Anomalous resonant Josephson tunneling between nonideal BCS condensates

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We calculate the critical supercurrent through a resonant scatterer between disordered BCS superconductors with Cooper pair breaking. Profound deviations from the case of ideal BCS superconductors are found in line with recent experiments on the Josephson effect in carbon nanotube quantum dots. The effect of pair breaking is accounted for within scattering theory via the relation between the Andreev scattering matrix and the quasiclassical Green functions of the superconductors in the Usadel limit.

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Since its discovery the Josephson effect [1] has been studied for a variety of superconducting weak links [2, 3, 4]. The research has recently entered a new phase with the experimental realization of quantum dot weak links exploiting electronic properties of finite-length carbon nanotubes coupled to superconducting leads [5, 6, 7]. In particular, the resonant Josephson current mediated by discrete energy levels in a carbon nanotube quantum dot was observed [6]. This work also demonstrated the feasibility of a quantum supercurrent transistor action where the critical current was modulated by means of a gate voltage tuning successive energy levels in the dot on- and off-resonance with the Fermi energy in the leads. Apart from possible applications, such systems are of fundamental interest, for weak Josephson coupling provided by a resonant level can serve as a sensitive and controllable means for studying the role of various perturbations in superconductors.

Experiment [6], conducted in the absence of Coulomb blockade, indeed reported critical currents well below the theoretical limit, $I_c = e\Delta/\hbar$ [8, 9, 10], for a single resonant level (Δ, e and $\hbar$ are the pairing energy, electron charge and Planck's constant). Moreover, both the resonance lineshape and the dependence of I_c on the normal-state conductance showed significant deviations from predictions of Refs. [9, 10], indicating the need for further theoretical studies.

One of the common reasons for the suppression of weak superconductivity, widely discussed in the literature, is the influence of the electromagnetic environment [3, 11]. Here we address another general mechanism of the supercurrent suppression that can be particularly effective in resonant weak links. It stems from the distortion of the Bardeen-Cooper-Schrieffer (BCS) superconducting state caused by pair breaking. Pair breaking, an essential attribute of real superconductors, can be induced by a number of factors, e.g. by the lack of time-reversal symmetry [12, 13] or by structural inhomogeneities producing spatial fluctuations of the superconducting coupling constant [14]. As well known, bulk superconductivity gets suppressed at rather high pair-breaking rates of order of $\Delta/\hbar$. The Josephson coupling can, on the other hand, be sensitive to much lower pair-breaking rates that should be compared with tunneling rates of Cooper pairs. In fact, we intend to show that any finite pair-breaking rate results in an anomalous single-level resonant supercurrent as compared to the case of ideal BCS superconductors [9, 10]: It has a non-Breit-Wigner resonance lineshape and shows no simple correlation with the normal-state conductance.

We employ the basic model of Refs. [9, 10] for a short superconducting constriction with a resonant scatterer. The Josephson current is calculated using the normal-state scattering matrix of the system and the Andreev scattering matrix. Unlike [9, 10] we focus on dirty superconductors for which the Andreev matrix can be quite generally expressed in terms of the quasiclassical Green functions [15], allowing us to generalize the scattering theory [9, 10] to superconductors subject to pair-breaking perturbations. Such a combination of the scattering and Green function approaches makes it possible to address a new regime of the Josephson coupling accounting simultaneously for both energy-dependent resonant scattering in the constriction and pair breaking in the leads. Previously, the pair-breaking effect has been studied in nonresonant diffusive junctions (see, e.g. Refs. [4, 16]).

