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Class a spacetimes

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CLASS A SPACETIMES

Abstract. We introduce class A spacetimes, i.e. compact vicious spacetimes (M,g) such that the Abelian cover $(\overline{M},\overline{g})$ is globally hyperbolic. We study the main properties of class A spacetimes using methods similar to the one introduced in [19] and [3]. As a consequence we are able to characterize manifolds admitting class A metrics completely as mapping tori. The set of class A spacetimes is shown to be open in the C^0-topology on the set of Lorentzian metrics. As an application we prove a coarse Lipschitz property for the time separation of the Abelian cover.

1. Introduction

The theory of compact Lorentzian manifolds is in large parts terra incognita. In opposition to Riemannian geometry, Lorentzian geometry is focused on noncompact manifolds, for well known reasons motivated by physical intuition in general relativity. The situation with compact Lorentzian manifolds is vague to the extent that there is no well established large subclass of compact Lorentzian manifolds with well understood geometric features. It is the purpose of these notes to propose one such class (class A) and study some of its properties. The main application for these spacetimes will be the study of homologically maximizing causal geodesics (Aubry-Mather theory) in subsequent publications.

A compact spacetime (M,g) is said to be *class A* if (M,g) is vicious and the Abelian cover is globally hyperbolic. A spacetime is called *vicious* if every point lies on a timelike loop. Equivalently one can suppose that the chronological past and future of every point are equal to the entire manifold. A spacetime (M,g) is *globally hyperbolic* if the there exists a subset $S\subseteq M$ such that every inextendable timelike curves intersects S exactly once.

First examples of class A spacetimes are flat Lorentzian tori, i.e. quotients of Minkowski space by a cocompact lattice. Other known examples are spacetime structures on 2-tori admitting either a timelike or spacelike conformal Killing vector field ([18]).

This simple definition in terms of causality conditions yields surprising restrictions on the topological and geometric structure of these spacetimes. The main results of these notes are theorem 4.8 and 4.13. Theorem 4.8 has two important corollaries (theorem 4.3, corollary 4.10). Theorem 4.3 states that the set of class A metrics (i.e. Lorentzian metrics on M such that (M,g) is class A) is open in the C^0-uniform topology on the space of Lorentzian metrics $\mathrm{Lor}(M)$ on M. This represents a uniform version of theorem 12 of [9]: For any globally hyperbolic spacetime (M,g) there exists an open neighborhood U of g in $Lor(M)$, equipped with the fine C^0-topology, such that any Lorentzian metric $g_1\in U$ is globally hyperbolic as well. Note that one cannot use Geroch's neighborhoods from [9] for $\overline{g}:=\overline{\pi}^*g$ directly, since the topology induced on $\mathrm{Lor}(M)$ by the canonical projection $\overline{\pi}\colon\overline{M}\to M$ of the Abelian cover $\overline{M}$ is finer than the uniform topology on $\mathrm{Lor}(M)$, and therefore $\overline{g}$ might be the only periodic Lorentzian metric in U.

Corollary 4.10 gives a precise characterization of manifolds that admit class A metrics. Like in the case of globally hyperbolic spacetimes, existence of class A spacetime structures induce strong restrictions on the topology of M, i.e. there exists a class A metric in $\mathrm{Lor}(M)$ iff M is diffeomorphic to a mapping torus. Note that this result can be seen as an analogue of the global splitting theorem for globally hyperbolic spacetimes ([9],[2]) for compact spacetimes.

The proof of theorem 4.8 incorporates several different constructions and methods, e.g. Sullivan's structure cycles ([19], see appendix A), a generalization of a methods introduced by D. Yu Burago ([3]) and the construction of the homological timecone $\mathfrak{T}$ (see section 4). The homological timecone can be seen as an asymptotic (i.e. stable) version of the causality relations in the Abelian cover, much in the same way the stable norm on $H_1(M,\mathbb{R})$ ([10] 4.19) can be seen as an asymptotic version of the Riemannian distance function on the Abelian cover. The example contructed in 4 shows that the result of theorem 4.8 is in some respect optimal.

The second main result theorem 4.13 claims the coarse Lipschitz property of the time separation (Lorentzian distance) of the Abelian cover of a class A spacetime. The Lipschitz continuity of the time separation has received very little attention in the literature so far. It made a short appearance in connection with the Lorentzian version of the Cheeger-Gromoll splitting theorem ([5], [6]). The idea we employ here is different from the approaches before and is based on socalled cut-and-paste arguments commonly used in Aubry-Mather theory ([1],[14]).

The text is structured as follows. In section 2 we collect the necessary notions from Lorentzian and Riemannian geometry and set the global notation. In section 3 we review previous work on Lorentzian surfaces and globally conformally flat tori. In section 4 we define class A spacetimes and introduce the stable time cone $\mathfrak{T}$, the homological equivalent of the causal future. Further the section discusses the main results and examples mentioned so far. Finally sections 5 and 6 contain the proofs of theorem 4.8 resp. 2.3.

2. Geometric Notions and Notation

Notation. $\mathcal{D}(M',M)$ denotes the group of deck transformations for a regular cover $\pi'\colon M'\to M$. By $\overline{M}$ we denote the quotient of the universal cover $\widetilde{M}$ by the commutator group of $\pi_1(M)$, i.e. $\overline{M}\cong\widetilde{M}/[\pi_1(M),\pi_1(M)]$. $\overline{M}$ be called the *Abelian cover* of M. Denote with $\overline{\pi}$ the canonical projection of $\overline{M}$ to M. Further we denote with $H_1(M,\mathbb{Z})_\mathbb{R}$ the image of the natural map $H_1(M,\mathbb{Z})\to H_1(M,\mathbb{R})$.

Lorentzian Geometry. Denote by $[g]$ the conformal class of the Lorentzian metric g sharing the same time-orientation, i.e. all Lorentzian metrics g' such that there exists a $u\in C^\infty(M)$ with $g'=e^u g$ and $v\in TM$ is future pointing for g if and only if v is future pointing for g'. Further define the sets

$$\mathrm{Time}(M,[g]):=\{\text{future pointing timelike vectors in } (M,g)\}$$

and

$$\mathrm{Light}(M,[g]):=\{\text{future pointing lightlike vectors in } (M,g)\}.$$

Both $\mathrm{Time}(M,[g])$ and $\mathrm{Light}(M,[g])$ are smooth fibre bundles over M (Recall that $0\in TM_p$ is not a causal vector). Denote by $\mathrm{Time}(M,[g])_p$ and $\mathrm{Light}(M,[g])_p$ the fibres of $\mathrm{Time}(M,[g])$ and $\mathrm{Light}(M,[g])$ over $p\in M$, respectively. For $\varepsilon>0$ we define

$$\mathrm{Time}(M,[g])^\varepsilon:=\{v\in\mathrm{Time}(M,[g])|\ \mathrm{dist}(v,\mathrm{Light}(M,[g])\geq\varepsilon|v|\}.$$

$\mathrm{Time}(M,[g])^\varepsilon$ is a smooth fibre bundle as well with fibre $\mathrm{Time}(M,[g])^\varepsilon_p$ over $p\in M$. The fibres are convex for every $p\in M$ according to the following lemma and corollary.

Lemma 2.1. *Let $(V,|.|)$ be a finite-dimensional normed vector space and $V\neq\mathfrak{K}\subseteq V$ a convex set. Then the function $v\in\mathfrak{K}\mapsto\mathrm{dist}_{|.|}(v,\partial\mathfrak{K})$ is concave.*

The proof is an exercise in convex geometry. See [4] theorem 1.10 for a proof in the more general case that (M, g_R) is Riemannian manifold of nonnegative curvature.

If $\mathfrak{K}$ is a convex cone we know that $v \in \mathfrak{K} \mapsto \operatorname{dist}_{|.|}(v, \partial\mathfrak{K})$ is positively homogenous of degree one, i.e. $\operatorname{dist}_{|.|}(\lambda v, \partial\mathfrak{K}) = \lambda \operatorname{dist}_{|.|}(v, \partial\mathfrak{K})$ for all $\lambda \geq 0$. Lemma 2.1 and the positive homogeneity then imply

$$\operatorname{dist}_{|.|}(v + w, \partial\mathfrak{K}) \geq \operatorname{dist}_{|.|}(v, \partial\mathfrak{K}) + \operatorname{dist}_{|.|}(w, \partial\mathfrak{K}).$$

Corollary 2.2. *Let $\mathfrak{K} \neq V$ be a convex cone and $\varepsilon > 0$. The cones $\mathfrak{K}_\varepsilon := \{v \in V |\ \operatorname{dist}_{|.|}(v, \partial\mathfrak{K}) \geq \varepsilon|v|\}$ are convex for all $\varepsilon > 0$.*

Riemannian structures. We will need the concept of *rotation vectors* from [14]. Let $k_1, \ldots, k_b$ ($b := dim\ H_1(M, \mathbb{R})$) be a basis of $H_1(M, \mathbb{R})$ consisting of integer classes, and $\alpha_1, \ldots, \alpha_b$ the dual basis with representatives $\omega_1, \ldots, \omega_b$. For two points $x, y \in \overline{M}$ we define the *difference* $y - x \in H_1(M, \mathbb{R})$ via a C^1-curve $\gamma\colon [a, b] \to \overline{M}$ connecting x and y, by

$$\alpha_i(y - x) := \int_\gamma \pi^* \omega_i$$

for all $i \in \{1, \ldots, b\}$. The rotation vector of γ as well as of $\overline{\pi} \circ \gamma$ is defined as

$$\rho(\gamma) = \rho(\overline{\pi} \circ \gamma) := \frac{1}{b - a}(y - x).$$

Note that the map $(x, y) \mapsto y - x$ is i.g. not surjective. But we know that the convex hull of the image is equal to $H_1(M, \mathbb{R})$. Just observe that by our choice of classes α_i we know that every $k \in H_1(M, \mathbb{Z})_\mathbb{R}$ is the image of $(x, x + k)$ for every $x \in \overline{M}$.

We choose a Riemannian metric g_R on M arbitrary but fixed once and for all. We denote the distance function relative to g_R by dist and the metric balls of radius r around $p \in M$ with $B_r(p)$. The metric g_R induces a norm on every tangent space of M which we denote by $|.|$, i.e. $|v| := \sqrt{g_R(v, v)}$ for all $v \in TM$. For convenience of notation we denote the lift of g_R to $\overline{M}$, and all objects associated to it, with the same letter. Set

$$\operatorname{diam}(M, g_R) := \max_{p \in M} \min_{k \in H_1(M,\mathbb{Z}) \setminus \{0\}} \{\operatorname{dist}(\overline{p}, \overline{p} + k) |\ \overline{p} \in \overline{\pi}^{-1}(p)\}$$

the homological diameter of (M, g_R).

We will constantly employ the following theorem.

Theorem 2.3 ([3], [13]). *Let (M, g_R) be a compact Riemannian manifold. Then there exists a unique norm $\|.\|\colon H_1(M, \mathbb{R}) \to \mathbb{R}$ and a constant $\operatorname{std}(g_R) < \infty$ such that*

$$|\operatorname{dist}(x, y) - \|y - x\|| \leq \operatorname{std}(g_R)$$

for any $x, y \in \overline{M}$.

Further denote with $\|.\|$ the stable norm of g_R on $H_1(M, \mathbb{R})$. The distance function on $H_1(M, \mathbb{R})$ relative to $\|.\|$ is written as $\operatorname{dist}_{\|.\|}$. By $\|.\|^*$ we denote its dual norm on $H^1(M, \mathbb{R})$.

