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Long-run Identification in a Fractionally Integrated System

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ABSTRACT. We propose an extension of structural fractionally integrated vector autoregressive models that avoids certain undesirable effects for impulse responses if long-run identification restrictions are imposed. We derive its Granger representation, investigate the effects of long-run restrictions and clarify their relation to finite-horizon schemes. It is illustrated by asymptotic analysis and simulations that enforcing integer integration orders can have severe consequences for impulse responses. In a system of US real output and aggregate prices effects of structural shocks strongly depend on integration order specification. In the statistically preferred fractional model the long-run restricted shock has only very short-lasting influence on GDP.

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1. INTRODUCTION

The analysis of impulse responses to structural, economic meaningful shocks provides an indispensable tool for studying the dynamics of economic models. Frequently, shocks to a dynamic system are classified with respect to their long-run effect. If their impact eventually vanishes, they are called *transitory* and *permanent* otherwise. In an econometric time series model shocks with permanent effects can only be present if the model includes at least one variable that is integrated of order one or higher. In contrast to parts of the literature we term a shock *persistent* if its impact is long-lasting independently whether it is permanent or transitory.

Structural vector autoregressive (structural VAR) models provide a common and widespread modeling framework for estimating impulse response functions. In order to determine economic meaningful shocks, identification restrictions are required. One way to impose such restrictions is to constrain the long-run effect of a specified shock such that it conforms to economic theory. This idea of imposing long-run zero constraints was introduced by Blanchard and Quah (1989) for identifying supply and demand shocks in a system of GDP growth and unemployment. Likewise, e.g. Bayoumi and Eichengreen (1994), among others, analyzed systems of GDP growth and inflation. In their stylized aggregate supply/aggregate demand framework, the permanent impact of demand innovations on GDP is constrained to zero, whereas prices may react permanently to both demand and supply shocks. Quah and Vahey (1995) provide an alternative theoretical interpretation of the transitory shock as an innovation to core inflation that can be recovered by decomposing the measured inflation variable.

In this literature, GDP and prices are frequently modeled as variables integrated of order one ($I(1)$) where the impact of transitory shocks vanishes at an exponential rate. However, this does not guarantee that shocks die out quickly, i.e. within few periods. For example, based on a bivariate structural vector autoregressive model of post-war US GDP and prices, the impulse responses of GDP to a demand shock remain substantial after 10 years. This will be shown in Section 4.5, see in particular the upper right graph in Figure 10. If, however, prices are assumed to be an $I(2)$ instead of an $I(1)$ process (Quah and Vahey 1995), the transitory shocks die out within very few periods, see the upper right graph in Figure 11. This suggests that the correct specification of the integration order is of crucial importance for impulse response estimation. Lately, the importance of correct specification of integration orders for long-run identification has received some attention. Christiano et al. (2003) and more recently Gospodinov (2010) call attention to a weak instrument problem if the variable on which both shocks have unrestricted effect has (near) unit root behavior but is modeled as $I(0)$. On the other hand, overdifferencing may result in a loss of low frequency information to which long-run identification is very sensitive (Gospodinov et al. 2010).

By restricting the integration orders to integer numbers, the analysis is limited to two competing specifications of integration orders with extremely different implications for impulse responses. None of these may be close to the true dynamics. Recent work on long memory provides evidence that the orders of integration of income and prices are less than and greater than one, respectively (e.g. Diebold and Rudebusch 1989; Caporale and Gil-Alana 2009; Hassler and Wolters 1995). Therefore, we propose to relax the requirement of integer orders of integration and to allow for fractional integration orders instead. Likewise, Gil-Alana and Moreno (2009) and Lovcha (2009) consider structural fractionally integrated VARs for studying the impact of technology shocks on hours worked. When using a standard VAR model for the fractionally differenced series, contributions of the long-run restricted shock to GDP are forced to be negatively integrated if income is integrated of an order less than one. This is likely to be problematic since then the sum of impulse responses over all periods is zero.

To avoid this restriction we suggest to suitably adapt the class of fractional vector time series models suggested by Johansen (2008) that contains the standard fractional VAR model as a special case. We derive the Granger representation for this model class. These models can be restricted in such a way that identification based on long-run zero constraints assures the restricted shocks to have only short-memory influence on GDP like in the classical structural VAR models. At the same time, the unrestricted shock exhibits persistent effects due to long memory without being forced to have a permanent impact like in the traditional case.

Especially in cases where a variable is driven exclusively by transitory shocks, other identification restrictions that limit the degree of persistence may be useful. Intuitively, constraints affecting medium-run reactions appear as a plausible alternative. Therefore we consider schemes that minimize different measures of variance contributions of the restricted shock to GDP over certain medium-run horizons. Interestingly, we can show that such procedures are asymptotically equivalent to formally imposing the conventional long-run zero restriction of Blanchard and Quah (1989). Furthermore, we investigate the effect of the traditional long-run zero restriction (Blanchard and Quah 1989) in the presence of fractional integration. In a bivariate system it implies that the rate of decay of the responses to the restricted shock is faster than to the unrestricted shock. This can be nicely seen from the Granger representation, as given below in (13).

Moreover, we study the consequences of order misspecification for a setting that is motivated by the estimated integration orders of GDP and prices in Section 4.2. If prices are $I(d)$, $1.5 < d < 2$ but are erroneously modeled as $I(1)$ process, we show for a stylized short memory setup that with an increasing sample size the impulse response estimates converge in probability to a constant which is independent of the prediction horizon. A small Monte Carlo study reveals that the speed of convergence is very slow and that for typical macroeconomic sample sizes impulse response estimates exhibit a slowly vanishing impact of transitory demand shocks.

In sum, misspecification of the orders of integration in combination with an imposed long-run zero constraint can have devastating effects for the impulse response analysis. Logically, if integration orders are indeed fractional, using correctly specified fractionally integrated VAR models have obvious advantages. If, on the other hand, integration has integer orders, the fractional model does perform satisfactorily as well in the sense that the estimation variance does not overly increase as our simulation results indicate.

Using our fractional model in the empirical investigation we reject integer integration orders of GDP and price level. The long-run restricted shocks have small and relatively short-living effect on real GDP, compared to the unit root specification. The reader can expect the following: The next section presents our model and identification of the structural disturbances. Section 3 discusses estimation and effects of misspecification, while Section 4 contains empirical results for U.S. data. The last section concludes. An electronic supplement that is available from the website of the University of Regensburg Publication Server `http://epub.uni-regensburg.de/16901/` contains additional proofs and simulation results.

2. AN APPROPRIATE STRUCTURAL FRACTIONAL VAR WITH LONG-RUN RESTRICTIONS

In this section we show how for structural analysis the careful adaption of the class of fractional vector time series models suggested by Johansen (2008) provides a natural and very useful way of circumventing the limitations of standard integrated VAR models. Furthermore, we explain how the interpretation of the traditional long-run zero restriction (Blanchard and Quah 1989) may change if integration orders are fractional. We motivate this identification procedure by studying its relation to three reasonable medium-run identification schemes.

For ease of presentation, we restrict our discussion to models for bivariate time series of GDP and prices denoted by $\boldsymbol{y}_t = (gdp_t \; p_t)'$. We assume that the stochastic process can be split according to $\boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{d}_t + \boldsymbol{x}_t$ where $\boldsymbol{x}_t$ is purely stochastic, $\boldsymbol{C}$ is an unknown parameter matrix and $\boldsymbol{d}_t$ contains all deterministic terms, i.e. constants and linear trends, where $\boldsymbol{d}_t = 0$ for $t \leq 0$. We allow $\boldsymbol{x}_t$ to be generated by a very general reduced form linear bivariate time series model

$$\boldsymbol{\Pi}(L)\boldsymbol{x}_t = \boldsymbol{u}_t, \qquad t = 1, 2, \dots\,, \tag{1}$$

where $\boldsymbol{\Pi}(L)$ is a (possibly infinite order) VAR polynomial, which will be further specified below, and $\boldsymbol{u}_t \sim IID(\boldsymbol{0}; \boldsymbol{\Omega})$ for $t \geq 1$. For notational convenience, since $\boldsymbol{x}_t$ is non-stationary in general, we set starting values to zero, $\boldsymbol{x}_t = \boldsymbol{u}_t = 0$ for $t \leq 0$, to obtain a well defined solution $\boldsymbol{x}_t = \boldsymbol{\Pi}(L)^{-1}\boldsymbol{u}_t$ without the use of truncated operators as in Johansen (2008) since then the coefficients of $\boldsymbol{\Pi}(L)^{-1}$ from lag t onwards are multiplied by zero. Fixed nonzero starting values of $\boldsymbol{x}_t$ which generate a deterministic term have been considered in the literature (e.g. Johansen and Nielsen 2010a), but for ease of exposition we model deterministic terms explicitly as described above.

In what follows we use a structural $\boldsymbol{B}$-model (e.g. Lütkepohl 2005, Section 9.1.2) to identify economically meaningful shocks, i.e.

$$\boldsymbol{u}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t. \tag{2}$$

Here $\boldsymbol{B}$ is the impact matrix which contains the contemporaneous effects of structural shocks to the economic variables. Identification assumptions are needed to obtain the four unknown coefficients of the impact matrix $\boldsymbol{B}$. Three identification constraints are imposed by normalizing $\text{Var}(\boldsymbol{\varepsilon}_t) = \boldsymbol{I}$. This yields three distinct equations in $\boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Omega}$. Therefore, one further restriction is needed to fully determine $\boldsymbol{B}$.

In the following this will be achieved by constraining the influence of ε_{2t} on $x_{1,t+h}$ for large horizons h. We therefore label ε_{2t} the long-run restricted shock (LRRS), though its effect on $x_{2,t+h}$ is unrestricted. In contrast, we call ε_{1t} the long-run unrestricted shock (LRUS) since it is allowed to have long-lasting effects on both variables.

The straightforward interpretation of LRUS and LRRS in a basic aggregate supply/aggregate demand model is that of shocks to aggregate supply and demand, respectively. Alternatively, grounding on Phillips curve considerations, they have been treated as shocks to the core inflation (LRRS) and non-core inflation (LRUS), see Quah and Vahey (1995).

2.1 Structural Integrated VAR Models

In an integrated VAR setting where both elements of $\boldsymbol{x}_t$ are integrated of order one but not cointegrated, first differences of the series follow a stable vector autoregression

$$\boldsymbol{A}(L)\Delta\boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t, \quad t = 1, 2, \ldots \ . \tag{3}$$

Here $\boldsymbol{A}(z) = \boldsymbol{I} - \boldsymbol{A}_1 z - \ldots - \boldsymbol{A}_p z^p$ denotes a finite order VAR polynomial with all solutions of $|\boldsymbol{A}(z)| = 0$ outside the unit circle, and $\Delta := 1 - L$ represents the first difference operator. We call (3) an IVAR(1,1) model. More generally, if differences of an integer order d_2 are taken to make $\Delta^{d_2} x_{2t}$ an $I(0)$ process, we call it an IVAR$(1, d_2)$ model.

In a bivariate model of GDP and unemployment, Blanchard and Quah (1989) introduced the concept of long-run restrictions. This states that certain shocks are restricted not to keep their influence over infinitely many periods. To obtain some intuition, recall that by the Granger representation theorem (e.g. Lütkepohl 2005, Proposition 6.1) the IVAR(1,1) model can be stated in terms of the multivariate version of the Beveridge-Nelson decomposition

$$\boldsymbol{x}_t = \boldsymbol{\Xi}(1) \sum_{i=1}^{t} \boldsymbol{\varepsilon}_i + \boldsymbol{\Xi}^*(L)\boldsymbol{\varepsilon}_t, \tag{4}$$

where

$$\boldsymbol{\Xi}(z) = \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j z^j = \boldsymbol{A}(z)^{-1}\boldsymbol{B}. \tag{5}$$

The matrix $\boldsymbol{\Xi}(1)$ has full rank and the coefficients of $\boldsymbol{\Xi}^*(L)$ are given by

$$\boldsymbol{\Xi}_0^* = \boldsymbol{B} - \boldsymbol{\Xi}(1), \qquad \boldsymbol{\Xi}_i^* = -\sum_{j=i+1}^{\infty} \boldsymbol{\Xi}_j \quad i = 1, 2, \ldots \ . \tag{6}$$

Then the first summand of (4) contains the stochastic trends of the variables, while the second term is integrated of order zero with geometrically decreasing MA coefficients. Thus the effect of a shock in $\boldsymbol{\varepsilon}_t$ on $\boldsymbol{x}_{t+h}$ in the distant future, $h \to \infty$, is given by $\boldsymbol{\Xi}(1)$. The long-run zero restriction (LRR) that the LRRS ε_{2t} has no permanent effect on the first variable can therefore be stated as

$$\boldsymbol{\Xi}(1) = \boldsymbol{A}(1)^{-1}\boldsymbol{B} = \begin{pmatrix} \xi_{11}(1) & 0 \\ \xi_{21}(1) & \xi_{22}(1) \end{pmatrix}. \tag{7}$$

Note that for all $\boldsymbol{B}$ the long-run covariance matrix $\boldsymbol{A}(1)^{-1}\boldsymbol{\Omega}[\boldsymbol{A}(1)']^{-1}$ is a function of reduced form parameters only. Using the Cholesky decomposition this matrix can be uniquely written as $\boldsymbol{\Xi}(1)\boldsymbol{\Xi}(1)'$ due to the triangularity imposed on $\boldsymbol{\Xi}(1)$. The impact matrix $\boldsymbol{B}$ is then straightforwardly calculated as $\boldsymbol{A}(1)\boldsymbol{\Xi}(1)$.

