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## A remark on the rigidity case of the positive energy theorem

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# A REMARK ON THE RIGIDITY CASE OF THE POSITIVE ENERGY THEOREM

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ABSTRACT. In their proof of the positive energy theorem, Schoen and Yau showed that every asymptotically flat spacelike hypersurface $M$ of a Lorentzian manifold which is flat along $M$ can be isometrically imbedded with its given second fundamental form into Minkowski spacetime as the graph of a function $\mathbb{R}^n \to \mathbb{R}$; in particular, $M$ is diffeomorphic to $\mathbb{R}^n$. In this short note, we give an alternative proof of this fact. The argument generalises to the asymptotically hyperbolic case, works in every dimension $n$, and does not need a spin structure.

## 1. INTRODUCTION

The *rigidity case* of the positive energy theorem is the situation when $E = |P|$ holds for the energy $E \in \mathbb{R}$ and the momentum $P \in \mathbb{R}^n$ of an asymptotically flat spacelike hypersurface $M$ of a Lorentzian $(n+1)$-manifold $(\overline{M}, \overline{g})$ with $n \geq 3$ which satisfies the dominant energy condition at every point of $M$. The positive energy theorem says that then the Riemann tensor of $\overline{g}$ vanishes at every point of $M$; we call this the *rigidity statement*.

This has been proved by Parker/Taubes [6] in the case when $M$ admits a spin structure — and under the assumption that $M$ is 3-dimensional, but the argument generalises to higher dimensions. (The original proof of Witten [10] made the slightly stronger assumption that $(\overline{M}, \overline{g})$ satisfies the dominant energy condition on a neighbourhood of $M$.)

Another proof of the positive energy theorem, in particular of the rigidity statement, had been given earlier by Schoen/Yau [7, 8, 9], without the spin assumption — again assuming $n = 3$, but the argument can be generalised to $n \leq 7$. More recently, Lohkamp extended their approach to higher dimensions [4]; the details for arbitrary fundamental forms have not been published yet, however. Schoen has announced a proof in a similar spirit.

Schoen/Yau proved actually more than Parker/Taubes: they showed that in the rigidity case the Riemannian $n$-manifold $M$ with its second fundamental form induced by the imbedding in $(\overline{M}, \overline{g})$ can be imbedded isometrically into Minkowski spacetime $\mathbb{R}^{n,1} = \mathbb{R}^n \times \mathbb{R}$ as the graph of a function $\mathbb{R}^n \to \mathbb{R}$, which implies in particular that $M$ is diffeomorphic to $\mathbb{R}^n$.

It is natural to ask whether one can decouple the proof of imbeddability into Minkowski spacetime from the proof of the rigidity statement: When we know already — for instance from the Parker/Taubes proof — that $\overline{g}$ is flat along $M$, can we deduce directly that $M$ with its second fundamental form admits an imbedding of the desired form and is in particular diffeomorphic to $\mathbb{R}^n$?

The aim of the present short article is to show how this can be done in a simple way, independently of the Schoen/Yau arguments, and with minimal assumptions. Locally, the desired imbeddability follows already from the fundamental theorem of hypersurface theory due to Bär/Gauduchon/Moroianu [1, Section 7] (which has a short elegant proof).

Since this theorem applies not only to flat metrics but to metrics of arbitrary constant sectional curvature, we can also consider the case of imbeddings into anti-de Sitter spacetime. An analogue of the Parker/Taubes proof in this situation is the work by Maerten [5], which requires a spin assumption. He shows in this case that the hypersurface with its second fundamental form imbeds isometrically into anti-de Sitter spacetime. As Schoen/Yau, he does this via an explicit construction which is a by-product of the specific method that is used to prove the positive energy theorem.

The result of the present article, Theorem 1.5 below, applies in a situation when it has already been proved somehow that along the hypersurface the Gauss and Codazzi equations of an ambient Lorentzian metric of constant curvature $c \leq 0$ are satisfied. The conclusion is that then a suitable isometric imbedding into Minkowski or anti-de Sitter spacetime exists and is essentially unique, which implies in particular that the hypersurface is diffeomorphic to $\mathbb{R}^n$. The proof does not require any spin assumption or dimensional restriction.

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Let us adopt the following conventions and terminology. All manifolds, bundles, metrics, maps, etc. are smooth. The sign convention for the Riemann tensor is $\mathrm{Riem}(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$. Lorentzian metrics on $(n+1)$-manifolds have signature $(n,1)$ (i.e. $n$ positive=spacelike dimensions, 1 negative=timelike dimension).

1.1. **Definition** (hypersurface data set)**.** A *hypersurface data set* is a quadruple $(M,g,N,K)$ such that $M$ is a manifold, $g$ is a Riemannian metric on $M$, $N$ is a Riemannian line bundle over $M$ (i.e. a real line bundle equipped smoothly with scalar products on the fibres), and $K$ is a section in $\mathrm{Sym}^2(T^*M)\otimes N \to M$.

