sigma q^X)q'' = 0$ , but there is no easy condition ensuring that such a diagram comes from a  $3 \times 3$  diagram. It means the existence of four (special, exact) octahedra as follows,

such that  $\varepsilon = \varepsilon'$  and  $\kappa = \kappa'$ . As Vaknin pointed out [Vak01b, Remark 5.3], it is always possible to construct four octahedra as above, but, in general, one cannot guarantee  $\varepsilon = \varepsilon'$  or  $\kappa = \kappa'$ , i.e. the third and fourth octahedra may contain different distinguished triangle completions of  $i^Z i = i'' i^Y : Y \rightarrow Z''$ . Notice that the first and second octahedra are always special.

In  $\bar{S}_\bullet(v\mathcal{T})$ , a  $3 \times 3$  diagram yields a diagram of virtual triangles as (4.1.12). In order for such diagram to come from a  $3 \times 3$  diagram there must be four virtual octahedra as above.

A  $3 \times 3$  diagram in  $\bar{S}_\bullet(\text{Gr}^b \mathcal{A})$  is a diagram formed by six long exact sequences,

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \rightarrow & X'_n & \longrightarrow & X_n & \longrightarrow & X''_n & \longrightarrow & X'_{n-1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \rightarrow & Y'_n & \longrightarrow & Y_n & \longrightarrow & Y''_n & \longrightarrow & Y'_{n-1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \rightarrow & C'_n & \longrightarrow & C_n & \longrightarrow & C''_n & \longrightarrow & C'_{n-1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \rightarrow & X'_{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X''_{n-1} & \longrightarrow & X'_{n-2} & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & & & & & \ominus & & & 
 \end{array}$$

where all squares are commutative, except from the squares labelled with  $-1$ , which are anticommutative. Following the notation in Remark 1.6.10, it can also be depicted as (4.1.12).

**Definition 4.1.13.** If  $\mathcal{T}$  is a (strongly) triangulated category, we say that a diagram like (4.1.12) is  $\blacklozenge$ -coherent if it comes from a  $3 \times 3$  diagram in  $\bar{S}_\bullet(\blacklozenge \mathcal{T})$ , if  $\blacklozenge = b, d, v$ , or in  $\bar{Q}_\bullet(\mathcal{T})$ , if  $\blacklozenge = s$ .

This kind of coherence condition was first introduced by Vaknin [Vak01b] for  $\bar{S}_\bullet(d\mathcal{T})$ .

**Proposition 4.1.14.** *Given a  $3 \times 3$  diagram in  $\mathcal{C}_\bullet$  the following equation holds in  $\mathcal{D}_1(\mathcal{C}_\bullet)$ ,*

$$\begin{aligned}
 \langle [d_0 d_1 \Theta_1], [d_0 d_3 \Theta_1] \rangle &= -[d_3 \Theta_1] - [d_0 \Theta_3]^{[d_2 d_3 \Theta_3]} - [d_2 \Theta_3] \\
 &\quad + [d_2 \Theta_4] + [d_0 \Theta_4]^{[d_2 d_3 \Theta_4]} + [d_3 \Theta_2].
 \end{aligned}$$

This result follows straightforwardly from (R8) and (R9). It is also a particular case of Proposition 4.1.21 below, just take  $w_1$ ,  $w_2$ ,  $w''$ , and  $w^C$  to be identity morphisms.



In an exact category  $\mathcal{C}$  we may have two  $3 \times 3$  diagrams over the same six objects,  $i = 1, 2$ ,

$$\begin{array}{ccccc}
 A' & \xrightarrow{j_i^A} & A & \xrightarrow{r_i^A} & A'' \\
 j_i' \downarrow & & \downarrow j_i & & \downarrow j_i'' \\
 B' & \xrightarrow{j_i^B} & B & \xrightarrow{q_i^B} & B'' \\
 r_i' \downarrow & & \downarrow r_i & & \downarrow r_i'' \\
 C' & \xrightarrow{j_i^C} & C & \xrightarrow{r_i^C} & C''
 \end{array}$$

We now extend this situation to the simplicial framework.

**Definition 4.1.15.** A pair of  $3 \times 3$  diagrams consists of two  $3 \times 3$  diagrams,  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$  and  $\Theta'_1, \Theta'_2, \Theta'_3, \Theta'_4$ , such that for  $i = 0, 1, 2$ ,

$$\begin{aligned}
 d_i d_3 \Theta_1 &= d_i d_3 \Theta'_1, & d_i d_3 \Theta_2 &= d_i d_3 \Theta'_2, & d_i d_0 \Theta_3 &= d_i d_0 \Theta'_3, \\
 d_i d_2 \Theta_3 &= d_i d_2 \Theta'_3, & d_i d_0 \Theta_4 &= d_i d_0 \Theta'_4, & d_i d_2 \Theta_4 &= d_i d_2 \Theta'_4.
 \end{aligned}$$

**Corollary 4.1.16.** For any pair of weak  $3 \times 3$  diagrams in  $\mathcal{C}_\bullet$ ,  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$  and  $\Theta'_1, \Theta'_2, \Theta'_3, \Theta'_4$ , the following relation between pairs of triangles in  $\pi_1 \mathcal{D}_*(\mathcal{C}_\bullet)$  holds,

$$\begin{aligned}
 &[d_3 \Theta_1; d_3 \Theta'_1] - [d_2 \Theta_4; d_2 \Theta'_4] + [d_0 \Theta_3; d_0 \Theta'_3] \\
 &= [d_3 \Theta_2; d_3 \Theta'_2] - [d_2 \Theta_3; d_2 \Theta'_3] + [d_0 \Theta_4; d_0 \Theta'_4].
 \end{aligned}$$

This is an easy consequence of Proposition 4.1.14, compare [MT08, Theorem 3.1].

**Corollary 4.1.17.** Suppose we have two diagrams of distinguished triangles over the same objects in a (strongly) triangulated category  $\mathcal{T}$ ,  $j = 1, 2$ ,

$$(4.1.18) \quad
 \begin{array}{ccccccc}
 X' & \xrightarrow{f_j^X} & X & \xrightarrow{i_j^X} & X'' & \xrightarrow{q_j^X} & \Sigma X' \\
 f_j' \downarrow & & \downarrow f_j & & \downarrow f_j'' & & \downarrow \Sigma f_j' \\
 Y' & \xrightarrow{f_j^Y} & Y & \xrightarrow{i_j^Y} & Y'' & \xrightarrow{q_j^Y} & \Sigma Y' \\
 i_j' \downarrow & & \downarrow i_j^j & & \downarrow i_j'' & & \downarrow \Sigma i_j' \\
 Z' & \xrightarrow{f_j^Z} & Z & \xrightarrow{i_j^Z} & Z'' & \xrightarrow{q_j^Z} & \Sigma Z' \\
 q_j' \downarrow & & \downarrow q_j^j & & \downarrow q_j'' & \ominus & \downarrow -\Sigma q_j' \\
 \Sigma X' & \xrightarrow{\Sigma f_j^X} & \Sigma X & \xrightarrow{\Sigma i_j^X} & \Sigma X'' & \xrightarrow{-\Sigma q_j^X} & \Sigma^2 X'
 \end{array}$$

commutative except from the right bottom squares, which are anticommutative. If both diagrams are  $\blacklozenge$ -coherent,  $\blacklozenge = b, d, s, v$ , then the following relation between pairs

of distinguished triangles holds in  $K_1(\bullet\mathcal{T})$ ,

(4.1.19)

$$\begin{aligned} & [X' \begin{array}{c} \xrightarrow{f_1^X} X \xrightarrow{i_1^X} X'' \xrightarrow{q_1^X} \Sigma X' \\ \xrightarrow{f_2^X} \xrightarrow{i_2^X} \xrightarrow{q_2^X} \end{array}] - [Y' \begin{array}{c} \xrightarrow{f_1^Y} Y \xrightarrow{i_1^Y} Y'' \xrightarrow{q_1^Y} \Sigma Y' \\ \xrightarrow{f_2^Y} \xrightarrow{i_2^Y} \xrightarrow{q_2^Y} \end{array}] + [Z' \begin{array}{c} \xrightarrow{f_1^Z} Z \xrightarrow{i_1^Z} Z'' \xrightarrow{q_1^Z} \Sigma Z' \\ \xrightarrow{f_2^Z} \xrightarrow{i_2^Z} \xrightarrow{q_2^Z} \end{array}] \\ &= [X' \begin{array}{c} \xrightarrow{f_1'} Y' \xrightarrow{i_1'} Z' \xrightarrow{q_1'} \Sigma X' \\ \xrightarrow{f_2'} \xrightarrow{i_2'} \xrightarrow{q_2'} \end{array}] - [X \begin{array}{c} \xrightarrow{f_1} Y \xrightarrow{i_1} Z \xrightarrow{q_1} \Sigma X \\ \xrightarrow{f_2} \xrightarrow{i_2} \xrightarrow{q_2} \end{array}] + [X'' \begin{array}{c} \xrightarrow{f_1''} Y'' \xrightarrow{i_1''} Z'' \xrightarrow{q_1''} \Sigma X'' \\ \xrightarrow{f_2''} \xrightarrow{i_2''} \xrightarrow{q_2''} \end{array}]. \end{aligned}$$

**Definition 4.1.20.** A weak  $3 \times 3$  diagram in  $\mathcal{C}_\bullet$  consists of a  $3 \times 3$  diagram  $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ ; two objects  $\Delta_1, \Delta_2$  in  $\mathcal{C}_2$  together with morphisms,

$$w_1: \Delta_1 \xrightarrow{\sim} d_0\Theta_3, \quad w_2: \Delta_2 \xrightarrow{\sim} d_0\Theta_4;$$

and a commutative diagram in  $\text{we}(\mathcal{C}_1)$ ,

$$\begin{array}{ccc} d_0d_1\Theta_3 & \xleftarrow{\sim} & d_0\Delta_2 \\ \uparrow \sim & & \sim \uparrow w'' \\ d_0\Delta_1 & \xleftarrow{\sim} & C'' \\ & w^C & \end{array}$$

A pair of weak  $3 \times 3$  diagrams consists of two weak  $3 \times 3$  diagrams, the first one as before and the second one given by  $\Theta'_1, \Theta'_2, \Theta'_3, \Theta'_4$ ,

$$\begin{array}{ccc} w'_1: \Delta'_1 \xrightarrow{\sim} d_0\Theta'_3, & w'_2: \Delta'_2 \xrightarrow{\sim} d_0\Theta'_4; & \begin{array}{ccc} d_0d_1\Theta'_3 & \xleftarrow{\sim} & d_0\Delta'_2 \\ \uparrow \sim & & \sim \uparrow w''' \\ d_0\Delta'_1 & \xleftarrow{\sim} & C''' \\ & (w^C)' & \end{array} \end{array}$$

such that, for  $i = 1, 2$ ,

$$\begin{aligned} d_i d_3 \Theta_1 &= d_i d_3 \Theta'_1, & d_i d_3 \Theta_2 &= d_i d_3 \Theta'_2, & d_i d_2 \Theta_3 &= d_i d_2 \Theta'_3, \\ d_i d_2 \Theta_4 &= d_i d_2 \Theta'_4, & d_i \Delta_1 &= d_i \Delta'_1, & d_i \Delta_2 &= d_i \Delta'_2. \end{aligned}$$

**Proposition 4.1.21.** Given a weak  $3 \times 3$  diagram in  $\mathcal{C}_\bullet$  as above, the following equation holds in  $\mathcal{D}_1(\mathcal{C}_\bullet)$ ,

$$\begin{aligned} \langle [d_2\Delta_1], [d_2\Delta_2] \rangle &= -[d_3\Theta_1, d_2\Delta_2] - [\Delta_1, w^C]^{[d_2d_3\Theta_3]} - [d_2\Theta_3, d_1w_1] \\ &\quad + [d_2\Theta_4, d_1w_2] + [\Delta_2, w'']^{[d_2d_3\Theta_4]} + [d_3\Theta_2, d_2w_1]. \end{aligned}$$

The proof of [MT08, Proposition 1.6] also works in this case.

**Corollary 4.1.22.** For any pair of  $3 \times 3$  diagrams in  $\mathcal{C}_\bullet$  as in the previous definition, the following relation between pairs of weak triangles in  $\pi_1\mathcal{D}_*(\mathcal{C}_\bullet)$  holds,

$$\begin{aligned} \text{(S1)} \quad & [d_3\Theta_1, d_2\Delta_2; d_3\Theta'_1, d_2\Delta'_2] - [d_2\Theta_4, d_1w_2; d_2\Theta'_4, d_1w'_2] + [\Delta_1, w^C; \Delta'_1, (w^C)'] \\ &= [d_3\Theta_2, d_2w_1; d_3\Theta'_2, d_2w'_1] - [d_2\Theta_3, d_1w_1; d_2\Theta'_3, d_1w'_1] + [\Delta_2, w''; \Delta'_2, w''']. \end{aligned}$$

This is an easy consequence of Proposition 4.1.21, compare [MT08, Theorem 3.1].

**Corollary 4.1.23.** *Given two pairs of weak triangles in  $\mathcal{C}_\bullet$ ,  $(\Delta_1, f_1; \Delta'_1, f'_1)$  and  $(\Delta_2, f_2; \Delta'_2, f'_2)$ , if we denote  $C$  the source of  $f_1$  and  $f'_1$ , and  $C'$  the source of  $f_2$  and  $f'_2$ , then the following relation holds in  $\pi_1 \mathcal{D}_*(\mathcal{C}_\bullet)$ ,*

$$[\Delta_1 \sqcup \Delta_2, f_1 \sqcup f_2; \Delta'_1 \sqcup \Delta'_2, f'_1 \sqcup f'_2] = [\Delta_1, f_1; \Delta'_1, f'_1] + [\Delta_2, f_2; \Delta'_2, f'_2].$$

*Proof.* Apply the previous corollary to the following pair of weak  $3 \times 3$  diagrams:

$$\begin{array}{ll} \Theta_1^i = s_0^2 d_2 \Delta_i \sqcup s_2 \Delta'_i, & \Theta_2^i = s_0 s_1 d_2 \Delta_i \sqcup s_1 \Delta'_i, \\ \Theta_3 = s_0 \Delta_i \sqcup s_1^2 d_1 \Delta'_i, & \Theta_4 = s_1 \Delta_i \sqcup s_2 \Delta'_i, \\ \Delta_1^i = d_0 \Theta_3^i, & w_1^i = 1_{d_0 \Theta_3^i}, \\ \Delta_2^i = s_0 C \sqcup s_1 C', & w_2 = s_0 f_i \sqcup s_1 f'_i, \\ w_i'' = 1_C, & w_i^C = f_i. \end{array}$$

□

The following result is completely new. It yields a smaller presentation of  $\mathcal{D}_*(\mathcal{C}_\bullet)$  which can be applied in some important situations.

**Proposition 4.1.24.** *If weak equivalences in  $\mathcal{C}_\bullet$  are isomorphisms, then  $\mathcal{D}_*(\mathcal{C}_\bullet)$  has a presentation with generators (G1) and (G3) and relations (R2), (R8), (R9) and  $[s_0^2(*)] = 0$ .*

*Proof.* This proof consists of an intensive use of the isomorphism lifting property in  $\mathcal{C}_\bullet$  assumed at the beginning of this section. Any isomorphism  $f: X \xrightarrow{\sim} X'$  in  $\mathcal{C}_1$  can be lifted to an isomorphism  $\Phi(f): s_1(X) \xrightarrow{\sim} \Delta(f)$  in  $\mathcal{C}_2$  such that  $d_0 \Phi(f)$  is degenerate,

$$d_1 \Phi(f) = f, \quad d_2 \Phi(f) = 1_X.$$

By (R7),  $[f] = [\Delta(f)]$ , therefore  $\mathcal{D}_*(\mathcal{C})$  is generated by (G1) and (G3). By Remark 3.2.2, we now just have to check that (R6) and (R7) are redundant.

