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Abstract. The linearized stability of stationary solutions for surface diffusion is studied. We consider hypersurfaces that lie inside a fixed domain, touch its boundary with a right angle and fulfill a no-flux condition. We formulate the geometric evolution law as a partial differential equation with the help of a parametrization from Vogel [Vog00], which takes care of a possible curved boundary. For the linearized stability analysis we identify as in the work of Garcke, Ito and Kohsaka [GIK05] the problem as an H^{-1}-gradient flow, which will be crucial to show self-adjointness of the linearized operator. Finally we study the linearized stability of some examples.

Keywords. partial differential equations on manifolds, surface diffusion, linearized stability of stationary solutions, gradient flow

AMS subject classification. 35G30, 35R35, 35B35

1 Introduction

We consider the geometric evolution law

$$V = -\Delta H \,, \tag{1.1}$$

called surface diffusion flow, for evolving hypersurfaces Γ in $\mathbb{R}^{n+1}$. Here V is the normal velocity of the evolving hypersurface, H is the mean curvature and Δ is the Laplace-Beltrami operator. Our sign convention is that H is negative for spheres provided with outer unit normal.

Surface diffusion flow (1.1) was first proposed by Mullins [Mu57] to model motion of interfaces where this motion is governed purely by mass diffusion within the interfaces. Davi and Gurtin [DG90] derived the above law within rational thermodynamics and Cahn, Elliott and Novick-Cohen [CEN96] identified it as the sharp interface limit of a Cahn-Hilliard equation with degenerate mobility. An existence result for curves in the plane and stability of circles has been shown by Elliott and Garcke [EG97] and this result was generalized to the higher dimensional case by Escher, Mayer and Simonett [EMS98]. Cahn and Taylor [CT94] showed that (1.1) is the H^{-1}-gradient flow of the area functional and we finally mention that for closed embedded hypersurfaces the enclosed volume is preserved and the surface area decreases in time as can be seen for example in [EG97] or [EMS98].

We will examine surface diffusion flow with boundary conditions by considering evolving hypersurfaces Γ that meet the boundary of a fixed bounded region Ω. These boundary conditions were derived by Garcke and Novick-Cohen [GN00] as the asymptotic limit of a Cahn-Hilliard system with a degenerate mobility matrix. At the outer boundary this yields natural boundary conditions given by a 90° angle condition and a no-flux condition, i.e. we require at $\Gamma(t) \cap \partial\Omega$

$$\Gamma(t) \perp \partial\Omega \,, \tag{1.2}$$

$$\nabla H \cdot n_{\partial\Gamma} = 0 \,. \tag{1.3}$$

Here ∇ is the surface gradient and $n_{\partial\Gamma}$ is the outer unit conormal of Γ at boundary points. The conditions (1.2) and (1.3) are the natural boundary conditions when viewing surface diffusion (1.1) with outer boundary contact as the H^{-1}-gradient flow of the area functional.

Smooth solutions Γ of the flow (1.1) together with the boundary conditions (1.2) and (1.3) are area-minimizing and volume-preserving in the sense that

$$\frac{d}{dt}A(t) \leq 0 \quad \text{and} \quad \frac{d}{dt}Vol(t) = 0\,,$$

where $A(t)$ indicates the surface area of $\Gamma(t)$ and $V(t)$ the volume of the region enclosed by $\Gamma(t)$ and $\partial\Omega$, see e.g. [Dep10].

For one evolving curve in the plane with boundary conditions (1.2) and (1.3) Garcke, Ito and Kohsaka gave in [GIK05] a linearized stability criterion for spherical arcs resp. lines, which are the stationary states in this case. In [GIK08] the same authors showed nonlinear stability results for the above situation.

We will introduce a linear stability criterion based on the work of Garcke, Ito and Kohsaka [GIK05] for curves in the plane and extend it to the case of hypersurfaces. One of the main difficulties lies in the very beginning of the work when we want to introduce a parametrization with good properties to rewrite the geometric evolution law as a partial differential equation for an unknown function. Therefore we use a curvilinear coordinate system as in the work of Vogel [Vog00] which accounts for a possible curved boundary. In this way we consider evolving hypersurfaces given as a graph over some fixed stationary reference hypersurface. It is very important that we can describe the linearized problem as in the curve case as an H^{-1}-gradient flow, because this is the main reason that the linearized operator is self-adjoint. Then we are in a good position to apply results from spectral theory. We can relate the asymptotic stability of the zero solution of the linearized problem to the fact that the eigenvalues of the linearized operator are negative. Since we can describe the largest eigenvalue with the help of a bilinear form arising due to the gradient flow structure, we can finally give a criterion for linearized stability of the original geometric problems around stationary states. At the end of the work we discuss some examples.