When $M$ is a spacelike hypersurface of a Lorentzian manifold $(\overline{M},\overline{g})$, then the *hypersurface data set induced by the inclusion* $M \to (\overline{M},\overline{g})$ is the hypersurface data set $(M,g,N,K)$ such that $g$ is the restriction of $\overline{g}$, such that $N$ is the normal bundle of $M$ in $(\overline{M},\overline{g})$ equipped with the restriction of $-\overline{g}$ as fibre metric, and such that $K$ is the second fundamental form of $M$ in $(\overline{M},\overline{g})$.

Let $(M,g,N,K)$ be a hypersurface data set. An *isometric imbedding of* $(M,g,N,K)$ into a Lorentzian manifold $(\overline{M},\overline{g})$ is a pair $(f,\iota)$ such that

- $f\colon (M,g)\to(\overline{M},\overline{g})$ is an isometric imbedding;
- $\iota$ is an isomorphism of Riemannian line bundles from $N$ to the normal bundle $N'$ of the spacelike hypersurface $M' := f(M)$ in $(\overline{M},\overline{g})$, where the fibre metric on $N'$ is the restriction of $-\overline{g}$;
- the second fundamental form $II \in \Gamma(\mathrm{Sym}^2T^*M'\otimes N')$ of $M'$ in $(\overline{M},\overline{g})$ is given by $II(f_*v,f_*w) = \iota(K(v,w))$ for all $x\in M$ and $v,w\in T_xM$.

An *isometric immersion of* $(M,g,N,K)$ into $(\overline{M},\overline{g})$ is a pair $(f,\iota)$ such that $f\colon M\to\overline{M}$ is an immersion, such that $\iota$ is a map whose domain is the total space of $N$, and such that every $x\in M$ has a neighbourhood $U$ for which $(f|U,\iota|(N|U))$ is an isometric imbedding of $(U,g|U,N|U,K|U)$ into $(\overline{M},\overline{g})$.

*Remark.* In most contexts where a spacelike hypersurface $M$ of a Lorentzian manifold $(\overline{M},\overline{g})$ is considered (e.g. in the positive energy theorem or discussions of the constraint equations in General Relativity), it is assumed that the normal bundle of $M$ is trivial (i.e. that $\overline{g}$ is time-orientable on a neighbourhood of $M$), and a unit normal vector field is fixed. This assumption is often unnecessary, in particular for the rigidity case of the positive energy theorem: We obtain the triviality of the normal bundle as a *conclusion*, we do not have to assume it.

1.2. **Definition.** Let $(M,g,N,K)$ be a hypersurface data set. We denote the fibre scalar product on $N$ by $\langle .,.\rangle_N$. We define a covariant derivative $\mathrm{d}^N$ on the Riemannian line bundle $N\to M$ by declaring every local unit-length section to be parallel. We define $\nabla^{g,N}$ to be the covariant derivative on the vector bundle $\mathrm{Sym}^2T^*M\otimes N\to M$ induced by the Levi-Civita connection of $g$ and $\mathrm{d}^N$.

Let $c\in\mathbb{R}$. $(M,g,N,K)$ *satisfies the Gauss and Codazzi equations for constant curvature* $c$ iff the equations

$$c\big(g(u,z)g(v,w)-g(u,w)g(v,z)\big) = \mathrm{Riem}_g(u,v,w,z) - \langle K(u,w),K(v,z)\rangle_N + \langle K(u,z),K(v,w)\rangle_N \ ,$$

$$0 = -\big\langle(\nabla^{g,N}_u K)(v,w),n\big\rangle_N + \big\langle(\nabla^{g,N}_v K)(u,w),n\big\rangle_N$$

hold for all $x\in M$ and $u,v,w,z\in T_xM$ and $n\in N_x$.

1.3. **Fact.** Let $(M,g,N,K)$ be the hypersurface data set induced by the inclusion of a spacelike hypersurface $M$ into a Lorentzian manifold $(\overline{M},\overline{g})$ which has constant (sectional) curvature $c$ at every point of $M$. Then $(M,g,N,K)$ satisfies the Gauss and Codazzi equations for constant curvature $c$. □

*Remark.* When the hypersurface data set $(M,g,N,K)$ induced by the inclusion of a spacelike hypersurface $M$ into a Lorentzian manifold $(\overline{M},\overline{g})$ satisfies the Gauss and Codazzi equations for constant curvature $c$, then $(\overline{M},\overline{g})$ does in general not have constant curvature $c$ at any point of $M$. The reason is that the Gauss and Codazzi equations do not yield information about the curvature components $\mathrm{Riem}_{\overline{g}}(n,v,w,n)$ with $v,w\in T_xM$ and $n\in N_x$.