3. Preceding Work

There exist two of preceeding studies addressing similar problems as studied here. [16] considers class A spacetimes in dimension 2, though he uses a different characterization using the lightlike distributions, which is in fact equivalent for time orientable Lorentzian surfaces (see [18]).

The second study [17], is concerned with the problem of Lipschitz continuity of the time separation in the Abelian cover of a globally conformally flat Lorentzian torus. Note that globally conformally flat Lorentzian tori are trivially of class A.
Lorentzian surfaces. For details of this exposition see [18].

Locally every Lorentzian surface gives rise to two transversal lightlike distributions. In general these distributions are not globally well defined. Note that they are globally well defined if and only if M^2 is orientable.

Assume that M^2 is orientable. To every nonsingular distribution $\mathfrak{D}$ on a surface M^2 we can canonically associate a class, called the *rotation class*, $m^{\mathfrak{D}} \in PH_1(M^2, \mathbb{R})$, the projective space over the first real homology vector space of M^2:

$$m^{\mathfrak{D}} = \lim_{\|\zeta(T)-\zeta(T')\|\to\infty} [span(\zeta(T) - \zeta(T'))] \in PH_1(M, \mathbb{R}),$$

where $\zeta\colon \mathbb{R} \to M$ is any piecewise regular curve tangential to $\mathfrak{D}$ (for the definition of $\zeta(T) - \zeta(T')$ see section 2).

Call (M^2, g) space orientable, if (M^2, g) admits a spacelike nonsingular vector field. This is equivalent to $(M^2, -g)$ being time orientable. In this notation the following conditions are equivalent:

(i) The lightlike distributions are orientable.
(ii) (M^2, g) is time and space orientable.
(iii) M is orientable and (M^2, g) is time orientable
(iv) M is orientable and (M^2, g) is space orientable

Recall that any compact Lorentzian manifold admits a twofold time orientable covering ([8]). Therefore any compact Lorentzian manifold admits a, at most, fourfold orientable and time orientable covering.

Assume now that the lightlike distributions are well defined and orientable, i.e. there exist two future pointing lightlike vector fields X^+ and X^- such that $\{X_p^+, X_p^-\}$ is a positive oriented basis of TM_p^2 for all $p \in M^2$. Define $\mathfrak{D}^+$ through $X^+ \in \mathfrak{D}^+$ and $\mathfrak{D}^-$ through $X^- \in \mathfrak{D}^-$. It is obvious that $\mathfrak{D}^\pm \in \Gamma^\infty(G_1TM)$. Abridge $m^\pm := m^{\mathfrak{D}^\pm}$.

Proposition 3.1 ([18])**.** *A closed 2-dimensional spacetime (M, g) is of class A if and only if $(m')^+ \neq (m')^-$ for one (hence every) finite orientable covering (M', g') of (M, g).*

Note that the condition "$m^+ \neq m^-$" is only sensible if the underlying closed surface is orientable, since otherwise, i.e. $M \cong$ Klein bottle, $H_1(M, \mathbb{R}) \cong \mathbb{R}$.

If $m^+ \neq m^-$ and the lightlike curve ζ is future pointing, all homology classes $\zeta(T_2) - \zeta(T_1)$ $(T_1 \leq T_2)$ lie in a bounded distance to a halfline $\overline{m}^{\mathfrak{D}}$ of $m^{\mathfrak{D}}$. This halfline again depends only on the chosen oriented lightlike distribution $\mathfrak{D}$. Consequently, instead of the projective class $m^{\mathfrak{D}}$, only a halfline of $m^{\mathfrak{D}}$ needs to be considered to distinguish the asymptotic direction of $\mathfrak{D}$. Denote by $\mathfrak{T}$ the convex hull of $\overline{m}^+ \cup \overline{m}^-$.

Lorentzian conformally-flat n-tori. Consider a real vector space V of dimension $m < \infty$ and $\langle .,. \rangle_1$ a nondegenerate symmetric bilinear form on V with signature $(-, +, \ldots, +)$. Further let $\Gamma \subseteq V$ be a co-compact lattice and $f\colon V \to (0, \infty)$ a smooth and Γ-invariant function. The Lorentzian metric $\overline{g} := f^2\langle .,. \rangle_1$ then descends to a Lorentzian metric on the torus V/Γ. Denote the induced Lorentzian metric by g. Choose a time orientation of $(V, \langle .,. \rangle_1)$. This time orientation induces a time orientation on $(V/\Gamma, g)$ as well. Note that $(V/\Gamma, g)$ is vicious and the universal cover $(V, \overline{g})$ is globally hyperbolic. According to [15] proposition 2.1, $(V/\Gamma, g)$ is geodesically complete in all three causal senses. Fix a norm $\|.\|$ on V and denote the dual norm by $\|.\|^*$. Note that $\|.\|$ induces a metric on V/Γ. Further denote by $\mathfrak{T}$ the positively oriented causal vectors of $(V, \langle .,. \rangle_1)$.

For $\varepsilon > 0$ set $\mathfrak{T}_\varepsilon := \{v \in \mathfrak{T} |\ \mathrm{dist}(v, \partial\mathfrak{T}) \geq \varepsilon\|v\|\}$. Choose an orthonormal basis $\{e_1, \ldots, e_m\}$ of $(V, \langle.,.\rangle_1)$. Note that the translations $x \mapsto x + v$ are conformal diffeomorphisms of $(V, \overline{g})$ for all $v \in V$. Then the $\overline{g}$-orthogonal frame field $x \mapsto (x, (e_1, \ldots, e_m))$ on V descends to a g-orthogonal frame field on V/Γ. Relative to this identification of $V \cong TV_p$ follows $\mathfrak{T} = \mathrm{Time}(V, [\overline{g}])_p \cup \mathrm{Light}(V, [\overline{g}])_p$ and $\mathfrak{T}_\varepsilon = \mathrm{Time}(V, [\overline{g}])_p^\varepsilon$.

[17] contains the following compactness result for future pointing maximizers in $(V/\Gamma, g)$.

Theorem 3.2 ([17])**.** *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\dot\gamma(t) \in \mathfrak{T}_\delta$$

for all future pointing maximizers $\gamma\colon I \to V/\Gamma$ with $\dot\gamma(t_0) \in \mathfrak{T}_\varepsilon$ for some $t_0 \in I$ and all $t \in I$.

Theorem 3.2 has the following immediate consequence.

Corollary 3.3 ([17])**.** *Let $\varepsilon > 0$. Then any limit curve of a sequence of future pointing maximizers $\gamma_n\colon I_n \to V/\Gamma$ with $\dot\gamma_n(t_n) \in \mathfrak{T}_\varepsilon$, for some $t_n \in I_n$, is timelike.*

The author then deduces, following [6], the Lipschitz continuity of the time separation d of $(V, \overline{g})$ on $\{(p, q) \in V \times V |\ q - p \in \mathfrak{T}_\varepsilon\}$ for every $\varepsilon > 0$. Using the standard argument that local Lipschitz continuity with a fixed Lipschitz constant implies Lipschitz continuity, one obtains the following theorem.

Theorem 3.4 ([17])**.** *For all $\varepsilon > 0$ there exists $L = L(\varepsilon) < \infty$ such that the time separation d of $(V, \overline{g})$ is L-Lipschitz on $\{(x, y) \in V \times V | \, y - x \in \mathfrak{T}_\varepsilon\}$.*

4. Causality Properties of Class A Spacetimes

Recall the definition of class A spacetimes.

Definition 4.1. *A compact spacetime (M, g) is of class A if (M, g) is vicious and the Abelian cover $\overline{\pi}\colon (\overline{M}, \overline{g}) \to (M, g)$ is globally hyperbolic. We call a metric $g \in \mathrm{Lor}(M)$ class A iff (M, g) is class A.*

For a spacetime to be of class A is purely a condition on the causal structure. So any spacetime globally conformal to a class A spacetime is class A as well.

Both conditions on class A spacetimes are independent of each other in the sense that neither viciousness of (M, g) implies the global hyperbolicity of $(\overline{M}, \overline{g})$ (even if $\dim H_1(M, \mathbb{R}) > 0$), nor does the global hyperbolicity of $(\overline{M}, \overline{g})$ imply the viciousness of (M, g).

Note that $b := \dim H_1(M, \mathbb{R}) > 0$ for any class A spacetime. Else $\overline{M}$ would be a finite cover of M and the causality of $(\overline{M}, \overline{g})$ would be violated. This is due to the fact that any finite cover of a non-causal spacetime is again non-causal. In fact even more is true, any finite cover of a vicious spacetime is again vicious.

The global hyperbolicity of $(\overline{M}, \overline{g})$ does not depend on the choice of a torsion free Abelian cover or the Abelian covering with torsion, i.e. if the group of deck transformations is isomorphic to $H_1(M, \mathbb{Z})$ or its image $H_1(M, \mathbb{Z})_\mathbb{R} \subseteq H_1(M, \mathbb{R})$ under the natural homomorphism $H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{R})$. In the subsequent discussion we will always assume that the group of deck transformations is given by the lattice $H_1(M, \mathbb{Z})_\mathbb{R}$.

Remark 4.2. *A cover (M', g') of a globally hyperbolic spacetime (M, g) is always globally hyperbolic. Conversely a spacetime (M, g) is globally hyperbolic if it is finitely covered by a globally hyperbolic spacetime.*

Proof. [12], proposition 1.4. □ □

Note that the global hyperbolicity of the universal cover $(\widetilde{M},\widetilde{g})$ i.g. does not imply the global hyperbolicity of the Abelian cover. An explicit example can be deduced from [11].

Natural examples for class A spacetimes are product manifolds $M = S^1 \times N$, where N is a compact manifold, and the Lorentzian metric is given by $g := -f^2 d\varphi^2 + \beta d\varphi + h$. Here f is a smooth non-vanishing function on M, $\beta \in \Lambda^1(S^1 \times N, TN)$ and h is a Riemannian metric on N periodic in the S^1-coordinate.

The S^1-coordinate loops are closed timelike curves by definition. This implies the viciousness of the spacetimes. Since the coverings $\mathbb{R} \times N \to S^1 \times N$ are globally hyperbolic, the spacetimes are of class A by remark 4.2.

Note that in these examples the set of class A metrics forms an open subset of $\mathrm{Lor}(S^1 \times N)$ in the C^0-topology. This observation is not limited to these examples.

Theorem 4.3. *For every compact manifold M the set*

$$\{g \in \mathrm{Lor}(M) |\ (M, g) \text{ is of class A}\}$$

is open in the C^0-topology on $\mathrm{Lor}(M)$.

The set of class A metrics in $\mathrm{Lor}(M)$ may be empty, even if $\chi(M) = 0$ (e.g. $M \cong S^3$).

Theorem 4.3 will be the consequence of another result giving a characterization of manifolds admitting a class A metric.

Next note the following simple technical fact about compact vicious spacetimes.

Fact 4.4. *Let M be compact and (M, g) vicious. Then there exists a constant $\mathrm{fill}(g, g_R) < \infty$ such that any two points $p, q \in M$ can be joined by a future pointing timelike curve with g_R-arclength less than $\mathrm{fill}(g, g_R)$.*

Next we introduce the main technical object of these notes. Recall for $x, y \in \overline{M}$ the definition of $y - x \in H_1(M, \mathbb{R})$ and $\rho(\gamma)$ for a Lipschitz curve $\gamma\colon [a, b] \to \overline{M}$ from section 2. Consider a future pointing curve $\gamma\colon [a, b] \to M$ parameterized by g_R-arclength. A sequence of such curves $\{\gamma_i\}_{i\in\mathbb{N}}$ is called admissible, if $L^{g_R}(\gamma_i) \to \infty$ for $i \to \infty$. $\mathfrak{T}^1$ is defined to be the set of all accumulation points of sequences $\{\rho(\gamma_i)\}_{i\in\mathbb{N}}$ in $H_1(M, \mathbb{R})$ of admissible sequences $\{\gamma_i\}_{i\in\mathbb{N}}$. $\mathfrak{T}^1$ is compact for any compact spacetime since the stable norm of any rotation vector is bounded by $1 + \mathrm{std(g_R)}$ (theorem 2.3). If (M, g) is vicious, $\mathfrak{T}^1$ is convex by note 4.4.

