THE PERIODIC MOTIONS OF THE MAGNETIZATION IN SUPERFLUID ^{3}He B

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Exact periodic solutions of the Leggett equations for ^{3}He B are found and compared with existing asymptotic solutions.

The periodic motions of the magnetization are of particular interest for investigating magnetic substances by means of magnetic resonance. In the superfluid phases of ^{3}He the motion of the magnetization is governed by the Leggett equations [1]. The periodic solutions of the Leggett equations in the limit of strong magnetic fields $\omega_L \gg \Omega$ (where ω_L is the Larmor frequency and Ω is the frequency of the longitudinal oscillations) were found by Brinkman and Smith [2] (cf. also ref. [3]). These solutions represent the stationary precession of the magnetization. In the case of zero magnetic field ($\omega_L = 0$) the Leggett equations for the B phase were solved analytically [4,5]. In particular, a solution corresponding to periodic motion of the magnetization as well as of the order parameter – the so called "wall-pinned mode" (WPM) – was found. In a recent paper, Novikov [6] has shown that the Leggett equations for the B phase must have periodic solutions at *arbitrary* strength of the magnetic field. In the present paper we construct these solutions explicitly.

In order to account for relaxation processes we start with the Leggett–Takagi equations in the hydrodynamic limit [7]. It will be convenient to express them in terms of Euler angles. The order parameter in the B phase is the rotation matrix $\mathbf{R}(\theta, \boldsymbol{n})$, θ being the angle of rotation and $\boldsymbol{n}$ the direction of the rotation axis. In the course of motion $\mathbf{R}$ remains to be the rotation matrix; so all the motions of the order parameter can be represented as rotations of the coordinates (ξ, η, ζ) with respect to the fixed frame of reference (x, y, z) and can be parametrized by the Euler angles α, β, γ: $\mathbf{R}(\alpha, \beta, \gamma) = \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta)\mathbf{R}_z(\gamma)$, where $\mathbf{R}_z(\alpha)$ is a rotation around the z axis through an angle α. The dipole energy can be expressed by α, β, γ as follows:

$$U = \tfrac{8}{15}(\Omega^2/\omega_L)(\tfrac{1}{4} + \cos\theta)^2$$
$$= \tfrac{2}{15}(\Omega^2/\omega_L)[\cos\beta - \tfrac{1}{2} + (1 + \cos\beta)\cos\phi]^2 , \quad (1)$$

where $\phi = \alpha + \gamma$. The equations of motion in terms of Euler angles were formulated by Maki [8]. Here, we introduce as independent variables (together with α, β, γ) the ζ-projection of the spin, S_ζ, the hamiltonian conjugate to β, S_β, and $P = S_z - S_\zeta$. Normalization of spin is chosen so that it is measured in s^{-1} and in the equilibrium $|\boldsymbol{S}| = \omega_L$. We also include relaxation. We then have the following system of equations:

$$\dot{S}_\zeta = -(\partial U/\partial\phi)\omega_L , \quad (2)$$

$$\dot{S}_\beta + \frac{\sin\beta}{(1+\cos\beta)^2}\left(\frac{P}{1-\cos\beta} + S_\zeta\right) \times \left(S_\zeta - \frac{\cos\beta}{1-\cos\beta}P\right) + \frac{\partial U}{\partial\beta}\omega_L$$
$$= \frac{\kappa}{2}\frac{\partial U}{\partial(\cos\theta)}\frac{\sin\phi}{\sin\beta}[P\cos\beta - (1-\cos\beta)S_\zeta] , \quad (3)$$

$$\dot{P} = \tfrac{1}{2}\kappa\,[\partial U/\partial(\cos\theta)]\{(1+\cos\phi)$$
$$\times\,[P + (1-\cos\beta)S_\xi] - \sin\phi\sin\beta\,S_\beta\}\,, \tag{4}$$
$$\dot{\alpha} + \omega_L = (\sin^2\beta)^{-1}[P + (1-\cos\beta)S_\xi]$$
$$+\tfrac{1}{2}\kappa\,[\partial U/\partial(\cos\theta)]\sin\phi\,, \tag{5}$$
$$\dot{\beta} = S_\beta + \tfrac{1}{2}\kappa\sin\beta\,(1+\cos\phi)\partial U/\partial(\cos\theta)\,, \tag{6}$$
$$\dot{\phi} = -\omega_L + (1+\cos\beta)^{-1}(P + 2S_\xi)$$
$$+\kappa\sin\phi\,\partial U/\partial(\cos\theta)\,. \tag{7}$$

It is essential that the dipole energy U does not depend on α. So one can look for solutions where only α changes with time. Therefore, we set $\dot{S}_\xi = \dot{S}_\beta = \dot{P} = \dot{\beta} = \dot{\phi} = 0$. From eq. (2) one obtains then

$$(1+\cos\beta)\sin\phi\,\partial U/\partial(\cos\theta) = 0\,. \tag{8}$$

$\cos\beta = -1$ is a singular point geometrically, so we consider only the cases (i) $\partial U/\partial(\cos\theta) = 0$ or (ii) $\sin\phi = 0$.

In the case (i) one has

$$\partial U/\partial\beta = -\tfrac{1}{2}\sin\beta\,(1+\cos\phi)\partial U/\partial(\cos\theta) = 0\,,$$

and therefore we find from eqs. (3), (6), (7):

$$S_\beta = 0\,,\quad S_\xi = \omega_L\,,\quad P = \omega_L(\cos\beta - 1)\,, \tag{9}$$

and from eq. (5) we obtain $\dot{\alpha} = -\omega_L$, so that this solution represents a precession at Larmor frequency, and it is essentially the solution of Brinkman and Smith [2].