1.4. **Notation.** Let $n,r\geq 0$, let $c\in\mathbb{R}_{\leq 0}$. Let $\mathbb{R}^{n,r}$ denote $\mathbb{R}^{n+r}$ equipped with the semi-Riemannian metric $g_{n,r} := \sum_{i=1}^n \mathrm{d}x_i^2 - \sum_{i=n+1}^{n+r}\mathrm{d}x_i^2$. We define $\mathcal{M}_0^{n,1}$ to be Minkowski spacetime $\mathbb{R}^{n,1}$. For $c<0$, we consider the pseudohyperbolic spacetime $\mathcal{H}_c^{n,1} := \big\{x\in\mathbb{R}^{n,2} \;\big|\; g_{n,2}(x,x) = \frac{1}{c}\big\}$ (which is a Lorentzian submanifold of $\mathbb{R}^{n,2}$) and its universal covering $\varpi\colon \mathbb{R}^n\times\mathbb{R}\to\mathcal{H}_c^{n,1}$ given by $(x,t)\mapsto(x,\cos t\sqrt{|x|^2-1/c},\sin t\sqrt{|x|^2-1/c})$, and we define the anti-de Sitter spacetime $\mathcal{M}_c^{n,1}$ to be $\mathbb{R}^n\times\mathbb{R}$ equipped with the $\varpi$-pullback metric of the metric on $\mathcal{H}_c^{n,1}$. (Both $\mathcal{H}_c^{n,1}$ and $\mathcal{M}_c^{n,1}$ have constant curvature $c$; sometimes $\mathcal{H}_c^{n,1}$ instead of $\mathcal{M}_c^{n,1}$ is called anti-de Sitter spacetime.) For $c\leq 0$, we define $\mathrm{pr}\colon \mathcal{M}_c^{n,1} = \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ to be the projection $(x,t)\mapsto x$.

Now we can state the main result (our definition of *simply connected* includes being connected):

1.5. **Theorem.** *Let $n \geq 0$ and $c \in \mathbb{R}_{\leq 0}$, let $M$ be a connected $n$-manifold which contains a simply connected non-compact $n$-dimensional submanifold-with-boundary that is closed in $M$ and has compact boundary, let $(M, g, N, K)$ be a hypersurface data set which satisfies the Gauss and Codazzi equations for constant curvature $c$. Assume that $(M, g)$ is complete. Then:*

(i) *$(M, g, N, K)$ admits an isometric imbedding $(f, \iota)$ into $\mathcal{M}_c^{n,1}$ such that* $\mathrm{pr} \circ f \colon M \to \mathbb{R}^n$ *is a diffeomorphism.*

(ii) *When $(\tilde{f}, \tilde{\iota})$ is an isometric immersion of $(M, g, N, K)$ into $\mathcal{M}_c^{n,1}$, then there is an isometry $A \colon \mathcal{M}_c^{n,1} \to \mathcal{M}_c^{n,1}$ with $\tilde{f} = A \circ f$; in particular, $\tilde{f}$ is an imbedding.*

*Remark 1.* In the rigidity case of (the asymptotically flat version of) the positive energy theorem, the assumptions of our theorem are satisfied: The hypersurface data set is induced by the inclusion of $M$ into a Lorentzian manifold which is flat along $M$, and thus satisfies the Gauss and Codazzi equations for constant curvature $0$. The Riemannian metric $g$ is complete (this follows from the definition of asymptotic flatness). $M$ contains a compact $n$-dimensional submanifold-with-boundary $C$ such that $M \backslash (C \backslash \partial C)$ is diffeomorphic to a nonempty disjoint union of copies of $\mathbb{R}^n \backslash$(open ball) each of which is closed in $M$ (this closedness follows from the completeness of the metric) and simply connected (because $n \geq 3$ is assumed in the positive energy theorem).

Similarly, the assumptions are satisfied in Maerten's theorem for asymptotically hyperbolic hypersurfaces [5, second half of the proof of the first theorem in Section 4].

*Remark 2.* Statement (i) shows that $f(M)$ is the spacelike graph of a function $\mathbb{R}^n \to \mathbb{R}$. This implies also that $f(M)$ is an acausal subset of $\mathcal{M}_c^{n,1}$. (Note that e.g. not every spacelike imbedding $f \colon \mathbb{R}^n \to \mathbb{R}^{n,1}$ is acausal: consider an imbedding that winds up, i.e. in the direction of increasing time, in a spacelike way like a spiral staircase.)

*Remark 3.* Theorem 1.5 would clearly be false without the simply-connectedness assumption, even in the case $K \equiv 0$: take e.g. $(M, g, N, K)$ to be the hypersurface data set induced by the inclusion of $M = \mathbb{R}^{n-1} \times S^1 \times \{0\}$ into the flat product Lorentzian manifold $\mathbb{R}^{n-1} \times S^1 \times \mathbb{R}$ with $\mathbb{R}$ as timelike factor. Then (i) is clearly not true.

The theorem would also be false without the completeness assumption: small subsets (e.g. diffeomorphic to a ball or an annulus) of a complete spacelike hypersurface in Minkowski spacetime yield counterexamples.

*Remark 4.* The theorem does not assume that the Riemannian line bundle $N$ is trivial. But it implies that $N$ is trivial, because every Riemannian line bundle over $\mathbb{R}^n$ is trivial. Note that also this triviality would in general not hold without the simply-connectedness assumption: flat $\mathbb{R}^{n-1} \times S^1$ admits an isometric imbedding (with $K \equiv 0$) into the flat Lorentzian manifold $\mathbb{R}^{n-1} \times \mathfrak{M}$, where $\mathfrak{M}$ is the Möbius strip, regarded as a line bundle over $S^1$ with timelike fibres. The normal bundle is not trivial in this case, but all assumptions of Theorem 1.5 except for the simply-connectedness are satisfied.

