Extended Complex Series Methods for the Convergence Acceleration of Fourier Series[*][†]

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Abstract

The acceleration of slowly convergent Fourier series is discussed. For instance, such series arise in spectral methods with irregular grids as are required to resolve shock waves or other narrow features, in the description of lattice-gas models, and in the Matsubara formalism. In extension of a method proposed by Sidi, it is shown that rather spectacular convergence acceleration can often be achieved by rewriting the given Fourier series as a sum of suitable power series for which the usual convergence accelerators can be applied separately. The sum of the Fourier series is then approximated by the sum of the extrapolated values of the power series. Methods that work in the vicinity of jumps and other singularities of the Fourier series are presented.

1 Introduction

There are many applications where slowly convergent or even divergent Fourier series arise. Thus the question of convergence acceleration of such series arises. Up to now, not many devices are known that allow to accelerate the convergence or to sum the divergence of Fourier series. In this contribution, we will describe some methods that have been developed for this purpose.

Potential fields of application of such methods are for instance the description of waves in optics and electromagnetic theory, quantum theory, astronomy and astrophysics, or in fluid mechanics. An example that arises in the realm of spectral methods for solution of convection-diffusion problems and Navier-Stokes problems, are evaluations of interpolatory sums over cardinal functions of sinc, Fourier, or Chebyshev type at off-grid points as may be used to describe shock waves and steep frontal zones (cp. [1] and references therein). Another example is given in [2], where slowly convergent Fourier series have to be evaluated in the description of a one-dimensional lattice-gas model. Some further applications like the Matsubara formalism for the computation of temperature-dependent Green functions are described in [3].

There are linear methods that still are useful nowadays. Examples are the method of Jones and Hardy [4] that was extended by Shaw, Johnson and Riess [5], Tasche [6], Baszenski and Delvos [7] and by Baszenski, Delvos and Tasche [8], the method of Kiefer and Weiss [9] that is based on partial summation, the methods of Longman (see [10] and references therein) that are based on representations of Fourier series as integrals of power series, and also the work of Boyd [11, 1, 12] who studied generalizations of the Euler transformation for Fourier series and for pseudospectral methods. As examples of nonlinear methods that may directly be applied to the Fourier series, we mention the ϵ algorithm [13], the nonlinear $d^{(m)}$ transformations [14], the Levin-type $\mathcal{H}$ transformation [15], and the iterative $\mathcal{I}$ transformation [3, 16, 17]. A review of methods for the convergence acceleration and summation of Fourier series is currently in preparation.

Here we concentrate on a certain rather general method that depends on a relation to power series. The reason is that there are many methods to accelerate the convergence of power series or to sum even violently divergent power series. For good introductions to these methods see for instance the books of Brezinski and Redivo Zaglia [18] and Wimp [19] and also the work of Weniger [20, 21, 22, 23]. Many of these methods are special cases of the recently introduced $\mathcal{J}$ transformation [3, 24, 25, 26, 16, 27]. Suitable variants of the $\mathcal{J}$ transformation belong to the most powerful nonlinear accelerators for linearly and logarithmically convergent sequences and are able to sum violently divergent power series [26].

It would be highly desirable to be able to rewrite the summation of Fourier series as that of some related power series. The present contribution studies some ways how to do this. In this way, all the powerful methods for power series become applicable.

Of these, for simplicity we mainly use in the sequel the well-known Levin transformation [28]

$$\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n) \;=\; \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{s_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{1}{\omega_{n+j}}}\,, \tag{1}$$

that can also be computed recursively [20], and in particular, its u variant defined by

$$u_k^{(n)}(\beta, s_n) = \mathcal{L}_k^{(n)}(\beta, s_n, (n+\beta)\Delta s_{n-1})\,. \tag{2}$$

But the use of many other convergence accelerators for power series would be equally possible, for instance of the ϵ algorithm of Wynn [13], of several transformations of Weniger [20], of the d transformation of Levin and Sidi [14] in combination with the $W^{(m)}$ algorithm of Ford and Sidi [29], or of the ${}_2\mathbf{J}$ transformation [26, 16] that is a special case of the $\mathcal{J}$ transformation mentioned above.

We first will sketch two approaches for the evaluation of Fourier series that are based on related complex series and that have been described in the literature. The second of these may be regarded as a generalization of the first. Both approaches allow the use of simpler acceleration algorithms as for the original Fourier series, but only for simple cases the obtained related series have the form of power series, and thus, only in these cases all the acceleration algorithms for power series become useful. This drawback can often be avoided as shown in this contribution: It is possible to extend the second of these approaches to more complicated cases in such a way that only power series need to be evaluated using acceleration algorithms.