Model.– We consider a junction between two superconductors S_1 and S_2 adiabatically narrowing into quasi-one-dimensional ballistic wires S_1' and S_2' coupled to a scattering region N [Fig. 1]. The transformation from the superconducting electron spectrum to the normal-metal one is assumed to take place abruptly at the boundaries S_1S_1' and $S_2'S_2$, implying the pairing potential of the form [2]: $\Delta(x) = \Delta e^{i\varphi_1}$ for $x < -L/2$, $\Delta(x) = 0$

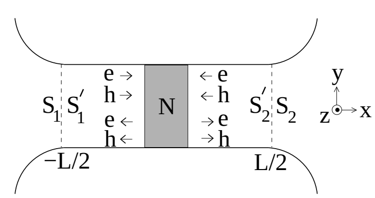

FIG. 1: Scheme of a superconducting constriction with a normal scattering region N. The arrows indicate the electrons (e) and holes (h) incident on and outgoing from N.

for $|x| \leq L/2$ and $\Delta(x) = \Delta e^{i\varphi_2}$ for $x > L/2$ where $\varphi_{1,2}$ are the order parameter phases in $S_{1,2}$, and the constriction length $L \ll \hbar v_F/\Delta$ (v_F is the Fermi velocity in $S_{1,2}$). The spatial variation of $\Delta(x)$ gives rise to bound electron-hole states that mediate a Cooper pair transfer between the leads $S_{1,2}$ driven by the phase difference $\varphi \equiv \varphi_2 - \varphi_1$ [17]. A short single-channel junction supports two bound states (BS) with energies $\pm E(\varphi)$ (with respect to the Fermi level) carrying a net supercurrent [10]:

$$I(\varphi) = -\frac{2e}{\hbar}\frac{\partial E(\varphi)}{\partial \varphi}\tanh\frac{E(\varphi)}{2\theta}, \quad (1)$$

where θ is the temperature in energy units.

Scattering theory [10, 17] links the formation of the BS to the Andreev process [18] whereby electrons are retroreflected as Fermi-sea holes (and vice versa) off the superconductors due to the high transparency of the boundaries S_1S_1' and $S_2'S_2$. Additional scattering, without coupling electrons and holes, is caused by the N region in the middle of the junction. In the general approach of Ref. [10] the BS energies are obtained from the following determinantal equation:

$$\mathrm{Det}\left[\hat{1} - \hat{s}_A(E)\hat{s}_N(E)\right] = 0. \quad (2)$$

Here $\hat{s}_N(E)$ is a 4×4 unitary matrix relating the incident electron and hole waves on the N region to the outgoing ones [Fig. 1]. It is diagonal in the electron-hole space:

$$\hat{s}_N = \begin{bmatrix} s_{ee}(E) & 0 \\ 0 & s_{hh}(E) \end{bmatrix},\ s_{ee}(E) = \begin{bmatrix} r_{11}(E) & t_{12}(E) \\ t_{21}(E) & r_{22}(E) \end{bmatrix}.$$

The matrix $s_{ee}(E)$ describes electron scattering in terms of the reflection and transmission amplitudes, $r_{jk}(E)$ and $t_{jk}(E)$, for a transition from S_k' to S_j' ($j,k = 1,2$). The hole scattering matrix is related to the electron one by $s_{hh}(E) = s_{ee}^*(-E)$. The Andreev scattering matrix $\hat{s}_A(E)$ is off-diagonal in the electron-hole space:

$$\hat{s}_A = \begin{bmatrix} 0 & s_{eh}(E) \\ s_{he}(E) & 0 \end{bmatrix}, \quad (3)$$

where the 2×2 matrices $s_{he}(E)$ and $s_{eh}(E)$ govern the electron-to-hole and hole-to-electron scattering off the superconductors. In Ref. [10] these matrices were obtained by matching the solutions of the Bogolubov-de Gennes equations in the wires $S_{1,2}'$ to the corresponding evanescent solutions in impurity-free leads. Gorkov's Green function formalism in combination with the quasiclassical theory [19] allows one to generalize the results of Ref. [10] to dirty leads with a short mean free path $\ell \ll \hbar v_F/\Delta$. In the latter case the matrices $s_{he}(E)$ and $s_{eh}(E)$ can be expressed in terms of the quasiclassical Green functions of the superconductors as follows [15]:

$$s_{eh} = \begin{bmatrix} \frac{f_1(E)}{g_1(E)+1} & 0 \\ 0 & \frac{f_2(E)}{g_2(E)+1} \end{bmatrix},\ s_{he} = \begin{bmatrix} \frac{-f_1^\dagger(E)}{g_1(E)+1} & 0 \\ 0 & \frac{-f_2^\dagger(E)}{g_2(E)+1} \end{bmatrix}.$$

