Rigid Syntomic Regulators and the p-adic L-function of a modular form

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Contents

Introduction

Fundamental objects studied in Arithmetic Geometry are schemes X of finite type over $\mathbb{Q}$. One way of obtaining interesting invariants of X is the following: Assume X is smooth projective of pure dimension d. For each $0 \le i \le 2d$, one can define the formal product over all primes

$$L(s, h^i X) := \prod_p \det(1 - \gamma_p \, q^{-s} \mid H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_p})^{-1} ,$$

where l is a prime $\neq p$, γ_p is a geometric Frobenius element at p and I_p is the inertia subgroup at p. The polynomials

$$\det(1 - \gamma_p T \mid H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_p})$$

have coefficients in $\mathbb{Q}$ for all $l \neq p$ such that p is a prime of good reduction and conjecturally this is true for all primes. Granted this, $L(s, h^i X)$ defines a holomorphic function in s in some right half-space of the complex plane. One expects that it can be continued meromorphically to a function on the whole of $\mathbb{C}$ and therefore it makes sense to consider the values $L^*(n, h^i X)$ for an arbitrary integer n. The superscript $*$ indicates that by value we mean the first nonvanishing coefficient in the Laurent series expansion at $s = n$. Motivated by the class number formula

$$L(0, h^0 \operatorname{Spec} K) = -\frac{hR}{w} , \qquad K/\mathbb{Q} \text{ a number field}$$

$$h = \text{ class number}, \quad w = \text{ number of roots of unity}, \quad R = \text{ regulator},$$

one hopes that also for higher dimensions, the analytic invariants $L^*(n, h^i X)$ are related to algebraic invariants of X. Conjectures of Beilinson [**Bei85**],[**DS91**] tell us more precisely what we should expect for these values, at least up to a rational number: He considers higher Chern classes, so-called regulator maps

$$r_{\mathcal{D}} : H^{i+1}_{\text{mot}}(X, n) \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$$

from rational motivic cohomology into Deligne cohomology. For simplicity, assume $n > \frac{i}{2} + 1$. Beilinson conjectures that

(1) The restriction of $r_{\mathcal{D}}$ to a certain $\mathbb{Q}$-subspace of "integral" elements is an isomorphism after tensoring with $\mathbb{R}$.
(2) The determinant of this isomorphism calculated relative to basis elements in $H^{i+1}_{\mathrm{mot}}(X,n)$ on the left hand side and a basis in a natural $\mathbb{Q}$-structure of Deligne cohomology on the right hand side, is equal to $L^*(i+1-n, h^iX)$.

The full conjecture is only known for $\dim X = 0$, where it is deduced from results of Borel [**Bor74**] by a comparison of two regulators. The problem for higher dimensions is that finite dimensionality of the motivic cohomology groups involved is not known. It is however still interesting to consider the weaker problem of finding a suitable subspace of elements of $H^{i+1}_{\mathrm{mot}}(X,n)$ whose determinant gives the desired L-value. Let us generalize the situation slightly and replace X by a (pure) motive M of weight i over $\mathbb{Q}$ which we think of as given by a pair (X,ρ), where $X/\mathbb{Q}$ is smooth and projective and ρ is a projector in a suitable ring of correspondences. For such M, we formally set

$$H^1_{\mathrm{mot}}(M(n)) := p_* H^{i+1}_{\mathrm{mot}}(X,n), \quad H^1_{\mathcal{D}}(M(n)) := p_* H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \ .$$

Here, we always assume $n > \dfrac{i}{2}+1$. The weak Beilinson conjecture as formulated above can now be extended to the case of motives in an obvious way and has been proven in a number of cases, for example for motives attached to Dirichlet characters [**Bei85**], Hecke characters of imaginary quadratic fields [**Den89**], and Hecke cusp eigenforms of weight $k \geq 2$ [**Bei86**], [**SS88**], [**DS91**, §5], [**Gea06**]. By the modularity theorem, the latter class of examples includes all elliptic curves over $\mathbb{Q}$.

One can ask if this philosophy relating the complex L-function to regulators can also be found in the p-adic world, where p is a fixed finite prime. For this, let M be a motive over $\mathbb{Q}$ and for simplicity let it have good reduction at p. One can attach to M p-adic invariants which are of algebraic nature like its p-adic étale realization or the crystalline realization of its reduction mod p. Conjecturally, there should also exist a p-adic analytic invariant of M, the p-adic L-function attached to M. The p-adic L-function should be a p-adic analytic function

$$L_{(p)}(\cdot, M) : \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \to \mathbb{C}_p$$

on the space of p-adic characters of $\mathbb{Z}_p^\times$ which is characterized by a certain interpolation property with respect to the complex L-function. $L_{(p)}(\cdot, M)$ is an important object in arithmetic and conjecturally is closely related to the Iwasawa theory of M. The interpolation property implies that for an integer n which is critical in the sense of Deligne, the number $L_{(p)}(y^n, M)$ (where

$y : \mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p$ is the obvious inclusion) is algebraic and essentially equal to $L(n, M)$ divided by a period coming from the comparison of Betti and de Rham cohomology. For a noncritical integer n, the value $L_{(p)}(y^n, M)$ is much more mysterious and is a priori just a possibly transcendental p-adic number. One can ask if it has an interpretation in terms of regulator maps as in the case of the complex L-function. For this one needs to find a good target space for a p-adic regulator map which is analogous to Deligne cohomology. Deligne cohomology can be thought of as "absolute Betti cohomology". This means roughly that a complex computing Deligne cohomology is obtained from a complex computing Betti cohomology by first taking the F^0-part of the Hodge filtration and then invariants under complex conjugation, the infinite Frobenius. (Here, taking invariants under a map ψ is used in the sophisticated sense of taking the shifted mapping cone of $1-\psi$.) Therefore, in order to get a p-adic analogue of Deligne cohomology, we should first look for a p-adic Betti cohomology, i.e. a "geometric" p-adic cohomology theory. Betti cohomology can be considered as the cohomology which is computed using real-analytic differential forms on $X(\mathbb{C})$. A natural candidate for p-adic Betti cohomology is therefore Berthelot's rigid cohomology which is computed using p-adic analytic (overconvergent) differential forms on the rigid analytic space associated to $X_{\mathbb{Q}_p}$. If one takes the F^0-part of the Hodge filtration and then the Frobenius invariants of suitable rigid cohmology complexes (this is much more complicated than we make it seem here) one obtains *rigid syntomic cohomology,* which has been developed by Besser in [**Bes00**]. For a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and any smooth scheme over $\mathcal{O}_K$, he defines rigid syntomic (or simply syntomic) cohomology groups $H^i_{\mathrm{syn}}(X, n)$ with Tate twist coefficients which are independent of auxiliary data. He also defines higher Chern classes with values in syntomic cohomology which give a syntomic regulator map

$$r_{\mathrm{syn}} : H^i_{\mathrm{mot}}(X, n) \to H^i_{\mathrm{syn}}(X, n)\ .$$

As in Deligne cohomology one can generalize this to a motive M and obtain a regulator map

$$r_{\mathrm{syn}} : H^1_{\mathrm{mot}}(M) \to H^1_{\mathrm{syn}}(M)\ .$$

The purpose of this thesis is to relate this regulator map to the p-adic L-function of M in case $M = M(f)(k+l)$, where $M(f)$ is the motive constructed by Scholl [**Sch90**] associated to a cusp newform of weight $k \geq 2$ and l is a natural number. We assume that f has good reduction mod p and that $p \geq 5$. Let us furthermore assume only for this introduction that f has rational Fourier coefficients. Our strategy for relating the p-adic L-function and the syntomic regulator is to imitate the proof of the complex weak Beilinson conjecture for $M = M(f)(k+l)$, which consists essentially of three steps:

(1) Describe the image of specific K-theory classes $\mathrm{Eis}_{\mathrm{mot}}(\varphi)$, (the Eisenstein symbols) under the regulator map.
(2) Compute explicitly the cup product of these images in order to get elements in the correct degree.
(3) Relate this product to the L-function using duality and the Rankin-Selberg method.

In the p-adic case, step 1) has been solved by Bannai-Kings [**BK**]. We build on their work and obtain step 2) as our first main result: Proposition II.7.1 gives an explicit description of the product of two syntomic Eisenstein clases in terms of p-adic modular forms. The harder part of this paper deals with step 3). We first derive a p-adic Rankin-Selberg method in the cyclotomic variable (Theorem V.2.1) from results of Panchishkin [**Pan02**], [**Pan03**]. Whereas usually the term "p-adic Rankin-Selberg method" refers to the p-adic interpolation of complex Rankin-Selberg convolutions, we use it in a stricter sense: Our method gives an interpretation of the p-adic L-function also at noncritical values, namely as a rigid-analytic Petersson inner product. Let us stress that Panchishkin's ideas are fundamental for our approach, in fact this thesis can be taken as a cohomological interpretation of Panchishkin's results. We use the explicit description from step 2) and the p-adic Rankin-Selberg method in order to relate the regulator to the p-adic L-function.
Before stating the main theorem, let us note that for $M = M(f)(k+l)$, there is a natural isomorphism

$$H^1_{\mathrm{syn}}(M) \cong H_{\mathrm{rig}}(M) = \text{ rigid realization of } M$$

and we will identify both spaces. Remember that $H_{\mathrm{rig}}M(f)$ has a Frobenius endomorphism Φ with characteristic polynomial

$$X^2 - a_p X + p^{k-1} = (X-\alpha)(X-\beta)\ , \quad v_p(\alpha) < p-1\ .$$

Because f is ordinary, $v_p(\alpha) = 0$ and $\alpha \neq \beta$. The p-adic L-function attached to the motive $M(f)$ will be written $L_{(p)}(\cdot, f, \alpha, \Omega)$, see chapter IV for details. For values at the n-fold power of the cyclotomic character we use the notation $L_{(p)}(n, f, \alpha, \Omega)$, this is normalized so that $n = 1, \ldots, k-1$ are the critical integers. We denote the map deduced from r_{syn} by tensoring with a finite extension F of $\mathbb{Q}$ still by r_{syn}.

THEOREM .0.1. *Let $p \geq 5$ be a prime and $l \geq 0$ an integer. Let $M = M(f)(k+l)$, where f is a cusp newform with good ordinary reduction mod p of level $N_f \geq 4$ and weight $k \geq 2$. There exists*

$$\kappa \in H^1_{\mathrm{mot}}(M) \otimes F, \qquad \mathbb{Q} \subset F \text{ a finite extension },$$

such that

$$L_{(p)}(-l, f, \alpha, \Omega) = A \cdot \big(v_\alpha \ , \ r_{\mathrm{syn}}(\kappa)\big)_{\mathrm{rig}} \cdot t^{l+1} \ ,$$

where A is a nonzero algebraic number and t is the p-adic analogue of $2\pi i$. Furthermore, $(\cdot,\cdot)_{\mathrm{rig}}$ is the rigid duality pairing and $v_\alpha \in H_{\mathrm{rig}}M(f)$ is a normalized Frobenius eigenvector with unit eigenvalue α, namely it satisfies:

$$\Phi \, v_\alpha = \alpha \, v_\alpha, \quad (v_\alpha, \omega_f) = t^{k-1} \ ,$$

where

$$\omega_f = \text{ cohomology class in } H_{\mathrm{rig}}M(f) \text{ defined by } f(q)\frac{dq}{q}(dz)^{k-2} \ .$$

The theorem is an incarnation of the p-adic Beilinson conjecture as formulated by Perrin-Riou [**Col00**, Conj.2.7] (the element v appearing there equals our v_α). Note that in loc. cit., the conjecture is stated not in terms of the syntomic regulator, but in terms of the étale regulator and the Bloch-Kato exponential map which amounts to the same by the compatibility of both regulators [**Bes00**, §9], [**Niz97**]. The constant A is explicitly calculated, cf. Corollary V.2.3. The field F is a cyclotomic extension which we use in order to decompose Eisenstein symbols according to Dirichlet characters. We stress that the result as such is not new: It was known to experts that Kato's Euler system combined with a reciprocity law of Perrin-Riou [**Kat04**, Thm. 16.4.(ii)], [**PR93**, 2.2] and work of Gealy on étale Eisenstein classes [**Gea06**, chap.10] would yield such formulas. The new content is that the proof of Thm. 0.1 does not use Kato's Euler System and in fact no comparison with étale cohomology at all. It stays completely on the rigid (or crystalline) side of p-adic Hodge theory. The main tools the proof uses are:

- The calculation of the syntomic Eisenstein class by Bannai-Kings [**BK**].
- Panchishkin's construction of the p-adic L-function [**Pan02**],[**Pan03**].

- The theory of p-adic modular forms as formulated by Coleman, Hida, Katz and others.
- Hida's ordinary projection operator on the space of p-adic modular forms.

Note that in contrast to the complex case, the theorem makes a statement about the honest value at $-l$ and not about the first derivative at $-l$. This is because in the p-adic case, there are no complex Gamma functions involved and therefore the p-adic L-function does not necessarily vanish at all negative integers. The reason why we are only able to prove the result for ordinary forms are technical problems with the spectral theory of the p-adic Hecke operator U acting on the space of (overconvergent) p-adic modular forms: The projection on U-eigenspaces of slope higher than 0 is *not* continuous in the q-expansion topology. This makes it hard for us in this case to give an interpretation of the measure constructed by Panchishkin [**Pan02**],[**Pan02**] outside of the critical integers.
We conclude this introduction with a speculative remark about how the p-adic Rankin Selberg method (Theorem V.2.1, Observation V.2.2) might be used in order to get more information about the mysterious p-adic L-function: The method expresses the value $L_{(p)}(\cdot, f, \alpha, \Omega)$ for any integer n (one could also take more general weight characters) as a nonzero multiple of

$$(\ \omega_{\bar{f},\alpha} \ , \ \mathscr{E}_n \ )_{\mathrm{rig}} \ ,$$

where $\omega_{\bar{f},\alpha}$, $\mathscr{E}_n$ are two rigid cohomology classes defined by overconvergent p-adic modular forms. The author hopes that, using reciprocity laws of Coleman for p-adic differentials on curves [**Col89**], this might be used to derive nonvanishing conditions for p-adic L-values, in particular at the non-critical integers where no direct comparison with the complex L-function is possible. However for this idea to work, one would have to understand the behavior of the overconvergent modular forms involved near (the lifts of) the supersingular points, which seems rather hard.

Overview

Let us explain the content of the paper in more detail: The first chapter gives a quick review of the results of Bannai-Kings [**BK**], which describe the image of the Eisenstein symbol under the syntomic regulator in terms of p-adic modular forms.
In the second chapter we begin by defining the syntomic cup product with coefficients over an unramified base K. Like in [**BK**], we only work with particularly well-behaved spaces, namely with smooth compactifications $X \hookrightarrow \overline{X}$ over $\mathcal{O}_K$ together with an overconvergent Frobenius lift. In this situation,

one can define syntomic cohomology using Cech resolutions [**Ban00**, Def 2.2], [**BK**, A.3] and we use a standard formula [**Bes00**, Lemma 3.2] in order to define the syntomic cup product in terms of the de Rham and the rigid cup product on the level of Cech complexes. We then define a product $\sqcup_M$ for the cohomology of the modular curve

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}(1)) .$$

We show that under the standard isomorphism

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}(1)) \cong H^{l+1}_{\mathrm{syn}}(X^l, l+1) ,$$

this coincides up to a sign with a product for the spaces

$$H^{l+1}_{\mathrm{syn}}(X^l, l+1)$$

which is suggested by work of Scholl [**DS91**, §5]. This is done to ensure that the $\sqcup_M$-product of two syntomic Eisenstein classes is in the image of the regulator map. We are forced to show this compatibility in a very direct way, as we could not find a reference for a rigid syntomic Leray spectral sequence and its behavior under cup product. The last part of the chapter deals with the explicit computation of the product of two Eisenstein classes. Technical problems arise because some of the p-adic Eisenstein series used in [**BK**] are not overconvergent. We use work of Coleman and others on the relationship between rigid cohomology and overconvergent modular forms [**Col95**], [**CGJ95**]. We discover that certain non-overconvergent forms still define rigid cohomology classes.

In the third chapter, we first collect some facts on Hecke operators and rigid cohomology. We then use the rigid Poincaré duality pairing in order to define a rigid-analytic analogue $l_{f,\mathrm{rig}}$ of a linear form l_f defined by Panchishkin [**Pan02**],[**Pan03**] via the classical Petersson inner product. We show that the linear forms coincide up to a nonzero p-adic number. This step is crucial later on in order to give an interpretation of Panchishkin's measure at noncritical weights.

Chapter IV gives a reworking of Panchishkin's [**Pan02**],[**Pan03**] construction of the p-adic L-function of f, which is adapted to our situation. One first constructs a measure μ_ξ with values in the space of p-adic modular forms, then projects onto the α-eigenspace for the U-operator, and finally applies the linear form l_f. One checks that this gives the p-adic L-function using the Rankin-Selberg method.

In the last chapter, we find by studying congruences of q-expansions, that after we project onto the α-eigenspace, the p-adic modular forms appearing in the product of two Eisenstein classes are essentially the same as the ones gotten from evaluating μ_ξ at a noncritical integer. We then use the rigid duality pairing to derive a p-adic Rankin-Selberg method (Thm. V.2.1) from the results of chapter IV. These two steps prove Cor. V.2.3 and by this, the main theorem Thm 0.1.

It is a pleasure to thank my advisor Guido Kings for introducing me to the p-adic point of view and for his constant help and support. I also would like to thank Kenichi Bannai who kindly answered all my questions.

Notations and Conventions. p is always assumed to be a prime ≥ 5. Let $\mathbb{Q}_p \subset K$ be finite and unramified. We use de Rham, rigid and rigid syntomic cohomology of smooth pairs $\mathscr{X} = (X, \overline{X})$ over $\mathcal{O}_K$ with overconvergent Frobenius ϕ_X with coefficients in filtered overconvergent F-isocrystals $\mathscr{M} = (M, \nabla, F, \Phi_M)$ as defined in [**BK**, A]. Like in loc. cit. we denote these cohomology groups by

$$H^n_{\mathrm{dR}}(\mathscr{X}, \mathscr{M}), H^n_{\mathrm{rig}}(\mathscr{X}, \mathscr{M}), H^n_{\mathrm{syn}}(\mathscr{X}, \mathscr{M}) .$$

When taking sections of the underlying coherent modules M and M_{rig} of $\mathscr{M}$, we often write $\mathscr{M}$ instead of M or M_{rig} and whether we mean algebraic or rigid-analytic sections is always clear from the space over which the sections are taken. A small difference in notation with respect to [**BK**] is that we denote the Frobenius structure of the coefficients $\mathscr{M}$ by Φ_M in order to distinguish it from the Frobenius endomorphism Φ on $H^n_{\mathrm{rig}}(\mathscr{X}, \mathscr{M})$ which is induced by ϕ_X and Φ_M. We often call rigid syntomic cohomology simply syntomic cohomology.
We use the following convention from [**BK**]: If X is a scheme over $\mathcal{O}_K$, we write $\mathcal{X}$ for its completion with respect to the special fiber and we denote the rigid analytic space associated to this formal scheme by $\mathcal{X}_K$. The rigid analytic space associated to the generic fiber X_K of X is denoted by X^{an}_K.

CHAPTER I

Syntomic Eisenstein classes

We give a quick sketch of the main result of [**BK**].

Before we start our discussion, let us introduce a variable T in order to keep track of Tate twists in rigid cohomology. Let V be a vector space over $\mathbb{Q}_p$ with Frobenius endomorphism. Consider the ring $\mathbb{Q}_p[T, T^{-1}]$ with Frobenius endomorphism

$$\Phi_{|\mathbb{Q}_p} = \mathrm{id}_{\mathbb{Q}_p}$$

$$\Phi(T) = p^{-1}T \ .$$

We identify the rigid Tate object $\mathbb{Q}_p(j) = \mathbb{Q}_p \cdot e_j$, cf. [**Ban00**, Def. 1.10 (i),(iv)], with $\mathbb{Q}_p \cdot T^j \subset \mathbb{Q}_p[T, T^{-1}]$ by sending $1 \cdot e_j$ to T^j. In the same way we identify the twisted Frobenius vector space

$$V(j)$$

with the the space

$$V \otimes T^j$$

inside

$$V \otimes \mathbb{Q}_p[T, T^{-1}] \ .$$

We write $V \cdot T^j$ instead of $V \otimes T^j$. A p-adic analogue t of $2\pi i$ is then given by $t := T^{-1}$. We work with T rather than with T^{-1} because we prefer to think in terms of geometric Frobenius weights.

Let $M/\mathbb{Z}_p$ be the pullback to $\mathbb{Z}_p$ of the modular curve of level $\Gamma(N)$, $N \geq 3$, prime to p. If $\overline{M}$ is the smooth compacitfication of M, then $\mathscr{M} := (M, \overline{M})$ is a smooth pair. As M is smooth and affine, there is a Frobenius lifting

$$\phi_M : \mathcal{M} \to \mathcal{M} \ .$$

on the level of formal schemes which overconverges on the associated rigid spaces. We denote by $X, \overline{X}$ the universal and the universal generalized elliptic curve over M and set

$$X^l := X_M \times \cdots {}_M X$$

where the fibre product is taken l times. $\widetilde{X^l}$ denotes the canonical desingularization of $\overline{X}^l$ constructed by Deligne. Then $\mathscr{X}^l := (X^l, \widetilde{X^l})$ is a smooth pair and one can define the higher direct images [**BK**, A.2]

$$R^i\pi_*^{(l)}\mathbb{Q}_p(n)$$

of the structure morphism

$$\pi^{(l)} : \mathscr{X}^l \to \mathscr{M} \ .$$

In particular, we can consider the "modular" cohomology vector space

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(n)) \,, \quad \mathscr{H}^\vee := R^1\pi_*\mathbb{Q}_p \ .$$

The motivic cohomology constructed from algebraic K-theory has only Tate twist coefficients. To consider the regulator map from motivic to syntomic cohomology, one therefore needs syntomic cohomology of X^l with only Tate twists as coeffiecents. Unfortunately, the definition of rigid syntomic cohomology we use [**BK**, A] cannot be directly applied to $\mathscr{X}^l$, because there is no obvious Frobenius lift on the formal scheme $\mathcal{X}$. There are different ways to fix this. We proceed like Bannai-Kings [**BK**] and define the cohomology groups in question using Besser's [**Bes00**] definition of rigid syntomic cohomology. To make the different definition apparent in notation, we denote these cohomology vector spaces like in [**BK**] by $H^*_{\mathrm{syn}}(X^l, n)$ and not by $H^*_{\mathrm{syn}}(\mathscr{X}^l, \mathbb{Q}_p(n))$. To compare $H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(n))$ and $H^*_{\mathrm{syn}}(X^l, n)$ one needs a Leray spectral sequence in rigid syntomic cohomology for which however there seems to be no reference. As we only need to compare $H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(l+1))$ with the eigenspace $H^*_{\mathrm{syn}}(X^l, l+1)(\epsilon)$, (here ϵ is as usual the character on the group $\mu_2^l \rtimes S_l$ that is the product map on μ_2^l and the sign character on S_l) there is a way to work around this [**BK**, Def. 2.7]: Assume $l \geq 1$, otherwise the cohomology groups are equal. Both ways of defining syntomic cohomology come with natural long exact sequences

$$\ldots \to H^i_{\mathrm{syn}}(X^l, n) \to F^0 H^i_{\mathrm{dR}}(X^l, n) \xrightarrow{1-\Phi} H^i_{\mathrm{rig}}(X^l, n) \to \ldots$$

$$\ldots \to H^i_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(n)) \to F^0 H^i_{\mathrm{dR}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(n)) \xrightarrow{1-\Phi} H^i_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(n)) \to \ldots$$

and these induce isomorphisms

$$H^{l+1}_{\mathrm{syn}}(X^l, l+1)(\epsilon) \cong H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^{l+1}_{\mathrm{rig}}(X^l, l+1)(\epsilon)) \ ,$$

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(l+1)) \cong H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(l+1)))$$

by [**BK**, Prop. 2.6]. Therefore, we can define an isomorphim

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(l+1)) \cong H^{l+1}_{\mathrm{syn}}(X^l, l+1)(\epsilon)$$

by requiring that this map makes the diagram

$$\begin{array}{ccc} H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) & \longrightarrow & H^{l+1}_{\mathrm{syn}}(X^l, l+1)(\epsilon) \\ \downarrow \cong & & \downarrow \cong \\ H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) & \underset{\cong}{\longrightarrow} & H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^{l+1}_{\mathrm{rig}}(X^l, l+1)(\epsilon)) \end{array}$$

commutative, where the lower map is induced by the isomorphism given by the Leray spectral sequence and the Künneth map in rigid cohomology. Note that this gives the "correct" map for any reasonable definition of a rigid syntomic Leray spectral sequence, because any such definition should be compatible with the analogous rigid spectral sequence.
We turn to the syntomic regulator constructed by Besser. By [**Bes00**], there is a natural regulator map

$$r_{\mathrm{syn}} : H^{l+1}_{\mathrm{mot}}(X^l, l+1) \to H^{l+1}_{\mathrm{syn}}(X^l, l+1)$$

which is compatible with the de Rham regulator map. Hence we get a map

$$H^{l+1}_{\mathrm{mot}}(X^l, l+1) \to H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1))$$

which we also denote by r_{syn}. For any field F we define

$$F[(\mathbb{Z}/N)] := \{\varphi : (\mathbb{Z}/N) \to F\}\ .$$

In case $F = \mathbb{C}$ one has the Fourier transform in the first variable

$$P_1\varphi(m,n) := \sum_{v=0}^{N-1} \varphi(v,n) \exp(\frac{2\pi i m v}{N})$$

and the symplectic Fourier transform

$$\widehat{\varphi}(m,n) := \frac{1}{N} \sum_{u,v} \varphi(u,v) \exp(\frac{2\pi i(un - mv)}{N})\ .$$

We also set

$$L(\varphi, s) := \sum_{m \geq 1} \frac{\varphi(m,0)}{m^s}$$

for large $\mathrm{Re}(s)$ and denote the meromorphic continuation of this function by the same symbol. For any rational linear combination

$$\varphi \in \mathbb{Q}[(\mathbb{Z}/N)] := \{\psi : (\mathbb{Z}/N) \to \mathbb{Q}\}$$

of torsion sections which satisfies $\varphi(0,0)=0$ in case $l=0$, there is an element

$$\mathrm{Eis}^{l+2}_{\mathrm{mot}}(\varphi)\in H^{l+1}_{\mathrm{mot}}(X^l,l+1)(\epsilon)$$

called an Eisenstein symbol, [**BK**, Def. 1.1]. In case $l=0$, we assume in addition that $\widehat{\varphi}(0,0)=0$ in order to be able to apply a q-expansion formula of Katz [**Kat76**, Lemma 3.3.1]. The main result of [**BK**] gives an explicit descrption of $\mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi):=r_{\mathrm{syn}}(\mathrm{Eis}^{l+2}_{\mathrm{mot}}(\varphi))$. Before stating this, we have to look at how elements of $H^1_{\mathrm{syn}}(\mathscr{M},\mathrm{Sym}^l\mathscr{H}^\vee(l+1))$ can be described in general [**BK**, Prop. A.16]. Essentially by definition of rigid syntomic cohomology a class in this vector space can be represented by a pair (α,η), where

$$\eta\in H^0(\overline{M}_{\mathbb{Q}_p},\omega^l\otimes\Omega^1(\log C))\ ,C=\mathrm{Cusp}$$

is an algebraic section and

$$\alpha\in H^0(M^{\mathrm{an}}_{\mathbb{Q}_p},j^\dagger\mathrm{Sym}^l\mathscr{H}^\vee(l+1))$$

is a rigid section which satisfies

$$\nabla\alpha=(1-\Phi)\eta_{\mathrm{rig}}\ .$$

Here Φ is the Frobenius on $\mathcal{M}_{\mathbb{Q}_p}$ composed with the Frobenius of the coefficients $\mathrm{Sym}^l\mathscr{H}^\vee(l+1)$ and η_{rig} is the rigid analytic section associated to η. We sometimes write η instead of η_{rig}. Because the F^0 part of the Hodge filtration of $\mathrm{Sym}^l\mathscr{H}^\vee(l+1)$ is zero, one shows that the pair (α,η) representing the cohomology class is unique.

Now let (α,η) be the pair representing $\mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi)$. It turns out that η is konwn: By compatibility of r_{syn} with the de Rham regulator map r_{dR} it is equal to the section representing the de Rham Eisenstein class $\mathrm{Eis}^{l+2}_{\mathrm{dR}}(\varphi):=r_{\mathrm{dR}}(\mathrm{Eis}^{l+2}_{\mathrm{mot}}(\varphi))$. This section is known to be

$$\frac{2E_{l+2,0,\varphi}}{l!N^l}(dz)^l\otimes\delta,$$

where $\delta=\dfrac{dq}{q}$ is the one-form dual to the Gauß-Manin connection and $2E_{l+2,0,\varphi}$ is the algebraic Eisenstein series of level $\Gamma(N)^{\mathrm{arith}}$ which has q-expansion (at ∞)

$$\frac{1}{2}L(-1-l,f(m,0)-(-1)^{l+1}f(-m,0))+$$
$$\sum_{n\geq 1}q^n\sum_{\substack{dd'=n\\ d,d'>0}}\Big(d^{l+1}f(d,d')-(-d)^{l+1}f(-d,-d')\Big),\quad f=P_1(\hat{\varphi})\ .$$

Note that because of $M(\Gamma(N))\cong M(\Gamma(N)^{\mathrm{arith}})\otimes_{\mathbb{Q}}\mathbb{Q}(\mu_N)$, any modular form on $M(\Gamma(N)^{\mathrm{arith}})$ defined over $\mathbb{Q}(\mu_N)$ by extension of scalars gives a modular form on $M(\Gamma(N))$ which is defined over $\mathbb{Q}$. The advantage of using $\Gamma(N)^{\mathrm{arith}}$-structures at this point lies in the fact that the fibre at p of

$M(\Gamma(N)^{\mathrm{arith}})/\mathbb{Z}$ is geometrically connected and therefore a modular form of this level is uniquely determined by its q-expansion at ∞.
In case $l \geq 1$, the de Rham part η determines the rigid part α : This is because of the isomorphism

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \cong H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)))$$

$$= \mathrm{Ker}(1 - \Phi : F^0 H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \to H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)))$$

from above, which on pairs (α, η) is given by

$$(\alpha, \eta) \mapsto \eta_{\mathrm{rig}} \ .$$

In case $l = 0$ the situation is hardly more complicated. From the long exact sequence, one deduces the short exact sequence

$$0 \to \mathbb{Q}_p \to H^1_{\mathrm{syn}}(\mathscr{M}, \mathbb{Q}_p(1)) \to H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}, \mathbb{Q}_p(1))) \to 0$$

in which the first map sends a constant $c \in \mathbb{Q}_p$ to the pair $(c, 0)$ and the second map is $(\alpha, \eta) \mapsto \eta_{\mathrm{rig}}$. Thus η determines α up to a constant.