2.2 Structural Fractionally Integrated VAR$_b$ Models

If the integration orders $\boldsymbol{d} = (d_1;\ d_2)'$ of x_{1t} and x_{2t} are allowed to be real-valued, the standard modeling approach has been to consider fractionally integrated VAR (FIVAR) models. Then, fractional rather than first differences of the variables are modeled as a stable vector autoregression, as e.g. discussed by Nielsen (2004),

$$\boldsymbol{A}(L)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t, \quad t = 1, 2, 3, \ldots, \tag{8}$$

where $\boldsymbol{\Delta}(L; d_1, d_2) = \text{diag}(\Delta^{d_j})$. The fractional differencing operator Δ^d is given by a power expansion as

$$\Delta^d = (1-L)^d = \pi_0 + \pi_1 L + \pi_2 L^2 + \ldots \qquad \text{with} \quad \pi_j = \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}, \tag{9}$$

where $\Gamma(\cdot)$ denotes the Gamma function. If $\boldsymbol{A}(z)$ is a stable VAR polynomial, the series in $\boldsymbol{x}_t$ are integrated of orders d_1 and d_2, respectively.

Technically, the restriction LRR (7) can be imposed exactly in the same way as in the $I(1)$ case since only $\boldsymbol{A}(\cdot)$ and $\boldsymbol{\Omega}$ are required for computing $\boldsymbol{B}$. To clarify the implications of LRR (7) in the FIVAR model we consider its Granger representation

$$\boldsymbol{x}_t = \begin{pmatrix} \Delta^{-d_1} & 0 \\ 0 & \Delta^{-d_2} \end{pmatrix} \boldsymbol{\Xi}(1)\boldsymbol{\varepsilon}_t + \begin{pmatrix} \Delta^{1-d_1} & 0 \\ 0 & \Delta^{1-d_2} \end{pmatrix} \boldsymbol{\Xi}^*(L)\boldsymbol{\varepsilon}_t \tag{10}$$

that we derive in Appendix A. Since $\boldsymbol{\Xi}^*(L)\boldsymbol{\varepsilon}_t$ is independent of integration orders, it remains integrated of order zero with coefficients given as above in (6). Imposing LRR (7) in this case allows to write the first variable as

$$x_{1t} = \underbrace{[\Delta^{-d_1}\xi_{11}(1) + \Delta^{1-d_1}\xi_{11}^*(L)]\varepsilon_{1t}}_{\text{LRUS: persistent and short-lasting component}} + \underbrace{\Delta^{1-d_1}\xi_{12}^*(L)\varepsilon_{2t}}_{\text{LRRS: short-lasting component}}.$$

The impulse responses with respect to the structural shocks are given by the coefficient matrices $\boldsymbol{\Theta}_h$ of the MA representation $\boldsymbol{x}_t = \boldsymbol{\Theta}(L)\boldsymbol{\varepsilon}_t$. Its sk-th element $\theta_{sk,h}$ denotes the impulse response of the sth variable to the kth shock at horizon h. By the first term in (10), the impulse responses of x_{1t} to the unrestricted shock eventually evolve as $\theta_{11,h} = O(h^{d_1-1})$. They decay slowly if GDP is nonstationary and $0.5 < d_1 < 1$, converge to a constant in the unit root case ($d_1 = 1$) and diverge for $d_1 > 1$. In contrast, when imposing LRR (7), by the second term in (10) the responses to the restricted shock $\theta_{12,h}$ are $O(h^{d_1-2})$ and hence converge to zero even when $1 < d_1 < 2$ and LRUS has an infinitely living effect.

For some income variables such as real GDP recent evidence suggests $d_1 < 1$. In this case neither LRUS nor LRRS permanently affect x_{1t}. The short-lasting component of the series is negatively integrated. As an implication the impulse responses of the identified LRRS sum up to zero, exhibiting at least one sign change. This is not reasonable, e.g. if LRRS is interpreted as a shock to aggregate demand. Moreover, the impulse response function approaches zero at the slow hyperbolic rate h^{d_1-2}. In sum, imposing LRR (7) in FIVAR models (8) leads to unsatisfactory and overly restrictive properties of the identified LRRS.

As a solution to these problems we propose a model for the short-run dynamics which stems from Johansen (2008). It generalizes standard (F)IVAR models discussed above. To begin with, we call $\boldsymbol{\eta}_t$ a VAR$_b$ process if it follows

$$\boldsymbol{A}(L_b)\boldsymbol{\eta}_t = \boldsymbol{u}_t, \tag{11}$$

where $\boldsymbol{A}(z)$ is again a matrix polynomial of order p as before. This is not a standard finite order VAR, however, because of the use of the fractional lag operator L_b that can be expanded to an infinite order polynomial in L,

$$L_b := 1 - \Delta^b = c_1 L + c_2 L^2 + \dots \qquad \text{with } b > 0,$$

so that a fractional lag $L_b\boldsymbol{\eta}_t$ generally depends on all past values of $\boldsymbol{\eta}_t$. The coefficients c_1, c_2, etc. in this expansion are computed using (9) and shown in Figure 1(a) for various values of b. Also note that $L_b^j\boldsymbol{\eta}_t$ is a weighted sum of $\boldsymbol{\eta}_{t-j}$, $\boldsymbol{\eta}_{t-j-1}$ etc. which underpins the notion of L_b as a lag operator. For $b = 1$ we have $L_1 = L$ and $\boldsymbol{\eta}_t$ follows a standard vector autoregression. Further note that if z_t is $I(d)$ then $L_b z_t$ keeps this property because it can be written as the sum of an $I(d)$ and an $I(d-b)$ variable where the larger order dominates.

Insert Figure 1 about here

Johansen (2008, Corollary 6) gives the condition under which $\boldsymbol{\eta}_t$ generated by (11) is an $I(0)$ process. It is required that the roots of $|\boldsymbol{A}(z)| = 0$ are outside $\mathbb{C}_b$, which is the image of the unit circle under the mapping $f : z \mapsto 1 - (1-z)^b$. Figure 1(b) plots $\mathbb{C}_b$ for various b. This condition depends both on $\boldsymbol{A}(\cdot)$ and on b, but is easily checked for given parameter values. It is worthwhile to observe that the parameter b adds some flexibility to the short-run properties of the process, as illustrated by the spectral density of an example process in Figure 2.

Insert Figure 2 about here

Inserting $\boldsymbol{\eta}_t = \Delta(L;\boldsymbol{d})\boldsymbol{x}_t$ into (11), we obtain and propose the fractionally integrated VAR$_b$ (FIVAR$_b$) model

$$\boldsymbol{A}(L_b)\Delta(L;\boldsymbol{d})\boldsymbol{x}_t = \boldsymbol{u}_t, \quad t = 1, 2, \dots \,. \tag{12}$$

The FIVAR$_b$ model differs from the fractional vector error correction model proposed by Johansen (2008) in two ways. First, like all models described so far, it does not contain nontrivial (fractional) cointegration relations. In our empirical application we justify this assumption by formal testing, see Section 4.1. Second, the FIVAR$_b$ model allows for different integration orders whereas the Johansen (2008) model restricts the integration order d to be the same for all series. The main motivation for using the $\boldsymbol{A}(L_b)$ filter in the Johansen (2008) setup has originally been to conveniently model autocorrelation in fractional VECMs. However, it turns out that the additional flexibility offered by the parameter b also has favorable implications for our structural model. Note that for $b = 1$ one obtains the FIVAR specification (8) as a special case.

For the FIVAR$_b$ model we also derive the Granger representation

$$\boldsymbol{x}_t = \begin{pmatrix} \Delta^{-d_1} & 0 \\ 0 & \Delta^{-d_2} \end{pmatrix} \boldsymbol{\Xi}(1)\boldsymbol{\varepsilon}_t + \begin{pmatrix} \Delta^{b-d_1} & 0 \\ 0 & \Delta^{b-d_2} \end{pmatrix} \boldsymbol{\Xi}^*(L_b)\boldsymbol{\varepsilon}_t, \tag{13}$$

see Appendix A. Under LRR (7), the short-lasting component of x_{1t}, $\Delta^{b-d_1}\xi^*_{12}(L)\varepsilon_{2t}$, is integrated of order $d_1 - b$ since $\boldsymbol{\Xi}^*(L_b)$ expands to an infinite order MA polynomial in L that generates $I(0)$ processes with nonsingular spectrum at the zero frequency (Johansen 2008, Theorem 8). Thus, in the FIVAR$_b$ model the parameter b influences the memory property of the short-lasting component related to the restricted shock through Δ^{b-d_1} and thus offers additional flexibility compared to the FIVAR specification.

Fluctuations in x_{1t} due to LRRS exhibit long memory behavior if $0 < b < d_1$, are negatively integrated if $b > d_1$ and have short memory only if $b = d_1$. This short memory restriction can be tested against the data, as we will do in Section 4.3. Note that a necessary condition for this short-lasting component to be asymptotically stationary is $d_1 - b < 0.5$. The IVAR(1,1) model is contained as a special case where a permanent shock is turned into a transitory shock with short memory effects only.

To illustrate the consequences of different values of b when imposing LRR (7), Figure 3 gives the impulse responses of a stylized FIVAR$_b$ process given by (14). Observe the undesirable sign change for the FIVAR$_1$ model

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -0.5 \\ 0 & 0.5 \end{pmatrix} L_b\right] \begin{pmatrix} \Delta^{0.7} & 0 \\ 0 & \Delta^{1.7} \end{pmatrix} \boldsymbol{x}_t = \boldsymbol{u}_t, \quad \boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{14}$$

Insert Figure 3 about here

2.3 Finite-horizon Motivation for the Long-run Restriction

In the fractional integration models shocks to GDP only have an ever lasting effect on future realizations if $d_1 \geq 1$. Whenever $d_1 < 1$, the impact of any shock to GDP vanishes with an increasing horizon so that in the terminology of Blanchard and Quah (1989) no "long-run" effect exists. Then, as explained above, the economic content of LRR (7) is based on different rates at which impulse responses to LRRS and LRUS decline. Interestingly, this identifying restriction can also be motivated by means of finite-horizon identification schemes, which are found to give the same result as LRR asymptotically as the horizon tends to infinity. We consider three identification schemes motivated by Faust (1998) who evaluated a similar one to evaluate the robustness of structural VARs with regard to identification.

The forecast error of $x_{s,t+h}$, $s = 1, 2$ based on known coefficients and information up to period t is given by $\sum_{j=0}^{h-1} \sum_{k=1}^{2} \theta_{sk,j} \varepsilon_{k,t+h-j}$. Orthonormality and the IID property of the shocks imply a forecast error variance

$$\mathrm{Var}_t(x_{s,t+h}) = \sum_{j=0}^{h-1} \left(\theta_{s1,j}^2 + \theta_{s2,j}^2\right) = \sum_{j=0}^{h-1} \theta_{s1,j}^2 + \sum_{j=0}^{h-1} \theta_{s2,j}^2, \quad s = 1, 2, \tag{15}$$

which can be decomposed into one variance component due to LRUS, $\varepsilon_{1,t}$, and another one due to LRRS, $\varepsilon_{2,t}$. Thus, the share of h-step forecast variance of variable s due to $\varepsilon_{k,t}$ is given by

$$\omega_{sk,h} = \frac{\sum_{i=0}^{h-1} \theta_{sk,i}^2}{\mathrm{Var}_t(x_{s,t+h})}. \tag{16}$$

If for (sufficiently) long horizons the restricted shock should have a small impact on the behavior of x_{1t}, one may choose an identification procedure that minimizes the forecast error variance share of this shock for suitably large h, i.e. FIN1

$$\min_{\boldsymbol{B}} \omega_{12,h} \qquad s.t. \quad \boldsymbol{BB'} = \boldsymbol{\Omega}. \tag{17}$$

Alternatively, schemes that minimize average variance shares over a suitable range $h \in [l; u]$, FIN2

$$\min_{\boldsymbol{B}} \frac{1}{u-l+1} \sum_{h=l}^{u} \omega_{12,h} \qquad s.t. \quad \boldsymbol{BB'} = \boldsymbol{\Omega}, \tag{18}$$

or, avoiding short-term influence in the objective function, FIN3

$$\min_{\boldsymbol{B}} \frac{\sum_{i=l}^{h} \theta_{12,i}^2}{\mathrm{Var}_t(x_{1,t+h})} \qquad s.t. \quad \boldsymbol{BB'} = \boldsymbol{\Omega} \tag{19}$$

can be used for identifying structural shocks.

Note that LRR (7) and the finite-horizon restrictions yield qualitatively different results: LRR (7) affects the memory property of the short-lasting component of x_{1t} and thus the rate of decay of impulse responses, while the finite-horizon conditions FIN1 (17), FIN2 (18) and FIN3 (19) do not. However, the results of LRR (7) are very similar to all finite-horizon restrictions if long horizons are considered since for $h, u \to \infty$ all three identification schemes approach the result of LRR (7), as shown in Section 2 in the electronic supplement. Imposing LRR (7) may therefore provide an approximation to economically relevant horizons as well as a reference to a hypothetical situation where all adjustment processes have finished.

Figure 4 shows how the impulse response function of the first variable to the second shock (LRRS) depends on the horizon h for the finite-horizon restriction FIN1 (17). For the illustration we again use the stylized process (14) with $b = 0.7$. With increasing horizon h, the FIN1 based impulse responses approach those resulting from LRR. For $h = 100$ and larger the differences are rather negligible. Similar results are found for FIN2 (18) and FIN3 (19). For the latter, differences between the various horizons are negligible if the first $l = 10$ periods are ignored and the short-run influence is kept apart from the objective function. Corresponding figures can be found in the electronic supplement.