Here $g_{1,2}$ and $f_{1,2}$ ($f_{1,2}^\dagger$) are, respectively, the normal and anomalous retarded Green functions in $S_{1,2}$. These matrices are diagonal in the electrode space due to a local character of Andreev reflection in our geometry.

Neglecting the influence of the narrow weak link on the bulk superconductivity, we can use the Green functions of the uncoupled superconductors $S_{1,2}$ described by the position-independent Usadel equation [19],

$$\left[E\hat{\tau}_3 + \hat{\Delta}_j + i\Delta\frac{\zeta}{2}\,\hat{\tau}_3\hat{g}_j\hat{\tau}_3\,,\,\hat{g}_j\right] = 0, \quad (4)$$

with the normalization condition $\hat{g}_j^2 = \hat{\tau}_0$ for the matrix Green function

$$\hat{g}_j = \begin{bmatrix} g_j & f_j \\ f_j^\dagger & -g_j \end{bmatrix},\ \hat{\Delta}_j = \begin{bmatrix} 0 & \Delta e^{i\varphi_j} \\ -\Delta e^{-i\varphi_j} & 0 \end{bmatrix},\ j = 1,2.$$

Here $\hat{\tau}_0$ and $\hat{\tau}_3$ are the unity and Pauli matrices, respectively, and $[...,...]$ denotes a commutator. Equation (4) accounts for a finite pair-breaking rate characterized by a dimensionless parameter ζ. Its microscopic expression depends on the nature of the pair-breaking mechanism [12, 13]. For instance, for magnetic impurities, $\zeta = \hbar/(\tau_s\Delta)$ is inversely proportional to the spin-flip time τ_s. For thin superconducting films in a parallel magnetic field, $\zeta = (B/B_*)^2$ where the characteristic field $B_* = (\Phi_0/\pi d)\sqrt{18\Delta/\hbar v_F \ell}$ depends on the film thickness d (Φ_0 is the flux quantum). In the case of the spatial fluctuations of the superconducting coupling, ζ is proportional to the variance of the fluctuations [14].

From Eq. (4) one obtains the Green functions [13]

$$g_j = \frac{u}{i\sqrt{1-u^2}} = u e^{-i\varphi_j} f_j,\ \frac{E}{\Delta} = u\left(1 - \frac{\zeta}{\sqrt{1-u^2}}\right), \quad (5)$$

and $f_j^\dagger = f_j^*$. Here the second equation has a real solution $|u(E)| \leq (1-\zeta^{2/3})^{1/2}$ for $\zeta < 1$ corresponding to the reduced energy gap $|E| \leq \Delta_{\mathbf{g}} = \Delta\left(1-\zeta^{2/3}\right)^{3/2}$ [12, 13]. Inside the gap the matrices s_{eh} and s_{he} can be expressed using Eqs. (5) as follows:

$$s_{eh} = e^{-i\beta(E)}\begin{bmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{bmatrix},\ s_{he} = e^{-2i\beta(E)}s_{eh}^*, \quad (6)$$

where $\beta(E) = \arccos u(E)$. The Andreev reflection amplitudes depend on the pair-breaking parameter ζ via $u(E)$ given implicitly by Eq. (5). For $\zeta = 0$ we recover the familiar result $\beta(E) = \arccos(E/\Delta)$ [10]. Inserting Eqs. (3) and (6) into Eq. (2) yields the equation for the BS energies:

$$\begin{aligned} &e^{-4i\beta(E)}\,\mathrm{Det}s_{ee}(E)\,\mathrm{Det}s_{ee}^*(-E) - \\ &e^{-2i\beta(E)}\,\left[r_{11}(E)r_{11}^*(-E) + r_{22}(E)r_{22}^*(-E) + \right. \\ &\left. e^{-i\varphi}t_{21}(E)t_{12}^*(-E) + e^{i\varphi}t_{12}(E)t_{21}^*(-E)\right] + 1 = 0. \end{aligned} \quad (7)$$

