On an Extension of the Complex Series Method for the Convergence Acceleration of Orthogonal Expansions

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Abstract

Recently, Sidi [Sidi, A. (1995): Acceleration of convergence of (generalized) Fourier series by the d-transformation. Ann. Numer. Math. **2**, 381–406] proposed a method for the convergence acceleration of certain orthogonal expansions. The present contribution shows that it is possible to extend the method proposed by Sidi to a wider class of problems by simple means. The extended method is both simpler and also more effective. The theoretical basis for the latter method is analyzed. An example is presented that shows that it is possible to obtain the same accuracy using only half of the number of terms that are required in the method of Sidi.

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1 Introduction

Trigonometric Fourier series and their generalizations occur in many branches of applied mathematics. Especially in the case of slow convergence, their usage can often be improved using nonlinear convergence acceleration methods. These methods are often also able to sum such series in the case of divergence.

Sidi [1] has proposed a method for infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)] \,, \tag{1}$$

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where the function $\phi_n(x)$ and $\psi_n(x)$ behave similarly to the sine and cosine functions for large n. Examples are Fourier-Bessel, Legendre, and Chebyshev series (compare also Table 1). This method will be described below in more detail. Here, it suffices to say that the method essentially rewrites the given problem series as a sum of certain complex series The value of each of these complex series is then approximated separately by extrapolation using the $d^{(m)}$ transformation of Levin and Sidi [2] with a suitable m. Finally, the problem series is calculated as the sum of these approximate values. This approach will be called the *method of the attached series*.

Let us remark here that, of course, it is possible to use nearly the same approach in combination with other sequence transformations instead of the $d^{(m)}$ transformation as pointed out also in Ref. [1] for the ϵ algorithm [3] and the u transformation of Levin [4]. The latter approach, for instance, will be called the *method of the attached series via the u transformation.* This naming convention then will be applied *mutatis mutandis* also to using other sequence transformations similarly. That such approaches yield also good results is demonstrated in the case of trigonometric series in [5, 6] for the Levin transformation [4] and the $\mathcal{H}$ transformation [7, 5, 8, 9].

As in the case of the $d^{(m)}$ transformation that produces reliable results in the vicinity of singularities operating on sequence elements s_{R_ℓ} with $R_\ell = \tau\ell$ with $\tau > 1$ [1], it it also possible to use a similar τ-fold frequency approach in combination with other sequence transformations as has been demonstrated recently [8, 9, 10, 6] for the case of the $\mathcal{H}$ transformation and the $\mathcal{I}$ transformation [5, 10] and will be shown in Sec. 6 also for the $\mathcal{K}$ transformation [11]. Then, results of similar quality as for the $d^{(2)}$ transformation are obtained near singularities with less numerical effort. Numerical examples will be given below.

In a different approach for the treatment of trigonometric Fourier series that is somewhat more complicated than the τ-fold frequency approach, Oleksy [12] has shown that the use of a preprocessing transformation, followed by a convergence acceleration of a number of related series, enhances the applicability of the Levin transformation, the ϵ algorithm and the $\mathcal{H}$ transformation near singularities enormously, similarly as the τ-fold frequency approach.

In Ref. [6], an *extended method of the attached series* was introduced that works also for more complicated trigonometric Fourier series of the type

$$s = \sum_{n=0}^{\infty} \prod_{j=1}^{L} \left(a_n^{(j)} \cos(n\alpha_j) + b_n^{(j)} \sin(n\alpha_j) \right) \tag{2}$$

that depend on several frequencies $\{\alpha_j\}_{j=1}^{L}$, assuming that all the coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are adequate, where n dependent quantities like $g(n)$ or λ_n are called *adequate* if they are asymptotically of the form

$$q^n n^\epsilon \sum_{j=0}^{\infty} \xi_j n^{-j} \,, \qquad (n \to \infty \,,\, \xi_0 \neq 0) \tag{3}$$

for some complex nonzero q and ϵ. Let us remark that (finite) products of adequate quantities are also adequate.

The basic idea is to rewrite the series as a sum of 4^L power series with adequate terms, for which then all the usual, highly developed nonlinear extrapolation methods for power series may be applied separately. Such methods are usually derived from model sequences or by iteration of simple transformations [13, 14, 15, 16] and can be used to accelerate convergent series and to sum divergent ones.

It will be shown in the present contribution that such an *extended method of the attached series* may also be used in the case of generalized Fourier series of the form

$$s = \sum_{n=0}^{\infty} \prod_{j=1}^{L} \left(a_n^{(j)} \phi_n^{(j)}(x^{(j)}) + b_n^{(j)} \psi_n^{(j)}(x^{(j)}) \right) \tag{4}$$

where the functions $\phi_n^{(j)}(x)$ and $\psi_n^{(j)}(x)$ generalize sine and cosine functions as in the method of the attached series. This will be discussed in more detail in Sec. 4.

2 Definitions and Basic Relations

First, we define the τ-fold-frequency method [8, 9, 10]. Consider a series s with terms u_n and partial sums s_n as given by

$$s = \sum_{j=0}^{\infty} u_j \ , \qquad s_n = \sum_{j=0}^{n} u_j \ . \tag{5}$$

By combining τ consecutive terms, one obtains a new series with terms $\check{u}_j$ and with partial sums $\check{s}_n = s_{\tau\, n}$ according to

$$s = \sum_{j=0}^{\infty} \check{u}_j \ , \qquad \check{s}_n = \sum_{j=0}^{n} \check{u}_j \tag{6}$$

where

$$\check{u}_0 = u_0 \ , \qquad \check{u}_j = \sum_{k=1}^{\tau} u_{\tau(j-1)+k} \text{ for } j > 0 \ . \tag{7}$$

Now, we will define some sequence transformations.

The Levin transformation [4] is defined by

$$\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n) \ = \ \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{s_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{1}{\omega_{n+j}}} \ . \tag{8}$$

It can also be computed recursively [16]. Its u variant is defined by

$$u_k^{(n)}(\beta, s_n) = \mathcal{L}_k^{(n)}(\beta, s_n, (n+\beta)\triangle s_{n-1}) \tag{9}$$

where the $\triangle$ denotes the forward difference operator with respect to the variable n acting as

$$\triangle f(n) = f(n+1) - f(n)\,, \qquad \triangle g_n = g_{n+1} - g_n\,. \tag{10}$$

If the Levin transformation is applied to , and if the remainder estimates $\omega_n = (n + \beta/\tau)(s_{\tau\, n} - s_{(\tau\, n)-1})$ are used, then one obtains nothing but the $d^{(1)}$ transformation [2] with $R_\ell = \tau\ell$ for $\tau \in \mathbb{N}$. This holds because the latter is given by the transformation (see [1, Eq. 4.12])

$$W_\nu^{(n)} = \frac{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1} s_{\tau\, n}/(s_{\tau\, n} - s_{(\tau\, n)-1})\right]}{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1}/(s_{\tau\, n} - s_{(\tau\, n)-1})\right]} \tag{11}$$

In fact, the identity

$$W_{\nu-1}^{(n)} = \mathcal{L}_\nu^{(n)}(\beta/\tau, s_{\tau\, n}, (n+\beta/\tau)(s_{\tau\, n} - s_{\tau\, n-1})) \tag{12}$$

holds. We remark that for $\tau \neq 1$ this is not identical to the u variant of the Levin transformation as applied to the partial sums $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ because in the case of the u variant one would have to use the remainder estimates $\omega_n = (n + \beta')(s_{\tau\, n} - s_{\tau\,(n-1)})$.