*Remark 5.* $A$ in (ii) is in general neither time orientation-preserving nor space orientation-preserving. (Every isometric imbedding can be composed with an isometry of $\mathcal{M}_c^{n,1}$ which is space and/or time orientation-reversing.)

*Remark 6.* In the case $c < 0$, the theorem holds also with $\mathcal{H}_c^{n,1} \cong \mathbb{R}^n \times S^1$ and the projection $\mathrm{pr}' \colon \mathbb{R}^n \times S^1 \ni (x, t) \mapsto x \in \mathbb{R}^n$ instead of $\mathcal{M}_c^{n,1}$ and pr. Similarly, Minkowski spacetime $\mathcal{M}_0^{n,1}$ is the universal cover of a Lorentzian manifold $\mathcal{H}_0^{n,1} = (\mathbb{R}^n \times S^1, g_0)$ via the covering $q \colon \mathbb{R}^n \times \mathbb{R} \ni (x, s) \mapsto (x, [s]) \in \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})$, and the theorem would hold with $\mathcal{H}_0^{n,1}$ and $\mathrm{pr}'$ instead of $\mathcal{M}_0^{n,1}$ and pr. One can see this either by checking that the proof of Theorem 1.5 remains valid with these modifications, or directly by applying the theorem and composing maps $M \to \mathcal{M}_c^{n,1}$ with $q$.

The rest of the article contains the proof of Theorem 1.5.

## 2. The fundamental theorem for hypersurfaces

We need the following special case of the fundamental theorem for hypersurfaces due to Bär/Gauduchon/Moroianu [1, Section 7]:

2.1. **Proposition.** *Let $n \geq 0$ and $c \in \mathbb{R}$, let $M$ be a simply connected $n$-manifold, let $(M, g, N, K)$ be a hypersurface data set which satisfies the Gauss and Codazzi equations for constant curvature $c$. Then $(M, g, N, K)$ admits an isometric immersion into $\mathcal{M}_c^{n,1}$. When $f_0, f_1$ are isometric immersions of $(M, g, N, K)$ into $\mathcal{M}_c^{n,1}$, then there exists an isometry $A\colon \mathcal{M}_c^{n,1} \to \mathcal{M}_c^{n,1}$ with $f_1 = A \circ f_0$.*

*Remarks on the proof.* Bär/Gauduchon/Moroianu (BGM) consider the situation when the metric on $M$ has arbitrary signature and trivial spacelike normal bundle in $(\overline{M}, \overline{g})$ (see the beginning of [1, Section 3]). Since every real line bundle over a simply connected manifold is trivial (the Stiefel/Whitney class $w_1(N) \in H^1(M; \mathbb{Z}_2)$ classifies real line bundles $N \to M$ up to isomorphism), so is our $N$. To apply the BGM result in our case, we reverse the signs of our $\overline{g}$ and $c$, then use their Corollary 7.5. We obtain existence, and uniqueness up to isometries, of isometric immersions of the sign-reversed version of $(M, g, N, K)$ into the sign-reversed version of $\mathcal{M}_c^{n,1}$. This yields existence and uniqueness up to isometries of isometric immersions of $(M, g, N, K)$ into $\mathcal{M}_c^{n,1}$.

In this argument we have not applied the BGM result literally, because the sign-reversed version of our $\mathcal{M}_c^{n,1}$ is the (nontrivial) universal cover of BGM's $\mathbb{M}_{-c}^{1,n}$. But the BGM Corollary 7.4, which makes only a local statement, does not care about the difference, and the BGM Corollary 7.5 then follows from a standard monodromy argument which works for every geodesically complete manifold of signature $(1, n)$ and constant curvature $-c$. □

## 3. Quasicoverings

Let us use the following terminology:

3.1. **Definition.** Let $M, B$ be $n$-manifolds. A map $\phi\colon M \to B$ is a *quasicovering* iff it has the following properties:

(i) $\phi$ is an immersion (equivalently: it is a local diffeomorphism, i.e., every $y \in M$ has an open neighbourhood $U$ such that $\phi|U$ is diffeomorphism onto its image).

(ii) The $\phi$-preimage of every connected component of $B$ is nonempty.

(iii) For all paths $\gamma\colon [0,1] \to B$ and $\tilde{\gamma}\colon [0,1[ \to M$ with $\phi \circ \tilde{\gamma} = \gamma|\,[0,1[$, there exists an extension of $\tilde{\gamma}$ to a path $[0,1] \to M$.

We will only be interested in the case $B = \mathbb{R}^n$.

It is easy to see that every covering map (in the smooth category) is a quasicovering. (Recall that a covering map is defined by the condition that every $x \in B$ has an open neighbourhood $U$ such that $\phi^{-1}(U)$ is the nonempty union of open disjoint sets $U_i$ each of which is mapped diffeomorphically onto $U$ by $\phi$.)