Note that these ideas can also be applied to more general orthogonal expansions. The corresponding results are presented elsewhere [30]. In the following, we treat only trigonometric Fourier series that involve sines and cosines (or complex exponentials) and non-oscillating coefficients but that may depend on several frequencies.

For all the approaches numerical examples will be presented to illustrate the methods, and also to show how to improve the convergence behavior and the stability near singularities and jumps by using a τ-fold frequency method [16, 31, 17].

2 The Method of Associated Complex Series

As is well-known, for instance from the work of Smith and Ford [32], Brezinski and Redivo Zaglia [18, Sec. 6.1.3], Homeier [3], and Sidi [33] real Fourier series of the form

$$s = s(\alpha) = a_0/2 + \sum_{j=1}^{\infty} \bigl(a_j \cos(j\alpha) + b_j \sin(j\alpha)\bigr) \tag{3}$$

Table I: Convergence acceleration of the Fourier series (8) of α in $(-\pi, \pi)$ for $\alpha = 0.9\pi$ by the method of the associated series via the Levin transformation

n	$-\lg\lvert s_n - \alpha\rvert$	$-\lg\lvert B_n - \alpha\rvert$	$-\lg\lvert C_n - \alpha\rvert$
10	0.3	4.8	4.8
16	0.7	7.4	7.4
20	0.6	8.9	8.3
22	0.8	9.9	8.2
26	0.8	11.9	6.7
32	1.0	15.2	4.6
36	1.0	16.5	3.9
40	0.9	19.3	3.1

$B_n = \Re(u_n^{(0)}(1, t_0))$, (`Digits`=32)
$C_n = \Re(u_n^{(0)}(1, t_0))$, (`Digits`=16)

with real coefficients a_j and b_j, real frequency α and partial sums

$$s_n = a_0/2 + \sum_{j=1}^{n} \big(a_j \cos(j\alpha) + b_j \sin(j\alpha)\big) \tag{4}$$

may be rewritten as the real part of an associated complex power series in the variable $z = \exp(\mathrm{i}\alpha)$ of the form

$$t = t(\alpha) = a_0/2 + \sum_{k=1}^{\infty} [a_k - \mathrm{i}b_k] \exp(\mathrm{i}k\alpha) \tag{5}$$

with partial sums

$$t_0 = a_0/2; \qquad t_n = a_0/2 + \sum_{k=1}^{n} [a_k - \mathrm{i}b_k] \exp(\mathrm{i}k\alpha)\,. \tag{6}$$

Thus, we have

$$s = \Re(t); \qquad s_n = \Re(t_n)\,. \tag{7}$$

If the coefficients a_j and b_j themselves are not oscillating as a function of j, a convergence acceleration can usually be achieved by approximating s as the real part of the result of the some convergence accelerator as applied to the power series t, e.g., $s \approx s'_n = \Re(u_n^{(0)}(1, t_0))$. We call this approach the *method of the associated series* and also *method of the associated series via the Levin transformation* in order to stress the application of the Levin transformations. In case that one uses another sequence transformation, of course, one may also speak of the *method of the associated series via that particular transformation.*

Table II: Convergence acceleration of the τ-fold frequency Fourier series related to the Fourier series (8) of α in $(-\pi, \pi)$ for $\alpha = 0.9\pi$ and $\tau = 5$ by the method of the associated series via the Levin transformation

n	$-\lg\|s_{n\tau} - \alpha\|$	$-\lg\|B_n - \alpha\|$	$-\lg\|C_n - \alpha\|$
6	0.7	6.4	6.4
12	1.0	12.0	12.0
13	1.4	14.3	14.2
18	1.2	17.6	14.2
24	1.3	23.5	14.0
30	1.4	29.4	13.1

$B_n = \Re(\mathcal{L}_n^{(0)}(1, [t_{m\tau}]|_{m=0}, [(\tau m + 1)(t_{\tau m} - t_{(\tau m)-1})]|_{m=0}))$, (`Digits`=32)
$C_n = \Re(\mathcal{L}_n^{(0)}(1, [t_{m\tau}]|_{m=0}, [(\tau m + 1)(t_{\tau m} - t_{(\tau m)-1})]|_{m=0}))$, (`Digits`=16)