The iteratively defined $\mathcal{J}$ transformation that was first introduced in [5] and was characterized in detail in a series of papers [14, 17, 15] also belongs to the class of Levin-type transformations. It may be defined via

$$\begin{aligned}
D_n^{(0)} &= 1/\omega_n\,, \qquad N_n^{(0)} = s_n/\omega_n\,, \\
D_n^{(k+1)} &= (D_{n+1}^{(k)} - D_n^{(k)})/\delta_n^{(k)}\,, \\
N_n^{(k+1)} &= (N_{n+1}^{(k)} - N_n^{(k)})/\delta_n^{(k)}\,, \\
N_n^{(k)}/D_n^{(k)} &= \mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{\delta_n^{(k)}\})\,.
\end{aligned} \tag{13}$$

Here, and in the following, we assume that $\delta_n^{(k)} \neq 0$ for all n and k, and that the remainder estimates $\{\omega_n\}$ are restricted by the conditions $\omega_n \neq 0$ and $D_n^{(k)} \neq 0$ for all n and k. Compare also [14, Theorem 5].

The ${}_p\mathbf{J}$ transformation may be regarded as the special case of the $\mathcal{J}$ transformation corresponding to

$$\delta_n^{(k)} = \frac{1}{\left(n + \beta + (p-1)k\right)_2}\,, \tag{14}$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$ denotes a Pochhammer symbol, that is,

$${}_p\mathbf{J}_n^{(k)}(\beta, \{s_n\}, \{\omega_n\}) = \mathcal{J}_n^{(k)}(\{s_n\}, \{\omega_n\}, \{1/(n+\beta+(p-1)k)_2\})\,. \tag{15}$$

The $\mathcal{H}$ transformation is defined by the recursive scheme [7]

$$\begin{aligned}
\mathcal{Z}_n^{(0)} &= (n+\beta)^{-1}\, s_n/\omega_n \,, \qquad \mathcal{N}_n^{(0)} = (n+\beta)^{-1}/\omega_n \,, \\
\mathcal{Z}_n^{(k)} &= (n+\beta)\mathcal{Z}_n^{(k-1)} + (n+2k+\beta)\mathcal{Z}_{n+2}^{(k-1)} \\
&\quad - 2\cos(\alpha)(n+k+\beta)\mathcal{Z}_{n+1}^{(k-1)} \,, \\
\mathcal{N}_n^{(k)} &= (n+\beta)\mathcal{N}_n^{(k-1)} + (n+2k+\beta)\mathcal{N}_{n+2}^{(k-1)} \\
&\quad - 2\cos(\alpha)(n+k+\beta)\mathcal{N}_{n+1}^{(k-1)} \,, \\
\frac{\mathcal{Z}_n^{(k)}}{\mathcal{N}_n^{(k)}} &= \mathcal{H}_n^{(k)}(\alpha,\beta,\{s_n\},\{\omega_n\}) \,.
\end{aligned} \tag{16}$$

and is exact for model sequences of the form

$$s_n = s + \omega_n \left(\mathrm{e}^{\mathrm{i}\alpha n} \sum_{j=0}^{k-1} c_j (n+\beta)^{-j} + \mathrm{e}^{-\mathrm{i}\alpha n} \sum_{j=0}^{k-1} d_j (n+\beta)^{-j} \right) \tag{17}$$

with arbitrary coefficients c_j and d_j and remainder estimates ω_n. A generalized $\mathcal{H}$ transformation has also been introduced [8, 9].

The $\mathcal{I}$ transformation may be defined as [5, 10]

$$\begin{aligned}
\mathbf{N}_n^{(0)} &= s_n/\omega_n \,, \qquad \mathbf{D}_n^{(0)} = 1/\omega_n \,, \\
\mathbf{N}_n^{(k+1)} &= \left(\mathbf{N}_{n+2}^{(k)} - 2\cos(\alpha)\, \mathbf{N}_{n+1}^{(k)} + \mathbf{N}_n^{(k)} \right) \Big/ \Delta_n^{(k)} \,, \\
\mathbf{D}_n^{(k+1)} &= \left(\mathbf{D}_{n+2}^{(k)} - 2\cos(\alpha)\, \mathbf{D}_{n+1}^{(k)} + \mathbf{D}_n^{(k)} \right) \Big/ \Delta_n^{(k)} \,, \\
\frac{\mathbf{N}_n^{(k)}}{\mathbf{D}_n^{(k)}} &= \mathcal{I}_n^{(k)}(\alpha,\{s_n\},\{\omega_n\},\{\Delta_n^{(k)}\}) \,.
\end{aligned} \tag{18}$$

The $\mathcal{I}$ transformation is a generalization of the transformation

$$\begin{aligned}
\mathrm{N}_n^{(0)} &= s_n/\omega_n \,, \mathrm{D}_n^{(0)} = 1/\omega_n, \\
\mathrm{N}_n^{(k+1)} &= (n+\beta)^{\gamma} \left(\mathrm{N}_{n+2}^{(k)} - 2\cos(\alpha)\, \mathrm{N}_{n+1}^{(k)} + \mathrm{N}_n^{(k)} \right) , \\
\mathrm{D}_n^{(k+1)} &= (n+\beta)^{\gamma} \left(\mathrm{D}_{n+2}^{(k)} - 2\cos(\alpha)\, \mathrm{D}_{n+1}^{(k)} + \mathrm{D}_n^{(k)} \right) , \\
\frac{\mathrm{N}_n^{(k)}}{\mathrm{D}_n^{(k)}} &= \mathrm{I}_n^{(k)}(\alpha,\beta,\gamma,\{s_n\},\{\omega_n\})
\end{aligned} \tag{19}$$

as seen by substituting $\Delta_n^{(k)} = (n+\beta)^{-\gamma}$. The transformation (19) was introduced in [5] and there, it was called the H transformation. The $\mathcal{K}$ transformation is a generalization of the $\mathcal{I}$ transformation given by [11]

$$\begin{aligned}
\mathrm{N}_n^{(0)} &= s_n/\omega_n \,, \qquad \mathrm{D}_n^{(0)} = 1/\omega_n \,, \\
\mathrm{N}_n^{(k+1)} &= \left(\zeta_{n+k}^{(0)} \mathrm{N}_n^{(k)} + \zeta_{n+k}^{(1)}\, \mathrm{N}_{n+1}^{(k)} + \zeta_{n+k}^{(2)} \mathrm{N}_{n+2}^{(k)} \right) \Big/ \delta_n^{(k)} \,, \\
\mathrm{D}_n^{(k+1)} &= \left(\zeta_{n+k}^{(0)} \mathrm{D}_n^{(k)} + \zeta_{n+k}^{(1)}\, \mathrm{D}_{n+1}^{(k)} + \zeta_{n+k}^{(2)} \mathrm{D}_{n+2}^{(k)} \right) \Big/ \delta_n^{(k)} \,, \\
\frac{\mathrm{N}_n^{(k)}}{\mathrm{D}_n^{(k)}} &= \mathcal{K}_n^{(k)}(\{\delta_n^{(k)}\},\{\zeta_n^{(j)}\},\{s_n\},\{\omega_n\}) \,.
\end{aligned} \tag{20}$$