Given two composable isomorphisms in  $\mathcal{C}_1$ ,

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

we can take an isomorphism  $\Xi^{f,g}: s_2 s_1(X) \xrightarrow{\sim} \Theta(f, g)$  in  $\mathcal{C}_3$  such that,  $d_0 \Xi^{f,g}$  is degenerate,

$$d_1 \Xi^{f,g} = \Phi(g), \quad d_2 \Xi^{f,g} = \Phi(gf), \quad d_3 \Xi^{f,g} = \Phi(f).$$

If we apply (R8) to  $\Theta(f, g)$  we obtain (R6).

Suppose now that  $\Phi: \Delta_1 \rightarrow \Delta_2$  is an isomorphism in  $\mathcal{C}_2$ . We choose two isomorphisms in  $\mathcal{C}_2$ ,

$$\Delta_1 \xrightarrow{\Psi^1} \Delta' \xrightarrow{\Psi^2} \Delta'',$$

with,

$$\begin{array}{lll} d_0(\Psi^1) = 1_{d_0 \Delta_1}, & d_1(\Psi^1) = d_1 \Phi, & d_2(\Psi^1) = 1_{d_2 \Delta_1}, \\ d_0(\Psi^2) = d_2 \Phi, & d_1(\Psi^2) = 1_{d_1 \Delta_2}, & d_2(\Psi^2) = 1_{d_2 \Delta_1}, \end{array}$$

and two isomorphisms in  $\mathcal{C}_3$ ,

$$\Theta_1(\Phi) \xleftarrow{\Xi^1} s_1(\Delta) \xrightarrow{\Xi^2} \Theta_2(\Phi),$$

with,

$$\begin{aligned} d_0\Xi^1 &= 1_{s_1 d_0 \Delta_1}, & d_1\Xi^1 &= \Phi(d_1\Phi), & d_2\Xi^1 &= \Psi^1, & d_3\Xi^1 &= 1_{\Delta_1}, \\ d_0\Xi^2 &= 1_{d_0 \Delta_1}, & d_1\Xi^2 &= \Phi, & d_2\Xi^2 &= \Psi^2\Psi^1, & d_3\Xi^2 &= \Phi(d_2\Phi). \end{aligned}$$

We also consider  $\Theta_3(\Phi) = s_2\Delta(d_1\Phi)$  and  $\Theta_4(\Phi) = s_2\Delta_2$ .

Now (R7) follows from Proposition 4.1.14 applied to the  $3 \times 3$  diagram  $\Theta_1(\Phi)$ ,  $\Theta_2(\Phi)$ ,  $\Theta_3(\Phi)$ ,  $\Theta_4(\Phi)$ . Recall that Proposition 4.1.14 only uses (R8) and (R9), hence we are done.  $\square$

**Corollary 4.1.25.** *Let  $\mathcal{W}$  be a Waldhausen category where weak equivalences are isomorphisms (e. g. an abelian or exact category),  $\mathcal{T}$  a (strongly) triangulated category, and  $\mathcal{A}$  an abelian category. Then  $\mathcal{D}_*(\mathcal{W})$ ,  $\mathcal{D}_*(\mathrm{Gr}^{[n,m]}\mathcal{A})$ , and  $\mathcal{D}_*(\bullet\mathcal{T})$ ,  $\blacklozenge = b, d, s, v$ , have a presentation with generators (G1) and (G3) and relations (R2), (R8), (R9) and  $[s_0^2(*)] = 0$ .*

**Definition 4.1.26.** A simplicial category with weak equivalences  $\mathcal{C}_\bullet$  has *functorial coproducts* if  $\mathcal{C}_n$ ,  $n \geq 0$ , is endowed with a symmetric monoidal structure  $+$  which is strictly associative,

$$(X + Y) + Z = X + (Y + Z),$$

strictly unital with unit object  $s_0^n(*)$ ,

$$s_0^n(*) + X = X = X + s_0^n(*),$$

but not necessarily strictly commutative, such that

$$X = X + s_0^n(*) \longrightarrow X + Y \longleftarrow s_0^n(*) + Y = Y$$

is always a coproduct diagram.

We define the stable quadratic module  $\mathcal{D}_*^+(\mathcal{C}_\bullet)$  as the quotient of  $\mathcal{D}_*(\mathcal{C}_\bullet)$  by the following extra relation,

$$(R10) \quad [s_0(X) + s_1(Y)] = 0 \text{ for any pair of objects } X \text{ and } Y \text{ in } \mathcal{D}_1.$$

**Theorem 4.1.27.** *Let  $\mathcal{C}_\bullet$  be a simplicial category with weak equivalences and a functorial coproducts. If the set of objects of  $\mathcal{C}_1$  is free as a monoid under  $+$ , then the natural projection,*

$$\mathcal{D}_*(\mathcal{C}_\bullet) \twoheadrightarrow \mathcal{D}_*^+(\mathcal{C}_\bullet),$$

*is a weak equivalence. It is actually the retraction of a strong deformation retraction.*

The proof is the same as the proof of [MT08, Theorem 4.2] with the obvious change of terminology. The hypothesis is not very strong.

**Proposition 4.1.28.** *For any simplicial category with weak equivalences  $\mathcal{C}_\bullet$  there is another one  $\mathcal{C}'_\bullet$  with functorial coproducts whose simplicial monoid of objects is freely generated by the simplicial set of objects in  $\mathcal{C}_\bullet$  mod  $*$  and its degeneracies, and such that the natural simplicial functor  $\mathcal{C}_\bullet \rightarrow \mathcal{C}'_\bullet$  is an equivalence levelwise and restricts to a levelwise equivalence  $\mathrm{we}(\mathcal{C}_\bullet) \rightarrow \mathrm{we}(\mathcal{C}'_\bullet)$ .*

For the proof of this proposition one applies levelwise the  $\mathrm{Sum}(-)$  construction in [MT08, Proposition 4.3].

**Lemma 4.1.29.** *Given two weak triangles  $(\Delta, f)$  and  $(\Delta', f')$  in a simplicial category with weak equivalences and a functorial coproducts  $\mathcal{C}_\bullet$ , if we denote  $C$  and  $C'$  the source of  $f$  and  $f'$ , respectively, then the following relation holds in  $\mathcal{D}_1^+(\mathcal{C}_\bullet)$ ,*

$$[\Delta + \Delta', f + f'] = [\Delta, f]^{[d_1 \Delta']} + [\Delta', f'] + \langle [d_2 \Delta], [C'] \rangle.$$

*Proof.* Apply Proposition 4.1.21 to

$$\begin{aligned} \Theta_1 &= s_0^2 d_2 \Delta + s_2 \Delta', & \Theta_2 &= s_0 s_1 d_2 \Delta + s_1 \Delta', \\ \Theta_3 &= s_0 \Delta + s_1^2 d_1 \Delta', & \Theta_4 &= s_1 \Delta + s_2 \Delta', \\ \Delta_1 &= d_0 \Theta_3, & w_1 &= 1_{d_0 \Theta_3}, \\ \Delta_2 &= s_0 C + s_1 C', & w_2 &= s_0 f + s_1 f', \\ w'' &= 1_C, & w^C &= f. \end{aligned}$$

□

**Corollary 4.1.30.** *Given two triangles  $\Delta, \Delta'$  in a simplicial category with weak equivalences and a functorial coproducts  $\mathcal{C}_\bullet$  and two weak equivalences  $f: X \xrightarrow{\sim} Y$ ,  $f': X' \xrightarrow{\sim} Y'$  in  $\mathcal{C}_1$ , the following relations hold in  $\mathcal{D}_1^+(\mathcal{C}_\bullet)$ ,*

$$\begin{aligned} [\Delta + \Delta'] &= [\Delta]^{[d_1 \Delta']} + [\Delta'] + \langle [d_2 \Delta], [d_0 \Delta'] \rangle, \\ [f + f'] &= [f]^{[Y']} + [f']. \end{aligned}$$

**Lemma 4.1.31.** *Let  $\mathcal{C}_\bullet$  be a simplicial category with weak equivalences and a functorial coproducts, and  $X_1, \dots, X_n$  objects in  $\mathcal{C}_1$ . Given a permutation of  $n$  elements,  $\sigma \in \text{Sym}(n)$ , we denote*

$$\sigma_{X_1, \dots, X_n}: X_{\sigma_1} + \dots + X_{\sigma_n} \longrightarrow X_1 + \dots + X_n$$

*the isomorphism permuting the factors of the coproduct. The following formula holds in  $\mathcal{D}_1^+(\mathcal{C}_\bullet)$ ,*

$$[\sigma_{X_1, \dots, X_n}] = \sum_{\substack{i > j \\ \sigma_i < \sigma_j}} \langle [X_{\sigma_i}], [X_{\sigma_j}] \rangle$$

This lemma can be proved as [MT08, Lemma 4.9].

*Proof of Theorem 4.1.6.* In this proof we translate the argument in the proof of [MT08, Theorem 2.1] to our unified framework. By Proposition 4.1.28 we can suppose that  $\mathcal{C}_\bullet$  has functorial coproducts in such a way that the monoid of objects of  $\mathcal{C}_1$  is freely generated by a set  $\mathbb{S}$  of non-degenerate objects, so we can work with  $\mathcal{D}_1^+(\mathcal{C}_\bullet)$  by Theorem 4.1.27.

Any  $x \in \mathcal{D}_1^+(\mathcal{C}_\bullet)$  is a sum of triangles and weak equivalences in  $\mathcal{C}_1$  with coefficients  $\pm 1$ . Therefore, by Corollary 4.1.30, the following equation holds,

$$\begin{aligned} x &= -[f: X \xrightarrow{\sim} Y] - [\Delta] + [\Delta'] + [f': X' \xrightarrow{\sim} Y'] && \text{mod } \langle \cdot, \cdot \rangle \\ &= -[f + 1_{X'}] - [\Delta + s_0 d_0 \Delta' + s_1 d_2 \Delta'] \\ &\quad + [s_0 d_0 \Delta + s_1 d_2 \Delta + \Delta'] + [1_X + f'] && \text{mod } \langle \cdot, \cdot \rangle. \end{aligned}$$

If  $\partial(x) = 0$  modulo commutators then,

$$\begin{aligned} 0 = & -[X + X'] + [Y + X'] - [d_2\Delta + d_2\Delta'] - [d_0\Delta + d_0\Delta'] + [d_1\Delta + d_0\Delta' + d_2\Delta'] \\ & - [d_0\Delta + d_2\Delta + d_1\Delta'] + [d_0\Delta + d_0\Delta'] + [d_2\Delta + d_2\Delta'] - [X + Y'] + [X + X'] \\ & \text{mod } [\cdot, \cdot], \end{aligned}$$

and therefore,

$$[Y + X' + d_1\Delta + d_0\Delta' + d_2\Delta'] = [X + Y' + d_0\Delta + d_2\Delta + d_1\Delta'] \text{ mod } [\cdot, \cdot].$$

The quotient of  $\mathcal{D}_0^+(\mathcal{C}_\bullet)$  by the commutator subgroup is the free abelian group with basis  $\mathbb{S}$ , hence there are objects  $S_1, \dots, S_n \in \mathbb{S}$  and a permutation  $\sigma \in \text{Sym}(n)$  with,

$$\begin{aligned} Y + X' + d_1\Delta + d_0\Delta' + d_2\Delta' &= S_1 + \dots + S_n, \\ X + Y' + d_0\Delta + d_2\Delta + d_1\Delta' &= S_{\sigma_1} + \dots + S_{\sigma_n}. \end{aligned}$$

In particular, there is an isomorphism,

$$\sigma_{S_1, \dots, S_n} : X + Y' + d_0\Delta + d_2\Delta + d_1\Delta' \longrightarrow Y + X' + d_1\Delta + d_0\Delta' + d_2\Delta'.$$

By the isomorphism lifting property, there exists an isomorphism in  $\mathcal{C}_2$ ,

$$\Phi : s_0X + s_0Y' + s_0d_0\Delta + s_1d_2\Delta + \Delta' \xrightarrow{\sim} \Delta_2,$$

such that  $d_0\Phi$  and  $d_2\Phi$  are identity morphisms and  $d_1\Phi = \sigma_{S_1, \dots, S_n}$ .

By Corollaries 4.1.30 and 4.1.31, modulo the image of  $\langle \cdot, \cdot \rangle$ ,

$$\begin{aligned} x = & -[f + 1_{X'} + 1_{d_0\Delta + d_0\Delta'}] - [s_0Y + s_0X' + \Delta + s_0d_0\Delta' + s_1d_2\Delta'] \\ & + [s_0X + s_0Y' + s_0d_0\Delta + s_1d_2\Delta + \Delta'] + [1_X + f' + 1_{d_0\Delta + d_0\Delta'}] \\ = & -[f + 1_{X'} + 1_{d_0\Delta + d_0\Delta'}] - [s_0Y + s_0X' + \Delta + s_0d_0\Delta' + s_1d_2\Delta'] \\ & + [\sigma_{S_1, \dots, S_n}] \\ & + [s_0X + s_0Y' + s_0d_0\Delta + s_1d_2\Delta + \Delta'] + [1_X + f' + 1_{d_0\Delta + d_0\Delta'}] \\ = & -[f + 1_{X'} + 1_{d_0\Delta + d_0\Delta'}] - [s_0Y + s_0X' + \Delta + s_0d_0\Delta' + s_1d_2\Delta'] \\ & + [\Delta_2] + [1_X + f' + 1_{d_0\Delta + d_0\Delta'}] \\ = & -[f + 1_{X'} + 1_{d_0\Delta + d_0\Delta'}]^{[d_2\Delta + d_2\Delta']} - [s_0Y + s_0X' + \Delta + s_0d_0\Delta' + s_1d_2\Delta'] \\ & + [\Delta_2] + [1_X + f' + 1_{d_0\Delta + d_0\Delta'}]^{[d_2\Delta + d_2\Delta']} \\ = & [s_0Y + s_0X' + \Delta + s_0d_0\Delta' + s_1d_2\Delta', f + 1_{X'} + 1_{d_0\Delta + d_0\Delta'}; \\ & \Delta_2, 1_X + f' + 1_{d_0\Delta + d_0\Delta'}], \end{aligned}$$

i.e.  $x$  is represented by a pair of weak triangles modulo the image of  $\langle \cdot, \cdot \rangle$ ,

$$x = [\Delta_1, f_1; \Delta_2, f_2] + y, \quad y \text{ in the image of } \langle \cdot, \cdot \rangle.$$

Assume now that  $\partial(x) = 0$ . Then  $\partial(y) = 0$  as well, therefore by [MT08, Lemma 5.1]  $y = \langle a, a \rangle$  for some  $a \in \mathcal{D}_0(\mathcal{C}_\bullet)$ , which is the free group of nilpotency class 2 with basis  $\mathbb{S}$ . Since  $y$  only depends on  $a \text{ mod } 2$ , we can suppose that  $a = [S'_1] + \dots + [S'_m] = [M]$ ,  $M = S'_1 + \dots + S'_m$ ,  $S'_i \in \mathbb{S}$ , therefore,

$$y = \langle [M], [M] \rangle = [s_0M + s_1M; s_1M + s_0M],$$

is a pair of triangles, in particular a pair of weak triangles, so  $x$  is also a pair of weak triangles by Corollary 4.1.23.  $\square$

We finish this section with some relevant results for the applications in the following sections.