We define the *stable time cone* $\mathfrak{T}$ to be the cone over $\mathfrak{T}^1$. Note that $\mathfrak{T}$ does not depend on the choice of g_R, $\{k_1, \ldots, k_b\}$ and $\omega_i \in \alpha_i$, whereas $\mathfrak{T}^1$ does. Reversing the time-orientation yields $-\mathfrak{T}$ as stable time cone. $\mathfrak{T}$ is invariant under global conformal changes of the metric and therefore depends only on the causal structure of (M, g). It coincides with the cone of rotation vectors of structure cycles defined in appendix A. As noted there, in this case the cone structure is given by the positively oriented causal vectors. Further it is easy to see that this definition of $\mathfrak{T}$ coincides with the ones given in section 3.

For compact and vicious spacetimes the stable time cone is characterized uniquely by the following property.

Proposition 4.5. *Let (M, g) be a compact and vicious spacetime. Then $\mathfrak{T}$ is the unique cone in $H_1(M, \mathbb{R})$ such that there exists a constant $\mathrm{err}(g, g_R) < \infty$ with $\mathrm{dist}_{\|.\|}(J^+(x) - x, \mathfrak{T}) \leq \mathrm{err}(g, g_R)$ for all $x \in \overline{M}$, where $J^+(x) - x := \{y - x |\ y \in J^+(x)\}$.*

Compare this result to theorem 2.3.

We will give a proof of proposition 4.5 in section 5.2. Note that by note 4.4 the distance of $\gamma(b) - \gamma(a)$ to $\mathfrak{T}$ is uniformly bounded by $\mathrm{fill}(g, g_R) + \mathrm{std(g_R)}$ (theorem 2.3) for any future pointing curve $\gamma\colon [a, b] \to M$. Therefore the $J^+(x) - x$ is contained in the $\text{fill}(g, g_R)+\text{std}(\text{g}_\text{R})$-neighborhood of $\mathfrak{T}$ for every $x \in \overline{M}$. It remains to show the existence of a real number $K < \infty$ such that $\mathfrak{T}$ is contained in the K-neighborhood of $J^+(x) - x$.

Proposition 4.6. *If (M, g) is vicious, $\mathfrak{T}^\circ$ is nonempty.*

We will give a proof of this proposition in section 6.
Structure results. We denote with $\mathfrak{T}^*$ the *dual stable time cone* of $\mathfrak{T}$, i.e.

$$\mathfrak{T}^* := \{\alpha \in H^1(M, \mathbb{R})|\ \alpha|_{\mathfrak{T}} \geq 0\}.$$

Definition 4.7. *A spacetime is cylindrical if it is globally hyperbolic and contains a compact Cauchy hypersurface.*

Theorem 4.8. *Let (M, g) be compact and vicious. Then the following statements are equivalent:*

(i) *(M, g) is of class A.*
(ii) *$0 \notin \mathfrak{T}^1$, especially $\mathfrak{T}$ is a compact cone (see appendix A).*
(iii) *$(\mathfrak{T}^*)^\circ \neq \emptyset$ and for every $\alpha \in (\mathfrak{T}^*)^\circ$ there exists a smooth 1-form $\omega \in \alpha$ such that $\ker \omega_p$ is spacelike in (TM_p, g_p) for all $p \in M$, i.e. ω is a closed transversal form for the cone structure of future pointing vectors in (M, g).*
(iv) *(M, g) admits a normal cylindrical covering $(\overline{M}, \overline{g}) \to (M', g') \to (M, g)$ such that $\mathcal{D}(M', M) \cong \mathbb{Z}$.*

The proof of theorem 4.8 will be given in section 5. Next we will discuss two applications of theorem 4.8 and show that theorem 4.3 follows from theorem 4.8. After that we construct an example showing that the assumption of viciousness is essential in theorem 4.8.

Corollary 4.9. *Let (M, g) be of class A. Then there exists a constant $C_{g,g_R} < \infty$ such that*

$$L^{\overline{g}_R}(\gamma) \leq C_{g,g_R} \operatorname{dist}(p, q)$$

for all $p, q \in \overline{M}$ and $\gamma \in \mathcal{C}(p, q)$.

Proof. Clear from theorem 4.8(iii). □ □

Corollary 4.10. *Let M be a closed manifold with $\chi(M) = 0$. Then the set of class A metrics in $\text{Lor}(M)$ is nonempty if and only if M is diffeomorphic to a mapping torus over a closed manifold N. Further any class A spacetime gives rise to a foliation by smooth compact spacelike hypersurfaces.*

Remark 4.11. *In the light of the differential splitting theorem for globally hyperbolic spacetimes ([2]), the corollary is not completely surprising. In fact one should expect a similar result for compact spacetimes which are covered by a globally hyperbolic one. That it fails if one drops the assumption of viciousness is the subject of Example 4.*

Corollary 4.10. (i) Let (M, g) be of class A. Choose a cohomology class α with representative ω according to theorem 4.8(iii). W.l.o.g. we can assume that we have $\alpha(H_1(M, \mathbb{Z})_\mathbb{R}) \subseteq \mathbb{Z}$. Let $f\colon \overline{M} \to \mathbb{R}$ be a primitive of $\overline{\pi}^*\omega$, $f^{-1}(\tau) \subseteq \overline{M}$ any level set of f and $x \in f^{-1}(\tau)$. By our choice of α every levelset $f^{-1}(\tau)$ descends to a compact hypersurface in M.

Denote with $\omega^\sharp$ the pointwise g_R-dual of ω and set

$$X^\omega := \frac{1}{g_R(\omega^\sharp, \omega^\sharp)} \omega^\sharp.$$

For the flow Φ^ω of X^ω we know that $\Phi(., t)\colon \Sigma_\tau \to \Sigma_{\tau+t}$ $(\omega(X^\omega) \equiv 1)$. Then M is diffeomorphic to the mapping torus $\Sigma_\tau \times_{\Phi(.,\alpha(k_1))} \mathbb{R}$ for all $\tau \in \mathbb{R}$.

Since we can choose ω such that $\ker \omega_p = T(\Sigma_\tau)_p$ is spacelike, we obtain a foliation of M by by compact spacelike hypersurfaces.

(ii) Let $N \times_\Phi \mathbb{R}$ be a mapping torus defined as the quotient of $N \times \mathbb{R}$ and the group of diffeomorphisms $\{(x,t) \mapsto (\Phi^n(x), t+n)\}_{n\in\mathbb{N}}$. Let g_R be a Riemannian metric on $N \times_\Phi \mathbb{R}$. We can assume that the vector field ∂'_t on $N \times_\Phi \mathbb{R}$ induced by the embeddings $\mathbb{R} \hookrightarrow N \times \mathbb{R}$ is orthogonal to N and of unit length. It is clear that

$$g := g_R - 2(\partial'_t)^\flat \otimes (\partial'_t)^\flat$$

is a Lorentzian metric on $N \times_\Phi \mathbb{R}$. Since $g|_{N\times N} \equiv g_R|_{N\times N}$, N is a spacelike submanifold of $N \times_\Phi \mathbb{R}$ under the natural embedding. ∂'_t is timelike for g and induces a time-orientation on $(N \times_\Phi \mathbb{R}, g)$. The spacetime $(N \times_\Phi \mathbb{R}, g)$ is vicious since any path $\gamma\colon [a,b] \to N \times_\Phi \mathbb{R}$ parameterized w.r.t. g_R-arclength can be "twisted" to a timelike curve in $(N \times_\Phi \mathbb{R}, g)$. Choose a lift $\overline{\gamma}$ of γ to $N \times \mathbb{R}$ and an integer $n > b-a$. Set

$$\widetilde{\gamma}(t) := \overline{\gamma}(t) + \frac{n}{b-a}(0, t-a).$$

The projection of $\widetilde{\gamma}$ is a timelike curve in $N \times_\Phi \mathbb{R}$ connecting $\gamma(a)$ with $\gamma(b)$. This yields $(N \times_\Phi \mathbb{R}, g)$ as a vicious spacetime.

The differential of the projection $\pi_2\colon N \times \mathbb{R} \to \mathbb{R}$ induces on $N \times_\Phi \mathbb{R}$ a smooth closed 1-form ω such that $\ker \omega_p$ is spacelike for all $p \in N \times_\Phi \mathbb{R}$. Then $(N \times_\Phi \mathbb{R}, g)$ is a class A spacetime, according to theorem 4.8 (iii). □ □

Theorem 4.3. The openness of the viciousness condition was already proven in fact 4.4. Consequently it remains to verify the condition $(\overline{M}, \overline{g})$ ***globally hyperbolic*** is open in the C^0 topology on $\mathrm{Lor}(M)$ in the case that (M,g) is vicious.

Consider a smooth and closed 1-form ω on M such that $\ker \omega_p$ is spacelike for all $p \in M$. Next consider the set $\mathcal{G}(\omega) \subseteq Lor(M)$ of metrics g_1 such that $\ker \omega_p$ is g_1-spacelike for all $p \in M$. $\mathcal{G}(\omega)$ is certainly an open neighborhood of g in $Lor(M)$.

Let $g_1 \in \mathcal{G}(\omega)$. We want to show that the lift $\overline{g}_1$ of g_1 to $\overline{M}$ is globally hyperbolic. Since $\ker \omega_p$ is g_1-spacelike for all $p \in M$, any primitive $\tau_\omega\colon \overline{M} \to \mathbb{R}$ of $\overline{\pi}^*\omega$ is a temporal function for $(\overline{M}, \overline{g}_1)$. By the compactness of M there exists $\varepsilon_1 > 0$ such that we have $|d\tau_\omega(v)| \geq \varepsilon_1 |v|$ for all $\overline{g}_1$-nonspacelike $v \in T\overline{M}$.

Let $\gamma\colon \mathbb{R} \to \overline{M}$ be an inextendible $\overline{g}_1$-nonspacelike curve parameterized w.r.t. $\overline{g}_R$-arclength. W.l.o.g. we can assume that $\tau_\omega \circ \gamma$ is increasing, i.e. we have $d\tau_\omega(\dot\gamma(t)) \geq \varepsilon_1 |\dot\gamma(t)|$ whenever $\dot\gamma(t)$ exists. Let $\Sigma := \tau_\omega^{-1}(\sigma)$ be any level set of τ_ω. We want to show that γ intersects Σ exactly once. Then we are done, since by that property Σ is a Cauchy hypersurface of $(\overline{M}, \overline{g}_1)$. This is equivalent to the global hyperbolicity of $(\overline{M}, \overline{g}_1)$.

Set $\sigma_0 := \tau_\omega(\gamma(0))$. For $r \geq \frac{|\sigma - \sigma_0|}{\varepsilon_1}$ we have

$$|\tau_\omega(\gamma(\pm r)) - \sigma_0| = |\int_0^{\pm r} d\tau_\omega(\dot\gamma)| \geq \varepsilon_1 r \geq |\sigma - \sigma_0|.$$

Then t is either contained in the interval $[\tau_\omega(\gamma(-r)), \sigma_0]$ or $[\sigma_0, \tau_\omega(\gamma(r))]$. By the intermediate value theorem γ has to intersect Σ. Since τ_ω is strictly increasing along γ, the intersection is unique. □

Example. The assumption of viciousness on (M,g) in theorem 4.8 cannot be dropped. Examples of compact spacetimes with globally hyperbolic Abelian covering space and no cylindrical covering or transversal closed 1-form can be constructed as follows.