As an example, we apply this method to the 2π-periodic Fourier series

$$s = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\alpha) \tag{8}$$

with partial sums

$$s_n = 2 \sum_{m=1}^{n+1} \frac{(-1)^{m+1}}{m} \sin(m\alpha) \tag{9}$$

that equals $f(\alpha) = \alpha$ in $(-\pi, \pi)$, i.e., the Fourier series of the "sawtooth". The results are displayed in Table I. The calculation was done using `MAPLE V`$^{\mathrm{TM}}$, once with 32 decimal digits accuracy corresponding to `Digits`=32, and once with 16 decimal digits accuracy corresponding to `Digits`=16. It is observed that for `Digits`=32 good acceleration is achieved while for `Digits`=16 rounding errors limit the attainable results, and best is the value obtained for $n = 20$ where the method of the associated series produces an estimate with an absolute error of less than 10^{-8}. This somewhat disappointing situation is related to the fact that one is close to the jump at $\alpha = \pi$, i.e., to a singularity of the function that is represented by the Fourier series.

As is explained in more detail elsewhere [16, 31, 17], it is better to accelerate the τ-fold frequency series with partial sums $\{\check{s}_n = s_{\tau n}\}$ where $\tau > 1$ is some integer. In Table II, we present results for $\tau = 5$. It is seen that now even with `Digits`=16, essentially full precision is reached. The price to pay, however, is that partial sums of the associated series up to $t_{13\tau} = t_{65}$ have to be used to achieve absolute errors of 10^{-14}.

3 The Method of Attached Complex Series

Somewhat more complicated is the situation in the case of complex Fourier series with truely complex coefficients a_j and/or b_j. In this situation, one can accelerate several complex power series and sum up the results: One may put

$$\begin{aligned} A_{\pm} &= a_0/2 + \sum_{j=1}^{\infty} a_j \exp(\pm \mathrm{i} j\alpha) \, , \\ B_{\pm} &= \sum_{j=1}^{\infty} b_j \exp(\pm \mathrm{i} j\alpha) \, , \end{aligned} \tag{10}$$

and

$$\begin{aligned} s_{\gamma} &= a_0/2 + \sum_{j=1}^{\infty} a_j \cos(j\alpha) = \frac{1}{2}[A_+ + A_-] \, , \\ s_{\sigma} &= \sum_{j=1}^{\infty} b_j \sin(j\alpha) = \frac{1}{2\mathrm{i}}[B_+ - B_-] \, . \end{aligned} \tag{11}$$

The series $A_{\pm}$ and $B_{\pm}$ are the above mentioned complex power series. Since

$$s = s_{\gamma} + s_{\sigma} \, , \tag{12}$$

one can try to extrapolate the complex power series with the methods mentioned in the previous section, and sum the extrapolated values to obtain an estimate for the (anti-)limit of the Fourier series under consideration. This method was essentially proposed by Sidi [33]. We call it the *method of the attached series.*

In the case of real Fourier series we have $A_- = [A_+]^*$, $B_- = [B_+]^*$, $s_{\gamma} = \Re(A_+)$, $s_{\sigma} = \Im(B_+)$ and $s = \Re(A_+ - \mathrm{i}\, B_+)$. Thus, one obtains the method of the associated power series as a special case of the method of the attached series.

There is a difficulty with this method of the attached power series if the coefficients a_j and b_j are oscillating functions of the index j. Then, the formal power series do not show the power series behavior that is required for most convergence acceleration methods to be effective. An example is given below. As a remedy, one could use the $d^{(m)}$ transformation with $m > 1$ to accelerate the computation of $A_{\pm}$ and $B_{\pm}$ as proposed by Sidi [33]. We mention that for Fourier coefficients of the form $a_j = a_{1,j}\cos(j\beta) + b_{1,j}\cos(j\beta)$, $b_j = a_{2,j}\cos(j\beta) + b_{2,j}\sin(j\beta)$ with nonoscillating coefficients $a_{\ell,j}, b_{\ell,j}$ one could also use the $\mathcal{H}$ transformation defined by the recursive scheme [15]

$$\begin{aligned} \mathcal{Z}_n^{(0)} &= (n+\beta)^{-1}\, s_n/\omega_n \, , \qquad \mathcal{N}_n^{(0)} = (n+\beta)^{-1}/\omega_n \, , \\ \mathcal{Z}_n^{(k)} &= (n+\beta)\mathcal{Z}_n^{(k-1)} + (n+2k+\beta)\mathcal{Z}_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)\mathcal{Z}_{n+1}^{(k-1)} \, , \\ \mathcal{N}_n^{(k)} &= (n+\beta)\mathcal{N}_n^{(k-1)} + (n+2k+\beta)\mathcal{N}_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)\mathcal{N}_{n+1}^{(k-1)} \, , \\ \frac{\mathcal{Z}_n^{(k)}}{\mathcal{N}_n^{(k)}} &= \mathcal{H}_n^{(k)}(\alpha, \beta, \{s_n\}, \{\omega_n\}) \, , \end{aligned} \tag{13}$$