Here, the ζ's are the coefficients of the three-term recurrence relation

$$\zeta_n^{(0)} v_n + \zeta_n^{(1)} v_{n+1} + \zeta_n^{(2)} v_{n+2} = 0 \tag{21}$$

of some system of orthogonal polynomials. The $\mathcal{K}$ transformation may be obtained by iterating the simple transformation

$$s_n^{(1)} = \frac{\zeta_n^{(0)} \dfrac{s_n}{\omega_n} + \zeta_n^{(1)} \dfrac{s_{n+1}}{\omega_{n+1}} + \zeta_n^{(2)} \dfrac{s_{n+2}}{\omega_{n+2}}}{\zeta_n^{(0)} \dfrac{1}{\omega_n} + \zeta_n^{(1)} \dfrac{1}{\omega_{n+1}} + \zeta_n^{(2)} \dfrac{1}{\omega_{n+2}}} \tag{22}$$

that is exact for model sequences of the form

$$s_n = s + \omega_n \left(c\, P_n + d\, Q_n \right), \tag{23}$$

with coefficients c and d and where P_n and Q_n are two linearly independent solutions of the recurrence relation (21).

Now, some sets of sequences are defined.

A sequence $\{u_n\}_{n=0}^{\infty}$ is called *nonvanishing* if $u_n \neq 0$ for all n. The set of all nonvanishing sequences is denoted as $\mathcal{V}$.

A sequence $\{u_n\}_{n=0}^{\infty}$ is called *(q, ϵ)-adequate* if it possesses the asymptotic expansion for large n of the form

$$u_n \sim q^n\, n^{\epsilon} \sum_{j=0}^{\infty} \nu_j n^{-j} \tag{24}$$

with $q \neq 0$ and $\nu_0 \neq 0$ and complex ϵ. In this case, the sequence $\{\nu_j\}_{j=0}^{\infty}$ is called the coefficient sequence of $\{u_n\}$. The set of all (q, ϵ)-adequate sequences is denoted as $\mathcal{A}(q, \epsilon)$.

Lemma 1 *If $\{u_n\} \in \mathcal{A}(q, \epsilon)$ with coefficient sequence $\{\nu_j\}$ and $\{u'_n\} \in \mathcal{A}(q', \epsilon')$ with coefficient sequence $\{\nu'_j\}$ then $\{u_n\, u'_n\} \in \mathcal{A}(qq', \epsilon + \epsilon')$ with coefficient sequence*

$$\{ \sum_{a+b=j} \nu_a \nu'_b \}_{j=0}^{\infty} .$$

Proof: This follows from the result [18, p.125]

$$v_n v'_n \sim \sum_{j=0}^{\infty} \left(\sum_{a+b=j} \nu_a \nu'_b \right) n^{-j} \tag{25}$$

for asymptotic expansions

$$v_n \sim \sum_{j=0}^{\infty} \nu_j n^{-j} \tag{26}$$

and

$$v'_n \sim \sum_{j=0}^{\infty} \nu'_j n^{-j} \tag{27}$$

on putting $v_n = u_n q^{-n} n^{-\epsilon}$ and $v'_n = u'_n q'^{-n} n^{-\epsilon'}$.

Induction leads to the following corollary:

Corollary 1 *If* $\{u_n^{(k)}\}_{n=0}^{\infty} \in \mathcal{A}(q^{(k)}, \epsilon^{(k)})$ *with coeffifient sequences* $\{\nu_j^{(k)}\}_{j=0}^{\infty}$ *for* $k = 1, 2, \ldots, L$ *then*

$$\left\{ \prod_{k=1}^{L} u_n^{(k)} \right\} \in \mathcal{A}\left(\prod_{k=1}^{L} q^{(k)}, \sum_{k=1}^{L} \epsilon^{(k)} \right)$$

with coefficient sequence

$$\left\{ \sum_{a_1+\cdots+a_L=j} \prod_{k=1}^{L} \nu_{a_k}^{(k)} \right\}_{j=0}^{\infty} .$$

These results will be used later.

3 The method of Sidi

Sidi [1] has proposed a method for infinite series of the form

$$F(x) = \sum_{n=0}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)] \,, \tag{28}$$

where the function $\phi_n(x)$ and $\psi_n(x)$ are assumed to satisfy

$$\rho_n^{\pm}(x) = \phi_n(x) \pm \mathrm{i}\,\psi_n(x) = \exp(\pm\,\mathrm{i}\,n\,\omega x) g_n^{\pm}(x) \,, \tag{29}$$

where ω is some fixed real parameter, and

$$g_n^{\pm}(x) \sim n^{\gamma} \sum_{j=0}^{\infty} \delta_j^{\pm}(x) n^{-j} \qquad \text{as } n \to \infty \tag{30}$$

for some fixed γ that can be complex, and $\delta_0^{\pm}(x) \neq 0$. Thus, $\{\rho_n^{\pm}(x)\} \in \mathcal{A}(\exp(\pm\,\mathrm{i}\,\omega x, \gamma)$ and $\{g_n^{\pm}(x)\} \in \mathcal{A}(1, \gamma)$. This class of series comprises many types of generalized Fourier series that arise in practical applications [1]. Examples are summarized in the Tab. 1 where T_n and U_n denote Chebychev polynomials of the first and second kind, P_n and Q_n denote Legendre polynomials and associated Legendre functions of order zero of the second kind, each of degree n, while J_ν and Y_ν denote Bessel functions of order ν of the first and second kind, respectively, and

$$\lambda_n \sim n \sum_{j=0}^{\infty} \alpha_j n^{-j} \,, \qquad (n \to \infty, \alpha_0 > 0) \,, \tag{31}$$

and thus, $\{\lambda_n\} \in \mathcal{A}(1, 1)$.