How can one describe a general section $\alpha \in H^0(M^{\mathrm{an}}_{\mathbb{Q}_p}, j^\dagger \mathrm{Sym}^l \mathscr{H}^\vee_{\mathrm{rig}}(l+1))$? Bannai-Kings answer this question by describing the image of α under two (injective) maps. The first map is just the map

$$H^0(M^{\mathrm{an}}_{\mathbb{Q}_p}, j^\dagger \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \hookrightarrow H^0((\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1))$$

which restricts an overconvergent section on the open modular curve $\mathcal{M}_{\mathbb{Q}_p}$ to a convergent section on the ordinary part of the modular curve $(\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p}$. The latter space receives a map

$$\widetilde{\mathcal{M}}_{\mathbb{Q}_p} \to (\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p} \ ,$$

where $\widetilde{\mathcal{M}}$ is the formal $\mathbb{Z}_p$-scheme which parametrizes trivialized elliptic curves with level $\Gamma(N)$-structure. Recall that a trivialization of an ordinary ellitpic E/B over a complete and separated $\mathbb{Z}_p$-Algebra B is given by an isomorphism of formal groups

$$\widehat{E} \cong \widehat{G}_m$$

over B. We denote the pullback of $\mathscr{H}^\vee$ to $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$ by $\widetilde{\mathscr{H}^\vee}$. The advantage the space $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$ is that $\widetilde{\mathscr{H}^\vee}$ is locally free: There is a unique element $\omega \in \Omega^1_{\widetilde{E}/\widetilde{M}}$ whose restriction coincides with the pullback of the differential $dT/(1+T)$ on

$\widehat{G_m}$ via the universal trivialization. It satisfies $\Phi\omega = p\,\omega$. Secondly, by work of Dwork and Katz there is a global section u of $\widetilde{\mathscr{H}^\vee}$ which is Frobenius invariant and satisfies $\nabla\omega = u \otimes \delta$. ($\nabla$ is the Gauß-Manin connection). Obviously, $\nabla u = 0$. Because of their simple behavior under Frobenius and the Gauß-Manin connection, ω, u is a basis of $\widetilde{\mathscr{H}^\vee}$ which is well suited for solving the equation

$$\nabla\alpha = (1-\Phi)\eta \ .$$

Note that we write ω, u for what was written $\tilde{\omega}, \tilde{u}$ in [**BK**].
Having a global basis, we can write

$$H^0(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathrm{Sym}^l \widetilde{\mathscr{H}^\vee}) = \Big\{ \sum_{n=0}^{l} c_n u^n \omega^{k-n} : c_n \in \Gamma(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathcal{O}) \Big\} \ .$$

By definition,

$$V(\Gamma(N), \mathbb{Q}_p) := \Gamma(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathcal{O})$$

is the space of Katz (p-adic) modular forms [**Kat76**, Chap. V]. For $w \in \mathbb{Z}$ there is a subspace $V_w(\Gamma(N), \mathbb{Q}_p) \subset V(\Gamma(N), \mathbb{Q}_p)$ of Katz modular forms that have (p-adic) weight w [**Kat76**, 5.3] and one checks that a section of the form

$$cu^n\omega^{l-n} \in H^0(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathrm{Sym}^l \widetilde{\mathscr{H}^\vee})$$

descends to $H^0((\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p}, \mathrm{Sym}^l \mathscr{H}^\vee)$ if and only if c is a Katz modular form of weight $l-2n$. One concludes that the natural pullback map

$$H^0((\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p}, \mathrm{Sym}^l \mathscr{H}^\vee) \to H^0(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathrm{Sym}^l \widetilde{\mathscr{H}^\vee})$$

identifies $H^0((\mathcal{M}^{\mathrm{ord}})_{\mathbb{Q}_p}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1))$ with the space

$$\Big\{ \sum_{n=0}^{l} c_n u^n \omega^{k-n}\, T^{l+1} : \ c_n \in V_{l-2n}(\Gamma(N), \mathbb{Q}_p) \Big\} \ .$$

We can therefore describe α by determining the associated Katz modular forms c_n. For this one needs certain non-classical p-adic Eisenstein series. For $m \geq 1$, $r \in \mathbb{Z}$, these are Katz modular forms $E^{(p)}_{m,r,\varphi}$ of level $\Gamma(N)^{\mathrm{arith}}$ and weight $m+r$ which are characterized by their q-expansion at ∞ which is

$$\frac{1}{2} \sum_{n \geq 1} q^n \sum_{\substack{dd'=n \\ d,d'>0 \\ p \nmid d'}} \Big(d^{m-1}(d')^r f(d,d') - (-d)^{m-1}(-d')^r f(-d,-d') \Big), \quad f = P_1(\hat{\varphi}) \ ,$$

if $m \geq 2$ and

$$\frac{1}{4}L(-r, f(0,m) - (-1)^{l+1} f(0,-m))$$
$$+\frac{1}{2}\sum_{n\geq 1} q^n \sum_{\substack{dd'=n\\ d,d'>0\\ p\nmid d'}} \Big((d')^r f(d,d') - (-d')^r f(-d,-d')\Big), \quad f = P_1(\hat{\varphi}) \ ,$$

in case $m = 1$.
Denote by $\widetilde{\alpha}$ the pullback of α to $\widetilde{\mathcal{M}}$. The main result of [**BK**] can now be stated:

Theorem I.0.2. *(Bannai-Kings,[**BK**, Thm. 5.11])*
Let $l \geq 1$. If $\mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi) = (\alpha, \eta)$, then the Katz modular forms c_n , $0 \leq n \leq l$ associated to α are given by

$$c_n = \frac{2}{(l-n)!N^l}\, E^{(p)}_{l+1-n,-1-n,\varphi} \ .$$

In other words,

$$\widetilde{\alpha} = \sum_{n=0}^{l} \frac{2}{N^l(l-n)!}\, E^{(p)}_{l+1-n,-1-n,\varphi} u^n \omega^{l-n} \cdot T^{l+1} \ .$$

Remark. The theorem is stated in a slightly different form in [**BK**]: There, the Tate module $\mathscr{H}$ rather than its dual is used and the formula reads:

$$\widetilde{\alpha} = \sum_{n=0}^{l} \frac{(-1)^n 2}{N^l(l-n)!} E^{(p)}_{l+1-n,-1-n,\varphi_2} \omega^{\vee n} u^{\vee l-n} T \ \in H^0((\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathrm{Sym}^l \widetilde{\mathscr{H}}(1)) \ .$$

(In loc. cit. there is no T to be precise, but we found it helpful in order to remember the right Frobenius action.) Here $\omega^\vee, u^\vee$ are the sections of $\widetilde{\mathscr{H}}$ dual to ω, u. Let us show that both formulas are equivalent: If $[\cdot,\cdot]$ denotes the relative rigid Poincaré pairing of the universal trivialized elliptic curve over $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$, we have (with $\eta_0 := xdx/y$, where x, y are the Weierstraß coordinates of the universal trivialized elliptic curve given by ω)

$$[\omega, u]_{\mathrm{rig}} = [\omega, \eta_0]_{\mathrm{rig}} = [\omega, \eta_0]_{\mathrm{dR}} = T^{-1}. \quad ([\textbf{Kat76}, \text{p.512}])$$

We can reformulate this by saying that

$$u^\vee = \omega T, \ \omega^\vee = -uT$$

under the identification

$$\mathscr{H} \cong \mathscr{H}^\vee(1) \ .$$

This shows the equivalence of the formulas.

Remark. The theorem stays true up to a constant if $l = 0$: In this case one has to solve the equation

$$d\, c_0 = 2E^{(p)}_{2,0,\varphi} \otimes \delta$$

and this is established by

$$c_0 = 2E^{(p)}_{1,-1,\varphi} .$$

One conclude

$$\widetilde{\alpha} = 2E^{(p)}_{1,-1,\varphi} \bmod \mathbb{Q}_p .$$

We believe that one has the equality $\widetilde{\alpha} = 2E^{(p)}_{1,-1,\varphi}$ "on the nose" and in the following will use this equality in all computations in order to avoid tedious case-by-case analysis. Our application does not depend on this assumption: once we pair the product of two Eisenstein classes with a parabolic cohomology class, the contributions coming from the constants vanish for weight reasons.

We conclude by introducing some notation. For convenience in later computations, we set

Definition I.0.3.

$$\mathcal{E}^{l+2}_{?}(\varphi) := \frac{N^l}{2} \mathrm{Eis}^{l+2}_{?}(\varphi) .$$

for $? \in \{\mathrm{mot}, \mathrm{dR}, \mathrm{syn}\}$.

Let $\mathbb{Q} \subset F$ be a finite extension. Denote the maps deduced from

$$\mathrm{Eis}^{l+2}_{\mathrm{mot}} : \mathbb{Q}[(\mathbb{Z}/N)^2] \to H^{l+1}_{\mathrm{mot}}(X^l, l+1),$$

and

$$r_? : H^{l+1}_{\mathrm{mot}}(X^l, l+1) \to H^1_?(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1))$$

by tensoring with F still by the same symbols. Then for a Dirichlet character ϵ mod N of parity l and F an extension containing the values of ϵ, set

Definition I.0.4.

$$\phi_\xi := \widehat{P_1^{-1}\varphi_\epsilon} ,$$

where

$$\varphi_\epsilon(m,n) = \begin{cases} \epsilon(m) & N \mid n \\ 0 & N \nmid n \end{cases}$$

and P_1^{-1} is the inverse Fourier transform in the first variable. Furthermore, we set

$$E_?^{l+2}(\epsilon) := \mathcal{E}_?(\phi_\epsilon) \; .$$

Note that $\phi_\xi(m,n)$ is just $\epsilon(n)$, but the above definition is better suited for computing the q-expansion of the de Rham realization: One has

$$E_{\mathrm{dR}}^{l+2}(\epsilon) = E_{l+2,0}(\epsilon)\frac{1}{l!}\omega^l \otimes \delta$$

where $E_{l+2,0}(\epsilon)$ is the Eisenstein series of level $\Gamma_0(N)$ with Dirichlet character ϵ that has q-expansion

$$\frac{L(-1-l,\epsilon)}{2} + \sum_{n\geq 1} q^n \sum_{d|n,d>0} \epsilon(d)d^{l+1}$$

at ∞. Here we mean q-expansion at the Tate curve of level $\Gamma_1(N)$ and not $\Gamma(N)$, so there is no exponentiation $q \mapsto q^N$. One easily checks that already $E_{\mathrm{mot}}^l(\epsilon)$ has level $\Gamma_0(N)$ (and not just level $\Gamma(N)$).

CHAPTER II

The product of syntomic Eisenstein classes

II.1. Syntomic cup product with coefficients

We define a cup product in rigid syntomic cohmology with coefficients over an unramified base.

For a finite unramified extension $\mathbb{Q}_p \subset K$, let $\mathscr{X} = (X, \overline{X})$, $\overline{X} - X = D$ be a smooth pair over $\mathcal{O}_K$ with overconvergent Frobenius ϕ_X. Furthermore, let $\mathcal{M} = (M, \nabla, F, \Phi_M)$ be a filtered overconvergent F- Isocrystal on $\mathscr{X}$ and $\mathbb{U} = (\overline{U}_i)_{i \in I}$ a finite Zariski-open covering of $\overline{X}$. We then have the de Rham complex of coherent $\mathcal{O}_{\bar{X}}$-modules associated to $\mathcal{M}$:

$$DR^\bullet(M) := M \otimes_{\mathcal{O}_{\overline{X}}} \Omega^\bullet(\log D)$$

and we define

$$R^\bullet_{DR}(\mathbb{U}, \mathcal{M})$$

to be the simple complex

$$s\mathcal{C}^\bullet(\mathbb{U}, DR^\bullet(M))$$

associated to the Czech double complex

$$\mathcal{C}^\bullet(\mathbb{U}, DR^\bullet(M)) \ .$$

In degree n, this complex is given by

$$\prod_{t+q=n} M \otimes \Omega^q(\log D)(\overline{U}_{i_0 \cdots i_t}) \ ,$$

where we are taking the product over all subsets of I of cardinality $t+1$ and all nonnegative integers q. In order to define the syntomic cup product, we need to define the de Rham and the rigid cup product on the level of Cech complexes. First, we do this in the de Rham case using the definition given in [**dJ**].

De Rham cup product. Let $\mathcal{N}$ be another overconvergent F-Isocrystal on $\mathscr{X}$ and let N be the associated $\mathcal{O}_{\bar{X}}$-Module with integrable connection with logarithmic singularities around D. We are going to define a map of complexes

$$\mathrm{tot}(R^\bullet_{DR}(\mathbb{U}, \mathcal{M}) \otimes R^\bullet_{DR}(\mathbb{U}, \mathcal{N})) \to R^\bullet_{DR}(\mathbb{U}, \mathcal{M} \otimes \mathcal{N})$$

which induces the cup product

$$H^i_{\mathrm{dR}}(\mathscr{X},\mathcal{M})\times H^j_{\mathrm{dR}}(\mathscr{X},\mathcal{N})\to H^{i+j}_{\mathrm{dR}}(\mathscr{X},\mathcal{M}\otimes\mathcal{N})$$

on cohomology. First consider the map of complexes

$$\mathrm{tot}(s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(M)))\otimes s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(N))\to s\mathcal{C}^\bullet(\mathbb{U},\mathrm{tot}(DR^\bullet(M)\otimes DR^\bullet(N)))$$

given as follows: If α,β are elements of $s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(M)), s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(N))$ in degrees n,m respectively, this map sends $\alpha\otimes\beta$ to

$$(\gamma)_{i_0\cdots i_t}:=\sum_{r=0}^{t}(-1)^{r(m-(t-r))}\alpha_{i_0\cdots i_r}\otimes\beta_{i_r\cdots i_t}$$

where for a complex of sheaves $D^\bullet$ on $\overline{X}$ and an element δ in degree d of $s\mathcal{C}^\bullet(\mathbb{U},D^\bullet)$, we denote by $\delta_{i_0\cdots i_t}$ its component in $D^{d-t}(\overline{U}_{i_0\cdots i_t})$. That this is a map of complexes is checked in [**dJ**]. For an explanation of the sign we also refer to de [**dJ**], who refers to Deninger [**Den95**].
Now consider the map of complexes

$$s\mathcal{C}^\bullet(\mathbb{U},\mathrm{tot}(DR^\bullet(M)\otimes DR^\bullet(N)))\to s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(M\otimes N))$$

which is induced by the map of complexes of $\mathcal{O}_{\bar{X}}$-modules

$$\mathrm{tot}(DR^\bullet(M)\otimes DR^\bullet(N))\to DR^\bullet(M\otimes N)$$

given by

$$m\otimes\omega_i\otimes n\otimes\omega_j\mapsto m\otimes n\otimes\omega_i\wedge\omega_j$$

on sections. We define the de Rham cup product on $\mathscr{X}$ with respect to $\mathcal{M},\mathcal{N}$ and $\mathbb{U}$ to be the composite of the two maps described above and denote it by the symbol $\cup$.
The Filtration F on M induces a Filtration on $DR^\bullet(M)$ which is given in degree q by

$$F^mDR^q(M)=F^{m-q}M\otimes\Omega^q$$

which in turn induces the filtration

$$F^ms\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(M)):=s\mathcal{C}^\bullet(\mathbb{U},F^mDR^\bullet(M))$$

on

$$R^\bullet_{DR}(\mathbb{U},\mathcal{M})=s\mathcal{C}^\bullet(\mathbb{U},DR^\bullet(M))$$

The cup product respects these filtrations in the sense that the image of

$$\mathrm{tot}(F^iR^\bullet_{DR}(\mathbb{U},\mathcal{M})\otimes F^jR^\bullet_{DR}(\mathbb{U},\mathcal{N}))\xrightarrow{\cup}R^\bullet_{DR}(\mathbb{U},\mathcal{M}\otimes\mathcal{N})$$

lands in

$$F^{i+j}R^\bullet_{DR}(\mathbb{U},\mathcal{M}\otimes\mathcal{N})\ .$$

Rigid cup product. Let as above be $\mathbb{U} = (\overline{U}_i)_{i \in I}$ a finite Zariski-open cover and put $U_i := \overline{U}_i \cap X$.Recall that $\overline{\mathcal{X}}, \mathcal{U}_i$ denote the completion of $\overline{X}, U_i$ with respect to the special fiber and $\overline{\mathcal{X}}_K, \mathcal{U}_{i,K}$ denote the associated rigid analytic spaces. Call the obvious inclusion

$$\mathcal{U}_{i_0 \cdots i_n K} \hookrightarrow \overline{\mathcal{X}}_K$$

$j_{i_0 \cdots i_n}$ and let

$$DR^\bullet_{\mathrm{rig}}(M_{\mathrm{rig}}) := M_{\mathrm{rig}} \otimes \Omega^\bullet_{\overline{\mathcal{X}}_K}$$

be the rigid de Rham complex [**BK**, A.1]. The complex

$$R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{M})$$

is then defined to be the simple complex

$$s\mathcal{C}^\bullet(\overline{\mathcal{X}}_K, j^\dagger_\bullet DR^\bullet_{\mathrm{rig}}(M_{\mathrm{rig}}))$$

associated to the Czech double complex

$$\mathcal{C}^\bullet(\overline{\mathcal{X}}_K, j^\dagger_\bullet DR^\bullet_{\mathrm{rig}}(M_{\mathrm{rig}}))\ .$$

The cup product

$$\mathrm{tot}(R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{M}) \otimes R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{N})) \to R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{M} \otimes \mathcal{N})$$

on the rigid complexes is now defined by the identical formulas used in the de Rham case and by abuse of notation also denoted $\cup$.
Let $\Phi_M, \Phi_N, \Phi_{M \otimes N}$ denote the Frobenii belonging to $\mathcal{M}, \mathcal{N}, \mathcal{M} \otimes \mathcal{N}$ respectively. ϕ_X and Φ_M induce a σ-linear endomorphism Φ_1 of $R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{M})$ and likewise we get Φ_2, Φ_3 for $\mathcal{N}, \mathcal{M} \otimes \mathcal{N}$ respectively, where we suppress the dependence of the Φ_i on $\mathbb{U}$. We claim that these Frobenius endomorphisms are compatible with the cup product in the following sense: If α, β denote elements in $R^i_{\mathrm{rig}}(\mathbb{U}, \mathcal{M}), R^j_{\mathrm{rig}}(\mathbb{U}, \mathcal{N})$ respectively, we have

$$\Phi_1 \alpha \cup \Phi_2 \beta = \Phi_3(\alpha \cup \beta)\ .$$

Going through our definition of the cup product as the composite of two maps, we see that checking this equality amounts to showing the equalities

$$(\Phi_1 \otimes \Phi_2) \sum_{r=0}^{t} (-1)^{r(m-(t-r))} \alpha_{i_0 \cdots i_r} \otimes \beta_{i_r \cdots i_t} =$$

$$\sum_{r=0}^{t} (-1)^{r(m-(t-r))} \Phi_1 \alpha_{i_0 \cdots i_r} \otimes \Phi_2 \beta_{i_r \cdots i_t}$$

and

$$\Phi_3(m \otimes n \otimes \omega_i \wedge \omega_j) =$$

$$\Phi_{M \otimes N}(m \otimes n) \otimes \phi_X(\omega_i \wedge \omega_j) =$$

$$\Phi_M(m) \otimes \Phi_N(n) \otimes \phi_X \omega_i \wedge \phi_X \omega_j$$

hold. (m, n are sections in some degree of $DR^\bullet_{\mathrm{rig}}(M_{\mathrm{rig}}), DR^\bullet_{\mathrm{rig}}(N_{\mathrm{rig}})$ respectively.)
But these hold by the definition of $\Phi_1 \otimes \Phi_2$, Φ_3 and $\Phi_{M\otimes N}$.

Cup product in syntomic cohomology. Consider the comparison homomorphism

$$\theta = \theta_{\mathbb{U}} : R^\bullet_{DR}(\mathbb{U}, \mathcal{M}) \to R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{M})$$

between the de Rham and the rigid complex. Given the analogous maps for the overconvergent F-Isocrystals $\mathcal{N}, \mathcal{M} \otimes \mathcal{N}$, we find that these comparison maps are compatible with the above cup products in the obvious sense by the construction of the products. (We used the same formula in the de Rham and the rigid case.) Because we have checked that the Frobenii are compatible with the rigid cup product, the same is true for the composition of the comparison maps with the Frobenii

$$\Phi \circ \theta : R^\bullet_{DR}(\mathbb{U}, \mathcal{A}) \to R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{A}); \quad \mathcal{A} = \mathcal{M}, \mathcal{N}, \mathcal{M} \otimes \mathcal{N};\ \Phi = \Phi_1, \Phi_2, \Phi_3\,.$$

To define a cup product on the complex which computes syntomic cohomology (in the limit over all coverings)

$$R^\bullet_{\mathrm{syn}}(\mathbb{U}, \mathcal{A}) := \mathrm{Cone}(F^0 R^\bullet_{DR}(\mathbb{U}, \mathcal{A}) \xrightarrow{\theta - \Phi\circ\theta} R^\bullet_{\mathrm{rig}}(\mathbb{U}, \mathcal{A}))[-1]\,,$$

we use the following special case of a general homological lemma of Besser [**Bes00**, Lemma 3.2 and formula (6.3)]:

LEMMA II.1.1. *Let $A_i^\bullet, B_i^\bullet$, $i = 1, 2, 3$, be complexes with homomorphisms*

$$\mathrm{tot}(A_1^\bullet \otimes A_2^\bullet) \xrightarrow{\cup} A_3^\bullet\,,$$

$$\mathrm{tot}(B_1^\bullet \otimes B_2^\bullet) \xrightarrow{\cup} B_3^\bullet\,.$$

Furthermore, let

$$f_i, g_i : A_i^\bullet \to B_i^\bullet$$

be homomorphisms that satisfy

$$f_3(a_1 \cup a_2) = f_1 a_1 \cup f_2 a_2 \text{ and } g_3(b_1 \cup b_2) = g_1 b_1 \cup g_2 b_2\,.$$

Set

$$C_i^\bullet := \mathrm{Cone}(A_i^\bullet \xrightarrow{f_i - g_i} B_i^\bullet)[-1]\,.$$

Then the formula

$$(b_1|a_1) \cup_C (b_2|a_2) := (b_1 \cup (\gamma f_2 a_2 + (1-\gamma) g_2 a_2) + (-1)^{\deg a_1}((1-\gamma) f_1 a_1 + \gamma g_1 a_1) \cup b_2 | a_1 \cup a_2)$$

defines a homomorphism of complexes

$$\mathrm{tot}(C_1^\bullet \otimes C_2^\bullet) \xrightarrow{\cup} C_3^\bullet\,,$$

and two such maps are homotopic for different choices of the parameter γ.

We apply the lemma to case

$$A_1^\bullet = F^0R^\bullet_{DR}(\mathbb{U},\mathcal{M}), A_2^\bullet = F^0R^\bullet_{DR}(\mathbb{U},\mathcal{N}), A_3^\bullet = F^0R^\bullet_{DR}(\mathbb{U},\mathcal{M}\otimes\mathcal{N})$$

and

$$B_1^\bullet = R^\bullet_{\mathrm{rig}}(\mathbb{U},\mathcal{M}), B_2^\bullet = R^\bullet_{\mathrm{rig}}(\mathbb{U},\mathcal{N}), B_3^\bullet = R^\bullet_{\mathrm{rig}}(\mathbb{U},\mathcal{M}\otimes\mathcal{N}),$$

$$f_i = \theta, g_i = \Phi_i \circ \theta.$$

and get a map of complexes

$$\cup_{\gamma,\mathbb{U}} : \mathrm{tot}(R^\bullet_{\mathrm{syn}}(\mathbb{U},\mathcal{M}) \otimes R^\bullet_{\mathrm{syn}}(\mathbb{U},\mathcal{N})) \to R^\bullet_{\mathrm{syn}}(\mathbb{U},\mathcal{M}\otimes\mathcal{N})$$

Definition II.1.2. The syntomic cup product

$$\cup : H^i_{\mathrm{syn}}(\mathscr{X},\mathcal{M}) \times H^j_{\mathrm{syn}}(\mathscr{X},\mathcal{N}) \to H^{i+j}_{\mathrm{syn}}(\mathscr{X},\mathcal{M}\otimes\mathcal{N})$$

is defined to be the map on cohomology induced by the maps $\cup_{\gamma,\mathbb{U}}$ in the limit over all coverings $\mathbb{U}$.

From the definition we see that the syntomic cup product is compatible with the de Rham cup product under the natural map

$$H^i_{\mathrm{syn}}(\mathscr{X},\mathcal{M}) \to H^i_{\mathrm{DR}}(\mathscr{X},\mathcal{M}) .$$

Relative cup products. Let

$$u : \mathscr{X} \to \mathscr{Y}$$

be a proper smooth morphism of smooth pairs. Then for the relative cohomology sheaves

$$\mathcal{H}^n_{\mathrm{dR}}(X_K/Y_K) := R^n u_{K*}K, \quad \mathcal{H}^n_{\mathrm{rig}}(\mathscr{X}/\mathscr{Y}) := j_Y^\dagger R^n u^{\mathrm{an}}_{K*}K$$

one has a canonical isomorphism of $j_Y^\dagger \mathcal{O}_{\overline{\mathcal{Y}}_K}$- modules

$$\mathcal{H}^n_{\mathrm{rig}}(\mathscr{X}/\mathscr{Y}) \cong j_Y^\dagger \mathcal{O}_{\overline{\mathcal{Y}}_K} \otimes_{\mathcal{O}_{Y_K}} \mathcal{H}^n_{\mathrm{dR}}(X_K/Y_K) ,$$

see [**BK**, A.2], [**Ger07**, p.8]. Like the absolute algebraic de Rham complex $\Omega^\bullet_{X_K/K}$, the relative de Rham complex $\Omega^\bullet_{X_K/Y_K}$ admits an acyclic Cech resolution. Therefore, the relative de Rham cup prodct can be defined by the same formulas we used above in the absolute case. This is also true for the relative rigid complex $j_X^\dagger \Omega^\bullet_{\overline{\mathcal{X}}_K/\overline{\mathcal{Y}}_K}$ and we obtain relative cup products which are compatible under the above isomorphism. This will be applied to the universal elliptic curve over a modular curve.

II.2. Product structures on modular cohomology groups

Let $M/\mathbb{Z}_p$ the pullback to $\mathbb{Z}_p$ of a modular curve of some representable level prime to p. In this section we construct a product map

$$\sqcup_M : H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l\mathscr{H}^\vee(l+1))$$

$$\to H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)) \ .$$

This map will later be applied to pairs of syntomic Eisenstein classes and related to special values of p-adic L-functions. There is also a product map on the cohomology groups $H^a_{\mathrm{syn}}(X^b, n)(\epsilon)$ defined by imitating a construction of Scholl [**DS91**, 5.7] in motivic cohomology. We show that both maps coincide (up to a sign) under the isomorphisms

$$H^a_{\mathrm{syn}}(X^b, n)(\epsilon) \cong H^a_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^b\mathscr{H}^\vee(n))$$

given by the Leray spectral sequence and the Künneth isomorphism. Our reason for working entirely on the modular curve (and not on self-products of the universal elliptic curve) is that in this setting one has the explicit formulas for the syntomic Eisenstein classes given by [**BK**]. The comparison of the product structures is needed to make sure that products in our sense of syntomic Eisenstein classes still lie in the image of the regulator map. This gives the justification for expecting a relationship between such products and special values of p-adic L-functions.

For shorter notation, set

$$\mathscr{L}_m := \mathrm{Sym}^m\mathscr{H}^\vee \ .$$

DEFINITION II.2.1. Let $k \geq 2, l \geq 0$ be integers. The map

$$\sqcup_M : H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k+l-2}(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_l(l+1)) \to H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k-2}(k+l))$$

is defined to be the composition of the following maps:

(1) The syntomic cup product on $\mathscr{M}$:

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k+l-2}(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_l(l+1)) \to H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k+l-2} \otimes \mathscr{L}_l(k+2l))$$

(2) The map induced by the map of syntomic coefficients

$$\mathscr{L}_{k+l-2} \otimes \mathscr{L}_l \to \mathscr{H}^{\vee \otimes k+l-2} \otimes \mathscr{H}^{\vee \otimes l}$$

given by the canonical section $\mathscr{L}_m \to \mathscr{H}^{\vee \otimes m}$.

(3) The map induced by the map of syntomic coefficients

$$\mathscr{H}^{\vee \otimes k+l-2} \otimes \mathscr{H}^{\vee \otimes l} \xrightarrow{\mathrm{id}^{\otimes k-2} \otimes [\ ,\ ]_{X/M}^{\otimes l}} \mathscr{H}^{\vee k-2}(-l) ,$$

in which

$$[\ ,\ ]_{X/M} : \mathscr{H}^{\vee} \otimes \mathscr{H}^{\vee} \to \mathbb{Q}_p(-1)$$

is the duality pairing on syntomic coefficients and

$$[\ ,\ ]_{X/M}^{\otimes l} : \mathscr{H}^{\vee \otimes l} \otimes \mathscr{H}^{\vee \otimes l} \to \mathbb{Q}_p(-l)$$

is the map

$$(\otimes_i a_i) \bigotimes (\otimes_i b_i) \mapsto \prod_i [a_i, b_i]_{X/M}$$

on sections.

(4) The map induced by the canonical projection

$$\mathscr{H}^{\vee k-2} \to \mathrm{Sym}^{k-2} \mathscr{H}^{\vee} = \mathscr{L}_{k-2} .$$

Syntomic Leray-Künneth-maps. In the following we define Leray-Künneth-isomorphisms

$$H^a_{\mathrm{syn}}(X^b, n)(\epsilon) \cong H^a_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^b \mathscr{H}^{\vee}(n))$$

in particular cases, namely in case syntomic cohomology is either isomorphic to de Rham or rigid cohomology via the long exact sequence. Remember from chapter I that for $a \geq 0$ we defined an isomorphism

$$H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_a(a+1)) \cong H^{a+1}_{\mathrm{syn}}(X^a, a+1)(\epsilon) ,$$

by requiring that this map makes the diagram

$$\begin{array}{ccc} H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_a(a+1)) & \longrightarrow & H^{a+1}_{\mathrm{syn}}(X^a, a+1)(\epsilon_a) \\ \downarrow\cong & & \downarrow\cong \\ H^0_{\mathrm{syn}}(\operatorname{Spec}\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L}_a(a+1)) & \underset{\cong}{\longrightarrow} & H^0_{\mathrm{syn}}(\operatorname{Spec}\mathbb{Z}_p, H^{a+1}_{\mathrm{rig}}(X^a, a+1)(\epsilon_a)) \end{array},$$

commutative, where the vertical maps come from the long exact sequence defining syntomic cohomology and the lower map is induced by the isomorphism given by the Leray spectral sequence and the Künneth map in rigid cohomology. We call this isomorphism λ. Furthermore, for $n \geq a+2$ we define (by abuse of notation)

$$\lambda : H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_a(n)) \xrightarrow{\cong} H^{a+2}_{\mathrm{syn}}(X^a, n)(\epsilon_a),$$

$$\lambda' : H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{H}^{\vee\otimes a}(n)) \to H^{a+2}_{\mathrm{syn}}(X^a, n),$$

by the commutative diagrams

$$\begin{array}{ccc} H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_a(n)) & \xrightarrow{\lambda} & H^{a+2}_{\mathrm{syn}}(X^a, n)(\epsilon_a) \\ \uparrow\cong & & \uparrow\cong \\ H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L}_a(n) & \underset{\cong}{\longrightarrow} & H^{a+1}_{\mathrm{rig}}(X^a, n)(\epsilon_a) \end{array},$$

and

$$\begin{array}{ccc} H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{H}^{\vee\otimes a}(n)) & \xrightarrow{\lambda'} & H^{a+2}_{\mathrm{syn}}(X^a, n) \\ \cong\uparrow & & \cong\uparrow \\ H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{H}^{\vee\otimes a}(n)) & \longrightarrow & H^{a+1}_{\mathrm{rig}}(X^a, n) \end{array}.$$

respectively. In the diagrams, the lower map is given by the Leray-Künneth maps in rigid cohomology and the vertical maps come from the long exact sequence relating syntomic, rigid, and de Rham cohomology. The vertical maps are isomorphisms because of

$$F^0 H^{a+2}_{\mathrm{dR}}(X^a, n) = 0, \; n \geq a+2 .$$

The definitions of the maps λ, λ' also work when we replace the universal elliptic curve $\mathscr{X} \to \mathscr{M}$ with the universal elliptic *ordinary* curve $\mathscr{X}_0 \to \mathscr{M}_0$, the crucial point being that by [**BK**, Lemma 4.2] the natural maps

$$H^1_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{L}_a(a+1)) \to H^0_{\mathrm{syn}}(\operatorname{Spec}\mathbb{Z}_p, H^1_{\mathrm{rig}}(\mathscr{M}_0, \mathscr{L}_a(a+1)) ,$$

$$H^{a+1}_{\mathrm{syn}}(X^a_0, a+1)(\epsilon_a) \to H^0_{\mathrm{syn}}(\mathrm{Spec}\,\mathbb{Z}_p, H^{a+1}_{\mathrm{rig}}(X^a_0, a+1)(\epsilon_a))$$

are isomorphisms.

Following [**DS91**, 5.7], one has the following product map on the constant-coefficient cohomology vector spaces

$$H^a_{\mathrm{syn}}(X^a, a+1)(\epsilon_a).$$

Let $k \geq 2, l \geq 0$ be integers. First, we take the maps

$$H^{k+l-1}_{\mathrm{syn}}(X^{k-2+l}, k+l-1) \to H^{k+l-1}_{\mathrm{syn}}(X^{k-2+2l}, k+l-1)\ ,$$

$$H^{l+1}_{\mathrm{syn}}(X^l, l+1) \to H^{l+1}_{\mathrm{syn}}(X^{k-2+2l}, l+1)\ ,$$

induced by the projection on the first $k-2+l$ and last l factors respectively. We compose with the syntomic cup product on X^{k-2+2l} and get

$$H^{k+l-1}_{\mathrm{syn}}(X^{k-2+l}, k{+}l{-}1)(\epsilon_{k+l-2}) \times H^{l+1}_{\mathrm{syn}}(X^l, l{+}1)(\epsilon_l) \to H^{k+2l}_{\mathrm{syn}}(X^{k-2+2l}, k{+}2l)\ .$$

The syntomic cup product for smooth schemes over $\mathbb{Z}_p$ (and more general bases) *without* auxiliary data (compactification and Frobenius lift) is defined for Tate twist coefficients in [**Bes00**, §3]. In case of a smooth pair $\mathscr{Y} = (Y, \overline{Y})$ with overconvergent Frobenius lift, the isomorphism

$$H^m_{\mathrm{syn}}(Y, n) \cong H^m_{\mathrm{syn}}(\mathscr{Y}, \mathbb{Q}_p(n))$$

is compatible with the cup products, because we use the same formula as Besser [**Bes00**, (6.3)] in order to define the product on the level of complexes. We now compose with the map

$$H^{k+2l}_{\mathrm{syn}}(X^{k-2+2l}, k+2l) \to H^{k+2l}_{\mathrm{syn}}(X^{k-2+l}, k+2l)$$

induced by

$$X^{k-2+l} \xrightarrow{\mathrm{id}_{X^{k-2}} \times \Delta_{X^l}} X^{k-2+2l}\ .$$

where Δ_{X^l} is the diagonal of X^l/M. We then compose with the syntomic

Gysin map

$$p_* : H^{k+2l}_{\mathrm{syn}}(X^{k-2+l}, k+2l) \to H^k_{\mathrm{syn}}(X^{k-2}, k+l)\ ,$$

given by the projection

$$X^{k-2+l} \to X^{k-2}$$

on the first $k-2$ coordinates. The syntomic Gysin map is defined in general in [**CCM10**], but for our purposes it is enough to define it as the map that makes the diagram

$$\begin{array}{ccc} H^{k+2l}_{\mathrm{syn}}(X^{k-2+l}, k+2l) & \longrightarrow & H^k_{\mathrm{syn}}(X^{k-2}, k+l) \\ \uparrow\cong & & \uparrow\cong \\ H^{k-1+2l}_{\mathrm{rig}}(X^{k-2+l}, k+2l) & \xrightarrow[p_{*,\mathrm{rig}}]{} & H^{k-1}_{\mathrm{rig}}(X^{k-2}, k+l) \end{array}$$

commutative where $p_{*,\mathrm{rig}}$ is the rigid Gysin map, see [**CCM10**, Rmk. 5.8]. Finally, we compose with the projection

$$\mathrm{pr}_\epsilon : H^k_{\mathrm{syn}}(X^{k-2}, k+l) \to H^k_{\mathrm{syn}}(X^{k-2}, k+l)(\epsilon_{k-2})$$

onto the ϵ_{k-2}-eigenspace.

Definition II.2.2. The map

$$H^{k+l-1}_{\mathrm{syn}}(X^{k+l-2}, k+l-1)(\epsilon_{k+l-2}) \times H^{l+1}_{\mathrm{syn}}(X^l, l+1)(\epsilon_l) \to H^k_{\mathrm{syn}}(X^{k-2}, k+l)(\epsilon_{k-2})$$

just constructed is called $\sqcup_X$.

We come to the main result of this section.

Theorem II.2.3. *The diagram*

$$\begin{array}{ccc} H^{k+l-1}_{\mathrm{syn}}(X^{k+l-2}, k+l-1)(\epsilon_{k+l-2}) \times H^{l+1}_{\mathrm{syn}}(X^l, l+1)(\epsilon_l) & \xrightarrow{\sqcup_X} & H^k_{\mathrm{syn}}(X^{k-2}, k+l)(\epsilon_{k-2}) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k+l-2}(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_l(l+1)) & \xrightarrow{\sqcup_M} & H^2_{\mathrm{syn}}(\mathscr{M}, \mathscr{L}_{k-2}(k+l))\ , \end{array}$$

in which the vertical maps are given by the Leray-Künneth maps described above, is commutative up to the sign $(-1)^{k+\frac{l(l+1)}{2}}$.