Consider $\mathbb{R}^3$ with the canonical coordinates $\{x, y, z\}$. Denote with $\overline{T}_i := x^{-1}(i)$ for $i = 1, \ldots, 6$. Choose a $7 \cdot \mathbb{Z}^3$-invariant Lorentzian metric $\overline{g}$ on $\mathbb{R}^3$ subject to the following conditions:

(i) $\overline{g}|_{\overline{T}_1 + (7\mathbb{Z})e_1} = \overline{g}|_{\overline{T}_4 + (7\mathbb{Z})e_1} = (dx + dz)dx + dy^2$,

(ii) $\overline{g}|_{\overline{T}_3+(7\mathbb{Z})e_1} = \overline{g}|_{\overline{T}_6+(7\mathbb{Z})e_1} = (dx - dz)dx + dy^2$,
(iii) $\overline{g}|_{\overline{T}_2+(7\mathbb{Z})e_1} = -dydz + dx^2$,
(iv) $\overline{g}|_{\overline{T}_5+(7\mathbb{Z})e_1} = dydz + dx^2$ and
(v) $\ker dz_p$ is spacelike for all $p \notin (\overline{T}_2 \cup \overline{T}_5) + 7\mathbb{Z}e_1$.
(vi) $(\mathbb{R}^3, \overline{g})$ contains a timelike periodic curve $\overline{\gamma}\colon [0,1] \to \mathbb{R}^3$ with $\overline{\gamma}(1) - \overline{\gamma}(0) = 7e_3$

Since $\mathbb{R}^3$ is simply connected we can choose a time-orientation for $(\mathbb{R}^3, \overline{g})$. Choose the time-orientation such that dz is nonnegative on future pointing vectors. Note that by condition (v) the real number $dz(v)$ is either positve or negative for every nonspacelike vector $v \in T\mathbb{R}^3$ for except $v = \partial_y$ and $\pi_{T\mathbb{R}^3}(v) \in \overline{T}_2$ or $v = -\partial_y$ and $\pi_{T\mathbb{R}^3}(v) \in \overline{T}_5$.

We can choose $\varepsilon > 0$ such that $\tau_1\colon \mathbb{R}^3 \to \mathbb{R}$, $p \mapsto \varepsilon y(p) + z(p)$ is a temporal function for $x(p) \in [-1, 4] + 7\mathbb{Z}$ and $\tau_2\colon \mathbb{R}^3 \to \mathbb{R}$, $p \mapsto -\varepsilon y(p) + z(p)$ is a temporal function for $x(p) \in [3, 8] + 7\mathbb{Z}$. Therefore there exists $\varepsilon' > 0$ such that $|d\tau_1(v)|$ or $|d\tau_2(v)| \geq \varepsilon'|v|$ for all nonspacelike vectors $v \in T\mathbb{R}^3_p$. We know that the existence of temporal functions is sufficient for global hyperbolicity and thus we see that $([-1, 4] + 7\mathbb{Z}, \overline{g})$ and $([5, 8] + 7\mathbb{Z}, \overline{g})$ are globally hyperbolic. Note that any future pointing curve starting in $x^{-1}([-1, 4])$ can never leave $x^{-1}([-1, 4])$. The same holds for future pointing curves starting in $x^{-1}([3, 8])$. Together with the periodicity of $\overline{g}$, these observations imply that $(\mathbb{R}^3, \overline{g})$ is globally hyperbolic.

Since we have chosen $\overline{g}$ invariant under translations in $7 \cdot \mathbb{Z}^3$, it descends to a Lorentzian metric g on $T^3 := \mathbb{R}^3/(7\cdot\mathbb{Z}^3)$. Note (T^3, g) is time-orientable but not vicious (recall the argument that future pointing curves can never leave $x^{-1}([-1, 4])$).

Now assume that there exists a cylindrical cover $\pi'\colon (Z, g') \to (T^3, g)$ with compact Cauchy hypersurface Σ. Any lift $\overline{\Sigma}$ of Σ to $\mathbb{R}^3$ has to be a Cauchy hypersurface of $(\mathbb{R}^3, \overline{g})$ ([7]). With [2] we can assume that Σ is spacelike. Note that $(\overline{T}_2, \overline{g}|_{\overline{T}_2})$ and $(\overline{T}_5, \overline{g}|_{\overline{T}_5})$ are Lorentzian submanifolds of $(\mathbb{R}^3, \overline{g})$. Denote the projections of $\overline{T}_2$ and $\overline{T}_5$ to Z with T_2' and T_5'. Then the intersections of T_2' and T_5' with Σ are transversal and compact, since Σ is compact and spacelike. Consequently they are compact spacelike curves in (Z, g') and the fundamental classes in $\pi_1(T_2')$ resp. $\pi_1(T_5')$ are nontrivial (The lifts to $\overline{T}_2$ and $\overline{T}_5$ cannot be closed). Therefore they intersect the projections of $\{x = 2, z = z_0\}$ and $\{x = 5, z = z_0\}$ for every $z \in \mathbb{R}$.

Choose a closed curves in each intersection. The fundamental classes of the projections are contained in $\mathrm{pos}_{\mathbb{Z}}\{-7e_2, 7e_3\} \subseteq \pi_1(T^3)$ on T_2 resp. in $\mathrm{pos}_{\mathbb{Z}}\{7e_2, 7e_3\} \subseteq \pi_1(T_5)$ on T_5. Denote them by $\sigma_1 \in \mathrm{pos}_{\mathbb{Z}}\{-7e_2, 7e_3\}$ resp. $\sigma_2 \in \mathrm{pos}_{\mathbb{Z}}\{7e_2, 7e_3\}$.

Since Σ is homotopic to the cylindrical covering space, $\pi_1(\Sigma)$ can be considered as a subgroup of $\pi_1(T^3)$. But then $\mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \subseteq \pi_1(\Sigma)$.

Thus any curve representing the fundamental class $7e_3$ is of finite order in the cylindrical cover. By condition (vi) there exists a closed timelike curve γ in T^3 with fundamental class $7e_3$. The lift γ' of γ to Z has finite order and there exists a closed iterative of γ'. This clearly contradicts the causality property of (Z, g').

To see why (T^3, g) doesn't contain any closed transversal 1-form, simply note that the sum of the causal future pointing closed curves

$$\gamma_{1,2}\colon t \mapsto [(2, t, 0)], [(5, -t, 0)]$$

are nullhomologous. Therefore no closed form can be transversal to both loops.

The coarse-Lipschitz property. When comparing Lorentzian geometry with Riemannian geometry the question of Lipschitz continuity of the time separation appears naturally. As Minkowski space shows this question has no general positive answer for neither the entire set J nor I. It received some attention in the literature, though, in connection with the Cheeger-Gromoll splitting theorem for Lorentzian manifolds (see [5]).

Definition 4.12. *For* $\varepsilon > 0$ *set* $\mathfrak{T}_\varepsilon := \{h \in \mathfrak{T} |\ \mathrm{dist}_{\|.\|}(h, \partial\mathfrak{T}) \geq \varepsilon\|h\|\}$.

Theorem 4.13. *Let* (M, g) *be of class A. Then for every* $\varepsilon > 0$ *there exists* $L_c(\varepsilon) < \infty$*, such that*

$$|d(x, y) - d(z, w)| \leq L_c(\varepsilon)(\mathrm{dist}(x, z) + \mathrm{dist}(y, w) + 1)$$

for all $(x, y), (z, w) \in \overline{M} \times \overline{M}$ *with* $y - x, w - z \in \mathfrak{T}_\varepsilon$.

The stronger question of Lipschitz continuity is unanswered at this point in this generality. Note that the assumptions of theorem 4.13 are not empty due to proposition 4.6.

The proof of theorem 4.13 consists of showing that future pointing curves γ from x to y can be used to "build" future pointing curves from z to w, with the additional property that the length of the part of γ, which has to be sacrificed in the construction, is congruent to $\mathrm{dist}(x, z) + \mathrm{dist}(y, w) + 1$. The arguments in the proof are similar to the socalled cut-and-paste arguments employed in [1], [14] et.al..

5. Proof of Theorem 4.8

The proof of theorem 4.8 will be divided into several steps. The first steps will prove the implications (ii)⇒ (iii)⇒(iv)⇒(i). The implication (i)⇒(ii) is the subject of subsection 5.2.

Recall theorem 4.8:

Theorem 5.1. *Let* (M, g) *be compact and vicious. Then the following statements are equivalent:*

(i) (M, g) *is of class A.*
(ii) $0 \notin \mathfrak{T}^1$*, especially* $\mathfrak{T}$ *is a compact cone.*
(iii) $(\mathfrak{T}^*)^\circ \neq \emptyset$ *and for every* $\alpha \in (\mathfrak{T}^*)^\circ$ *there exists a smooth* 1*-form* $\omega \in \alpha$ *such that* $\ker \omega_p$ *is a spacelike in* (TM_p, g_p) *for all* $p \in M$*, i.e.* ω *is a closed transversal form for the cone structure of future pointing vectors in* (M, g).
(iv) (M, g) *admits a normal cylindrical covering* $(\overline{M}, \overline{g}) \to (M', g') \to (M, g)$ *such that* $\mathcal{D}(M', M) \cong \mathbb{Z}$.

5.1. *(ii)* ⇒ *(iii)* ⇒ *(iv)* ⇒ *(i)*.

(ii) ⇒ *(iii).* By elementary convex geometry we see that $(\mathfrak{T}^*)^\circ \neq \emptyset$. The rest is a consequence of theorem I.7(ii) and (iv) in [19]. More precisely, since (M, C_g) contains no null homologous structure cycles there is a closed transversal 1-form. Since (M, g) contains closed causal curves there are structure cycles of (M, C_g). Thus by (iv) of theorem I.7 the interior of $\mathfrak{T}^*$ consists of classes of closed transversal 1-forms. □ □

Lemma 5.2. *Consider a rational supporting hyperplane* H *of* $\mathfrak{T}$ *with* $H \cap \mathfrak{T} = \{0\}$. *Define* $\Gamma := H_1(M, \mathbb{Z})_\mathbb{R} \cap H$. *Then the covering* $M' := \overline{M}/\Gamma$ *with the induced Lorentzian metric* g' *is cylindrical, i.e. contains a compact Cauchy hypersurface. Further* $\mathcal{D}(M', M)$ *is isomorphic to* $\mathbb{Z}$.

Proof. Choose a $\mathbb{Z}$-basis $k_1, \ldots, k_{b-1} \in H_1(M, \mathbb{Z})_\mathbb{R}$ of $H \cap H_1(M, \mathbb{Z})_\mathbb{R}$ and define

$$M' := \overline{M}/ < k_1, \ldots, k_{b-1} >_\mathbb{Z} .$$

For the group of deck transformation of $\pi' \colon M' \to M$ we have

$$\mathcal{D}(M', M) \cong H_1(M, \mathbb{Z})_\mathbb{R}/ < k_1, \ldots, k_{b-1} >_\mathbb{Z} \cong \mathbb{Z}.$$

Consider $p', q' \in M'$ and a lift $\overline{p}$ of p' to $\overline{M}$. The set of lifts $\overline{q}$ of q' such that $\overline{q} \in J^+(\overline{p})$ is finite, since for any $h \in H_1(M,\mathbb{R})$ the intersection of $(h+H) \cap \mathfrak{T}$ is bounded ($H \cap \mathfrak{T} = \{0\}$). Now $J^+(p') \cap J^-(q')$ is the image of a finite union of compact sets and therefore compact itself.