The method consists of the following steps [1]:

Table 1: Examples of generalized Fourier series

Type	$\phi_n(x)$	$\psi_n(x)$	ω	γ
Classical	$\cos(n\omega x)$	$\sin(n\omega x)$	ω	0
“Nonclassical”	$\cos(\lambda_n x)$	$\sin(\lambda_n x)$	α_0	0
Chebyshev	$T_n(\cos x)$	$\sin x \, U_{n-1}(\cos x)$	1	0
Legendre	$P_n(\cos x)$	$-(2/\pi)\, Q_n(\cos x)$	1	$-1/2$
Bessel	$J_\nu(\lambda_n x)$	$Y_\nu(\lambda_n x)$	α_0	$-1/2$

Step 1 Define

$$A^{\pm}(x) = \sum_{n=0}^{\infty} a_n \rho^{\pm}(x) \, , \qquad B^{\pm}(x) = \sum_{n=0}^{\infty} b_n \rho^{\pm}(x) \tag{32}$$

and observe that

$$\begin{aligned} F_\phi(x) &= \sum_{n=0}^{\infty} a_n \phi(x) = \frac{1}{2}[A^+(x) + A^-(x)] \, , \\ F_\psi(x) &= \sum_{n=0}^{\infty} b_n \psi(x) = \frac{1}{2\mathrm{i}}[B^+(x) - B^-(x)] \, , \\ F(x) &= F_\phi(x) + F_\psi(x) \, . \end{aligned} \tag{33}$$

Step 2 Apply the $d^{(m)}$ transformation of Levin and Sidi [2] with a suitable m to approximate the four series $A^{\pm}(x)$ und $B^{\pm}(x)$.

Step 3 Use the results of Step 2 in combination with (33) to approximate $F(x)$.

We will call this method the *method of the attached series* or, more precisely, the *method of the attached series via the* $d^{(m)}$ *transformation*. In case of real functions ϕ_n and ψ_n it suffices to extrapolate only the two series $A^+(x)$ and $B^+(x)$ because in this case the series $A^-(x)$ and $B^-(x)$ can be obtained from $A^+(x)$ and $B^+(x)$ by complex conjugation.

4 The extended method of the attached series

We consider for $j = 1, \ldots, L$ complex functions of the form

$$\rho_n^{(j,\pm)}(x) = \phi_n^{(j)}(x) \pm \mathrm{i}\, \psi_n^{(j)}(x) = \exp(\pm \mathrm{i}\, n\, \omega^{(j)} x) g_n^{(j,\pm)}(x) \, , \tag{34}$$

where $\omega^{(j)}$ is some fixed real parameter, and

$$g_n^{(j,\pm)}(x) \sim n^{\epsilon(j)} \sum_{m=0}^{\infty} \delta_m^{(j,\pm)}(x) n^{-m} \qquad \text{as } n \to \infty \tag{35}$$

for some fixed $\epsilon(j)$ that can be complex. Substituting

$$\begin{aligned}
\phi_n^{(j)}(x^{(j)}) &= \frac{1}{2}[\rho_n^{(j,+)}(x^{(j)}) + \rho_n^{(j,-)}(x^{(j)})]\,, \\
\psi_n^{(j)}(x^{(j)}) &= \frac{1}{2\mathrm{i}}[\rho_n^{(j,+)}(x^{(j)}) - \rho_n^{(j,-)}(x^{(j)})]
\end{aligned} \tag{36}$$

into Eq. (4) and putting

$$\begin{aligned}
c_n^{(1,j)} = c_n^{(3,j)} &= a_n^{(j)}/2 \\
c_n^{(2,j)} = -c_n^{(4,j)} &= b_n^{(j)}/(2\,\mathrm{i}\,) \\
\chi_n^{(1,j)} = \chi_n^{(2,j)} &= \rho_n^{(j,+)}(x^{(j)}) \\
\chi_n^{(3,j)} = \chi_n^{(4,j)} &= \rho_n^{(j,-)}(x^{(j)})
\end{aligned}$$

we obtain

$$s = \sum_{\mu_1=1}^{4} \cdots \sum_{\mu_L=1}^{4} \sum_{n=0}^{\infty} c_n^{(\mu_1,1)} \cdots c_n^{(\mu_L,L)}\, \chi_n^{(\mu_1,1)} \cdots \chi_n^{(\mu_L,L)}\,. \tag{37}$$

Thus, the series s can be represented as a sum of 4^L infinite series with terms that are products of functions $\rho_n^{(j,\pm)}(x^{(j)})$ and suitable coefficients $a_n^{(j)}$ and $b_n^{(j)}$. As a consequence of Eqs. (34) and (35), each of these series can (asymptotically) be regarded as power series in some variable $\exp(\,\mathrm{i}\,(\pm\omega^{(1)}x^{(1)} + \cdots \pm \omega^{(L)}x^{(L)}))$. Such series will be called *attached series*. A further consequence is that if the coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are adequate, then the coefficients

$$p_n = c_n^{(\mu_1,1)} \cdots c_n^{(\mu_L,L)}\, h_n^{(\mu_1,1)} \cdots h_n^{(\mu_L,L)} \tag{38}$$

of the power series are also adequate. Here, $h_n^{(1,j)} = h_n^{(2,j)} = g_n^{(j,+)}(x^{(j)})$, and $h_n^{(3,j)} = h_n^{(4,j)} = g_n^{(j,-)}(x^{(j)})$.

Each of these power series can then be extrapolated with the usual methods for power series. If an extrapolation method $\mathcal{M}$ is used, we call the resulting the *extended method of the attached series via* $\mathcal{M}$. Whether the method $\mathcal{M}$ is accelerative or not, depends on both the method and the nature of the problem that is determined by the asymptotic behavior of the coefficients and the position of singularities of the function to be calculated.

Before we discuss examples of such methods, let us note that series of the form

$$s = \sum_{n=0}^{\infty} A_n \prod_{j=1}^{L} \theta_n^{(j)} \tag{39}$$

with $\theta_n^{(j)} \in \{\phi_n^{(j)}(x^{(j)}), \psi_n^{(j)}(x^{(j)})\}$ are special cases of Eq. (4). If the coefficients A_n are adequate, then the extended method of the attached series can be simplified because in this case, the computation of 2^L attached series suffice. To see this, one only has to rewrite the product of the $\theta_n^{(j)}$ as a linear combination of 2^L terms where each term is a product of L suitable factors $\rho_n^{(j,\pm)}(x^{(j)})$. Also, savings in the number of attached series to be computed are possible if the coefficients of the original series are real. This will be demonstrated for an example below.

We now come back to the question of suitable extrapolation methods for the extended method of the attached series.

One of these methods is the Levin transformation. In combination with the τ-fold-frequency method, the convergence and stability results of Sidi [1, Theorems 4.3, 4.4] hold. This will be analyzed later in more detail.

Alternatively, one could use also the ${}_p\mathbf{J}$ transformations for $p = 1, 2, 3$ that are a special cases of the $\mathcal{J}$ transformation and have proven to be powerful convergence accelerators for linearly and logarithmically convergent sequences and series and that may also be used as summation methods for alternating divergent series[14, 15]. Regarding stability, they behave similarly to the Levin transformation and may also be combined with the τ-fold frequency approach as shown in [19].

As a further method that could be used in the extended method of the attached series, we mention the well-known ϵ algorithm [3].