Bound states in a resonant junction.– Let us assume that the N region is a quantum dot (QD) and electrons

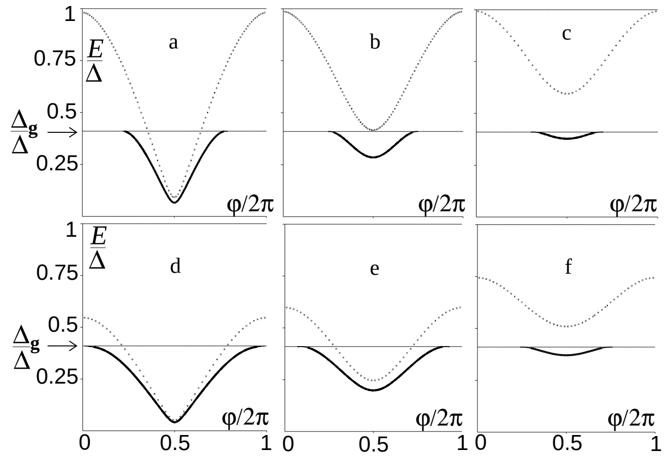

FIG. 2: Phase dependence of the bound state for zero (dotted curves) and finite ($\zeta = 0.3$, solid curves) pair-breaking parameter. Panels (a), (b) and (c) are for a broad QD level ($\Delta = 0.1\Gamma$) positioned at $E_r = 0.1\Gamma$, $E_r = 0.5\Gamma$, and $E_r = 0.8\Gamma$, respectively. Panels (d), (e) and (f) are for a modestly broad level ($\Delta = \Gamma$) positioned at $E_r = 0.1\Gamma$, $E_r = 0.5\Gamma$, and $E_r = 1.1\Gamma$; arrows show the normalized gap for $\zeta = 0.3$.

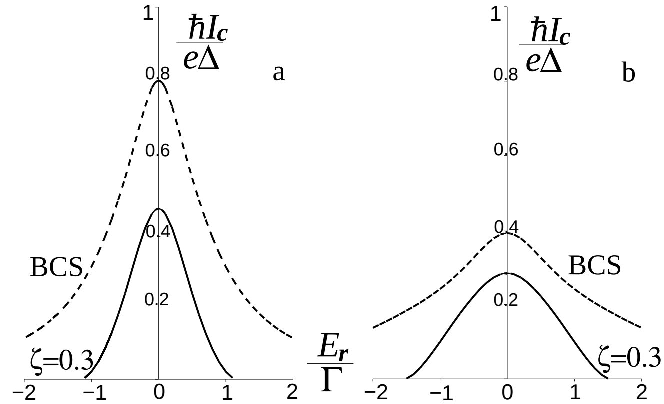

FIG. 3: Critical current vs. resonant level position: (a) $\Delta = 0.1\Gamma$ and (b) $\Delta = \Gamma$; $\theta = 0.1\Delta$.