Less obviously, every quasicovering is a covering; i.e., the two concepts are equal. I do not know a reference where this elementary fact is stated explicitly, although I suspect that some exists. In the proof of Theorem 1.5 below we will be in a situation where it is easy to check that a certain map $\phi\colon M \to \mathbb{R}^n$ is a quasicovering. If we knew a priori that it is a covering, then covering theory would imply that it is a diffeomorphism (because $\mathbb{R}^n$ is simply connected); this is what we need.

But the covering property of $\phi$ is hard to verify directly: For every $x \in B$, every $y \in \phi^{-1}(\{x\})$ has an open neighbourhood $U_y$ which is mapped diffeomorphically to an open neighbourhood $V_y$ of $x$. But $\phi^{-1}(\{x\})$ could a priori be infinite, and we would have to show that the sets $U_y$ can be chosen such that the intersection of the sets $V_y$ is a neighbourhood of $x$.

However, one can show directly that every quasicovering $\phi\colon M \to \mathbb{R}^n$ is a diffeomorphism just by going through the standard proofs of covering theory and checking that they remain valid, essentially word by word, for a quasicovering. One can even verify in this way that the classifications of coverings and quasicoverings coincide in general, which implies that every quasicovering is a covering; but we are not interested in doing that.

3.2. **Lemma.** *Let $M, B$ be connected $n$-manifolds with $B$ simply connected, let $\phi\colon M \to B$ be a quasicovering. Then $\phi$ is a diffeomorphism.*

*Sketch of proof.* As mentioned, we just have to go through some of the standard proofs of covering theory, e.g. as in [2, Sections III.3–8]. The main steps are as follows.

*Step 1: For every path $\gamma\colon [0,1] \to B$ and every $z \in M$ with $\phi(z) = \gamma(0)$, there exists a unique path $\tilde{\gamma}\colon [0,1] \to M$ with $\phi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = z$.* In order to prove this, consider the set $I$ of all $t \in [0,1]$ such that there exists a unique path $\tilde\gamma\colon [0,t] \to M$ with $\phi \circ \tilde\gamma = \gamma|[0,t]$ and $\tilde\gamma(0) = z$. Clearly $0 \in I$. Property (i) in the quasicovering definition implies that $I$ is open in $[0,1]$. The closedness of $I$ follows easily from property (iii). Hence $I = [0,1]$.

*Step 2: There exists a continuous map $\xi\colon B \to M$ with $\phi \circ \xi = \mathrm{id}_B$.* This is a standard monodromy argument: By property (ii) in the quasicovering definition, there exists a point $z_0 \in M$; let $x_0 = \phi(z_0)$. Every point $x_1 \in B$ can be connected to $x_0$ by a path $\gamma$, and Step 1 yields a unique path $\tilde\gamma$ in $M$ with $\phi \circ \tilde\gamma = \gamma$ and $\tilde\gamma(0) = z_0$. We have to prove that $\xi(x_1) := \tilde\gamma(1)$ does not depend on the choice of $\gamma$. This follows from the simply-connectedness of $B$, because it is straightforward to verify that homotopic choices of $\gamma$ yield the same $\tilde\gamma(1)$. It remains to check that the resulting map $\xi\colon B \to M$ is continuous, which is also straightforward. (Cf. e.g. [2, proof of Theorem III.4.1].)

*Step 3: $\xi \circ \phi = \mathrm{id}_M$ holds.* The set $S := \{z \in M \mid \xi(\phi(z)) = z\}$ is nonempty because it contains $z_0$.

Let $z \in M$. There exists an open neighbourhood $U_0$ of $z$ in $M$ such that $\phi|U_0$ is a diffeomorphism onto its image. There exists an open neighbourhood $U_1$ of $\xi(\phi(z))$ in $M$ such that $\phi|U_1$ is a diffeomorphism onto its image. Since $W' := \phi(U_0) \cap \phi(U_1)$ is a neighbourhood of $\phi(z) = \phi(\xi(\phi(z)))$ in $B$, there exists a connected open neighbourhood $W$ of $\phi(z)$ whose closure in $B$ is contained in $W'$. The sets $V_i := (\phi|U_i)^{-1}(W)$ are nonempty, connected, and open in $\phi^{-1}(W)$. They are also closed in $\phi^{-1}(W)$: the closure of $V_i$ in $M$ is contained in $(\phi|U_i)^{-1}(W')$, and we have $(\phi|U_i)^{-1}(W') \cap \phi^{-1}(W) = (\phi|U_i)^{-1}(W)$. Thus $V_0$ and $V_1$ are connected components of the manifold $\phi^{-1}(W)$, hence either equal or disjoint.

The set $V := V_0 \cap (\xi \circ \phi)^{-1}(V_1)$ is an open neighbourhood of $z$ in $M$. If $x = \xi(\phi(x))$ holds for some $x \in V$, then $\xi(\phi(x)) \in V_0 \cap V_1$ and thus $V_0 = V_1$. In that case $y = \xi(\phi(y))$ holds for every $y \in V$: the points $y$ and $\xi(\phi(y))$ lie both in $V_1$ and have the same $\phi$-image, and $\phi|V_1$ is injective.