5 Theoretical Results

We consider only series of the form (39). In order to find out whether the Levin transformation can be applied profitably to the attached series, we have to investigate whether the terms are adequate. For this end, we need some preliminary results.

An immediate consequence of Corollary 1 is the following corollary:

Corollary 2 *If $\{A_n\} \in \mathcal{A}(q, \epsilon)$ with coefficient sequence $\{\alpha_j\}$ and if*

$$\rho_n^{(m,k_m)}(x) = \exp(k_m\, i n\, \omega^{(m)} x) g_n^{k_m}(x) \,, \tag{40}$$

with $\{g_n^{k_m}(x)\} \in \mathcal{A}(1, \epsilon^{(m)})$ with coefficient sequence $\{\delta_j^{(m,k_m)}(x)\}_{j=0}^{\infty}$ and $k_m \in \{+, -\}$ for $m = 1, \ldots, L$, then

$$\left\{ A_n \prod_{m=1}^{L} \rho_n^{(m,k_m)}(x^{(m)}) \right\}_{n=0}^{\infty} \in \mathcal{A}(qQ(\vec{k}), \mathcal{E}) \tag{41}$$

with coefficient sequence $\{\Xi_j(\vec{k})\}_{j=0}^{\infty}$ *where* $\vec{k} = (k_1, \dots, k_L) \in \{+, -\}^L$*, and*

$$\begin{aligned} Q(\vec{k}) &= \exp(i \sum_{m=1}^{L} k_m \omega^{(m)} x^{(m)}) \\ \mathcal{E} &= \epsilon + \sum_{m=1}^{L} \epsilon^{(m)} \\ \Xi_j(\vec{k}) &= \sum_{a+b=j} \alpha_a \Delta_b(\vec{k}) \\ \Delta_j(\vec{k}) &= \sum_{a_1+\cdots+a_L=j} \prod_{m=1}^{L} \delta_{a_m}^{(m,k_m)}(x^{(m)}) \end{aligned} \tag{42}$$

Lemma 2 *Suppose* $\vec{k}$*,* A_n*,* $\rho_n^{(m,k_m)}(x)$*,* $Q(\vec{k})$*, and* $\mathcal{E}$ *are given as in Corollary 2. Assume*

$$\theta_n^{(m)} = \beta^{(m,+)} \rho_n^{(m,+)}(x^{(m)}) + \beta^{(m,-)} \rho_n^{(m,-)}(x^{(m)}) \tag{43}$$

with constants $\beta^{(m,\pm)}$*. Define*

$$s_N = \sum_{n=0}^{N} A_n \prod_{m=1}^{L} \theta_n^{(m)} \, .$$

Then

$$s_N = \sum_{\vec{k} \in \{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} P_N(\vec{k}) \tag{44}$$

where $P_N(\vec{k})$ *are partial sums*

$$P_N(\vec{k}) = \sum_{n=0}^{N} p_n(\vec{k}) \tag{45}$$

of attached series with terms that are given by

$$p_n(\vec{k}) = A_n \prod_{m=1}^{L} \rho_n^{(m,k_m)}(x^{(m)}) \tag{46}$$

and satisfy

$$\{p_n(\vec{k})\}_{n=0}^{\infty} \in \mathcal{A}(qQ(\vec{k}), \mathcal{E}) \, .$$

Proof: This follows from Corollary 2 once Eq. (44) is established. But equation (44) follows from

$$\prod_{m=1}^{L} \sum_{k_m \in \{+,-\}} \beta^{(m,k_m)} \rho_n^{(m,k_m)}(x^{(m)}) = \sum_{\vec{k} \in \{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} \rho_n^{(m,k_m)}(x^{(m)})$$

by multiplication with A_n, summation over n from 0 to N, and interchanging the order of summation.

Theorem 1 *Let* $\{A_n\} \in \mathcal{A}(q,\epsilon)$, $\rho_n^{(m,\pm)}(x)$, $P_N(\vec{k})$, $p_n(\vec{k})$, $Q(\vec{k})$, *and* $\mathcal{E}$ *be defined as in Lemma 2. Suppose* $|q| \le 1$ *and assume that* $qQ(\vec{k}) \neq 1$ *for all* $\vec{k} \in \{+,-\}^L$. *Then, for all* $\vec{k} \in \{+,-\}^L$, *the asymptotic expansion*

$$P_N(\vec{k}) \sim P(\vec{k}) + p_N(\vec{k}) \sum_{j=0}^{\infty} \xi_j(\vec{k}) N^{-j} \tag{47}$$

with $\xi_0(\vec{k}) \neq 0$ *holds for large* N, *whether* $\lim_{N\to\infty} P_N(\vec{k})$ *exists or not. If this limit exists, it equals* $P(\vec{k})$, *otherwise* $P(\vec{k})$ *is the antilimit that can be obtained from*

$$P(\vec{k}) = \lim_{z\to 1-} \sum_{j=0}^{\infty} p_j(\vec{k}) z^j \ .$$

Proof: This is a direct consequence of Lemma 2 and a result of Sidi [1, Theorem 4.2].

Theorem 2 *Using the notations of Theorem 1, assume that for each* $\vec{k} \in \{+,-\}^L$

$$p_n(\vec{k}) = [qQ(\vec{k})]^n \, n^{\sigma(\vec{k})} \, w_{\vec{k}}(n)$$

where $\sigma(\vec{k})$ *is a nonnegative integer and* $w_{\vec{k}}(n)$ *is a Laplace transform given by*

$$w_{\vec{k}}(n) = \int_0^{\infty} \exp(-n\,t) f_{\vec{k}}(t)\, dt$$

such that $f_{\vec{k}}(t)$ *is continuous in a neighborhood of 0 except possibly at 0, and satisfies*

$$\int_0^{\infty} \exp(-t) |f_{\vec{k}}(t)|\, dt < \infty$$

and

$$f_{\vec{k}}(t) \sim \sum_{a=0}^{\infty} \mu_a(\vec{k}) t^{\eta(\vec{k})+a-1}$$

with $\mu_0(\vec{k}) \neq 0$ *and* $\Re(\eta(\vec{k})) > 0$. *Further, assume that*

$$qQ(\vec{k}) \notin [1,\infty)$$

in the complex plane. Then Eq. (47) holds with

$$P(\vec{k}) = p_0(\vec{k}) + \int_0^{\infty} f_{\vec{k}}(t) \left[\left(z \frac{d}{dz} \right)^{\sigma(\vec{k})} \frac{z}{1-z} \right]_{z = qQ(\vec{k}) \exp(-t)} dt$$

whether $\lim_{N\to\infty} P_N(\vec{k})$ *exists or not. Here,* $P(\vec{k})$ *is the limit or antilimit of* $\{P_N(\vec{k})\}_{N=0}^{\infty}$ *and is analytic in the* $qQ(\vec{k})$*-plane cut along* $[1,\infty)$.

Proof: This is a direct consequence of Lemma 2 and a result of Sidi [1, Theorem 4.1].