**Proposition 4.1.32.** *Assume that  $\mathcal{C}_\bullet$  is equipped with an additive functor*

$$\Gamma: \mathcal{C}_1 \longrightarrow \mathcal{C}_2: X \mapsto \Gamma_X,$$

such that,

$$d_2\Gamma = 1_{\mathcal{C}_2}; \quad d_1\Gamma = 0.$$

Moreover, suppose that for any object  $\Delta$  in  $\mathcal{C}_2$  there is an object  $\Theta$  in  $\mathcal{C}_3$  and weak equivalences  $\Phi$  and  $\Phi'$  in  $\mathcal{C}_2$ ,

$$\Phi: d_2\Theta \xrightarrow{\sim} s_0d_0\Delta \sqcup \Gamma_{d_2\Delta}, \quad \Phi': s_0d_0\Delta \sqcup s_1d_0\Gamma_{d_2\Delta} \xrightarrow{\sim} s_0d_0\Delta \sqcup s_1d_0\Gamma_{d_2\Delta},$$

such that,

$$\begin{aligned} d_0\Theta &= s_1d_0\Delta \sqcup s_0d_0\Gamma_{d_2\Delta}, & d_3\Theta &= \Delta, & d_0\Phi &= d_1\Phi', \\ d_1\Phi &= 1_{d_0\Delta} = d_0\Phi', & d_2\Phi &= 1_{d_2\Delta}, & d_2\Phi' &= 1_{d_0\Gamma_{d_2\Delta}}. \end{aligned}$$

Then the following equation is satisfied in  $\mathcal{D}_1(\mathcal{C}_\bullet)$ ,

$$[d_1\Theta] + [\Delta] = [\Gamma_{d_2\Delta}] + \langle [d_0\Delta], [d_0\Gamma_{d_2\Delta}] \rangle.$$

*Proof.* The object  $\Theta$  and the weak equivalences  $\Phi$  and  $\Phi'$  yield the following relations,

$$\begin{aligned} [d_1\Theta] + [\Delta] &= [d_2\Theta] + [s_1d_0\Delta \sqcup s_0d_0\Gamma_{d_2\Delta}]^{[d_2\Delta]}, \\ 0 &= [d_0\Phi]^{[d_0\Delta]} = -[s_0d_0\Delta \sqcup \Gamma_{d_2\Delta}] + [d_2\Theta]. \end{aligned}$$

Moreover, the object  $s_0^2d_0\Delta \sqcup s_2\Gamma_{d_2\Delta}$  in  $\mathcal{C}_3$  yields,

$$[\Gamma_{d_2\Delta}] = [s_0d_0\Delta \sqcup \Gamma_{d_2\Delta}] + [s_0d_0\Delta \sqcup s_1d_0\Gamma_{d_2\Delta}]^{[d_2\Delta]}.$$

Now the equation in the statement follows.  $\square$

*Remark 4.1.33.* If  $\mathcal{T}$  is a triangulated category this proposition can be applied to  $\bar{S}_\bullet(\bullet\mathcal{T})$ ,  $\blacklozenge = b, d, v$ . The functor  $\Gamma$  is defined for any object  $X$  in  $\mathcal{T}$  by the following distinguished triangle,

$$\Gamma_X: \quad X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{-1} \Sigma X.$$

In particular  $d_0\Gamma = \Sigma$ . If  $\Delta: X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X$  is a distinguished (resp. virtual) triangle, then  $\Theta$  is the following special (resp. virtual) octahedron,

$$\Theta: \quad \begin{array}{ccccc} & & C^f & \xrightarrow{-q} & \Sigma X \\ & & \uparrow \scriptstyle{f} & \searrow \scriptstyle{(-1)} & \uparrow \scriptstyle{(0,1)} \\ & & C^f \oplus \Sigma X & & \\ & & \downarrow \scriptstyle{i} & \swarrow \scriptstyle{-1} & \downarrow \scriptstyle{\Sigma f} \\ & & Y & \xrightarrow{f} & C^f \\ & & \downarrow \scriptstyle{f} & \swarrow \scriptstyle{(q,1)} & \downarrow \scriptstyle{(1/0)} \\ X & \xleftarrow{q} & 0 & \xrightarrow{-1} & C^f \end{array}$$

Indeed, in the distinguished case, we have the following isomorphisms with obviously distinguished lower rows,

$$\begin{array}{ccccccc}
Y & \xrightarrow{\begin{pmatrix} i \\ -i \end{pmatrix}} & C^f \oplus C^f & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -q & 0 \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(0, \Sigma f)} & \Sigma Y \\
\downarrow 1 & & \downarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & & \downarrow 1 & & \downarrow 1 \\
Y & \xrightarrow{\begin{pmatrix} 0 \\ i \end{pmatrix}} & C^f \oplus C^f & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(0, \Sigma f)} & \Sigma Y
\end{array}$$
  

$$\begin{array}{ccccccc}
C^f \oplus \Sigma X & \xrightarrow{\begin{pmatrix} q & 1 \\ 0 & -1 \end{pmatrix}} & \Sigma X \oplus \Sigma X & \xrightarrow{(\Sigma f, \Sigma f)} & \Sigma Y & \xrightarrow{\begin{pmatrix} \Sigma i \\ 0 \end{pmatrix}} & \Sigma C^f \oplus \Sigma^2 X \\
\downarrow 1 & & \downarrow \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} & & \downarrow 1 & & \downarrow 1 \\
C^f \oplus \Sigma X & \xrightarrow{\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}} & \Sigma X \oplus \Sigma X & \xrightarrow{(\Sigma f, 0)} & \Sigma Y & \xrightarrow{\begin{pmatrix} \Sigma i \\ 0 \end{pmatrix}} & \Sigma C^f \oplus \Sigma^2 X
\end{array}$$

This shows that  $\Theta$  is special. Moreover, the isomorphisms  $\Phi$  and  $\Phi'$  are, respectively,

$$\begin{array}{ccccccc}
\Phi: & X & \xrightarrow{0} & C^f & \xrightarrow{\begin{pmatrix} 1 \\ -q \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(q, 1)} & \Sigma X \\
& \downarrow 1 & & \downarrow 1 & & \downarrow \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} & & \downarrow 1 \\
& X & \xrightarrow{0} & C^f & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(0, 1)} & \Sigma X
\end{array}$$
  

$$\begin{array}{ccccccc}
\Phi': & \Sigma X & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(1, 0)} & C^f & \xrightarrow{0} & \Sigma^2 X \\
& \downarrow 1 & & \downarrow \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} & & \downarrow 1 & & \downarrow 1 \\
& \Sigma X & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & C^f \oplus \Sigma X & \xrightarrow{(1, 0)} & C^f & \xrightarrow{0} & \Sigma^2 X
\end{array}$$

Notice that

$$d_1\Theta: \quad Y \xrightarrow{i} C^f \xrightarrow{-q} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

is (isomorphic to) the usual translation of  $\Delta$ . In this case the formula in Proposition 4.1.32 is as follows,

$$[\Gamma_X] + \langle [C^f], [\Sigma X] \rangle = [Y \xrightarrow{i} C^f \xrightarrow{-q} \Sigma X \xrightarrow{\Sigma f} \Sigma Y] + [X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{-q} \Sigma X].$$

If  $\mathcal{A}$  is an abelian category the proposition can also be applied to  $\bar{S}_\bullet(\mathrm{Gr}^b \mathcal{A})$ , and to  $\bar{S}_\bullet(\mathrm{Gr}^{[n, m]} \mathcal{A})$  if  $X, Y$  and  $C^f$  are concentrated in  $[n, m-1]$ . The notation above also makes sense in this case, see Remark 1.6.10.

For the following proposition one argues as in [MT07, Corollaries 1.9 and 1.10].

**Proposition 4.1.34.** *For any object  $X$  in  $\mathcal{C}_1$  the following equation is satisfied in  $\mathcal{D}_1(\mathcal{C}_\bullet)$ ,*

$$\langle [X], [X] \rangle = [X \sqcup X \xrightarrow{\text{twist}} X \sqcup X].$$

Moreover, if  $\mathcal{C}_1$  is additive,

$$\langle [X], [X] \rangle = [X \xrightarrow{-1} X].$$



*Remark 4.1.35.* Using this proposition one can easily check that the formula in Remark 4.1.33 is equivalent to

$$\begin{aligned} [\Gamma X] + \langle [C^f] + [\Sigma X], [\Sigma X] \rangle &= [Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y] \\ &\quad + [X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X]. \end{aligned}$$

#### 4.2. Graded and ungraded determinant functors on an abelian category.

Let  $\mathcal{A}$  be an abelian category. Objects in  $\mathcal{A}$  are regarded as graded objects concentrated in degree 0. Given a bounded graded object  $X$  in  $\mathcal{A}$  concentrated in  $[n, +\infty)$  we denote  $X_{\geq n+1}$  the graded object concentrated in  $[n+1, +\infty)$  which is equal to  $X$  within this interval. Moreover,  $\Sigma X$  is the graded object with  $(\Sigma X)_n = X_{n-1}$ . Let  $A$  be an object in  $\mathcal{A}$ . We consider the long exact sequences,

$$\begin{aligned} \Delta_n^X &= X_{\geq n+1} \longrightarrow X \longrightarrow \Sigma^n X_n \longrightarrow \Sigma X_{\geq n+1}, \\ \Gamma_A^n &= \Gamma_{\Sigma^n A} = \Sigma^n A \longrightarrow 0 \longrightarrow \Sigma^{n+1} A \longrightarrow \Sigma^{n+1} A, \end{aligned}$$

whose morphisms are either identities or trivial.

**Theorem 4.2.1.** *The natural morphism  $j$  below fits into a strong deformation retraction,*

$$\alpha \circlearrowleft \mathcal{D}_*(\mathrm{Gr}^b \mathcal{A}) \xleftarrow[j]{p} \mathcal{D}_*(\mathcal{A}),$$

where  $p$  is the unique morphism with  $p[\Delta_n^X] = 0 = p[\Gamma_A^n]$  for any  $n \in \mathbb{Z}$ , any graded object  $X$  concentrated in  $[n, +\infty)$ , and any object  $A$  in  $\mathcal{A}$ , and  $\alpha$  is the unique homotopy  $\alpha: jp \Rightarrow 1$  such that  $\alpha j = 0$ .

The fact that  $j$  is a weak equivalence also follows from [Nee05, Theorem 3 (i)].

The theorem above is a direct consequence of the following proposition, since,

$$\mathcal{D}_*(\mathrm{Gr}^b \mathcal{A}) = \operatorname{colim}_{n \rightarrow +\infty} \mathcal{D}_*(\mathrm{Gr}^{[-n, n]} \mathcal{A}).$$

This formula is true as a levelwise colimit of groups as well as a colimit in the category of stable quadratic modules.

**Proposition 4.2.2.** *Given  $n < m$ , the natural morphism  $j$  below fits into a strong deformation retraction,*

$$\alpha \circlearrowleft \mathcal{D}_*(\mathrm{Gr}^{[n, m]} \mathcal{A}) \xleftarrow[j]{p} \mathcal{D}_*(\mathrm{Gr}^{[n+1, m]} \mathcal{A}),$$

where  $p$  is the unique morphism with  $p[\Delta_n^X] = 0 = p[\Gamma_A^n]$ , and  $\alpha$  is the unique homotopy  $\alpha: jp \Rightarrow 1$  such that  $\alpha j = 0$ . Similarly, there is a strong deformation retraction,

$$\alpha \circlearrowleft \mathcal{D}_*(\mathrm{Gr}^{[n, m]} \mathcal{A}) \xleftarrow[j]{p} \mathcal{D}_*(\mathrm{Gr}^{[n, m-1]} \mathcal{A}),$$

where  $p$  is the unique morphism with  $p[\Delta_n^X] = 0 = p[\Gamma_A^{m-1}]$ , and  $\alpha$  is the unique homotopy  $\alpha: jp \Rightarrow 1$  such that  $\alpha j = 0$ .

In the proof of this proposition we use several lemmas. Given a graded object  $X$  concentrated in  $[n, +\infty)$  and a morphism  $f_n: X_n \rightarrow A$  in  $\mathcal{A}$  we define  $X(f_n)$  by the long exact sequence,

$$X(f_n) \longrightarrow X \longrightarrow \Sigma^n \operatorname{Im} f_n \longrightarrow \Sigma X(f_n),$$

i.e.  $X(f_n)$  coincides with  $X$  in all dimensions except from  $n$ ,  $X_n(f_n) = \text{Ker } f_n$ .

**Lemma 4.2.3.** *Given  $n < m$ , three graded objects  $X, Y, C^f$  in  $\mathcal{A}$  concentrated in  $[n, m]$ , and a long exact sequence,*

$$\cdots \rightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{i_n} C_n^f \xrightarrow{q_n} X_{n-1} \rightarrow \cdots,$$

the following formula holds in  $\mathcal{D}_*(\text{Gr}^{[n, m]} \mathcal{A})$ ,

$$\begin{aligned} & [X \rightarrow Y \rightarrow C^f \rightarrow \Sigma X] \\ &= [\Delta_n^Y] + [\Sigma^n \text{Ker } i_n \rightarrow \Sigma^n Y_n \rightarrow \Sigma^n C_n^f \rightarrow \Sigma^{n+1} \text{Ker } i_n]^{[Y_{\geq n+1}]} \\ &\quad - [\Delta_n^{Y(i_n)}] + [X \rightarrow Y(i_n) \rightarrow C_{\geq n+1}^f \rightarrow \Sigma X] + [\Delta_n^{C^f}]^{[X]}. \end{aligned}$$

*Proof.* This lemma follows from the following diagrams,

□

Using this lemma, Proposition 4.1.32, and Remarks 4.1.33 and 4.1.35, we derive the following result.

**Lemma 4.2.4.** *Given  $n < m$ , three graded objects  $X, Y, C^f$  in  $\mathcal{A}$  concentrated in  $[n, m]$ , and a long exact sequence,*

$$\cdots \rightarrow X_n \xrightarrow{f_n} Y_n \xrightarrow{i_n} C_n^f \xrightarrow{q_n} X_{n-1} \rightarrow \cdots,$$

we have the following relation in  $\mathcal{D}_1(\text{Gr}^{[n, m]} \mathcal{A})$ ,

$$\begin{aligned} & [X \rightarrow Y \rightarrow C^f \rightarrow \Sigma X] \\ &= [\Delta_n^Y] + [\Sigma^n \text{Ker } i_n \rightarrow \Sigma^n Y_n \rightarrow \Sigma^n C_n^f \xrightarrow{0} \Sigma^{n+1} \text{Ker } i_n]^{[Y_{\geq n+1}]} \\ &\quad + [\Gamma_{\text{Ker } f_n}^n]^{[Y_{\geq n+1}]} + [X_{\geq n+1} \rightarrow Y_{\geq n+1} \rightarrow C_{\geq n+1}^f(q_{n+1}) \rightarrow \Sigma Y_{\geq n+1}] \\ &\quad - [C_{\geq n+1}^f(q_{n+1}) \rightarrow C_{\geq n+1}^f \rightarrow \Sigma^{n+1} \text{Ker } f_n \rightarrow \Sigma C_{\geq n+1}^f(q_{n+1})]^{[\Sigma^n \text{Ker } f_n] + [X_{\geq n+1}]} \\ &\quad - [\Sigma^n \text{Ker } f_n \rightarrow \Sigma^n X_n \rightarrow \Sigma^n \text{Ker } i_n \xrightarrow{0} \Sigma^{n+1} \text{Ker } f_n]^{[X_{\geq n+1}]} \\ &\quad - [\Delta_n^X] - [\Delta_n^{C^f}]^{[X]} + \langle [\Sigma^n \text{Ker } i_n], [C_{\geq n+1}^f] \rangle + \langle [\Sigma^n \text{Ker } f_n], [C_{\geq n+1}^f(q_{n+1})] \rangle. \end{aligned}$$

**Corollary 4.2.5.** *Given  $n < m$ , the stable quadratic module  $\mathcal{D}_*(\text{Gr}^{[n, m]} \mathcal{A})$  is generated by the generators of  $\mathcal{D}_*(\text{Gr}^{[n+1, m]} \mathcal{A})$  together with  $[\Delta_n^X]$  and  $[\Gamma_A^n]$  in degree 1, for each object  $A$  in  $\mathcal{A}$  and each bounded graded object  $X$  concentrated in  $[n, m]$ .*

**Corollary 4.2.6.** *The stable quadratic module  $\mathcal{D}_*(\mathrm{Gr}^b \mathcal{A})$  is generated by:*

- (1)  $j[A]$ , in degree 0,  $A$  an object in  $\mathcal{A}$ ;
- (2)  $[\Gamma_A^n]$ , in degree 1,  $A$  an object in  $\mathcal{A}$ ,  $n \in \mathbb{Z}$ ;
- (3)  $[\Delta_X^n]$ , in degree 1,  $X$  a graded object in  $\mathcal{A}$  concentrated in  $[n, +\infty)$ ,  $n \in \mathbb{Z}$ ;
- (4)  $j[A \twoheadrightarrow B \twoheadrightarrow C]$ , in degree 1,  $A \twoheadrightarrow B \twoheadrightarrow C$  a short exact sequence in  $\mathcal{A}$ .