In the case (ii) we neglect relaxation terms ($\kappa = 0$) for a beginning. From eqs. (3), (6), (7) we have

$$(\omega_L - S_\xi)(S_\xi - \omega_L\cos\beta)$$
$$= (1-\cos\beta)^2\omega_L\,\partial U/\partial(\cos\beta)\,, \tag{10}$$
$$P = \omega_L(1+\cos\beta) - 2S_\xi\,, \tag{11}$$
$$S_\beta = 0\,. \tag{12}$$

Since $\sin\phi = 0$ we have $\cos\phi = \pm 1$. Further analysis shows that only $\cos\phi = +1$ gives a stable stationary solution. Using the relation

$$1 + 2\cos\theta = \cos\beta + \cos\phi + \cos\beta\cos\phi$$

we obtain for $\cos\phi = 1$

$$\cos\theta = \cos\beta\,. \tag{13}$$

From eqs. (1), (5) and (10) we find the frequency of the periodic motion

$$\dot{\alpha}(\dot{\alpha} + \omega_L) = -\tfrac{16}{15}\Omega^2(\cos\beta + \tfrac{1}{4})\,. \tag{14}$$

In the limit of strong fields ($\dot{\alpha} \approx -\omega_L$) one recovers from (14) the known formula for the shift of the transverse NMR frequency [2,3]. In zero field ($\omega_L = 0$) eq. (14) coincides with the ringing frequency of the WPM [9]. In the intermediate region it gives the exact expression for the angular velocity of periodic motion as a function of the angle β, defined by the initial conditions.

It is interesting to perturb the stationary solutions and to find the frequencies of the resulting small oscillations. The full analysis is lengthy and will be published separately together with a more complete account of the analytical results [10]. Here, we will only make use of the system (2)–(7) to correct for an earlier error [11] in the calculation of the damping of the small oscillations having a characteristic frequency of the order of Ω. Linearizing (2)–(7) in the vicinity of both solutions (i) and (ii) shows that the frequency of this mode in the limit of strong fields and small κ is given by

$$\omega = \omega_\parallel(0)[1 - \mathrm{i}\omega_\parallel(0)\kappa/2\omega_L]\,, \tag{15}$$

with

$$\omega_\parallel(0) = \Omega(\tfrac{1}{4} + \cos\beta)^{1/2}$$

for solution (i) and $\cos\beta > -\tfrac{1}{4}$, and with

$$\omega_\parallel(0) = \Omega[-\tfrac{8}{15}(1+\cos\beta)(\tfrac{1}{4} + \cos\beta)]^{1/2}$$

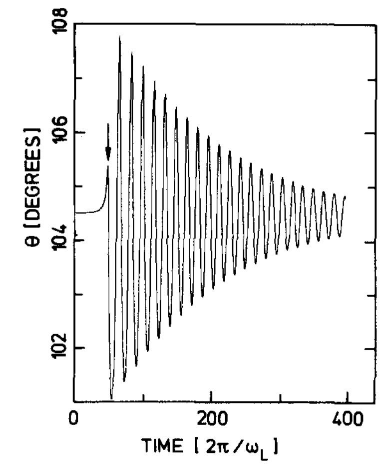

Fig. 1. Oscillations of the rotation angle θ after a $\beta = 100°$ tipping pulse. The arrow indicates the end of the tipping pulse. The frequency and the damping are given by eq. (15). Parameters: $\omega_L = 3.3 \times 10^6$ Hz, $\Omega = 8.6 \times 10^5$ Hz, $\kappa = 0.83$.

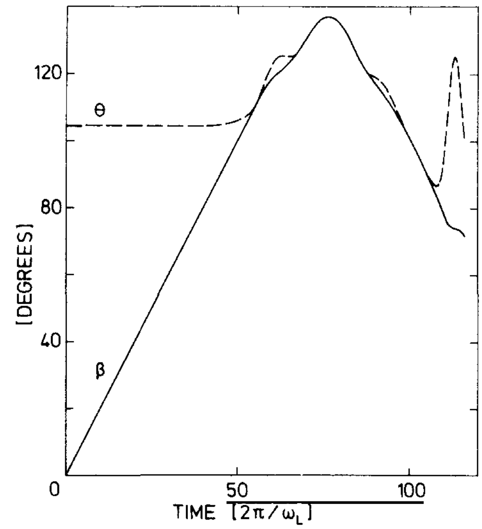

Fig. 2. Time dependence of the tipping angle β and the rotation angle θ during the tipping process. Parameters: $H_1/H_0 = 1/180$; $\omega_L = 3.3 \times 10^6$ Hz, $\Omega = 5.5 \times 10^5$ Hz.

for solution (ii) and $\cos\beta < -\frac{1}{4}$. Formula (15) differs by the factor $(1 + \cos\beta)/2$ in the damping term from the previously found expression [11].

The mistake was first found by a numerical analysis of the Leggett–Takagi equations. In these calculations pulsed NMR experiments were numerically simulated and it was found that after the tipping pulse the variables are oscillating until they reach the stationary solution. As an example, oscillations of the rotation angle θ about its equilibrium value of arccos$(-\frac{1}{4}) \approx 104°$ after a 100° pulse are shown in fig. 1. Numerical calculations are particularly useful for studying transient phenomena such as the behavior of the magnetization during the tipping process. Fig. 2 shows the time dependence of the tipping angle $\beta = \arccos(S_z/S)$ and of the rotation angle θ. One can see substantial deviations from the stationary solution in the region where $\omega_\| = 0$.

The results presented here make it possible to interpret pulsed NMR experiments in ^{3}He B not only in high or zero magnetic field but also in the intermediate region as well.

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