Table III: Acceleration of the Fourier series (14) using the method of the attached series for $\alpha_1 = 8\pi/10$ and $\alpha_2 = \alpha_1/2^{1/2}$

n	A_n	B_n	C_n	D_n
10	1.4	1.1	1.9	4.2
14	1.3	1.5	1.5	6.7
18	1.4	1.4	1.4	7.7
22	1.8	1.1	1.2	9.5
26	2.1	2.3	2.3	11.5
30	2.0	1.7	1.8	14.1
34	2.0	2.4	2.2	15.0
38	2.3	1.7	1.7	16.7
40	2.6	1.9	2.6	17.7
45	3.6	2.5	2.5	19.9
50	2.7	2.7	2.4	22.3

Number of exact digits (defined as the negative decadic logarithm of the relative error). (`Digits`=32)
A_n: Number of exact digits of $f_n(\alpha_1, \alpha_2)$
B_n: Number of exact digits of $\Re(u_n^{(0)}(1, A_{+,0}))$,
C_n: Number of exact digits of $\Re(\mathcal{L}_n^{(0)}(1, A_{+,0}, [A_{+,m} - A_{+,m-1}]|_{m=0}))$
D_n: Number of exact digits of $\Re(\mathcal{H}_{n-2[\![n/2]\!]}^{([\![n/2]\!])}(\alpha_2, 1, \{A_{+,n}\}, \{\exp(\mathrm{i}\alpha_1 n)/n^2\}))$

and the $\mathcal{I}$ transformations [3, 16, 31, 17], while for coefficients of the form $a_j = a_{1,j}P_j(\beta) + b_{1,j}Q_j(\beta)$, $b_j = a_{2,j}P_j(\beta) + b_{2,j}Q_j(\beta)$ with nonoscillating coefficients $a_{\ell,j}, b_{\ell,j}$ and orthogonal polynomials P_j and second solutions Q_j of the corresponding three-term recurrence relation, one could also use the $\mathcal{K}$ transformation [34, 16, 30]. A further possibility would be to try the ϵ algorithm [13]. The computation of the $\mathcal{H}$, the $\mathcal{I}$, the $\mathcal{K}$ transformations, and especially of the $d^{(m)}$ transformations with $m > 1$ requires more computational work, however, and using all these algorithms including the ϵ algorithm, more Fourier coefficients are needed to achieve a given accuracy than in the method described in the next section.

But before we study an example. A simple Fourier series is given by

$$\begin{aligned} f(\alpha_1, \alpha_2) &= \sum_{m=1}^{\infty} \frac{1}{m^2} \cos(\alpha_1\, m) \cos(\alpha_2\, m) \\ &= -\pi^2/12 + (\min(|\alpha_1|, |\alpha_2|)^2 + (\max(|\alpha_1|, |\alpha_2|) - \pi)^2)/4 \end{aligned} \tag{14}$$

for $-\pi \le \alpha_1 \le \pi$, and $-\pi \le \alpha_2 \le \pi$, with partial sums

$$f_n(\alpha_1,\alpha_2) = \sum_{m=1}^{n+1} \frac{1}{m^2} \cos(\alpha_1\, m)\cos(\alpha_2\, m)\;, \quad n \in \mathbb{N}_0\;. \tag{15}$$

The function $f(\alpha_1,\alpha_2)$ represented by the series is a solution of the one-dimensional wave equation

$$\left[\frac{\partial^2}{\partial\,{\alpha_1}^2} - \frac{\partial^2}{\partial\,{\alpha_2}^2}\right] f(\alpha_1,\alpha_2) = 0 \tag{16}$$

that is 2π–periodic in each argument. Fourier series of similar type occur for instance also in the description of Brownian motion for unsteady-state viscous flows in pores [35].