*Proof of Proposition 4.2.2.* We concentrate on the first part of the statement and use the presentations given by Corollary 4.1.25.

Since  $p$  must satisfy  $pj = 1$  and

$$\begin{aligned} 0 &= \partial p[\Delta_n^X] = -p[X] + p[\Sigma^n X_n] + pj[X_{\geq n+1}], \\ 0 &= \partial p[\Gamma_A^n] = pj[\Sigma^{n+1} A] + p[\Sigma^n A], \end{aligned}$$

we must define  $p$  on degree 0 generators as

$$p[X] = -[\Sigma^{n+1} X_n] + [X_{\geq n+1}].$$

Lemma 4.2.4, Proposition 4.1.32, and Remarks 4.1.33 and 4.1.35 force us to define  $p$  on degree 1 generators as follows,

$$\begin{aligned} & p[X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X] \\ &= -[\Sigma^{n+1} \mathrm{Ker} i_n \rightarrow \Sigma^{n+1} Y_n \rightarrow \Sigma^{n+1} C_n^f \xrightarrow{0} \Sigma^{n+2} \mathrm{Ker} i_n]^{p[Y]} \\ &+ [X_{\geq n+1} \rightarrow Y_{\geq n+1} \rightarrow C_{\geq n+1}^f(q_{n+1}) \rightarrow \Sigma X_{\geq n+1}] \\ &- [C_{\geq n+1}^f(q_{n+1}) \rightarrow C_{\geq n+1}^f \rightarrow \Sigma^{n+1} \mathrm{Ker} f_n \rightarrow \Sigma C_{\geq n+1}^f(q_{n+1})]^{p[A(f_n)]} \\ &+ [\Sigma^{n+1} \mathrm{Ker} f_n \rightarrow \Sigma^{n+1} X_n \rightarrow \Sigma^{n+1} \mathrm{Ker} i_n \xrightarrow{0} \Sigma^{n+2} \mathrm{Ker} f_n]^{p[X]} \\ &+ \langle p[C^f], [\Sigma^{n+1} \mathrm{Ker} i_n] \rangle - \langle [\Sigma^{n+1} \mathrm{Ker} f_n], [C_{\geq n+1}^f(q_{n+1})] + [\Sigma^{n+1} \mathrm{Ker} i_n] \rangle. \end{aligned}$$

Let us check in detail compatibility with (R2) and (R9) in order to guide the reader through some standard computations with stable quadratic modules,

$$\begin{aligned}
& \partial p[X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X] \\
&= -p[Y] - [\Sigma^{n+1} \text{Ker } i_n] - [\Sigma^{n+1} C_n^f] + [\Sigma^{n+1} Y_n] - [\Sigma^{n+1} Y_n] + [Y_{\geq n+1}] \\
&\quad - [Y_{\geq n+1}] + [C_{\geq n+1}^f(q_{n+1})] + [X_{\geq n+1}] - [X_{\geq n+1}] + [\Sigma^{n+1} \text{Ker } f_n] \\
&\quad - [C_{\geq n+1}^f(q_{n+1})] - [\Sigma^{n+1} \text{Ker } f_n] + [C_{\geq n+1}^f] - [\Sigma^{n+1} \text{Ker } f_n] + [X_{\geq n+1}] \\
&\quad - [X_{\geq n+1}] + [\Sigma^{n+1} X_n] - [\Sigma^{n+1} X_n] + [\Sigma^{n+1} \text{Ker } i_n] + [\Sigma^{n+1} \text{Ker } f_n] + p[X] \\
&\quad - [p[C^f], [\Sigma^{n+1} \text{Ker } i_n]] + [[\Sigma^{n+1} \text{Ker } f_n], [C_{\geq n+1}^f(q_{n+1})] + [\Sigma^{n+1} \text{Ker } i_n]] \\
&= -p[Y] - [\Sigma^{n+1} \text{Ker } i_n] - [\Sigma^{n+1} C_n^f] + [C_{\geq n+1}^f(q_{n+1})] + [\Sigma^{n+1} \text{Ker } f_n] \\
&\quad - [C_{\geq n+1}^f(q_{n+1})] - [\Sigma^{n+1} \text{Ker } f_n] + [C_{\geq n+1}^f] - [\Sigma^{n+1} \text{Ker } f_n] \\
&\quad + [\Sigma^{n+1} \text{Ker } i_n] + [\Sigma^{n+1} \text{Ker } f_n] + p[X] \\
&\quad - [p[C^f], [\Sigma^{n+1} \text{Ker } i_n]] + [[\Sigma^{n+1} \text{Ker } f_n], [C_{\geq n+1}^f(q_{n+1})] + [\Sigma^{n+1} \text{Ker } i_n]] \\
&= -p[Y] - [\Sigma^{n+1} \text{Ker } i_n] - [\Sigma^{n+1} C_n^f] + [C_{\geq n+1}^f] - [\Sigma^{n+1} \text{Ker } f_n] \\
&\quad + [\Sigma^{n+1} \text{Ker } i_n] + [\Sigma^{n+1} \text{Ker } f_n] + p[X] \\
&\quad - [-[\Sigma^{n+1} C_n^f] + [C_{\geq n+1}^f], [\Sigma^{n+1} \text{Ker } i_n]] + [[\Sigma^{n+1} \text{Ker } f_n], [\Sigma^{n+1} \text{Ker } i_n]] \\
&= -p[Y] - [\Sigma^{n+1} C_n^f] + [C_{\geq n+1}^f] - [\Sigma^{n+1} \text{Ker } i_n] - [\Sigma^{n+1} \text{Ker } f_n] \\
&\quad + [\Sigma^{n+1} \text{Ker } i_n] + [\Sigma^{n+1} \text{Ker } f_n] + p[X] + [[\Sigma^{n+1} \text{Ker } f_n], [\Sigma^{n+1} \text{Ker } i_n]] \\
&= -p[Y] + p[C^f] + p[X]
\end{aligned}$$

Moreover,  $p[0 \rightarrow 0 \rightarrow 0 \rightarrow 0] = 0$ .

Given two graded objects  $X, Y$  concentrated in  $[n, m]$ ,

$$\begin{aligned}
& p[X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{\binom{0,1}{0}} Y \xrightarrow{0} \Sigma X] \\
&= -[\Sigma^{n+1} X_n \xrightarrow{\binom{1}{0}} \Sigma^{n+1} X_n \oplus \Sigma^{n+1} Y_n \xrightarrow{\binom{0,1}{0}} \Sigma^{n+1} Y_n \xrightarrow{0} \Sigma^{n+2} X_n]^{p[X \oplus Y]} \\
&\quad + [X_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{\binom{0,1}{0}} Y_{\geq n+1} \xrightarrow{0} \Sigma X_{\geq n+1}] \\
&\quad + \langle p[Y], [\Sigma^{n+1} X_n] \rangle, \\
& p[Y \xrightarrow{\binom{0}{1}} X \oplus Y \xrightarrow{\binom{1,0}{0}} X \xrightarrow{0} \Sigma Y] \\
&= -[\Sigma^{n+1} Y_n \xrightarrow{\binom{0}{1}} \Sigma^{n+1} X_n \oplus \Sigma^{n+1} Y_n \xrightarrow{\binom{1,0}{0}} \Sigma^{n+1} X_n \xrightarrow{0} \Sigma^{n+2} Y_n]^{p[X \oplus Y]} \\
&\quad + [Y_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{\binom{0,1}{0}} X_{\geq n+1} \xrightarrow{0} \Sigma Y_{\geq n+1}] \\
&\quad + \langle p[X], [\Sigma^{n+1} Y_n] \rangle,
\end{aligned}$$

therefore,

$$\begin{aligned}
& -p[Y \xrightarrow{\binom{0}{1}} X \oplus Y \xrightarrow{(1,0)} Y_{\geq n+1} \longrightarrow \Sigma Y] \\
& + p[X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(0,1)} Y_{\geq n+1} \longrightarrow \Sigma X] \\
& = -\langle p[X], [\Sigma^{n+1} Y_n] \rangle \\
& \quad - [Y_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{(0,1)} X_{\geq n+1} \longrightarrow \Sigma Y_{\geq n+1}] \\
& \quad + [\Sigma^{n+1} Y_n \xrightarrow{\binom{1}{0}} \Sigma^{n+1} X_n \oplus \Sigma^{n+1} Y_n \xrightarrow{(0,1)} \Sigma^{n+1} X_n \longrightarrow \Sigma^{n+2} Y_n]^{p[X \oplus Y]} \\
& \quad - [\Sigma^{n+1} X_n \xrightarrow{\binom{0}{1}} \Sigma^{n+1} X_n \oplus \Sigma^{n+1} Y_n \xrightarrow{(1,0)} \Sigma^{n+1} Y_n \longrightarrow \Sigma^{n+2} X_n]^{p[X \oplus Y]} \\
& \quad + [X_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{(0,1)} Y_{\geq n+1} \longrightarrow \Sigma X_{\geq n+1}] \\
& \quad + \langle p[Y], [\Sigma^{n+1} X_n] \rangle \\
& = -\langle p[X], [\Sigma^{n+1} Y_n] \rangle \\
& \quad - [Y_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{(0,1)} X_{\geq n+1} \longrightarrow \Sigma Y_{\geq n+1}] \\
& \quad + \langle [\Sigma^{n+1} Y_n], [\Sigma^{n+1} X_n] \rangle \\
& \quad + [X_{\geq n+1} \xrightarrow{\binom{1}{0}} X_{\geq n+1} \oplus Y_{\geq n+1} \xrightarrow{(0,1)} Y_{\geq n+1} \longrightarrow \Sigma X_{\geq n+1}] \\
& \quad + \langle p[Y], [\Sigma^{n+1} X_n] \rangle \\
& = -\langle -[\Sigma^{n+1} X_n] + [X_{\geq n+1}], [\Sigma^{n+1} Y_n] \rangle + \langle [X_{\geq n+1}], [Y_{\geq n+1}] \rangle \\
& \quad + \langle [\Sigma^{n+1} Y_{n+1}], [\Sigma^{n+1} X_{n+1}] \rangle + \langle -[\Sigma^{n+1} Y_n] + [Y_{\geq n+1}], [\Sigma^{n+1} X_n] \rangle \\
& = \langle -[\Sigma^{n+1} X_n] + [X_{\geq n+1}], -[\Sigma^{n+1} Y_n] + [Y_{\geq n+1}] \rangle \\
& = \langle p[X], p[Y] \rangle.
\end{aligned}$$

This proves compatibility with (R9).

Given six bounded graded objects  $X, Y, Z, C^f, C^g$  and  $C^{gf}$ , concentrated in  $[n, m]$ , and a commutative diagram containing four long exact sequences,

we must show compatibility with (R8), i.e.

$$\begin{aligned} & p[Y \rightarrow Z \rightarrow C^g \rightarrow \Sigma Y] + p[X \rightarrow Y \rightarrow C^f \rightarrow \Sigma X] \\ &= p[X \rightarrow Z \rightarrow C^{gf} \rightarrow \Sigma Y] + p[C^f \rightarrow C^{gf} \rightarrow C^g \rightarrow \Sigma C^f]^{p[X]}. \end{aligned}$$

This is a lengthy but straightforward computation which only uses the formulas in  $\mathcal{D}_*(\text{Gr}^{[n+1, m]}\mathcal{A})$  derived from the six diagrams depicted at the end of this proof.

In order to be a homotopy,  $\alpha$  must satisfy,

$$\begin{aligned} \text{(a)} \quad & -[\Gamma_A^n] = \alpha\partial[\Gamma_A^n] \\ &= \alpha(j[\Sigma^{n+1}A] + [\Sigma^n A]) \\ &= \alpha(j[\Sigma^{n+1}A])^{[\Sigma^n A]} + \alpha[\Sigma^n A] \\ &= \alpha[\Sigma^n A], \\ \text{(b)} \quad & -[\Delta_n^X] = \alpha\partial[\Delta_n^X] \\ &= \alpha(-[X] + [\Sigma^n X_n] + j[X_{\geq n+1}]) \\ &= -\alpha([X])^{-[X] + [\Sigma^n X_n] + [X_{\geq n+1}]} + \alpha([\Sigma^n X_n])^{[X_{\geq n+1}]} + \alpha j[X_{\geq n+1}] \\ &= -[\Delta_n^X] + \alpha[X] + [\Delta_n^X] + \alpha([\Sigma^n X_n])^{[X_{\geq n+1}]}. \end{aligned}$$

This forces as to define  $\alpha$  on generators as

$$\alpha[X] = [\Gamma_{X_n}^n]^{[X_{\geq n+1}]} - [\Delta_n^X].$$

The homotopy defined in this way satisfies

$$\begin{aligned} \partial\alpha[X] &= -[X_{\geq n+1}] + [\Sigma^{n+1}X_n] + [\Sigma^n X_n] + [X_{\geq n+1}] - [X_{\geq n+1}] - [\Sigma^n X_n] + [X] \\ &= -p[X] + [X]. \end{aligned}$$

We now have to check that the following formula holds on degree 1 generators,

$$\alpha\partial[X \rightarrow Y \rightarrow C \rightarrow \Sigma X] = -p[X \rightarrow Y \rightarrow C \rightarrow \Sigma X] + [X \rightarrow Y \rightarrow C \rightarrow \Sigma X].$$

We can restrict ourselves to the generators given by Corollary 4.2.5. But this is exactly formulas (a) and (b), hence we are done for the first part of the statement.