Insert Figure 4 about here

3. ESTIMATION AND SPECIFICATION

3.1 Maximum Likelihood Estimation of Structurally Integrated VAR$_b$ Models

The parameters of the FIVAR$_b$ model (12) can either be estimated by a two-step procedure or by maximum likelihood methods. Two-step procedures that estimate $\boldsymbol{d}$ by a semiparametric procedure (e.g. local Whittle estimators) first and the short-memory dynamics by standard VAR methods in a second step are widely used in practice and involve straightforward calculations. However, they suffer from a lack of efficiency and do not yield standard $\sqrt{n}$-asymptotics. Additionally, estimation of b would not be possible in the two-step routine. Therefore, we use a maximum likelihood estimator.

Treating starting values as fixed implies that the FIVAR$_b$ process (12) is nonstationary (although asymptotically stationary for $d < 0.5$). Therefore, the exact maximum likelihood estimator designed for covariance stationary long memory processes is not applicable here. Under the assumption of fixed, potentially nonzero starting values asymptotic properties of maximum likelihood estimators are available for related models. These estimators also avoid prior differencing of the data. In the FIVAR framework with $b = 1$, Nielsen (2004) discusses maximum likelihood estimation, testing and efficiency. Johansen and Nielsen (2010a) treat the estimation of the Johansen (2008) VAR$_{d,b}$ specification (with potential cointegration relations but equal d_s). Since none of these works explicitly handles our case with different integration orders and VAR$_b$ short-run dynamics, asymptotic results are not available for the newly suggested FIVAR$_b$ model (12). Due to its similarity to the models in the aforementioned references, we expect the estimators of $\boldsymbol{d}$, b and $\boldsymbol{A}_j$ to exhibit standard asymptotic properties as well.

Leaving aside deterministic terms the Gaussian log-likelihood of the FIVAR$_b$ model is

$$L(\boldsymbol{d}, b, \boldsymbol{\alpha}, \boldsymbol{\Omega}) = -\frac{n}{2}\log|\boldsymbol{\Omega}| - \frac{1}{2}\sum_{t=1}^{n}[\boldsymbol{A}(L_b)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t]'\boldsymbol{\Omega}^{-1}[\boldsymbol{A}(L_b)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t]$$

with $\boldsymbol{\alpha} = \text{vec}(\boldsymbol{A}_1; \ldots; \boldsymbol{A}_p)$. The high-dimensional numerical optimization problem can be reduced to a 3-dimensional problem by concentrating the likelihood function. First, concentrate the likelihood with respect to $\boldsymbol{\Omega}$ by plugging in the maximum likelihood estimator $\widehat{\boldsymbol{\Omega}}(\boldsymbol{d}, b, \boldsymbol{\alpha}) = \frac{1}{n}\sum_{t=1}^{n}[\boldsymbol{A}(L_b)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t][\boldsymbol{A}(L_b)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t]'$ for given $\boldsymbol{d}$, b and $\boldsymbol{\alpha}$ so that the profile likelihood becomes $L^*(\boldsymbol{d}, b, \boldsymbol{\alpha}) = -\frac{n}{2}\log|\widehat{\boldsymbol{\Omega}}(\boldsymbol{d}, b, \boldsymbol{\alpha})|$. For given $\boldsymbol{d}$ and b the maximum likelihood estimator of $\boldsymbol{\alpha}$ is given by the multivariate least squares estimator from a regression of $\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t$ on $\boldsymbol{\Delta}(L;\boldsymbol{d})L_b\boldsymbol{x}_t, \ldots, \boldsymbol{\Delta}(L;\boldsymbol{d})L_b^p\boldsymbol{x}_t$ (see Johansen and Nielsen 2010a). Denote by $\hat{\boldsymbol{u}}_t(\boldsymbol{d}, b)$ the corresponding residual series. Then a further parameter concentration simplifies the log likelihood to

$$L^{**}(\boldsymbol{d}, b) = -\frac{n}{2}\log\left|\frac{1}{n}\sum_{t=1}^{n}\hat{\boldsymbol{u}}_t(\boldsymbol{d}, b)\hat{\boldsymbol{u}}_t(\boldsymbol{d}, b)'\right|. \tag{20}$$

3.2 Semiparametric Estimation and Treatment of Deterministic Terms

In line with Section 2. we allow for deterministic terms, i.e. nonzero mean and linear time trends, and therefore the purely stochastic components, $\boldsymbol{x}_t = \boldsymbol{y}_t - \boldsymbol{C}\boldsymbol{d}_t$ in (20) are not observable. Treating deterministic terms within the likelihood approach appears possible, as Nielsen (2004) notes for his model. For computational convenience, we instead estimate the parameters $\boldsymbol{C}_{s\boldsymbol{\cdot}}$ of each equation $s = 1, 2$ in a first-step where for given d_s

$$\Delta^{d_s} y_{st} = \boldsymbol{C}_{s\boldsymbol{\cdot}}\Delta^{d_s}\boldsymbol{d}_t + \text{error}, \qquad s = 1, 2, \tag{21}$$

is estimated by least squares. Integration orders d_s are estimated with the semiparametric exact local Whittle estimator of Shimotsu (2010) that allows for deterministic trends. The detrended series can then be computed as $\hat{x}_{st} = y_{st} - \hat{\boldsymbol{C}}_{s\boldsymbol{\cdot}}\boldsymbol{d}_t$.

3.3 Effects Of Misspecified Integration Orders

Christiano et al. (2003) show that insufficient differencing in IVAR models may have serious consequences when using the long-run restriction. If in an instrumental variable framework x_{2t} is $I(1)$, but not differenced and an IVAR(1,0) model fitted, the impact coefficients are not consistently estimated. The estimates converge to a limiting distribution involving functionals of Brownian motion. Similar results hold for a near unit-root specification for x_{2t}, see Gospodinov (2010). We obtain a different result for $x_{2t} \sim I(d)$ with $1.5 < d < 2$, when an IVAR(1,1) process is fitted. For ease of exposition we limit our attention to the process

$$\left(\boldsymbol{I} - \begin{pmatrix} 0 & \gamma \\ 0 & \alpha \end{pmatrix} L\right)\begin{pmatrix} \Delta & 0 \\ 0 & \Delta^{d_2} \end{pmatrix}\boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t, \tag{22}$$

with $1.5 < d_2 < 2$ and, for simplicity, $\boldsymbol{\Omega} = \boldsymbol{I}$. This setup is related to the stylized FIVAR$_b$ process (14). Falsely assuming $d_2 = 1$ with known $\boldsymbol{\Omega}$ we estimate α and γ by least squares using first differences of x_{1t} and x_{2t}. Imposing LRR (7) yields $\hat{\theta}_{12,i} = -[\hat{\gamma}\hat{\alpha}^i]/[(\hat{\alpha}-1)^2 + \hat{\gamma}^2]^{1/2}$. We show in Appendix B that for $i = 0, 1, 2, \ldots$

$$\hat{\theta}_{12,i} = -\frac{\hat{\gamma}\hat{\alpha}^i}{\sqrt{(\hat{\alpha}-1)^2 + \hat{\gamma}^2}} \xrightarrow{p} -\frac{C_1}{\sqrt{C_2^2 + C_1^2}}, \quad C_2 < 0. \tag{23}$$

Here C_1 and C_2 are constants depending on α, γ and d_2. Interestingly these limiting impulse responses do not depend on i. In contrast to Christiano et al. (2003) the limit is nonstochastic. Long-run identification fails, because for large n estimates suggest a permanent effect of LRRS to x_{1t}.

The impact of result (23) in finite samples is illustrated in Figure 5, where 1000 replications are generated according to (22) with $\alpha = -\gamma = 0.5$, and $d_2 = 1.7$ for four different sample sizes $n \in \{250, 1000, 10000, 50000\}$ in order to estimate $\theta_{12,i}$. Extremely slow convergence to the nonstochastic limit is found, while for relevant sample sizes slowly decaying impulse-responses and large sampling-uncertainty is observed.

Insert Figure 5 about here

3.4 A Small Monte Carlo Study

A simulation study shall illustrate the behavior of the maximum likelihood based estimation of impulse responses in scenarios that are more realistic than the FIVAR process (22). We present results for two data generating processes, each with one lag. Results on two processes with four lags are contained in the electronic supplement. The stochastic processes considered in this section are:

FIVAR$_b$1. This specification matches integration orders, the reduced form error covariance matrix and $\boldsymbol{A}(1)$ matrix of a FIVAR$_{d_1}$ model with four lags fitted to GDP and prices in Section 4.2 such that the impact matrix $\boldsymbol{B}$ and the long-run characteristics given by the first term in the Granger representation (13) are the same

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & -1.5 \\ 0.18 & 0.2 \end{pmatrix} L_{0.83}\right] \begin{pmatrix} \Delta^{0.83} & 0 \\ 0 & \Delta^{1.77} \end{pmatrix} \boldsymbol{x}_t = \boldsymbol{u}_t, \quad \boldsymbol{\Omega} = \begin{pmatrix} 6.9 & -0.11 \\ -0.11 & 0.71 \end{pmatrix}. \tag{24}$$

IVAR1. This specification matches the reduced form error covariance matrix and $\boldsymbol{A}(1)$ matrix of an IVAR(1,1) model with four lags fitted to GDP and prices in Section 4.2

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.26 & -0.24 \\ 0.12 & 0.96 \end{pmatrix} L\right] \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \boldsymbol{x}_t = \boldsymbol{u}_t, \quad \boldsymbol{\Omega} = \begin{pmatrix} 7.4 & -0.2 \\ -0.2 & 0.77 \end{pmatrix}. \tag{25}$$

For each data generating process, 5000 realizations are simulated with Gaussian innovations and sample size $n = 250$, the latter corresponding to the empirical application in Section 4. The parameters are estimated by assuming either an IVAR(1,1), IVAR(1,2) or FIVAR$_{d_1}$ model with correct lag length and LRR (7) imposed.

The FIVAR$_{d_1}$ specification is estimated in the following way. First, the deterministic terms are removed as outlined in Section 3.2 where the exact local Whittle estimation (Shimotsu 2010) of integration orders is based on $\lfloor n^{0.5} \rfloor = 15$ frequencies. Then, based on the detrended series $\hat{x}_{st}$ resulting from the regression (21), the log likelihood (20) is computed on a grid of d_1 and d_2 values with step size 0.1. The grid point delivering the largest log likelihood is used as starting value for a nonlinear optimization routine in order to further maximize the log likelihood. 28 starting values are used so that the sum in (20) is taken over 232 observations.

Figures 6 and 7 show boxplots of the estimated impact coefficients b_{12} in the upper left graphs. All other graphs display the true impulse responses (grey solid line) of the data generating mechanism along with the mean (dashed line) as well as 5%, 10%, 25%, 50%, 75%, 90% and 95% quantiles (dotted lines) of the estimates for different models. Because of space limitations we restrict attention to responses of the first variable to LRRS. Not surprisingly, the estimates from the FIVAR$_{d_1}$ model provide a reliable answer for the FIVAR$_b$1 process (24), with estimates centered at the true values as Figures 6 shows. As expected, the estimated IVAR(1,1) impulse responses are misleading due to their slow convergence to zero as already seen in Figure 5. Interestingly, sampling uncertainty is larger than in the fractional model despite the fact that integration orders are fixed rather than estimated. The IVAR(1,2) model understates the impact coefficient and impulse responses which follows from discarding zero-frequency information by over-differencing.

To assess the efficiency loss of using a fractional model when the unit root assumption holds, we estimate all three specifications for the IVAR1 process (25). As can be seen from Figure 7, the increase in variance is noticeable but rather small when compared to the strong misspecification effects shown in the lower left panel of Figure 6 that can be avoided.

Insert Figures 6 and 7 about here

These results become less clear cut if processes and specifications with more lags are considered. We conducted additional Monte Carlo simulations of which detailed results and figures are provided in the electronic supplement. They suggest that more lags increase the variance of estimated impulse responses. Furthermore, including only one lag in a FIVAR$_b$ model when the true process is an IVAR(1,1) process with four lags leads to severe underestimation of the responses of the first variable to LRRS.

4. STRUCTURAL ANALYSIS OF GDP AND PRICES

In this section our structural fractionally integrated VAR_b model (12) is fitted to output and price data. We use quarterly time series of US real gross domestic product and its implicit price deflator (series GDPC1 and GDPDEF from the Federal Reserve Bank St. Louis, `http://research.stlouisfed.org/fred2/`, downloaded on August 23, 2009) ranging from 1947Q1 to 2009Q2, both in natural logarithms.

4.1 Semiparametric Estimates of Memory Parameters and Deterministic Terms

The semiparametric estimation of the memory parameters and deterministic terms is carried out as outlined in Section 3.2. In order to check the robustness of the semiparametric estimator, we compute exact local Whittle estimates (Shimotsu 2010) and asymptotic confidence intervals using a minimum of 9 up to a total of 30 Fourier frequencies.

Over this range we find real GDP to be integrated of order less than one, though the difference to integration order one is not found statistically significant for any frequency. This confirms findings in the literature. Estimates for the integration order of the price level are found between 1.5 and 1.8. The $I(1)$ hypothesis is rejected for any number of frequencies and the $I(2)$ hypothesis is rejected if more than 11 frequencies are used. This mirrors the fact that modeling inflation as $I(1)$ or $I(0)$ are coexisting strategies throughout the empirical literature.