can only tunnel via one of its levels characterized by its position E_r with respect to the Fermi level and broadening Γ. Following [9, 10] we take the simplest Breit-Wigner scattering matrix with $r_{11} = r_{22} = (E-E_r)/(E-E_r+i\Gamma)$ and $t_{12} = t_{21} = \Gamma/i(E - E_r + i\Gamma)$. Equation (7) then is

$$u^2 + \frac{2\mathcal{T}(E/\Gamma)}{1-\mathcal{T}(E/\Gamma)^2} u\sqrt{1-u^2} = 1 - \frac{\mathcal{T}\sin^2(\varphi/2)}{1-\mathcal{T}(E/\Gamma)^2}, \quad (8)$$

where $\mathcal{T} = \Gamma^2/(E_r^2+\Gamma^2)$ is the Breit-Wigner transmission probability that gives the normal-state tunneling rate through the dot in units of the on-resonance ($E_r = 0$) tunneling rate. The ratio $E/\Gamma \sim \Delta/\Gamma$ accounts for the energy dependence of the resonant scattering.

We start our analysis with an analytically accessible case $\Delta/\Gamma \to 0$ where Eq. (8) reduces to $u^2 = 1 - \mathcal{T}\sin^2(\varphi/2)$. Along with Eq. (5) this yields the BS energies $\pm E(\varphi)$,

$$E(\varphi) = \Delta\sqrt{1-\mathcal{T}\sin^2(\varphi/2)}\left[1 - \frac{\zeta}{\sqrt{\mathcal{T}}|\sin(\varphi/2)|}\right], \quad (9)$$

provided that $\sin^2(\varphi/2) \geq \zeta^{2/3}/\mathcal{T}$ and $\zeta^{2/3} \leq \mathcal{T}$. The latter condition implies that BS exist only if the pair-breaking rate is sufficiently low in comparison with the tunneling rate through the dot. This condition can be rewritten in terms of the resonant level position as

$$|E_r| \leq \tilde{E}_r, \quad \tilde{E}_r = \Gamma\sqrt{\zeta^{-2/3}-1}. \quad (10)$$

The BS is lost when E_r exceeds the threshold $\tilde{E}_r$ driven, say, by the gate voltage. An equation of the same form as Eq. (9) was derived earlier for a nonresonant system and by a different method [16].

Solving Eqs. (5) and (8) numerically we extend our analysis to a QD level of finite width $\Gamma \geq \Delta$, a case more realistic and relevant to the experiment of Ref. [6]. Figures 2(a), (b) and (c) show the phase dependence of the BS for three off-resonance values of E_r. The detuning from resonance results in a weaker $E(\varphi)$ dependence and eventually leads to the loss of the BS for $\zeta \neq 0$. The situation close to that is captured in Fig. 2(c). These three panels correspond to a rather broad QD level ($\Delta/\Gamma = 0.1$). Panels (d), (e) and (f) are for a modestly broad QD level with $\Delta/\Gamma = 1$. In this case the BS vanishes at a certain value of $E_r \sim \Gamma$, too [see (f)], however this cannot be described by the simple formulas (9) and (10).

Critical current.– The interplay of the pair breaking and tunneling is reflected in the magnitude of the critical Josephson current $I_c \equiv \max I(\varphi)$. We use the numerical data for $E(\varphi)$ to find the maximum of $I(\varphi)$ [Eq. (1)] and analyze I_c as a function of E_r/Γ, Δ/Γ and ζ.

Figure 3(a) shows I_c as a function of the QD level position, which models the gate voltage dependence with ($\zeta = 0.3$) and without (BCS) pair breaking. Apart from a nearly 50% reduction of the on-resonance current, pair breaking profoundly modifies the resonance lineshape resulting in a much stronger decrease than in the BCS case where $I_c(E_r)$ has the standard Breit-Wigner asymptotics $\sim (\Gamma/E_r)^2 \to 0$ [9, 10]. Figure 3(b) also demonstrates an overall current supression as the relative broadening Γ/Δ decreases.