For the second part of the statement one would like to use the following type of long exact sequence rather than  $\Delta_n^X$ ,

$$\Sigma^m X_m \longrightarrow X \longrightarrow X_{\leq m-1} \longrightarrow \Sigma^{m+1} X_m.$$

Both long exact sequences are related by the following diagram,

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & \Sigma^n X_n & & \\ & \searrow & & \nearrow & \\ & & X_{\leq m-1} & & \\ & & \swarrow & \searrow & \\ & & & X_{\geq n+1} & \\ & \swarrow & & \swarrow & \\ \Sigma^m X_m & \xleftarrow{\quad} & (X_{\geq n+1})_{\leq m-1} & & \end{array}$$

(Note: The diagram includes additional arrows: a vertical arrow from  $X$  to  $\Sigma^m X_m$ , a vertical arrow from  $\Sigma^n X_n$  to  $(X_{\geq n+1})_{\leq m-1}$ , a diagonal arrow from  $X$  to  $(X_{\geq n+1})_{\leq m-1}$ , a diagonal arrow from  $\Sigma^n X_n$  to  $X_{\leq m-1}$ , and a diagonal arrow from  $X_{\leq m-1}$  to  $(X_{\geq n+1})_{\leq m-1}$ . All diagonal arrows are labeled  $-1$ . Dotted lines connect  $X$  to  $X_{\geq n+1}$  and  $\Sigma^n X_n$  to  $X_{\leq m-1}$ .)

which yields,

$$\begin{aligned} & [\Delta_n^X] + [\Sigma^m X_m \rightarrow X_{\geq n+1} \rightarrow (X_{\geq n+1})_{\leq m-1} \rightarrow \Sigma^m X_m] \\ &= [\Sigma^m X_m \rightarrow X \rightarrow X_{\leq m-1} \rightarrow \Sigma^m X_m] + [\Delta_n^{X_{\leq m-1}}][\Sigma^m X_m]. \end{aligned}$$

This forces,

$$\begin{aligned} & p[\Sigma^m X_m \rightarrow X \rightarrow X_{\leq m-1} \rightarrow \Sigma^m X_m] \\ &= [\Sigma^m X_m \rightarrow X_{\geq n+1} \rightarrow (X_{\geq n+1})_{\leq m-1} \rightarrow \Sigma^m X_m]. \end{aligned}$$

Moreover,

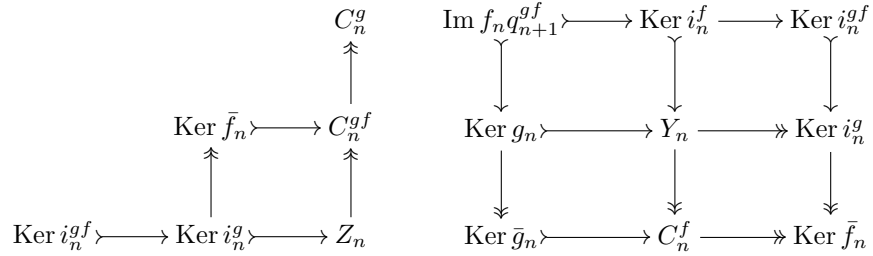
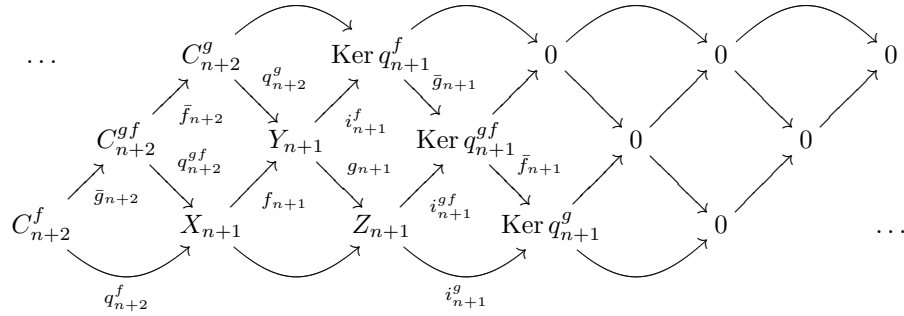
$$\begin{aligned} -p[X] + [X_{\leq m-1}] + p[\Sigma^m X_m] &= \partial p[\Sigma^m X_m \rightarrow X \rightarrow X_{\leq m-1} \rightarrow \Sigma^m X_m] \\ &= -[X_{\geq n+1}] + [(X_{\geq n+1})_{\leq m-1}] + p[\Sigma^m X_m], \end{aligned}$$

therefore we must define,

$$p[X] = [X_{\leq m-1}] - [(X_{\geq n+1})_{\leq m-1}] + [X_{\geq n+1}].$$

The rest of the proof goes now along the same lines as the first case. We leave the details to the reader.

The following list of diagrams contains three diagrams in  $\mathcal{A}$  which are to be regarded as diagrams in  $\text{Gr}^{[n+1, m]} \mathcal{A}$  concentrated in degree  $n+1$ .



$$\begin{array}{c}
 \text{Ker } f_n \longrightarrow \text{Ker } g_n f_n \longrightarrow X_n \longrightarrow \text{Ker } i_n^f \longrightarrow \text{Ker } i_n^{gf} \\
 \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\
 \text{Im } f_n q_{n+1}^{gf} \\
 \uparrow \\
 \text{C}_{\geq n+1}^g \longrightarrow \Sigma^{n+1} \text{Ker } \bar{g}_n \\
 \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\
 \text{C}_{\geq n+1}^f \longrightarrow \Sigma^{n+1} \text{Ker } g_n \xrightarrow{-1} \Sigma^{n+1} \text{Im } f_n q_{n+1}^{gf} \\
 \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\
 \text{C}^g \longrightarrow \tilde{\text{C}}^g \\
 \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\
 \text{C}^g \longrightarrow \Sigma^{n+1} \text{Im } f_n q_{n+1}^{gf} \xrightarrow{-1} \Sigma^{n+1} \text{Ker } \bar{g}_n
 \end{array}$$

$$\begin{array}{ccccccc}
 \tilde{\text{C}}^f & \longrightarrow & \tilde{\text{C}}^{gf} & \longrightarrow & \tilde{\text{C}}^g & \longrightarrow & \Sigma \tilde{\text{C}}^f \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{C}_{\geq n+1}^f & \longrightarrow & \text{C}_{\geq n+1}^{gf} & \longrightarrow & \tilde{\text{C}}^g & \longrightarrow & \Sigma \text{C}_{\geq n+1}^f \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma^{n+1} \text{Ker } f_n & \longrightarrow & \Sigma^{n+1} \text{Ker } g_n f_n & \longrightarrow & \Sigma^{n+1} \text{Im } f_n q_{n+1}^{gf} & \longrightarrow & \Sigma^{n+2} \text{Ker } f_n \\
 \downarrow & & \downarrow & & \downarrow \oplus & & \downarrow \\
 \Sigma \tilde{\text{C}}^f & \longrightarrow & \Sigma \tilde{\text{C}}^{gf} & \longrightarrow & \Sigma \tilde{\text{C}}^g & \longrightarrow & \Sigma^2 \tilde{\text{C}}^f
 \end{array}$$

□

#### 4.3. Low-dimensional $K$ -theory of a triangulated category with a $t$ -structure.

**Definition 4.3.1.** A  $t$ -structure on a triangulated category  $\mathcal{T}$  is a pair of full replete subcategories  $\mathcal{T}_{\geq 0}$  and  $\mathcal{T}_{\leq 0}$  satisfying the following axioms. If  $\mathcal{T}_{\geq n} = \Sigma^n \mathcal{T}_{\geq 0}$  and  $\mathcal{T}_{\leq n} = \Sigma^n \mathcal{T}_{\leq 0}$ , then:

- $\mathcal{T}_{\geq 1} \subset \mathcal{T}_{\geq 0}$  and  $\mathcal{T}_{\leq 0} \subset \mathcal{T}_{\leq 1}$ .
- $\mathcal{T}(X, Y) = 0$  if  $X \in \mathcal{T}_{\geq 1}$  and  $Y \in \mathcal{T}_{\leq 0}$ .
- For any object  $X$  in  $\mathcal{T}$  there is a distinguished triangle,

$$X_{\geq 1} \longrightarrow X \longrightarrow X_{\leq 0} \longrightarrow \Sigma X_{\geq 1},$$

with  $X_{\leq 0}$  in  $\mathcal{T}_{\leq 0}$  and  $X_{\geq 1}$  in  $\mathcal{T}_{\geq 1}$ .

This distinguished triangle turns out to be natural. Moreover, the full inclusion  $\mathcal{T}_{\leq n} \subset \mathcal{T}$  has a left adjoint,

$$\mathcal{T} \longrightarrow \mathcal{T}_{\leq n} : X \mapsto X_{\leq n},$$

and  $\mathcal{T}_{\geq n} \subset \mathcal{T}$  has a right adjoint,

$$\mathcal{T} \longrightarrow \mathcal{T}_{\geq n} : X \mapsto X_{\geq n},$$

such that the two first arrows of the previous natural distinguished triangle are the counit and the unit of two of these adjunctions, respectively. In general there are natural distinguished triangles

$$\Delta_n^X : X_{\geq n+1} \longrightarrow X \longrightarrow X_{\leq n} \longrightarrow \Sigma X_{\geq n+1}.$$



The *heart* of the  $t$ -structure is the abelian category  $\mathcal{A} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$ . We can associate with any  $t$ -structure a homology,

$$H_n: \mathcal{T} \longrightarrow \mathcal{A}, \quad \Sigma^n H_n X = (X_{\geq n})_{\leq n}, \quad n \in \mathbb{Z},$$

which satisfies properties similar to those of homology in the derived category of complexes in  $\mathcal{A}$ .

A  $t$ -structure is *bounded* if any object  $X$  has bounded homology, i.e.  $H_n X = 0$  for  $|n| \gg 0$ , and it is *non-degenerate* if any object  $X$  with trivial homology  $H_n X = 0$ ,  $n \in \mathbb{Z}$ , is trivial  $X = 0$ . An object  $X$  in  $\mathcal{T}$  is said to be  $n$ -*connected* if  $X_{\leq n} = 0$ .

We refer the reader to [BBD82, 1.3] and [GM03, IV.4] for further details about  $t$ -structures.

**Theorem 4.3.2.** *For any triangulated category  $\mathcal{T}$  with a bounded non-degenerate  $t$ -structure with heart  $\mathcal{A}$ , the morphisms*

$$\begin{array}{ccccc} & & \mathcal{D}_*({}^b\mathcal{T}) & & \\ & \nearrow & & \searrow & \\ \mathcal{D}_*(\mathcal{A}) & \longrightarrow & \mathcal{D}_*({}^d\mathcal{T}) & & \mathcal{D}_*(\mathrm{Gr}^b\mathcal{A}) \\ & \searrow & & \nearrow & \\ & & \mathcal{D}_*({}^v\mathcal{T}) & & \end{array}$$

are weak equivalences.

The fact that  $\mathcal{D}_*(\mathcal{A}) \rightarrow \mathcal{D}_*({}^b\mathcal{T})$  is a weak equivalence is equivalent to [Bre06, Theorem 5.2]. We here give a different proof. In this proof we use the following versions of Lemma 4.2.3 and Corollary 4.2.6.

**Lemma 4.3.3.** *Given  $(n-1)$ -connected objects  $X, Y, C^f$ , and a distinguished (resp. virtual) triangle in  $\mathcal{T}$ ,*

$$\Delta: X \xrightarrow{f} Y \xrightarrow{i} C^f \xrightarrow{q} \Sigma X,$$

the following formula holds in  $\mathcal{D}_*(\diamond\mathcal{T})$ ,  $\diamond = b, d$  (resp.  $v$ ),

$$\begin{aligned} & [X \rightarrow Y \rightarrow C^f \rightarrow \Sigma X] \\ &= [\Delta_n^Y] + [\Sigma^n \mathrm{Ker} H_n i \rightarrow \Sigma^n H_n Y \rightarrow \Sigma^n H_n C^f \rightarrow \Sigma^{n+1} \mathrm{Ker} i_n]^{[Y_{\geq n+1}]} \\ &\quad - [\Delta_n^{Y(i_n)}] + [X \rightarrow Y(i_n) \rightarrow C_{\geq n+1}^f \rightarrow \Sigma X] + [\Delta_n^{C^f}]^{[X]}. \end{aligned}$$

Here  $X \rightarrow Y(i_n) \rightarrow C_{\geq n+1}^f \rightarrow \Sigma X$  is the truncation of  $\Delta$  [Vak01c, Definition 1.12].

*Proof.* This lemma follows from the following (special, virtual) octahedra,

□

We write  $\Gamma_A^n = \Gamma_{\Sigma^n A}$  for any object  $A$  in  $\mathcal{A} \subset \mathcal{T}$ . Using this lemma, Proposition 4.1.32, and Remarks 4.1.33 and 4.1.35, we derive the following result.

**Corollary 4.3.4.** *The stable quadratic module  $\mathcal{D}_*(\diamond \mathcal{T})$ ,  $\diamond = b, d, v$ , is generated by:*

- (1)  $j[A]$ , in degree 0,  $A$  an object in  $\mathcal{A}$ ;
- (2)  $[\Gamma_A^n]$ , in degree 1,  $A$  an object in  $\mathcal{A}$ ,  $n \in \mathbb{Z}$ ;
- (3)  $[\Delta_X^n]$ , in degree 1,  $X$  an  $(n-1)$ -connected object in  $\mathcal{T}$ ,  $n \in \mathbb{Z}$ ;
- (4)  $j[A \twoheadrightarrow B \twoheadrightarrow C]$ , in degree 1,  $A \twoheadrightarrow B \twoheadrightarrow C$  a short exact sequence in  $\mathcal{A}$ .

Now we are ready to tackle the proof of Theorem 4.3.2.

*Proof of Theorem 4.3.2.* Theorem 4.2.1 gives a strong deformation retraction,

$$\alpha \circlearrowleft \mathcal{D}_*(\text{Gr}^b \mathcal{A}) \xleftarrow[j]{p} \mathcal{D}_*(\mathcal{A}), .$$

We here construct a strong deformation retraction,

$$\beta \circlearrowleft \mathcal{D}_*(^d \mathcal{T}) \xleftarrow[j^d]{p H_*} \mathcal{D}_*(\mathcal{A}).$$

We define the homotopy,

$$\beta: \mathcal{D}_0(^d \mathcal{T}) \longrightarrow \mathcal{D}_1(^d \mathcal{T}),$$

on generators by the following recursive formulas. Given an object  $A$  in  $\mathcal{A}$  and an  $(n-1)$ -connected object  $X$  in  $\mathcal{T}$ , then,

$$\begin{aligned} \beta j^d[A] &= 0, \\ -[\Gamma_A^n] &= \beta([\Sigma^{n+1} A])^{[\Sigma^n A]} + \beta[\Sigma^n A], \\ \beta[X] &= [\Delta_X^n] + \beta[X_{\geq n+1}]. \end{aligned}$$

We now check that  $p' = 1_{\mathcal{D}_*(^d \mathcal{T})} + \alpha$ , in the sense of Lemma 2.3.4, coincides with  $j^d p H_*$ . It is enough to check this on the generators given by Corollary 4.3.4. This is indeed equivalent to the three previous equations.

The morphism,

$$\mathcal{D}_*(^d \mathcal{T}) \longrightarrow \mathcal{D}_*(^b \mathcal{T}),$$

is the identity in dimension zero and surjective in dimension 1. Using Theorem 4.2.1 and the previous strong deformation retraction we obtain that it is an isomorphism in  $\pi_1$ , therefore it is an isomorphism of stable quadratic modules.

It is only left to check that

$$\mathcal{D}_*(^d\mathcal{T}) \longrightarrow \mathcal{D}_*(^v\mathcal{T}),$$

is an isomorphism. As in the previous case it is enough to check surjectivity in dimension 1, this follows from Corollary 4.3.4 for  $\mathcal{D}_*(^d\mathcal{T})$  and  $\mathcal{D}_*(^v\mathcal{T})$ , which gives common sets of generators for stable quadratic modules.  $\square$

**4.4. On additivity and localization for low-dimensional  $K$ -theory of triangulated categories.** We begin this section by proving an additivity theorem for low dimensional  $K$ -theories of (strongly) triangulated categories. In the statement we use the various notions of coherence introduced in Definition 4.1.13. In order not to overload the notation, in this section we will use the symbol  $\blacklozenge$  instead of  $b, d, s, v$ .