It remains to confirm the causality property of (M', g'), i.e. (M', g') contains no closed causal curves. Assume that (M', g') contains a closed causal curve γ'. Project γ' to M and consider the homology class $h_{\gamma'}$ defined by the projection. By definition we have $h_{\gamma'} \in \mathfrak{T}$. The homology class $h_{\gamma'}$ has to be contained in H as well, since γ' is closed in M'. Therefore we get $h_{\gamma'} \in \mathfrak{T} \cap H = \{0\}$. This shows that any lift $\overline{\gamma}$ of γ' to $\overline{M}$ is closed, which contradicts the causality of $(\overline{M}, \overline{g})$. □ □

(iii) ⇒ (iv). Since $(\mathfrak{T}^*)^\circ \neq \emptyset$ there exists a cohomology class $\alpha \in (\mathfrak{T}^*)^\circ$ with $\alpha(H_1(M,\mathbb{Z})_\mathbb{R} \subseteq \mathbb{Q}$. The kernel $\ker \alpha$ is a rational supporting hyperplane of $\mathfrak{T}$ with $\ker\alpha \cap \mathfrak{T} = \{0\}$. Now apply lemma 5.2. □ □

(iv) ⇒ (i). We have seen in remark 4.2 that any cover of a globally hyperbolic spacetime is globally hyperbolic. Consequently $(\overline{M}, \overline{g})$ is globally hyperbolic. □ □

5.2. *(i) ⇒ (ii)*. In order to prove the implication (i)⇒(ii) in theorem 4.8, we use proposition 4.5. The proof of proposition 4.5 consists of a modification of a method introduced by D. Yu Burago in [3].

Definition 5.3. *Let (M, g) be compact and vicious. For $h \in H_1(M,\mathbb{Z})_\mathbb{R}$ and $x \in \overline{M}$ define*

$$\mathfrak{f}_x(h) := \min\{\operatorname{dist}(x+h, z) |\ z \in J^+(x)\} \text{ and } \mathfrak{f}(h) := \min\{\mathfrak{f}_x(h) |\ x \in \overline{M}\}.$$

Note that $x \mapsto \mathfrak{f}_x(h)$ is invariant under the action of $\mathcal{D}(\overline{M}, M)$ for all $h \in H_1(M,\mathbb{Z})_\mathbb{R}$. Consequently $\mathfrak{f}$ is well defined. Recall the statement of proposition 4.5.

Proposition 5.4. *Let (M, g) be a compact and vicious spacetime. Then $\mathfrak{T}$ is the unique cone in $H_1(M,\mathbb{R})$ such that there exists a constant $\operatorname{err}(g, g_R) < \infty$ with $\operatorname{dist}_{\|.\|}(J^+(x) - x, \mathfrak{T}) \leq \operatorname{err}(g, g_R)$ for all $x \in \overline{M}$, where $J^+(x) - x := \{y - x |\ y \in J^+(x)\}$.*

As we have seen before there exists $K < \infty$ such that $J^+(x) - x \subseteq B_K^{\|.\|}(0)$ for all $x \in \overline{M}$. The other inclusion is more involved. First we prove that $\mathfrak{f}$ has the coarse-Lipschitz property.

Lemma 5.5. *There exists $C < \infty$ such that*

$$|\mathfrak{f}(h_1) - \mathfrak{f}(h_2)| \leq \|h_1 - h_2\| + C$$

for all $h_1, h_2 \in H_1(M,\mathbb{Z})_\mathbb{R}$.

Proof. Let $h_1, h_2 \in H_1(M,\mathbb{Z})_\mathbb{R}$. Choose $x, y \in \overline{M}$ with $\mathfrak{f}(h_1) = \mathfrak{f}_x(h_1)$, $\mathfrak{f}(h_2) = \mathfrak{f}_y(h_2)$ and $\operatorname{dist}(x, y) \leq \operatorname{diam}(M, g_R)$. Since $\mathfrak{f}_x(h_2) \leq \mathfrak{f}_y(h_2) + \operatorname{diam}(M, g_R)$ we have

$$|\mathfrak{f}_x(h_2) - \mathfrak{f}_y(h_2)| \leq \operatorname{diam}(M, g_R),$$

Further we have $\mathfrak{f}_x(h_1) \leq \mathfrak{f}_x(h_2) + \operatorname{dist}_{\|.\|}(x+h_1, x+h_2)$ where $x+h := \{z |\ z-x = h\}$. An immediate consequence of theorem 2.3 is

$$|\operatorname{dist}_{\|.\|}(x+h_1, x+h_2) - \|h_1 - h_2\|| \leq D'$$

for some constant $D' < \infty$. Now we get

$$\begin{aligned}
|\mathfrak{f}(h_1) - \mathfrak{f}(h_2)| &\leq |\mathfrak{f}_x(h_1) - \mathfrak{f}_x(h_2)| + |\mathfrak{f}_x(h_2) - \mathfrak{f}_y(h_2)| \\
&\leq \operatorname{dist}_{\|.\|}(x+h_1, x+h_2) + \operatorname{diam}(M, g_R) \\
&\leq \|h_1 - h_2\| + D' + \operatorname{diam}(M, g_R).
\end{aligned}$$

□ □

The following lemma differs slightly from the statement of lemma 1 in [3]. We leave the proof to the reader since it is an almost literally transcription of the proof given in therein.

Lemma 5.6. *Let $C < \infty$ and $F\colon \mathbb{N} \to [0, \infty)$ be a coarse-Lipschitz function with*

(1) $2F(s) - F(2s) \leq C$ and

(2) $F(\kappa s) - \kappa F(s) \leq C$ for $\kappa = 2, 3$

and all $s \in \mathbb{N}$. Then there exists an $\mathfrak{a} \in \mathbb{R}$ such that $|F(s) - \mathfrak{a}s| \leq 2C$ for all $s \in \mathbb{N}$.

Now we want to apply this lemma to $\mathfrak{f}$. First we fix a trivial fact.

Fact 5.7. *Consider $\mathfrak{f}$ as in definition 5.3. Then we have $\mathfrak{f}(2h) \leq 2\mathfrak{f}(h)$ and $\mathfrak{f}(3h) \leq 3\mathfrak{f}(h)$ for all $h \in H_1(M, \mathbb{Z})_\mathbb{R}$.*

The next lemma requires more attention.

Lemma 5.8. *Consider $\mathfrak{f}$ as in definition 5.3. Then there exists a constant $C = C(g, g_R) < \infty$ such that $\mathfrak{f}(2h) \geq 2\mathfrak{f}(h) - C$ for all $h \in H_1(M, \mathbb{Z})_\mathbb{R}$.*

We will need the following lemma contained in [3].

Lemma 5.9. *Let V be a real vector space of dimension $b < \infty$ and $\gamma\colon [a, b] \to V$ a continuous curve. Then there exist no more than $[b/2]$-many essentially disjoint subintervals $[a_i, b_i] \subseteq [a, b]$ $(1 \leq i \leq k \leq [b/2])$ such that*

$$\sum_{i=1}^{k} [\gamma(b_i) - \gamma(a_i)] = \frac{1}{2}[\gamma(b) - \gamma(a)].$$

The proof is a nontrivial application of the theorem of Borsuk-Ulam and can be found in [3].

Lemma 5.8. We have already seen above that

$$|\mathfrak{f}_x(h) - \mathfrak{f}_y(h)| \leq 2 \operatorname{diam}(M, g_R)$$

for all $x, y \in \overline{M}$ and $h \in H_1(M, \mathbb{Z})_\mathbb{R}$. Let $h \in H_1(M, \mathbb{Z})_\mathbb{R}$ be given. Fix $x \in \overline{M}$. Further choose a future pointing curve $\gamma\colon [0, T] \to \overline{M}$ with $\gamma(0) = x$ and $\operatorname{dist}(\gamma(T), x + 2h) = \mathfrak{f}_x(2h)$. Now consider the curve $\gamma_D\colon [0, T] \to H_1(M, \mathbb{R})$, $t \mapsto \gamma(t) - \gamma(0)$. The pair $(H_1(M, \mathbb{R}), \gamma_D)$ obviously meets the assumptions of lemma 5.9. Consequently there exist at most $[b/2]$-many intervals $[s_i, t_i] \subseteq [0, T]$ $(1 \leq i \leq k \leq [b/2])$ with

$$\sum [\gamma_D(t_i) - \gamma_D(s_i)] = \frac{1}{2}[\gamma_D(T) - \gamma_D(0)].$$

W.l.o.g. we can assume that $a_1 = 0$. In the other case simply consider the complementary intervals $[t_{i-1}, s_i]$. Note that

$$\| \sum [\gamma_D(t_i) - \gamma_D(s_i)] - h\| \leq \frac{1}{2}(\operatorname{std}(\mathrm{g_R}) + \mathfrak{f}_x(2h)),$$

since $\|[\gamma(T) - \gamma(0)] - 2h\| \leq \operatorname{std}(\mathrm{g_R}) + \mathfrak{f}_x(2h)$. Choose inductively deck transformations k_i starting with $k_1 := 0 \in H_1(M, \mathbb{Z})_\mathbb{R}$ and for $i \geq 2$ $k_i \in H_1(M, \mathbb{Z})_\mathbb{R}$ such that $\gamma(s_i) + k_i \in J^+(\gamma(t_{i-1}) + k_{i-1})$ and $\operatorname{dist}(\gamma(t_{i-1}) + k_{i-1}, \gamma(s_i) + k_i) \leq \operatorname{fill}(g, g_R)$. Join $\gamma(t_{i-1}) + k_{i-1}$ and $\gamma(s_i) + k_i$ by a future pointing curve length at most $\operatorname{fill}(g, g_R)$. The resulting future poiniting curve $\zeta\colon [0, T'] \to \overline{M}$ then satisfies

$$\|\zeta(T') - \zeta(0) - h\| \leq [b/2] \operatorname{fill}(g, g_R) + \frac{1}{2}(\operatorname{std}(\mathrm{g_R}) + \mathfrak{f}_x(2h)).$$

Since by theorem 2.3 we have $\operatorname{dist}(\zeta(T'), x + h) \leq \|\zeta(T') - \zeta(0) - h\| + \operatorname{std}(\mathrm{g_R})$, the lemma follows for $C := 2[b/2] \operatorname{fill}(g, g_R) + 3 \operatorname{std}(\mathrm{g_R})$. □ □

Now we can apply lemma 5.6 to the function $n \mapsto \mathfrak{f}(nh)$ for every $h \in H_1(M,\mathbb{Z})_{\mathbb{R}}$. As a result we get $\mathfrak{a}(h) \in \mathbb{R}$ with $|\mathfrak{a}(h)n - \mathfrak{f}(nh)| \leq 2C$ for all $n \in \mathbb{N}$. This immediately implies positive homogeneity of $\mathfrak{a}$. Combining this we get the following fact.

Fact 5.10. *There exists a map $\mathfrak{a}\colon H_1(M,\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}$ and $C < \infty$ such that*

(1) $\mathfrak{a}$ is positively homogenous of degree one, i.e. $\mathfrak{a}(nh) = n\mathfrak{a}(h)$ for all $n \in \mathbb{N}$ and

(2) $|\mathfrak{f}(h) - \mathfrak{a}(h)| \leq 2C$

for every $h \in H_1(M,\mathbb{Z})_{\mathbb{R}}$.