Therefore $S$ and $M\backslash S$ are open in $M$: if one of these sets contains $z$, then it contains the neighbourhood $V$ of $z$. Since $M$ is connected, we obtain $S = M$. This completes the proof of Step 3.

The steps 2 and 3 show that $\phi$ is a homeomorphism. Since it is a local diffeomorphism, it is a diffeomorphism. □

## 4. A PROPOSITION

Recall that a map $f\colon M \to N$ from a manifold $M$ to a Lorentzian manifold $(N,h)$ is *spacelike* iff for every $x \in M$ the image of $T_xf\colon T_xM \to T_{f(x)}N$ is spacelike; here the subspace $\{0\}$ of $T_{f(x)}N$ counts as spacelike.

4.1. **Lemma.** *Let $n \geq 0$ and $c \in \mathbb{R}_{\leq 0}$, let $w\colon\ [0,1[\ \to \mathcal{M}^{n,1}_c$ be a spacelike path such that* $\mathrm{pr} \circ w\colon\ [0,1[\ \to \mathbb{R}^n$ *has finite euclidean length. Then $w$ has finite length.*

*Proof.* For $y \in \mathcal{M}^{n,1}_c = \mathbb{R}^n \times \mathbb{R}$, the map $T_y\mathrm{pr}\colon T_y\mathcal{M}^{n,1}_c = \mathbb{R}^n \times \mathbb{R} \to T_{\mathrm{pr}(y)}\mathbb{R}^n = \mathbb{R}^n$ is given by $(u,w) \mapsto u$. We claim that $|v|_{\mathcal{M}^{n,1}_c} \leq |(T_y\mathrm{pr})(v)|_{\mathrm{eucl}}$ holds for all $\mathcal{M}^{n,1}_c$-spacelike $v$. This is obvious for $c = 0$: $|(u,w)|^2_{\mathcal{M}^{n,1}_0} = |u|^2_{\mathrm{eucl}} - w^2 \leq |u|^2_{\mathrm{eucl}} = |(T_y\mathrm{pr})(u,w)|^2_{\mathrm{eucl}}$. For $c < 0$, we have $|(u,w)|^2_{\mathcal{M}^{n,1}_c} = g_{n,2}\big(T_y\varpi(u,w), T_y\varpi(u,w)\big)$ (cf. Notation 1.4), where $T_y\varpi(u,w) \in T_{\varpi(y)}\mathcal{H}^{n,1}_c \subseteq \mathbb{R}^n \times \mathbb{R}^2$ has the form $(u, b(y,u,w))$ for some $b(y,u,w) \in \mathbb{R}^2$. Thus $|(u,w)|^2_{\mathcal{M}^{n,1}_c} = |u|^2_{\mathrm{eucl}} - |b(y,u,w)|^2_{\mathrm{eucl}} \leq |u|^2_{\mathrm{eucl}} = |(T_y\mathrm{pr})(u,w)|^2_{\mathrm{eucl}}$. This proves our claim.

We obtain $\mathrm{length}(w) = \int_0^1 |w'(t)|\mathrm{d}t \leq \int_0^1 \big|T_{w(t)}\mathrm{pr}(w'(t))\big|_{\mathrm{eucl}}\mathrm{d}t = \int_0^1 |(\mathrm{pr}\circ w)'(t)|_{\mathrm{eucl}}\mathrm{d}t = \mathrm{length}(\mathrm{pr}\circ w)$. □

We say that a map $f\colon (M,g) \to (N,h)$ from a Riemannian manifold to a Lorentzian manifold is *long* iff it is spacelike and for every interval $I \subseteq \mathbb{R}$ and every path $w\colon I \to M$, the $g$-length of $w$ is finite if the $h$-length of $f \circ w$ is finite. For example, every spacelike isometric immersion is long.

4.2. **Proposition.** *Let $n \geq 0$ and $c \in \mathbb{R}_{\leq 0}$, let $(M,g)$ be a nonempty connected complete Riemannian $n$-manifold, let $f\colon (M,g) \to \mathcal{M}^{n,1}_c$ be a long immersion. Then $f$ is a smooth imbedding, and* $\mathrm{pr} \circ f\colon M \to \mathbb{R}^n$ *is a diffeomorphism.*

*Proof.* The map $\phi := \mathrm{pr} \circ f$ is an immersion, because for every $x \in M$ the image of $T_xf\colon T_xM \to T_{f(x)}\mathcal{M}^{n,1}_c$ is spacelike and $T_{f(x)}\mathrm{pr}$ maps every spacelike subspace of $T_{f(x)}\mathcal{M}^{n,1}_c$ injectively to $T_{\mathrm{pr}(f(x))}\mathbb{R}^n$ (since $\ker(T_{f(x)}\mathrm{pr}) = \{0\} \times \mathbb{R} \subseteq \mathbb{R}^n \times \mathbb{R} = T_{f(x)}\mathcal{M}^{n,1}_c$ is timelike). We claim that $\phi$ is a quasicovering.