Now, we use these results to derive a theorem regarding the convergence of the extended method of the attached series.

Theorem 3 *Suppose* $\vec{k}$, A_n, q, ϵ, $\rho_n^{(m,k_m)}(x)$, $Q(\vec{k})$, $\Xi_j(\vec{k})$ *and* $\mathcal{E}$ *are given as in Corollary 2. Let* s_N *be defined by Eq. (44) with* $P_N(\vec{k})$ *and* $p_n(\vec{k})$ *as in Eqs. (45) and (46) such that* $\{p_n(\vec{k})\}_{n=0}^{\infty} \in \mathcal{A}(qQ(\vec{k}),\mathcal{E})$ *with coefficient sequence* $\{\Xi_j(\vec{k})\}$. *Define for some* $\alpha > 0$

$$W_\nu^{(j)}(\vec{k}) = \frac{\triangle^{\nu+1}\left[(\tau j+\alpha)^{\nu-1}P_{\tau j}(\vec{k})/p_{\tau j}(\vec{k})\right]}{\triangle^{\nu+1}\left[(\tau j+\alpha)^{\nu-1}/p_{\tau j}(\vec{k})\right]} \tag{48}$$

Assume $|q| \le 1$ *and* $qQ(\vec{k}) \neq 1$ *or that the additional assumptions of Theorem 2 are satisfied such that Eq. (47) holds, or equivalently*

$$P_N(\vec{k}) \sim P(\vec{k}) + p_N(\vec{k}) \sum_{j=0}^{\infty} \tilde{\xi}_j(\vec{k})(N+\alpha)^{-j} \tag{49}$$

with $\tilde{\xi}_0(\vec{k}) = \xi_0(\vec{k}) \neq 0$, *and also* $q^\tau Q(\vec{k})^\tau \neq 1$, *for all* $\vec{k} \in \{+,-\}^L$. *Define*

$$s = \sum_{\vec{k}\in\{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} P(\vec{k}) \tag{50}$$

and

$$s'_n = \sum_{\vec{k}\in\{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} W_\nu^{(n)}(\vec{k})\,. \tag{51}$$

Then, whether the limits $\lim_{N\to\infty} P_N(\vec{k})$ *and* $\lim_{N\to\infty} s_N$ *exist or not, we have*

$$s'_N - s \sim N^{\mathcal{E}-2\nu-1} q^{\tau N} \sum_{\vec{k}\in\{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} Q(\vec{k})^{\tau N}(K_\nu(\vec{k}) + O(1/N))/N^{\mu(\vec{k})} \tag{52}$$

where $\mu(\vec{k})$ *is the smallest nonnegative integer such that* $\tilde{\xi}_{\nu+\mu(\vec{k})}(\vec{k}) \neq 0$ *and*

$$K_\nu(\vec{k}) = \Xi_0(\vec{k})\tilde{\xi}_{\nu+\mu(\vec{k})}(\vec{k})(\mu(\vec{k})+1)_{\nu+1}\tau^{\mathcal{E}-\nu-\mu(\vec{k})}\left[\frac{q^\tau Q(\vec{k})^\tau}{q^\tau Q(\vec{k})^\tau - 1}\right]^{\nu+1} \tag{53}$$

This may be compared to

$$s_{\tau(N+\nu+1)} - s \sim [\tau(N+\nu+1)]^{\mathcal{E}} q^{\tau(N+\nu+1)} \sum_{\vec{k}\in\{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} Q(\vec{k})^{\tau(N+\nu+1)}(\Xi_0(\vec{k}) + O(1/N)). \tag{54}$$

If for sufficiently large N we have

$$\sum_{\vec{k}\in\{+,-\}^L} \prod_{m=1}^{L} \beta^{(m,k_m)} Q(\vec{k})^{\tau(N+\nu+1)} \Xi_0(\vec{k}) \neq 0 \,,$$

then on putting $\mu = \min_{\vec{k}} \mu(\vec{k})$ the asymptotical relation

$$\frac{s_N - s}{s_{\tau(N+\nu+1)-s}} \sim \mathcal{C} N^{-(2\nu+1+\mu)} \tag{55}$$

holds for some constant $\mathcal{C}$.

In particular, this means that this method provides stable convergence acceleration if for each attached series the terms t_n have the asymptotic expansion

$$t_n \sim \zeta^n n^\sigma \sum_{j=0}^{\infty} e_j n^{-j} \,, \qquad (n \to \infty \,,\, e_0 \neq 0 \,,\, \zeta \in \mathbb{C} \setminus \{1\} \,,\, \sigma \in \mathbb{C}) \tag{56}$$

and, if $|\zeta| > 1$, additionally $\zeta \notin [1, \infty)$ and also $t_n = \zeta^n n^p w(n)$ holds, where p is a nonnegative integer and $w(n)$ is a Laplace transform

$$w(n) = \int_0^\infty \exp(-nt)\, \phi(t)\, dt \tag{57}$$

such that $\phi(t)$ is continuous in a neighborhood of 0 except possibly at 0, and satisfies

$$\int_0^\infty e^{-nt}\, |\phi(t)|\, dt < \infty \,; \qquad \phi(t) \sim \sum_{j=0}^{\infty} \mu_j t^{\eta+j-1} \\ (t \to 0+ \,,\, \mu_0 \neq 0 \,,\, \Re\eta > 0) \,. \tag{58}$$

The quantity $\tau \in \mathbb{N}$ then should be chosen in such a way that ζ^τ is sufficiently different from 1.

6 A Numerical Example

We treat the expansion

$$\begin{aligned} s = G(\theta, \alpha) &= \sum_{j=0}^{\infty} \cos((j+1/2)\alpha)\, P_j(\cos\theta) \\ &= \begin{cases} [2(\cos\alpha - \cos\theta)]^{-1/2} & \text{für } 0 \leq \alpha < \theta < \pi \\ 0 & \text{für } 0 < \theta < \alpha \leq \pi \end{cases} \end{aligned} \tag{59}$$

with a singularity at $\alpha = \theta$ and with partial sums given by

$$s_n = \sum_{j=0}^{n} \cos((j+1/2)\alpha)\, .P_j(\cos\theta) \tag{60}$$

This example was also treated by Sidi [1, Tabs. 6, 7]. There, machine precision was about 33 decimal digits. Here, we use `MAPLE V`$^{\mathrm{TM}}$ Release 3 with `Digits`=32 whence a direct comparison of the results is possible.