Fact 5.11. *We have $\mathfrak{a}(h) = \operatorname{dist}_{\|.\|}(h, \mathfrak{T})$ for all $h \in H_1(M,\mathbb{Z})_{\mathbb{R}}$.*

Proof. Let $h \in H_1(M,\mathbb{Z})_{\mathbb{R}}$. For $n \in \mathbb{N}$ let $\gamma_n\colon [0,T] \to \overline{M}$ be a future pointing curve with

$$\operatorname{dist}(\gamma_n(0) + nh, \gamma_n(T)) = \mathfrak{f}(nh).$$

Then with theorem 2.3 and fact 5.10 we get

$$\begin{aligned}
&|\|nh - (\gamma_n(T) - \gamma_n(0))\| - \mathfrak{a}(h)n| \\
&\qquad \leq |\operatorname{dist}(\gamma_n(0) + nh, \gamma_n(T)) - \mathfrak{a}(h)n| + D \\
&\qquad \leq 2C + D.
\end{aligned}$$

Now we have

$$\lim_{n\to\infty} \frac{1}{n}\|nh - (\gamma_n(T) - \gamma_n(0))\| = \operatorname{dist}_{\|.\|}(h, \mathfrak{T})$$

since otherwise the distance between $\gamma_n(0) + nh$ and $\gamma_n(T)$ would not be minimal. □ □

To prove the remaining inclusion in the proof of proposition 4.5 observe that by fact 5.10, 5.11 and the fact that $H_1(M,\mathbb{Z})_{\mathbb{R}}$ is a cocompact lattice in $H_1(M,\mathbb{R})$, the Hausdorff distance between $\mathfrak{T} = \operatorname{dist}_{\|.\|}(.,\mathfrak{T})^{-1}(0)$ and

$$\mathfrak{f}^{-1}(0) = \{h \in H_1(M,\mathbb{Z})_{\mathbb{R}} |\ \exists x \in \overline{M} \text{ with } x + h \in J^+(x)\}$$

is bounded by $2C$. Further observe that by fact 4.4 there exists a constant $C' = C'(g, g_R) < \infty$ such that

$$\operatorname{dist}_{\|.\|}(J^+(x) - x, J^+(y) - y) \leq C'$$

for all $x, y \in \overline{M}$. Thus the Hausdorff distance of $\mathfrak{f}^{-1}(0)$ and $J^+(x) - x$ is uniformly bounded in x. Now combining these arguments we get the claim of proposition 4.5.

Theorem 4.8 (i) ⇒ (ii). The first step is to confirm that $\mathfrak{T}$ does not contain a nontrivial linear subspace. This is done by contradiction.

Assume $\mathfrak{T}$ contains a linear subspace $V \neq \{0\}$. Choose $h \in V \setminus \{0\}$. By proposition 4.5 there exists for any $h' \in V$ a homology class $h'_x \in J^+(x) - x$ with $\|h' - h'_x\| \leq \operatorname{err}(g, g_R)$ for any $x \in \overline{M}$. We can choose future pointing curves $\gamma^+, \gamma^-\colon [0,1] \to \overline{M}$ with

$$\begin{aligned}
&\|\gamma^+(1) - \gamma^+(0) - h\|,\ \|\gamma^-(1) - \gamma^-(0) + h\| \leq \operatorname{err}(g, g_R), \\
&\operatorname{dist}(\gamma^+(1), \gamma^-(0)) \leq \operatorname{fill}(g, g_R) \text{ and } \gamma^-(0) \in J^+(\gamma^+(1)).
\end{aligned}$$

Then $\operatorname{dist}(\gamma^+(0), \gamma^-(1)) \leq 2C + \operatorname{fill}(g, g_R) + \operatorname{std}(\mathrm{g_R})$ and we can construct a future pointing curve ζ_h connecting $\gamma^+(0)$ with $\gamma^-(1)$ of g_R-length at least $2\|h\| - \operatorname{std}(\mathrm{g_R})$. Choose a sequence of future pointing curves $\zeta_n := \zeta_{h_n}\colon [0, T_n] \to \overline{M}$ for an unbounded sequence $h_n \in V$. By passing to a subsequence we can assume $\zeta_n(0) \to p'$ and $\zeta_n(1) \to q'$. Choose any point $p \in I^-(p')$ and $q \in I^+(q')$. Then $J^+(p) \cap J^-(q)$ is not compact, thus contradicting the global hyperbolicity of $(\overline{M}, \overline{g})$. Consequently $\mathfrak{T}$ cannot contain any nontrivial linear subspaces.

If $\mathfrak{T}$ doesn't contain a nontrivial linear subspace we can choose a cohomology class α with $\ker\alpha\cap\mathfrak{T}=\{0\}$. Consequently there exists $\varepsilon>0$ such that $\alpha(h)\geq\varepsilon\|h\|$ for all $h\in\mathfrak{T}$. Assume that there exists an admissible sequence of future pointing curves $\gamma_n\colon[a_n,b_n]\to M$ with $\|\rho(\gamma_n)\|\leq n^{-1}$. Partition $[a_n,b_n]$ into subintervals $[a_{n,i},b_{n,i}]$ such that $b_{n,i}-a_{n,i}\in[n,2n]$. We have

$$\frac{1}{n}(b_n-a_n)\geq\|\gamma_n(b_n)-\gamma_n(a_n)\|\geq\varepsilon\sum_i\|\gamma_n(b_{n,i})-\gamma_n(a_{n,i})\|.$$

Since $b_n-a_n=\sum_i(b_{n,i}-a_{n,i})$ there exists an index i with $\varepsilon\|\gamma_n(b_{n,i})-\gamma_n(a_{n,i})\|\leq 2$. Consequently we have constructed an admissible sequence of future pointing curves $\gamma'_n\colon[a_n,b_n]\to M$ with $\|\gamma'_n(b_n)-\gamma'_n(a_n)\|\leq 2\varepsilon^{-1}$. By the previous arguments γ'_n has to stay in a uniformly compact subset of $\overline{M}$. But this contradicts the compactness of the sets $\mathcal{C}(p,q)$. □ □

6. Proof of theorem 4.13

Proposition 4.6 is a necessary ingredient in the proof of theorem 4.13.

6.1. **Proof of Proposition 4.6.** For $p\in M$ let $\mathfrak{T}_p$ be the set of classes $k\in H_1(M,\mathbb{Z})_{\mathbb{R}}$ which contain a timelike future pointing curve through p. $\mathfrak{T}_p$ is obviously a positively homogenous subset of $\mathfrak{T}\cap H_1(M,\mathbb{Z})_{\mathbb{R}}$. A homology class $h\in H_1(M,\mathbb{R})$ is called $\mathfrak{T}_p$-rational if $nh\in\mathfrak{T}_p$ for some $n\in\mathbb{N}$.

Lemma 6.1. *Let (M,g) be compact and vicious. Then for every $p\in M$ the set of $\mathfrak{T}_p$-rational homology classes is dense in $\mathfrak{T}$.*

Proof. This is a direct consequence of fact 4.4 and proposition 4.5. □ □

Proposition 6.2. *The set of $(\cap_{p\in M}\mathfrak{T}_p)$-rational homology classes is dense in $\mathfrak{T}$.*

Lemma 6.3. *There exists $C<\infty$ such that for all $p,q\in M$ there are $\overline{p}\in\overline{\pi}^{-1}(p)$ and $\overline{q}\in\overline{\pi}^{-1}(q)$ with $\operatorname{dist}(\overline{p},\overline{q})<C$ and $\varepsilon_{p,q},\delta_{p,q}>0$, such that for all $\overline{r}\in B_{\varepsilon_{p,q}}(\overline{p})$ and all $\overline{s}\in B_{\varepsilon_{p,q}}(\overline{q})$, we have*

$$B_{\delta_{p,q}}(\overline{r})\subseteq I^-(\overline{s}),B_{\delta_{p,q}}(\overline{s})\subseteq I^+(\overline{r}).$$

Proof. Choose any timelike future pointing curve γ connecting p with q of g_R-length less than $\operatorname{fill}(g,g_R)$. Considering a lift of γ to $\overline{M}$ with endpoints $\overline{p}$ resp. $\overline{q}$, the claim follows when considering normal neighborhoods around $\overline{p}$ and $\overline{q}$. □ □

Proposition 6.2. Let $x\in M$ and $\varepsilon>0$ be the minimum of the Lebesgue numbers of the coverings

$$\{B_{\varepsilon_{p,q}}(p)\times B_{\varepsilon_{p,q}}(q)\}_{p,q\in M}\text{ and }\{B_{\delta_{p,q}}(p)\times B_{\delta_{p,q}}(q)\}_{p,q\in M}$$

of $M\times M$. Then for all $p,q\in M$ there exist $\overline{p}\in\overline{\pi}^{-1}(p)$ and $\overline{q}\in\overline{\pi}^{-1}(q)$ with $\operatorname{dist}(\overline{p},\overline{q})\leq\operatorname{fill}(g,g_R)$ such that

$$B_\varepsilon(\overline{r})\subseteq I^-(\overline{s}),B_\varepsilon(\overline{s})\subseteq I^+(\overline{r})$$

for all $\overline{r}\in B_\varepsilon(\overline{p})$ and all $\overline{s}\in B_\varepsilon(\overline{q})$. Take a finite subcover $\{B_\varepsilon(p_1),\ldots,B_\varepsilon(p_N)\}$ of M and choose timelike future pointing curves $c_1\colon[0,N]\to M$, with $c_1(n)=p_{N-n}$ for $0\leq n\leq N-1$ and $c_1(N)=x$, and $c_2\colon[0,N]\to M$, with $c_2(0)=x$ and $c_2(n)=p_n$ for $1\leq n\leq N$ such that for one (hence every) lift $\overline{c}_1$ resp. $\overline{c}_2$ of c_1 resp. c_2 we have $B_\varepsilon(\overline{c}_i(n+1))\subseteq I^+(\overline{c}_i(n))$ $(i=1,2)$. The g_R-arclength of both curves can be bounded by $(N+1)\operatorname{fill}(g,g_R)$. By joining a timelike future pointing representative of $k\in\mathfrak{T}_x$ with c_1 and c_2, we obtain $k+[c_1*c_2]\in\cap_{1\leq i\leq N}\mathfrak{T}_{p_i}$. The assertion follows if we can show that $k+[c_1*c_2]\in\cap_{p\in M}\mathfrak{T}_p$. This can be seen as follows: For $y\in M$ choose p_i with $y\in B_\varepsilon(p_i)$. Let $\overline{y}$ be a lift of y to $\overline{M}$, $\overline{p}_i$ a lift of p_i with $\overline{y}\in B_\varepsilon(\overline{p}_i)$ and $\overline{c_1*c_2}$ a lift of c_1*c_2 through $\overline{p}_i$. We can choose a timelike future pointing curve β_1 from $\overline{c_1 * c_2}(i-1)$ to $\overline{c_1 * c_2}(i+1)$ via $\overline{y}$ and homotopic to $\overline{c_1 * c_2}|_{[i-1,i+1]}$ if $\overline{p}_i = \overline{c_1 * c_2}(i)$. In the same manner choose a future pointing curve β_2 from $\overline{c_1 * c_2}(2N-i-1)$ to $\overline{c_1 * c_2}(2N-i+1)$ via $\overline{y}$ if $\overline{p}_i = \overline{c_1 * c_2}(2N-i)$. If we substitute $\overline{c_1 * c_2}|_{[i-1,i+1]}$ with β_1 and $\overline{c_1 * c_2}|_{[2N-i-1,2N-i+1]}$ with β_2, we obtain a timelike future pointing curve homologous to $c_1 * c_2$. Thus $k+[c_1 * c_2] \in \mathfrak{T}_y$. $\square$ $\square$