**Theorem 4.4.1** (Additivity). *Let  $F, G, H: \mathcal{T} \rightarrow \mathcal{T}'$  be exact functors between (strongly) triangulated categories. Suppose we have a natural distinguished (or virtual) triangle,*

$$(4.4.2) \quad F(X) \xrightarrow{f(X)} G(X) \xrightarrow{i(X)} H(X) \xrightarrow{q(X)} \Sigma F(X),$$

such that for any distinguished (or virtual) triangle  $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X$ , the diagram

$$(4.4.3) \quad \begin{array}{ccccccc} F(X) & \xrightarrow{f(X)} & G(X) & \xrightarrow{i(X)} & H(X) & \xrightarrow{q(X)} & \Sigma F(X) \\ F(f) \downarrow & & \downarrow G(f) & & \downarrow H(f) & & \downarrow \Sigma F(f) \\ F(Y) & \xrightarrow{f(Y)} & G(Y) & \xrightarrow{i(Y)} & H(Y) & \xrightarrow{q(Y)} & \Sigma F(Y) \\ F(i) \downarrow & & \downarrow G(i) & & \downarrow H(q) & & \downarrow \Sigma F(q) \\ F(Z) & \xrightarrow{f(Z)} & G(Z) & \xrightarrow{i(Z)} & H(Z) & \xrightarrow{q(Z)} & \Sigma F(Z) \\ F(q) \downarrow & & \downarrow G(q) & & \downarrow H(q) \oplus \ominus & & \downarrow \Sigma F(q) \\ \Sigma F(X) & \xrightarrow{\Sigma f(X)} & \Sigma G(X) & \xrightarrow{\Sigma i(X)} & \Sigma H(X) & \xrightarrow{\Sigma q(X)} & \Sigma^2 F(X) \end{array}$$

is  $\blacklozenge$ -coherent. Then the following induced homomorphisms coincide,

$$\begin{aligned} K_0(\blacklozenge F) + K_0(\blacklozenge H) &= K_0(\blacklozenge G): K_0(\blacklozenge \mathcal{T}) \longrightarrow K_0(\blacklozenge \mathcal{T}'), \\ K_1(\blacklozenge F) + K_1(\blacklozenge H) &= K_1(\blacklozenge G): K_1(\blacklozenge \mathcal{T}) \longrightarrow K_1(\blacklozenge \mathcal{T}'). \end{aligned}$$

*Proof.* For  $K_0$  the result follows from the following equation,

$$\partial[F(X) \xrightarrow{f(X)} G(X) \xrightarrow{i(X)} H(X) \xrightarrow{q(X)} \Sigma F(X)] = -[G(X)] + [H(X)] + [F(X)].$$

Any element in  $K_1$  is represented by a pair of distinguished (resp. virtual) triangles,

$$[X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{f'} \end{array} Y \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{i'} \end{array} Z \begin{array}{c} \xrightarrow{q} \\ \xrightarrow{q'} \end{array} \Sigma X],$$

see Corollary 4.1.7. If we apply Corollary 4.1.17 to the induced diagrams (4.4.3) we obtain the following equation in  $K_1$ ,

$$\begin{aligned}
0 &= [F(X) \begin{smallmatrix} f(X) \\ \rightrightarrows \\ f(X) \end{smallmatrix} G(X) \begin{smallmatrix} i(X) \\ \rightrightarrows \\ i(X) \end{smallmatrix} H(X) \begin{smallmatrix} q(X) \\ \rightrightarrows \\ q(X) \end{smallmatrix} \Sigma F(X)] - [F(Y) \begin{smallmatrix} f(Y) \\ \rightrightarrows \\ f(Y) \end{smallmatrix} G(Y) \begin{smallmatrix} i(Y) \\ \rightrightarrows \\ i(Y) \end{smallmatrix} H(Y) \begin{smallmatrix} q(Y) \\ \rightrightarrows \\ q(Y) \end{smallmatrix} \Sigma F(Y)] \\
&\quad + [F(Z) \begin{smallmatrix} f(Z) \\ \rightrightarrows \\ f(Z) \end{smallmatrix} G(Z) \begin{smallmatrix} i(Z) \\ \rightrightarrows \\ i(Z) \end{smallmatrix} H(Z) \begin{smallmatrix} q(Z) \\ \rightrightarrows \\ q(Z) \end{smallmatrix} \Sigma F(Z)] \\
&= [F(X) \begin{smallmatrix} F(f) \\ \rightrightarrows \\ G(f') \end{smallmatrix} F(Y) \begin{smallmatrix} F(i) \\ \rightrightarrows \\ F(i') \end{smallmatrix} F(Z) \begin{smallmatrix} F(q) \\ \rightrightarrows \\ F(q') \end{smallmatrix} \Sigma F(X)] - [G(X) \begin{smallmatrix} G(f) \\ \rightrightarrows \\ G(f') \end{smallmatrix} G(Y) \begin{smallmatrix} G(i) \\ \rightrightarrows \\ G(i') \end{smallmatrix} G(Z) \begin{smallmatrix} G(q) \\ \rightrightarrows \\ G(q') \end{smallmatrix} \Sigma G(X)] \\
&\quad + [H(X) \begin{smallmatrix} H(f) \\ \rightrightarrows \\ G(f') \end{smallmatrix} H(Y) \begin{smallmatrix} H(i) \\ \rightrightarrows \\ H(i') \end{smallmatrix} H(Z) \begin{smallmatrix} H(q) \\ \rightrightarrows \\ H(q') \end{smallmatrix} \Sigma H(X)],
\end{aligned}$$

hence we are done.  $\square$

Additivity for derivator  $K$ -theory has been fully proved by Cisinski and Neeman [CN08]. Notice that this additivity theorem does not contradict [Sch02, Remark 2.3].

*Remark 4.4.4.* The hypothesis is satisfied if the natural distinguished triangle has models, e.g. if  $\mathcal{T}$  and  $\mathcal{T}'$  are the categories of perfect complexes over two rings  $R$  and  $R'$ , and the exact functors  $F, G, H$  are given by the derived tensor product with perfect complexes of  $R'$ - $R$ -bimodules  $F_*, G_*, H_*$  fitting into an exact triangle  $F_* \rightarrow G_* \rightarrow H_* \rightarrow \Sigma F_*$ .

Now we concentrate on localization exact sequences. We exclude the strongly triangulated case  $\diamond = s$ , for which there is no notion of Verdier quotient. We first define a relative low-dimensional  $K$ -theory of triangulated categories.

**Definition 4.4.5.** Given a triangulated category  $\mathcal{T}$  and a thick subcategory  $\mathcal{T}' \subset \mathcal{T}$  we define the stable quadratic module  $\mathcal{D}_*(\diamond \mathcal{T}, \mathcal{T}')$  as the stable quadratic module obtained from  $\mathcal{D}_*(\diamond \mathcal{T})$  by adding generators:

- $\llbracket X \rrbracket$ , in dimension 1, for any object  $X$  in  $\mathcal{T}'$ ;

and relations:

- $\partial \llbracket X \rrbracket = [X]$ ;
- $\llbracket X \rightarrow Y \rightarrow Z \rightarrow \Sigma X \rrbracket = -\llbracket Y \rrbracket + \llbracket Z \rrbracket + \llbracket X \rrbracket$ , for  $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$  any distinguished (resp. virtual) triangle in  $\mathcal{T}'$ .

We define the *relative  $K$ -theory* of  $\mathcal{T}' \subset \mathcal{T}$  in dimensions 1 and 2 as,

$$K_0(\diamond \mathcal{T}, \mathcal{T}') = \pi_0 \mathcal{D}_*(\diamond \mathcal{T}, \mathcal{T}'), \quad K_1(\diamond \mathcal{T}, \mathcal{T}') = \pi_1 \mathcal{D}_*(\diamond \mathcal{T}, \mathcal{T}').$$

We can also consider the  $K$ -theory of the Verdier quotient  $\mathcal{T}/\mathcal{T}'$ . The relation between them is determined by the following result.

**Proposition 4.4.6.** *There is a natural morphism of stable quadratic modules  $\varphi: \mathcal{D}_*(\diamond \mathcal{T}, \mathcal{T}') \rightarrow \mathcal{D}_*(\diamond \mathcal{T}/\mathcal{T}')$  such that  $\varphi_0$  is an isomorphism and, for  $\diamond = b, d$ ,  $\varphi_1$  is surjective, in particular it induces an isomorphism on  $\pi_0$  and an epimorphism on  $\pi_1$ ,*

$$K_0(\diamond \mathcal{T}, \mathcal{T}') \cong K_0(\diamond \mathcal{T}/\mathcal{T}'), \quad K_1(\diamond \mathcal{T}, \mathcal{T}') \twoheadrightarrow K_1(\diamond \mathcal{T}/\mathcal{T}').$$

*Proof.* Recall from [Nee01, 2.1] the explicit construction of  $\mathcal{T}/\mathcal{T}'$ . The composite

$$\mathcal{D}_*(\blacklozenge \mathcal{T}') \rightarrow \mathcal{D}_*(\blacklozenge \mathcal{T}) \rightarrow \mathcal{D}_*(\blacklozenge \mathcal{T}/\mathcal{T}')$$

is null-homotopic. An explicit homotopy is given by,

$$\alpha[X] = [X \xrightarrow{\sim} 0],$$

and the morphism  $\varphi$  is defined by Proposition 4.4.11 below,  $\varphi_0$  is the identity and  $\varphi_1$  satisfies,

$$\varphi_1[X \rightarrow Y \rightarrow Z \rightarrow \Sigma X] = [X \rightarrow Y \rightarrow Z \rightarrow \Sigma X], \quad \varphi_1[[X]] = [X \xrightarrow{\sim} 0].$$

We have to check, for  $\blacklozenge = d$  and the empty symbol, that  $[X' \rightarrow Y' \rightarrow Z' \rightarrow \Sigma X'] \in \mathcal{D}_1(\blacklozenge \mathcal{T}/\mathcal{T}')$  is in the image of  $\varphi_1$  for any distinguished triangle  $X' \rightarrow Y' \rightarrow Z' \rightarrow \Sigma X'$  in  $\mathcal{T}/\mathcal{T}'$ . Any such triangle is isomorphic in  $\mathcal{T}/\mathcal{T}'$  to a distinguished triangle in the image of the natural functor  $p: \mathcal{T} \rightarrow \mathcal{T}/\mathcal{T}'$ , which is the identity on objects, therefore, by (R7), it is enough to check that  $[Y \xrightarrow{\sim} Y'] \in \mathcal{D}_1(\blacklozenge \mathcal{T}/\mathcal{T}')$  is in the image of  $\varphi_1$  for any isomorphism in  $\mathcal{T}/\mathcal{T}'$ . Any such isomorphism is of the form  $p(g)p(f)^{-1}$  for

$$Y \xleftarrow{f} X \xrightarrow{g} Y'$$

morphisms in  $\mathcal{T}$  with mapping cone in  $\mathcal{T}'$ . By (R6),  $[X \rightarrow X'] = [p(g)] - [p(f)]$ , hence, by analogy, it is enough to show that  $[p(f)]$  is in the image of  $\varphi_1$ . For this purpose, complete  $f$  to a distinguished triangle in  $\mathcal{T}$ ,

$$X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X$$

and consider the following isomorphism of distinguished triangles in  $\mathcal{T}/\mathcal{T}'$ ,

$$\begin{array}{ccccccc} X & \xrightarrow{p(f)} & Y & \xrightarrow{p(i)} & Z & \xrightarrow{p(q)} & \Sigma X \\ p(f) \downarrow & & \downarrow 1 & & \downarrow & & \downarrow \Sigma f \\ Y & \xrightarrow{1} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma X \end{array}$$

If we apply (R7) we obtain,

$$[Z \xrightarrow{\sim} 0]^{[X]} = [p(f)] + [X \xrightarrow{p(f)} Y \xrightarrow{p(i)} Z \xrightarrow{p(q)} \Sigma X],$$

and then,

$$\begin{aligned} [p(f)] &= [Z \xrightarrow{\sim} 0]^{[X]} - [X \xrightarrow{p(f)} Y \xrightarrow{p(i)} Z \xrightarrow{p(q)} \Sigma X] \\ &= \varphi_1([Z]^{[X]} - [X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} \Sigma X]). \end{aligned}$$

□

*Remark 4.4.7.* We seriously doubt that relative  $K_1$  is in general isomorphic to  $K_1(\blacklozenge \mathcal{T}/\mathcal{T}')$ , nevertheless we do not have any example of such behaviour. If one tries to prove that they are isomorphic for  $\blacklozenge = d$  or the empty symbol, one faces the following problem: a map of distinguished triangles in  $\mathcal{T}$ ,

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

such that the mapping cones of the vertical arrows are in  $\mathcal{T}'$ , need not extend to a  $3 \times 3$  diagram. The axioms of triangulated categories do not seem to prevent us from this situation, but we have not found a concrete example.

In the virtual case we cannot even prove that the comparison homomorphism in  $K_1$  is surjective since we do not know whether any virtual triangle in  $\mathcal{T}/\mathcal{T}'$  is isomorphic to the image of a virtual triangle in  $\mathcal{T}$ .

Relative  $K_1$  is better than  $K_1$  of the quotient in order to obtain exact sequences.

**Theorem 4.4.8.** *Given a triangulated category  $\mathcal{T}$  and a thick subcategory  $\mathcal{T}' \subset \mathcal{T}$  there is a natural exact sequence,  $\diamond = b, d, v$ ,*

$$K_1(\diamond \mathcal{T}') \rightarrow K_1(\diamond \mathcal{T}) \rightarrow K_1(\diamond \mathcal{T}, \mathcal{T}') \xrightarrow{\delta} K_0(\diamond \mathcal{T}') \rightarrow K_0(\diamond \mathcal{T}) \rightarrow K_0(\diamond \mathcal{T}, \mathcal{T}') \rightarrow 0.$$

This theorem follows from Lemma 4.4.10 and Proposition 4.4.12 below.

**Definition 4.4.9.** The *cofiber* of a stable quadratic module morphism  $f: C_* \rightarrow D_*$  is the stable quadratic module  $\text{cof}(f)$  defined as follows:  $\text{cof}(f)_0 = D_0$ , in order to define  $\text{cof}(f)_1$  we consider the following push-out in the category of groups of nilpotency class 2,

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ \partial \downarrow & \text{push} & \downarrow \\ C_0 & \longrightarrow & P \end{array}$$

We define  $\text{cof}(f)_1$  as the quotient of  $P$  by the following relations,

$$\langle f_0(c_0), \partial(d_1) \rangle = [d_1, c_0], \quad c_0 \in C_0, d_1 \in D_1.$$

The homomorphism  $\partial: \text{cof}(f)_1 \rightarrow \text{cof}(f)_0$  is defined by  $f_0$  and  $\partial_{D_*}$ , and the bracket is defined by the bracket in  $D_*$  composed with  $D_1 \rightarrow P \rightarrow \text{cof}(f)_1$ .

**Lemma 4.4.10.** *Given a triangulated category  $\mathcal{T}$  and a thick subcategory  $\mathcal{T}' \subset \mathcal{T}$ , the stable quadratic module  $\mathcal{D}_*(\mathcal{T}, \mathcal{T}')$  is the cofiber of the morphism  $\mathcal{D}_*(\mathcal{T}') \rightarrow \mathcal{D}_*(\mathcal{T})$  induced by the inclusion  $\mathcal{T}' \subset \mathcal{T}$ .*

This is just the way in which we have found the presentations of  $\mathcal{D}_*(\diamond \mathcal{T}, \mathcal{T}')$ .

There is a diagram in the 2-category of stable quadratic modules,

$$\begin{array}{ccccc} & & 0 & & \\ & & \curvearrowright & & \\ C_* & \xrightarrow{f} & D_* & \xrightarrow{i} & \text{cof}(f) \\ & & \uparrow \alpha & & \\ & & 0 & & \end{array}$$

defined as follows:  $i_0$  is the identity and  $i_1$  is the composite  $D_1 \rightarrow P \rightarrow \text{cof}(f)_1$ . Moreover,  $\alpha$  is  $C_0 \rightarrow P \rightarrow \text{cof}(f)_1$ . This diagram satisfies the following universal property.

**Proposition 4.4.11.** *For any diagram in the 2-category of stable quadratic modules,*

$$\begin{array}{ccccc} & & 0 & & \\ & & \curvearrowright & & \\ C_* & \xrightarrow{f} & D_* & \xrightarrow{j} & E_* \\ & & \uparrow \beta & & \\ & & 0 & & \end{array}$$

*there exists a unique morphism  $l: \text{cof}(f) \rightarrow E_*$  such that  $j = li$  and  $\beta = l\alpha$ .*

*Proof.* The universal property of the push-out yields a unique homomorphism  $l'_1$ ,

$$\begin{array}{ccc}
 C_1 & \xrightarrow{f_1} & D_1 \\
 \partial \downarrow & \text{push} & \downarrow \\
 C_0 & \longrightarrow & P \\
 & \searrow \beta & \downarrow j_1 \\
 & & E_1
 \end{array}$$

$l'_1$  (arrow from  $P$  to  $E_1$ )

The homomorphism  $l'_1$  factors through a unique homomorphism  $l_1: \text{cof}(f)_1 \rightarrow E_1$ . Indeed, given  $c_0 \in C_0$ ,  $\partial\beta(c_0) = j_0 f_0(c_0)$ , hence, given  $d_1 \in D_1$ ,

$$j_1 \langle f_0(c_0), \partial(d_1) \rangle = \langle j_0 f_0(c_0), j_0 \partial(d_1) \rangle = \langle \partial\beta(c_0), \partial j_1(d_1) \rangle = [j_1(d_1), \beta(c_0)].$$

The homomorphisms  $l_0 = 1_{D_0}$  and  $l_1$  define the morphism  $l$ .  $\square$

Cofibers satisfy the following fundamental property.