Like in the simulation study in Section 3.4, the fractional differences for running the regressions (21) are computed with semiparametric estimates based on $\lfloor n^{0.5} \rfloor = 15$ frequencies. We obtain the estimates $\hat{d}_1 = 0.77$ and $\hat{d}_2 = 1.54$, delivering the detrended series

$$\hat{x}_{1t} = gdp_t - 7.4729 - 0.0082t, \quad \hat{x}_{2t} = p_t - 2.5813 - 0.0104t. \tag{26}$$

Validity of our multivariate fractional model rests on the assumption that no fractional cointegration relationship is present. Cointegration between GDP and price level is not possible due to the inequality of integration orders. To treat the possibility of unbalanced cointegration explicitly, we consider potential cointegration between GDP and the quarterly inflation rate. Nielsen (2010) introduced a nonparametric test which tests the null hypothesis of no cointegration without requiring knowledge of integration orders and allowing for deterministic terms. Accounting for linear trends in the data as well as in the cointegration relations and using the proposed partial summation order ($d_1 = 0.1$ in the notation of Nielsen (2010)) we obtain a trace-statistic of 3.9372. For this test, the critical value depends on the assumed integration order and critical values are decreasing with increasing order of integration. Even for an assumed common integration order $d_1 = d_2 - 1 = 0.9$ the 10%-critical value is larger than the test statistic. Since the null of no cointegration cannot be rejected if the test statistic is smaller than the critical value, we unambiguously fail to reject the null hypothesis of no cointegration.

4.2 Maximum Likelihood Estimates

We proceed with maximum likelihood estimation based on (20) as derived in Section 3.1, where $\hat{\boldsymbol{x}}_t$ is used instead of $\boldsymbol{x}_t$. Different values of p and restricted and unrestricted b are considered. A range of presample values from 1947Q1 to 1953Q4 is used to compute fractional differences appearing in the likelihood so that the sum in (20) is taken over 232 observations. Estimates are obtained for different specifications of b, namely for freely estimated b and under the constraints $b = 1$ or $b = d_1$.

Without restrictions such as $\boldsymbol{A}_p \neq \boldsymbol{0}$ or stability of the VAR$_b$ polynomial there is an indeterminacy between $\boldsymbol{d}$, b and p, as illustrated by Johansen and Nielsen (2010b) in their univariate model. For some p we find different local maxima with approximately the same likelihood, which empirically reflects this identification problem. Closer inspection shows that stability of the VAR$_b$ polynomial is (nearly) violated at some of the local maxima. For $p = 4$ and $b = d_1$ we show the profile likelihood in Figure 8 as well as the corresponding $\mathbb{C}_b$ and the characteristic roots at two different local optima in Figure 9. For the global maximum likelihood estimates $\hat{\boldsymbol{d}} = (0.83\ 1.77)'$ the roots of $\boldsymbol{A}(z)$ are clearly outside $\mathbb{C}_b$, while the local maximum $\tilde{\boldsymbol{d}} = (0.17\ 1.85)'$ almost violates the stability condition. For all lag lengths our estimates are obtained by excluding such constellations from the parameter space.

Insert Figures 8 and 9 about here

Estimation results for various specifications are given in Table 1. The lag order p may be chosen by the information criteria AIC or SC. Regardless of the specification of b, the AIC criterion favors $p = 4$ while SC suggests $p = 1$, as can be seen from Table 1. Since we want to allow for enough flexibility to model dynamic interactions, we prefer $p = 4$. Given the quarterly sampling frequency, this seems to be a plausible choice. Except for the case $p = 0$, where evidence for misspecification is overwhelming, estimated integration orders are mostly close to the estimated $\hat{\boldsymbol{d}} = (0.83\ 1.77)'$. In this sense we find relative robustness with respect to lag length.

Insert Table 1 about here

4.3 Tests on b and Nonfractionality Hypotheses

We are interested whether the data significantly reject the standard fractionally integrated VAR model (8) with $b = 1$. Secondly, we check whether $b = d_1$ holds. Only in the latter case the short-lasting part of GDP has short memory due to imposing LRR (7), see Section 2.2.

For small p the implied infinite order VAR structure, which depends on b, is needed to capture higher order autocorrelation in the differenced series. As a consequence one or both of the hypotheses is rejected for $p \leq 3$. Once one has allowed for sufficiently rich short-run dynamics through a reasonably large lag length p it is hard to distinguish statistically between different values of the parameter b. For $p = 4$, which will be the model we use from now on, we obtain LR-statistics of 0.6410 ($b = 1$) and 0.4419 ($b = d_1$), respectively, and neither of the two hypotheses can be rejected at conventional significance levels.

Our goal to generalize structural models to the fractional case is only useful if less flexible models with prespecified integer-valued integration orders are rejected by the data. We thus test these IVAR specifications using the likelihood-ratio approach. Results for $p = 1$ and $p = 4$ are shown in Table 2.

Unit root behavior of GDP and price level has been assumed in prior studies. With fractional alternatives we find clear evidence against $H_0: \ d_1 = d_2 = 1$ for most lag lengths p. For our preferred specification with $p = 4$ and $b = d_1$ imposed we reject this hypothesis at conventional significance levels.

A widespread alternative specification with a unit root inflation process, the IVAR(1,2) model, is also nested and thus testable in our fractional setting. For the specifications with restrictions on b there is clear evidence against the hypothesis $H_0: \ d_1 = 1$ and $d_2 = 2$ as well. Both hypotheses are rejected at the 5% level given our baseline specification.

Carrying out tests of nonfractional values of d_1 and d_2 individually, there is evidence that $d_1 \neq 1$ and $d_2 \neq 1$ (with p values 0.0444 and 0.0023, respectively). The $I(2)$ hypothesis for prices cannot be rejected for $p > 1$, however. To summarize the results we find that the IVAR(1,1) model which is the workhorse in the related literature is misspecified. This also holds if prices are assumed to be $I(2)$ instead. Our fractional specification seems suitable for such situations.

Insert Table 2 about here

The structural analysis is carried out by imposing the long-run zero constraint LRR (7) to identify shocks with short-lasting effect on output, LRRS, and shocks with persistent effect, LRUS. As reference models we take an IVAR(1,1) model, where the VAR in first differences is estimated by OLS, as well as the IVAR(1,2) specification where prices are differenced twice. As a result from the previous section we impose $b = d_1$ in the FIVAR$_b$ model. Lag length is set to $p = 4$ for all specifications.

4.4 Contemporaneous Effects

Depending on the model choice, imposing LRR (7) leads to differing estimation results of the impact matrix $\boldsymbol{B}$ that measures the contemporaneous effects of short-lasting and persistent shocks. Table 3 displays the results. Theoretical predictions of the aggregate supply/aggregate demand model are in line with all of the three specifications regarding the signs of the estimated effects. While the signs of both LRRS and LRUS disturbances are normalized to give them positive impact on GDP, the aggregate supply interpretation of LRUS is supported by the fact that the impact on prices is opposite to the real effect. Also in line with aggregate supply/aggregate demand theory, LRRS has a positive impact on inflation.

Insert Table 3 about here

It can be seen from Table 3, however, that moving from the IVAR(1,1) to the IVAR(1,2) model drastically reverses the quantitative importance of the shocks to each variable. In the latter specification GDP disturbances are almost exclusively determined by LRUS with an insignificant impact coefficient of LRRS. Prices are hit only by the restricted shocks. The fractional setting helps to avoid a difficult decision between the two models and points towards an in-between scenario. The result is relatively similar to the specification with prices modeled as $I(2)$ in that LRUS plays the most prominent role in GDP fluctuations while prices react mostly to the LRRS.

4.5 Dynamic Responses to Structural Shocks

This section considers the dynamic reactions (impulse responses) of output and prices after the occurrence of a structural disturbance. The estimates are given by the coefficients $\hat{\theta}_{sk,h}$ which are plotted against the horizon h in the four panels of Figures 10, 11 and 12. Bootstrapped 90% confidence intervals are simulated by both methods discussed in Christiano et al. (2007), namely the standard error method (dotted lines) and the percentile method (dashed lines).

Insert Figures 10, 11 and 12 about here

Responses of output Consider the effects of structural shocks on GDP first. In the IVAR(1,1) model, see Figure 10, responses to LRUS converge to a positive constant, the long-run effect $\xi_{11}(1)$. Despite the fact that no infinitely-living effect of LRRS exists, there is a significant response found for intermediate horizons. It reaches its maximum value after two quarters and decays slowly, whereby even after 20 quarters about half of the maximum impact remains. Both impulse response functions of GDP are very similar to those estimated in a bivariate output-unemployment system by Blanchard and Quah (1989). Associating LRRS with a demand shock, the observed medium-run real effects would have far-reaching consequences for the understanding of business cycles and policy making.

These results contrast sharply to those of the IVAR(1,2) specification. While the long run effect of LRUS is of about the same magnitude, estimated effects of LRRS are negligible even at short horizons, see Figure 11. We find a considerable difference across specifications and it may be hard to decide which results to rely upon.

As discussed above, the FIVAR$_{d_1}$ specification is a way to avoid such a tough choice. Figure 12 gives the corresponding impulse responses. Since GDP is estimated to be integrated of an order less than one, both structural shocks are transitory so that the impulse responses eventually decrease for both LRUS and LRRS. While the decay of responses to LRUS is very slow, the effect of LRRS on output is small and short-living compared to the IVAR(1,1). The responses are significant only for a small interval of horizons and turn insignificantly negative after about three years.

Thus, based on the FIVAR$_{d_1}$ model the impact of the restricted shock is truly short-lasting and indeed very different from the persistent effects found by using the IVAR(1,1) specification. This is in line with theoretical and simulation evidence for misspecified IVAR processes presented in Sections 3.3 and 3.4.

Responses of prices We now turn to price reactions, shown in the lower panels of Figures 10 to 12. The pattern of responses to LRUS has caught interest in different contexts. Consistency with aggregate demand/supply theory is tested by Bayoumi and Eichengreen (1994) who use an IVAR(1,1) model. Estimated long-run effects of LRUS – interpreted as aggregate supply shock in their setup – on prices are found to be negative in almost all countries, backing the aggregated supply/aggregate demand interpretation of the identified shocks. In contrast, following the approach of Quah and Vahey (1995), significant long-run effects of LRUS (shocks to non-core inflation in their terminology) on inflation would indicate misspecification. Interestingly, in the IVAR(1,2) setup of the latter a positive but insignificant influence of LRUS is found. Consequently, the findings heavily depend on the specification of integration orders, and each of the contradictory hypotheses are supported in the respective setting.

With both the IVAR(1,1) and the IVAR(1,2) specification previous results in the literature are reproduced in our analysis. In the unit root model the effects of LRUS on the price level are negative, while LRRS has a strong positive effect. While impulse responses converge to a constant in the unit root specification, this differs in both the IVAR(1,2) and the fractional model. Here, as an effect of the integration orders being larger than one, both shocks have rising impact over increasing horizons. For both models we observe positive long-run price responses to both shocks. Contrary to the fractional model, the confidence intervals of the LRUS impulse responses based on the IVAR(1,2) model exclude zero for larger horizons, providing evidence against the aggregate supply/aggregate demand interpretation.

Historical Decomposition Given the estimated parameters it is possible to decompose the observed series in components which are driven by each of the identified shocks, $\hat{\boldsymbol{x}}_t^s$, where $s \in \{\text{LRUS}, \ \text{LRRS}\}$. These are computed as

$$\hat{\boldsymbol{x}}_t^s = \hat{\boldsymbol{\Pi}}(L)^{-1}\hat{\boldsymbol{B}}\hat{\boldsymbol{\varepsilon}}_t^s \tag{27}$$

where $\hat{\boldsymbol{\varepsilon}}_t^s$ is the bivariate structural disturbance series with the other shock set to zero and $\hat{\boldsymbol{\Pi}}(L)^{-1}$ contains estimated reduced form parameters. We find considerable differences between specifications, most notably for the decomposition of GDP. In Figures 13 and 14 we show the detrended output series (solid grey) as well as its LRUS (dashed) and LRRS (solid black) components,

$\hat{x}_{1t}^{LRUS}$ and $\hat{x}_{1t}^{LRRS}$, respectively. In the IVAR model the component which is due to LRRS is very persistent, contrary to the notion of a short-lasting component. This is not the case for the FIVAR$_{d1}$ model, where relatively fast reversion of GDP to the LRUS-component is observed.

Insert Figures 13 and 14 about here

Effect of individual integration orders As already pointed out, misspecified orders of integration have substantial effects on impulse responses to structural shocks. We therefore address the question how changing individual integration orders affect our results. Impulse response surfaces are computed where one integration parameter is varied on a specified grid, while all other coefficients are estimated by restricted maximum likelihood.

Besides direct effects of integration orders on the shape of persistent impulse responses over longer horizons, results are notably sensitive in other respects: Both varying over d_1 and d_2 leads to sign changes in the impulse responses of prices to LRUS. Both smaller d_1 and increased d_2 turn this effect positive, so that the difference between unit root and estimated fractional model inherits both these effects. The medium-run responses of GDP to LRRS do not change their shape, but their relative importance in determining GDP is heavily reduced in more persistent specifications of price level (higher d_2), see Figure 15. This is the main driving force leading to the smaller impact of LRRS found in the fractional model.

Insert Figure 15 about here

5. CONCLUSION

In this paper we analyzed structural time series models of fractionally integrated variables. Identification is achieved by long-run constraints drawn from economic theory. In the framework of standard fractionally integrated VAR models this identification strategy causes problems which are resolved by our approach based on adapting the class of fractional vector time series models suggested by Johansen (2008). It is shown that the interpretation of the long-run zero constraint changes if the variable on which one shock has restricted impact is integrated of order less than one. In this case, all structural shocks have transitory effects although their impulse responses may decline very slowly. Imposing the long-run zero constraint causes a faster decay of responses to the restricted shock. We show that identification based on the well known long-run zero constraint of Blanchard and Quah (1989) is identical to identification schemes based on medium-run horizons if one lets the horizon go to infinity. We discuss maximum likelihood estimation of our model and find it computationally convenient.