Let as before $M_0 := M^{\mathrm{ord}}, X_0 := X^{\mathrm{ord}}$ be the ordinary locus of M and the universal ordinary elliptic curve respectively. Because the restriction map

$$H^k_{\mathrm{syn}}(X^{k-2}, k+l)(\epsilon_{k-2}) \to H^k_{\mathrm{syn}}(X_0^{k-2}, k+l)(\epsilon_{k-2})$$

is isomorphic to the restriction map

$$H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L}_{k-2}(k+l)) \to H^1_{\mathrm{rig}}(\mathscr{M}_0, \mathscr{L}_{k-2}(k+l)) ,$$

it is injective and it suffices to prove the theorem for (X, M) replaced with (X_0, M_0). The latter pair has the advantage that, by lifting the canonical subgroup, one has compatible (overconvergent) Frobenius lifts on the respective formal schemes. After factoring the diagram in the theorem into three diagrams corresponding to cup product, Gysin map and projection on the ϵ_{k-2}-eigenspace, the theorem follows directly from the following result:

LEMMA II.2.4. *In the following statements, all vertical maps are given by the Leray Künneth-maps defined above.*

a) Let $a, b \geq 0$. The diagram

$$\begin{array}{ccc} H^{a+1}_{\mathrm{syn}}(X_0^a, a+1)(\epsilon_a) \times H^{b+1}_{\mathrm{syn}}(X_0^b, b+1)(\epsilon_b) & \xrightarrow{\cup} & H^{a+b+2}_{\mathrm{syn}}(X_0^{a+b}, a+b+2) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{L}_a(a+1)) \times H^1_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{L}_b(b+1)) & \longrightarrow & H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes a} \otimes \mathscr{H}^{\vee\otimes b}(a+b+l)) \end{array}$$

in which the lower map is given by composition of the cup product on $\mathscr{M}_0$ and the map induced by the inclusion $\mathscr{L}_m \hookrightarrow \mathscr{H}^{\vee\otimes m}$, $m = a, b$, is commutative up to the sign $(-1)^a$.

b) The diagram

$$\begin{array}{ccc} H^{k+2l}_{\mathrm{syn}}(X_0^{k-2+2l}, k+2l) & \xrightarrow{p_* \circ (\mathrm{id}_{X_0^{k-2}} \times \Delta_{X_0^l})^*} & H^k_{\mathrm{syn}}(X_0^{k-2}, k+l) \\ \uparrow & & \uparrow \\ H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2+l} \otimes \mathscr{H}^{\vee\otimes l}(k+2l)) & \xrightarrow{\mathrm{id}^{\otimes k-2} \otimes [\ ,\ ]^{\otimes l}_{X/M}} & H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2}(k+l)) \end{array}$$

is commutative up to the sign $(-1)^{\frac{l(l-1)}{2}}$.

c) The diagram

$$\begin{array}{ccc} H^k_{\mathrm{syn}}(X_0^{k-2}, k+l) & \xrightarrow{\mathrm{pr}_\epsilon} & H^k_{\mathrm{syn}}(X_0^{k-2}, k+l)(\epsilon_{k-2}) \\ \uparrow & & \uparrow \\ H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2}(k+l)) & \longrightarrow & H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{L}^\vee_{k-2}(k+l)) \end{array}$$

in which the lower map is induced by the natural projection $\mathscr{H}^{\vee\otimes k-2} \to \mathrm{Sym}^{k-2}\mathscr{H}^\vee$, is commutative.

PROOF.
a) Via the Künneth map on the coefficients, we regard $H^1_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{L}_a(a+1))$

as the subspace $H^1_{\text{syn}}(\mathscr{M}_0, R^a\pi^{(a)}_*\mathbb{Q}_p(a{+}1))(\epsilon_a)$ of $H^1_{\text{syn}}(\mathscr{M}_0, R^a\pi^{(a)}_*\mathbb{Q}_p(a{+}1))$. (Here $\pi^{(a)}\mathscr{X}^a_0 \to \mathscr{M}_0$ is the a-fold self-product of $\mathscr{X}_0$ over $\mathscr{M}_0$. An element in

$$H^1_{\text{syn}}(\mathscr{M}_0, R^a\pi^{(a)}_*\mathbb{Q}_p(a+1))(\epsilon_a)$$

is then given by a pair (α_1, ω_1), where

$$\omega_1 \in H^0(\overline{M}_{\mathbb{Q}_p}, \big(\pi^{(a)}_*\Omega^a_{\widetilde{X}^a/M}\big) \otimes \Omega^1_{M/\mathbb{Q}_p}(\log C \cup SS))(\epsilon_a)$$

($\widetilde{X}^a$ is the Deligne compactification, SS is the supersingular divisor) and

$$\alpha_1 \in H^0(M^{\text{an}}_{0,\mathbb{Q}_p}, j^\dagger R^a\pi^{(a)}_*\mathbb{Q}_p(a+1))(\epsilon_a)$$

is the unique ([**BK**, Lemma 4.2]) solution of the differential equation

$$\nabla\alpha_1 = (1-\Phi)\omega_1 \ ,$$

in which ∇, Φ are the Gauß-Manin connection and Frobenius. We now describe the image of (α_1, ω_1) under the Leray map λ. Using our definition of λ above and the isomorphism between rigid and de Rham cohomology, we first have to understand the image of ω_1 under the de Rham Leray map

$$H^1_{\text{dR}}(M_{0,\mathbb{Q}_p}, R^a\pi^{(a)}_*\mathbb{Q}_p(a+1))(\epsilon_a) \to H^{a+1}_{\text{dR}}(X^a_{0,\mathbb{Q}_p}, a+1)(\epsilon_a) \ .$$

For this, we assume that $a > 0$, otherwise λ is the indentity map. Because M_0 is étale over (the ordinary part of) the j-line, ω_1 has the form $\beta_1 \otimes \mu_1$ (on $M_{0,\mathbb{Q}_p}$) where

$$\beta_1 \in H^0(M_{0,\mathbb{Q}_p}\ ,\ \pi^{(a)}_*\Omega^a_{X^a_0/M_0}) = H^0(X^a_0, \Omega^a_{X^a_0/M_0}) \ , \quad \mu_1 \in \Omega_{M^1_0/\mathbb{Q}_p} \ .$$

According to the construction of the Leray spectral sequence in de Rham cohomology [**KO68**], the image of $\beta_1 \otimes \mu_1$ in

$$H^{a+1}_{\text{dR}}(X^a_{0,\mathbb{Q}_p}, a+1)(\epsilon_a)$$

is the cohomology class given by the form

$$\mu_1 \wedge \widetilde{\beta}_1 \ ,$$

(see [**KO68**, p.202]) where

$$\widetilde{\beta}_1 \in H^0(X^a_0, \Omega^a_{X^a_0/\mathbb{Q}_p})$$

is a lift of β_1. Because $\Omega^2_{M_0/\mathbb{Q}_p} = 0$ the term $\mu_1 \wedge \widetilde{\beta}_1$ does not depend on the choice of the lift. Now take a finitely indexed open affine cover of $X_{0,\mathbb{Q}_p}$ and let (U_i) be the affine cover of $X^a_{0,\mathbb{Q}_p}$ given by a-fold products of the former cover. Let

$$j_{i_0 \dots i_r} : \mathcal{U}_{i_0 \dots i_r} \hookrightarrow X^{\mathrm{an}}_{0,\mathbb{Q}_p}$$

denote the obvious inclusions of rigid analytic spaces. Consider the complexes

$$(C^\bullet_{X^a_0/\mathbb{Q}_p}, d_X) := sH^0(X^{a,\mathrm{an}}_{0,\mathbb{Q}_p}, j^\dagger_\bullet \Omega^\bullet_{X^a_0/\mathbb{Q}_p}(a+1))$$

$$(C^\bullet_{X^a_0/M_0}, d_{X/M}) := sH^0(X^{a,\mathrm{an}}_{0,\mathbb{Q}_p}, j^\dagger_\bullet \Omega^\bullet_{X^a_0/M_0}(a+1)) \ .$$

Here the prefix s means that we take the associated simple complex of a double complex. The complexes have compatible Frobenii $\Phi_X, \Phi_{X/M}$: There is a diagram of overconvergent Frobenius lifts

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{\phi_{X_0}} & \mathcal{X}_0 \\ \downarrow & & \downarrow \\ \mathcal{M}_0 & \xrightarrow{\phi_{M_0}} & \mathcal{M}_0 \end{array}$$

given by taking the quotient by lifts of the canonical subgroup [**Col95**, p.336]. ϕ_{X_0} induces endomorphisms of the rigid de Rham complexes, and multiplying these by p^{-a-1} gives the desired Frobenii. The cohmology of the complexes is the absolute

$$H^\bullet_{\mathrm{rig}}(\mathscr{X}^a_0, a+1)$$

and the relative rigid cohomology

$$H^\bullet_{\mathrm{rig}}(\mathscr{X}^a_0/\mathscr{M}_0\,,\, a+1) := H^0(M^{\mathrm{an}}_{0,\mathbb{Q}_p}, j^\dagger R^\bullet \pi^{(a)}_* \mathbb{Q}_p(a+1))$$

respectively. The algebraic differential form $\lambda\omega_1 = \mu_1 \wedge \widetilde{\beta}_1$ defines a rigid analytic form on X^{an}_0, and therefore a class in $C^{a+1}_{X^a_0/\mathbb{Q}_p}$. We claim that there exists an element $\widetilde{\alpha}_1$ of $C^a_{X^a_0/\mathbb{Q}_p}$ which satisfies

(1) $\widetilde{\alpha}_1$ lifts $\alpha_1 \in C^a_{X^a_0/M_0}$
(2) We have

$$d_X\, \widetilde{\alpha}_1 = (1 - \Phi_{X_0})\, \lambda\omega_1 \ .$$

Indeed, if

$$\varphi : C^a_{X^a_0/M_0} \to C^a_{X^a_0/\mathbb{Q}_p}$$

denotes the section of the cannonical projection

$$C^a_{X^a_0/\mathbb{Q}_p} \to C^a_{X^a_0/M_0}$$

constructed in [**KO68**, p.207], we set

$$\widetilde{\alpha}_1 := \varphi(\alpha_1) \ .$$

(Katz-Oda deal with the algebraic de Rham case but the same definition works in the rigid setting.) By Lemma 5 of [**KO68**] (and the construction of the Gauß-Manin connection ∇ on p.208-210) we have the identiy

$$d_X \circ \varphi \ \alpha_1 = \lambda(\nabla\alpha_1) + \varphi \circ d_{X/M} \ \alpha_1 \ ,$$

but α_1 being a $d_{X/M}$-cocycle, the last term is zero (beware that the notation of Katz-Oda differs from ours, for example their d_X denotes the exterior and not the total Cech differential; also the map λ appearing on p. 209 is not our λ). We compute:

$$d_X \circ \varphi \ \alpha_1 = \lambda(\nabla\alpha_1) = \lambda(1 - \Phi_{X/M})\omega_1 = (1 - \Phi_X)\, \lambda\omega_1$$

where the last equality follows because Frobenius compatibility implies

$$\begin{aligned} \lambda\Phi_{X/M}(\omega_1) &= \lambda\Phi_{X/M}(\beta_1 \otimes \mu_1) = \lambda\big(\Phi_{X/M}(\beta_1) \otimes \Phi_M(\mu_1)\big) \\ &= \Phi_M(\mu_1) \wedge \widetilde{\Phi_{X/M}(\beta_1)} = \Phi_M(\mu_1) \wedge \Phi_X(\widetilde{\beta}_1) = \Phi_X(\lambda\omega_1) \ . \end{aligned}$$

Because φ is a section, $\widetilde{\alpha}_1 = \varphi(\alpha_1)$ is a lift of α_1. By [**BK**, Lemma 4.2], any class

$$(\alpha, \omega) \in H^{a+1}_{\mathrm{syn}}(\mathscr{X}^a_0, a+1)(\epsilon_a)$$

is uniquely determined by its de Rham part ω. We conclude that the pair

$$(\widetilde{\alpha}_1, \lambda\omega_1) = (\varphi\alpha_1, \mu_1 \wedge \widetilde{\beta}_1)$$

is the image of (α_1, ω_1) under the Leray map.
We now consider the cup product. According to the previous section, we have products

$$\begin{aligned} \mathrm{tot}^\bullet(C^\bullet_{X^{a+b}_0/\mathbb{Q}_p} \otimes C^\bullet_{X^{a+b}_0/\mathbb{Q}_p}) &\xrightarrow{\cup} C^\bullet_{X^{a+b}_0/\mathbb{Q}_p} \\ \mathrm{tot}^\bullet(C^\bullet_{X^{a+b}_0/M_0} \otimes C^\bullet_{X^{a+b}_0/M_0}) &\xrightarrow{\cup} C^\bullet_{X^{a+b}_0/M_0} \ , \end{aligned}$$

given by the formula

$$(A \cup B)_{i_0 \dots i_t} := \sum_{r=0}^{t} (-1)^{r(\deg B - (t-r))} A_{i_0 \cdots i_r} \wedge B_{i_r \cdots i_t}$$

on Cech cocycles. These products induce the cup product and the relative cup product on cohomology respectively. Via the projection on the first a (resp. last b) coordinates, we view $C^\bullet_{X_0^a/\mathbb{Q}_p}$ (resp. $C^\bullet_{X_0^b/\mathbb{Q}_p}$) as subcomplex of $C^\bullet_{X_0^{a+b}/\mathbb{Q}_p}$. We now take classes

$$\begin{aligned}(\alpha_1, \omega_1) &= (\alpha_1, \beta_1 \otimes \mu_1) \in H^1_{\mathrm{syn}}(\mathscr{M}_0, R^a \pi_*^{(a)} \mathbb{Q}_p(a+1))(\epsilon_a) \\ (\alpha_2, \omega_2) &= (\alpha_2, \beta_2 \otimes \mu_2) \in H^1_{\mathrm{syn}}(\mathscr{M}_0, R^b \pi_*^{(b)} \mathbb{Q}_p(b+1))(\epsilon_b),\end{aligned}$$

and consider the diagram

$$\begin{array}{ccc} H^{a+1}_{\mathrm{syn}}(X_0^a, a+1)(\epsilon_a) \times H^{b+1}_{\mathrm{syn}}(X_0^b, b+1)(\epsilon_b) & \xrightarrow{\cup} & H^{a+b+2}_{\mathrm{syn}}(X_0^{a+b}, a+b+2) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{syn}}(\mathscr{M}_0, R^a \pi_*^{(a)} \mathbb{Q}_p(a+1))(\epsilon_a) \times H^1_{\mathrm{syn}}(\mathscr{M}_0, R^b \pi_*^{(b)} \mathbb{Q}_p(b+1))(\epsilon_b) & \longrightarrow & H^2_{\mathrm{syn}}(\mathscr{M}_0, R^{a+b} \pi_*^{(a+b)} \mathbb{Q}_p(a+b+2)) \end{array}$$

in which the lower map is given by composition of the cup product on $\mathscr{M}_0$ and the relative cup product on the higher direct images. It is easily checked that claim a) is equivalent to showing that this diagram commutes up to the sign $(-1)^a$. We specialize the formula for the syntomic cup product (Lemma II.1.1) by setting $\gamma = 1$. Going first horizontal and then vertical in the diagram, we find:

$$\begin{aligned}(\alpha_1, \omega_1), (\alpha_2, \omega_2) &\mapsto \alpha_1 \otimes \beta_2 \otimes \mu_2 - (\Phi_{X/M} \beta_1) \otimes \alpha_2 \otimes (\Phi_M \mu_1) \\ &\mapsto [\alpha_1 \cup \beta_2] \otimes \mu_2 - [(\Phi_{X/M} \beta_1) \cup \alpha_2] \otimes (\Phi_M \mu_1) \\ &\mapsto \mu_2 \wedge (\widetilde{\alpha_1 \cup \beta_2}) - (\Phi_M \mu_1) \wedge (\widetilde{\Phi_{X/M} \beta_1}) \cup \alpha_2 \ .\end{aligned}$$

Going first vertical and then horizontal, we have:

$$\begin{aligned}(\alpha_1, \omega_1), (\alpha_2, \omega_2) &\mapsto (\widetilde{\alpha}_1, \mu_1 \wedge \widetilde{\beta}_1), (\widetilde{\alpha}_2, \mu_2 \wedge \widetilde{\beta}_2) \\ &\mapsto \widetilde{\alpha}_1 \cup (\mu_2 \wedge \widetilde{\beta}_2) + (-1)^{a+1} \Phi_X(\mu_1 \wedge \widetilde{\beta}_1) \cup \widetilde{\alpha}_2 \ .\end{aligned}$$

Let us note the following:

- For a global one form μ, one has $\mu \wedge \gamma = \mu \cup \gamma$ and $\mu \cup \gamma = (-1)^{\deg \gamma} \gamma \cup \mu$ (this identity is on the level of complexes) . This is easily checked using the above formula for $\cup$.

- $\cup$ is associative on the level of complexes. (see [**dJ**].)

- For

$$\mu \in \Omega^1_{M_0^{\mathrm{an}}/\mathbb{Q}_p},\ \alpha \in H^{a+1}_{\mathrm{rig}}(\mathscr{X}_0^a/\mathscr{M}_0\,,\, a+1),\ \beta \in H^{b+1}_{\mathrm{rig}}(\mathscr{X}_0^b/\mathscr{M}_0\,,\, b+1)\ ,$$

one has

$$\mu \wedge (\widetilde{\alpha \cup \beta}) = \mu \wedge (\widetilde{\alpha} \cup \widetilde{\beta})$$

because both $(\widetilde{\alpha \cup \beta})$ and $\widetilde{\alpha} \cup \widetilde{\beta}$ are lifts of $\alpha \cup \beta$.

With these remarks, we compute

$$\widetilde{\alpha}_1 \cup (\mu_2 \wedge \widetilde{\beta}_2) = \widetilde{\alpha}_1 \cup \mu_2 \cup \widetilde{\beta}_2$$

$$= (-1)^a \mu_2 \wedge (\widetilde{\alpha}_1 \cup \widetilde{\beta}_2) = (-1)^a \mu_2 \wedge (\widetilde{\alpha \cup \beta})\ .$$

This shows that the first summands match up to the sign $(-1)^a$. As for the second summands, we have

$$\Phi_X(\mu_1 \wedge \widetilde{\beta}_1) \cup \widetilde{\alpha}_2 = (\Phi_M \mu_1 \wedge \Phi_X \widetilde{\beta}_1) \cup \widetilde{\alpha}_2$$

$$= \Phi_M \mu_1 \wedge (\Phi_X \widetilde{\beta}_1 \cup \widetilde{\alpha}_2) = \Phi_M \mu_1 \wedge (\widetilde{\Phi_{X/M} \beta_1 \cup} \alpha_2),$$

because both $\Phi_X \widetilde{\beta}_1 \cup \widetilde{\alpha}_2$ and $\widetilde{\Phi_{X/M} \beta_1 \cup} \alpha_2$ lift $\Phi_{X/M} \beta_1 \cup \alpha_2$. This proves a).

b) Consider the diagram in question

$$\begin{array}{ccc}
H^{k+2l}_{\mathrm{syn}}(X_0^{k-2+2l}, k+2l) & \xrightarrow{p_* \circ (\mathrm{id}_{X_0^{k-2}} \times \Delta_{X_0^l})^*} & H^k_{\mathrm{syn}}(X_0^{k-2}, k+l) \\
\uparrow & & \uparrow \\
H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee \otimes k-2+l} \otimes \mathscr{H}^{\vee \otimes l}(k+2l)) & \xrightarrow{\mathrm{id}^{\otimes k-2} \otimes [\ ,\ ]^{\otimes l}_{X/M}} & H^2_{\mathrm{syn}}(\mathscr{M}_0, \mathscr{H}^{\vee \otimes k-2}(k+l))
\end{array}$$

Using the long exact sequence relating syntomic, de Rham and rigid cohomology and the fact that by definition the syntomic Gysin map is compatible with the rigid Gysin map, we see that the diagram is isomorphic to

$$\begin{array}{ccc} H^{k-1+2l}_{\mathrm{rig}}(X_0^{k-2+2l}, k+2l) & \xrightarrow{p_{*,\mathrm{rig}}\circ(\mathrm{id}_{X_0^{k-2}}\times\Delta_{X_0^l})^*} & H^{k-1}_{\mathrm{rig}}(X_0^{k-2}, k+l) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{rig}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2+l}\otimes\mathscr{H}^{\vee\otimes l}(k+2l)) & \xrightarrow{\mathrm{id}^{\otimes k-2}\otimes[\ ,\ ]^{\otimes l}_{X/M}} & H^1_{\mathrm{rig}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2}(k+l)) \end{array} .$$

Because the comparison isomorphism between rigid and de Rham cohomology is compatible with Gysin maps, this diagram is isomorphic to

$$\begin{array}{ccc} H^{k-1+2l}_{\mathrm{dR}}(X_0^{k-2+2l}, k+2l) & \xrightarrow{p_{*,\mathrm{dR}}\circ(\mathrm{id}_{X_0^{k-2}}\times\Delta_{X_0^l})^*} & H^{k-1}_{\mathrm{dR}}(X_0^{k-2}, k+l) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{dR}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2+l}\otimes\mathscr{H}^{\vee\otimes l}(k+2l)) & \xrightarrow{\mathrm{id}^{\otimes k-2}\otimes[\ ,\ ]^{\otimes l}_{X/M}} & H^1_{\mathrm{dR}}(\mathscr{M}_0, \mathscr{H}^{\vee\otimes k-2}(k+l)) \end{array} .$$

This diagram is already defined with $\mathbb{Q}$-coeffiecients and it is enough to prove the claim for the respective diagram of $\mathbb{Q}$-vector spaces. After tensoring with $\mathbb{C}$ we may realize the Gysin map on $\mathcal{C}^\infty$ differential forms as integration over the fibers

$$\frac{1}{(2\pi i)^l}\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^l} .$$

The relative de Rham pairing

$$\mathscr{H}^\vee_{\mathrm{dR},\mathbb{C}}\otimes\mathscr{H}^\vee_{\mathrm{dR},\mathbb{C}}\to\mathbb{C}$$

is given by

$$(\omega_1,\omega_2)\mapsto\frac{1}{2\pi i}\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})}\omega_1\wedge\omega_2 .$$

The claim now follows from the fact that for $\mathcal{C}^\infty$ one forms $\omega_1,\ldots,\omega_l,\omega'_1,\ldots,\omega'_l$, one has

$$\omega_1\wedge\ldots\omega_l\wedge\omega'_1\wedge\ldots\omega'_l=(-1)^{\frac{l(l-1)}{2}}\,\omega_1\wedge\omega'_1\wedge\ldots\wedge\omega_l\wedge\omega'_l .$$

c) This is true because the Leray-Künneth map

$$H^1_{\mathrm{rig}}(\mathscr{M}_0,\mathscr{H}^{\vee\otimes k-2}(k+l))\to H^{k-1}_{\mathrm{rig}}(X_0^{k-2},k+l)$$

is equivariant with respect to the S_{k-2}-action. □

As noted before, the construction of $\sqcup_X$, which was the composition of cup-product, Gysin map and projection on the $\epsilon_{k.2}$-eigenspace, was just a repetition of what is done for motivic cohomology by Scholl in [**DS91**]. We can therefore ask if the regulator map is compatible with $\sqcup_X$. Compatibility with the syntomic cup product is proved by Besser [**Bes00**, Prop. 7.7] and compatibility with the syntomic Gysin map follows from the compatibility of the regulator with the rigid (or, by comparison, the de Rham) Gysin map. The regulator is compatible with $\mathrm{pr}_{\epsilon_{k-2}}$ because it is equivariant with respect ot the S_{k-2}-action. We conclude that $\sqcup_X$ is compatible with the regulator map and get as a corollary of the theorem:

COROLLARY II.2.5. *Let $\mathscr{M} = \mathscr{M}(\Gamma(N))$. Under the identificaiton*

$$H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)) \cong H^k_{\mathrm{syn}}(X^{k-2}, k+l)(\epsilon_{k-2}) ,$$

the element

$$\mathrm{Eis}^{k+l}_{\mathrm{syn}}(\varphi_1) \sqcup_M \mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi_2) \in H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l))$$

is in the image of the regulator map.

The remaining part of this chapter will deal with the explicit computation of this product. Let us explain what we exactly mean by this. We have seen that there is a product map

$$\sqcup_M : \; H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{l}\mathscr{H}^\vee(l+1))$$

$$\to H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)) \; .$$

By the long exact sequence, the latter vector space is canonically isomorphic to

$$H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)) = H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee)(k+l).$$

This space has a natural restriction map r^*_{ord} to

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)).$$

It is the image of

$$(\mathrm{Eis}^{k+l}_{\mathrm{syn}}(\varphi_1), \mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi_2))$$

under the above maps in the space

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)).$$

that we want to describe.

DEFINITION II.2.6.

$$\mathrm{Eis}^{k+l}_{\mathrm{syn}}(\varphi_1) \cup_l \mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi_2) :=$$

$$r^*_{\mathrm{ord}}\Big[\mathrm{Eis}^{k+l}_{\mathrm{syn}}(\varphi_1) \sqcup_M \mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi_2)\Big] \in H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)) \ .$$

The purpose of the index l in $\cup_l$ is to remind us that this is not the cup product, but rather the composition of the cup product with a map which "integrates" over l copies of $\mathscr{H}^\vee$.

Using the space

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee) \ ,$$

we can reformulate our problem in terms of overconvergent p-adic modular forms:

As we will review in II.5, Coleman has shown [**Col95**] that there is an isomorphism

$$M^\dagger_k/\theta^{k-1}M^\dagger_{2-k} \cong H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee) \ ,$$

where $M^\dagger_l$ denotes the space of overconvergent p-adic modular forms of weight l and θ is the operator $q\frac{d}{dq}$ on q-expansions. We can therefore ask the following

QUESTION II.2.7. By which elements of $M^\dagger_k$ can we represent the cohomology class

$$\mathrm{Eis}^{k+l}_{\mathrm{syn}}(\varphi_1) \cup_l \mathrm{Eis}^{l+2}_{\mathrm{syn}}(\varphi_2) \cdot T^{-k-l} \ ?$$

The question will be answered in II.7.

II.3. The product of two Eisenstein classes

Recall the renormalization

$$\mathcal{E}^m_{\mathrm{syn}}(\varphi) := \frac{N^m}{2}\,\mathrm{Eis}^m_{\mathrm{syn}}(\varphi),\ m \geq 2 \ .$$

According to chapter I, the syntomic cohomology class $\mathcal{E}^{k+l}_{\mathrm{syn}}(\varphi_1)$ is represented by a pair

$$(\alpha_1, \eta_1)\ .$$

α_1 is the overconvergent rigid analytic section of $\mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1)$ on $\mathcal{M}_{\mathbb{Q}_p}$ whose pullback $\widetilde{\alpha_1}$ to the space of trivialized elliptic curves $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$ is given by the formula

$$\widetilde{\alpha}_1 = \sum_{n=0}^{k+l-2} \frac{1}{(k+l-2-n)!} E^{(p)}_{k+l-1-n,-1-n,\varphi_1} u^n \omega^{k+l-2-n} \cdot T^{k+l-1}.$$

η_1 is the de Rham section of $\mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1) \otimes \Omega^1_M$ whose pullback to $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$ is given by a single algebraic Eisenstein series, namely by the formula

$$\widetilde{\eta}_1 = \frac{1}{(k+l-2)!} E_{k+l,0,\varphi_1} \omega^{(k+l-2)} \otimes \delta \cdot T^{k+l-1}$$

where δ denotes the 1- form on $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$ dual to the Gauss-Manin connection. For the sake of completeness, we also record that (α_2, η_2) is given in the same way by the pair of sections

$$\widetilde{\alpha}_2 = \sum_{n=0}^{l} \frac{1}{(l-n)!} E^{(p)}_{l+1-n,-1-n,\varphi_2} u^n \omega^{l-n} T^{l+1}\ ,$$

$$\widetilde{\eta}_2 = \frac{1}{l!} E_{l+2,0,\varphi_2} \omega^l \otimes \delta \cdot T^{l+1}\ .$$

Going through the definition of the syntomic cup product and specializing the parameter γ to $\gamma = 1$, we see that the image of

$$[(\alpha_1, \eta_1), (\alpha_2, \eta_2)]$$

under

$$\begin{aligned} &H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1)) \times H^1_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \\ &\quad \xrightarrow{\cup} H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1) \otimes \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \end{aligned}$$

is given by

$$\begin{aligned} &(\alpha_1 \cup \eta_2 + (-1)^1 \Phi\eta_1 \cup \alpha_2), 0) = \\ &(\alpha_1\eta_2 - (\Phi\eta_1)\alpha_2, 0). \end{aligned}$$

Here, by abuse of notation, we do not distinguish between the de Rham section η_i and its associated rigid analytic section. The section

$$\widetilde{\alpha_1\eta_2}$$

in

$$\Gamma(\widetilde{\mathcal{M}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1) \otimes \mathrm{Sym}^l \mathscr{H}^\vee(l+1))$$

is given by

$$\sum_{n=0}^{k+l-2} \frac{1}{(k+l-2-n)!l!} E^{(p)}_{k+l-1-n,-1-n,\varphi_1} E_{l+2,0,\varphi_2} u^n \omega^{k+l-2-n} T^{k+l-1} \otimes \omega^l T^{l+1} \otimes \delta\ .$$

and the section

$$\widetilde{(\Phi\eta_1)\alpha_2}$$

is given by a similar formula.
In order to compute $\cup_l$, we have to compute the image of $(\alpha_1\eta_2 - (\Phi\eta_1)\alpha_2, 0)$ under

$$p_l : H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1) \otimes \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \to H^2_{\mathrm{syn}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l))$$

where p_l is the map induced by

$$id^{\otimes k-2} \otimes [\cdot,\cdot]^{\otimes l} : \mathscr{H}^{\vee k+l-2} \otimes \mathscr{H}^{\vee l} \to \mathscr{H}^{\vee k-2}(-l)$$

and the fact that Sym^m is a direct summand of $(\cdot)^{\otimes m}$. Equivalently, we have to compute the image of $\alpha_1\eta_2 - \alpha_2(\Phi\eta_1)$ under the analogous map in rigid cohomology

$$p_l : H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^{k+l-2}\mathscr{H}^\vee(k+l-1) \otimes \mathrm{Sym}^l \mathscr{H}^\vee(l+1)) \to H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l))\ .$$

In order to lighten notation, let us still write $\mathscr{H}^\vee$ for its pullback $\widetilde{\mathscr{H}^\vee}$ to $\widetilde{\mathcal{M}}_{\mathbb{Q}_p}$.

Now consider the section

$$u^n \omega^{k+l-2-n}$$

of $\mathrm{Sym}^{k+l-2}\mathscr{H}^\vee$. The splitting

$$\mathrm{Sym}^{k+l-2}\mathscr{H}^\vee \hookrightarrow \mathscr{H}^{\vee \otimes k+l-2}$$

maps it to

$$\kappa = \frac{n!(k+l-2-n)!}{(k+l-2)!} \sum_{\substack{S \subset \{1,\ldots,k+l-2\} \\ \#S = n}} v_1 \otimes \cdots \otimes v_{k+l-2}, \quad v_i = \begin{cases} u & i \in S \\ \omega & i \notin S \end{cases}$$

In the same way, the section ω^l is mapped to $\omega^{\otimes l}$. The image of

$$\kappa \otimes \omega^{\otimes l}$$

under

$$id^{\otimes k-2} \otimes [\cdot,\cdot]^{\otimes l} : \mathscr{H}^{\vee k+l-2} \otimes \mathscr{H}^{\vee l} \to \mathscr{H}^{\vee k-2}(-l)$$

is given by

$$\frac{n!(k+l-2-n)!}{(k+l-2)!}\sum_{\substack{S\subset\{1,\dots,k+l-2\}\\ \#S=n}} v_1\otimes\cdots v_{k-2}\cdot[v_{k-1}\,,\,\omega]\cdots[v_{k+l-2}\,,\,\omega]\,.$$

Projecting on

$$\mathrm{Sym}^{k-2}\mathscr{H}^\vee(-l),$$

we get the section

$$\frac{\binom{k-2}{n-l}}{\binom{k+l-2}{n}}u^{n-l}\omega^{k-2-(n-l)}\cdot(-T^{-1})^l=(-1)^l\frac{\binom{n}{l}}{\binom{k+l-2}{l}}u^{n-l}\omega^{k-2-(n-l)}\cdot T^l$$

With this formula, we get that $p_l(\widetilde{\alpha_1\eta_2})$ equals

$$\frac{(-1)^l}{\binom{k+l-2}{l}\,l!}\sum_{n=0}^{k+l-2}\frac{\binom{n}{l}}{(k+l-2-n)!}E^{(p)}_{k+l-1-n,-1-n,\varphi_1}E_{l+2,0,\varphi_2}u^{n-l}\omega^{k-2-(n-l)}\otimes\delta\cdot T^{-l}\cdot T^{k+2l}$$

and therefore we have

Proposition II.3.1.

$$p_l(\widetilde{\alpha_1\eta_2})$$

$$=\frac{(-1)^l}{\binom{k+l-2}{l}\,l!}\sum_{n=0}^{k-2}\frac{\binom{n+l}{l}}{(k-2-n)!}E^{(p)}_{k-1-n,-l-1-n,\varphi_1}E_{l+2,0,\varphi_2}u^n\omega^{k-2-n}\otimes\delta\cdot T^{k+l}$$

We now turn to the computation of $p_l((\Phi\eta_1)\alpha_2)$. Recall that the Frobenius Φ on a section is given by the composition of the Frobenius on the space of trivialized elliptic curves ϕ with the Frobenius on the coefficients which we call Φ' in the following. If Frob denotes the endomorphism of $\widetilde{\mathcal{M}}$ which induces $q\mapsto q^p$ on q-expansions [**BK**, 4.3], we have $\phi=(\mathrm{Frob}\otimes\sigma)$ in case $M=M(\Gamma(N))$ and $\phi=\mathrm{Frob}$ if $M=M(\Gamma_1(N))$. By definition of ω,u,T we have the identities

$$\Phi'\omega=p\omega,\ \Phi'u=u,\ \Phi'T=\frac{1}{p}T,$$

and because, on the Tate curve, $\delta=\frac{dq}{q}$, we also have

$$\phi\delta=p\delta.$$

As a result, we compute that

$$\Phi(E_{k+l,0,\varphi_1}\,\omega^{k+l-2}\otimes\delta\cdot T^{k+l-1})$$

$$=\phi^*(E_{k+l,0,\varphi_1})\,\omega^{k+l-2}\otimes\delta\cdot T^{k+l-1}\ ,$$

thus

$$\widetilde{(\Phi\eta_1)}\alpha_2$$

$$= \sum_{n=0}^{l} \frac{1}{(l-n)!(k+l-2)!}\phi^*(E_{k+l,0,\varphi_1})E^{(p)}_{l+1-n,-1-n,\varphi_2}\omega^{k+l-2}\otimes\delta\cdot T^{k+l-1}\otimes u^n\omega^{l-n}\cdot T^{l+1}$$

As before, we need to know the image κ' of a section $u^n\omega^{l-n}$ under the splitting

$$\mathrm{Sym}^l \mathscr{H}^\vee \hookrightarrow \mathscr{H}^{\vee \otimes l}$$

and then apply

$$id^{\otimes k-2} \otimes [\cdot,\cdot]^{\otimes l} : \mathscr{H}^{\vee k+l-2} \otimes \mathscr{H}^{\vee l} \to \mathscr{H}^{\vee k-2}(-l)$$

to the section $\omega^{\otimes k+l-2} \otimes \kappa'$. But because of $[\omega\,,\,\omega] = 0$, this can only be nonzero if $n = l$. In this case, $\omega^{\otimes k+l-2} \otimes \kappa'$ is simply mapped to $\omega^{k-2}\, T^{-l}$. As a result, we get:

Proposition II.3.2.

$$p_l(\widetilde{(\Phi\eta_1)}\alpha_2) = \frac{1}{(k+l-2)!}\phi^*(E_{k+l,0,\varphi_1})E^{(p)}_{1,-l-1,\varphi_2}\omega^{k-2} \otimes \delta \cdot T^{k+l}\ .$$

Let us remark that later on, we will only be interested in the projection of this element onto a certain eigenspace, and we will also show that this projection of $p_l((\Phi\eta_1)\alpha_2)$ differs from the one of $p_l(\eta_1\alpha_2)$ only by a constant factor.