The series (14) can be regarded as a Fourier series of the form (3) of frequency $\alpha = \alpha_1$, with coefficients that oscillate with frequency α_2 (with $a_j = \cos(\alpha_2 j)/j^2$ for $j \neq 0$, $a_0 = b_j = 0$). In the method of the attached series, one thus has to extrapolate essentially the series

$$A_+ = \sum_{m=1}^{\infty} \frac{1}{m^2} \exp(\mathrm{i}\,\alpha_1\, m)\cos(\alpha_2\, m) \tag{17}$$

with partial sums

$$A_{+,n} = \sum_{m=1}^{n+1} \frac{1}{m^2} \exp(\mathrm{i}\,\alpha_1\, m)\cos(\alpha_2\, m) \tag{18}$$

that is only formally a power series in the variable $\exp(\mathrm{i}\alpha_1)$, but can also be regarded as cosine Fourier series with frequency α_2. Thus, it is to be expected that straightforward application of the Levin transformation will fail. As discussed above, the $\mathcal{H}$ transformation is expected to be applicable to the series A_+ whence an estimate for $f(\alpha_1,\alpha_2)$ should be obtainable taking real parts. Indeed, this is the case as is plain from the last column of Tab. III that demonstrates that the $\mathcal{H}$ transformation performs well in this case. Results of similar quality should be obtainable by an analogous application of the $d^{(2)}$ transformation instead of $\mathcal{H}$ transformation. The data displayed in Tab. III also show that neither the u transform of Levin nor that variant of the Levin transformation that corresponds to the choice $\omega_m = A_{+,m} - A_{+,m-1}$, are able to produce any acceleration effect, as expected.

4 The Extended Method of Attached Complex Series

Here, we consider Fourier series of the form

$$s = \sum_{j=0}^{\infty} \prod_{\ell=1}^{L} \left(a_{\ell,j}\cos(j\alpha_\ell) + b_{\ell,j}\sin(j\alpha_\ell)\right) \tag{19}$$

Table IV: Acceleration of the Fourier series (14) using the extended method of the attached series for $\alpha_1 = 8\pi/10$ and $\alpha_2 = \alpha_1/2^{1/2}$

n	$-\lg\lvert f_n(\alpha_1,\alpha_2)/f(\alpha_1,\alpha_2)-1\rvert$	$-\lg\lvert f'_n(\alpha_1,\alpha_2)/f(\alpha_1,\alpha_2)-1\rvert$
10	1.4	6.5
14	1.3	9.3
18	1.4	12.6
22	1.8	15.3
26	2.1	17.9
30	2.0	20.8
34	2.0	25.2
38	2.3	23.6
40	2.6	23.7

that depend on several known frequencies $\{\alpha_\ell\}_{\ell=1}^{L}$.

Regarding a Fourier series of the form (19) as a Fourier series in the variable $\alpha = \alpha_1$ of the form (3) it follows that the Fourier coefficients

$$a_j = a_{1,j} \prod_{\ell=2}^{L} (a_{\ell,j}\cos(j\alpha_\ell) + b_{\ell,j}\sin(j\alpha_\ell)) \tag{20}$$

and

$$b_j = b_{1,j} \prod_{\ell=2}^{L} (a_{\ell,j}\cos(j\alpha_\ell) + b_{\ell,j}\sin(j\alpha_\ell)) \tag{21}$$

oscillate for $j > 0$.

For series of this type, one may use the generalized $\mathcal{H}$ transformation [16, 31] and also the $d^{(m)}$ transformation and the ϵ algorithm. This requires, however, relatively many terms of the series.

The idea for an alternative approach is very simple and consists of expanding the cosine and sine functions that are parts of the coefficients again in terms of complex exponential functions. In this way, the original series may rewritten as a sum of a larger number of (formal) power series. If the $a_{\ell,j}$ and $b_{\ell,j}$ are nonoscillating, one may then use all the well-known extrapolation methods that are useful for power series to accelerate each of the resulting power series separately, and finally, add up the resulting estimates, to obtain an estimate for the (anti-)limit of the original Fourier series, in a way that is a clearcut extension of the method of the attached power series. This method will be called the *extended method of the attached series*.