We apply our model to quarterly postwar US aggregate price and GDP data. Our empirical results provide strong evidence that non-fractional structural models for this data are misspecified. We find that the structural and dynamic properties of the more general fractional model differ substantially from those of a unit root approach. These differences can be mainly attributed to misspecified integration orders in non-fractional models, as we clarify theoretically and by simulation.

Since uncertainty about the orders of integration is an issue in most applied structural VAR analyses, reconsideration of previous findings with fractional integration techniques may be enlightening. Further research for handling structural models with more variables, potential fractional cointegration and different types of identifying restrictions seems promising.

Appendix A GRANGER REPRESENTATION FOR FRACTIONAL MODELS

To obtain (13), write the VAR(∞) polynomial of the FIVAR$_b$ model using $\Delta^b = 1 - L_b$ and $\boldsymbol{\Delta}(L;\gamma_1,\gamma_2) = \mathrm{diag}(\Delta^{\gamma_1},\Delta^{\gamma_2})$ as

$$\boldsymbol{\Pi}(L) = \boldsymbol{A}(L_b)\boldsymbol{\Delta}(L;\boldsymbol{d}) = \left[\boldsymbol{I}(1-L_b) - \boldsymbol{A}_1 L_b(1-L_b) - \ \ldots - \boldsymbol{A}_p L_b^p(1-L_b)\right]\boldsymbol{\Delta}(L;d_1-b,d_2-b).$$

Substituting u for L_b, define the polynomial in square brackets as $\boldsymbol{\Pi}^*(u) = (1-u)\boldsymbol{I} - \boldsymbol{A}_1 u(1-u) - \ldots - \boldsymbol{A}_p u^p(1-u)$, which is the characteristic polynomial of a multivariate IVAR process without cointegration. The inverse of $\boldsymbol{\Pi}^*(u)$ is as in the standard $I(1)$ case, $\boldsymbol{\Pi}^*(u)^{-1} = \boldsymbol{A}(1)^{-1}(1-u)^{-1} + \boldsymbol{H}^*(u),\ 0 < |u-1| < \delta, \delta > 0$, where $|\boldsymbol{H}^*(u)|$ has no root on the unit circle (Johansen 2008, p.665). This allows to write the process $\boldsymbol{x}_t$ as $\boldsymbol{\Pi}^*(L_b)\boldsymbol{\Delta}(L;d_1-b,d_2-b)\boldsymbol{x}_t = \boldsymbol{u}_t$ and hence $\boldsymbol{\Delta}(L;d_1-b,d_2-b)\boldsymbol{x}_t = (1-L)^{-b}\boldsymbol{A}(1)^{-1}\boldsymbol{u}_t + \boldsymbol{H}^*(L_b)\boldsymbol{u}_t$, noting that the right hand side is well defined by the assumption $u_t = 0$ for $t \leq 0$. Next, by inverting the matrix difference operator, plugging in $\boldsymbol{u}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t$, and using (5) as well as (6), one obtains $\boldsymbol{x}_t = \boldsymbol{\Delta}(L;-d_1,-d_2)\boldsymbol{\Xi}(1)\boldsymbol{\varepsilon}_t + \boldsymbol{\Delta}(L;b-d_1,b-d_2)\boldsymbol{\Xi}^*(L_b)\boldsymbol{\varepsilon}_t$. The process $\boldsymbol{z}_t := \boldsymbol{\Xi}^*(L_b)\boldsymbol{\varepsilon}_t$ is an $I(0)$ process, see Johansen (2008, eqn. 21). To show this, he considers the stationary variable $\boldsymbol{z}_t^* = \boldsymbol{\Xi}_0^*\boldsymbol{\varepsilon}_t^* + \boldsymbol{\Xi}_1^* L_b\boldsymbol{\varepsilon}_t^* + \boldsymbol{\Xi}_2^* L_b^2\boldsymbol{\varepsilon}_t^* + \ldots = \boldsymbol{F}_0\boldsymbol{\varepsilon}_t^* + \boldsymbol{F}_1\boldsymbol{\varepsilon}_{t-1}^* + \boldsymbol{F}_2\boldsymbol{\varepsilon}_{t-2}^* + \ldots$, where $\boldsymbol{\varepsilon}_t^* = \boldsymbol{\varepsilon}_t$ for $t \geq 1$ and $\boldsymbol{\varepsilon}_t^* \sim IID(\boldsymbol{0};\boldsymbol{I})$, and shows that its spectral density matrix $f_z(\lambda) = (2\pi)^{-1}\boldsymbol{F}(\mathrm{e}^{-i\lambda})\boldsymbol{F}(\mathrm{e}^{i\lambda})'$ is bounded and bounded away from zero at zero frequency. This delivers (13). The Granger representation (10) follows as a special case by setting $b = 1$.

Appendix B LIMITING BEHAVIOR OF IMPULSE RESPONSES WITH MISSPECIFIED INTEGRATION ORDERS

Presume that the data $\boldsymbol{x}_t$ is generated by the stylized process (22) with $1.5 < d_2 < 2$. First, we derive the estimator for the impulse responses, $\hat{\theta}_{12,j}$, based on the misspecified model (22) with $d_2 = 1$. Note that in the case $\boldsymbol{\Omega} = \boldsymbol{I}$ the impact matrix $\boldsymbol{B}$ is orthonormal and can thus be indexed by a single number β (up to sign). Then its columns may be denoted by $\boldsymbol{B}_{\bullet 1}(\beta) = (\sqrt{1-\beta^2},\ -\beta)'$ and $\boldsymbol{B}_{\bullet 2}(\beta) = (\beta,\ \sqrt{1-\beta^2})'$.

For given β we have $\boldsymbol{\Xi}(L;\beta) = |\boldsymbol{A}(L)|^{-1}\boldsymbol{A}^{adj}(L)\boldsymbol{B}(\beta)$, which yields $\xi_{12,0}(\beta) = \beta$ and, for $i = 1, 2, \ldots$, $\xi_{12,i} = \sqrt{1-\beta^2}\gamma\alpha^{i-1}$. Imposing LRR (7) requires $\xi_{12}(1;\beta) = \beta + \sqrt{1-\beta^2}\gamma(1-\alpha)^{-1} = 0$ which is solved for $\beta = -\gamma/[(1-\alpha)^2+\gamma^2]^{1/2}$. We find from (13) and (7) that the impulse responses are given by $\theta_{12,i} = \xi^*_{12,i}$. From (6) the impulse responses are $\theta_{12,0} = b_{12} = \beta = -\gamma/[(1-\alpha)^2+\gamma^2]^{1/2}$ and, for $i = 1, 2, \ldots$, $\theta_{12,i} = -\sum_{j=i+1}^{\infty}\xi_{12,j} = -\sum_{j=i+1}^{\infty}\sqrt{1-\beta^2}\gamma\alpha^{j-1} = -\gamma\alpha^i/[(1-\alpha)^2+\gamma^2]^{1/2}$. For known $\boldsymbol{\Omega} = \boldsymbol{I}$ the impulse responses are estimated by replacing α and γ by the respective estimates. Defining $\boldsymbol{z}_t := \Delta\boldsymbol{x}_t$, the maximum likelihood estimates correspond to $\hat{\alpha} = (\sum_{t=1}^{n} z_{2,t}z_{2,t-1})/(\sum_{t=1}^{n} z_{2,t-1}^2)$ and $\hat{\gamma} = (\sum_{t=1}^{n} z_{1,t}z_{2,t-1})/(\sum_{t=1}^{n} z_{2,t-1}^2)$.

Now suppose that $1.5 < d_2 < 2$ (as suggested by our estimates in Section 4.2), while for estimating the impulse responses the IVAR(1,1) model is used. Denoting $C_{1n} := n^{-1}\sum_{t=1}^{n} z_{1,t}z_{2,t-1}$, $C_{2n} := n^{-1}\sum_{t=1}^{n}\Delta z_{2,t}z_{2,t-1}$ and $C_{3n} := n^{-1}\sum_{t=1}^{n} z_{2,t-1}^2$ one can rewrite the estimated impulse response at horizon i as

$$\hat{\theta}_{12,i} = -\frac{\hat{\alpha}^i\hat{\gamma}}{[(\hat{\alpha}-1)^2+\hat{\gamma}^2]^{1/2}} = -\frac{\hat{\alpha}^i C_{1n}/C_{3n}}{[(C_{2n}/C_{3n})^2+(C_{1n}/C_{3n})^2]^{1/2}} = -\frac{\hat{\alpha}^i C_{1n}}{[C_{2n}^2+C_{1n}^2]^{1/2}}.$$

A law of large numbers can be applied to each of the terms C_{1n}, C_{2n} and $\hat{\alpha}$. In the following we will apply convergence results of Marinucci and Robinson (1999, 2000) although they were proven within a slightly different setting where only fractional difference operators are truncated. Our setting corresponds to truncating all lag operators for the solution of the processes considered.

First note that $z_{1,t} \sim I(0)$, $z_{2,t} \sim I(d_2-1)$, $d_2 - 1 < 1$ and thus $C_{1n} \xrightarrow{p} C_1$, where $C_1 := \frac{1}{2\pi}\int_{-\pi}^{\pi} f_{12}(\lambda;\alpha,\gamma)(1-\mathrm{e}^{i\lambda})^{-d_2}\,\mathrm{d}\lambda$, see Robinson and Marinucci (2001, Theorems 4.1 and 5.1). Here $f_{12}(\lambda;\alpha,\gamma) = [\gamma(1-\exp(-i\lambda)) + \alpha\gamma\mathrm{e}^{-i\lambda}]/|1-\alpha\exp(i\lambda)|^2$ denotes the upper right entry of the (limiting) spectral density matrix of $(\Delta x_{1t},\ \Delta^{d_2}x_{2,t-1})'$, replacing that of $(\eta_t,\ \theta_t)'$ in Robinson and Marinucci (2001).

Further observe $C_{2n} = 0.5n^{-1}z_{2n}^2 - 0.5n^{-1}\sum_{t=1}^{n}(\Delta z_{2,t})^2$. Note that $z_{2n} = O_p(n^{d_2-1.5})$ (Marinucci and Robinson 2000, eqn. 6) and hence $n^{-1}z_{2n}^2 \xrightarrow{p} 0$. Consider the variable $\eta_t^* = \Delta^{2-d_2}(1-\alpha L)^{-1}u_{2t}^*$ with $u_t^* = u_t$ for $t \geq 1$ and $u_t^* \sim IID(\boldsymbol{0};\boldsymbol{\Omega})$ for $t < 1$, which follows a stationary negatively dependent ARFIMA(1,$\tilde{d}$,0) process with $-0.5 < d_2 - 2 = \tilde{d} < 0$. As a result of Marinucci and Robinson (1999) we have $\mathrm{Var}(\eta_t^* - \Delta z_{2t}) = O(t^{2d_2-5})$. Therefore $C_{2n} =$

$-0.5n^{-1}\sum_{t=1}^{n}(\Delta z_{2,t})^2 + o_p(1) = -0.5n^{-1}\sum_{t=1}^{n}(\eta_t^*)^2 + o_p(1) \xrightarrow{p} C_2$ with $C_2 := -0.5\,\mathrm{Var}(\eta_t^*) = -(4\pi)^{-1}\int_{-\pi}^{\pi}|1-\mathrm{e}^{i\lambda}|^{-2(d_2-2)}|1-\alpha\mathrm{e}^{i\lambda}|^{-2}\,\mathrm{d}\lambda$.

Turning to $\hat{\alpha}$ observe $\hat{\alpha} = 1 + C_{2n}/C_{3n} = 1 + [n^{1-2d_2}C_{2n}]/[n^{1-2d_2}C_{3n}]$. By similar arguments as above $n^{1-2d_2}C_{2n} \xrightarrow{p} 0$, and, by Marinucci and Robinson (2000), $n^{1-2d_2}C_{3n} = O_p(1)$. Thus $\hat{\alpha} \xrightarrow{p} 1$ follows by Davidson (1994, Theorem 18.12).

To summarize we find $(C_{1n},\ C_{2n},\ \hat{\alpha}) \xrightarrow{p} (C_1,\ C_2,\ 1)$. Since $C_2 \neq 0$ the mapping $f : \boldsymbol{x} \mapsto -x_1 x_3^i/\sqrt{x_1^2 + x_2^2}$ is continuous at $\boldsymbol{x} = (C_1,\ C_2,\ 1)'$ and hence by Slutsky's theorem (Davidson 1994, Theorem 18.10(ii)) we obtain $\hat{\theta}_{12,i} \xrightarrow{p} -C_1/[C_2^2 + C_1^2]^{1/2}$ which is a constant with its sign depending on that of γ.

SUPPLEMENTAL MATERIALS

Supplement: It contains i) an algorithm to compute the impact matrix under finite-horizon restrictions, ii) an analysis of the relation between long-run and finite-horizon identification restrictions, and iii) additional Monte Carlo results on processes with four lags.

The supplemental materials are available from the website of the University of Regensburg Publication Server `http://epub.uni-regensburg.de/16901/`.