Lemma 6.4. *Let (M,g) be compact and vicious. Then there exists $C < \infty$ such that for every future pointing curve $\gamma\colon [a,b] \to \overline{M}$ there exists $k \in \cap_{p\in M}\mathfrak{T}_p$ with $\|\gamma(b)-\gamma(a)-k\| \le C$ and $\varepsilon_k > 0$, such that $B_{n\varepsilon_k}(p+nk) \subseteq I^+(p)$ for all $p \in \overline{M}$ and all $n \in \mathbb{N}$.*

Proof. The same argument used in the proof of proposition 6.2 shows: There exists $\varepsilon_k > 0$ such that $B_{\varepsilon_k}(p+k) \subseteq I^+(p)$ for all $p \in \overline{M}$. The claim then follows inductively. $\square$ $\square$

Proposition 4.6. The proof is an easy consequence of lemma 6.4. Take any $k \in \cap_{p\in M}\mathfrak{T}_p$ and $n \in \mathbb{N}$ such that $n\varepsilon_k \ge \operatorname{diam}(M, g_R)$. Then $B_{\operatorname{diam}(M,g_R)}(\overline{p}+nk) \subseteq I^+(\overline{p})$ for all $\overline{p} \in \overline{M}$. This implies directly that $\cap_{p\in M}\mathfrak{T}_p$ contains a basis of $H_1(M,\mathbb{R})$. $\square$ $\square$

6.2. **Proof of Theorem 4.13.**

Proposition 6.5. *For every $R > 0$ there exists a constant $0 < K = K(R) < \infty$ such that*

$$B_R(q) \subseteq I^+(p)$$

for all $p, q \in \overline{M}$ with $q - p \in \mathfrak{T}$ and $\operatorname{dist}_{\|.\|}(q-p, \partial\mathfrak{T}) \ge K$.

Note that there exists a $K < \infty$ such that for every $p \in \overline{M}$ the intersection $B_K(p) \cap I^+(p)$ contains a fundamental domain of the Abelian covering $\overline{\pi}\colon \overline{M} \to M$.

Proof. Choose a basis $\{k_1, \dots, k_b\} \subseteq \cap_{p\in M}\mathfrak{T}_p$ of $H_1(M,\mathbb{R})$ such that there exists an $\varepsilon_0 > 0$ with $B_{\varepsilon_0}(q+k_i) \subseteq J^+(q)$ for all $q \in \overline{M}$ and all $1 \le i \le b$. The existence of the k_i is ensured by lemma 6.4.

Set $K' := \sup_{p\in\overline{M}} \sup_{1\le i\le b} \operatorname{dist}(p, p+k_i)$ and $K'' := (\frac{R+bK'}{\varepsilon_0} + b)(K' + \operatorname{std}(\mathrm{g_R}))$. For $h = \sum r^i k_i \in H_1(M,\mathbb{R})$ with $r^i \ge 0$ and $\|h\| > K''$ we have

$$\sum r^i \ge \frac{R+bK'}{\varepsilon_0} + b.$$

Because of $\sum [r^i] \ge \sum r^i - b$ we obtain $\sum [r^i] \ge \frac{R+bK'}{\varepsilon_0}$. By the choice of K' and ε_0 we conclude

$$B_R(x+h) \subseteq B_{R+bK'}(x + \sum [r^i]k_i) \subseteq B_{\sum [r^i]\varepsilon_0}(x + \sum [r^i]k_i) \subseteq I^+(x)$$

with lemma 6.4 for every point $x \in \overline{M}$. Now if we have

$$\operatorname{dist}_{\|.\|}(q-p, \partial\mathfrak{T}) \ge K'' + \operatorname{err}(g, g_R) + \operatorname{std}(\mathrm{g_R}) =: K$$

there exists $r \in I^+(p)$ with $q - r \in \operatorname{pos}\{k_1, \dots, k_b\}$ and $\|q - r\| \ge K''$ (proposition 4.5). Since $B_R(q) \subseteq I^+(r)$ we conclude

$$B_R(q) \subseteq I^+(p).$$

$\square$ $\square$

Remark 6.6. *Lemma 6.4 implies: For every $\varepsilon > 0$ there exist $N(\varepsilon) \in \mathbb{N}$ and $k_1, \dots, k_N \in \cap_{p\in M}\mathfrak{T}_p$ with $\mathfrak{T}_\varepsilon \subseteq \operatorname{pos}\{k_1, \dots, k_N\}$. Since $\mathfrak{T}^\circ \neq \emptyset$ we know that for $\varepsilon > 0$ sufficiently small, $\{k_1, \dots, k_N\}$ necessarily contains a basis of $H_1(M,\mathbb{R})$.*

Recall from proposition 4.5 that $\mathrm{dist}_{\|.\|}(\mathfrak{T}, J^+(x)-x) \leq \mathrm{err}(g, g_R)$ for all $x \in \overline{M}$. Further recall that $\mathfrak{T}$ is a compact cone. Consequently we can choose $0 < \delta < 1$ and $K_0 < \infty$ such that

$$\|\sum h_i\| \geq \delta \sum \|h_i\|$$

for any finite set $\{h_i\}_{i=1,\dots n} \subseteq B_{\mathrm{err}(g,g_R)}(\mathfrak{T}) \setminus B_{K_0}(0)$.

Lemma 6.7. *Set* $K_1(\varepsilon) := \max\{K_0, \frac{4b\,\mathrm{err}(g,g_R)}{\delta\varepsilon}\}$ *and let* $\varepsilon > 0$ *be given. Further let* $\{h_i\}_{1\leq i\leq N} \subseteq \mathfrak{T}$ *with* $\|h_i\| \geq K_1$, $\frac{1}{2} \leq \frac{\|h_i\|}{\|h_j\|} \leq 2$ *and* $\sum h_i \in \mathfrak{T}_\varepsilon$ *for all* $1 \leq i, j \leq N$. *Then there exists a subset* $\{i_1, \dots, i_b\} \subseteq \{1, \dots, N\}$ *with* $\sum_j h_{i_j} \in \mathfrak{T}_\eta$ *for* $\eta := \frac{\delta}{8b}\varepsilon$.

Proof. The assumption $\sum h_i \in \mathfrak{T}_\varepsilon$ implies that

$$\mathrm{conv}\{h_1, \dots, h_N\} \cap \mathfrak{T}_\varepsilon \neq \emptyset.$$

With the theorem of Caratheodory follows: There exist $1 \leq i_1, \dots, i_b \leq N$ and $\lambda_1, \dots, \lambda_b \geq 0$ with $\sum_j \lambda_j = 1$ such that $\sum \lambda_j h_{i_j} \in \mathfrak{T}_\varepsilon$. For every $\alpha \in \mathfrak{T}^*$ with $\|\alpha\|^* = 1$ and every $j \in \{1, \dots, b\}$ we have

$$\max_m\{\alpha(h_{i_m})\} \geq \alpha(\sum_m \lambda_m h_{i_m}) \geq \varepsilon\|\sum_m \lambda_m h_{i_m}\| \geq \delta\varepsilon\sum_m \lambda_m\|h_{i_m}\| \geq \frac{\delta}{2}\varepsilon\|h_{i_j}\|.$$

And therefore for $\alpha(h_{i_k}) = \max_m\{\alpha(h_{i_m})\}$ we get

$$\begin{aligned}\frac{1}{\|\sum_j h_{i_j}\|}\alpha(\sum_j h_{i_j}) &\geq \sum_j \frac{1}{2b\|h_{i_j}\|}\alpha(h_{i_j}) \\ &\geq \frac{1}{2b\|h_{i_k}\|}\alpha(h_{i_k}) - \sum_j \frac{1}{2b\|h_{i_j}\|}\,\mathrm{err}(g, g_R) \\ &\geq \frac{\delta}{4b}\varepsilon - \frac{\mathrm{err}(g,g_R)}{2K_1} \geq \frac{\delta}{8b}\varepsilon.\end{aligned}$$

□ □

Theorem 4.13. (i) First we reduce the claim to the following special case: *For every* $\varepsilon > 0$ *there exists* $C(\varepsilon) < \infty$ *such that* $|d(x, y) - d(z, w)| \leq C(\varepsilon)$ *for all* $x, y, z, w \in \overline{M}$ *with* $y - x, w - z \in \mathfrak{T}_\varepsilon$ *and*

$$\mathrm{dist}(x, z), \mathrm{dist}(y, w) < K_2 := \max\{\mathrm{fill}(g, g_R), 2\} + \mathrm{std}(\mathrm{g_R})\,.$$

Let $x, y, z, w \in \overline{M}$ be given with $y - x, w - z \in \mathfrak{T}_\varepsilon$. Choose $k_{x,z} \in H_1(M, \mathbb{Z})_{\mathbb{R}}$ with $\mathrm{dist}(x - k_{x,z}, z) \leq \mathrm{diam}(M, g_R)$. For every $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$ we have

$$\begin{aligned}\mathrm{dist}(x + k, z) + \mathrm{dist}(y + k, w) &\geq \|(x - z) - (y - w)\| - 2\,\mathrm{std}(\mathrm{g_R}) \\ &\geq \|(y - w) - k_{x,z}\| - \mathrm{diam}(M, g_R) - 3\,\mathrm{std}(\mathrm{g_R}) \\ &\geq \mathrm{dist}(y - k_{x,z}, w) - \mathrm{diam}(M, g_R) - 4\,\mathrm{std}(\mathrm{g_R})\,.\end{aligned}$$

Since we have $d(x + k, y + k) = d(x, y)$ for every $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$, we can assume that $\mathrm{dist}(x, z) < \mathrm{fill}(g, g_R)$. Note that $\mathrm{diam}(M, g_R) \leq \mathrm{fill}(g, g_R)$.

If we have $\|y - x\| \geq \frac{2+\varepsilon}{\varepsilon}K_2 \geq \frac{2+\varepsilon}{\varepsilon}\|x - z\|$ then

$$\mathrm{dist}_{\|.\|}(y - z, \partial\mathfrak{T}) \geq \mathrm{dist}_{\|.\|}(y - x, \partial\mathfrak{T}) - \|x - z\| \geq \frac{\varepsilon}{2}\|y - z\|.$$

The special case then yields $|d(x, y) - d(z, y)| \leq C(\frac{\varepsilon}{2})$.

For any integer $1 \leq i \leq n := [\|y - w\|]$ set $h_i := (w - z) + \frac{i}{n}(y - w)$. Since $\mathfrak{T}_{\varepsilon/2}$ is convex we have $h_i \in \mathfrak{T}_{\varepsilon/2}$ for $1 \leq i \leq n$. Choose points $w_i \in \overline{M}$ with $w_i - z = h_i$ for $1 \leq i \leq n$. With the special case we have (Note that $\mathrm{dist}(w_i, w_{i+1}) \leq 2 + \mathrm{std}(\mathrm{g_R})$)

$$|d(z, w) - d(z, w_1)|, |d(z, w_i) - d(z, w_{i+1})|, |d(z, w_n) - d(z, y)| \leq C(\varepsilon/2)$$

for all $1 \leq i \leq n-1$. With the triangle inequality we get

$$\begin{aligned}|d(x,y)-d(z,w)| &\leq (n+2)C(\varepsilon/2)\\ &\leq C(\varepsilon/2)(\mathrm{std}(\mathrm{g_R})+2)(\mathrm{dist}(x,z)+\mathrm{dist}(y,w)+1)\\ &=: L_c(\varepsilon)(\mathrm{dist}(x,z)+\mathrm{dist}(y,w)+1).\end{aligned}$$

The case $\|y-x\| < \frac{2+\varepsilon}{\varepsilon}K_2$ can be absorbed into the constant $L_c(\varepsilon)$ since the time separation is bounded on any compact subset of $\overline{M}\times\overline{M}$. This shows that the general claim follows from the special case.