**Proposition 4.4.12.** *For any morphism of stable quadratic modules  $f: C_* \rightarrow D_*$  there is an exact sequence,*

$$\pi_1 C_* \xrightarrow{\pi_1 f} \pi_1 D_* \xrightarrow{\pi_1 i} \pi_1 \text{cof}(f) \xrightarrow{\delta} \pi_0 C_* \xrightarrow{\pi_0 f} \pi_0 D_* \xrightarrow{\pi_0 i} \pi_0 \text{cof}(f) \rightarrow 0.$$

*Proof.* The universal property of the push-out yields a unique homomorphism  $\delta''$ ,

$$\begin{array}{ccc}
 C_1 & \xrightarrow{f_1} & D_1 \\
 \partial \downarrow & \text{push} & \downarrow \\
 C_0 & \longrightarrow & P \\
 & \searrow \text{projection} & \downarrow 0 \\
 & & \pi_0 C_*
 \end{array}$$

$\delta''$  (arrow from  $P$  to  $\pi_0 C_*$ )

This morphism factors uniquely through  $\delta': \text{cof}(f)_1 \rightarrow \pi_0 C_*$ . We define  $\delta$  as the composite,

$$\delta: \pi_1 \text{cof}(f) \subset \text{cof}(f)_1 \xrightarrow{\delta'} \pi_0 C_*.$$

Obviously  $\delta(\pi_1 i) = 0$ . Any  $x \in \text{cof}(f)_1$  can be written as,

$$x = c_0 + d_1 + \sum_i [d_1^i, c_0^i], \quad c_0, c_0^i \in C_0, \quad d_1, d_1^i \in D_1.$$

If  $x \in \pi_1 \text{cof}(f)$ ,  $\delta(x) \in \pi_0 C_*$  is represented by  $c_0$ , so  $\delta(x) = 0$  if and only if  $c_0 = \partial(c_1)$  for some  $c_1 \in C_1$ . Let,

$$y = f_1(c_1) + d_1 + \sum_i \langle f_0(c_0^i), d_1^i \rangle \in D_1.$$

We have

$$\partial(y) = \partial(x) = 0 \in D_0 = \text{cof}(f)_0,$$

and  $i(y) = x$ , hence  $x \in \pi_1 \text{cof}(f)$  is the image of  $y \in \pi_1 D_*$ . Exactness at  $\pi_0 C_*$  and  $\pi_1 D_*$  is left as an exercise.  $\square$

**4.5. Derived and non-derived determinant functors on a Waldhausen category.** Grothendieck asked in a letter to Knudsen whether determinant functors on an exact category  $\mathcal{E}$  coincide essentially with determinant functors in the bounded derived category  $D^b(\mathcal{E})$  regarded as triangulated category equipped with a ‘category of true triangles’ [Knu02a, Appendix B]. We interpret this ‘category of true triangles’ to be the bounded derived category of the exact category  $S_2(\mathcal{E})$  of short exact sequences in  $\mathcal{E}$ , which coincides with the homotopy category of the Waldhausen category  $S_2C^b(\mathcal{E})$  of short exact sequences of bounded complexes in  $\mathcal{E}$ . With this interpretation, determinant functors in the triangulated category equipped with a ‘category of true triangles’ are the derived determinant functors in  $C^b(\mathcal{E})$ , or equivalently, the determinant functors on the triangulated derivator  $\mathbb{D}_{\mathcal{E}}$  constructed by Keller in [Kel07].

The Waldhausen category  $C^b(\mathcal{E})$  has cylinders and a saturated class of weak equivalences, therefore the two following result answer Grothendieck’s question positively.

**Corollary 4.5.1.** *If we regard  $\mathcal{E}$  as the full subcategory of complexes in  $C^b(\mathcal{E})$  concentrated in degree 0, then any determinant functor on  $\mathcal{E}$  factors through a determinant functor in  $C^b(\mathcal{E})$  in an essentially unique way.*

This follows from Theorems 3.2.4 and 3.5.3 and from the Gillet–Waldhausen theorem [Cis02]. A direct proof of this result can be found in [Knu02a, Knu02b].

**Corollary 4.5.2.** *Let  $\mathcal{W}$  be a Waldhausen category with cylinders and a saturated class of weak equivalences, i.e. weak equivalences are exactly those maps in  $\mathcal{W}$  which become invertible in  $\mathrm{Ho}\mathcal{W}$ . Then any determinant functor on  $\mathcal{W}$  factors through a derived determinant functor in an essentially unique way.*

This follows from Theorems 3.2.4 and 3.5.3 and [Mur08, Theorem 6.1].

*Remark 4.5.3.* If the class of weak equivalences in  $\mathcal{W}$  is not saturated then Weiss’s Whitehead group  $\mathrm{Wh}(\mathcal{W})$  may not vanish [Wei99], and in this case the universal determinant functor  $\det: \mathcal{W} \rightarrow \mathcal{D}_*(\mathcal{W})$  need not factor through a derived determinant functor, compare [Mur08, Remark 6.3].

**4.6. A counterexample to two conjectures by Maltiniotis.** Here we disprove the conjectures mentioned in the introduction giving a counterexample which goes back to Deligne, Vaknin and Breuning [Vak01a, Bre08].

Let  $\mathcal{E}$  be the category of finitely generated free modules over the ring of dual numbers  $R = k[\varepsilon]/(\varepsilon^2)$  over a field  $k$ . We regard  $D^b(\mathcal{E})$  as a strongly triangulated category. It is well known that  $K_1(\mathcal{E}) \cong K_1(R) \cong R^\times$  is the group of units. There is an isomorphism,

$$\begin{aligned} k \times k^\times &\xrightarrow{\cong} R^\times, \\ (x, u) &\mapsto u(1 + x\varepsilon). \end{aligned}$$

Given  $x \in k^\times$ , the element  $1 + x\varepsilon \in R^\times$  corresponds to

$$[1 + x\varepsilon: R \xrightarrow{\sim} R] \in K_1(\mathcal{E}).$$



This element is in the kernel of  $K_1(\mathcal{E}) \rightarrow K_1({}^sD^b(\mathcal{E}))$  since we have an automorphism of distinguished triangles, see (R7),

$$\begin{array}{ccccccc} R & \xrightarrow{\varepsilon} & R & \longrightarrow & C & \longrightarrow & \Sigma R \\ \downarrow 1 & & \downarrow 1+x\varepsilon & & \downarrow 1 & & \downarrow 1 \\ R & \xrightarrow{\varepsilon} & R & \longrightarrow & C & \longrightarrow & \Sigma R \end{array}$$

Indeed,  $C$  is the complex  $\cdots \rightarrow 0 \rightarrow R \xrightarrow{1+x\varepsilon} R \rightarrow 0 \rightarrow \cdots$ , and the square in the middle commutes in the derived category since we have a homotopy defined by the homomorphism  $R \rightarrow R: 1 \mapsto x\varepsilon$ .

The same example shows that, if we only regard  $D^b(\mathcal{E})$  as a triangulated category, then the comparison homomorphisms,

$$K_1(\mathcal{E}) \longrightarrow K_1({}^dD^b(\mathcal{E})), \quad K_1(\mathcal{E}) \longrightarrow K_1({}^vD^b(\mathcal{E})), \quad K_1(\mathcal{E}) \longrightarrow K_1({}^bD^b(\mathcal{E})),$$

are not isomorphisms.

Moreover, if  $\mathbb{D}_{\mathcal{E}}$  is the triangulated derivator associated to  $\mathcal{E}$ , the comparison homomorphism

$$K_1(\mathbb{D}_{\mathcal{E}}) \longrightarrow K_1({}^s\mathbb{D}_{\mathcal{E}}(*))$$

is neither an isomorphism because the composite,

$$K_1(\mathcal{E}) \xrightarrow{\cong} K_1(\mathbb{D}_{\mathcal{E}}) \longrightarrow K_1({}^s\mathbb{D}_{\mathcal{E}}(*)) \cong K_1({}^sD^b(\mathcal{E}))$$

is the previous comparison homomorphism between Quillen's  $K$ -theory and Maltzin's  $K$ -theory of a strongly triangulated category, which is not injective. The first arrow is the natural comparison homomorphism, which is an isomorphism by [Mur08, Theorem 1].

We can actually compute Neeman's  $K_1({}^dD^b(\mathcal{E}))$  and Breuning's  $K_1({}^bD^b(\mathcal{E}))$ . This improves and generalizes some computations in [Bre08].

**Proposition 4.6.1.** *Let  $k$  be a field. For  $\diamond = b, d$ , the stable quadratic modules  $\mathcal{D}_*({}^\diamond D^b(k))$  and  $\mathcal{D}_*({}^\diamond D^b(k[\varepsilon]/(\varepsilon^2)))$  are weakly equivalent to*

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\langle \cdot, \cdot \rangle} k^\times \xrightarrow{\partial} \mathbb{Z}, \quad \langle m, n \rangle = (-1)^{mn}, \quad \partial = 0.$$

Moreover, the comparison homomorphism,

$$k \times k^\times \cong K_1(k[\varepsilon]/(\varepsilon^2)) \longrightarrow K_1({}^\diamond D^b(k[\varepsilon]/(\varepsilon^2))) \cong k^\times,$$

is the natural projection.

*Proof.* The proof is exactly the same in both cases. For  $\mathcal{D}_*({}^dD^b(k))$  the result follows from Theorem 4.3.2 and well-known facts about the low-dimensional  $K$ -theory of a field. We have already seen that the subgroup  $k \subset k \times k^\times$  is in the kernel of the comparison homomorphism in the statement, which is known to be surjective, therefore it induces an epimorphism,

$$k^\times \twoheadrightarrow K_1({}^dD^b(k[\varepsilon]/(\varepsilon^2))).$$

This epimorphism is also injective since the following composite is the identity,

$$k^\times \twoheadrightarrow K_1({}^dD^b(k[\varepsilon]/(\varepsilon^2))) \longrightarrow K_1({}^dD^b(k)) \cong k^\times.$$

Here the second arrow is induced by the change of coefficients along the  $k$ -algebra projection  $k[\varepsilon]/(\varepsilon^2) \rightarrow k: \varepsilon \mapsto 0$ . This finishes the proof of the proposition.  $\square$

**4.7. The  $K$ -theory of some unusual triangulated categories.** Let  $R$  be a commutative local ring with maximal ideal  $(\varepsilon) \neq 0$  such that  $\varepsilon^2 = 0$  and with residue field  $k = R/(\varepsilon)$  of characteristic 2. This ring is quasi-Frobenius. Notice that either  $\varepsilon = 2$  or  $R = k[\varepsilon]/(\varepsilon^2)$ . Recall from [MSS07] that the category  $\mathcal{F}(R)$  of finitely generated free  $R$ -modules admits a unique structure of triangulated category such that the suspension functor  $\Sigma = 1_{\mathcal{F}(R)}$  is the identity and the following triangle is distinguished,

$$R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R.$$

This triangulated category does not admit models if  $\varepsilon = 2$ . Otherwise it is the compact derived category of a certain differential graded algebra, in particular it can be described as the homotopy category of a Waldhausen category.

**Theorem 4.7.1.** *Neeman's  $K$ -theories of the triangulated category  $\mathcal{F}(R)$  satisfy:*

$$K_0({}^d\mathcal{F}(R)) \cong K_0({}^v\mathcal{F}(R)) \cong 0, \quad K_1({}^d\mathcal{F}(R)) \cong 0.$$

Moreover, there is a surjective homomorphism  $K_1({}^v\mathcal{F}(R)) \rightarrow k^\times / (k^\times)^2$ .

Notice that  $k^\times / (k^\times)^2 \neq 0$  as long as  $k$  is non-perfect, thus we obtain examples of triangulated categories  $\mathcal{T}$  such that  $K_1({}^d\mathcal{T})$  is not isomorphic to  $K_1({}^v\mathcal{T})$ .

An acyclic 3-periodic complex in  $\mathcal{F}(R)$ ,

$$T: \quad X_0 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0,$$

fits into a natural short exact sequence of complexes,

$$\varepsilon \cdot T \rightarrow T \xrightarrow{\varepsilon} \varepsilon \cdot T,$$

which induces isomorphisms in homology,

$$\sigma_n^T: H_{n+1}(\varepsilon \cdot T) \longrightarrow H_n(\varepsilon \cdot T), \quad n \in \mathbb{Z}/3.$$

A distinguished triangle in  $\mathcal{F}(R)$  is the same as an acyclic 3-periodic chain complex  $T$  such that the automorphism

$$\rho_n^T = \sigma_n^T \sigma_{n+1}^T \sigma_{n+2}^T: H_n(\varepsilon \cdot T) \longrightarrow H_n(\varepsilon \cdot T)$$

is the identity for some, and hence all,  $n \in \mathbb{Z}/3$ , see [MSS07, Remark 7]. Notice that all these isomorphisms are natural in  $T$ .

**Definition 4.7.2.** We define the *determinant* of an acyclic 3-periodic complex  $T$  in  $\mathcal{F}(R)$  as  $\det(T) = \det(\rho_n^T) \in k^\times$ , which is independent of  $n \in \mathbb{Z}/3$ .

The determinant is clearly invariant under shifts of the complex  $T$  and isomorphisms. Notice that the determinant of an exact triangle is  $1 \in k^\times$ .