Direct acceleration of the real series can be done using the $d^{(4)}$ transformation, while in the above described method of Sidi, there are two possibilities: One may regard the series as Fourier series

$$G(\alpha) = \sum_{j=0}^{\infty} f_j \cos((j+1/2)\alpha) \tag{61}$$

with coefficients $f_j = P_j(\cos\theta)$ and then, one has to accelerate the attached series

$$F_{\pm} = \sum_{j=0}^{\infty} f_j \exp(\pm \mathrm{i}(j+1/2)\alpha) \tag{62}$$

with partial sums

$$F_{n,\pm} = \sum_{j=0}^{n} f_j \exp(\pm \mathrm{i}(j+1/2)\alpha) \tag{63}$$

either via the $d^{(2)}$-Transformation or via the $\mathcal{K}$ transformation in combination with the τ-fold frequency approach, producing approximations

$$\mathcal{K}_{n,\pm}^{(\tau)} = \Re\left(\mathcal{K}_{n-2[\![n/2]\!]}^{([\![n/2]\!])}(\{(n+1)^{-1}\},\{\gamma_n^{(j)}\},\{F_{\tau\, n,\pm}\},\{\omega_n^{(\tau)}\})\right) \tag{64}$$

using $\gamma_n^{(0)} = n+2$, $\gamma_n^{(1)} = -(2n+5)\,x_\tau$, and $\gamma_n^{(2)} = n+3$ corresponding to the (shifted) three-term recurrence relation [20, p. 736]

$$(n+2)\,P_{n+2}(x) - (2n+3)\,x\,P_{n+1}(x) + (n+1)\,P_n(x) = 0 \tag{65}$$

of Legendre polynomials at the argument

$$x_\tau = \cos(\tau\theta) \tag{66}$$

and remainder estimates $\omega_n^{(\tau)} = (n\tau+1)^{\nu}\exp(\pm \mathrm{i}(n\tau+1/2)\alpha)$. The parameter $\nu \in \{0,1\}$ defines two variants of the $\mathcal{K}$ transformation, analogous to t and u variants of the Levin transformation.

Alternatively, one may regard the series as an expansion in Legendre polynomials

$$G(\theta) = \sum_{j=0}^{\infty} a_j\, P_j(\cos\,\theta) \tag{67}$$

coefficients $a_j = \cos((j+1/2)\alpha)$ and then, one has to accelerate the attached series

$$A^{\pm}(\theta) = \sum_{j=0}^{\infty} a_j (P_j(\cos\theta) \mp \mathrm{i}\frac{2}{\pi} Q_j(\cos\theta)) \tag{68}$$

with partial sums

$$A_n^{\pm}(\theta) = \sum_{j=0}^{n} a_j (P_j(\cos\theta) \mp \mathrm{i}\frac{2}{\pi} Q_j(\cos\theta)) \tag{69}$$

either via the $d^{(2)}$ transformation, or via the $\mathcal{H}$ transformation, or via the $\mathcal{I}$ transformation. In combination with the τ-fold frequency approach, this produces the approximations

$$\mathcal{H}_{n,\pm}^{(\tau)} = \Re\left(\mathcal{H}_{n-2[\![n/2]\!]}^{([\![n/2]\!])}(\tau\,\alpha, 1, \{A_{\tau n}^{\pm}(\theta)\}, \{\omega_n^{(\tau)}\})\right) \tag{70}$$

and

$$\mathcal{I}_{n,\pm}^{(\tau)} = \Re\left(\mathrm{I}_{n-2[\![n/2]\!]}^{([\![n/2]\!])}(\tau\,\alpha, 1, 2, \{A_{\tau n}^{\pm}(\theta)\}, \{\omega_n^{(\tau)}\})\right) \tag{71}$$

with remainder estimates $\omega_n^{(\tau)} = (n\tau+1)(P_{n\tau}(\cos\theta) \mp \mathrm{i}\frac{2}{\pi}Q_{n\tau}(\cos\theta))$.

In both cases, one can approximately halve the number of coefficients that are required to achieve a certain accuracy when using the $d^{(2)}$ transformation in comparison to using the $d^{(4)}$ transformation on the real series. Using the $\mathcal{H}$, $\mathcal{I}$, and $\mathcal{K}$ transformations, one obtains in this present example even better results as shown below.

In the newly introduced extended method of the attached series, one may represent the series (59) as sum $s = p_1 + p_2 + p_3 + p_4$ of the four series

$$\begin{aligned} p_1 &= \sum_{j=0}^{\infty} \frac{1}{4}\,\exp(\mathrm{i}(j+1/2)\alpha)\,\rho_j^{+}(\theta)\,, \\ p_2 &= \sum_{j=0}^{\infty} \frac{1}{4}\,\exp(-\mathrm{i}(j+1/2)\alpha)\,\rho_j^{-}(\theta)\,, \\ p_3 &= \sum_{j=0}^{\infty} \frac{1}{4}\,\exp(\mathrm{i}(j+1/2)\alpha)\,\rho_j^{-}(\theta)\,, \\ p_4 &= \sum_{j=0}^{\infty} \frac{1}{4}\,\exp(-\mathrm{i}(j+1/2)\alpha)\,\rho_j^{+}(\theta)\,, \end{aligned} \tag{72}$$

with

$$\rho_j^{\pm}(\theta) = P_j(\cos\theta) \mp \mathrm{i}(2/\pi)Q_j(\cos\theta)\,. \tag{73}$$

Since the coefficients and the arguments α and θ are real, it may be observed that $p_2 = p_1^*$ and $p_4 = p_3^*$ such that the extrapolation of only two attached series suffices.

As discussed above, for the extrapolation of the attached series, one may use for instance the Levin transformation. In the vicinity of the singularity, one may additionally use the τ-fold frequency approach. Then, one obtains approximations

$$P_{n,j}^{(\tau)} = \mathcal{L}_n^{(0)}(1, [p_{j,\tau n}]|_{n=0}, [(\tau n+1)(p_{j,\tau n} - p_{j,(\tau n)-1})]|_{n=0}) \tag{74}$$

to p_j that are based on using the n-th partial sums $p_{j,n}$. This variant of the Levin transformation corresponds to the $d^{(1)}$ transformation with $R_l = \tau\, l$ as noted above. The value $\tau = 1$ corresponds to using the original frequency. For the original series, the approximate results $G_n^{(\tau)} = P_{n,1}^{(\tau)} + P_{n,2}^{(\tau)} + P_{n,3}^{(\tau)} + P_{n,4}^{(\tau)}$ or equivalently, $G_n^{(\tau)} = 2\Re(P_{n,1}^{(\tau)} + P_{n,3}^{(\tau)})$ are obtained.

We treat two different pairs (α, θ) of arguments. In each case, the number of exact digits defined as negative decadic logarithms of the relative errors are displayed for a number of methods as a functions of the number of terms of the original series. The results of the following methods are compared in Table 2 and Table 3, respectively: In the column $\mathcal{L}$, the results for the extended method of the attached series via the Levin transformation, ie., for the approximations $G_n^{(\tau)}$ are presented. In the column $\mathcal{H}$, the results for the method of the attached series via the $\mathcal{H}$ transformation, ie., for the approximations $\mathcal{H}_{n,\pm}^{(\tau)}$ are presented. In the column $\mathcal{I}$, the results for the method of the attached series via the $\mathcal{I}$ transformation, ie., for the approximations $\mathcal{I}_{n,\pm}^{(\tau)}$ are presented. In the column $\mathcal{K}_u$, the results for the method of the attached series via the $\mathcal{K}$ transformation with $\nu = 1$, ie., for the approximations $\mathcal{K}_{n,\pm}^{(\tau)}$ with $\nu = 1$ are presented. In the column $\mathcal{K}_t$, the results for the method of the attached series via the $\mathcal{K}$ transformation with $\nu = 0$, ie., for the approximations $\mathcal{K}_{n,\pm}^{(\tau)}$ with $\nu = 0$ are presented.