(ii) The special case follows from

$$d(x,y) \geq d(z,w) - C(\varepsilon) \tag{1}$$

for $x,y,z,w \in \overline{M}$ with $\mathrm{dist}(x.z), \mathrm{dist}(y,w) < K_2$ and $y-x, w-z \in \mathfrak{T}_\varepsilon$. Exchanging (x,y) and (z,w) in (1) we get $d(z,w) \geq d(x,y) - C(\varepsilon)$. Consequently we have

$$|d(x,y)-d(z,w)| \leq C(\varepsilon)$$

and with it the special case.

Recall the definition of $K(.)$ from proposition 6.5. Set

$$K_3 := \max\left\{\frac{1}{\eta\delta b}\left(K(K_2)+b(\mathrm{fill}(g,g_R)+\mathrm{std}(\mathrm{g_R}))\right), K_1\right\}.$$

To prove (1) we can assume that

$$z \in J^+(x) \text{ and } \mathrm{dist}(z,w) \geq 2\delta b K_3.$$

By fact 4.4 there exists $k \in H_1(M,\mathbb{Z})_\mathbb{R}$ with $\mathrm{dist}(x, z+k) < \mathrm{fill}(g,g_R)$ and $z+k \in J^+(x)$. The case $\mathrm{dist}(z,w) < 2\delta b K_3$ can be absorbed into the constant $C(\varepsilon)$ since the time separation is bounded on compact subsets of $\overline{M}\times\overline{M}$.

Choose a maximal future pointing curve $\gamma\colon [0,T] \to \overline{M}$ connecting z with w. With our assumption that $\mathrm{dist}(z,w) \geq 2\delta b K_3$, we can partition $[0,T]$ into at least b-many mutually disjoint intervals $[s_i,t_i]$ with $K_3 \leq \|\gamma(t_i)-\gamma(s_i)\| \leq 2K_3$. Then by lemma 6.7 there exist intervals $[s_{m_j}, t_{m_j}] \subseteq [0,T]$ $(1 \leq j \leq b)$ with $\sum \gamma(t_{m_j}) - \gamma(s_{m_j}) \in \mathfrak{T}_\eta$ $(\eta := \frac{\delta}{8b}\varepsilon)$. After relabeling we can assume $t_{m_i} \leq s_{m_{i+1}}$. We want to build a future pointing curve from z to y using pieces of γ. Choose $k_i \in H_1(M,\mathbb{Z})_\mathbb{R}$ such that

$$\gamma(t_{m_i}) + k_i \in J^+(\gamma(s_{m_i})) \cap B_{\mathrm{fill}(g,g_R)}(\gamma(s_{m_i}))$$

and future pointing curves $\zeta_i\colon [s_{m_i}, t_{m_i}] \to \overline{M}$ from $\gamma(s_{m_i})$ to $\gamma(t_{m_i})+k_i$. Define the future pointing curve $\gamma'\colon [0,T] \to \overline{M}$ as

$$\gamma' := \gamma|_{[0,s_{m_1}]} * \zeta_1 * (\gamma|_{[t_{m_1},s_{m_2}]} + k_1) * \zeta_2 * \ldots * \left(\gamma|_{[t_{m_b},T]} + \sum_{i=1}^b k_i\right).$$

Set $h_i' := \gamma(t_{m_i}) - \gamma(s_{m_i})$ and $l_i := \zeta(t_{m_i}) - \zeta(s_{m_i})$. By construction we have

$$w - \gamma'(T) = \sum_{i=1}^b h_i' - l_i.$$

Note that $\sum \|l_i\| \leq b(\mathrm{fill}(g,g_R) + \mathrm{std}(\mathrm{g_R}))$. We have

$$\begin{aligned}\mathrm{dist}_{\|.\|}(w-\gamma'(T), \partial\mathfrak{T}) &\geq \mathrm{dist}_{\|.\|}\left(\sum h_i', \partial\mathfrak{T}\right) - \sum\|l_i\|\\ &\geq \eta\|\sum h_i'\| - \sum\|l_i\| \geq \eta\delta\sum\|h_i'\| - \sum\|l_i\|\\ &\geq \eta\delta b K_3 - b(\mathrm{fill}(g,g_R)+\mathrm{std}(\mathrm{g_R}))\\ &\geq K(K_2) > 0.\end{aligned}$$

Since $\sum h_i' \in \mathfrak{T}$ and $\mathrm{dist}_{\|.\|}(\sum h_i' - l_i, \partial\mathfrak{T}) > 0$, we get $w - \gamma'(T) \in \mathfrak{T}$. With proposition 6.5 we have $y \in B_{K_2}(w) \subseteq I^+(\gamma'(T))$. Therefore we can choose future pointing curves ζ_0 and ζ_{b+1} connecting x with z resp. $\gamma'(T)$ with y and obtain

$$d(x,y) \geq L^{\overline{g}}(\zeta_0 * \gamma' * \zeta_{b+1}) \geq L^{\overline{g}}(\gamma) - \sum_{i=1}^{b} L^{\overline{g}}\left(\gamma|_{[s_{m_i},t_{m_i}]}\right).$$

Choose $\Lambda_{g,g_R} < \infty$ such that $|g(v,v)| \leq \Lambda_{g,g_R} g_R(v,v)$ for all $v \in TM$. With corollary 4.9 we have

$$\begin{aligned} L^{\overline{g}}(\gamma|_{[s_{m_i},t_{m_i}]}) &\leq \Lambda_{g,g_R} C_{g,g_R} \operatorname{dist}(\gamma(s_{m_i}), \gamma(t_{m_i})) \\ &\leq \Lambda_{g,g_R} C_{g,g_R} (2K_3 + \operatorname{std}(\mathrm{g_R})). \end{aligned}$$

Therefore we conclude

$$d(x,y) \geq d(z,w) - \Lambda_{g,g_R} C_{g,g_R} b(2K_3 + \operatorname{std}(\mathrm{g_R})) =: d(z,w) - C(\varepsilon).$$

□ □

Appendix A. Sullivan's struture cycles

In this section we briefly recall the main definitions and results for structure currents and structure cycles of a cone structure C from [19].

Definition A.1 ([19], definition I.1, I.2)**.** *(1) A compact convex cone C in a (locally convex topological) vector space over $\mathbb{R}$ is a convex cone which for some (continuous) linear functional L satisfies $L(x) > 0$ for $x \neq 0$ in C and $L^{-1}(x) \cap C$ is compact.*

(2) A cone structure on a closed subset X of a smooth manifold M is a continuous field of compact convex cones $\{C_x\}$ in the vector spaces $\Lambda_p M_x$ of tangent p-vectors on M, $x \in X$. Continuity of cones is defined by the Hausdorff metric on the compact subsets of the rays in $\Lambda_p M$.

Remark A.2. *For the present paper the application is restricted to the case $p = 1$.*

Obviously the set of future pointing tangent vectors in a time oriented Lorentzian manifold is an example of a cone structure. This connection is discussed briefly in [19] p. 248/249.

Definition A.3 ([19], definition I.3)**.** *A smooth differential p-form ω on M is transversal to the cone structure C if $\omega(v) > 0$ for each $v \neq 0$ in C_x, $x \in M$.*

Definition A.4 ([19], definition I.4)**.** *A Dirac current is one determined by the evaluation of 1-forms on a single vector at one point. The cone of structure currents $\mathcal{C}$ associated to the cone structure C is the closed convex cone of currents generated by the Dirac currents associated to elements of C_x, $x \in M$.*

Proposition A.5 ([19], proposition I.4, I.5)**.** *(1) A cone structure C admits transversal 1-forms.*

(2) If X is compact the cone of structure currents $\mathcal{C}$ associated to a cone structure C on X is a compact convex cone.

Definition A.6 ([19], definition I.6)**.** *If C is a cone structure, the structure cycles of C are the structure currents which are closed as currents.*

Theorem A.7 ([19], theorem I.7)**.** *Suppose C is a cone structure of p-vectors defined on a compact subspace X in the interior of M which is also compact (with or without boundary).*

(i) There are always non-trivial structure cycles in X or closed p-forms on M transversal to the cone structure.

(ii) If no closed transverse form exists some no-trivial structure cycle in X is homologous to zero in M.
(iii) If no non-trivial structure cycle exists some transversal closed form is cohomologous to zero.
(iv) If there are both structures cycles and transversal closed forms then
(a) the natural map

$$(\text{structure cycles on } X \to \text{homology classes in } M)$$

is proper and the image is a compact cone $\mathbb{C} \subseteq H_p(M, \mathbb{R})$
(b) the interior of the dual cone $\mathbb{C}' \subseteq H^p(M, \mathbb{R})$ consists precisely of the classes of closed forms transverse to C.

References

[1] Bangert, V., Minimal geodesics, Ergodic Theory Dynam. Systems, 10, 263–286 (1990)
[2] Bernal, A. N. and Sánchez, M., Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes, Comm. Math. Phys., 257, 43–50 (2005)
[3] Burago, D. Yu., Periodic metrics, Representation theory and dynamical systems (Adv. Soviet Math.), 9, 205–210 (1992)
[4] Cheeger, Jeff and Gromoll, Detlef, On the structure of complete manifolds of nonnegative curvature, Ann. of Math. (2), 96, 413–443, (1972)
[5] Eschenburg, J.-H., The splitting theorem for space-times with strong energy condition, J. Differential Geom., 27, 477–491, (1988)
[6] , Galloway, G. J. and Horta, A., Regularity of Lorentzian Busemann functions, Trans. Amer. Math. Soc., 348, 2063–2084 (1996)
[7] Garfinkle, D. and Harris, S. G., Ricci fall-off in static and stationary, globally hyperbolic, non-singular spacetimes, Classical Quantum Gravity, 14, 139–151 (1997)
[8] Geroch, R. P., Topology in general relativity, J. Mathematical Phys., 8, 782–786 (1967)
[9] Geroch, R. P., Domain of dependence, J. Mathematical Phys., 11, 437–449 (1970)
[10] Gromov, M., Metric structures for Riemannian and Non-Riemannian Spaces, XX+586 p., Birkhäuser, Boston (2007)
[11] Guediri, M., On the nonexistence of closed timelike geodesics in flat Lorentz 2-step nilmanifolds, Trans. Amer. Math. Soc., 355, 775–786 (electronic) (2003)
[12] , Harris, S. G., Discrete group actions on spacetimes: causality conditions and the causal boundary, Classical Quantum Gravity, 21, 1209–1236 (2004)
[13] Ketterer, C., Periodische Metriken und die stabile Norm, Diplomarbeit, Freiburg, (2008)
[14] Mather, J. N., Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z., 207, 169–207 (1991)
[15] Romero, A. and Sánchez, M., On completeness of certain families of semi-Riemannian manifolds, Geom. Dedicata, 53, 103–117 (1994)
[16] Schelling, E., Maximale Geodätische auf Lorentzmannigfaltigkeiten, Diplomarbeit, Freiburg, (1995)
[17] Suhr, S., Homologically Maximizing Geodesics in Conformally Flat Tori, arXiv:/1003.2322v1
[18] Suhr, S., Closed Geodesics in Lorentzian Surfaces, arXiv:/1011.4878v1
[19] Sullivan, D., Cycles for the dynamical study of foliated manifolds and complex manifolds, Invent. Math., 36, 225–255 (1976)