II.4. Rigid cohomology and overconvergent modular forms

We will start with generalities about the rigid cohomology of open curves. Let

$$\mathscr{U} = (U, C)$$

be a smooth pair over $\mathrm{Spec}\,\mathbb{Z}_p$ that has relative dimension one and let

$$C - U = Z_1 \cup Z_2$$

be two disjoint divisors with Z_2 nonempty. Put $W := U \cup Z_1$ and denote by

$$j_1 : \mathcal{U}_{\mathbb{Q}_p} \hookrightarrow U^{\mathrm{an}}_{\mathbb{Q}_p},\ j_2 : \mathcal{W}_{\mathbb{Q}_p} \hookrightarrow W^{\mathrm{an}}_{\mathbb{Q}_p}$$

the standard inclusions. Furthermore let $\mathscr{L} = (L, \nabla, F, \Phi)$ be an admissible filtered overconvergent F-Isocrystal on $\mathscr{U}$. Assume that $U_{\mathbb{Q}_p}, C_{\mathbb{Q}_p}$ and (L, ∇) are already defined over a number field. (This condition ensures that the comparison map between de Rham and rigid cohomology with coefficients is an isomorphism.) Then it follows from the main result of [**BC94**] that both

$$H^i\Gamma(U^{\mathrm{an}}_{\mathbb{Q}_p}, j_1^\dagger\mathscr{L} \otimes \Omega^\bullet_{(U_{\mathbb{Q}_p})^{an}})$$

and

$$H^i\Gamma(W^{\mathrm{an}}_{\mathbb{Q}_p}, j_2^\dagger \mathscr{L} \otimes \Omega^\bullet_{(W_{\mathbb{Q}_p})^{an}}(\log Z_1))$$

are isomorphic to

$$H^1_{\mathrm{rig}}(\mathscr{U}, \mathscr{L}) \ .$$

(Note that because Z_2 is nonempty, U and W are affine.) We can express this in words by saying that rigid cohomology can be computed by either demanding logarithmic singularities or overconvergence at each divisor.
Applying the above to the situation

$$C = \overline{M},\ U = M^{\mathrm{ord}}\,,$$

(Remember that $p \geq 5$ and that M^{ord} is the nonzero locus of the rational algebraic Eisenstein series E_{p-1})

$$Z_1 = Cusps \text{ (we abbreviate it by } C \text{ in the following)},$$

we get that there is a diagram

$$\frac{H^0_{\mathrm{rig}}(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \mathscr{L} \otimes \Omega^1(\log C))}{\nabla H^0_{\mathrm{rig}}(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \mathscr{L})} \xrightarrow{\cong} \frac{H^0_{\mathrm{rig}}(M^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_1^\dagger \mathscr{L} \otimes \Omega^1)}{\nabla H^0_{\mathrm{rig}}(M^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_1^\dagger \mathscr{L})} \xrightarrow{\cong} H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L})$$

The value of this is that we can represent H^1_{rig} as a quotient of a smaller space. Note that in case the coefficients $\mathscr{L}$ are $\mathrm{Sym}^{k-2}\mathscr{H}^\vee(k+l)$, the sections $p_l(\alpha_1\eta_2), p_l(\eta_1\alpha_2)$, which from our definition lie in

$$H^0_{\mathrm{rig}}(M^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_1^\dagger \mathscr{L} \otimes \Omega^1)\ ,$$

even lie in

$$H^0_{\mathrm{rig}}(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \mathscr{L} \otimes \Omega^1(\log C))\ .$$

This can be seen from the explicit formulas for these sections, in which all terms are defined at the cusp, except for the term δ which has a pole of order one.

DEFINITION II.4.1. Let $k \in \mathbb{Z}$.

$$M_k^\dagger := H^0_{\mathrm{rig}}(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \omega^{k-2} \otimes \Omega^1(\log C))$$

is called the space of overconvergent p-adic modular forms of weight k.

The inclusion

$$M_k^\dagger \hookrightarrow H^0_{\mathrm{rig}}(M^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_1^\dagger \omega^{k-2} \otimes \Omega^1)$$

can be thought of the inclusion of overconvergent p-adic modular forms of weight k, which are holomorphic at the cups, into the space of such forms that are only meromorphic at the cusps. We often write f for an element $f\,\omega^{k-2} \otimes \delta$ of $M_k^\dagger$.

II.5. A theorem of Coleman

From now on in this chapter, M denotes the pullback to $\mathbb{Z}_p$ of the modular curve of level $\Gamma_1(N)$, where $(N,p)=1$ and $N\geq 4$. Also all Eisenstein classes $\mathrm{Eis}^m_{\mathrm{syn}}(\varphi)$ appearing are understood to have level $\Gamma_1(N)$. This restriction is unessential but has a couple of advantages for our exposition: We can directly refer to results which are only treated for level $\Gamma_1(N)$ and not for $\Gamma(N)$ [**Col95**], [**CGJ95**]. Furthermore, we do not have to distinguish between the derivations $\theta, N\theta$ (cf. [**Kat76**]) and between the different Frobenii $\mathrm{Frob}, \mathrm{Frob}\otimes\sigma$. For our applications to p-adic L-values, the level $\Gamma_1(N)$ case is enough. However one can check that the results we refer to in order to prove the product formula for the Eisenstein classes stay true for level $\Gamma(N)$.
In this section we briefly recall a result of Coleman, which says that the cohomology vector space

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee)$$

can be written as a quotient of certain spaces of p-adic modular forms.

The operators F and U. Using the Frobenius on $X_0/M_0 = X^{\mathrm{ord}}/M^{\mathrm{ord}}$ given by taking the quotient by the canonical subgroup, one can define an Endomorphism

$$F: M_k^\dagger \to M_k^\dagger$$

which on q-expansions is given by

$$(Ff)(q) = f(q^p)$$

([**Col95**, §3]). Therefore F is the restriction of the Frobenius Operator on the space of Katz modular forms that we have called ϕ^* in section II.3.
There is also an analogue

$$U: M_k^\dagger \to M_k^\dagger$$

of the classical U_p-Operator which is defined essentially as the trace of F (see [**Col95**, §3]) and is normalized such that it maps

$$\sum_{n=0}^{\infty} a_n q^n$$

to

$$\sum_{n=0}^{\infty} a_{np}\, q^n\ .$$

One checks $UF = \mathrm{id}$.

The operator θ. Recall that on the space $V := V(\Gamma_1(N), \mathbb{Q}_p)$ of Katz modular forms, there is a derivation

$$\theta : V \to V$$

which shifts weight by two and on q-expansions is given by

$$\sum_{n=0}^{\infty} a_n q^n \mapsto \sum_{n=0}^{\infty} n a_n q^n \ ,$$

see for example [**Kat76**, Lemma 5.8.1]. This is in contrast to the case of classical modular forms, in which the derivation $q \cdot d/dq$ destroys modularity. However, θ does not preserve overconvergence: In [**CGJ95**], Coleman, Gouvea and Jochnowitz were able to show that if f is a overconvergent modular form of nonzero weight k, then $\theta(f)$ will *not* be overconvergent. It is therefore surprising that we have the

Proposition II.5.1. *(Coleman),* [**Col95**, Prop. 4.3] *Let $k \geq 2$ be an integer and let $f \in M_{2-k}^{\dagger}$. Then $\theta^{k-1}(f)$ is again overconvergent, i.e. an element of $M_k^{\dagger}$.*

Here is an example which illustrates this phenomenon. For $k \geq 3$ consider the Eisenstein series

$$^{(p)}E_{2-k,0}^{(p)} = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \sum_{dd'=n} d^{1-k}\, q^n$$

It is of weight $2-k$ and overconvergent because we can write

$$^{(p)}E_{2-k,0}^{(p)} = (1-F)\ ^{(p)}E_{2-k,0}$$

where

$$^{(p)}E_{2-k,0} = \frac{\zeta_p^*(k-1)}{2} + \sum_{n=1}^{\infty} \sum_{\substack{dd'=n \\ p \nmid d}} d^{1-k}\, q^n$$

is overconvergent by [**Col97**, B1] (the essential input is Hida's ordinary projection and the fact that $^{(p)}E_{2-k,0}$ is fixed by U). Note that F preserves overconvergence by [**Gou88**]. We find

$$\theta^{k-1}\ ^{(p)}E_{2-k,0}^{(p)} = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \sum_{dd'=n} d^{1-k} n^{k-1}\, q^n = \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \sum_{dd'=n} (d')^{k-1}\, q^n = {}^{(p)}E_{k,0}^{(p)}$$

and this last Eisenstein series is algebraic, therefore overconvergent.

We can now state:

Theorem II.5.2. *(Coleman),* [**Col95**, Thm. 5.4] *The map*

$$M_k^{\dagger} \hookrightarrow H^0_{\mathrm{rig}}(\overline{M}_{\mathbb{Q}_p}^{\mathrm{ord},an}, j_2^{\dagger}\mathrm{Sym}^{k-2}\mathscr{H}^{\vee} \otimes \Omega^1(\log C))$$

given by the inclusion

$$\omega^{k-2} \otimes \Omega^1(\log C) \hookrightarrow \mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C)$$

of sheaves induces an isomorphism

$$M_k^\dagger/\theta^{k-1}M_{2-k}^\dagger \xrightarrow{\cong} H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee)$$

and the endomorphism F on left hand side corresponds to $\frac{1}{p^{k-1}}\Phi$ on the right hand side, where Φ denotes the cohomological rigid Frobenius.

II.6. Rigid cohomology and non-overconvergent forms

In order to have more flexibility in representing elements of

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee)\ ,$$

we now want to explain how we can associate cohomology classes to p-adic modular forms that are "mildly non-overconvergent".
We start by explicating why θ does not preserve overconvergence in more detail. Consider the q-expansion

$$1 - 24\sum_{n=1}^{\infty}(\sum_{d|n} d)\, q^n\ ,$$

the so-called Ramanujan series. A classical holomorphic modular form with this q-expansion does not exist, but there is such a Katz modular form Q (also called E_2 or P) [**Kat73**, A 2.4] of weight 2. On p-adic test objects $(E/B, \varphi) =$ (ordinary elliptic curve, trivialization) it can be defined as the "direction of the unit root subspace:"

$$Q(E/B,\varphi) := 12\frac{[\eta_0, u]_{\mathrm{rig}}}{[\omega, u]_{\mathrm{rig}}} = [\eta_0, u]_{\mathrm{rig}} \cdot T$$

where $\omega = \varphi^*(dT/(1{+}T))$, $\eta_0 = xdx/dy$, u is the unique Frobenius invariant section of $H^1_{\mathrm{rig}}(E)$ such that $[\omega, u]_{\mathrm{rig}} = T^{-1}$ and $x = x(\omega), y = y(\omega)$ are the usual meromorphic sections determined by $\omega = dx/y$ and the equation

$$y^2 = 4x^3 - g_2x - g_3.$$

(The index 0 of η_0 is just for distinguishing it from the de Rham section η of the syntomic Eisenstein class (α, η)). From this definition we deduce the equality

$$u = -\frac{Q}{12}\omega + \eta_0$$

of sections of $\mathscr{H}^\vee$ on the space of trivialized elliptic curves. Q is related to the θ-operator in the following way: For $k \geq 0$, there exists a derivation

$$\delta_k : M_k^\dagger \to M_{k+2}^\dagger,$$

which increases weight by two and is given by the formula

$$\delta_k f = \theta f - k\frac{Q}{12}f,$$

see [**Col95**, Proof of Prop. 4.3]. The proof of the non-overconvergence of θ now proceeds in three steps. First, one proves that Q is not overconvergent by cohomological considerations and a result of Serre on congruences of classical modular forms [**CGJ95**, Theorem 1, Lemma 4]. Coleman et al. define the ring

Definition II.6.1.

$$M^\dagger := \bigoplus_{k\in\mathbb{Z}} M_k^\dagger \ .$$

They show that if Q would satisfy a monic polynomial equation over $M^\dagger$, it would itself be overconvergent, [**CGJ95**, Cor.7]. Finally one applies the above formula for δ_k to conclude that an overconvergent form of nonzero weight is mapped to a non-overconvergent form by θ. Therefore the non-overconvergence of θ is essentially equivalent to that of Q.

An interesting consequence of the non-overconvergence of Q is that a section

$$\beta \in \Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C)),\ k \geq 3$$

of the form

$$\beta = fu^n\omega^{k-2-n} \otimes \delta \ , 1 \leq n \leq k-2 \ ,$$

will *not* be overconvergent if $f \neq 0$, even if f is overconvergent, because of

$$u = -\frac{Q}{12}\omega + \eta_0.$$

This poses a problem as, by definition of rigid cohomology, only overconvergent sections represent cohomology classes. For the term $p_l(\alpha_1\eta_2)$ which appeared in our computation of the product of two Eisenstein classes, this means that although we know that the sum

$$\frac{(-1)^l}{\binom{k+l-2}{l}l!}\sum_{n=0}^{k-2}\frac{\binom{n+l}{l}}{(k-2-n)!}E^{(p)}_{k-1-n,-l-1-n,\varphi_1}E_{l+2,0,\varphi_2}u^n\omega^{k-2-n}\otimes\delta\cdot T^{k+l}$$

is overconvergent, the single terms will be not in general, and as a result we cannot consider this to be a sum of cohomology classes. Even in the Term

$$E^{(p)}_{k-1,-l-1,\varphi_1}E_{l+2,0,\varphi_2}\omega^{k-2}\otimes\delta\cdot T^{k+l}$$

in which ω^{k-2} does not destroy overconvergence, the coefficient function

$$E^{(p)}_{k-1,-l-1,\varphi_1}E_{l+2,0,\varphi_2}$$

is in general not overconvergent, as the following argument shows: Let $k \geq 3$, take for example

$$\varphi_1 := \widehat{P_1^{-1}\varphi_0}$$

where

$$\varphi_0(m,n) = \begin{cases} 1 & N \mid m \\ 0 & \text{else} \end{cases}$$

(This gives us an Eisenstein series $E^{(p)}_{k-1,-l-1,\varphi_1}$ of level $\Gamma_1(N)$ with trivial Dirichlet character, the essential point being that it is a U-eigenform with unit eigenvalue.)
and put, for $m \leq -2$,

$$E^{(p)}_{1,m} := \frac{\zeta_p^*(-m)}{2} + \sum_{n\geq 1} \sum_{\substack{dd'=n \\ p\nmid d'}} (d')^m \, q^n \; .$$

which obviously is equal to

$${}^{(p)}E_{m+1,0} := \frac{\zeta_p^*(-m)}{2} + \sum_{n\geq 1} \sum_{\substack{dd'=n \\ p\nmid d}} d^m \, q^n \; .$$

The constant term is the Kubota-Leopoldt p-adic L-function which is not important here. Then one checks that on q-expansions

$$E^{(p)}_{k-1,-l-1,\varphi_1}(q) = \theta^{k-2} E^{(p)}_{1,-k-l+1}(q^N) = \theta^{k-2} \; {}^{(p)}E_{-k-l+2,0}(q^N)$$

and, as stated in the prevoius section, the Eisenstein series

$${}^{(p)}E_{-k-l+2,0}$$

is overconvergent by Hida's theory of the ordinary projection. In the case $k = 3$ this already shows that

$$\theta^{k-2} \; {}^{(p)}E_{-k-l+2,0}$$

cannot be overconvergent and in fact this is true for all $k \geq 3$ by a lemma which we will state in a minute.
How do we cope with this problem? We will show that sections of the form

$$f\omega^{k-2} \otimes \delta$$

do represent cohomology classes, as long as their degree as polynomials in Q over the ring of overconvergent modular forms is small enough (is $\leq k-2$ to be precise). This will then be applied to our Eisenstein sections. We will also deal with sections of the form

$$fu^n\omega^{k-2-n} \otimes \delta \; , n \geq 1$$

in a similar way.
Consider the ring $M^\dagger[Q]$ of polynomials in Q over $M^\dagger$, which we regad as a subring of the ring of Katz modular forms. Denote by

$$M^\dagger[Q]_l$$

the subspace of $M^\dagger[Q]$ of forms of weight l and by

$$M[Q]_l^{\leq d}$$

the subspace of $M^\dagger[Q]_l$ which consists of forms which are polynomials in Q of degree less or equal to d. We write $\deg(f)$ for the degree in Q, and $w(f)$ for the weight of a form. The next lemma tells us how the degree changes when we apply θ.

LEMMA II.6.2. a) *Let $f \in M^\dagger[Q]_k$. If $\deg(f) \neq k$, then*

$$\deg(\theta f) = \deg(f) + 1 \ .$$

If $\deg(f) = k$, then

$$\deg(\theta f) \leq \deg(f) \ .$$

b) *Let $f \in M^\dagger[Q]_k$ be of degree d and let*

$$f = Q^d f_0 + \text{ (terms of degree } \leq d-1) \ , f_0 \in M^\dagger_{k-2d} \ .$$

Then

$$\theta f = \text{(constant)} \cdot Q^{d+1} f_0 + \text{ (terms of degree } \leq d) \ ,$$

where the constant may or may not be zero depending on $d = k$ or $d \neq k$.

PROOF. Part a) is [**CGJ95**, Prop.11]. Both a) and b) follow from the formula (see[**CGJ95**, p.33])

$$\theta(GQ^d) = \delta_{k-2d}(G)Q^d - \frac{d}{12} G E_4 Q + \frac{k-d}{12} G Q^{d+1},$$

where $G \in M^\dagger_{k-2d}$ and E_4 is the algebraic Eisenstein series of weight 4 and level $\mathrm{Sl}_2(\mathbb{Z})$. It can be verified by direct calculation. □

We apply part a) of the lemma to

$$\theta^{i} \, {}^{(p)}E_{-k-l+2,0} \ , i \geq 0$$

and find that

$$\deg(\theta^{k-2} \, {}^{(p)}E_{-k-l+2,0}) = k-2 \ ,$$

therefore it is not overconvergent for $k \geq 3$, but at least we have

$$(\theta^{k-2} \, {}^{(p)}E_{-k-l+2,0}) \cdot E_{l+2,0,\varphi_2} \in M^\dagger[Q]_k^{\leq k-2} \ .$$

We can in fact argue the same way for any choice of the coeffient function φ_1 : One extends coefficients, decomposes φ_1 according to Dirichlet characters and writes

$$E^{(p)}_{k-1,-l-1,\varphi_1}$$

as a linear combination of p-adic Eisenstein series

$$E^{(p)}_{k-1,-l-1,\varphi_1}(q) = \theta^{k-2} \sum_i \lambda_i E^{(p)}_{1,-k-l+1}(\chi_i)(q^{n_i}) \quad , (p, n_i) = 1 \ ,$$

where we have

$$U(E^{(p)}_{1,-k-l+1}(\chi_i)) = \chi_i(p) \, E^{(p)}_{1,-k-l+1}(\chi_i)$$

and the χ_i are Dirichlet characters of conductor prime to p. Therefore $E^{(p)}_{k-1,-l-1,\varphi_1}$ can be written as $\theta^{k-2} h$ with h overconvergent and we get

$$E^{(p)}_{k-1,-l-1,\varphi_1} \cdot E_{l+2,0,\varphi_2} \in M^\dagger[Q]_k^{\leq k-2}$$

for *all* φ_1.
Another consequence of the previous lemma is that θ maps the space

$$M^\dagger[Q]_{k-2}^{\le k-2}$$

into

$$M^\dagger[Q]_{k}^{\le k-2} .$$

The next proposition is the key in relating non-overconvergent forms to rigid cohomology.

PROPOSITION II.6.3. *The inclusion*

$$M_k^\dagger \subset M^\dagger[Q]_k^{\le k-2}$$

induces an isomorphism

$$M_k^\dagger/\theta^{k-1}M_{2-k}^\dagger \xrightarrow{\cong} M^\dagger[Q]_k^{\le k-2}/\theta M^\dagger[Q]_{k-2}^{\le k-2}$$

PROOF. *Injectivity:* Let $f \in M_k^\dagger$,

$$f = \theta g$$

where $g \in M^\dagger[Q]_{k-2}^{\le k-2}$. According to the previous lemma this implies that $\deg(g) = k-2$. Let

$$g = g_0 Q^{k-2} + (\text{terms of lower degree}) \ , 0 \ne g_0 \in M_{2-k}^\dagger \ .$$

Denote by $w(j)$ the weight of a modular form j. There is an $0 \le i \le k-2$ such that

$$\deg(\theta^i g_0) = w(\theta^i g_0) \ ,$$

because otherwise, by the previous lemma, $\deg(\theta^i g_0)$ would strictly increase for $0 \le i \le k-1$ contradicting $\deg(\theta^{k-1} g_0) = 0$. Let i be the smallest such number. By this minimality, we have $\deg(\theta^i g_0) = i$ and

$$\deg(\theta^i g_0) = w(\theta^i g_0)$$

becomes

$$i = 2i + 2 - k.$$

Hence $i = k-2$ and we conclude that

$$\deg(\theta^{k-2} g_0) = k-2 \ ,$$

By the previous lemma, $\theta^{k-2} g_0$ has as degree $k-2$ term $g_0 Q^{k-2}$ times a constant, which we now know to be nonzero. Therefore there exists a nonzero constant c such that

$$\deg(g - c\theta^{k-2} g_0) < k-2 \ .$$

If $h = g - c\theta^{k-2} g_0$ is nonzero, then its weight $k-2$ is different from its degree and thus $\deg(\theta h) > \deg(h) \ge 0$. But this contradicts

$$\theta h = \theta g - c\theta^{k-1} g_0 = f - c\theta^{k-1} g_0, \ \text{which is overconvergent.}$$

Therefore $h = 0$, i.e.

$$\theta^{k-2} c g_0 = g \ \text{,and finally } \theta^{k-1} c g_0 = f.$$

Surjectivity: Let $f \in M^\dagger[Q]_k^{\leq k-2}$ be of degree d with highest term $Q^d f_0$ and put $w = w(f_0)$. We show that modulo θ it is in the image of $M_k^\dagger$ by induction on d. If $d = 0$ there is nothing to prove. If $d > 0$ we have by definition of d that

$$2d + w = k \ .$$

Assume that

$$w(Q^{d-1} f_0) = \deg(Q^{d-1} f_0) \ .$$

This implies

$$2d - 2 + w = d - 1 \ ,$$

hence we get that $d = k - 1$, a contradiction to $d \leq k - 2$. Therefore, $Q^{d-1} f_0$ has different weight and degree and thus, by the previous lemma, the highest term of $\theta(Q^{d-1} f_0)$ is equal to $Q^d f_0$ times a nonzero constant. We get a constant c such that

$$f - c\theta(Q^{d-1} f_0)$$

has degree $\leq d - 1$ and are done by induction hypothesis. □

We get as a corollary that via the isomorhisms

$$H^1_{\mathrm{rig}}(\mathscr{M}, \mathrm{Sym}^{k-2} \mathscr{H}^\vee) \cong M_k^\dagger / \theta^{k-1} M_{2-k}^\dagger \cong M^\dagger[Q]_k^{\leq k-2} / \theta M^\dagger[Q]_{k-2}^{\leq k-2}$$

every element of $M^\dagger[Q]_k^{\leq k-2}$ represents a cohomology class. This allows us to associate a cohomology class to the "highest term" appearing in $p_l(\alpha_1 \eta_2)$, which is

$${}^{(p)}E_{k-1,-l-1,\varphi_1} \cdot E_{l+2,0,\varphi_2} \omega^{k-2} \otimes \delta = (\theta^{k-2} \ {}^{(p)}E_{-k-l+2,0,\varphi_1}) \cdot E_{l+2,0,\varphi_2} \omega^{k-2} \otimes \delta \ .$$

Next we want to deal with sections of the form

$$g u^n \omega^{k-2-n} \otimes \delta \ , n \geq 1$$

and in fact we will show that their contribution is zero on the level of cohomology. For this purpose we will formally define a vector space which contains sections of this kind and will then show that this space is isomorphic to

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2} \mathscr{H}^\vee) \ .$$

As cocycles, we take the vector Z space spanned by the set

$$\left\{ f_n \, u^n \omega^{k-2-n} \otimes \delta : w(f_n) = k - 2n, 0 \leq n \leq k-2 \ , \deg(f_n) \leq \begin{cases} k-2 & n = 0 \\ k-1-n & n \geq 1 \end{cases} \right\}$$

inside the space of only *convergent* sections

$$\Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2} \mathscr{H}^\vee \otimes \Omega^1(\log C)) \ .$$

Similarly, Y is defined to be the space spanned by the set

$$\left\{ f_n \, u^n \omega^{k-2-n} : w(f_n) = k - 2 - 2n \ , \deg(f_n) \leq k - 2 - n \ , 0 \leq n \leq k-2 \right\}$$

inside

$$\Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2} \mathscr{H}^\vee) \ .$$

The Gauß-Manin connection ∇ induces a map

$$\nabla : Y \longrightarrow Z$$

given by

$$\nabla f\, u^n \omega^{k-2-n} = \theta f\, u^n \omega^{k-2-n} \otimes \delta + (k-2-n) f\, u^{n+1} \omega^{k-2-n-1} \otimes \delta \ ,$$

and using Lemma 4.1, this is easily seen to be well-defined. Finally we define T to be the subspace

$$\left\{ (\theta g)\, \omega^{k-2} \otimes \delta : g \in M^\dagger[Q]^{\leq k-2}_{k-2} \right\}$$

of

$$\Gamma(\overline{\mathcal{M}}^{\text{ord}}_{\mathbb{Q}_p}, \operatorname{Sym}^{k-2} \mathscr{H}^\vee \otimes \Omega^1(\log C))$$

and put

$$B := (\nabla Y) + T \ .$$

The "virtual cohomology group" H_v is then defined as the quotient

$$Z/B \ .$$

PROPOSITION II.6.4. *The map*

$$M^\dagger[Q]^{\leq k-2}_k \longrightarrow Z \ ,$$

$$f \mapsto f\, \omega^{k-2} \otimes \delta$$

induces an isomorphism

$$M^\dagger[Q]^{\leq k-2}_k / \theta M^\dagger[Q]^{\leq k-2}_{k-2} \xrightarrow{\cong} H_v \ .$$

PROOF. By defintition of H_v, $\theta M^\dagger[Q]^{\leq k-2}_{k-2}$ maps to zero. Next we show

Injectivity: For this, let $f \in M^\dagger[Q]^{\leq k-2}_k$ and assume that

$$f\, \omega^{k-2} \otimes \delta = (\theta g)\, \omega^{k-2} \otimes \delta + \nabla(\sum_{n=0}^{k-2} g_n\, u^n \omega^{k-2-n}) \ ,$$

where

$$g \in M^\dagger[Q]^{\leq k-2}_{k-2} \ , \ (\sum_{n=0}^{k-2} g_n\, u^n \omega^{k-2-n}) \in Y \ .$$

We have

$$\nabla(\sum_{n=0}^{k-2} g_n\, u^n \omega^{k-2-n}) = (\theta g_0)\, \omega^{k-2} \otimes \delta + \sum_{n=1}^{k-2} h_n\, u^n \omega^{k-2-n} \otimes \delta$$

for certain h_n. Linear independence implies that

$$f = \theta g + \theta g_0$$

and we are done.
Surjectivity: Let

$$z = \sum_{n=0}^{k-2} f_n\, u^n \omega^{k-2-n} \otimes \delta \in Z$$

be given. Because for $1 \leq n \leq k-2$ we have

$$\nabla(f_n\, u^{n-1}\omega^{k-2-n+1})$$

$$= \theta f_n\, u^{n-1}\omega^{k-2-n+1} \otimes \delta + (k-2-n+1) f_n\, u^n \omega^{k-2-n} \otimes \delta\ ,$$

we see that (note that $k-2-n+1 \neq 0$) there exists an $h \in M^\dagger[Q]^{\leq k-2}_{k-2}$ such that

$$z = f_0\, \omega^{k-2} \otimes \delta + (\theta h)\, \omega^{k-2} \otimes \delta \text{ (modulo } \nabla Y) \ .$$

Thus,

$$z = f_0\, \omega^{k-2} \otimes \delta \text{ (modulo } \nabla Y + T)\ ,$$

which is clearly in the image. □

Consider the natural maps

$$i : \Gamma(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C)) \to \Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C))$$

$$i' : \Gamma(\overline{M}^{\mathrm{ord},an}_{\mathbb{Q}_p}, j_2^\dagger \mathrm{Sym}^{k-2}\mathscr{H}^\vee) \to \Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee)$$

which restrict overconvergent sections to convergent sections.

Lemma II.6.5. *i has image in Z and i' has image in Y.*

Proof. An overconvergent section s of

$$\mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C))$$

over $\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}$ can be written as

$$s = \sum_{n=0}^{k-2} f_n\, \eta_0^n \omega^{k-2-n} \otimes \delta\ ,\ f_n \in M^\dagger_{k-2n}\ .$$

After restricting to a convergent section, we can use the section u of $\mathscr{H}^\vee$ and apply the identity

$$\eta_0 = u + \frac{Q}{12}\omega$$

in order to expand

$$f_n\, \eta_0^n \omega^{k-2-n} \otimes \delta = f_n\, (u + \frac{Q}{12}\omega)^n \omega^{k-2-n} \otimes \delta$$

in terms of the basis elements $u^n\omega^{k-2-n} \otimes \delta$. It is a sum of terms of the form

$$(\text{const.}) \cdot Q^m f_n \, u^{n-m}\omega^{k-2-(n-m)} \otimes \delta \ , \ 0 \le m \le n \ .$$

Because of

$$\deg(Q^m f_n) \le m \le m + (k-2-n) = k-2-(n-m) \ ,$$

s lies in Z. The identical argument shows the claim for i'. □

As the Gauß-Manin connection commutes with restricting sections, i induces a map

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee) \longrightarrow H_v$$

which we also denote by i. We finally come to the main result of this section:

THEOREM II.6.6.

$$i : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee) \longrightarrow H_v$$

is an isomorphism.

PROOF. Consider the diagram

$$\begin{array}{ccc} M^\dagger_k/\theta^{k-1}M^\dagger_{2-k} & \longrightarrow & M^\dagger[Q]^{\le k-2}_k/\theta M^\dagger[Q]^{\le k-2}_{k-2} \\ \downarrow & & \downarrow \\ H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee) & \xrightarrow{\ i\ } & H_v \end{array}$$

in which the vertical maps are given by

$$f \mapsto [f\,\omega^{k-2} \otimes \delta]$$

and the top map is induced by inclusion. It is clearly commutative and i is an isomorphism because the other three maps are. □

II.7. A formula for the product of two Eisenstein classes

With notation as in the previous section, let us check that the element

$$p_l(\alpha_1\eta_2) \cdot T^{-k-l}$$

of

$$\Gamma(\overline{\mathcal{M}}^{\mathrm{ord}}_{\mathbb{Q}_p}, \mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \Omega^1(\log C))$$

actually lies in Z : We have already seen that

$$E^{(p)}_{k-1,-l-1,\varphi_1} = \theta^{k-2}h \ , \ h \text{ overconvergent}$$

and the identical argument (decomposition according to Dirichlet characters and applying Hida's ordinary projection) shows that

$$E^{(p)}_{k-1-n,-l-1-n,\varphi_1} = \theta^{k-2-n}h \ , \ h \text{ overconvergent}$$

for $1 \le n \le k-2$. It follows that

$$\deg(E^{(p)}_{k-1-n,-l-1-n,\varphi_1}) \le k-2-n < k-1-n$$

which is what we wanted to show. This observation and the theorem give us the justification to regard

$$p_l(\alpha_1\eta_2)\cdot T^{-k-l} =$$

$$\frac{(-1)^l}{\binom{k+l-2}{l}\,l!}\sum_{n=0}^{k-2}\frac{\binom{n+l}{l}}{(k-2-n)!}E^{(p)}_{k-1-n,-l-1-n,\varphi_1}E_{l+2,0,\varphi_2}u^n\omega^{k-2-n}\otimes\delta$$

as a sum of rigid cohomology classes by considering each term as an element in H_v. It is clear from the formula for ∇ that all terms of the form

$$f\,u^n\omega^{k-2-n}\otimes\delta\ ,\ n\geq 1\ ,$$

have cohomology class zero (see the surjectivity argument in the proof of Prop. II.6.4.) Therefore we get that (writing $[s]$ for the cohomology class of a section s):

$$[p_l(\alpha_1\eta_2)]\cdot T^{-k-l} = \Big[\frac{(-1)^l}{\binom{k+l-2}{l}\,(k-2)!\,l!}\;E^{(p)}_{k-1,-l-1,\varphi_1}E_{l+2,0,\varphi_2}\,\omega^{k-2}\otimes\delta\Big]\ .$$

We bring this together with our earlier computation of the element $p_l((\Phi\eta_1)\alpha_2)$ in Prop. II.3.2 and conclude:

Proposition II.7.1. *Under the identification*

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathrm{Sym}^{k-2}\mathscr{H}^\vee)\cong M^\dagger[Q]^{\leq k-2}_k/\theta M^\dagger[Q]^{\leq k-2}_{k-2}\ ,$$

the class

$$\mathcal{E}^{k+l}_{\mathrm{syn}}(\varphi_1)\cup_l\mathcal{E}^{l+2}_{\mathrm{syn}}(\varphi_2)\cdot T^{-k-l}$$

is represented by the p-adic modular form

$$\frac{(-1)^l}{(k+l-2)!}\,E^{(p)}_{k-1,-l-1,\varphi_1}E_{l+2,0,\varphi_2}\ -\ \frac{1}{(k+l-2)!}\,F(E_{k+l,0,\varphi_1})E^{(p)}_{1,-l-1,\varphi_2}\ .$$

Note that this does not fully answer our initial question which asked by which *overconvergent* p-adic modular form we could represent the product, because only the second term is overconvergent. However, the proof of surjectivity of the isomorphism

$$M^\dagger_k/\theta^{k-1}M^\dagger_{2-k}\cong M^\dagger[Q]^{\leq k-2}_k/\theta M^\dagger[Q]^{\leq k-2}_{k-2}$$

gives an algorithm for obtaining such an overconvergent modular form in $k-2$ steps. Also note that the issue of non-overconvergence does not arise in the case $k=2$.