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TABLES

Table 1: Maximum likelihood results of the FIVAR_b model (12) for quarterly postwar US GDP and prices with different specifications for b and lag length p.

	p	LogLik	AIC	SC	d_1	d_2	b
	0	2349.0768	-21.1448	-21.1142	1.3034	1.5953	1.0000
b free	1	2363.7856	-21.2323	**-21.1250**	0.4682	1.7501	0.9486
	2	2370.1967	-21.2540	-21.0854	1.0816	1.8360	0.0234
	3	2374.9499	-21.2608	-21.0309	1.1870	2.1555	0.2274
	4	2380.3073	**-21.2730**	-20.9818	0.7240	1.7532	0.8978
	5	2382.5904	-21.2576	-20.9050	0.7066	1.8368	0.6446
	6	2386.3916	-21.2558	-20.8419	0.7793	1.8198	0.7443
$b = 1$	1	2363.6710	-21.2403	**-21.1483**	0.4425	1.7409	1.0000
	2	2365.6726	-21.2223	-21.0690	0.5066	1.8064	1.0000
	3	2371.9018	-21.2424	-21.0278	0.5897	1.9864	1.0000
	4	2379.9868	**-21.2792**	-21.0033	0.6593	1.7520	1.0000
	5	2380.1454	-21.2446	-20.9074	0.6420	1.7420	1.0000
	6	2385.1049	-21.2532	-20.8547	0.5907	1.7041	1.0000
$b = d_1$	1	2361.4947	-21.2207	**-21.1287**	0.6825	1.8225	0.6825
	2	2367.3646	-21.2375	-21.0842	0.5874	1.8089	0.5874
	3	2373.4640	-21.2564	-21.0418	0.6829	1.8614	0.6829
	4	2380.0863	**-21.2801**	-21.0042	0.8258	1.7723	0.8258
	5	2382.5725	-21.2664	-20.9292	0.6696	1.8038	0.6696
	6	2386.3870	-21.2647	-20.8662	0.7594	1.8041	0.7594

NOTE: For each specification of b the minimum of AIC and SC are in boldface.

Table 2: Likelihood ratio tests of nonfractional hypotheses about d_1 and d_2.

H_0	p	$b = 1$		$b = d_1$		b free	
		LR	p value	LR	p value	LR	p value
$d_1 = d_2 = 1$,	1	52.1221	**0.0000**	47.7695	**0.0000**	21.3168	**0.0000**
	4	16.7521	**0.0002**	16.9512	**0.0002**	10.6868	**0.0048**
$d_1 = 1$, $d_2 = 2$	1	22.3089	**0.0000**	17.9562	**0.0001**	13.9783	**0.0009**
	4	8.5724	**0.0138**	8.7715	**0.0125**	3.3064	0.1914
$d_1 = 1$	1	14.3683	**0.0002**	10.0157	**0.0016**	12.8102	**0.0003**
	4	3.8434	**0.0499**	4.0424	**0.0444**	2.0750	0.1497
$d_2 = 1$	1	48.3734	**0.0000**	11.3220	**0.0008**	15.5094	**0.0001**
	4	6.7125	**0.0096**	9.2694	**0.0023**	6.0274	**0.0141**
$d_2 = 2$	1	13.2040	**0.0003**	5.6926	**0.0170**	10.1641	**0.0014**
	4	2.8717	0.0902	1.7934	0.1805	2.2352	0.1349

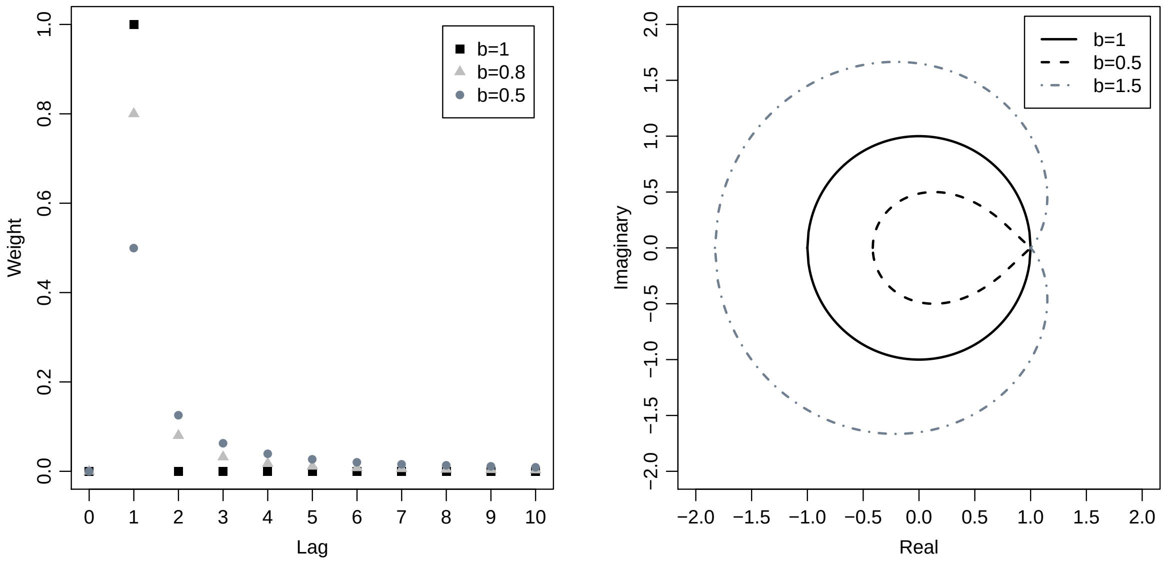
NOTE: p values smaller than 0.05 are in boldface.

Table 3: Estimated impact matrices $\boldsymbol{B}$ ($\times 10^3$) for different models.

	IVAR(1,1) model		IVAR(1,2) model		FIVAR$_{d_1}$ model	
	LRUS	LRRS	LRUS	LRRS	LRUS	LRRS
GDP	4.3717	7.4120	8.4460	1.0330	7.7323	3.0335
	(2.1474)	(3.1813)	(0.5210)	(0.8961)	(1.3501)	(1.9935)
Prices	-2.5051	1.2025	-0.4580	2.7209	-1.0977	2.4241
	(0.8674)	(0.6966)	(0.3825)	(0.1576)	(0.6464)	(0.4351)

NOTE: Bootstrap standard errors in parentheses.

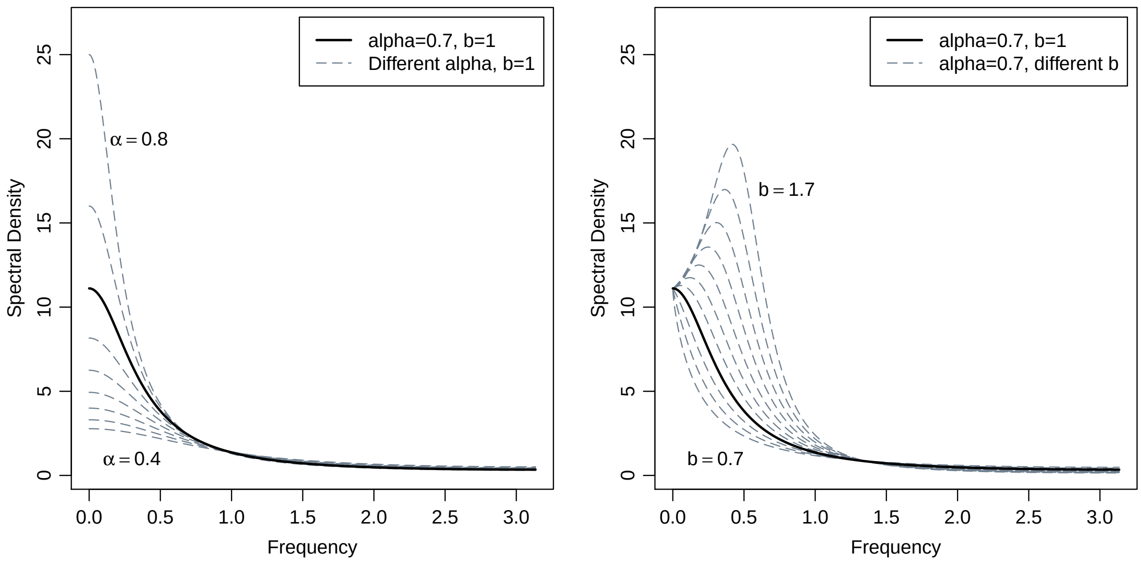
FIGURES



(a) Coefficients in the expansion of the fractional lag operator $L_b = c_1 L + c_2 L^2 + \ldots$ for different b.

(b) $\mathbb{C}_b$: Image of the unit disk under the mapping $f : z \mapsto 1 - (1 - z)^b$.

Figure 1: Illustration of the fractional lag operator and the stability condition of a VAR$_b$ process.



(a) Different α.

(b) Different b.

Figure 2: Spectral densities of an $\text{AR}_b(1)$ process $x_t = \alpha L_b x_t + u_t$. The parameter b adds flexibility to the spectral density and does not change the $I(0)$ property if the stability condition holds.

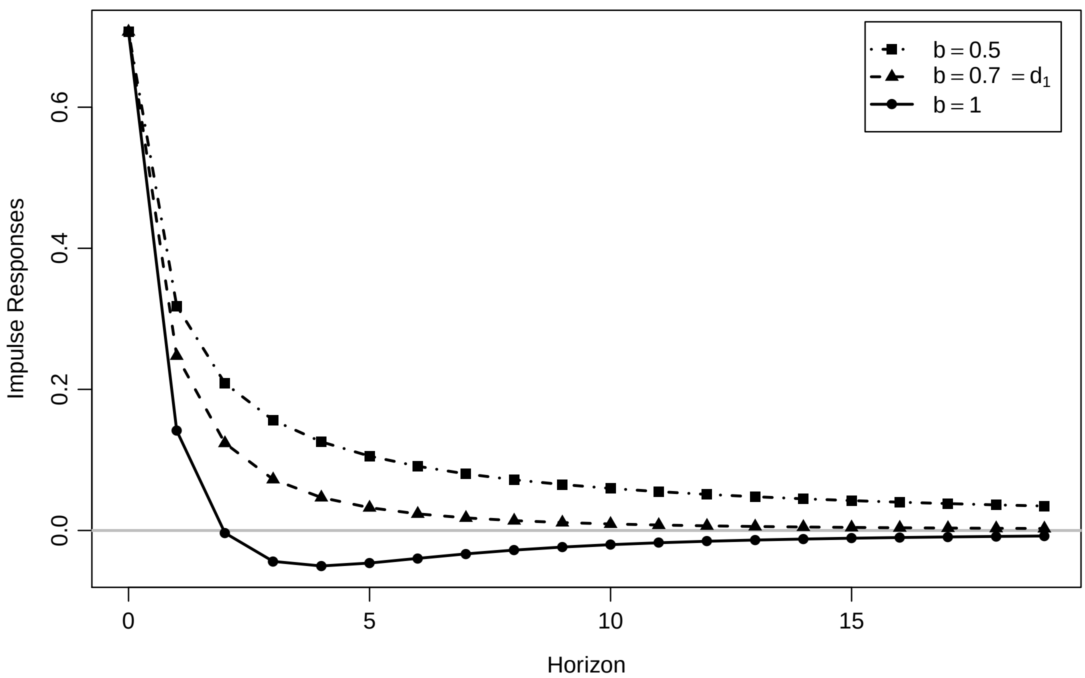


Figure 3: The impulse response function of the first variable to the second shock (LRRS) for FIVAR_b processes (14) with $b \in \{0.5, 0.7, 1\}$ and with the long-run restriction (LRR) (7) imposed.

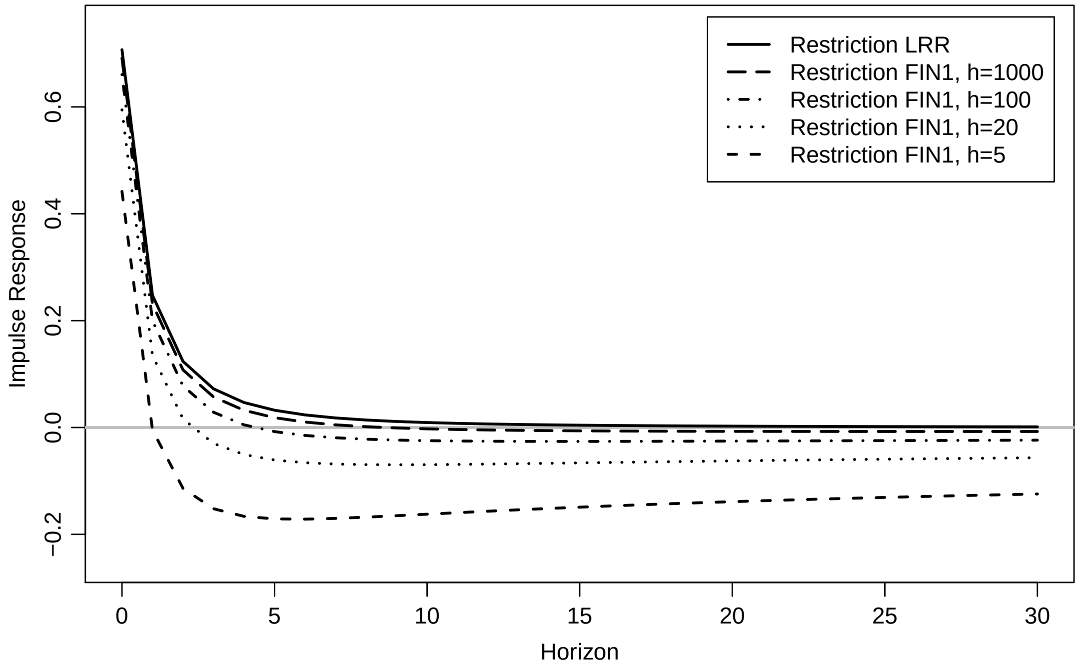


Figure 4: The impact of the long-run zero restriction (LRR) (7) and finite-horizon identification procedure FIN1 (17) on the impulse response function of the first variable to the second shock (LRRS) for the stylized FIVAR_b process (14) with $b = 0.7$.