The essence of this new method will become clearer by applying it to examples. The first example is the series (14). In the extended method of the

Table V: Acceleration of the Fourier series (26) using the extended method of the attached series for $\alpha_1 = \pi/10$, $\alpha_2 = 2^{1/2}\pi/10$ and $\alpha_3 = -4\pi/10$

n	$g_n(\alpha_1, \alpha_2, \alpha_3)$	$g'_n(\alpha_1, \alpha_2, \alpha_3)$
20	1.0668379118338712	1.0641155230966898
22	1.0646319010644110	1.0641155354240192
24	1.0624181211129220	1.0641155366967048
26	1.0616186381598774	1.0641155361516434
28	1.0621618930948200	1.0641155361058156
30	1.0629126952360728	1.0641155361297312
32	1.0631221548598504	1.0641155361313168
34	1.0630565825407210	1.0641155361302746
36	1.0633125461160056	1.0641155361302230
38	1.0639479826292972	1.0641155361302684
40	1.0644978847766702	1.0641155361302696
42	1.0646351478259112	1.0641155361302674
44	1.0645166041509708	1.0641155361302666
46	1.0644648067090496	1.0641155361302608
48	1.0645413732209338	1.0641155361302672
50	1.0645758026076632	1.0641155361303654

attached series, this series is represented by a sum

$$f(\alpha_1, \alpha_2) = \sum_{j=1}^{4} p_j \tag{22}$$

of four complex power series of the form

$$p_j = \sum_{m=1}^{\infty} \frac{1}{4\,m^2} \exp(\mathrm{i}\, m\, \tilde{\alpha}_j) \tag{23}$$

mit $\tilde{\alpha}_1 = \alpha_1 + \alpha_2$, $\tilde{\alpha}_2 = -\alpha_1 - \alpha_2$, $\tilde{\alpha}_3 = \alpha_1 - \alpha_2$ und $\tilde{\alpha}_4 = -\alpha_1 + \alpha_2$. The partial sums of these series

$$p_{j,n} = \sum_{m=1}^{n+1} \frac{1}{4\,m^2} \exp(\mathrm{i}\, m\, \tilde{\alpha}_j)\,, \quad n \in \mathbb{N}_0 \tag{24}$$

of these series are the input for a sequence transformation. For simplicity, we choose the same transformation for each of the power series, for instance the u transformation of Levin. The result is a sequence of approximations for f of the form

$$f'_n(\alpha_1, \alpha_2) = \sum_{j=1}^{4} u_n^{(0)}(1, p_{j,0})\,. \tag{25}$$

Table VI: Acceleration of the many-fold-frequency series with $\tau = 2$ corresponding to the Fourier series (26) using the extended method of the attached series for $\alpha_1 = \pi/10$, $\alpha_2 = 2^{1/2}\pi/10$ und $\alpha_3 = -4\pi/10$

n	$g_{\tau\,n}(\alpha_1, \alpha_2, \alpha_3)$	$g_n^{(2)}(\alpha_1, \alpha_2, \alpha_3)$
10	1.0668379118338712	1.0641157035479450
12	1.0624181211129220	1.0641156130271904
14	1.0621618930948200	1.0641155229665988
16	1.0631221548598504	1.0641155371290900
18	1.0633125461160056	1.0641155361420236
20	1.0644978847766702	1.0641155361166546
22	1.0645166041509708	1.0641155361321010
24	1.0645413732209338	1.0641155361301646
26	1.0644963120188820	1.0641155361302608
28	1.0644865387938350	1.0641155361302698
30	1.0643097008097368	1.0641155361302674
32	1.0635498655776198	1.0641155361302676
34	1.0637189077158576	1.0641155361302676

In Table IV we compare the partial sums f_n of the Fourier series with the approximations f_n'. Plotted are the number of exact digits, defined as the negative decadic logarithm of the relativ error. A pronounced convergence acceleration is achieved using the extended method of the attached series. The accuracy of the extrapolated values is best for $n = 34-36$ and deteriorates slowly for larger values of n. The results in Table IV have been computed using `MAPLE V`$^{\rm TM}$ with `Digits`=32. Doubling the number of relevant decimal digits by putting `Digits`=64, one obtains 27.9 exact digits for $n = 40$ and 35.7 exact digits for $n = 50$. Halving the number of relevant digits using `Digits`=16, one obtains 12.0 exact digits for $n = 17$ and 14.4 exact digits for $n = 18$ and, again, the accuracy drop for larger n due to rounding errors.

Comparing Tables III and IV, it is plain that the extended method of the attached series based on the Levin transformation produces the same accuracy of approximation using fewer Fourier coefficients, as compared to the $\mathcal{H}$ transformation (the $d^{(2)}$ transformation is expected to perform similarly as the $\mathcal{H}$ transformation). For example, to achieve approximately 18 digits requires partial sums up to s_{40} in the case of the $\mathcal{H}$ transformation, but only partial sums up to s_{26} are required in the case of extended method of the attached series.