CHAPTER III

The rigid realization of modular motives

The purpose of this chapter is to introduce the rigid realization

$$H_{\mathrm{rig}}(M(f))$$

of the Grothendieck motive

$$M(f)$$

associated to a Hecke Eigenform f of level prime to p, and to introduce certain cohomological linear forms that will be used later on. We apologize in advance to the reader that our convention for defining (the rigid realization of) the motive $M(f)$ is *dual* to the common one, for example the one used in [**Sch90**]. This has the disadvantage that the Frobenius endomorphism of $H_{\mathrm{rig}}M(f)$ has characteristic polynomial

$$X^2 - \bar{a}_p X + \bar{\psi}(p)p^{k-1}$$

instead of

$$X^2 - a_p X + \psi(p)p^{k-1} \ .$$

The reasons we still define it this way are twofold: First of all we can use classical Hecke operators instead of their transpose. Secondly, with our definition the differential form $f\omega^{k-2} \otimes \delta$ defines a class in $H_{\mathrm{rig}}M(f)$ and not in $H_{\mathrm{rig}}M(\bar{f})$. Both of this will make it more convenient for us to adapt certain constructions and computations of Panshishkin in the sequel. We remark that our normalization of the Hecke operators *coincides* with the one used in [**DFG04**].

Denote by

$$\pi : X \to M$$

the pullback to $\mathbb{Z}_p$ of the universal elliptic curve with level $\Gamma_1(N)$-structure, $N \geq 4$, $(N,p) = 1$. As has already become clear in the previous sections, the study of p-adic modular forms of weight k is closely related to the study of the cohomology vector space

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathrm{Sym}^{k-2} R^1\pi_*\mathbb{Q}_p(1)) \ .$$

This motivates the following discussion of basic properties of this space such as duality, weight decomposition and Hecke operators. All of this is well-known but we include it for lack of reference.

III.1. Rigid cohomology and Hecke operators

Duality. We begin by introducing cohomology with compact support. We intend no general theory and stick to the case of curves with locally free coefficients in which things are particularly convenient. Let

$$\mathscr{U} = (U, C)$$

be a smooth pair over $\operatorname{Spec}\mathbb{Z}_p$ which has relative dimension one. Let $\mathscr{U}$ have an overconvergent Frobenius ϕ and denote by

$$Z = C - U$$

the complement. The sheaf of ideals $I(Z)$ of Z is a coherent sheaf of $\mathcal{O}_C$-modules. We define

$$\mathscr{I}$$

to be the filtered overconvergent F-Isocrystal on $\mathscr{U}$ given by the 4-tuple

$$(I(Z)\ ,\ d\ (\text{exterior differential})\ , F^m = \begin{cases} I(Z) & m \leq 0 \\ 0 & m > 0 \end{cases}\ ,\ \mathrm{id})\ .$$

Let $\mathscr{N}$ be a filtered overconvergent F-Isocrystal on $\mathscr{U}$ which is locally free which means that this is true for the underlying coherent $\mathcal{O}_C$-module.

Definition III.1.1.

$$H^i_{c,\,\mathrm{rig}}(\mathscr{U}, \mathscr{N})$$

is defined as

$$H^i_{\mathrm{rig}}(\mathscr{U}, \mathscr{N} \otimes \mathscr{I})\ ,$$

and called rigid cohomology of $\mathscr{U}$ with coefficients in $\mathscr{N}$ with compact support.

We define de Rham cohomology with compact support

$$H^i_{c,\,\mathrm{dR}}(\mathscr{U}, \mathscr{N}) = H^i_{c,\,\mathrm{dR}}(U_{\mathbb{Q}_p}, N)$$

in the same way.

If $\mathscr{M}$ is another filtered overconvergent F-Isocrystal which is locally free, the rigid cup product gives a pairing

$$H^i_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \times H^j_{\mathrm{rig}}(\mathscr{U}, \mathscr{N} \otimes \mathscr{I}) \to H^{i+j}_{\mathrm{rig}}(\mathscr{U}, \mathscr{M} \otimes \mathscr{N} \otimes \mathscr{I})\ .$$

Hence by definition we get

$$\cup : H^i_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \times H^j_{c,\,\mathrm{rig}}(\mathscr{U}, \mathscr{N}) \to H^{i+j}_{c,\,\mathrm{rig}}(\mathscr{U}, \mathscr{M} \otimes \mathscr{N})$$

and similarly for the de Rham vector spaces. Now assume that all of $U_{\mathbb{Q}_p}, C_{\mathbb{Q}_p}, (N, \nabla_N), (M, \nabla_M)$ are already defined over a number field. This implies a certain p-adic monodromy condition which ensures that we have comparison isomorphisms betweeen de Rham and rigid cohomology with coefficients, cf. [**BC94**].

THEOREM III.1.2.

a) The pairings

$$\cup : H^i_{\mathrm{dR}}(U_{\mathbb{Q}_p}, M) \times H^{2-i}_{c,\mathrm{dR}}(U_{\mathbb{Q}_p}, M^\vee) \to H^2_{c,\mathrm{dR}}(U, \mathbb{Q}_p(0))$$

$$\cup : H^i_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) \times H^{2-i}_{c,\mathrm{rig}}(\mathscr{U}, \mathscr{M}^\vee) \to H^2_{c,\mathrm{rig}}(\mathscr{U}, \mathbb{Q}_p(0)) \ ,$$

induced by the cup product are perfect and compatible with the comparison isomorphism from de Rham to rigid cohomology.
b) There is an isomorphism ("the trace map")

$$\mathrm{tr} : H^2_{c,\mathrm{rig}}(\mathscr{U}, \mathbb{Q}_p(0)) \xrightarrow{\cong} \mathbb{Q}_p(-1)$$

which is compatible with the Frobenius structure and respects the $\mathbb{Q}$-structure coming from rational de Rham cohomology.

PROOF. Compatiblity of the pairing with the comparison isomorphisms follows from the compatiblity of the usual (=non-compactly supported) cup product with coefficients with the comparison isomorphisms. The pairing is nondegenerate because this is true in the de Rham case.
The rigid trace map is defined as composition of the comparison isomorphism with the the de Rham trace map, see [**Ber97**, Prop. 2.1,2.6] for details. Therefore it respects the de Rham $\mathbb{Q}$-structure. It commutes with Frobenius structure by [**Tsu99**, (6.2)]. □

We now apply this to the case

$$\mathscr{U} = \mathscr{M}^{\mathrm{ord}} = (M^{\mathrm{ord}}, \overline{M}), \quad \text{coefficients} = \mathrm{Sym}^{k-2}\mathscr{H}^\vee \ .$$

Let us first compute the dual of

$$\mathscr{L} := \mathrm{Sym}^{k-2}\mathscr{H}^\vee \ .$$

Recall that there was a perfect pairng

$$\mathscr{H}^\vee \otimes \mathscr{H}^\vee \to \mathbb{Q}_p(-1)$$

of filtered overconvergent F-isocrystals which induces a perfect pairing

$$\mathrm{Sym}^{k-2}\mathscr{H}^\vee \otimes \mathrm{Sym}^{k-2}\mathscr{H}^\vee \to \mathbb{Q}_p(2-k) \ .$$

We conclude that

$$\mathscr{L}^\vee \cong \mathscr{L}(k-2) \ .$$

COROLLARY III.1.3.
The cup product induces a perfect pairing

$$\cup : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}) \times H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}) \to \mathbb{Q}_p(1-k) \ .$$

Parabolic cohomology. Let

$$\widetilde{\mathscr{X}}^{k-2} := (\tilde{X}^{k-2}, \tilde{X}^{k-2}) \ ,$$

where $\tilde{X}^{k-2}$ is the pullback to $\mathbb{Z}_p$ of the desingularization constructed by Deligne and let ϵ be the projector constructed by Scholl. The rigid parabolic cohomology group $\tilde{H}_{\text{rig}}$ is defined as

$$\tilde{H}_{\text{rig}} := H^{k-1}(\widetilde{\mathscr{X}}^{k-2}, \mathbb{Q}_p)(\epsilon) \ .$$

and, following [**BK**, 2.2], there is a short exact sequence

$$0 \to \tilde{H}_{\text{rig}} \to H^1_{\text{rig}}(\mathscr{M}, \mathscr{L}) \to H^0_{\text{rig}}(\text{Isom}, \mathbb{Q}_p(1-k))^{(k-2)} \to 0$$

where the rightmost term is non-canonically isomorphic to

$$H^0_{\text{rig}}(\text{Cusp}, \mathbb{Q}_p(1-k)) \ .$$

We now want to identify $\tilde{H}_{\text{rig}}$ as a direct summand of

$$H^1_{\text{rig}}(\mathscr{M}^{\text{ord}}, \mathscr{L}) \ .$$

Proposition III.1.4. *There is a decomposition*

$$H^1_{\text{rig}}(\mathscr{M}^{\text{ord}}, \mathscr{L}) \cong \tilde{H}_{\text{rig}} \oplus R$$

of Frobenius modules, where R has Frobenius weight 2 if $k = 2$, and is a direct sum of Fobenius modules having weight $2k-2$ and k for $k > 2$.

Proof. Let us first remark that in order to define rigid cohomology, one needs less data than a smooth pair

$$\mathscr{X} = (X, \overline{X})$$

and a filtered overconvergent F-isocrystal

$$\mathscr{N} = (N, \nabla, F, \Phi).$$

In fact, one has rigid cohomology groups

$$H^i_{\text{rig}}(Y, \mathscr{F})$$

for any scheme Y of finite type over $\mathbb{F}_p$ with coeffiecients $\mathscr{F}$ in a suitable category

$$\text{Isoc}^\dagger(Y/\mathbb{Q}_p)$$

which only depends on Y and not on (the existence of) a smooth lift. Furthermore there is a natural isomorphism

$$H^i_{\text{rig}}(\mathscr{X}, \mathscr{N}) \cong H^i_{\text{rig}}(X_{\mathbb{F}_p}, \mathscr{N}_{\text{rig}}) \ .$$

(Here, $\mathscr{N}_{\text{rig}}$ is the object of $\text{Isoc}^\dagger(Y/\mathbb{Q}_p)$ defined by "analytification" of the coherent $\mathcal{O}_{\bar{X}}$-module N.)
In rigid cohomology one has most of the usual cohomological formalism, in particular an excision sequence and Poincaré duality, see [**Ked06a**] and [**Ked06b**]. We apply the excision sequence in the case

$$X = M_{\mathbb{F}_p},\ U = M^{\text{ord}}_{\mathbb{F}_p},\ Z = SS \text{ (the supersingular divisor) }, \mathscr{F} = \mathscr{L}_{\text{rig}};$$

and get the exact sequence

$$H^1_{SS,\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^1_{\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^1_{\mathrm{rig}}(M^{\mathrm{ord}}_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to$$
$$\to H^2_{SS,\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^2_{\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) .$$

The sequence is Frobenius-equivariant by [**Tsu99**, Prop.2.1.1]. The last term is zero by comparison with de Rham cohomology and the fact that $M_{\mathbb{Q}_p}$ is affine. The first term is zero by rigid Poincaré duality [**Ked06a**, Thm. 1.2.3]. Also by Poincaré duality we conclude that

$$H^2_{SS,\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \cong H^0_{\mathrm{rig}}(SS,\mathscr{L}_{\mathrm{rig}}(-1)) .$$

That this isomorphism is Frobenius compatible follows from [**Tsu99**, (6.2)]. We deduce the exact sequence

$$0 \to H^1_{\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^1_{\mathrm{rig}}(M^{\mathrm{ord}}_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^0_{\mathrm{rig}}(SS,\mathscr{L}_{\mathrm{rig}}(-1)) \to 0$$

Recall that we also have

$$0 \to \tilde{H}_{\mathrm{rig}} \to H^1_{\mathrm{rig}}(M_{\mathbb{F}_p},\mathscr{L}_{\mathrm{rig}}) \to H^0_{\mathrm{rig}}(\mathrm{Cusp},\mathbb{Q}_p(1-k)) \to 0 .$$

The claim now follows from the fact that parabolic cohomology has Frobenius weight $k-1$, whereas

$$H^0_{\mathrm{rig}}(\mathrm{Cusp},\mathbb{Q}_p(1-k)) \text{ and } H^0_{\mathrm{rig}}(SS,\mathscr{L}_{\mathrm{rig}}(-1))$$

have weights $2k-2$ and k respectively. □

We conclude by duality that there is a direct sum composition

$$H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) = H_1 \oplus H_2$$

where H_1 has weight $k-1$ and is dual to $\tilde{H}_{\mathrm{rig}}$ and H_2 has weights 0 and $k-2$. On the other hand, because of

$$H^{k-1}_{\mathrm{rig}}(\widetilde{\mathscr{X}}^{k-2},\mathbb{Q}_p)^\vee \cong H^{k-1}_{\mathrm{rig}}(\widetilde{\mathscr{X}}^{k-2},\mathbb{Q}_p)(k-1) ,$$

the injection

$$\tilde{H}_{\mathrm{rig}} \hookrightarrow H^1_{\mathrm{rig}}(\mathscr{M},\mathscr{L})$$

induces a surjection

$$H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \to \tilde{H}_{\mathrm{rig}}$$

and for weight and dimension reasons, this map identifies H_1 with $\tilde{H}_{\mathrm{rig}}$. We get:

Corollary III.1.5. *The cup-product pairing*

$$(\cdot,\cdot) : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \times H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \to \mathbb{Q}_p(1-k)$$

induces a nondegenerate self-duality

$$(\cdot,\cdot) : \tilde{H}_{\mathrm{rig}} \times \tilde{H}_{\mathrm{rig}} \to \mathbb{Q}_p(1-k) .$$

and is compatible with Frobenius, i.e.

$$(\Phi(\cdot),\Phi(\cdot)) = p^{k-1}(\cdot,\cdot) .$$

(a p-adic analogue of the Petersson inner product.)

Hecke Operators. We come to Hecke operators and their behavior under duality. Denote by $\pi : \overline{X} \to \overline{M}$ the pullback to $\mathbb{Z}_p$ of the universal generalized elliptic curve with $\Gamma_1(N)$ level structure. We collect most of what we need in the following theorem.

THEOREM III.1.6. *For a prime l, $l \nmid N$, there exist finite flat correspondences $T_{l,M}$, $\langle l \rangle_M$ on $\overline{M}$ and $T_{l,X}$, $\langle l \rangle_X$ on $\overline{X}$ which are compatible with the structure morphism*

$$\pi : \overline{X} \to \overline{M}$$

and which satisfy the following properties :

1) The correspondences restrict to correspondences on M,M^{ord} and X, X^{ord}.
2) The induced operators $T_l, \langle l \rangle$ on the cohomology vector spaces

$$H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L}),\ H^1_{c,\,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}),\ \tilde{H}_{\mathrm{rig}}$$

respect the decompositions

$$H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L}) = \tilde{H}_{\mathrm{rig}} \oplus R,\ \ H^1_{c,\,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}) = \tilde{H}_{\mathrm{rig}} \oplus R^{\vee}$$

and in particular induce operators on $\tilde{H}_{\mathrm{rig}}$.
3) The operators $T_l, \langle l \rangle$, $l \nmid N$ commute with each other.
4) Let

$$[g]$$

denote the cohomology class associated to a section

$$g\,\omega^{k-2} \otimes \delta$$

in

$$M_k^{\dagger} = H^0((\overline{M}^{\mathrm{ord}}_{\mathbb{Q}_p})^{\mathrm{an}}, j^{\dagger}\omega^{k-2} \otimes \Omega^1(\log C))$$

via the (surjective) map

$$H^0((\overline{M}^{\mathrm{ord}}_{\mathbb{Q}_p})^{\mathrm{an}}, j^{\dagger}\omega^{k-2} \otimes \Omega^1(\log C)) \to H^1_{\mathrm{rig}}(\mathscr{M}, \mathscr{L})\ .$$

Then:
a) for $l \nmid Np$

$$T_l\ [g] = [T_{l,\mathrm{cl}}\, g]$$

and

$$\langle l \rangle\ [g] = [\langle l \rangle_{\mathrm{cl}}\, g],$$

$\forall g \in M_k^{\dagger}$,where $T_{l,\mathrm{cl}}, \langle l \rangle_{\mathrm{cl}}$ are the Hecke Operators on p-adic modular forms which extend the classical Hecke Operators on algebraic modular forms [**Gou88**, chap.2].
b) In case $l = p$, one has

$$T_p\ [g] = [T_{p,\mathrm{cl}}\, g], \qquad \langle p \rangle\ [g] = [\langle p \rangle_{\mathrm{cl}}\, g]$$

for all classical *forms g of level* $\Gamma_1(N)$. *Furthermore, one has the formula (Eichler-Shimura relation)*

$$T_p[g] = [Ug] + \Phi[\langle p\rangle g]$$

for any $g \in M_k^\dagger$.
5) $T_l, \langle l\rangle$ *commute with the endomorphisms* Φ *and* U *on each of*

$$H^1_{\mathrm{rig}}(\mathscr{M},\mathscr{L}),\ H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}),\ \tilde{H}_{\mathrm{rig}}$$

for each l, $l \nmid N$. *(including* p*).*
6)With respect to the pairing

$$(\cdot,\cdot) : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \times H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \to \mathbb{Q}_p(1-k)\ ,$$

we have the formulas

$$(T_l\alpha,\beta) = (\alpha, \langle l\rangle^{-1}T_l\beta)$$
$$(\langle l\rangle\alpha,\beta) = (\alpha,\langle l\rangle^{-1}\beta)\ .$$

Note that

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}},\mathscr{L}) \ = \ H^1_{\mathrm{rig}}(\mathscr{M}(\Gamma_1(N))^{\mathrm{ord}},\mathscr{L})$$

has the remarkable property that the Hecke Operators T_p *and* $U = U_p$ are defined on it, whereas on the classical space

$$H^1_{\mathrm{betti}}(M(\Gamma_1(n))(\mathbb{C})\ ,\ \mathrm{Sym}^{k-2}\mathcal{H}^\vee)$$

only T_p *or* U is defined, depending on $p \nmid n$ or $p \mid n$.

Proof. Over $\mathbb{Q}_p$, define $T_{l,M}, T_{l,X}$ to be the transpose of the correspondences defined in [**Sch90**, §4]. The claimed $\mathbb{Z}_p$-integrality and in fact $\mathbb{Z}$-integrality follows from [**Con07**, Thm. 1.2.2]. If $(\overline{X},\alpha)$ denotes the universal generalilzed elliptic curve with level $\Gamma_1(N)$-structure α, the diamond correspondence (or morphism) $\langle l\rangle_M$ is defined as the classifying map associated to $(\overline{X}\,, l\cdot\alpha)$ and $\langle l\rangle_X$ as the pullback of $\langle l\rangle_M$ via π. That the correspondences restrict to the open and the ordinary locus is easily seen from their interpretation in terms of moduli problems. To prove claim 4 a), one applies the modular definition of the Hecke correpondences to the universal elliptic curve over $\mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p}$, where as usual $\mathcal{M}$ denotes the p-adic completion of M, see [**Gou88**, II.1.1]. The first statement in 4 b) follows in the same way with $\mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p}$ replaced by $M_{\mathbb{Q}_p}$. In order to prove the second statement in 4 b), it is enough to check this formula on overconvergent p-adic modular forms, and even on their q-expansions. Take an elliptic curve E/B over a p-adic ring B which is flat over $\mathbb{Z}_p$. Tensoring with $\mathbb{Q}_p$ we get an elliptic curve $E_{\mathbb{Q}_p}/B_{\mathbb{Q}_p}$ that has étale p-torsion. One applies this to the Tate curve at ∞, and standard computations with Tate curves as in [**Kat73**, 1.11] show the desired formula. The induced operators on cohomology respect the decompositions in 2) because they respect weights. The claims on commutativity of the operators are well-known. To prove 6), it is enough to check this on rational de Rham cohomology. After tensoring with $\mathbb{C}$ we find that the de Rham pairing is a constant multiple of the corresponding betti paring. We are thus reduced to the case of the classical Petersson inner product, where the claimed identities are well-known. □

As mentioned in the beginning of this chapter, our convention for the Hecke operators is dual to the one used in [**Sch90**]. Therefore the same is true for our Eichler-Shimura relation and the one in [**Sch90**, 4.2.2]. It is however the same used in [**DFG04**], see p. 684.
Let K be an extension of $\mathbb{Q}_p$. Via the diamond operators, the spaces

$$H^1_{\mathrm{rig}}(\mathscr{M}(\Gamma_1(N))^{\mathrm{ord}}, \mathscr{L}) \otimes K,\ H^1_{c,\mathrm{rig}}(\mathscr{M}(\Gamma_1(N))^{\mathrm{ord}}, \mathscr{L}) \otimes K,\ \tilde{H}_{\mathrm{rig}} \otimes K$$

receive a $(\mathbb{Z}/N)^{\times}$-action.

Definition III.1.7. For a Dirichlet character ψ mod N, we denote the respective ψ-eigenspaces by

$$H^1_{\mathrm{rig}}(\mathscr{M}(N,\psi)^{\mathrm{ord}}, \mathscr{L}) \otimes K,\ H^1_{c,\mathrm{rig}}(\mathscr{M}(N,\psi)^{\mathrm{ord}}, \mathscr{L}) \otimes K,\ (\tilde{H}_{\mathrm{rig}} \otimes K)^{\psi}\ .$$

This notation only makes sense if K contains the values of ψ and when we use it we will always assume this to be the case. Next, we renormalize the duality pairing:

Definition III.1.8.

$$(\cdot,\cdot)_{\mathrm{rig}} := \frac{1}{[\Gamma_0(N):\Gamma_1(N)]}\,(\cdot,\cdot)$$

If the level is not clear from the context, we will sometimes write $(\cdot,\cdot)_{\mathrm{rig},N}$ for $(\cdot,\cdot)_{\mathrm{rig}}$.

We deduce perfect pairings

$$(\cdot,\cdot)_{\mathrm{rig}} : H^1_{\mathrm{rig}}(\mathscr{M}(N,\psi)^{\mathrm{ord}}, \mathscr{L}) \otimes K \times H^1_{c,\mathrm{rig}}(\mathscr{M}(N,\bar{\psi})^{\mathrm{ord}}, \mathscr{L}) \otimes K \to K(1-k)\ ,$$

$$(\cdot,\cdot)_{\mathrm{rig}} : (\tilde{H}_{\mathrm{rig}} \otimes K)^{\psi} \times (\tilde{H}_{\mathrm{rig}} \otimes K)^{\bar{\psi}} \to K(1-k)\ ,$$

where $K(m) := \mathbb{Q}_p(m) \otimes K$ is the one-dimensional vector space K with Frobenius = multiplication by p^{-m}. $(\cdot,\cdot)_{\mathrm{rig}}$ is normalized in such a way that in case $\psi = 1$, it gives the duality pairing of level $\Gamma_0(N)$.

Rigid realization of $M(f)$. Let f be a classical cusp newform of level $\Gamma_0(N_f)$, $p \nmid N_f$ with character ψ and q-expansion

$$f(q) = \sum_{n\geq 1} a_n\, q^n \ .$$

We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ which does not appear in our notation. Via this embedding, we set $K := \mathbb{Q}_p(a_n, n \geq 1)(= \mathbb{Q}_p(a_n, n \geq 1, \psi))$. According to [**Sch90**], the following definition makes sense:

Definition III.1.9. $H_{\mathrm{rig}}(M(f))$ is defined to be the (two dimensional) K-eigenspace of $\tilde{H}_{\mathrm{rig}} \otimes K$ for the operators $T_l, \langle l \rangle,\ l \nmid N$ with eigenvalues $a_l, \psi(l)$ respectively.

Because the Hecke operators commute with Φ, the vector space $H_{\mathrm{rig}}(M(f))$ inherits a Frobenius structure which is still denoted Φ. We sometimes write Φ_f, U_f to emphasize the domain of these endomorphisms.
By transposition of Hecke operators, the field generated by the Fourier coefficients of f inherits a conjugation automorphism which we denote by $a \mapsto \bar{a}$. It is the identity iff the field is totally real and has order two iff the field is CM. Consider the modular form $\bar{f}$. It has q-expansion

$$\bar{f}(q) = \sum_{n\geq 1} \bar{a}_n\, q^n$$

and is a Hecke eigenform having as eigenvalues the conjugate eigenvalues of f. By definition of the conjugation, we have

Proposition III.1.10. *The p-adic Petersson inner product induces a perfect K-linear pairing*

$$(\cdot,\cdot)_{\mathrm{rig}} : H_{\mathrm{rig}}M(\bar{f}) \times H_{\mathrm{rig}}M(f) \to K(1-k) \ .$$

Definition III.1.11. Denote by ω_f the image of the differential form

$$f\,\omega^{k-2} \otimes \delta \in H^0(M_K^{\mathrm{ord}}, \omega^{k-2} \otimes \Omega^1(\log C)) \hookrightarrow H^1_{\mathrm{dR}}(M_K^{\mathrm{ord}}, L)$$

under the comparison isomorphism

$$H^1_{\mathrm{dR}}(M_K^{\mathrm{ord}}, L) \xrightarrow{\cong} H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}) \otimes K \ .$$

ω_f even belongs to the subspace $\tilde{H}_{\mathrm{rig}} \otimes K$. According to our normalization of the Hecke operators,

$$T_l\,\omega_f = a_l\,\omega_f, \quad \langle l \rangle\,\omega_f = \psi(l)\,\omega_f \ .$$

Therefore, $\omega_f \in H_{\mathrm{rig}}M(f)$.
Denote by α, β the roots of the p-th Hecke polynomial

$$X^2 - a_p X + \psi(p) p^{k-1} \ .$$

We enlarge K by requiring that $\alpha, \beta \in K$. From now on, we assume that *α and β are different.* This is known for $k = 2$ and for $k > 2$ would follow from the Tate conjecture on algebraic cycles, see[**EC98**]. The condition will be automatic later on because we will assume f to be ordinary. (This means that one of the roots is a p-adic unit.)

LEMMA III.1.12. *The endomorphism $\Phi_{\bar{f}}$ of $H_{\mathrm{rig}}M(\bar{f})$ has characteristic polynomial*

$$X^2 - a_p X + \psi(p) p^{k-1} .$$

PROOF. From the Eichler-Shimura relation, we have

$$U_f^2 - T_p U_f + \langle p \rangle p^{k-1} = 0$$

on $H_{\mathrm{rig}}M(f)$. Because of

$$(\Phi_{\bar{f}} \,\cdot,\ \cdot)_{\mathrm{rig}} = (\ \cdot, U_f\ \cdot)_{\mathrm{rig}} ,$$

the claim follows. □

COROLLARY III.1.13.
a) The elements

$$(\Phi_{\bar{f}} - \beta)\omega_{\bar{f}},\ (\Phi_{\bar{f}} - \alpha)\omega_{\bar{f}}$$

of $H_{\mathrm{rig}}M(\bar{f})$ are eigenvectors for $\Phi_{\bar{f}}$ with eigenvalues α, β respectively.
b) $H_{\mathrm{rig}}M(\bar{f})$ is spanned by

$$\{\Phi_{\bar{f}}\,\omega_{\bar{f}}, \omega_{\bar{f}}\} .$$

c) We have:

$$(\Phi_{\bar{f}}\,\omega_{\bar{f}}, \omega_f)_{\mathrm{rig}} \neq 0 .$$

PROOF. a) is clear and b) follows form a) and $\alpha \neq \beta$. For c) note that one has

$$(\omega_{\bar{f}}, \omega_f)_{\mathrm{rig}} = 0$$

by compatibility with the de Rham pairing and the fact that both forms are holomorphic. But the pairing is nondegenerate, so

$$(\Phi_{\bar{f}}\,\omega_{\bar{f}}, \omega_f)_{\mathrm{rig}} \neq 0 .$$

□

We conclude this section by defining a cohomology class which will be used to construct a certain linear form in the sequel.

DEFINITION III.1.14. $\omega_{\bar{f},\alpha}$ is defined as the image of

$$(\Phi_{\bar{f}} - \beta)\omega_{\bar{f}}$$

in

$$H^1_{c,\,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L})$$

under the inclusion

$$\tilde{H}_{\mathrm{rig}} \hookrightarrow \tilde{H}_{\mathrm{rig}} \oplus R^\vee = H^1_{c,\,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}, \mathscr{L}) .$$

III.2. Classical and p-adic modular forms

We introduce some notation for spaces of modular forms. Assum $N \geq 4$.

Definition III.2.1.
a) For any extension K of $\mathbb{Q}$, define

$$M_k(\Gamma_1(N,p),K) := H^0(\overline{M}(\Gamma_1(N,p))_K, \omega^{k-2} \otimes \Omega^1(\log C)) = M_k(\Gamma_1(N,p),\mathbb{Q}) \otimes K\ ,$$

where $\Gamma_1(N,p) := \Gamma_1(N) \cap \Gamma_0(p)$. If K is a subfield of $\mathbb{C}$, this coincides with the classical space of modular forms of level $\Gamma_1(N,p)$ which are defined over K.
b) For a finite extension K of $\mathbb{Q}_p$, denote by

$$M_k^\dagger(\Gamma_1(N),K)$$

the space

$$H^0((\overline{M}(\Gamma_1(N))^{\mathrm{ord}}_{\mathbb{Q}_p})^{\mathrm{an}}, j^\dagger\omega^{k-2} \otimes \Omega^1(\log C)) \otimes_{\mathbb{Q}_p} K$$

of overconvergent modular forms of weight k, level $\Gamma_1(N)$ and coefficients in K.
c) Let

$$V'(\Gamma_1(N),\mathbb{Q}_p) \subset H^0(\widetilde{\mathcal{M}_{\mathbb{Q}_p}}, \mathcal{O})$$

be the space of Katz modular forms that are holomorphic at the cusps and

$$V_k'(\Gamma_1(N),\mathbb{Q}_p) := H^0(\overline{\mathcal{M}}^{\mathrm{ord}}(\Gamma_1(N))_{\mathbb{Q}_p}, \omega^{k-2} \otimes \Omega^1(\log C))$$

the subspace of weight k Katz modular forms. For a finite extension K of $\mathbb{Q}_p$, we define

$$V'(\Gamma_1(N),K) := V'(\Gamma_1(N),\mathbb{Q}_p) \otimes K,$$

$$V_k'(\Gamma_1(N),K) := V_k'(\Gamma_1(N),\mathbb{Q}_p) \otimes K.$$

d) All of the above spaces have a $(\mathbb{Z}/N)$-action via the diamond operators and for F or K containing the values of a Dirichlet character ψ mod N we denote the ψ-eigenspaces by

$$M_k(Np,\psi,F),\ M_k(Np,\psi,K),\ M_k^\dagger(N,\psi,K),\ V'(N,\psi,K),\ V_k'(N,\psi,K)$$

respectively.