Let $\gamma\colon [0,1] \to \mathbb{R}^n$ and $\tilde\gamma\colon\ [0,1[\ \to M$ be paths with $\phi \circ \tilde\gamma = \gamma|\,[0,1[$. The path $\mathrm{pr} \circ f \circ \tilde\gamma = \gamma|\,[0,1[$ in $\mathbb{R}^n$ has finite euclidean length because $\gamma$ has finite euclidean length. By Lemma 4.1, $f \circ \tilde\gamma$ has finite length. Since $f$ is long, $\tilde\gamma$ has finite $g$-length.

We choose a sequence $(t_k)_{k\in\mathbb{N}}$ in $[0,1[$ which converges to 1. Since $\tilde{\gamma}$ has finite $g$-length, there is no $\varepsilon > 0$ such that $\forall k_0 \in \mathbb{N}\colon \exists k, l \geq k_0\colon \mathrm{dist}_g(\tilde{\gamma}(t_k), \tilde{\gamma}(t_l)) \geq \varepsilon$. Thus $(\tilde{\gamma}(t_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $(M, g)$. Completeness implies that it converges to some point $x \in M$. We extend $\tilde{\gamma}$ to $[0,1]$ by $\tilde{\gamma}(1) = x$. Using that $\phi$ maps a neighbourhood of $x \in M$ diffeomorphically to its image, we obtain $\phi(\tilde{\gamma}(1)) = \phi(\lim_{k\to\infty} \tilde{\gamma}(t_k)) = \lim_{k\to\infty} \phi(\tilde{\gamma}(t_k)) = \lim_{k\to\infty} \gamma(t_k) = \gamma(1)$ and deduce the smoothness of the extended $\tilde{\gamma}$ from $\gamma = \phi \circ \tilde{\gamma}$.

This shows that $\phi$ is a quasicovering, as claimed. By Lemma 3.2, $\phi$ is a diffeomorphism. Since $\phi$ is injective, so is $f$. Moreover, $f$ is proper, i.e., $f^{-1}(C)$ is compact for every compact set $C \subseteq \mathcal{M}^{n,1}_c$. That's because $\mathrm{pr}(C)$ and thus $(\mathrm{pr} \circ f)^{-1}(\mathrm{pr}(C))$ are compact and $f^{-1}(C)$ is a closed subset of $(\mathrm{pr} \circ f)^{-1}(\mathrm{pr}(C))$.

Since every proper injective immersion is a smooth imbedding, the proof is complete. □

*Remark.* We will apply Proposition 4.2 only in a situation where we know already that $M$ is simply connected. But that information would not simplify the proof.

## 5. Proof of Theorem 1.5

5.1. **Lemma.** *Let $n \geq 0$, let $M$ be a connected $n$-manifold which contains a simply connected noncompact $n$-dimensional submanifold-with-boundary that is closed in $M$ and has compact boundary. Then every covering map $\pi\colon \mathbb{R}^n \to M$ is a diffeomorphism.*

*Proof.* When a connected 1-manifold $M$ contains a noncompact subset which is closed in $M$, then $M$ is diffeomorphic to $\mathbb{R}$. Thus the lemma is true for $n = 1$. The case $n = 0$ is even simpler. Now we assume $n \geq 2$. Let $Z$ be a simply connected noncompact $n$-submanifold-with-boundary of $M$ which is closed in $M$ and has compact boundary.

Since $Z$ is simply connected, the submanifold-with-boundary $\pi^{-1}(Z)$ of $\mathbb{R}^n$ is the disjoint union of connected components $\tilde{Z}_i$ such that $\pi|\tilde{Z}_i\colon \tilde{Z}_i \to Z$ is a diffeomorphism. In particular, each $\tilde{Z}_i$ has compact boundary. Thus the boundary of $\pi^{-1}(Z)$ is a disjoint union of countably many compact nonempty connected $(n-1)$-manifolds $\Sigma_j$. No connected component $\tilde{Z}_i$ of $\pi^{-1}(Z)$ is compact, because otherwise $\pi(\tilde{Z}_i) = Z$ would be compact.

For each $j$, the Jordan/Brouwer separation theorem (cf. [3] for a simple proof) implies that $\mathbb{R}^n \backslash \Sigma_j$ has precisely two connected components. Since $n \geq 2$, precisely one of these two components is relatively compact in $\mathbb{R}^n$ (namely the unique component whose closure in the one-point compactification $S^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ does not contain the point $\infty$); we call it $\mathit{interior}_j$ and denote the closure of the other component by $\mathit{exterior}_j$.

We claim that for each $j$, $\pi^{-1}(Z)$ is contained in $\mathit{exterior}_j$. Assume not. Then $\pi^{-1}(Z) \cap \mathit{interior}_j \neq \varnothing$. Either a connected component of $\pi^{-1}(Z)$ is contained in $\mathit{interior}_j$, or $\pi^{-1}(Z)$ touches $\Sigma_j$ from the interior (that is, $U \cap \mathit{interior}_j \cap \pi^{-1}(Z) \neq \varnothing$ holds for every neighbourhood $U$ of $\Sigma_j$ in $\mathbb{R}^n$). Since $\Sigma_j$ is a boundary component of $\pi^{-1}(Z)$, the latter alternative implies that $\Sigma_j$ has a neighbourhood $U$ with $U \cap (\mathit{exterior}_j \backslash \partial\, \mathit{exterior}_j) \cap \pi^{-1}(Z) = \varnothing$. In each case, there exists a connected component $\tilde{Z}_i$ of $\pi^{-1}(Z)$ which is contained in the closure of $\mathit{interior}_j$. Since $\pi^{-1}(Z)$ is closed in $\mathbb{R}^n$ (because $Z$ is closed in $M$), this $\tilde{Z}_i$ is compact. This contradiction proves our claim.