In the first case, we put $\alpha = \pi/6$ and $\theta = 2\pi/3$. This corresponds to a relatively large distance from the singularity at $\alpha = \theta$. The corresponding results are given in Table 2. It is seen that the newly proposed method produces the best results and reaches nearly machine precision (i.e., 32 decimal digits) for about 30 terms (see column $\mathcal{L}$). The $\mathcal{H}$, $\mathcal{I}$, and $\mathcal{K}$ transformation reach approximately the same accuracy using 10 to 15 terms more.

The new method approximately yields one additional exact digit per additional term of the series. The ratio "number of digits/number of terms" is approximately 1. The method proposed by Sidi on the basis of the $d^{(2)}$ transformation is considerably less efficient, and the corresponding ratio is approximately 1/2. [1, Tab. 6] A saving of about half the terms results when the new method is used. The $d^{(4)}$ transformation on the real series is much worse since its corresponding ratio is approximately 1/4.

In the second case, we put $\alpha = 6\pi/10$ and $\theta = 2\pi/3$. This is already rather close to the singularity at $\alpha = \theta$. A value of $\tau = 10$ is thus chosen that also allows direct comparison to the results of Sidi with $R_l = 10l$ [1, Tab. 7]. The corresponding results are displayed in Table 3. The original series converges very slowly and yields less than two exact decimal digits using 301 terms. The overall results are rather similar to the case treated in Table 2. The extended method of the attached series using the Levin transformation for the τ-fold frequency series needs slightly less than 30 terms of the latter series, corresponding to about 280 terms of the original series. The $d^{(4)}$ transformation on the real series is worst, the method of the attached series works worst with the $d^{(2)}$ transformation that requires about 700 terms to reach 30 decimal digit accuracy, while the other transformations require between about 350 to 400 terms to reach this accuracy.

Table 2: Number of exact digits ($\alpha = \pi/6, \theta = 2\pi/3, \tau = 1$)

Terms	$\mathcal{L}$	$\mathcal{H}$	$\mathcal{I}$	$\mathcal{K}_u$	$\mathcal{K}_t$	$d^{(4)}$ [1, Tab. 6]	$d^{(2)}$ [1, Tab. 6]
10	9.8	6.7	5.4	5.7	6.5		2.5
18	17.6	13.5	10.8	11.7	11.9		5.8
20	19.3	14.9	13.0	13.3	13.2	2.5	
26	25.4	20.9	16.6	16.7	18.5		8.8
30	29.3	23.2	19.6	20.0	20.0		
32	29.7	25.5	21.6	21.2	21.3		
34	30.1	26.7	22.8	22.2	22.8		13.6
36	30.1	28.2	24.4	23.4	24.5	4.9	
38	29.7	29.2	25.9	24.8	26.0		
40	29.8	29.8	27.2	26.4	26.9		
42	30.3	29.6	28.2	28.9	28.1		16.5
44	29.2	29.6	29.5	29.2	29.5		
46	29.0	29.4	31.0	30.3	30.7		
48	28.9	29.3	29.5	31.0	31.0		
50	29.3	29.6	29.9	31.8	30.6		19.5
52	29.2	29.2	29.6	31.1	30.8	10.1	
58	28.0	29.4	30.1	32.0	31.1		22.9
66	27.8	29.3	29.3	30.6	31.1		26.7
68	27.3	28.6	28.7	30.6	30.9	15.6	
74	27.1	28.2	28.8	30.5	31.3		29.7

Table 3: Number of exact digits ($\alpha = 6\pi/10, \theta = 2\pi/3, \tau = 10$)

Terms	$\mathcal{L}$	$\mathcal{H}$	$\mathcal{I}$	$\mathcal{K}_u$	$\mathcal{K}_t$	$d^{(4)}$ [1, Tab. 7]	$d^{(2)}$ [1, Tab. 7]
161	18.1	11.9	14.0	10.3	12.4		
162							7.6
164						4.7	
241	27.3	19.7	20.8	18.2	18.0		
242							12.0
271	31.2	22.6	23.7	19.6	20.1		
281	31.2	23.3	24.3	20.7	20.9		
301	29.9	25.3	26.4	21.8	22.5		
321	30.3	27.1	28.0	23.2	24.1		
322							16.4
324						7.2	
341	30.1	28.2	29.6	25.0	25.3		
361	30.9	29.5	30.9	26.0	27.8		
381	30.3	29.9	30.2	28.5	28.2		
401	30.9	30.3	30.0	29.0	29.9		
402							18.2
481	30.2	29.7	29.6	30.7	31.1		
482							21.0
484						9.7	
561	29.4	29.1	30.4	32.0	30.6		
562							23.8
641		28.6	29.6	30.9	30.5		
642							27.0
644						11.5	
721		27.8	27.6	30.9	30.6		
722							30.0
801		27.2	27.2	30.9	32.0		
802							32.2
804						14.2	

For the extended method of the attached series using the Levin transformation the ratio "number of digits/number of terms" is about 0.11. The method of Sidi, i.e. using the method of the attached series via the $d^{(2)}$ transformation is considerably less efficient also in this case, the corresponding ratio is about 0.04. Thus, for a given accuracy, the new method uses only about 35-40% of the terms as required by the method of Sidi, i.e. savings of 60-65% are possible.

For the treated orthogonal expansion (59), one may conclude that the extended method of attached series via the Levin transformation, suitably combined with the τ-fold frequency approach requires about half of the terms in comparison with the method of the attached series via the $d^{(2)}$ transformation while this method is more effective when combined with the $\mathcal{H}$, $\mathcal{I}$ and $\mathcal{K}$ transformation. But even then, the extended method is superior. We remark that it is to be expected that the relative performance of the accelerators for the method of the attached series will vary with the example that is considered.

Again, we remark that besides the Levin transformation there are further methods that can be combined with the extended method of the attached series and hence, it is neither claimed nor probable that the Levin transformation is always superior to the other accelerators like the ϵ algorithm or the ${}_p\mathbf{J}$ transformation in this context.

Thus, the possibility to use other accelerators in place of the $d^{(2)}$ transformation in the method of the attached series and in place of the $d^{(1)}$ transformation with $R_\ell = \tau\ell$ in the extended method exists, and it should be regarded as a valuable addition to the toolbox of numerical methods.

In summary, one may say that the extended method of the attached series, suitably combined with a τ-fold frequency approach is a powerful method for the convergence acceleration of a class of complicated orthogonal expansions depending on several frequencies with adequate coefficients.

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