Following Coleman [**Col95**], we will now describe an important map from classical forms of level $\Gamma_1(N,p)$ to overconvergent forms of level $\Gamma_1(N)$. Recall that the moduli problem $\Gamma_1(N,p)$ has a model

$$\overline{M}(\Gamma_1(N,p))$$

over $\mathbb{Z}_p$, (sometimes called Katz-Mazur model) which, following [**Con07**], parametrizes isomorphism classes of generalized elliptic curves E/B with level $\Gamma_1(N)$-structure together with a finite flat subgroup scheme of E^{sm}/B of rank p. The formal scheme

$$\overline{\mathcal{M}}(\Gamma_1(N))^{\mathrm{ord}}$$

parametrizes isomorphism classes pairs $(E/B, \alpha)$ of generalized ordinary elliptic curves E over p-adically complete $\mathbb{Z}_p$-schemes B, together with a level $\Gamma_1(N)$-structure α. By completeness of B, it is possible to uniquely lift the Frobenius kernel of $E^{\mathrm{sm}}_{\mathbb{F}_p}$, and get the so called canonical $\Gamma_0(p)$ structure

$$\beta_{\mathrm{can}} : H \subset E/B\ .$$

Furthermore, this lifting overconverges in the sense that it can also be done for elliptic curves with Hasse invariant close to a unit. The construction commutes with base change. See [**Kat73**, Thm 3.1]. Now let

$$(\overline{X}, \alpha \times \beta)$$

denote the universal generalized elliptic curve over $\overline{M}(\Gamma_1(N, p))$ with level structure and let

$$(\overline{X}, \alpha)$$

denote the analogous object over $\overline{\mathcal{M}}(\Gamma_1(N))$. We apply the previous in the case

$$B = \overline{\mathcal{M}}(\Gamma_1(N))^{\mathrm{ord}},\ (E, \alpha) = (\overline{X}^{\mathrm{ord}}, \alpha)$$

and deduce a cartesian diagram

$$\begin{array}{ccc} (\overline{X}^{\mathrm{ord}}, \alpha \times \beta_{\mathrm{can}}) & \longrightarrow & (\overline{X}, \alpha \times \beta) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(\Gamma_1(N))^{\mathrm{ord}} & \longrightarrow & \overline{M}(\Gamma_1(N, p))\ . \end{array}$$

From this, we get a diagram of ringed spaces (recall that for a formal scheme $\mathcal{Y}$, we denote by $(\mathcal{Y})_{\mathbb{Q}_p}$ the associated rigid analytic space)

$$\begin{array}{ccc} (\overline{\mathcal{X}}^{\mathrm{ord}}, \alpha \times \beta_{\mathrm{can}})_{\mathbb{Q}_p} & \longrightarrow & (\overline{X}, \alpha \times \beta)_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}(\Gamma_1(N))^{\mathrm{ord}}_{\mathbb{Q}_p} & \longrightarrow & \overline{M}(\Gamma_1(N, p))_{\mathbb{Q}_p}\ . \end{array}$$

By the overconvergence of the lifting construction, we can extend the horizontal maps to strict neighborhoods of

$$\begin{array}{c} (\overline{\mathcal{X}}^{\mathrm{ord}}, \alpha \times \beta_{\mathrm{can}})_{\mathbb{Q}_p} \\ \downarrow \\ \overline{\mathcal{M}}(\Gamma_1(N))^{\mathrm{ord}}_{\mathbb{Q}_p} \end{array}$$

inside

$$\begin{array}{c}(\overline{\mathcal{X}}, \alpha \times \beta_{\mathrm{can}})_{\mathbb{Q}_p} \\ \big\downarrow \\ \overline{\mathcal{M}}(\Gamma_1(N))_{\mathbb{Q}_p}\ ,\end{array}$$

and by taking higher direct images we eventually get the commutative diagram

$$\begin{array}{ccc} M_k(\Gamma_1(N,p),K) & \longrightarrow & M_k^\dagger(\Gamma_1(N),K) \\ \big\downarrow & & \big\downarrow \\ H^1_{\mathrm{dR}}(M(\Gamma_1(N,p))_K, \mathscr{L}) & \longrightarrow & H^1_{\mathrm{rig}}(\mathscr{M}(\Gamma_1(N))^{\mathrm{ord}}, \mathscr{L}) \otimes K\ . \end{array}$$

(Remember $\mathscr{L} = \mathrm{Sym}^{k-2}\mathscr{H}^\vee$.) Here, the vertical maps are as usual induced by the obvious inclusion

$$\omega^{k-2} \hookrightarrow \mathrm{Sym}^{k-2}\mathscr{H}\ .$$

Proposition III.2.2. *The map*

$$M_k(\Gamma_1(N,p),K) \to M_k^\dagger(\Gamma_1(N),K)$$

induced by lifing the Frobenius kernel is the identity on q-expansions. In particular it is injective and equivariant with respect to the classical U_p-operator on the left hand side and the p-adic U-operator on the right hand side.

Proof. The Tate curve at ∞ in $M(N,p)$ has as level $\Gamma_0(p)$-structure the roots of unity

$$\mu_p \subset \mathbb{G}_m/q\ .$$

But this lifts the Frobenius kernel. □

Using the modular definition of Hecke operators, it is furthermore not hard to see that:

Proposition III.2.3. *The commutative diagram induced by lifting the Frobenius kernel*

$$\begin{array}{ccc} M_k(\Gamma_1(N,p),K) & \longrightarrow & M_k^\dagger(\Gamma_1(N),K) \\ \big\downarrow & & \big\downarrow \\ H^1_{\mathrm{dR}}(M(\Gamma_1(N,p))_K, \mathscr{L}) & \longrightarrow & H^1_{\mathrm{rig}}(\mathscr{M}(\Gamma_1(N))^{\mathrm{ord}}, \mathscr{L}) \otimes K\ . \end{array}$$

is equivariant with respect to the Hecke operators $T_l, \langle l\rangle$, $l \nmid Np$.

Via the $\Gamma_1(N)$ part of the level structure, the spaces

$$M_k(\Gamma_1(N,p),K), H^1_{\mathrm{dR}}(M(\Gamma_1(N,p))_K, \mathscr{L})$$

have a $(\mathbb{Z}/N)^\times$-action. By taking ψ-eigenspaces, we deduce the diagram

$$\begin{array}{ccc} M_k(Np,\psi,K) & \longrightarrow & M_k^\dagger(N,\psi,K) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{dR}}(M(Np,\psi)_K,\mathscr{L}) & \longrightarrow & H^1_{\mathrm{rig}}(\mathscr{M}(N,\psi)^{\mathrm{ord}},\mathscr{L})\otimes K\ . \end{array}$$

III.3. Definition of the linear form $l_{f,\mathrm{rig}}$

Let us first explain the purpose of the following sections. In [**Pan02**], [**Pan03**], Panchishkin constructs the p-adic L-function of a Hecke eigenform. This is done in two steps: First, he defines a p-adic measure with values in certain spaces of modular forms which is essentially given by products of Eisenstein series. Then, he applies a linear form which is defined as taking the (classical) Petersson inner product of a modular form with a cusp form coming from the eigenform. The idea of using a suitable linear form for p-adic interpolation already goes back to Hida [**Hid85**]. Panchishkin uses algebraic Eisenstein series in order to get the p-adic L-function at *critical* values. Our goal is to explain the p-adic L-function at *noncritical* values and we will need non-classical p-adic Eisenstein series for this. We will also want to apply a duality pairing to (products of) these Eisentein series. However, in order to have cohomology classes associated to these non-classical series, we use *rigid* cohomology and the *rigid* duality pairing instead of the de Rham analogues. In the following we define a linear form $l_{f,\mathrm{rig}}$ on modular forms whose construction uses rigid duality.

We fix some Notation:

N is an integer prime to p and f is a normalized Hecke cusp eigenform of primitive level N_f *dividing* N. Let K be a finite extension containing all Fourier coefficients of f as well as the roots α,β of the p-th Hecke polynomial.

Recall the finite dimensional K-vector space $M_k(Np,\psi,K)$ of classical modular forms. It comes with an action of the $U_p=U$-operator. We define

$$M_k(Np,\psi,K)^\alpha$$

to be the generalized eigenspace

$$\bigcup_n \mathrm{Ker}(U-\alpha)^n$$

with respect to α. Consider the projection

$$\pi^N_{N_f}: M_k(Np,\psi,K)\to M_k(N_fp,\psi,K)$$

which is adjoint to the inclusion

$$M_k(N_fp,\psi,K)\subset M_k(Np,\psi,K)$$

w.r.t. the Petersson inner product and denote the induced map

$$M_k(Np,\psi,K)^\alpha \to M_k(N_fp,\psi,K)^\alpha$$

by the same symbol $\pi^N_{N_f}$. Lifting the Frobenius kernel gave us a map

$$M_k(N_fp,\psi,K) \to M_k^\dagger(N_f,\psi,K)$$

and thus we get a map

$$M_k(N_fp,\psi,K) \to H^1_{\mathrm{rig}}(\mathscr{M}(N_f,\psi)^{\mathrm{ord}},\mathscr{L})\otimes K$$

which we call

$$g \mapsto [g]\ .$$

Recall the rigid duality pairing (or p-adic Petersson inner product)

$$(\cdot,\cdot)_{\mathrm{rig}}\ : H^1_{c,\,\mathrm{rig}}(\mathscr{M}(N_f,\bar\psi)^{\mathrm{ord}},\mathscr{L})\otimes K\times H^1_{\mathrm{rig}}(\mathscr{M}(N_f,\psi)^{\mathrm{ord}},\mathscr{L})\otimes K \to K(1-k)$$

of level N_f. Finally, we need the cohomology class

$$\omega_{\bar f,\alpha} \in H^1_{c,\,\mathrm{rig}}(\mathscr{M}(N_f,\bar\psi)^{\mathrm{ord}},\mathscr{L})\otimes K$$

defined in the previous section. We are now ready to define l_{rig}.

Definition III.3.1. The linear form

$$l_{f,\mathrm{rig}} : M_k(Np,\psi,K)^\alpha \to K(1-k)$$

is defined as

$$g \mapsto (\omega_{\bar f,\alpha},[\pi^N_{N_f}\,g])_{\mathrm{rig},N_f}\ .$$

III.4. Panchishkin's linear form l_f

We introduce (a slight modification of) what is called $l_{f,\alpha}$ in [**Pan03**]. To emphasize that this construction is analogous to that of $l_{f,\mathrm{rig}}$ it would be natural to define l_f in terms of the duality pairing in algebraic de Rham cohomology. We chose to stick to the classical Petersson inner product because it is more convenient in later computations.
We begin with some notation. f is like in the previous section, in particular $\alpha \neq \beta$. Let M be an integer prime to p and ψ a Dirichlet character mod M. Set as before For

$$g,h \in M_k(Mp^v,\psi,\mathbb{C})$$

denote by

$$\langle g,h\rangle_{Mp^v} := \int\limits_{\Gamma_0(Mp^v)\backslash\mathbb{H}} \overline{g}\,h\,y^k\frac{dxdy}{y^2}$$

the Petersson inner product of level $\Gamma_0(Mp^v)$. We also use the common notation

$$g_{|_k\gamma} := (\det\gamma)^{k/2}(c\tau+d)^{-k}\,g\left(\frac{a\tau+b}{c\tau+d}\right),\ \gamma=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in \mathrm{Gl}_2(\mathbb{R})^+\ .$$

Consider the following "modifications" of f:

$$f_0 := f - \beta F f = f(\tau) - \beta f(p\tau)$$

$$f^0 := \bar{f}_{0|_k W},\ W = W_{(N_f p)} = \begin{bmatrix} 0 & -1 \\ N_f p & 0 \end{bmatrix} .$$

Then

$$f_0, f^0 \in S_k(N_f p, \psi, \mathbb{C})$$

and

$$U f_0 = \alpha f_0 \ ,$$

$$U^* f^0 = \bar{\alpha} f^0$$

where U^* is adjoint to U with respect to $\langle \cdot, \cdot \rangle_{N_f p}$. Therefore

$$\langle f^0, U g \rangle_{N_f p} = \alpha \langle f^0, g \rangle_{N_f p}$$

for any g. One also checks that

$$\langle f^0, T_l g \rangle_{N_f p} = a_l \langle f^0, g \rangle_{N_f p}$$

for any $l : l \nmid N_f p$. Because f is a N_f-newform and Eisenstein series are orthogonal to cusp forms, we conclude that

$$\langle f^0, g \rangle_{N_f p} = 0$$

for all g which are not N_f (cusp-)newforms. The space of N_f-newforms has a basis of eigenforms for the T_l, $l \nmid N_f p$. But because of $\langle f^0, T_l g \rangle_{N_f p} = a_l \langle f^0, g \rangle_{N_f p}$, one has $\langle f^0, g \rangle = 0$ *except* possibly for

$$g \in \operatorname{span}\{f, Ff\} \ .$$

Now

$$\operatorname{span}\{f, Ff\} = \operatorname{span}\{f_0, f_1\}$$

where

$$f_1 = f - \alpha F f \ .$$

because $\alpha \neq \beta$. From

$$U f_1 = \beta f_1$$

we conclude that

$$\langle f^0, f_1 \rangle_{N_f p} = 0 \ .$$

But the Petersson inner product is nondegenerate and therefore

$$\langle f^0, f_0 \rangle_{N_f p} \neq 0 \ .$$

Finally, note that f_0, f_1 are defined over the number field $E := \mathbb{Q}((a_n)_n, \alpha)$. We deduce from the discussion:

PROPOSITION III.4.1. *Let* $E := \mathbb{Q}((a_n)_n, \alpha)$*. Then the rule*

$$g \mapsto \frac{\langle f^0, g\rangle_{N_f p}}{\langle f^0, f_0\rangle_{N_f p}}$$

defines a nondegenerate linear form

$$\tilde{l}_f : M_k(N_f p, \psi, E) \to E \ .$$

Now let $K = \mathbb{Q}_p((a_n)_n, \alpha)$ (remember that we fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$) . Denote by $\tilde{l}_f \otimes K$ the linear form deduced from $\tilde{l}_f$ by extension of scalars. Remember that N is an integer prime to p which is divided by N_f. Also recall the basis element T of the Frobenius vector space

$$K(1) \ .$$

DEFINITION III.4.2. The linear form

$$l_f : M_k(Np, \psi, K)^\alpha \to K(1-k)$$

is defined as

$$l_f := T^{1-k} \cdot (\tilde{l}_f \otimes K) \circ \pi^N_{N_f} \ .$$

III.5. Comparison of the linear forms.

Here we show that l_f and $l_{f,\mathrm{rig}}$ only differ by a nonzero p-adic number which is the p-adic analogue of the real number $\langle f, f\rangle$, the Petersson inner product. We set

$$\mathrm{Pet}_p := (\,\Phi\,\omega_{\bar{f}}, \omega_f\,)_{\mathrm{rig}} \cdot T^{k-1} \in K \ .$$

We will see in a minute that this is nonzero.

PROPOSITION III.5.1. *The linear forms* $l_f, l_{f,\mathrm{rig}}$ *satisfy*

$$l_{f,\mathrm{rig}} = d_p \cdot l_f$$

where

$$d_p = (1 - \frac{\beta}{\alpha})\ \mathrm{Pet}_p$$

is nonzero.

PROOF. By the definition of both maps, it is enough to show the claim for $N = N_f$. Abbreviate $l_{\mathrm{rig}} := l_{f,\mathrm{rig}}$. We claim that l_{rig} satisfies

$$l_{\mathrm{rig}}(T_l g) = a_l\ l_{\mathrm{rig}}(g), \quad l_{\mathrm{rig}}(Ug) = \alpha\ l_{\mathrm{rig}}(g)$$

for l, $l \nmid N_f p$ and any g : Using the definition

$$l_{\mathrm{rig}}(g) = (\omega_{\bar{f},\alpha}\ ,\ [g])_{\mathrm{rig}}$$

we compute:

$$(\omega_{\bar f,\alpha}\ ,\ [T_l\, g])_{\mathrm{rig}} = (\langle l\rangle^{-1}\, T_l\, \omega_{\bar f,\alpha}\ ,\ [g])_{\mathrm{rig}} = \psi(l)\,\bar a_l(\omega_{\bar f,\alpha}\ ,\ [g])_{\mathrm{rig}} = a_l\,(\omega_{\bar f,\alpha}\ ,\ [g])_{\mathrm{rig}}$$

and

$$(\omega_{\bar f,\alpha}\ ,\ U\,[g])_{\mathrm{rig}} = (\Phi\,\omega_{\bar f,\alpha}\ ,\ [g])_{\mathrm{rig}} = \alpha\,(\omega_{\bar f,\alpha}\ ,\ [g])_{\mathrm{rig}}\ .$$

Therefore, by identical reasoning as in the previous section, we deduce that l_{rig} vanishes on all T_l-eigenforms except f_0. This implies that l_{rig} is a constant multiple of l_f, i.e.

$$l_{\mathrm{rig}} = d_p \cdot l_f$$

for some $d_p \in K$. Because of $l_f(f_0) = 1 \cdot T^{1-k}$,

$$l_{\mathrm{rig}}(f_0) = d_p \cdot T^{1-k}.$$

Recall that f_0 was defined as $f - \beta F f$. Because of the formula

$$\Phi[g] = p^{k-1}[Fg]\ ,$$

we conclude that

$$[f_0] = (1 - \frac{\beta}{p^{k-1}}\,\Phi)\,\omega_f$$

and compute

$$\begin{aligned} l_{\mathrm{rig}}(f_0) = (\omega_{\bar f,\alpha}\ ,\ (1 - \frac{\beta}{p^{k-1}}\,\Phi)\,\omega_f)_{\mathrm{rig}} &= (\Phi\omega_{\bar f} - \beta\omega_{\bar f}\ ,\ (1-\beta U^{-1})\omega_f)_{\mathrm{rig}} \\ &= \frac{1}{\alpha}\,(\Phi\omega_{\bar f} - \beta\omega_{\bar f}\ ,\ (U-\beta)\omega_f)_{\mathrm{rig}}\ . \end{aligned}$$

Furthermore,

$$\begin{aligned} &(\Phi\omega_{\bar f} - \beta\omega_{\bar f}\ ,\ (U-\beta)\omega_f)_{\mathrm{rig}} \\ &= \alpha(\Phi\omega_{\bar f} - \beta\omega_{\bar f}\ ,\ \omega_f)_{\mathrm{rig}} - \beta(\Phi\omega_{\bar f} - \beta\omega_{\bar f}\ ,\ \omega_f)_{\mathrm{rig}} \\ &= (\alpha-\beta)(\Phi\omega_{\bar f}\ ,\ \omega_f)_{\mathrm{rig}}\ = (\alpha-\beta)\,\mathrm{Pet}_p \cdot T^{1-k} \end{aligned}$$

where in the last step we used that two holomorphic forms pair to zero. This proves the claimed formula for d_p. It remains to show that d_p is not zero. The first factor is nonzero because $\alpha \neq \beta$ and $(\Phi\omega_{\bar f}\ ,\ \omega_f)_{\mathrm{rig}}$ is nonzero by Corollary 1.11 c). □

CHAPTER IV

Panchishkin's measure

To a large extent this chapter reviews Panchishkin's [**Pan02**], [**Pan03**] construction of the p-adic L-function attached to a normalized Hecke eigenform using Eisenstein series. The differences are as follows: Whereas Panchishkin's main focus is on the case of positive slope, i.e. $v_p(\alpha) > 0$, we only need the easier ordinary case in which this method already goes back to Hida [**Hid85**]. A more subtle difference is the following: In [**Pan03**], a certain auxiliary Dirichlet character ξ is used and chosen to have conductor p. This character corresponds to the choice of a complex period. However when one tries to relate the p-adic L-function to Eisenstein classes, the complex period appearing in the formulas turns out to be dependent on the specific choice of our Eisenstein classes. Because we can only deal with Eisenstein classes having conductor *prime* to p, we therefore have to redo Panchishkin's construction for a character ξ having conductor prime to p. This changes the precise formula for the critical L-values by a finite product of Euler factors and some additional elementary factors. Finally, whereas Panchishkin is able to only use *classical* modular forms in his construction, we extend the target of the Eisenstein measure used in [**Pan02**] and [**Pan03**] to the (huge) space of Katz modular forms (of some fixed weight and level). This space has the advantage of being a a Banach space in the q-expansion topology. Because $v_p(\alpha) = 0$, by work of Hida [**Hid86**] there is a projection operator π_α onto the generalized α-eigenspace for the U-operator which is continuous in the q-expansion topology. This fact will eventually allow us to evaluate the p-adic L-function at noncritical values.
In everythig that follows, N denotes an integer prime to p.

IV.1. Review of p-adic measures

We review some notation and facts from the language of p-adic measures closely following [**Pan02**, §1]. Let F be an extension of $\mathbb{Q}_p$ which is complete with respect to a valuation $|\cdot|$ extending $|\cdot|_p$ on $\mathbb{Q}_p$. For example, $F = \mathbb{C}_p$ or a finite extension of $\mathbb{Q}_p$. Let M denote a natural number prime to p. We will need the abelian group

$$Y := \varprojlim_v (\mathbb{Z}/Mp^v)^\times$$

with its profinite topology. We also define

$$LC(Y, F) := \{\varphi : Y \to F, \text{ locally constant}\} .$$

It is a normed vector space with norm

$$||\varphi|| := \max_Y |\varphi(x)| \ .$$

Definition IV.1.1. Let B be a normed vector space over F with ultrametric norm $|\cdot|_B$. A distribution μ on Y with values in B is a linear map

$$\mu : LC(Y, F) \to B \ .$$

μ is called measure if it is bounded, i.e. if for every φ one has

$$|\mu(\varphi)|_B \leq C||\varphi||$$

for a constant C independent of φ.

For a distribution

$$\mu : LC(Y, F) \to B$$

denote by

$$\mu(a + (Mp^v))$$

the value of μ on the characteristic function of the set $a + (Mp^v)$. μ defines a system of elements of B

$$\mu(a + (Mp^v)), \ v \geq 0, \ a \in (\mathbb{Z}/Mp^v)^\times,$$

which is compatible in the sense that

$$\sum_{\substack{a' \, (Mp^{v+1}): \\ a' \equiv a \, (p^v)}} \mu(a' + (Mp^{v+1})) = \mu(a + (Mp^v)) \ , v \geq 0.$$

Conversely it is straightforward that any such compatible system defines a distribution. The distribution is bounded iff

$$\mu(a + (Mp^v))$$

is bounded independently of v and a. This is because the norm on B is nonarchimedean. Consider the space

$$\mathscr{C}(Y, F) := \{\varphi : Y \to F, \ \text{continuous}\}$$

which is a Banach space under the max-norm (Y is compact) and which has

$$LC(Y, F)$$

as dense subspace. If B a normed space which is even a Banach space one shows that

Proposition IV.1.2. *A measure*

$$\mu : LC(Y, F) \to B \ .$$

extends uniquely to a bounded linear map

$$\mu : \mathscr{C}(Y, F) \to B \ .$$

We denote the extension by the same letter and also call it a measure.

IV.2. Convolution of Eisenstein measures

In this section we again follow Panchishkin and define a distribution on $\mathbb{Z}_p^\times$ with values in spaces of classical modular forms. This distribution is essentially given by products of classical Eisenstein series. It turns out to be bounded and therefore extends to a functional from $\mathscr{C}(\mathbb{Z}_p^\times, F)$ to the space of Katz modular forms of some weight and level. For later purposes, we evaluate this measure at powers of the cyclotomic character.

We need the existence of certain Eisenstein series.

Proposition IV.2.1. *Let ξ be a nontrivial Dirichlet character modulo N and let $a + (Np^v)$, $b + (p^v)$ denote residue classes in $(\mathbb{Z}/Np^v)^\times, (\mathbb{Z}/p^v)^\times$ respectively. Then there exist classical Eisenstein series $E_m(\xi, b)_v$, $E_m(a)_v$ of weight $m \geq 1$ and level $\Gamma_1(Np^v)$ (if $v \geq 1$) which have q-expansions*

$$E_m(\xi,b)_v\ (q) = \begin{cases} \sum_{n\geq 1} q^n \sum_{\substack{d\mid n \\ \frac{n}{d}\equiv b\,(p^v)}} \xi(d)\mathrm{sgn}(d)d^{m-1} & v \geq 1 \\ \\ \sum_{n\geq 1} q^n \sum_{\substack{d\mid n \\ p\nmid \frac{n}{d}}} \xi(d)\mathrm{sgn}(d)d^{m-1} & v = 0. \end{cases}$$

and, if $m \neq 2$,

$$E_m(a)_v\ (q) = \begin{cases} L_{Np}(1-m,a) + \sum_{n\geq 1} q^n \sum_{\substack{d\mid n \\ d\equiv a\,(Np^v)}} \mathrm{sgn}(d)d^{m-1} & v \geq 1. \\ \\ L_{Np}(1-m) + \sum_{n\geq 1} q^n \sum_{\substack{d\mid n \\ (d,Np)=1}} \mathrm{sgn}(d)d^{m-1} & v = 0. \end{cases}$$

Proof. We use a standard basis for the space of Eisenstein series of level $\Gamma_1(\cdot)$, [**DS05**, Thm. 4.5.2,Thm. 4.6.2, Thm. 4.8.1]. In the notation of Diamond-Shurman, for two primitive Dirichlet characters χ, χ' satisfying $\chi\chi'(-1) = (-1)^m$ of level $N_\chi, N_{\chi'}$ and a natural number t, there is an Eisenstein series

$$E_m^{\chi',\chi,t} = \delta(\chi') \cdot L(1-m,\chi) + 2\sum_{n\geq 1}\sum_{d\mid n, d>0} \chi(d)\chi'(\frac{n}{d})\, d^{m-1}\, q^{nt}\ ,$$

of level $\Gamma_0(N_\chi\, N_{\chi'}\, t)$ and Nebentypus $\chi\chi'$ where

$$\delta(\chi') = \begin{cases} 1 & \chi' = \mathbf{1} \\ 0 & else \end{cases}$$

and $m \neq 2$ in case $\chi = \chi' = \mathbf{1}$. For convenience, let us define $E_m^{\chi',\chi,t}$ to be zero if $\chi\chi'$ has the "wrong" parity $m-1$. We can produce such Eisenstein series even when the Dirichlet characters are not primitve [**Hid85**, §7] : Let χ, χ' be characters mod N_χ and mod $N_{\chi'}$ respectively, satisfying $\chi\chi'(-1) = (-1)^m$

and having associated primitive characters χ_0, χ_0' of conductor $N_{\chi_0}, N_{\chi_0'}$ respectively. We set

$$E_m^{\chi',\chi,1} := \sum_{t|\frac{N_\chi}{N_{\chi_0}}} \sum_{t'|\frac{N_{\chi'}}{N_{\chi_0'}}} \mu(t)\chi_0(t)t^{m-1}\,\mu(t')\chi_0'(t')\;E_m^{\chi_0',\chi_0,t\cdot t'}$$

where μ is the Moebius function and get an Eisenstein series

$$E_m^{\chi',\chi,1} = \delta(\chi')\cdot L(1-m,\chi) + 2\sum_{n\geq 1}\sum_{d|n,d>0} \chi(d)\chi'(\frac{n}{d})\,d^{m-1}\,q^n\ ,$$

of level $\Gamma_0(N_\chi\,N_{\chi'})$ and Nebentypus $\chi\chi'$ where

$$\delta(\chi') = \begin{cases} 1 & N_{\chi'} = N_{\chi_0'} = 1 \\ 0 & else \end{cases}$$

and $m \neq 2$ in case χ and χ' are 1 mod N_χ, mod $N_{\chi'}$ respectively. Again, we define $E_m^{\chi',\chi,1}$ to be zero if $\chi\chi'$ has the wrong parity. To prove the first statement, we set

$$\varphi(p^v)\,E_m(\xi,b)_v := \sum_{\chi \bmod p^v} \bar{\chi}(b)\,E_m^{\chi,\xi,1} + \xi(-1)(-1)^m \sum_{\chi \bmod p^v} \bar{\chi}(-b)\,E_m^{\chi,\xi,1}$$

in case $v \geq 1$ where the sums are taken over all Dirichlet characters mod p^v. The n-th Fourier coefficient of this sum is then zero if $n = 0$ and for $n \geq 1$ is equal to

$$\sum_{\substack{\chi \bmod p^v \\ \chi(-1)=\xi(-1)(-1)^m}} \bar{\chi}(b) \sum_{d|n,d>0} \xi(d)\chi(\frac{n}{d})\,d^{m-1}$$

$$+\,\xi(-1)(-1)^m \sum_{\substack{\chi \bmod p^v \\ \chi(-1)=\xi(-1)(-1)^m}} \bar{\chi}(-b) \sum_{d|n,d>0} \xi(d)\chi(\frac{n}{d})\,d^{m-1}$$

$$= \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{d|n,d>0} \xi(d)\chi(\frac{n}{d})\,d^{m-1}$$

$$+\,\xi(-1)(-1)^m \sum_{\chi \bmod p^v} \bar{\chi}(-b) \sum_{d|n,d>0} \xi(d)\chi(\frac{n}{d})\,d^{m-1}$$

which proves the claim, because $\sum_{\chi \bmod p^v} \bar{\chi}(b)\,\chi$ is $\varphi(p^v)$ times the characteristic function of the set $b + (p^v)$. As for $v = 0$, we set

$$E_m(\xi,b)_0 := \sum_{b\in(\mathbb{Z}/p)^\times} E_m(\xi,b)_1\ .$$

In order to prove the second statement, we set

$$\varphi(Np^v)\, E_m(a)_v := \sum_{\chi \mathrm{mod} Np^v} \bar{\chi}(a)\, E_m^{\mathbf{1},\chi,1} + (-1)^m \sum_{\chi \mathrm{mod} Np^v} \bar{\chi}(-a)\, E_m^{\mathbf{1},\chi,1}$$

in case $v \geq 1$ and

$$E_m(a)_0 := \sum_{a \in (\mathbb{Z}/Np)^\times} E_m(a)_1 \ .$$

□

Two Eisenstein measures. As before, we denote by

$$M_m(\Gamma_1(L), K)$$

the space of classical modular forms of weight m and level L. We set

$$M_m(\Gamma_1(Np^\infty), K) := \underset{v}{\cup} M_m(\Gamma_1(Np^v), K) \ .$$

This injects naturally into $V'(\Gamma_1(N), K)$, the space of Katz modular forms of tame level $\Gamma_1(N)$ with coefficients in K. It is the inclusion of a p-adic normed vector space into a p-adic Banach space when we take the max $|\cdot|_p$-norm on q-expansions on both spaces.

Proposition IV.2.2. *Let K be a finite extension of $\mathbb{Q}_p$ that contains the values of ξ. There is a unique measure $\mu_{1,m}$ on $\mathbb{Z}_p^\times$ with values in $M_m(\Gamma_1(Np^\infty), K)$, which is given on subsets of the form $a + (p^v) \subset \mathbb{Z}_p^\times$ by*

$$E_m(\xi, a)_v$$

Proof. It is clear that μ_1 defines a distribution. It is bounded because all Fourier coefficients are algebraic integers. □

We need a second Eisenstein measure, this time on $\mathbb{Z}_{p,N}^\times = \varprojlim_v (\mathbb{Z}/Np^v)^\times$.

Proposition IV.2.3. *There is a unique measure $\mu_{2,m}$ on $\mathbb{Z}_{p,N}^\times$ with values in $M_m(\Gamma_1(Np^\infty), K)$, which is given on subsets of the form $a + (Np^v) \subset \mathbb{Z}_{p,N}^\times$, $v \geq 0$, by*

$$E_m^{(p)}(a)_v := (1-F) E_{m,0}(a)_v \quad = \begin{cases} \sum_{\substack{n\geq 1 \\ p\nmid n}} q^n \sum_{\substack{d|n \\ d\equiv a\,(Np^v)}} \mathrm{sgn}(d) d^{m-1} & v \geq 1 \\ \sum_{\substack{n\geq 1 \\ p\nmid n}} q^n \sum_{\substack{d|n \\ (d,N)=1}} \mathrm{sgn}(d) d^{m-1} & v = 0 \end{cases}$$

Here, F denotes the Frobenius operator $(Fg)(q) = g(q^p)$.

Proof. Again the distribution property is clear and boundedness follows from integrality of the coefficients. □

We remark that for our applications we could also work with the Eisenstein distribution given by $E_m(a)_v$. However the constant term of the corresponding Fourier expansion is the Kubota-Leopoldt pseudo-measure, (the Mellin transform of) which has a pole at $\chi = 1$. This introduces unnecessary complications. Working with $E_m^{(p)}(a)_v$ or $E_m(a)_v$ will in the end only differ by one Euler factor.

From now on, we will fix our coefficient field K : As in the previous chapter f is a normalized cusp newform of weight k with character ψ which has level N_f prime to p. Let ξ be a nontrival Dirichlet character of conductor N_ξ prime to p and set $N := \mathrm{lcm}(N_f, N_\xi)$. We set (via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$)

$$K := \mathbb{Q}_p((a_n)_n, \alpha, \xi) .$$

We use the following convention for p-adic measures:

Convention. If μ is a K-valued measure and χ is a Dirichlet character mod p^v, we denote by $\mu(\chi)$ the value at χ of the measure $\mu \otimes K(\chi)$ obtained from μ by an extension of coefficients to $K(\chi)$.

We come to the main goal of this section.