**Lemma 4.7.3.** *Given a short exact sequence of acyclic 3-periodic complexes  $T' \rightarrow T \rightarrow T''$  in  $\mathcal{F}(R)$  we have  $\det(T) = \det(T') \det(T'') \pmod{(k^\times)^2}$ .*

*Proof.* The short exact sequence in the statement splits levelwise, so we have a short exact sequence  $\varepsilon \cdot T' \rightarrow \varepsilon \cdot T \rightarrow \varepsilon \cdot T''$  which induces a long exact sequence in homology,

$$\cdots \rightarrow H_n(\varepsilon \cdot T') \longrightarrow H_n(\varepsilon \cdot T) \longrightarrow H_n(\varepsilon \cdot T'') \longrightarrow H_{n-1}(\varepsilon \cdot T') \rightarrow \cdots .$$

Moreover, the following diagram of short exact sequences of complexes,

$$\begin{array}{ccccc}
\varepsilon \cdot T' & \longrightarrow & \varepsilon \cdot T & \twoheadrightarrow & \varepsilon \cdot T'' \\
\downarrow & & \downarrow & & \downarrow \\
T' & \longrightarrow & T & \twoheadrightarrow & T'' \\
\downarrow & & \downarrow & & \downarrow \\
\varepsilon \cdot T' & \longrightarrow & \varepsilon \cdot T & \twoheadrightarrow & \varepsilon \cdot T''
\end{array}$$

shows that the following diagram is commutative since we are in characteristic 2,

$$\begin{array}{ccccccc}
H_n(\varepsilon \cdot T') & \longrightarrow & H_n(\varepsilon \cdot T) & \longrightarrow & H_n(\varepsilon \cdot T'') & \longrightarrow & H_{n-1}(\varepsilon \cdot T') \\
\downarrow \sigma_n^{T'} & & \downarrow \sigma_n^T & & \downarrow \sigma_n^{T''} & & \downarrow \sigma_{n-1}^{T'} \\
H_{n-1}(\varepsilon \cdot T') & \longrightarrow & H_{n-1}(\varepsilon \cdot T) & \longrightarrow & H_{n-1}(\varepsilon \cdot T'') & \longrightarrow & H_{n-2}(\varepsilon \cdot T')
\end{array}$$

Therefore we have an automorphism of a 9-periodic long exact sequences,  $n \in \mathbb{Z}/3$ ,

$$\begin{array}{ccccccc}
H_n(\varepsilon \cdot T') & \xrightarrow{\phi} & H_n(\varepsilon \cdot T) & \longrightarrow & H_n(\varepsilon \cdot T'') & \longrightarrow & H_{n-1}(\varepsilon \cdot T') \\
\downarrow \rho_n^{T'} & & \downarrow \rho_n^T & & \downarrow \rho_n^{T''} & & \downarrow \rho_{n-1}^{T'} \\
H_n(\varepsilon \cdot T') & \xrightarrow{\phi} & H_n(\varepsilon \cdot T) & \longrightarrow & H_n(\varepsilon \cdot T'') & \longrightarrow & H_{n-1}(\varepsilon \cdot T')
\end{array}$$

Using the multiplicative property of determinants with respect to automorphisms of short exact sequences, if  $\rho' : \text{Ker } \phi \cong \text{Ker } \phi$  is the automorphism induced by  $\rho_n^{T'}$  we get

$$\begin{aligned}
\det(\rho')^2 &= \det(\rho_n^{T'}) \det(\rho_n^T)^{-1} \det(\rho_n^{T''}) \\
&\quad \det(\rho_{n-1}^{T'})^{-1} \det(\rho_{n-1}^T) \det(\rho_{n-1}^{T''})^{-1} \\
&\quad \det(\rho_{n-2}^{T'}) \det(\rho_{n-2}^T)^{-1} \det(\rho_{n-2}^{T''}).
\end{aligned}$$

Since these determinants are independent of  $n \in \mathbb{Z}/3$  we deduce, as desired, that

$$\det(\rho')^2 \det(\rho_n^T) = \det(\rho_n^{T'}) \det(\rho_n^{T''}).$$

□

**Lemma 4.7.4.** *A virtual triangle is the same as an acyclic 3-periodic complex  $T$ . Moreover,  $T$  is the direct sum of a contractible triangle and*

$$R^d \xrightarrow{\varepsilon} R^d \xrightarrow{\varepsilon} R^d \xrightarrow{\varepsilon \cdot \bar{\rho}} R^d,$$

where  $d = \dim_k H_n(\varepsilon \cdot T)$  and  $\bar{\rho}$  is any automorphism of  $R^d$  with  $\bar{\rho} \otimes_R k = \rho_n^T$  for some basis of  $H_n(\varepsilon \cdot T)$ ,  $n \in \mathbb{Z}/3$ .

*Proof.* It is clear that a virtual triangle is an acyclic 3-periodic complex. Consider an acyclic 3-periodic complex in  $\mathcal{F}(R)$ ,

$$T: \quad X_0 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0.$$

Let  $X'_n \subset \text{Ker } d_n \subset X_n$  be an injective envelope of  $\varepsilon \cdot \text{Ker } d_n$ . Since  $T$  is acyclic, we can factor this inclusion as

$$X'_n \rightarrow X_{n+1} \xrightarrow{d_n} X_n.$$

This allows to split  $T = T' \oplus T''$  as the direct sum of a contractible factor  $T'$  and a second factor  $T''$ ,

$$T': \quad X'_0 \oplus X'_2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} X'_2 \oplus X'_1 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} X'_1 \oplus X'_0 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} X'_0 \oplus X'_2,$$

$$T'': \quad X''_0 \xrightarrow{d''_2} X''_2 \xrightarrow{d''_1} X''_1 \xrightarrow{d''_0} X''_0,$$

with  $\text{Im } d''_n = \text{Ker } d''_{n-1} \subset \varepsilon \cdot X_{n-1}$ , so  $d''_n = \varepsilon \cdot \bar{d}_n$  for some  $\bar{d}_n: X_{n+1} \rightarrow X_n$ . One can easily check that  $\sigma_n^T = \bar{d}_n \otimes_R k$ , therefore  $\bar{d}_n \otimes_R k$ , and hence  $\bar{d}_n$ , is an isomorphism. Now the following isomorphism of 3-periodic complexes proves the lemma

$$\begin{array}{ccccccc} X''_0 & \xrightarrow{\varepsilon} & X''_0 & \xrightarrow{\varepsilon} & X''_0 & \xrightarrow{\varepsilon \cdot \bar{d}_0 \bar{d}_1 \bar{d}_2} & X''_0 \\ \downarrow 1 & & \downarrow \bar{d}_2 & & \downarrow \bar{d}_1 \bar{d}_2 & & \downarrow 1 \\ X''_0 & \xrightarrow{\varepsilon \cdot \bar{d}_2} & X''_2 & \xrightarrow{\varepsilon \cdot \bar{d}_1} & X''_1 & \xrightarrow{\varepsilon \cdot \bar{d}_0} & X''_0 \end{array}$$

□

**Lemma 4.7.5.** *Given a distinguished triangle  $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{q} X$  in  $\mathcal{F}(R)$ , the following diagram is specially coherent,*

$$\begin{array}{ccccccc} X & \xrightarrow{\varepsilon} & X & \xrightarrow{\varepsilon} & X & \xrightarrow{\varepsilon} & X \\ f \downarrow & & f \downarrow & & f \downarrow & & f \downarrow \\ Y & \xrightarrow{\varepsilon} & Y & \xrightarrow{\varepsilon} & Y & \xrightarrow{\varepsilon} & Y \\ i \downarrow & & i \downarrow & & i \downarrow & & i \downarrow \\ Z & \xrightarrow{\varepsilon} & Z & \xrightarrow{\varepsilon} & Z & \xrightarrow{\varepsilon} & Z \\ q \downarrow & & q \downarrow & & q \downarrow & & q \downarrow \\ X & \xrightarrow{\varepsilon} & X & \xrightarrow{\varepsilon} & X & \xrightarrow{\varepsilon} & X \end{array}$$

*Proof.* It is enough to check the lemma for the following distinguished triangles,

$$X \xrightarrow{1} X \rightarrow 0 \rightarrow X, \quad 0 \rightarrow X \xrightarrow{1} X \rightarrow 0, \quad X \rightarrow 0 \rightarrow X \xrightarrow{1} X, \quad X \xrightarrow{\varepsilon} X \xrightarrow{\varepsilon} X \xrightarrow{\varepsilon} X,$$

since any distinguished triangle is a direct sum of triangles of this kind. One can easily construct an appropriate  $3 \times 3$  diagram in each case. □

Without loss of generality, we can consider  $\mathcal{F}(R)$  to be the skeletal category whose objects are  $R^n$ ,  $n \geq 0$ , which has functorial coproducts  $R^m \oplus R^n = R^{m+n}$ , hence Theorem 4.1.27 applies.

**Lemma 4.7.6.** *Given an  $n \times n$  upper-triangular invertible matrix  $A = \begin{pmatrix} \lambda & v \\ 0 & B \end{pmatrix}$ , where  $\lambda \in R^\times$  and  $B$  is an  $(n-1) \times (n-1)$  upper-triangular matrix, the following*

diagram is virtually coherent,

$$\begin{array}{ccccccc}
R & \xrightarrow{\varepsilon} & R & \xrightarrow{\varepsilon} & R & \xrightarrow{\lambda \cdot \varepsilon} & R \\
\downarrow \binom{1}{0} & & \downarrow \binom{1}{0} & & \downarrow \binom{1}{0} & & \downarrow \binom{1}{0} \\
R^n & \xrightarrow{\varepsilon} & R^n & \xrightarrow{\varepsilon} & R^n & \xrightarrow{\varepsilon \cdot A} & R^n \\
\downarrow (0,1) & & \downarrow (0,1) & & \downarrow (0,1) & & \downarrow (0,1) \\
R^{n-1} & \xrightarrow{\varepsilon} & R^{n-1} & \xrightarrow{\varepsilon} & R^{n-1} & \xrightarrow{\varepsilon \cdot B} & R^{n-1} \\
\downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
R & \xrightarrow{\varepsilon} & R & \xrightarrow{\varepsilon} & R & \xrightarrow{\lambda \cdot \varepsilon} & R
\end{array}$$

In particular the following formula holds in  $\mathcal{D}_1^+(v\mathcal{F}(R))$ ,

$$[R^n \xrightarrow{\varepsilon} R^n \xrightarrow{\varepsilon} R^n \xrightarrow{\varepsilon \cdot A} R^n] = \sum_{i=1}^n [R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon \cdot a_{ii}} R].$$

*Proof.* One can easily construct an appropriate  $3 \times 3$  diagram showing the first part of the statement. Notice that the bracket operation in  $\mathcal{D}_*^+(v\mathcal{F}(R))$  is trivial. Indeed, the monoid of objects in  $\mathcal{F}(R)$  is freely generated by  $R$ , hence  $\mathcal{D}_0^+(v\mathcal{F}(R))$  is free on  $[R]$ , and

$$\langle [R], [R] \rangle = [-1: R \rightarrow R] = 0,$$

since we are in characteristic 2. Now the formula follows by induction in  $n$  applying Proposition 4.1.14 to the virtually coherent diagram.  $\square$

**Lemma 4.7.7.** *Given a virtual octahedron*

$$\begin{array}{ccccc}
Z & & & & C_g \\
\uparrow & \searrow & & & \downarrow \\
& & C_{gf} & & \\
& \nearrow & \swarrow & & \\
X & & & & C_f \\
\downarrow & \nearrow & \swarrow & & \downarrow \\
& & Y & & \\
& \searrow & & & \swarrow \\
& & & & \\
& & & & 
\end{array}$$

formed by virtual triangles  $T_f, T_g, T_{gf}, T_C$ , the following formula holds,

$$\det(T_g) \det(T_f) = \det(T_{gf}) \det(T_C) \pmod{(k^\times)^2}.$$

*Proof.* The octahedron contains morphisms of complexes,

$$\begin{array}{ccc}
T_f: & X \xrightarrow{f} Y \xrightarrow{i^f} C_f \xrightarrow{q^f} X & \\
\varphi \downarrow & \downarrow 1 \quad \downarrow g \quad \downarrow \bar{g} \quad \downarrow 1 & \\
T_{gf}: & X \xrightarrow{gf} Z \xrightarrow{i^{gf}} C_{gf} \xrightarrow{q^{gf}} X & \\
\psi \downarrow & \downarrow i^f \quad \downarrow i^{gf} \quad \downarrow 1 \quad \downarrow g & \\
T_C: & C_f \xrightarrow{\bar{g}} C_{gf} \xrightarrow{\bar{f}} C_g \xrightarrow{i^f q^g} C_f & 
\end{array}$$

The mapping cones of these morphisms fit in the middle of well known short exact sequences of complexes involving the target and a translation of the source, hence by Lemma 4.7.3,

$$\begin{aligned} \det(\text{Cone}(\varphi)) &= \det(T_{gf}) \det(T_f) \pmod{(k^\times)^2}, \\ \det(\text{Cone}(\psi)) &= \det(T_C) \det(T_g) \pmod{(k^\times)^2}. \end{aligned}$$

In this case they also fit into the following other short exact sequences,

$$\begin{array}{ccccccc} \Gamma_X: & X & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{1} & X \\ \downarrow & \downarrow \scriptstyle \left(\begin{smallmatrix} 1 \\ -f \end{smallmatrix}\right) & & \downarrow & & \downarrow \scriptstyle \left(\begin{smallmatrix} -i^{gf} \\ 1 \end{smallmatrix}\right) & & \downarrow \scriptstyle \left(\begin{smallmatrix} 1 \\ -f \end{smallmatrix}\right) \\ \text{Cone}(\varphi): & X \oplus Y & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} gf & g \\ 0 & -i^f \end{smallmatrix}\right)} & Z \oplus C_f & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} i^{gf} & \bar{g} \\ 0 & -q^f \end{smallmatrix}\right)} & C_{gf} \oplus X & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} q^{gf} & 1 \\ 0 & -i^f \end{smallmatrix}\right)} & X \oplus Y \\ \downarrow & \downarrow \scriptstyle (f,1) & & \downarrow \scriptstyle 1 & & \downarrow \scriptstyle (1, i^{gf}) & & \downarrow \scriptstyle (f,1) \\ T': & Y & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} g \\ -i^f \end{smallmatrix}\right)} & Z \oplus C_f & \xrightarrow{\scriptstyle (i^{gf}, \bar{g})} & C_{gf} & \xrightarrow{\scriptstyle fq^{gf}} & Y \end{array}$$
  

$$\begin{array}{ccccccc} \Gamma'_X: & 0 & \longrightarrow & X & \xrightarrow{1} & X & \longrightarrow & 0 \\ \downarrow & \downarrow & & \downarrow \scriptstyle \left(\begin{smallmatrix} -i^{gf} \\ 1 \end{smallmatrix}\right) & & \downarrow \scriptstyle \left(\begin{smallmatrix} 1 \\ -f \end{smallmatrix}\right) & & \downarrow \\ \text{Cone}(\psi): & Z \oplus C_f & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} i^{gf} & \bar{g} \\ 0 & -q^f \end{smallmatrix}\right)} & C_{gf} \oplus X & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} q^{gf} & 1 \\ 0 & -i^f \end{smallmatrix}\right)} & X \oplus Y & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} gf & g \\ 0 & -i^f \end{smallmatrix}\right)} & Z \oplus C_f \\ \downarrow & \downarrow \scriptstyle 1 & & \downarrow \scriptstyle (1, i^{gf}) & & \downarrow \scriptstyle (f,1) & & \downarrow \scriptstyle 1 \\ T'': & Z \oplus C_f & \xrightarrow{\scriptstyle (i^{gf}, \bar{g})} & C_{gf} & \xrightarrow{\scriptstyle fq^{gf}} & Y & \xrightarrow{\scriptstyle \left(\begin{smallmatrix} g \\ -i^f \end{smallmatrix}\right)} & Z \oplus C_f \end{array}$$

Moreover,  $T''$  is the translation of  $T'$ , therefore, using again Lemma 4.7.3,

$$\begin{aligned} \det(\text{Cone}(\varphi)) &= \det(\Gamma_X) \det(T') = \det(T') \pmod{(k^\times)^2}, \\ \det(\text{Cone}(\psi)) &= \det(\Gamma'_X) \det(T'') = \det(T') \pmod{(k^\times)^2}, \end{aligned}$$

so we are done.  $\square$

*Proof of Theorem 4.7.1.* Let  $\alpha: \mathcal{D}_0(d\mathcal{F}(R)) \rightarrow \mathcal{D}_1(d\mathcal{F}(R))$  be the homotopy with target the trivial morphism defined by

$$\alpha[X] = [X \xrightarrow{\varepsilon} X \xrightarrow{\varepsilon} X \xrightarrow{\varepsilon} X].$$

Using Lemma 4.7.5 and the presentation in Corollary 4.1.25 one notices that  $\alpha$  is a homotopy from the identity  $\alpha: 1 \Rightarrow 0$ , i.e. the stable quadratic module  $\mathcal{D}_*(d\mathcal{F}(R))$  is contractible, and therefore  $K_0(v\mathcal{F}(R)) \cong K_0(d\mathcal{F}(R)) \cong 0 \cong K_1(d\mathcal{F}(R))$ .

Let us regard the abelian group  $k^\times / (k^\times)^2$  as a stable quadratic module concentrated in degree 1. By Lemma 4.7.7 the determinant of virtual triangles defines a virtual determinant functor from  $\mathcal{F}(R)$  to  $k^\times / (k^\times)^2$ , and hence a morphism  $p: \mathcal{D}_*^+(v\mathcal{F}(R)) \rightarrow k^\times / (k^\times)^2$  with  $p[T] = \det(T)$  for any virtual triangle  $T$ . The induced morphism  $\pi_1(p): K_1(v\mathcal{F}(R)) \rightarrow k^\times / (k^\times)^2$  is surjective since

$$\begin{aligned} p[R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R] \\ = \det(R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R)^{-1} \det(R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon \cdot \lambda} R) = \lambda. \end{aligned}$$

$\square$

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