Thus $\pi^{-1}(Z)$ is contained in $\bigcap_j \mathit{exterior}_j$ (which is by definition equal to $\mathbb{R}^n$ if the index set is empty). The two sets are even equal, for otherwise a boundary component $\Sigma_j$ of $\pi^{-1}(Z)$ would meet the interior of $\bigcap_j \mathit{exterior}_j$, which is not possible because $\Sigma_j = \partial\, \mathit{exterior}_j$ is contained in the boundary of $\bigcap_j \mathit{exterior}_j$.

We claim that $\bigcap_j \mathit{exterior}_j$ is connected. To show this, consider $x, y \in \bigcap_j \mathit{exterior}_j$. We modify the straight path $\gamma$ in $\mathbb{R}^n$ from $x$ to $y$ on each interval $[a, b]$ it spends in $\mathit{interior}_j$ for some $j$: since $\gamma(a), \gamma(b)$ lie in $\Sigma_j$, we can replace $\gamma|[a, b]$ by a path in $\Sigma_j$ from $\gamma(a)$ to $\gamma(b)$. This yields a path from $x$ to $y$ in $\bigcap_j \mathit{exterior}_j$ and thus proves our claim.

Hence $\pi^{-1}(Z)$ is connected, and $\pi$ maps $\pi^{-1}(Z)$ diffeomorphically to $Z$. The connectedness of $M$ implies that $\pi$ is a one-sheeted covering, i.e. a diffeomorphism. □

*Remark.* In applications to positive energy theorems, one has much more information than is assumed in Lemma 5.1: one knows that $M$ (of dimension $n \geq 3$) is noncompact and contains a compact $n$-dimensional submanifold-with-boundary $C$ such that each connected component $Y$ of $M \backslash C$ is diffeomorphic to $S^{n-1} \times ]0, 1[$; the closure $Z$ in $M$ of each of these ends $Y$ is a submanifold-with-boundary of $M$ which is diffeomorphic to $S^{n-1} \times [0, 1[$ and thus satisfies the assumptions of the lemma. But all this additional information would not help much in the proof. For instance, $\pi^{-1}(C)$ could a priori still be noncompact; this makes arguments involving ends difficult.

*Proof of Theorem 1.5.* Let $\pi\colon \tilde{M} \to M$ be the universal covering of $M$, let $\tilde{g} := \pi^* g$, let $\tilde{N}$ be the pullback bundle $\pi^* N$ over $\tilde{M}$, and define $\tilde{K} = \pi^* K \in \Gamma(\mathrm{Sym}^2 T^* \tilde{M} \otimes \tilde{N})$ by $\tilde{K}(v, w) = K(\pi_* v, \pi_* w) \in N_{\pi(x)} = (\pi^* N)_x$ for all $x \in \tilde{M}$ and $v, w \in T_x \tilde{M}$. Since $(M, g, N, K)$ satisfies the Gauss and Codazzi equations for constant curvature $c$, so does $(\tilde{M}, \tilde{g}, \tilde{N}, \tilde{K})$. Being the pullback of a complete metric by a covering map, $\tilde{g}$ is complete.

Proposition 2.1 tells us that there exists an isometric immersion $(f, \iota)$ of $(\tilde{M}, \tilde{g}, \tilde{N}, \tilde{K})$ into $\mathcal{M}_c^{n,1}$; and that any two such immersions differ by an isometry of $\mathcal{M}_c^{n,1}$. Proposition 4.2 implies that $f$ is an isometric imbedding and that $\mathrm{pr} \circ f\colon \tilde{M} \to \mathbb{R}^n$ is a diffeomorphism. We identify $\tilde{M}$ with $\mathbb{R}^n$ via $\mathrm{pr} \circ f$.

Lemma 5.1 shows that the covering $\pi\colon \mathbb{R}^n \to M$ is a diffeomorphism. $(\tilde{M}, \tilde{g}, \tilde{N}, \tilde{K})$ and $(M, g, N, K)$ can be identified via $\pi$, and the theorem follows. □

*Remark 1.* The proof here is similar to the work of Maerten [5, second half of the proof of the first theorem in Section 4] (which deals with the case $c < 0$ on a spin manifold) insofar as both employ the universal covering of $M$ and argue that it is one-sheeted. Maerten uses apparently a statement similar to Lemma 5.1 at the end of his proof, but does not give a reference or spell out the details.

*Remark 2.* The proof of the positive energy theorem in [6] yields already the information that the hypersurface $M$ has only one end in the rigidity case. The arguments above provide a second, independent proof that $M$ has only one end.

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