PROPOSITION IV.2.4. *There is a unique measure μ on $\mathbb{Z}_p^\times$ with values in $M_k(\Gamma_1(Np^\infty), K)$ which is given on subsets of the form $a + (p^v) \subset \mathbb{Z}_p^\times$ by*

$$\mu(a + (p^v)) := \frac{1}{4} \sum_{b \in (\mathbb{Z}/Np^v)^\times} \psi\bar{\xi}(b) \;\; \mu_{1,k-1}(ab + (p^v)) \cdot \mu_{2,1}(b + (Np^v))$$

$$= \frac{1}{4} \sum_{b \in (\mathbb{Z}/Np^v)^\times} \psi\bar{\xi}(b) E_{k-1}(\xi, ab)_v \; E_1(b)_v \; .$$

Here, ab is the product of a with the reduction of b mod p^v. For a Dirichlet character χ mod $p^v, v \geq 0,$ one has

$$\mu(\chi) = \frac{1}{2}\, \mu_{1,k-1}(\chi) \;\; \frac{1}{2}\, \mu_{2,1}(\psi\overline{\xi\chi})$$

$$= E_{k-1}(\xi, \chi)\, E_1^{(p)}(\psi\overline{\xi\chi})$$

where

$$E_{k-1}(\xi, \chi)\,(q) = \begin{cases} \sum_{n \geq 1} q^n \sum_{\substack{d|n, d>0 \\ p \nmid \frac{n}{d}}} \xi(d)\, \chi(\frac{n}{d})\, d^{k-2}, & \xi\chi(-1) = (-1)^{k-1} \\ 0 \; \textit{else} \end{cases}$$

and

$$E_1^{(p)}(\psi\overline{\xi\chi})\,(q) = \begin{cases} \sum_{\substack{n \geq 1 \\ p \nmid n}} q^n \sum_{d|n, d>0} \psi\overline{\xi\chi}(d)\; . & \xi\chi(-1) = (-1)^{k-1} \\ 0 \; \textit{else} \end{cases} .$$

Furthermore the values of μ even lie in

$$M_k(\Gamma_0(Np^\infty),\psi,K) := \underset{v}{\cup} M_k(Np^v,\psi,K)\ .$$

PROOF. Checking the distribution property is completely formal: Write

$$\mu_1(a)_v := \frac{1}{2}\mu_{1,k-1}(a+(p^v)),\quad \mu_2(a)_v := \frac{1}{2}\mu_{2,1}(a+(Np^v)),\quad \epsilon := \psi\bar{\xi}\ .$$

We now have (let $v \geq 0$)

$$\sum_{\substack{a'(p^{v+1}):\\ a'\equiv a(p^v)}} \mu(a'+(p^{v+1})) = \sum_{\substack{a'(p^{v+1}):\\ a'\equiv a(p^v)}} \sum_{b\in(\mathbb{Z}/Np^{v+1})^\times} \epsilon(b)\ \mu_1(a'b)_{v+1}\,\mu_2(b)_{v+1}$$

$$= \sum_{b\in(\mathbb{Z}/Np^{v+1})^\times} \epsilon(b)\,\mu_2(b)_{v+1}\Big(\sum_{\substack{a'(p^{v+1}):\\ a'\equiv a(p^v)}} \mu_1(a'b)_{v+1}\Big)$$

$$= \sum_{b\in(\mathbb{Z}/Np^{v+1})^\times} \epsilon(b)\,\mu_2(b)_{v+1}\ \mu_1(a[b])_v \text{ (here } [b] \text{ denotes the reduction mod } p^v)$$

$$= \sum_{b_0\in(\mathbb{Z}/Np^v)^\times} \epsilon(b_0)\ \mu_1(ab_0)_v \sum_{\substack{b\in(\mathbb{Z}/Np^{v+1})^\times\\ b\equiv b_0(p^v)}} \mu_2(b)_{v+1}$$

$$= \sum_{b\in(\mathbb{Z}/Np^v)^\times} \epsilon(b)\,\mu_2(b)_v\ \mu_1(ab)_v = \mu(a+(p^v))\ .$$

The distribution is bounded because all Fourier coefficients are algebraic integers. For a Dirichlet character χ with values in K we have

$$\mu(\chi) = \frac{1}{4}\sum_{a\in(\mathbb{Z}/p^v)^\times} \chi(a) \sum_{b\in(\mathbb{Z}/Np^v)^\times} \psi\bar{\xi}(b)\ \ \mu_{1,k-1}(ab+(p^v))\cdot\mu_{2,1}(b+(Np^v))\ .$$

Interchanging summation and substituting ab^{-1} for a we get

$$\frac{1}{2}\Big(\sum_{a\in(\mathbb{Z}/p^v)^\times} \chi(a)\,\mu_{1,k-1}(a+(p^v))\Big)\ \ \frac{1}{2}\Big(\sum_{b\in(\mathbb{Z}/Np^v)^\times} \psi\overline{\xi\chi}(b)\,\mu_{2,1}(b+(Np^v))\Big)$$

and the claim follows. It remains to show the level condition and it is enough to prove this after we extend coefficients. Consider the residue class $a+(p^v)$, $v \geq 1$ and assume that K contains the values of all Dirichlet characters mod p^v. Then the characteristic function of $a+(p^v)$ is a linear combination of Dirichlet characters, namely

$$\frac{1}{\varphi(p^v)}\sum_{\chi}\bar{\chi}(a)\chi\ ,$$

the sum taken over all Dirichlet characters mod p^v. Therefore it is enough to show that

$$E_{k-1}(\xi,\chi)\,E_1^{(p)}(\psi\overline{\xi\chi}) \in M_k(Np^\infty,\psi,K)\ .$$

for all χ mod p^v of parity $\xi(-1)(-1)^{k-1}$. This follows from

$$2\,E_{k-1}(\xi,\chi) = E_{k-1}^{\chi,\xi,1} \in M_{k-1}(Np^v, \xi\chi, K)$$

and

$$2\,E_1^{(p)}(\psi\overline{\xi\chi}) = (1-F)E_1^{\mathbf{1},\psi\overline{\xi\chi},1} \in M_1(Np^{v+1}, \psi\overline{\xi\chi}, K)\ .$$

where we have used notation from the proof of Prop. IV.2.1. □

The natural inclusion

$$M_k(Np^\infty, \psi, K) \hookrightarrow V'(N, \psi, K)$$

factors throught the space $V'_k(N,\psi,K)$ of Katz modular forms that have (p-adic) weight k. As this is a closed subspace of the Banach space $V'(N,\psi,K)$, it is itself a Banach space and we conclude:

Corollary IV.2.5. *There exists a unique continuous linear functional*

$$\mathscr{C}(\mathbb{Z}_p^\times, K) \to V'_k(N,\psi,K)$$

whose restriction to $LC(\mathbb{Z}_p^\times, K)$ *is equal to* μ.

Definition IV.2.6. The functional in the previous corollary will be denoted by μ_ξ.

Important examples of continuous functions are integer powers of the indentity

$$y^n : \mathbb{Z}_p^\times \to K\ ,\ a \mapsto a^n\ ,\ n \in \mathbb{Z}\ .$$

Let us to evaluate μ_ξ at these. For this recall certain special cases of Eisenstein series of possibly negative weight defined by Katz in [**Kat76**, §6.11] and similar Eisenstein series defined in [**BK**, Def.5.5]. We only need those that have a Dirichlet character ϵ modulo N as coefficient function (and therefore have tame level $\Gamma_0(N)$ with character ϵ) and that have no constant term. We use the following notation: (ϵ has parity $a+b$)

$$E_{a,b}^{(p)}(\epsilon) = \sum_{n\geq 1} q^n \sum_{\substack{d|n, d>0 \\ p\nmid \frac{n}{d}}} \epsilon(d)\, d^{a-1} (\frac{n}{d})^b \quad , a \geq 1,\ b \in \mathbb{Z}$$

$${}^{(p)}E_{a,b}^{(p)}(\epsilon) = \sum_{\substack{n\geq 1 \\ p\nmid n}} q^n \sum_{d|n, d>0} \epsilon(d)\, d^{a-1} (\frac{n}{d})^b, \quad a \in \mathbb{Z},\ b \in \mathbb{Z}$$

Unfortunately this convention looks non-symmetric on q-expansions in a and b. Its advantage is that the above modular forms have weight $a+b$.
We extend this notation to an arbitrary continuous character

$$\chi : \mathbb{Z}_p^\times \to K^\times\ ,\ \chi(-1) = \epsilon(-1)(-1)^m\ ,$$

and let ${}^{(p)}E^{(p)}_{m,\chi}(\epsilon)$ be the p-adic Eisenstein series (of weight $m \cdot \chi$) with q-expansion

$$\sum_{\substack{n\geq 1\\ p\nmid n}} q^n \sum_{d|n,d>0} \epsilon(d)\, d^{m-1}\chi(\frac{n}{d}),$$

Analogously for ${}^{(p)}E^{(p)}_{\chi,m}(\epsilon)$. If no confusion can arise, we abbreviate the character $\chi\, y^n$ by $\chi + n$.

With these conventions, we have

$$\mu_\xi(\chi) = E_{k-1}(\xi,\chi)\, E_1^{(p)}(\psi\overline{\xi\chi}) = E^{(p)}_{k-1,\chi}(\xi)\cdot {}^{(p)}E^{(p)}_{\overline{\chi}+1,0}(\psi\bar{\xi})$$

for a Dirichlet character χ of correct parity. We now evaluate the measure μ_ξ at powers of the cyclotomic character.

$$y^n : \mathbb{Z}_p^\times \to K, \qquad a \mapsto a^n\ .$$

Proposition IV.2.7. *Let $n \in \mathbb{Z}$. If $\xi(-1) = (-1)^{k-1-n}$, then*

$$\mu_\xi(y^n) = E^{(p)}_{k-1,n}(\xi)\cdot {}^{(p)}E^{(p)}_{1-n,0}(\psi\bar{\xi})\ .$$

Proof. Take a continuous function

$$\lambda_v : \mathbb{Z}_p^\times \to K$$

which is congruent to y^n mod p^v and constant mod p^v. Then by the same convolution trick as in the proof of IV.2.4 (interchanging summation and using a substitution) we conclude that

$$\mu_\xi(\lambda_v) \equiv E^{(p)}_{k-1,n}(\xi)\cdot {}^{(p)}E^{(p)}_{1-n,0}(\psi\bar{\xi}) \bmod p^v\ .$$

The claim follows after we let v tend to infinity. □

Note that when $l \geq 0$ is a natural number s.t. $\xi(-1) = (-1)^{k+l}$, we get

$$\mu_\xi(y^{-l-1}) = E^{(p)}_{k-1,-l-1}(\xi)\cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})$$

and this looks similar to one of of the terms that showed up in the computation of the product of two Eisenstein symbols of weight $k+l$ and $l+2$ respectively.

Remark. With the identical arguments as in this section one can also construct a measure ν_ξ which on y^{-l-1} is equal to

$$^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi}) \ .$$

This measure has an obvious symmetry which is particularly striking when $k = 2$. We chose to work with μ_ξ instead because it is closer to what we get from the Eisenstein symbol. However we will introduce a projection operator π_α in the next section and after composing with it, both measures become equal.
Secondly, as was already remarked above when we defined μ_2 , one also has the option of defining a functional which has the value

$$\mu_\xi(y^{-l-1}) = E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E_{l+2,0}(\psi\bar{\xi}),$$

$$^{(p)}E_{l+2,0}(\psi\bar{\xi}) = \frac{L_p(-l-1,\psi\bar{\xi})}{2} + \sum_{n\geq 1} \sum_{\substack{d|n, d>0 \\ p\nmid d}} \psi\bar{\xi}(d)\, d^{l+1} \quad ,$$

on y^{-l-1}. This has the disadvantage of only giving a pseudo-measure. In our application, we will eventually compose our measure with the projection operator π_α and then evalute the projection against a modular form coming from the cusp form f. The resulting number one gets for the pseudo-measure and μ_ξ only differ by the Euler factor

$$(1 - \xi(p)\frac{p^{k-2}}{\alpha}) \ .$$

(This is nonzero, by considering p-adic absloute values for $k > 2$ and complex absolute values for $k = 2$.)

IV.3. Hida's ordinary projection

We briefly state a well-known variant of a theorem of Hida [**Hid85**, Prop. 4.1] which enables one to "go back" from the (infinite-dimensional) space of Katz modular forms to the finite-dimensional subspace of classical forms, as long as one is only interested in slope-0 eigenforms.
Let $k \geq 2$ be an integer, $K := \mathbb{Q}_p((a_n)_n, \alpha, \xi)$ as before. Recall that there is a natural inclusion

$$M_k(\Gamma_1(N,p), K) \hookrightarrow M^\dagger_k(\Gamma_1(N), K) \hookrightarrow V'_k(\Gamma_1(N), K) \ .$$

This induces

$$M_k(\Gamma_1(N,p), K)^\alpha \hookrightarrow V'_k(\Gamma_1(N), K)^\alpha$$

and

$$M_k(Np, \psi, K)^\alpha \hookrightarrow V'_k(N, \psi, K)^\alpha$$

on the generalized α-eigenspaces for the U-operator.

THEOREM IV.3.1. *(Hida) Let α have valuation 0, i.e. be a p-adic unit.*
a) There exists a linear idempotent

$$\pi_\alpha : V'_k(N,\psi,K) \to V'_k(N,\psi,K)^\alpha$$

which is the identity on $V'_k(N,\psi,K)^\alpha$ and is continuous in the q-expansion topology.

b)The natural inclusion

$$M_k(Np,\psi,K)^\alpha \hookrightarrow V'_k(N,\psi,K)^\alpha$$

is an isomorphism. In other words every Katz modular form of weight $k \geq 2$ and slope zero is classical.

PROOF. a) follows from [**Gou88**, top of p.69] and [**Gou88**, Prop. II 4.1].
b): This is a direct consequence of [**Hid85**, Prop.4.1]. □

COROLLARY IV.3.2. *The composition $\mu_\xi^\alpha := \pi_\alpha \circ \mu_\xi$ gives a continuous linear map*

$$\mu_\xi^\alpha : \mathscr{C}(\mathbb{Z}_p^\times, K) \to M_k(Np,\psi,K)^\alpha \ .$$

The same is of course true for the "symmetric measure" ν_ξ from the Remark after Prop. IV.2.7. As mentioned in the previous section, one has the

OBSERVATION IV.3.3.

$$\mu_\xi^\alpha = \nu_\xi^\alpha \ .$$

PROOF. It is enough to show this on Dirichlet charcters $\chi \bmod p^v,\ v \geq 1$. Put

$$g = E_{k-1}(\xi,\chi),\ h = E_1^{(p)}(\psi\overline{\xi\chi})$$

and note that

$${}^{(p)}E_{k-1}(\xi,\chi) = (1-\xi(p)p^{k-2}F)g \ .$$

Then

$$\nu_\xi^\alpha(\chi) = \pi_\alpha\left[(1-\xi(p)p^{k-2}F)g\cdot h\right]$$

$$= U^{-1}\pi_\alpha U\left[(1-\xi(p)p^{k-2}F)g\cdot h\right]$$

$$= U^{-1}\pi_\alpha U\left[g\cdot h\right] = \mu_\xi^\alpha(\chi)$$

using $U(Fg\cdot h) = g\cdot(Uh),\ Uh = 0.$ □

IV.4. Relation to the p-adic L-function

In this section we relate μ_ξ to the p-adic L-function attached to a p-ordinary Hecke eigenform f of tame conductor. We proceed exactly like Panchishkin in [**Pan03**], the only difference being that our ξ has different conductor and that we use two different tame levels N_f and N. In this method, one composes the measure μ_ξ^α with a (continuous) linear form l_f. Then one shows that the resulting measure evaluated at Dirichlet characters gives twists of the critical values of $L(f,\cdot)$. The key point is that the interpolation has already been done at the level of p-adic modular forms and as a result the congruences between twists of critical values are *automatic*. Note that, strictly speaking, $l_f \circ \mu_\xi^\alpha$ only will give "half" of the p-adic L-function in the sense that it vanishes on Dirichlet characters of the wrong parity. To fix this, one could choose *two* auxiliary characters ξ, ξ' of different parity and add up $l_f \circ \mu_\xi^\alpha$ and$l_f \circ \mu_{\xi'}^\alpha$. However we will not pursue this point of view, because there is no obvious benefit from it for the formulation of our main results. We will get back to the parity issue at the end of this section.

As before, $K = \mathbb{Q}_p((a_n)_n, \alpha, \xi)$. Remember that we regard $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}_p$ by the choice of an embedding which does not appear in our notation. (We will sometimes use an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ but the statements will be independent of this choice.) Let χ be a Dirichlet character of conductor p^v, $v \geq 1$, which satisfies $\chi\xi(-1) = (-1)^{k-1}$. Our goal is to show that

$$l_f \circ \mu_\xi^\alpha(\chi)$$

equals $L(f, \bar{\chi}, 1)$ divided by a transcendental period, up to a certain algebraic factor. This will be done in two steps: First we show that the above term equals a Rankin convolution and then we have to unfold this convolution explicitly, including the bad Euler factors. For two modular forms

$$f' = \sum_{n \geq 1} a'(n) q^n, \quad g' = \sum_{n \geq 1} b'(n) q^n$$

we define

$$D_{Np}(s, f', g')$$

to be the meromorphic continuation of what for large $\mathrm{Re}(s)$ is given by

$$L_{Np}(2s + 2 - k - l, \psi\overline{\xi\chi}) \sum_{n \geq 1} a'(n) b'(n) n^{-s} \ .$$

Set

$$\widetilde{N} := \frac{N}{N_f} \ .$$

The following computations are like in [**Pan03**] but we present most of the details for the reader's convenience.

PROPOSITION IV.4.1. [**Pan03**, Prop. 7.3]. *We have the equality*

$$l_f \circ \mu_\xi^\alpha(\chi) \;=\; e_\xi \cdot \lambda \cdot \alpha^{-v} \cdot \widetilde{N}^{k/2} \cdot \;\frac{D_{Np}(k-1, f', g')}{\pi^k \langle f^0, f_0\rangle}\; \cdot T^{1-k}$$

where

$$f' = V_{\tilde{N}} f_0 = f_0(\widetilde{N}\tau), \quad g' = E_{k-1}(\xi,\chi)_{|W_{Np^{v+1}}} \ ,$$

$$\lambda = i^{2k-1} 2^{-2k+2} p^{(v+1)(k-2)/2} (Np^{v+1})^{1/2} \Gamma(k-1) \ , \quad e_\xi = (1 - \frac{\xi(p)p^{k-2}}{\alpha}) \ ,$$

and

$$\langle f^0, f_0\rangle := \langle f^0, f_0\rangle_{N_f p} = \int\limits_{\Gamma_0(N_f p)\backslash \mathbb{H}} \overline{f^0}\, f_0\, y^k \frac{dxdy}{y^2}$$

is the unnormalized Petersson inner product on $M(\Gamma_0(N_f p))(\mathbb{C})$.

In the statement of the theorem we seperated the constant λ for easier comparison with Panchishkin's result, in which the constant is called T.

PROOF. By definition, $l_f \circ \mu_\xi^\alpha(\chi)$ is equal to

$$\frac{\langle f^0, \pi_{N_f}^N \circ \pi_\alpha(g h^{(p)})\rangle_{N_f p}}{\langle f^0, f_0\rangle_{N_f p}} \cdot T^{1-k},$$

where

$$g = E_{k-1}(\xi,\chi) \ , \quad h^{(p)} = E_1^{(p)}(\psi\overline{\xi\chi}) \ .$$

Set

$$h = E_1(\psi\overline{\xi\chi}) \ .$$

We have

$$h^{(p)} = (1-F)h, \quad Ug = \xi(p)p^{k-2} \ ,$$

and from this we deduce that

$$\pi_\alpha(g h^{(p)}) = (1 - \xi(p)p^{k-2}U^{-1})\pi_\alpha(gh) \ .$$

(U is invertible on the generalized α-eigenspace.) Hence,

$$\langle f^0, \pi_{N_f}^N \circ \pi_\alpha(g h^{(p)})\rangle_{N_f p}$$

$$= (1 - \frac{\xi(p)p^{k-2}}{\alpha})\ \langle f^0, \pi^N_{N_f} \circ \pi_\alpha(gh)\rangle_{N_f p} = e_\xi\ \langle f^0, \pi^N_{N_f} \circ \pi_\alpha(gh)\rangle_{N_f p}\ .$$

Furthermore,

$$\langle f^0, \pi^N_{N_f} \circ \pi_\alpha(gh)\rangle_{N_f p} = \langle f^0, \pi_\alpha(gh)\rangle_{Np}$$

$$= \langle f_0, U^{-v}\pi_\alpha U^v(gh)\rangle_{Np}$$

$$\alpha^{-v}\langle f^0, \pi_\alpha U^v(gh)\rangle_{Np} = \alpha^{-v}\langle f^0, U^v(gh)\rangle_{Np}$$

$$= \alpha^{-v}p^{v(k-1)}\langle F^v f^0, gh\rangle_{Np^{v+1}}\ .$$

Now

$$\langle F^v f^0, gh\rangle_{Np^{v+1}} = \langle (F^v f^0)_{|W_{Np^{v+1}}}, (gh)_{|W_{Np^{v+1}}}\rangle_{Np^{v+1}}$$

and

$$(F^v f^0)_{|W_{Np^{v+1}}} = p^{-vk/2}\, f^0\Big|\begin{bmatrix} p^v & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 0 & -1 \\ Np^{v+1} & 0\end{bmatrix}\ .$$

By definition of f^0 (ch.III §4) this equals

$$p^{-vk/2}\,(\bar{f}_0)_{|A}\ ,$$

where

$$A = \begin{bmatrix} 0 & -1 \\ N_f p & 0\end{bmatrix}\begin{bmatrix} p^v & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} 0 & -1 \\ Np^{v+1} & 0\end{bmatrix} = \begin{bmatrix} -Np^{v+1} & 0 \\ 0 & -N_f p^{v+1}\end{bmatrix}$$

and the last matrix acts like

$$\begin{bmatrix} -\widetilde{N} & 0 \\ 0 & -1\end{bmatrix}\ .$$

We conclude that

$$(F^v f^0)_{|W_{Np^{v+1}}} = (-1)^k p^{-vk/2}(\widetilde{N})^{k/2}\, V_{\widetilde{N}}(\bar{f}_0)\ .$$

As a result:

$$l_f \circ \mu^\alpha_\xi(\chi)\ =\ \alpha^{-v}(-1)^k p^{v(k/2-1)}(\widetilde{N})^{k/2}\,\frac{\langle V_{\widetilde{N}}(\bar{f}_0)\ ,\ g'h'\rangle_{Np^{v+1}}}{\langle f^0, f_0\rangle_{N_f p}}\ \cdot T^{1-k},$$

where

$$g' = E_{k-1}(\xi,\chi)_{|W_{Np^{v+1}}} \qquad h' = E_1(\psi\overline{\xi\chi})_{|W_{Np^{v+1}}} .$$

Now it follows by precisely the same Rankin-Selberg argument as on p.604 of [**Pan03**] that

$$\langle V_{\tilde{N}}(\bar{f}_0)\ ,\ g'h'\rangle_{Np^{v+1}} = \pi^{-k}\,\lambda\, p^{-v(k/2-1)}\, D_{Np}(k-1,f',g')\ ,$$

where

$$f' = V_{\tilde{N}} f_0\ .$$

This proves the claim. □

In order to relate $D_{Np}(k-1,f',g')$ to special values of the L-function of f, we now have to determine the Fourier coefficients of f' and g' explicitly and then use a general lemma of Shimura on Euler products.
We first turn to the Eisenstein series g'.

LEMMA IV.4.2. *g' has q-expansion*

$$(-1)^{k-1}\, N^{\frac{k-1}{2}}\, N_\xi^{-1}\, p^{-\frac{k-1}{2}(v-1)}\, G(\chi)G(\xi) \sum_{n\geq 1} q^{\frac{N\cdot p}{(N_\xi)}\, n} \sum_{d|n, d>0} \bar{\chi}(d)\,\bar{\xi}(\frac{n}{d})\, d^{k-2}\ .$$

PROOF. We prove the following: Let $l \geq 1$. For nontrivial Dirichlet characters χ,ξ of relatively prime conductors u,v which satisfy $\chi\xi(-1) = (-1)^l$, one has the formula

$$\Big(\sum_{d>0,d'>0} \xi(d)\chi(d')d^{l-1}q^{dd'}\Big)_{|W_{u\cdot v}}$$

$$= (-1)^l v^{\frac{l}{2}-1}\cdot u^{-\frac{l}{2}}\cdot G(\chi)\cdot G(\xi)\cdot \sum_{d>0,d'>0} \bar{\chi}(d)\bar{\xi}(d')d^{l-1}q^{dd'}$$

for the Atkin-Lehner involution on Eisenstein series. The claim follows from this once we specialize to $u = p^v, v = N_\xi$ and observe that

$$\begin{bmatrix} 0 & -1 \\ Np^{v+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ N_\xi\, p^v & 0 \end{bmatrix} \begin{bmatrix} \frac{N}{N_\xi}\, p & 0 \\ 0 & 1 \end{bmatrix} .$$

Let us now prove the formula in case $l \geq 3$: According to [**Kat76**, 3.2.5, 3.3.1, 3.4.1], the Eisenstein series

$$\frac{(uv)^l(l-1)!}{(-2\pi i)^l\cdot 2} \sum_{(m,n)\neq 0} \frac{\chi(m)\cdot\bar{\xi}(n)}{(mv\tau+n)^l}$$

has Fourier expansion = q-expansion at ∞ on $\Gamma_1(uv)$ (this is different from the q-expansion at ∞ on $\Gamma(uv)$ because we do not multiply τ by uv)

$$\sum_{d,d'>0} G_{uv}(\bar{\xi}, d) \cdot \chi(d') d^{l-1} q^{\frac{dd'}{u}}$$

where

$$G_{uv}(\bar{\xi}, d) = \sum_{t=0}^{uv-1} \bar{\xi}(t) \exp(\frac{2\pi i}{uv} td) .$$

We claim that

$$G_{uv}(\bar{\xi}, d) = \begin{cases} u \cdot G(\bar{\xi}) \cdot \xi(\frac{d}{u}) & u \mid d \\ 0 & u \nmid d \end{cases}$$

Indeed, if $u \nmid d$ we have

$$\sum_{t=0}^{uv-1} \bar{\xi}(t) \exp(\frac{2\pi i}{uv} td) = \sum_{a(v)} \bar{\xi}(a) \sum_{\substack{t(uv): \\ t \equiv a(v)}} \exp(\frac{2\pi i}{uv} td)$$

and

$$\sum_{\substack{t(uv): \\ t \equiv a(v)}} \exp(\frac{2\pi i}{uv} td) = \exp(\frac{2\pi i}{uv} a'd) \cdot \sum_{\substack{t(uv): \\ t \equiv 0(v)}} \exp(\frac{2\pi i}{uv} td)$$

$$= \exp(\frac{2\pi i}{uv} a'd) \cdot \sum_{t=0}^{u-1} \exp(\frac{2\pi i}{uv} tvd) = 0 \quad (a' \text{ is any lift of } a \bmod uv \text{) .}$$

If $u \mid d$,

$$G_{uv}(\bar{\xi}, d) = G_{uv}(\bar{\xi}, \frac{d}{u} \cdot u) = u \cdot G_v(\bar{\xi}, \frac{d}{u}) = u \cdot G(\bar{\xi}) \cdot \xi(\frac{d}{u}) .$$

We conclude that

$$\frac{(uv)^l (l-1)!}{(-2\pi i)^l \cdot 2} \sum_{(m,n) \neq 0} \frac{\chi(m) \cdot \bar{\xi}(n)}{(mv\tau + n)^l} = u \cdot G(\bar{\xi}) \cdot \sum_{\substack{d,d'>0 \\ u|d}} \xi(\frac{d}{u}) \cdot \chi(d') d^{l-1} q^{\frac{dd'}{u}}$$

$$= u^l G(\bar{\xi}) \cdot \sum_{d,d'>0} \xi(d) \cdot \chi(d') d^{l-1} q^{dd'} .$$

Hence,

$$\frac{v^l (l-1)!}{(-2\pi i)^l G(\bar{\xi}) \cdot 2} \sum_{(m,n) \neq 0} \frac{\chi(m) \cdot \bar{\xi}(n)}{(mv\tau + n)^l} = \sum_{d>0, d'>0} \xi(d) \chi(d') d^{l-1} q^{dd'} .$$

We now compute

$$\sum_{(m,n) \neq 0} \frac{\chi(m) \cdot \bar{\xi}(n)}{(mv\tau + n)^l} \Big| \begin{bmatrix} 0 & -1 \\ uv & 0 \end{bmatrix} = (uv)^{l/2} (uv\tau)^{-l} \sum_{(m,n) \neq 0} \frac{\chi(m) \cdot \bar{\xi}(n)}{(mv(-\frac{1}{uv\tau}) + n)^l}$$

$$= (uv)^{l/2} \sum_{(m,n)\neq 0} \frac{\chi(m)\cdot\bar\xi(n)}{(-mv+uvn\tau)^l} = (uv)^{l/2} v^{-l}\cdot\chi(-1) \sum_{(m,n)\neq 0} \frac{\chi(m)\cdot\bar\xi(n)}{(un\tau+m)^l}\ .$$

By the same computation as above,

$$\frac{u^l(l-1)!}{(-2\pi i)^l\cdot 2\cdot G(\chi)}\cdot \sum_{(m,n)\neq 0} \frac{\chi(m)\cdot\bar\xi(n)}{(un\tau+m)^l} = \sum_{d,d'>0} \bar\chi(d)\cdot\bar\xi(d')d^{l-1}q^{dd'}$$

and the claimed formula follows from this and the identities

$$G(\xi)\cdot\overline{G(\xi)} = v, \qquad \overline{G(\xi)} = \xi(-1)G(\bar\xi)\ .$$

In case $l = 1, 2$ the same argument works: The Eisenstein series with the right Fourier expansion is the limit of the function defined by the real analytic series

$$\sum_{(m,n)\neq 0} \frac{\chi(m)\cdot\bar\xi(n)}{(mv\tau+n)^l|mv\tau+n|^{2s}}$$

as $s\to 0$. The action of the Atkin-Lehner involution can be computed on this series for $\mathrm{Re}(s)$ sufficiently large to make the sum converge absolutely, and one obtains the claimed identity by letting s approach zero. □

We move on to the promised lemma on Euler products, see [**Pan03**, p.607f], [**Shi76**, Lemma 1].

Lemma IV.4.3. *Consider two formal Dirichlet series with Euler products*

$$F = \sum_{n\geq 1} A(n)n^{-s} = \prod_q \left[(1-\alpha_q q^{-s})(1-\alpha'_q q^{-s}\right]^{-1},$$

$$G = \sum_{n\geq 1} B(n)n^{-s} = \prod_q \left[(1-\beta_q q^{-s})(1-\beta'_q q^{-s}\right]^{-1},$$

and let $M(F), M(G)$ two positive integers. If we put

$$D = \gcd(M(F), M(G))\ , M'(F) = M(F)/D\ , M'(G) = M(G)/D$$

$$t = \mathrm{ord}_q(M'(G))\ , t' = \mathrm{ord}_q(M'(F))\ ,$$

then the Euler product of

$$\sum_{n\geq 1} A\Big(\frac{n}{M(F)}\Big)\, B\Big(\frac{n}{M(G)}\Big)\, n^{-s}\ \ (\textit{If } d\nmid n,\ A(\frac{n}{d}) := 0)$$

is given by

$$(M(F)M(G)/D)^{-s} \prod_q X^*_q(s)/Y_q(s)\ ,$$

where

$$X_q^*(s) = \begin{cases} 1 - \alpha_q\alpha_q'\beta_q\beta_q' q^{-2s}, & q \nmid M'(F)M'(G), \\ \\ A(q) - B(q)\alpha_q\alpha_q' q^{-s} & t = \mathrm{ord}_q(M'(G)) = 1, \\ \\ A(q^t) - A(q^{t-1})B(q)\alpha_q\alpha_q' q^{-s} & \\ +A(q^{t-2})(\alpha_q\alpha_q')^2\beta_q\beta_q' q^{-2s} & q^2 \mid M'(G), \\ \\ B(q) - A(q)\beta_q\beta_q' q^{-s} & t' = \mathrm{ord}_q(M'(F)) = 1, \\ \\ B(q^{t'}) - B(q^{t'-1})A(q)\beta_q\beta_q' q^{-s} & \\ +B(q^{t'-2})(\beta_q\beta_q')^2\alpha_q\alpha_q' q^{-2s}\,, & q^2 \mid M'(F)\,, \end{cases}$$

$$Y_q(s) = (1 - \alpha_q\beta_q q^{-s})(1 - \alpha_q\beta_q' q^{-s})(1 - \alpha_q'\beta q^{-s})(1 - \alpha_q'\beta_q' q^{-s})\,.$$

Let

$$L(s, f'g') := \sum_{n\geq 1} a'(n)b'(n)n^{-s}\,.$$

Then

$$D_{Np}(s, f'g') = L_{Np}(2s + 2 - k - l, \psi\overline{\xi\chi})\, L(s, f'g')$$

and

$$L(s, f'g') = \sum_{n\geq 1} a\Big(\frac{n}{\widetilde{N}}\Big)b'(n)n^{-s} - \beta \sum_{n\geq 1} a\Big(\frac{n}{\widetilde{N}p}\Big)b'(n)n^{-s}\,.$$

If we set

$$B(n) := \sum_{d|n, d>0} \bar{\chi}(d)\bar{\xi}(\frac{n}{d})\, d^{k-2}\,,$$

we can write

$$g'(q) = G_v \sum_{n\geq 1} B(n)\,(npN/N_\xi)^{-s} = G_v \sum_{n\geq 1} B\Big(\frac{n}{pN/N_\xi}\Big)\, n^{-s}\,,$$

where

$$G_v = (-1)^{k-1}\, N^{\frac{k-1}{2}}\, N_\xi^{-1}\, p^{\frac{k-1}{2}(v-1)}\, G(\chi)G(\xi)\,.$$

We therefore have to compute the Euler product of

$$R_1 := \sum_{n\geq 1} a\Big(\frac{n}{\widetilde{N}}\Big)B\Big(\frac{n}{pN/N_\xi}\Big)n^{-s}$$

and of

$$R_2 := \sum_{n\geq 1} a\Big(\frac{n}{\widetilde{N}p}\Big)B\Big(\frac{n}{pN/N_\xi}\Big)n^{-s}\,.$$

In the case of R_1, we set

$$F = f, G = \sum_{n\geq 1} B(n)n^{-s}, M(F) = \widetilde{N}(= N/N_f), M(G) = pN/N_\xi$$

in the above lemma, which implies

$$D = 1, M'(F) = \widetilde{N}, M'(G) = pN/N_\xi, \quad \text{(remember } N = \mathrm{lcm}(N_f, N_\xi))$$

In the case of R_2, we set

$$F = f,\ G = \sum_{n\geq 1} B(n)n^{-s},\ M(F) = p\widetilde{N},\ M(G) = pN/N_\xi$$

in the above lemma, which implies

$$D = p,\ M'(F) = \widetilde{N},\ M'(G) = N/N_\xi.$$

Therefore, according to the lemma, both Dirichlet series have the same Euler factor

$$X_q^*(s)/Y_q(s)$$

for $q \neq p$. For $q = p$, denote the respective Euler factors by

$$X_{p,1}^*(s)/Y_p(s),\ X_{p,2}^*(s)/Y_p(s)\ .$$

Plugging into the formulas of the lemma, one has

$$X_{p,1}^*(s) = a_p - \psi\bar{\xi}(p)p^{k-1-s}\ ,$$

$$X_{p,2}^*(s) = 1\ .$$

We conclude that

$$L(s, f'g') = G_v\ (\widetilde{N}\frac{Np}{N_\xi})^{-s} \left(\prod_q 1/Y_q(s)\right) \left(\prod_{q\neq p} X_q^*(s)\right) (a_p - \psi\bar{\xi}(p)p^{k-1-s} - \beta)$$

$$= G_v\ (\widetilde{N}\frac{Np}{N_\xi})^{-s} \left(\prod_q 1/Y_q(s)\right) \left(\prod_{q\neq p} X_q^*(s)\right) (\alpha - \psi\bar{\xi}(p)p^{k-1-s})\ .$$

Furthermore we have

$$\prod_{q|\widetilde{N}} X_q^*(s) = \bar{\chi}(\widetilde{N})(\widetilde{N})^{k-2}\ ,$$

and

$$\prod_{q\nmid Np} X_q^*(s) = \prod_{q\nmid Np}(1-\psi\overline{\xi\chi}(q)q^{2k-3-2s})$$

$$= L_{Np}(2s-2k+3,\psi\overline{\xi\chi})^{-1}\ .$$

Finally,

$$\prod_q 1/Y_q(s) = L(s-k+2,f,\bar\chi)\cdot L(s,f,\bar\xi)\ .$$

Collecting the above and specializing at $s=k-1$, we have:

Proposition IV.4.4.

$$D_{Np}(k-1,f',g') =$$

$$G_v\,(\widetilde N\frac{Np}{N_\xi})^{1-k}\ \bar\chi(\widetilde N)(\widetilde N)^{-1}\,(\alpha-\psi\bar\xi(p))\Big(\prod_{q|N/N_\xi} X_q^*(k-1)\Big)\ L(1,f,\bar\chi)\cdot L(k-1,f,\bar\xi)\ .$$

Together with Prop. 4.1, this implies the main result of this chapter:

Theorem IV.4.5. *Let χ be a Dirichlet character with conductor p^v, $v\geq 1$ and such that $\chi\xi(-1)=(-1)^{k-1}$.*

We have the equality

$$l_f\circ\mu_\xi^\alpha(\chi)\ =$$

$$= R\left(1-\frac{\xi(p)p^{k-2}}{\alpha}\right)\left(1-\frac{\psi\bar\xi(p)}{\alpha}\right)\bar\chi(\widetilde N)\,G(\chi)\,\alpha^{-v}\cdot L(1,f,\bar\chi)\cdot\left(\frac{L(k-1,f,\bar\xi)\,\Gamma(k-1)}{(-2\pi i)^k\cdot G(\bar\xi)\cdot i^{1-k}\cdot\langle f^0,f_0\rangle}\right)\cdot T^{1-k},$$

where

$$R=\alpha\,\xi(-1)2^{-k+2}\,N^{-k+2}\,(N_f)^{\frac{k}{2}-1}\,(N_\xi)^{k-1}\Big(\prod_{q|N/N_\xi} X_q^*(k-1)\Big)$$

is an algebraic constant which does not depend on χ. Here, $\widetilde N = N/N_f$ and

$$X_q^*(k-1)=\begin{cases} a_q-(\bar\chi(q)q^{k-2}+\bar\xi(q))\psi(q) & t:=\mathrm{ord}_q(N/N_\xi)=1.\\ \\ a_{q^t}-a_{q^{t-1}}(\bar\chi(q)q^{k-2}+\bar\xi(q))\psi(q) & \\ +a_{q^{t-2}}(\psi(q)q^{k-1})^2\,\overline{\chi\xi}(q)q^{-1} & q^2\mid N/N_\xi\end{cases}$$

Consider the nonvanishing condition

$$L(k-1, f, \bar{\xi}) \neq 0$$

on ξ. This condition is automatic for $k \geq 3$ [**Shi76**, top of p.800]. For $k = 2$, such ξ (nontrivial, with $(N_\xi, p) = 1$) always exist due to a result of Shimura [**Shi77**, Thm.2 and Rmk. on p.213]. If the condition $L(k-1, f, \bar{\xi}) \neq 0$ is satisfied, the theorem says that $l_f \circ \mu_\xi^\alpha$ is "almost" a constant multiple of the p-adic L-function, or rather its plus or minus part depending on ξ. To make this statement more precise, we recall the exact definition of the p-adic L-function attached to f, α and the choice of a complex period. We content ourselves with evaluating the p-adic L-function at characters of p-power conductor.