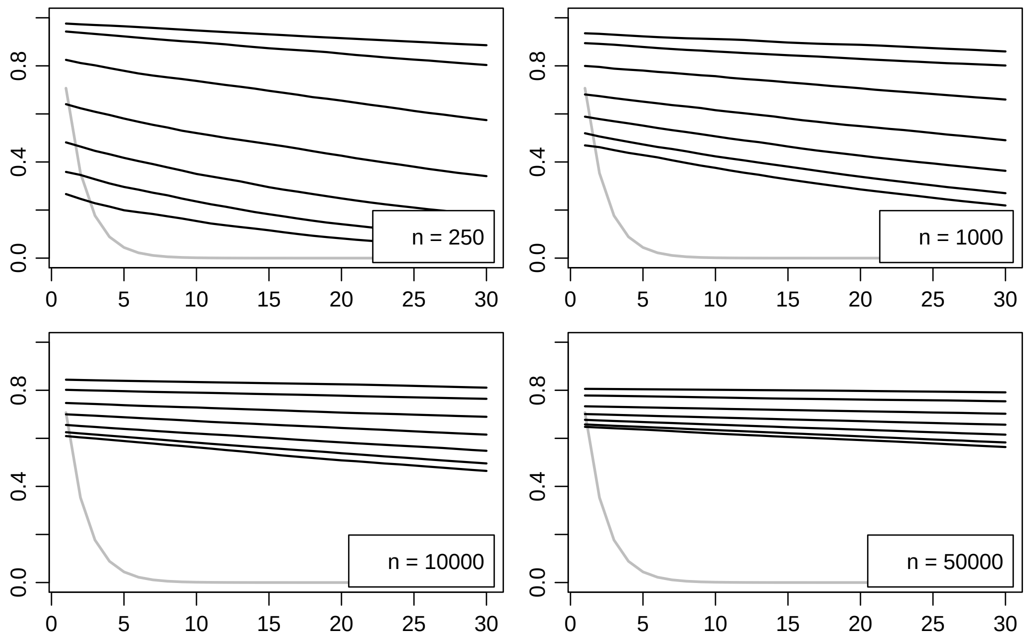


Figure 5: Simulated 5%, 10%, 25%, 50%, 75%, 90% and 95% quantiles of estimated impulse responses (black) and true impulse response function (grey) of the first variable to LRRS. The processes are generated by (22) with $\alpha = -\gamma = 0.5$, $d_1 = 1$, $d_2 = 1.7$ for different sample sizes n, while for the estimation $d_2 = 1$ is imposed.

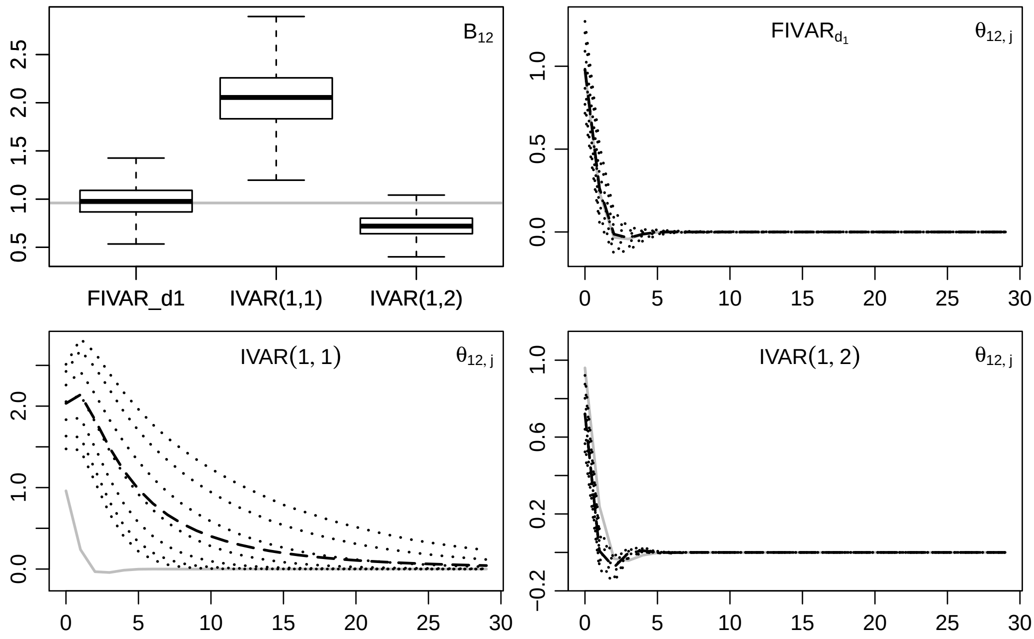


Figure 6: Upper left: Boxplots of estimated impact coefficients. All other: Mean (dashed) and quantiles (dotted) of estimated impulse responses of the first variable to LRRS according to a FIVAR$_{d_1}$, IVAR(1,1), and IVAR(1,2) specification. The grey solid lines correspond to the true impact coefficients and impulse responses of the FIVAR$_b$1 process (24) of which 5000 replications with 250 observations were drawn.

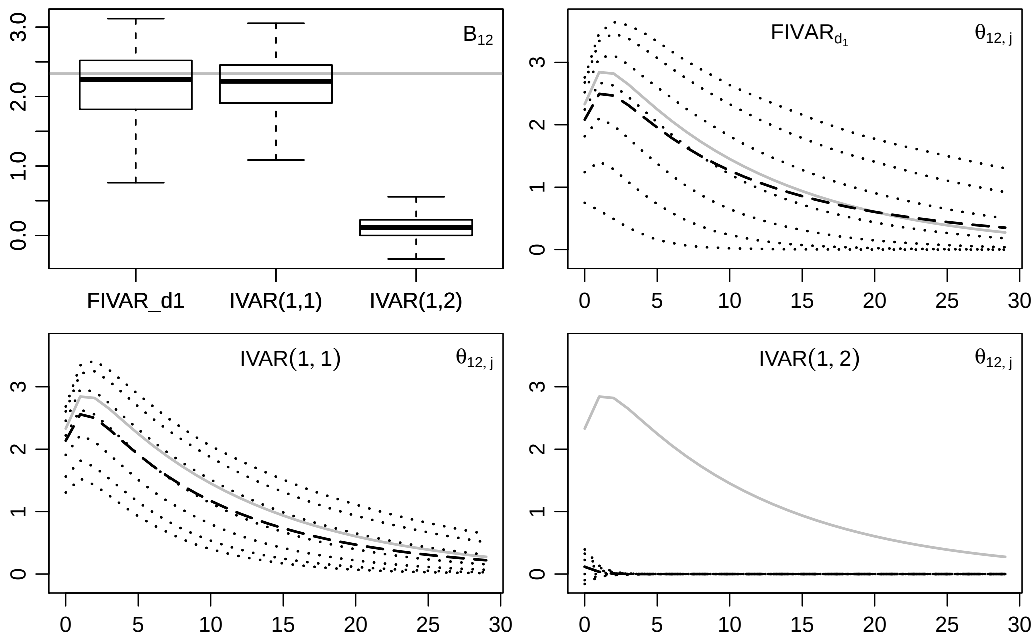


Figure 7: Upper left: Boxplots of estimated impact coefficients. All other: Mean (dashed) and quantiles (dotted) of estimated impulse responses of the first variable to LRRS according to a FIVAR$_{d_1}$, IVAR(1,1), and IVAR(1,2) specification. The grey solid lines correspond to the true impact coefficients and impulse responses of the IVAR1 process (25) of which 5000 replications with 250 observations were drawn.

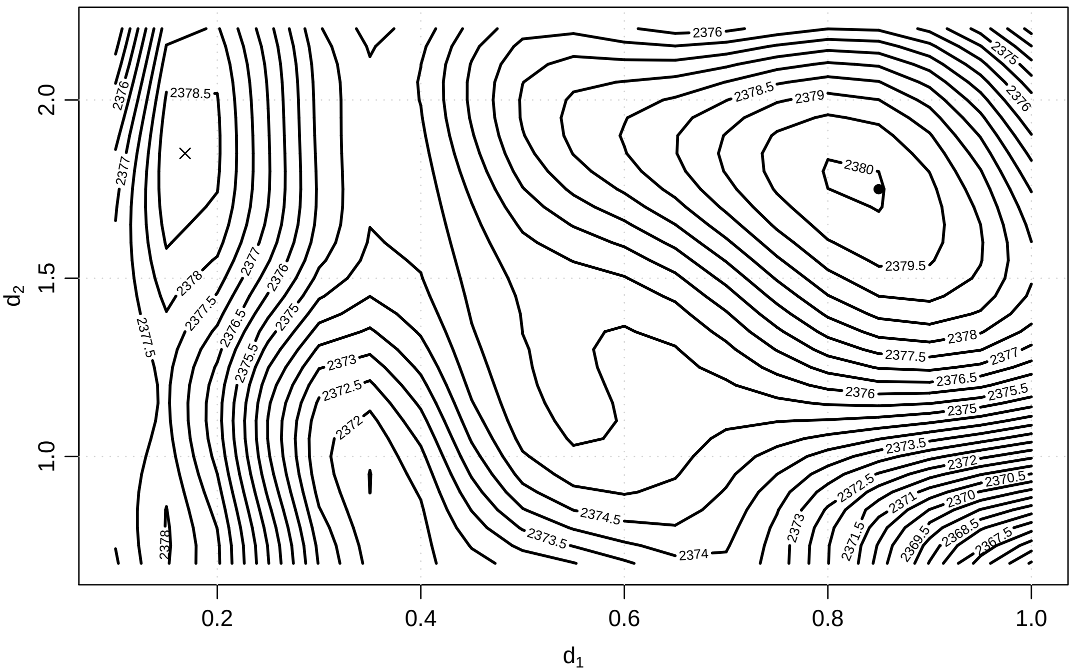
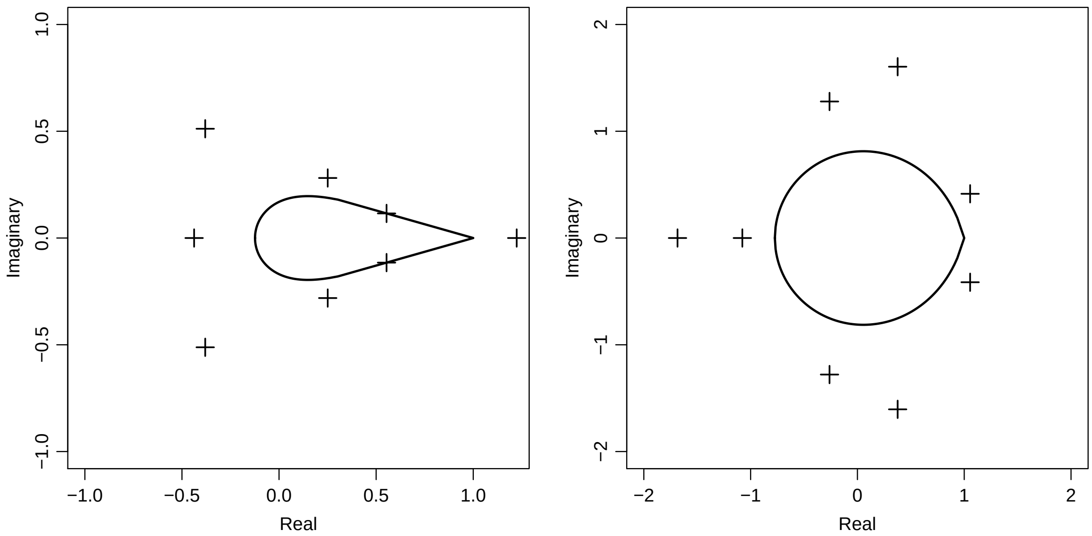


Figure 8: Profile Likelihood (20) of the FIVAR$_{d_1}$ model (12) with $p = 4$. The solid point indicates the global maximum, the cross denotes a local maximum.



(a) Local maximum at $\boldsymbol{d} = (0.17\ 1.85)'$

(b) Global maximum at $\boldsymbol{d} = (0.83\ 1.77)'$

Figure 9: Roots of $|\boldsymbol{A}(z)| = 0$ (denoted by $+$) and $\mathbb{C}_b$ (with boundary indicated by the solid line) from estimated FIVAR$_{d_1}$ models with $p = 4$. The stability condition of the VAR$_{d_1}$ polynomial requires the roots to lie outside $\mathbb{C}_{d_1}$.

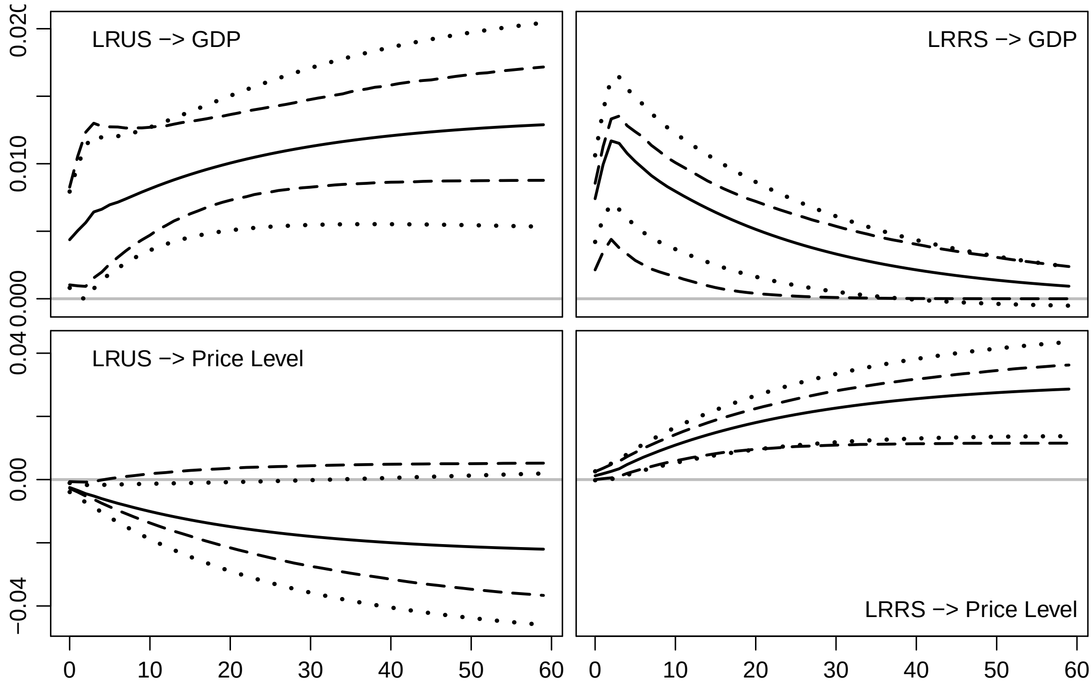


Figure 10: Estimated impulse responses of an IVAR(1,1) model with $p = 4$. Dotted lines are 90% confidence intervals based on bootstrap standard errors, dashed lines are bootstrap percentile confidence intervals.