A second, more complicated example is the Fourier series

$$g(\alpha_1, \alpha_2, \alpha_3) = \sum_{m=1}^{\infty} \frac{1}{m^2} \cos(\alpha_1 m) \cos(\alpha_2 m) \cos(\alpha_3 m) \tag{26}$$

Table VII: Acceleration of the τ-fold-frequency series with $\tau = 3$ corresponding to the Fourier series (26) using the extended method of the attached series for $\alpha_1 = \pi/10$, $\alpha_2 = 2^{1/2}\pi/10$ and $\alpha_3 = -4\pi/10$

n	$g_{\tau\, n}(\alpha_1, \alpha_2, \alpha_3)$	$g_n^{(3)}(\alpha_1, \alpha_2, \alpha_3)$
7	1.0658521479576092	1.0641110700608030
9	1.0617854023532320	1.0641158521306604
11	1.0630811611593686	1.0641155150537964
13	1.0642649507655460	1.0641155374226974
15	1.0644700418511348	1.0641155360594698
17	1.0645450137450582	1.0641155361334964
19	1.0645183650097528	1.0641155361301756
21	1.0636953781393952	1.0641155361302648
23	1.0638506348956798	1.0641155361302688
25	1.0640718910621728	1.0641155361302674
26	1.0640686861206392	1.0641155361302676
27	1.0640483438234590	1.0641155361302676

with partial sums

$$g_n(\alpha_1, \alpha_2, \alpha_3) = \sum_{m=1}^{n+1} \frac{1}{m^2} \cos(\alpha_1 m) \cos(\alpha_2 m) \cos(\alpha_3 m) \, . \tag{27}$$

The function represented by the series is a solution of the two-dimensional wave equation

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial \alpha_1{}^2} - \left(\frac{\partial^2}{\partial \alpha_2{}^2} + \frac{\partial^2}{\partial \alpha_3{}^2} \right) \right] g(\alpha_1, \alpha_2, \alpha_3) = 0 \, , \quad c = 2^{-1/2} \tag{28}$$

that is 2π–periodic in each argument. In the extended method of the attached series this series is represented by a sum

$$g(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^{8} q_j \tag{29}$$

of eight complex power series of the form

$$q_j = \sum_{m=1}^{\infty} \frac{1}{8\, m^2} \exp(\mathrm{i}\, m\, \tilde{\alpha}_j) \tag{30}$$

with $\tilde{\alpha}_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\tilde{\alpha}_2 = -\tilde{\alpha}_1$, $\tilde{\alpha}_3 = \alpha_1 + \alpha_2 - \alpha_3$, $\tilde{\alpha}_4 = -\tilde{\alpha}_3$, $\tilde{\alpha}_5 = \alpha_1 - \alpha_2 + \alpha_3$, $\tilde{\alpha}_6 = -\tilde{\alpha}_5$ and $\tilde{\alpha}_7 = -\alpha_1 + \alpha_2 + \alpha_3$, $\tilde{\alpha}_8 = -\tilde{\alpha}_7$. As before, the partial sums of these power series

$$q_{j,n} = \sum_{m=1}^{n+1} \frac{1}{8\,m^2} \exp(\mathrm{i}\, m\, \tilde{\alpha}_j)\,, \quad n \in \mathbb{N}_0 \tag{31}$$

are used as input for the u transformation of Levin and thus, a sequence of approximations for g according to

$$g'_n(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^{8} u_n^{(0)}(1, q_{j,0}) \tag{32}$$

is obtained.

This example is studied in the Tables V – VII. The data have been obtained using `MAPLE V`$^{\mathrm{TM}}$ with `Digits`=32.

In Table V, the partial sums g_n of the Fourier series are compared with the approximations g'_n. Also for this more complicated series there is a very pronounced convergence acceleration using the extended method of the attached series.

The convergence is relatively uniform up to $n = 42$, for larger n there is a deterioration of the results due to runding errors.

In order to increase the number of exact digits one may increase the accuracy of the calculation using higher values for the variable Digits. Alternatively, one can use the τ-fold frequency series with partial sums $\check{s}_n = s_{\tau\, n}$. [16, 31, 17] For this end, we define for given natural number τ the approximations

$$g_n^{(\tau)}(\alpha_1, \alpha_2, \alpha_3) = \sum_{j=1}^{8} \mathcal{L}_n^{(0)}(1, [q_{j,\tau m}]|_{m=0}, [(\tau m + 1)(q_{j,\tau m} - q_{j,(\tau m)-1})]|_{m=0})\,. \tag{33}$$