The p-adic L-function of f. Fix a sign $\delta \in \{\pm 1\}$. By work of Manin, there exists a nonzero complex number Ω_∞^δ, (unique up to multiplication by an algebraic number) such that

$$L(1, f, \chi)/\Omega_\infty^\delta$$

is algebraic for any primitive Dirichlet character χ mod p^v, $v \geq 1$, which satisfies $\chi(-1) = (-1)^k \delta$. One even has a unique measure

$$\mu(f, \alpha, \Omega_\infty^\delta) : \mathscr{C}(\mathbb{Z}_p^\times, \mathbb{C}_p) \to \mathbb{C}_p$$

which has the property that for a primitive Dirichlet character χ mod p^v, $v \geq 1$, one has

$$\mu(f, \alpha, \Omega_\infty^\delta)(\chi) = \begin{cases} G(\chi)\, \alpha^{-v}\, L(1, f, \bar{\chi})/\Omega_\infty^\delta & \chi(-1) = (-1)^k \delta \\ 0 & \chi(-1) = (-1)^{k+1}\delta \end{cases} .$$

Compare with [**Col00**, Prop.1.15], [**Kat04**, Rmk.16.3(1)] .

Definition IV.4.6. Let

$$\epsilon : \mathbb{Z}_p^\times \to \mathbb{C}_p$$

be any continuous character (i.e. of possibly infinite conductor) and let $\epsilon(-1) = (-1)^k \delta$ for some $\delta \in \pm 1$. Choose an Ω_∞^δ as above. Then the p-adic L-function of f evaluated at ϵ is defined as

$$L_{(p)}(\epsilon, f, \alpha, \Omega_\infty^\delta) := \mu(f, \alpha, \Omega_\infty^\delta)(\epsilon) .$$

Note that the function

$$L_{(p)}(\cdot, f, \alpha, \Omega_\infty^\delta) : \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \to \mathbb{C}_p$$

is not the whole p-adic L-function, but only the δ-part of it. One could choose two periods $\Omega_\infty = (\Omega_\infty^+, \Omega_\infty^-)$ of different parity and define

$$L_{(p)}(\cdot, f, \alpha, \Omega_\infty) := L_{(p)}(\cdot, f, \alpha, \Omega_\infty^+) + L_{(p)}(\cdot, f, \alpha, \Omega_\infty^-) \ .$$

As mentioned in the introduction to this section we will not pursue this, because we think that in the formulation of our main theorem, Cor.V.2.3, it is more natural to fix one parity.

We want to use the above definition of the p-adic L-function in order to reformulate the theorem. For this, choose ξ such that $L(k-1, f, \bar{\xi}) \neq 0$ and define

$$\Omega_\xi := \left(\frac{L(k-1, f, \bar{\xi})\, \Gamma(k-1)}{(-2\pi i)^k \cdot G(\bar{\xi}) \cdot i^{1-k} \cdot \langle f^0, f_0 \rangle} \right)^{-1} .$$

(It is a plus or minus period depending on $\xi(-1) = -1$ or $\xi(-1) = 1$.) Denote the extension of $l_f \circ \mu_\xi^\alpha$ to a $\mathbb{C}_p$-valued measure by the same symbol. Then with the above notation we have:

Corollary IV.4.7. *Let*

$$\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p, \quad \chi\xi(-1) = (-1)^{k-1}$$

be a (continuous) character. Then

$$l_f \circ \mu_\xi^\alpha(\chi) \ = \ R \left(1 - \frac{\xi(p) p^{k-2}}{\alpha} \right) \left(1 - \frac{\psi\bar{\xi}(p)}{\alpha} \right) \bar{\chi}(\widetilde{N})\, L_{(p)}(\chi, f, \alpha, \Omega_\xi) \ \cdot T^{1-k}.$$

In other words,

$$l_f \circ \mu_\xi^\alpha = \ R \left(1 - \frac{\xi(p) p^{k-2}}{\alpha} \right) \left(1 - \frac{\psi\bar{\xi}(p)}{\alpha} \right) \left((\widetilde{N})^{-(\cdot)} \star \mu(f, \alpha, \Omega_\xi) \right) \ \cdot T^{1-k},$$

where $(\widetilde{N})^{-(\cdot)}$ denotes the measure which is given on characters by

$$\chi \mapsto \bar{\chi}(\widetilde{N}) \ .$$

(Note that $\widetilde{N}$ is prime to p.) Here, the symbol "$\star$" denotes the convolution of measures.

CHAPTER V

The main theorem

In this chapter we prove our main theorems (Cor.V.2.3, Thm.0.1) which relate syntomic Eisenstein classes to the values of the p-adic L-function of f at noncritical integers. The connection will be established using the measure μ_ξ^α. We already know from the previous chapter that μ_ξ^α is essentially the p-adic L-function after composing with a suitable linear form coming from f. On the other hand, for $l \geq 0$, $\mu_\xi^\alpha(y^{-l-1})$ (here y is the identity on $\mathbb{Z}_p^\times$) is a product of two p-adic Eisenstein series. This, together with the comparison of the two linear forms $l_f, l_{f,\mathrm{rig}}$ will prove Thm. V.2.1, a p-adic Rankin-Selberg method. The product equals one of the two terms appearing in the computation of the product of two Eisenstein classes. It will turn out to also be equal to the other term up to a constant. This proves Corollary V.2.3 and Theorem 0.1.

V.1. Euler factors and the α-projection

In the previous chapters, we already have used a variety of different (p-adic) Eisenstein series

$$E_{k,r},\ E_{k,r}^{(p)},\ {}^{(p)}E_{k,r},\ {}^{(p)}E_{k,r}^{(p)}$$

where the left (right) superscript (p) refers to the divisibility condition $p \nmid d$ ($p \nmid \frac{n}{d}$) in the q-coefficients of the Eisenstein series. The next lemma says that when we consider π_α of certain products of these Eisenstein series, removing or adding a superscript corresponds to adding or removing an Euler factor.

Lemma V.1.1. *Let ψ, ξ be Dirichlet characters* $\bmod N$ *with ξ nontrivial and let l, $k \geq 2$ be integers satisfying $\xi(-1) = (-1)^{k+l}$. Let α denote a nonzero eigenvalue of the U-operator and write $1/U$ for the inverse of U on the generalized α-eigenspace. Then*
a)

$$\pi_\alpha\left[{}^{(p)}E_{k-1,-l-1}^{(p)}(\xi) \cdot {}^{(p)}E_{l+2,0}^{(p)}(\psi\bar{\xi})\right]$$

$$= \left(1 - \frac{\xi(p)p^{k-2}}{U}\right)\left(1 - \frac{\psi(p)p^{k+l-1}}{U}\right)\pi_\alpha\left[E_{k-1,-l-1}^{(p)}(\xi) \cdot E_{l+2,0}(\psi\bar{\xi})\right].$$

b)

$$\pi_\alpha\Big[{}^{(p)}E^{(p)}_{k+l,0}(\xi)\cdot{}^{(p)}E^{(p)}_{1,-l-1}(\psi\bar{\xi})\Big]$$

$$=\Big(1-\frac{\psi\bar{\xi}(p)}{U}\Big)\Big(1-\frac{\psi(p)p^{k+l-1}}{U}\Big)\;\pi_\alpha\Big[E_{k+l,0}(\xi)\cdot E^{(p)}_{1,-l-1}(\psi\bar{\xi})\Big]\;.$$

PROOF. a) Plugging the character y^{-l-1} into the equality of measures (see Observation IV.3.3)

$$\mu^\alpha_\xi=\nu^\alpha_\xi$$

we conclude that

$$\pi_\alpha\Big[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi)\cdot{}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})\Big]=\pi_\alpha\Big[E^{(p)}_{k-1,-l-1}(\xi)\cdot{}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})\Big]\;.$$

Now let

$$g:=E^{(p)}_{k-1,-l-1}(\xi),\quad h:=E_{l+2,0}(\psi\bar{\xi}),$$

$${}^{(p)}h:=(1-\psi\bar{\xi}(p)p^{l+1}F)h,\quad {}^{(p)}h^{(p)}:=(1-F)\,{}^{(p)}h\;.$$

With this notation we have to show that

$$\pi_\alpha\,[\;g\;\big({}^{(p)}h^{(p)}\big)\;]=\Big(1-\frac{\xi(p)p^{k-2}}{U}\Big)\Big(1-\frac{\psi(p)p^{k+l-1}}{U}\Big)\,\pi_\alpha\,[\;g\,h\;]\;.$$

Note that

$$Ug=\xi(p)p^{k-2}g\;.$$

We compute

$$\pi_\alpha\,[\;g\;\big({}^{(p)}h^{(p)}\big)\;]=\pi_\alpha\,[\;g\;(1-F)\big({}^{(p)}h\big)\;]$$

$$=\pi_\alpha\,[\;g\;\big({}^{(p)}h\big)\;]-U^{-1}\pi_\alpha\,U\,[g\;F\big({}^{(p)}h\big)\;]$$

$$=\Big(1-\frac{\xi(p)p^{k-2}}{U}\Big)\,[\;g\;\big({}^{(p)}h\big)\;]\;,$$

using the "projection formula"

$$U((Ff_1)\,f_2)=f_1\,(Uf_2)\;.$$

Repeating the argument, we pull out the second factor. The proof of b) proceeds the same way once one notes that

$$UE^{(p)}_{1,-l-1}(\psi\bar{\xi}) = \psi\bar{\xi}(p)\, E^{(p)}_{1,-l-1}(\psi\bar{\xi}) \,.$$

□

We will use the above lemma to relate one of the two terms appearing in the formula for the cup product of two Eisenstein classes to the measure μ^{α}_{ξ}. The following proposition will be used to relate the two terms to each other.

PROPOSITION V.1.2. *With notation as above, we have the equality*

$$\pi_\alpha\left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi)\cdot{}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})\right] = (-1)^{l+1}\,\pi_\alpha\left[{}^{(p)}E^{(p)}_{k+l,0}(\xi)\cdot{}^{(p)}E^{(p)}_{1,-l-1}(\psi\bar{\xi})\right] .$$

PROOF. Set

$$g_1 := {}^{(p)}E^{(p)}_{k-1,-l-1}(\xi)\cdot{}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi}),\quad g_2 := {}^{(p)}E^{(p)}_{k+l,0}(\xi)\cdot{}^{(p)}E^{(p)}_{1,-l-1}(\psi\bar{\xi}) \,.$$

The q-expansions of these elements are given by

$$g_1(q) = \sum_{n\geq 1} q^n \sum_{\substack{n_1+n_2=n\\ p\nmid n_1,\, p\nmid n_2}} \Big(\sum_{\substack{d_1|n_1\\ d_1>0}} \xi(d_1)\, d_1^{k-2}(\frac{n_1}{d_1})^{-l-1}\Big)\cdot\Big(\sum_{\substack{d_2|n_2\\ d_2>0}} \psi\bar{\xi}(d_2)\, d_2^{l+1}\Big) \;,$$

$$g_2(q) = \sum_{n\geq 1} q^n \sum_{\substack{n_1+n_2=n\\ p\nmid n_1,\, p\nmid n_2}} \Big(\sum_{\substack{d_1|n_1\\ d_1>0}} \xi(d_1)\, d_1^{k+l-1}\Big)\cdot\Big(\sum_{\substack{d_2|n_2\\ d_2>0}} \psi\bar{\xi}(d_2)\,(\frac{n_2}{d_2})^{-l-1}\Big) \;.$$

Now for $v \geq 0$ we have

$$U^v g_1(q) = \sum_{n\geq 1} q^n \sum_{\substack{n_1+n_2=p^v n\\ p\nmid n_1,\, p\nmid n_2}} \Big(\sum_{\substack{d_1|n_1\\ d_1>0}} \xi(d_1)\, d_1^{k-2}(\frac{n_1}{d_1})^{-l-1}\Big)\cdot\Big(\sum_{\substack{d_2|n_2\\ d_2>0}} \psi\bar{\xi}(d_2)\, d_2^{l+1}\Big)$$

$$= \sum_{n\geq 1} q^n \sum_{\substack{n_1+n_2=p^v n\\ p\nmid n_1,\, p\nmid n_2}} (\frac{n_2}{n_1})^{l+1}\Big(\sum_{\substack{d_1|n_1\\ d_1>0}} \xi(d_1)\, d_1^{k+l-1}\Big)\cdot\Big(\sum_{\substack{d_2|n_2\\ d_2>0}} \psi\bar{\xi}(d_2)\,(\frac{n_2}{d_2})^{-l-1}\Big) \,.$$

Because of $p \nmid n_1$ and $p \nmid n_2$,

$$\Big(\frac{n_2}{n_1}\Big)^{l+1} = \Big(\frac{-n_1+np^v}{n_1}\Big)^{l+1} = \Big(\frac{-n_2+np^v}{n_2}\Big)^{-l-1} \equiv (-1)^{l+1} \bmod p^v \,,$$

and hence

$$U^v(g_1)(q) \equiv (-1)^{l+1}\, U^v(g_2)(q) \bmod p^v.$$

Let m be an arbitrary positve integer. As π_α is continuous in the q-expansion topology, there exists a v for which

$$\pi_\alpha(U^v(g_1)) \equiv (-1)^{l+1}\, \pi_\alpha(U^v(g_2)) \bmod p^m \ ,$$

thus

$$U^v\pi_\alpha(g_1) \equiv (-1)^{l+1}\, U^v\pi_\alpha(g_2) \bmod p^m \ .$$

U is invertible on the generalized α-eigenspace, but need not be semisimple. It has a decomposition

$$U = \alpha + N$$

where N is nilpotent, say $N^{d+1} = 0$. Furthermore, N is p-integral because U is.
From

$$(\alpha+N)^{-1} = \alpha^{-1}(1+\frac{N}{\alpha})^{-1} = \alpha^{-1}\cdot\Big(1-\Big(\frac{N}{\alpha}\Big)+\cdots+(-1)^d\Big(\frac{N}{\alpha}\Big)^d\Big)$$

and the fact that α is a p-adic unit we conclude that U^{-1} is p-integral. This implies

$$\pi_\alpha(g_1) \equiv (-1)^{l+1}\, \pi_\alpha(g_2) \bmod p^m$$

for arbitrary m and thus both sides are equal. □

Set

$$M^\dagger[Q]_m := M^\dagger(\Gamma_1(N),K)[Q]_m \ ,$$

see II.6 for a definition of this space, which is a subspace of $V'_m(\Gamma_1(N),K)$ and which is stable under the U-operator [**CGJ95**, Cor. 9]. The U-operator also acts on the cohomology vector spaces

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)),\mathscr{L})\otimes K\ ,\ \mathscr{L} = \mathrm{Sym}^{k-2}\mathscr{H}^\vee\ ,$$

by writing

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)),\mathscr{L})\otimes K$$

as the quotient of spaces of modular forms (see II.6)

$$M^\dagger[Q]_k^{\leq k-2}/\theta M^\dagger[Q]_{k-2}^{\leq k-2}$$

and letting the action of U descend to the quotient. By this definition of the action of U on cohomology, the natural map

$$M^\dagger[Q]_k^{\leq k-2} \to H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)), \mathscr{L})$$

is tautologically U-equivariant. Because of the formulae $UF = \mathrm{id}$ and $\Phi = p^{k-1}F$, U induces the Verschiebung on the level of cohomology. From the U-equvariance, we deduce:

Lemma V.1.3. *Let $g \in M^\dagger[Q]_k^{\leq k-2}$ and let $[g]$ denote its cohomology class in $H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)), \mathscr{L}) \otimes K$. Furthermore, let*

$$\pi_\alpha : M^\dagger[Q]_k^{\leq k-2} \to \left(M^\dagger[Q]_k^{\leq k-2}\right)^\alpha,$$

$$\pi'_\alpha : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)), \mathscr{L}) \otimes K \to \left(H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(\Gamma_1(N)), \mathscr{L}) \otimes K\right)^\alpha$$

be the projections on the generalized α-eigenspaces for U. Then

$$\pi'_\alpha[g] \;=\; [\pi_\alpha g]\ .$$

The lemma gives the justification for using the symbol π_α instead of π'_α.

V.2. Proof of the main theorem

Before stating the first theorem, we recall the setup:

We let

$$f = \sum_{n\geq 1} a_n\, q^n$$

be a Hecke eigenform which has weight $k \geq 2$, is primitive of level $\Gamma_0(N_f)$, $N_f \geq 4$, $p \nmid N_f$ and has character ψ. We furthermore require f to be p-ordinary with respect to our chosen embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, which means that there exists a root α of the p-th Hecke polynomial

$$X^2 - a_pX + \psi(p)p^{k-1} \;=\; (X-\alpha)(X-\beta)$$

which is a p-adic unit. Finally, ξ is a nontrivial Dirichlet character of conductor N_ξ prime to p and $N := \mathrm{lcm}(N_f, N_\xi)$. In the following we always require that $L(k-1, f, \bar{\xi}) \neq 0$ is satisfied and as discussed in the previous chapter, such ξ exist. As in the previous chapter, $K = \mathbb{Q}_p((a_n)_n, \alpha, \xi)$. Set $\widetilde{N} := \frac{N}{N_f}$. Recall from chapter III that f defines a cohomology class

$$\omega_{\bar{f},\alpha} = \Phi\,\omega_{\bar{f}} - \beta\,\omega_{\bar{f}} \in H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \bar{\psi}), \mathscr{L}) \otimes K$$

and that the rigid cup product induces a nondegenerate pairing $(\cdot,\cdot)_{\mathrm{rig}}$:

$$H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \bar{\psi}), \mathscr{L}) \otimes K \times H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \psi), \mathscr{L}) \otimes K \to K(1-k)\ .$$

Also, recall from chapter IV the p-adic L-function

$$L_{(p)}(\chi, f, \alpha, \Omega^{\delta}_{\infty})$$

evaluated at a character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of parity $\chi(-1) = (-1)^k\delta$. If y denotes the inclusion

$$\mathbb{Z}_p^\times \to \mathbb{C}_p^\times,$$

we set

$$L_{(p)}(m, f, \alpha, \Omega^{\delta}_{\infty}) := L_{(p)}(y^{m-1}, f, \alpha, \Omega^{\delta}_{\infty}) \in K\ .$$

for an integer m of parity $(-1)^{m-1} = (-1)^k\,\delta$. The shift by one corresponds to the fact that in the definition of the p-adic L-function we interpolate at the critical value $s = 1$.

For any $l \in \mathbb{Z}$ with $\xi(-1) = (-1)^{k+l}$ consider the cohomology class

$$\left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})\right]$$

in

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N, \psi), \mathscr{L}) \otimes K\ .$$

Applying the projection

$$\pi^N_{N_f} : H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N, \psi), \mathscr{L}) \otimes K \to H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \psi), \mathscr{L}) \otimes K$$

(the adjoint for $(\cdot,\cdot)_{\mathrm{rig}}$ to the natural map

$$H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \bar{\psi}), \mathscr{L}) \otimes K \to H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N, \bar{\psi}), \mathscr{L}) \otimes K\ ),$$

we get the class

$$\pi^N_{N_f}\left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar{\xi})\right]$$

of level $\Gamma_0(N_f)$ and character ψ.

THEOREM V.2.1. *Let $l \in \mathbb{Z}$ satisfy $\xi(-1) = (-1)^{k+l}$. The element*

$$\Big(\omega_{\bar f,\alpha} \, , \, \pi^N_{N_f} \left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi) \right] \Big)_{\rm rig} \in K(1-k)$$

is equal to

$$A \cdot \mathrm{Pet}_p \cdot L_{(p)}(-l, f, \alpha, \Omega_\xi) \cdot T^{1-k} \quad ,$$

where A is the algebraic number

$$\Big(1 - \frac{\beta}{\alpha}\Big) \Big(1 - \frac{\xi(p)p^{k-2}}{\alpha}\Big) \Big(1 - \frac{\psi\bar\xi(p)}{\alpha}\Big) (\widetilde N)^{l+1} \cdot R,$$

$$R = \alpha\, \xi(-1) 2^{-k+2}\, N^{-k+2}\, (N_f)^{\frac{k}{2}-1}\, (N_\xi)^{k-1} \Big(\prod_{q|N/N_\xi} X^*_q(k-1) \Big),$$

and Pet_p *is the nonzero p-adic number*

$$(\Phi\, \omega_{\bar f}, \omega_f)_{\rm rig} \cdot T^{k-1} \in K \ .$$

Note that T^{1-k} can be thought of as a number inside $\mathbb{C}_p[T, T^{-1}]$ instead of merely a vector in $K(1-k)$. From this viewpoint, $t := T^{-1}$ is the p-adic analogue of $2\pi i$ and the number $(\Phi\, \omega_{\bar f}, \omega_f)_{\rm rig}\, T^{1-k}$ is the p-adic analogue of the complex number $\langle f, f\rangle\, (2\pi i)^{k-1}$, which appears in the complex period attached to an eigenform.

PROOF. Because of

$$\Phi\, \omega_{\bar f,\alpha} = \alpha\, \omega_{\bar f,\alpha} \ ,$$

and the fact that U and Φ are adjoint with respect to $(\cdot,\cdot)_{\rm rig}$, one has

$$\Big(\omega_{\bar f,\alpha} \, , \, \pi^N_{N_f} \left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi) \right] \Big)_{\rm rig}$$

$$= \Big(\omega_{\bar f,\alpha} \, , \, \pi_\alpha \circ \pi^N_{N_f} \left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi) \right] \Big)_{\rm rig}$$

$$= \Big(\omega_{\bar f,\alpha} \, , \, \pi^N_{N_f} \left[\pi_\alpha \left({}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi) \right) \right] \Big)_{\rm rig} \ .$$

By §2 of chapter IV and the definition of $l_{f,\text{rig}}$, this is equal to

$$l_{f,\text{rig}}(\mu_\xi^\alpha(y^{-l-1}))\ .$$

From chapter III, §5 we know that

$$l_{f,\text{rig}} = d_p \cdot l_f\ , \quad d_p = \Big(1 - \frac{\beta}{\alpha}\Big)\text{Pet}_p\ .$$

Therefore,

$$\Big(\ \omega_{\bar f,\alpha}\ ,\ \pi^N_{N_f}\left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi)\cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi)\right]\ \Big)_{\text{rig}} = d_p \cdot l_f(\mu_\xi^\alpha(y^{-l-1}))$$

and the theorem follows after applying Corollary 4.7. of chapter IV. □

Note that if $\prod_{q|N/N_\xi} X_q^*(k-1) \neq 0$, the number A is nonzero. This is of course the case if $N_f \mid N_\xi$ because then $N = N_\xi$. The theorem can be thought of as a p-adic Rankin-Selberg method: Usually the term p-adic Rankin-Selberg convolution is used for p-adic functions that interpolate special values of classical complex Rankin-Selberg convolutions. In this sense,

$$l_{f,\text{rig}} \circ \mu_\alpha^\xi = \frac{1}{d_p}\, l_f \circ \mu_\alpha^\xi : \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \to \mathbb{C}_p$$

is a p-adic R.-S. convolution as was shown in the previous chapter. The content of the theorem is that if we replace the de Rham pairing by a p-adic (i.e. rigid-analytic) pairing, the interpretation as a Rankin-Selberg integral in which a cusp form is paired with the product of two Eisenstein series extends to integers were there is *no* interpolation.
Another way of rephrasing the theorem is to say that a "universal p-adic L-function" for ordinary eigenforms is given by the product of two Eisenstein series. Proposition V.1.2 leads to the following observation which we find interesting but which will not be used in the sequel.

Observation V.2.2. In the situation of Theorem V.2.1., one can replace

$${}^{(p)}E^{(p)}_{k-1,-l-1}(\xi)\cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi)$$

by

$$\pm\,{}^{(p)}E^{(p)}_{k+l,0}(\xi)\cdot {}^{(p)}E^{(p)}_{1,-l-1}(\psi\bar\xi)$$

which is overconvergent.

Theorem 2.1 allows us to relate the product of syntomic Eisenstein classes to the p-adic L-function. For nonnegative l recall the class

$$E^{k+l}_{\mathrm{syn}}(\xi) \cup_l E^{l+2}_{\mathrm{syn}}(\psi\bar{\xi})$$

in

$$H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \psi), \mathscr{L}(k+l)) \otimes K$$

that was defined in chapter II and note that $(\cdot,\cdot)_{\mathrm{rig}}$ induces a pairing

$$H^1_{c,\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \bar{\psi}), \mathscr{L}) \otimes K \times H^1_{\mathrm{rig}}(\mathscr{M}^{\mathrm{ord}}(N_f, \psi, \mathscr{L}(k+l)) \otimes K \to K(l+1) \ .$$

which we also denote by $(\cdot,\cdot)_{\mathrm{rig}}$.

Corollary V.2.3. *Let l be a nonnegative integer satisfying $\xi(-1) = (-1)^{k+l}$. The element*

$$\Big(\, \omega_{\bar{f},\alpha} \ , \ \pi^N_{N_f} \left[E^{k+l}_{\mathrm{syn}}(\xi) \cup_l E^{l+2}_{\mathrm{syn}}(\psi\bar{\xi}) \right] \Big)_{\mathrm{rig}} \in K(l+1)$$

is equal to

$$A_0 \cdot \mathrm{Pet}_p \cdot L_{(p)}(-l, f, \alpha, \Omega_\xi) \cdot T^{l+1} \quad ,$$

where A_0 is the algebraic number

$$A_0 = \frac{(-1)^l}{(k+l-2)!} \cdot \left(1 - \frac{\beta}{\alpha}\right) \left(1 - \frac{\psi(p) p^{k+l-1}}{\alpha}\right)^{-1} \left(1 - \frac{\psi(p) p^{k-2}}{\alpha^2}\right) (\widetilde{N})^{l+1} \cdot R,$$

$$R = \alpha \, \xi(-1) 2^{-k+2} \, N^{-k+2} \, (N_f)^{\frac{k}{2}-1} \, (N_\xi)^{k-1} \Big(\prod_{q | N/N_\xi} X^*_q(k-1) \Big) ,$$

and Pet_p is the nonzero p-adic number

$$(\Phi \, \omega_{\bar{f}}, \omega_f)_{\mathrm{rig}} \cdot T^{k-1} \ .$$

Proof. According to II.7.1,

$$E^{k+l}_{\mathrm{syn}}(\xi) \cup_l E^{l+2}_{\mathrm{syn}}(\psi\bar{\xi})$$

equals

$$\frac{T^{k+l}}{(k+l-2)!} \Big[(-1)^l E^{(p)}_{k-1,-l-1}(\xi) \cdot E_{l+2,0}(\psi\bar{\xi}) \ - \ (F \, E_{k+l,0}(\xi)) \cdot E^{(p)}_{1,-l-1}(\psi\bar{\xi}) \Big] \ .$$

Set

$$Z := \Big(\; \omega_{\bar f,\alpha} \;,\; \pi^N_{N_f} \left[{}^{(p)}E^{(p)}_{k-1,-l-1}(\xi) \cdot {}^{(p)}E^{(p)}_{l+2,0}(\psi\bar\xi) \right] \Big)_{\mathrm{rig}} \cdot T^{k+l} \; .$$

It follows from Lemma V.1.1. a) that

$$\Big(\; \omega_{\bar f,\alpha} \;,\; \pi^N_{N_f} \left[E^{(p)}_{k-1,-l-1}(\xi) \cdot E_{l+2,0}(\psi\bar\xi) \right] \Big)_{\mathrm{rig}} T^{k+l}$$

$$= \Big(1 - \frac{\xi(p)p^{k-2}}{\alpha}\Big)^{-1} \Big(1 - \frac{\psi(p)p^{k+l-1}}{\alpha}\Big)^{-1} Z \; .$$

As for the second term, one has

$$\Big(\; \omega_{\bar f,\alpha} \;,\; \pi^N_{N_f} \left[(F\, E_{k+l,0}(\xi))\; E^{(p)}_{1,-l-1}(\psi\bar\xi) \;\right] \Big)_{\mathrm{rig}} T^{k+l}$$

$$= \Big(\; \Phi^{-1}\, \omega_{\bar f,\alpha} \;,\; U\, \pi^N_{N_f} \left[(F\, E_{k+l,0}(\xi))\; E^{(p)}_{1,-l-1}(\psi\bar\xi) \;\right] \Big)_{\mathrm{rig}} T^{k+l}$$

$$= \frac{\psi\bar\xi(p)}{\alpha} \Big(\; \omega_{\bar f,\alpha} \;,\; \pi^N_{N_f} \left[\; E_{k+l,0}(\xi) E^{(p)}_{1,-l-1}(\psi\bar\xi) \;\right] \Big)_{\mathrm{rig}} T^{k+l} \; .$$

By Lemma 1.1b) and Proposition 1.2 this equals

$$(-1)^{l+1}\, \frac{\psi\bar\xi(p)}{\alpha} \Big(1 - \frac{\psi\bar\xi(p)}{\alpha}\Big)^{-1} \Big(1 - \frac{\psi(p)p^{k+l-1}}{\alpha}\Big)^{-1} Z \; .$$

Collecting the above yields

$$\Big(\; \omega_{\bar f,\alpha} \;,\; \pi^N_{N_f} \left[\; \mathcal{E}^{k+l}_{\mathrm{syn}}(\xi) \cup_l \mathcal{E}^{l+2}_{\mathrm{syn}}(\psi\bar\xi) \right] \Big)_{\mathrm{rig}}$$

$$= \frac{(-1)^l}{(k+l-2)!} \Big(1 - \frac{\psi(p)p^{k+l-1}}{\alpha}\Big)^{-1} Z \left[\Big(1 - \frac{\xi(p)p^{k-2}}{\alpha}\Big)^{-1} + \Big(1 - \frac{\psi\bar\xi(p)}{\alpha}\Big)^{-1} \frac{\psi\bar\xi(p)}{\alpha} \;\right] .$$

A short computation shows that

$$\left[\Big(1 - \frac{\xi(p)p^{k-2}}{\alpha}\Big)^{-1} + \Big(1 - \frac{\psi\bar\xi(p)}{\alpha}\Big)^{-1} \frac{\psi\bar\xi(p)}{\alpha} \;\right] = \frac{\Big(1 - \frac{\psi(p)p^{k-2}}{\alpha^2}\Big)}{\Big(1 - \frac{\xi(p)p^{k-2}}{\alpha}\Big) \cdot \Big(1 - \frac{\psi\bar\xi(p)}{\alpha}\Big)}$$

and the Corollary follows after plugging in for Z the formula of Theorem 2.1 . □

Proof of Thm 0.1. Let us deduce Theorem 0.1 from the Corollary. For this we first fix $l \in \mathbb{Z}$ and then make our choice of the character ξ : Take an arbitrary character ϵ of conductor N_f. According to [**Shi77**, Rmk.p.213] there exists a character ξ_0 of conductor N_{ξ_0}, prime to $p \cdot N_f$ and of parity $\xi_0(-1) = \epsilon(-1)(-1)^{k+l}$ which satisfies $L(k-1, f, \epsilon\xi_0) \neq 0$. Therefore, $\xi := \epsilon\xi_0$ is a character of conductor divided by N_f and prime to p. Shimura's method even gives infinitely many such characters and we can and will assume that $\xi \neq \psi$ in case $l = 0$. For such ξ the algebraic number A_0 in V.2.3 does not vanish. A quick computation shows that

$$v := \frac{\omega_{\bar{f},\alpha}}{\mathrm{Pet}_p}$$

satisfies

$$(v\ ,\ \omega_f)_{\mathrm{rig}} = T^{1-k} = t^{k-1}\ .$$

Therefore v equals v_α from the statement of Theorem 0.1 and the claim follows directly from V.2.3. Note that all choices of periods $\Omega^{\pm}$ (of the same parity) differ by a nonzero algebraic number and therefore we can choose Ω_ξ in order to prove the theorem.

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