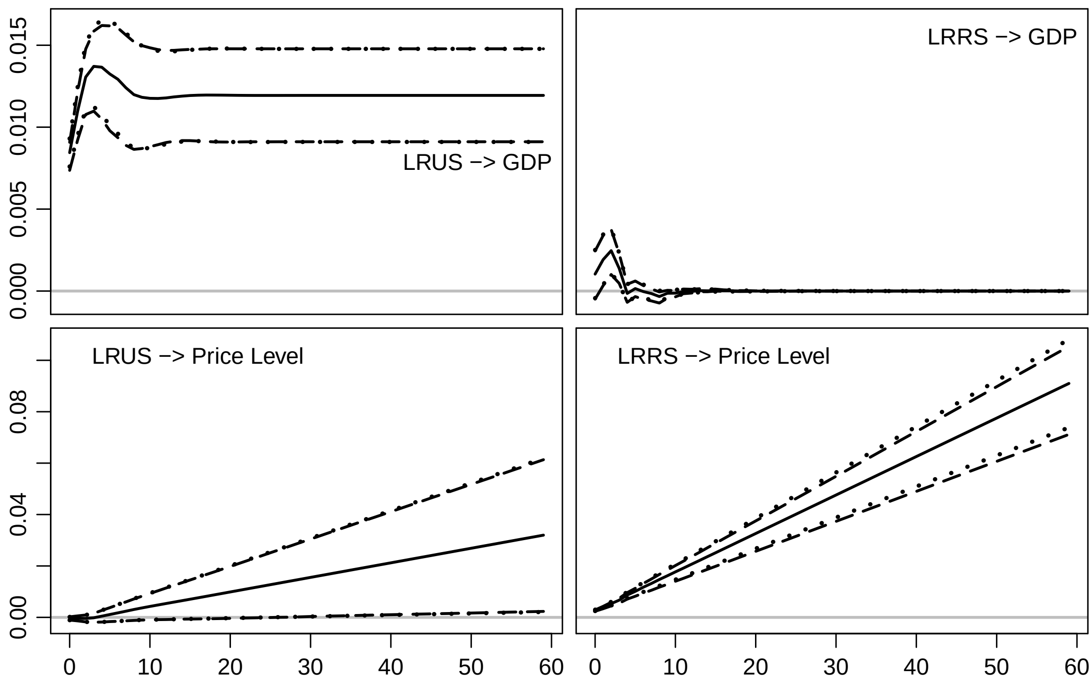


Figure 11: Estimated impulse responses of an IVAR(1,2) model with $p = 4$. Dotted lines are 90% confidence intervals based on bootstrap standard errors, dashed lines are bootstrap percentile confidence intervals.

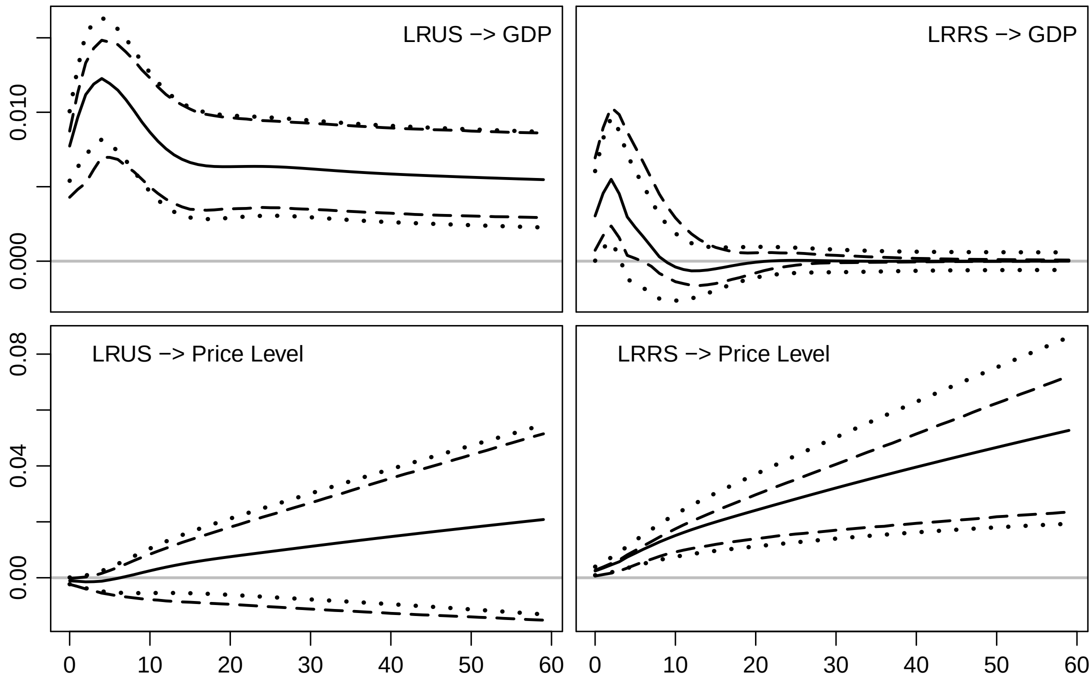


Figure 12: Estimated impulse responses of a FIVAR_{d_1} model with $p = 4$. Dotted lines are 90% confidence intervals based on bootstrap standard errors, dashed lines are bootstrap percentile confidence intervals.

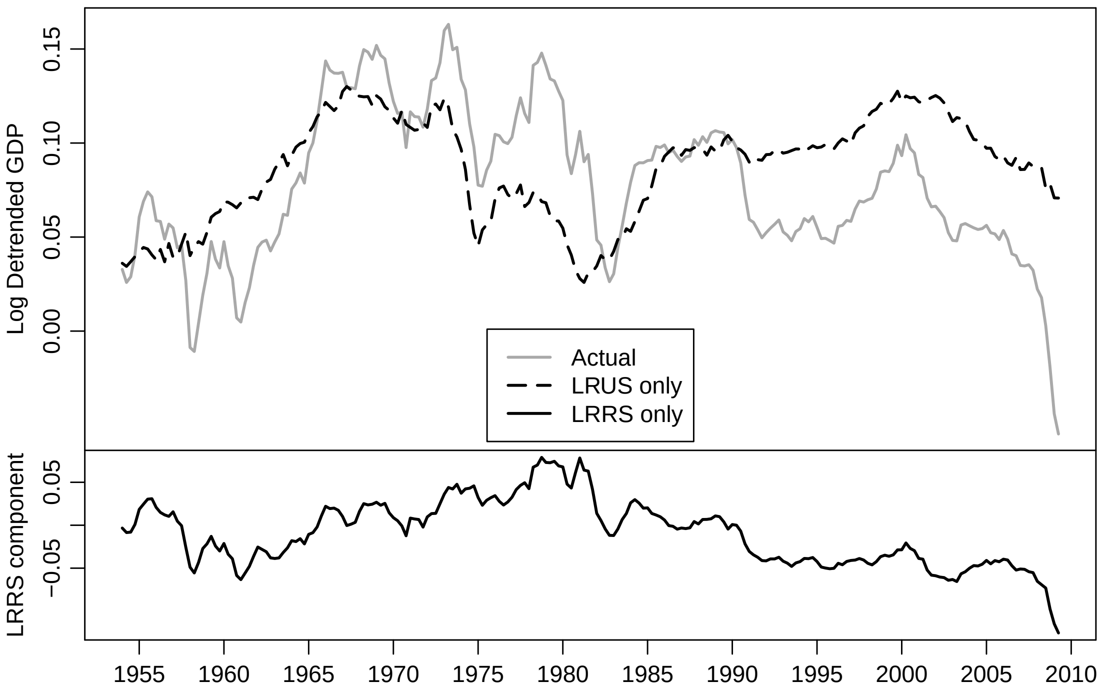


Figure 13: Decomposition (27) of detrended log GDP (solid grey) in LRUS (dashed) and LRRS (solid black) according to the estimated IVAR(1,1) model.

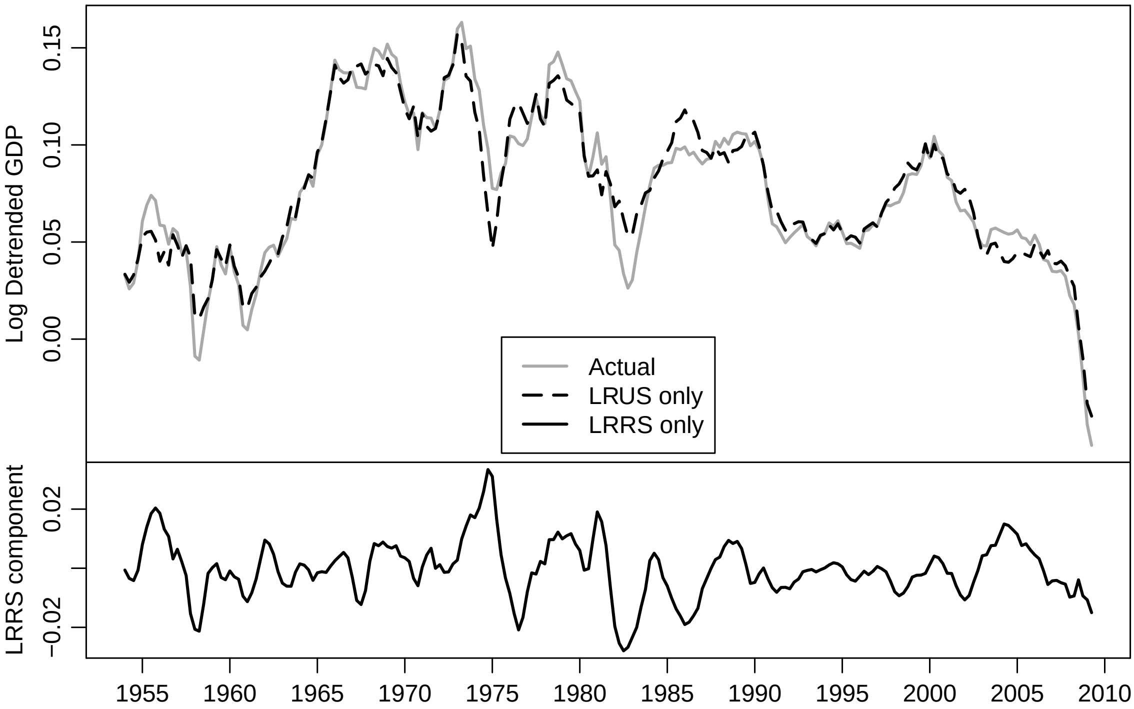


Figure 14: Decomposition (27) of detrended log GDP (solid grey) in LRUS (dashed) and LRRS (solid black) components according to the estimated FIVAR$_{d_1}$ model.

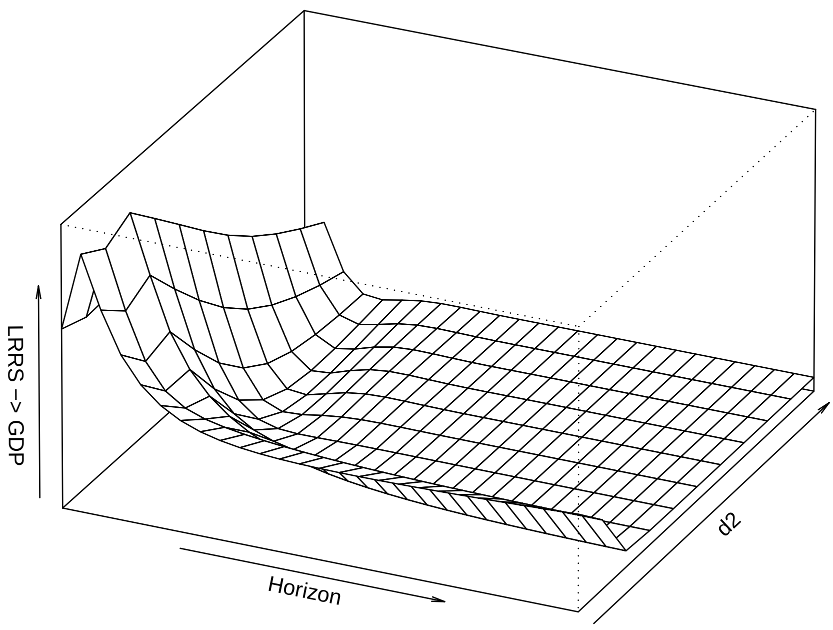


Figure 15: Impulse responses of the FIVAR$_{d_1}$ model with $p = 4$, where d_2 varies over [1;2] and all other parameters are estimated by maximum likelihood.