In Fig. 4(a) we compare the dependence of I_c on the normal-state transmission probability $\mathcal{T}$ for $\zeta = 0.15$ with the prediction of the BCS-based theory [9, 10]. There is a pronounced difference, especially at smaller $\mathcal{T}$, due to the pair-breaking effect on the BS even for such a small ζ. Experiment [6] reports similar deviations from theory [9, 10] (see, e.g. Fig. 3c and discussion in Ref. [6]). Interestingly, the reduction of the relative broadening Γ/Δ is accompanied by the change in the curvature of $I_c(\mathcal{T})$ [Fig. 4(b)]. This previously unnoticed

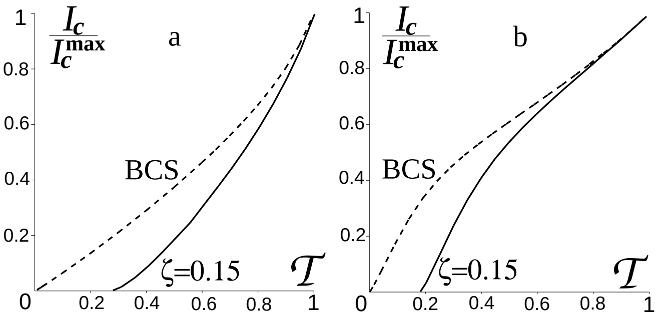

FIG. 4: Critical current normalized to its maximum $I_c^{\max}$ vs. normal-state transmission probability: (a) $\Delta = 0.1\Gamma$ and (b) $\Delta = \Gamma$; $\theta = 0.1\Delta$.

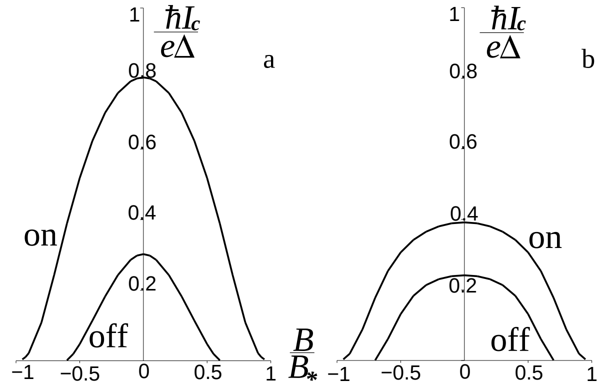

FIG. 5: Critical current vs. magnetic field on- and off ($E_r = \Gamma$)-resonance: (a) $\Delta = 0.1\Gamma$ and (b) $\Delta = \Gamma$; $\theta = 0.1\Delta$.

behavior can be used in practice as a simple indicator of the "dot-lead" coupling strength.

Applying an external magnetic field allows one to study the pair-breaking effect in a controllable way. With $\zeta = (B/B_*)^2$ one can obtain the dependence $I_c(B)$, shown in Fig. 5 for on- and off-resonance cases. Not only does the current decrease, but also the range of fields where $I_c(B) \neq 0$ shrinks as the QD level is driven off-resonance. A similar behavior was again found in the experiment (see, Supplementary information to Ref. [6]). In the limit $\Delta/\Gamma \to 0$, when Eq. (9) holds, the range of relevant fields is given by the criterion $\zeta \leq \mathcal{T}^{3/2}$ for the existence of BS:

$$|B| \leq \tilde{B}, \quad \tilde{B} = \mathcal{T}^{3/4} B_*, \tag{11}$$

where the characteristic field $\tilde{B}$ can be much smaller than B_* if the Breit-Wigner probability $\mathcal{T} \ll 1$.

In conclusion, we have found that a resonant Josephson current, measured as a function of a gate voltage, normal-state conductance or a magnetic field, can serve as an effective probe of weakly disturbed BCS condensates with pair-breaking rates much lower than the critical values. Based on the standard scattering and pair-breaking theories, our findings are consistent with the recently observed performance of quantum Josephson transistor devices [6]. As in Ref. [6] we considered no Coulomb blockade effects, assuming the charging energy in the dot much smaller than Δ. The condition $\Gamma \geq \Delta$, used in this work, guarantees then that the quantum dot is in the open regime. Our model can be extended to other regimes of quantum dot transport.

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