Note that each of its eight terms can also be interpreted as the result of applying the $d^{(1)}$ transformation of Levin and Sidi with $R_l = \tau l$ to the corresponding attached series. This holds because the $d^{(1)}$ transformation with $R_\ell = \tau\ell$ for $\tau \in \mathbb{N}$ is nothing but the transformation (see [33, Eq. 4.12])

$$W_\nu^{(n)} = \frac{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1} s_{\tau\, n}/(s_{\tau\, n} - s_{(\tau\, n)-1})\right]}{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1}/(s_{\tau\, n} - s_{(\tau\, n)-1})\right]} \tag{34}$$

where the $\triangle$ denotes the forward difference operator with respect to the variable n acting as

$$\triangle f(n) = f(n+1) - f(n)\,, \qquad \triangle g_n = g_{n+1} - g_n\,. \tag{35}$$

But this is identical to the Levin transformation when applied to the sequence $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ with remainder estimates $\omega_n = (n + \beta/\tau)(s_{\tau\, n} - s_{(\tau\, n)-1})$, i.e.,

$$W_{\nu-1}^{(n)} = \mathcal{L}_\nu^{(n)}(\beta/\tau, s_{\tau\, n}, (n+\beta/\tau)(s_{\tau\, n} - s_{\tau\, n-1}))\,. \tag{36}$$

We remark that for $\tau \neq 1$ this is not identical to the u variant of the Levin transformation as applied to the partial sums $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ because in the case of the u variant one would have to use the remainder estimates $\omega_n = (n+\beta')(s_{\tau\, n} - s_{\tau\,(n-1)})$.

In Table VI, we choose $\tau = 2$ and in Table VII, we choose $\tau = 3$. The data show that about 60 coefficients of the original Fourier series for $\tau = 2$ and somewhat more than 70 coefficients for $\tau = 3$ are needed to achieve 16 exact digits. This may be compared with the number of 44 coefficients, that are needed in the case of Table V corresponding to $n = 43$ and $\tau = 1$. However, in the latter case, one cannot really decide from the corresponding data alone which are the last three digits of the result.

Decreasing the accuracy requirements to `Digits`=16, one obtains 9 digits for $\tau = 1$ and $n = 21$, 11 digits for $\tau = 2$ and $n = 18$ corresponding to $\tau\, n + 1 = 37$ coefficients, 11 digits for $\tau = 3$ and $n = 15$ corresponding to 46 coefficients, 14 digits for $\tau = 6$ and $n = 16$ corresponding to 97 coefficients and, finally 13 digits for $\tau = 10$ and $n = 13$ corresponding to 131 coefficients. This implies that by a suitable choice of τ, one may stabilize the algorithm in the extended method of the attached series and decrease the importance of rounding errors. Similar results have been obtained in the case of the $\mathcal{H}$ and $\mathcal{I}$ transformations [16, 31, 17].

Of course, one should keep in mind that the extended method of the attached series can only be applied for series for which the coefficients $a_{\ell,j}$ and $b_{\ell,j}$ in Eq. (19) are given individually. This is a similar limitation as in the method of the attached series [33, Sec. 3.2, item 3. (b), p. 391].

A further point is that — since using the Levin transformation for the τ-fold frequency series as in Eq. (33) is equivalent to the use of the $d^{(1)}$ transformation with $R_l = \tau l$ as noted above — one may apply all the known convergence analysis and stability results for the latter [33] to the extended method of the attached series.

Loosely speaking, this means that the extended method of the attached series, in combination with the τ-fold frequency approach near singularities, will provide stable extrapolation results if the terms $c_{j,n} = d_n \exp(\mathrm{i}\, n\, \tilde{\alpha}_j)$ of the jth attached series asymptotically obey

$$c_{j,n} \sim \zeta_j^n n^{\sigma_j} \sum_{k=0}^{\infty} e_{j,k} n^{-k} \,, \quad \text{as } n \to \infty,\ e_{j,0} \neq 0,\ \zeta_j \text{ and } \sigma_j \text{ complex}, \tag{37}$$

for each relevant j. The value of τ has to be chosen such that ζ^τ is sufficiently different from 1. in order to ensure stability. Readers that are interested in more details of these convergence and stability results are referred to the literature [33, Sec. 4].

5 Conclusions

From the data presented we can conclude that the newly introduced extended method of the attached series, suitably combined with the τ-fold frequency approach in the vicinity of singularities, is an effective and relatively cheap method for the convergence acceleration of slowly convergent Fourier series with several frequencies and coefficients that are smooth.

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