The linearized equations are given through

$$\left\{\begin{array}{rcll} \partial_t \rho & = & -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{in } \Gamma^* \quad \text{for all } t>0\,, \\ 0 & = & \Big(\partial_\mu - S(n^*,n^*)\Big)\,\rho & \text{on } \partial\Gamma^* \quad \text{for all } t>0\,, \\ 0 & = & \partial_\mu\Big(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\Big) & \text{on } \partial\Gamma^* \quad \text{for all } t>0\,, \end{array}\right.$$

and the zero solution is asymptotically stable if and only if

$$I(\rho,\rho) = \int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right)d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*,n^*)\rho^2\, d\mathcal{H}^{n-1}$$

is positive for all $\rho \in H^1(\Gamma^*)\backslash\{0\}$ with $\int_{\Gamma^*}\rho = 0$. Herein σ^* is the second fundamental form of Γ^* with respect to the unit normal n^* and S is the second fundamental form of the boundary $\partial\Omega$ of the fixed region with respect to the inwards pointing unit normal $(-\mu)$ of Ω.

This work is part of the thesis [Dep10] of the author, where also the case of three evolving hypersurfaces that meet each other at a triple line, is considered. This problem will be the subject of a forthcoming publication.

2 Parametrization

In this section we present a suitable parametrization in order to formulate a partial differential equation out of the geometric evolution law (1.1)-(1.3).

In detail the problem consists in finding an evolving hypersurface $\Gamma = \bigcup_{t\in[0,T)}\{t\}\times\Gamma(t)$ with $\Gamma(t) \subset \mathbb{R}^{n+1}$ evolving due to surface diffusion flow, such that $\Gamma(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the boundary $\partial\Gamma(t)$ of each of the hypersurfaces intersects the boundary $\partial\Omega$ of the fixed region at a right angle. In formulas, the problem reads as follows. Find Γ as above, such that

$$\left\{\begin{array}{rcll} V & = & -\Delta_{\Gamma(t)} H & \text{in } \Gamma(t) \quad \text{for all } t>0\,, \\ \nabla_{\Gamma(t)} H \cdot n_{\partial\Gamma}(t) & = & 0 & \text{on } \partial\Gamma(t) \quad \text{for all } t>0\,, \\ \Gamma(t) & \subset & \Omega & \quad \text{for all } t>0\,, \\ \partial\Gamma(t) & \subset & \partial\Omega & \quad \text{for all } t>0\,, \\ n(t)\cdot\mu & = & 0 & \text{on } \partial\Gamma(t) \quad \text{for all } t>0\,, \\ \Gamma(0) & = & \Gamma_0\,. & \end{array}\right. \tag{2.1}$$

Here V, H, n, $n_{\partial\Gamma}(t)$ and μ are the normal velocity, the mean curvature, a unit normal of the evolving hypersurface Γ, the outer unit conormal of $\Gamma(t)$ at $\partial\Gamma(t)$ and the outer unit normal to $\partial\Omega$. $\nabla_{\Gamma(t)}$ is the surface gradient and $\Delta_{\Gamma(t)}$ the Laplace-Beltrami operator on $\Gamma(t)$. Γ_0 is a given starting surface, which lies in Ω and intersects the boundary $\partial\Omega$ at a right angle.

Now we fix a stationary hypersurface Γ^* of (2.1), i.e. Γ^* lies in Ω, intersects $\partial\Omega$ at a right angle, fulfills the natural boundary condition $\nabla_{\Gamma^*} H^* \cdot n_{\partial\Gamma^*} = \nabla_{\Gamma^*} H^* \cdot \mu = 0$ on $\partial\Gamma^*$ and the surface diffusion equation with $V=0$, resulting in constant mean curvature H^*.

As a first step to describe the hypersurfaces $\Gamma(t)$ that we want to consider, we set up a specific curvilinear coordinate system as in the work of Vogel [Vog00], that takes into account a possible curved boundary $\partial\Omega$ and the fact, that the considered hypersurfaces have to stay inside Ω and their boundary has to lie on $\partial\Omega$. Therefore, we postulate for small $d>0$ the existence of a smooth mapping

$$\Psi : \Gamma^* \times (-d,d) \longrightarrow \Omega\,, \qquad (q,w) \longmapsto \Psi(q,w)\,, \tag{2.2}$$

such that

$$\Psi(q,0) = q \quad \text{for all } q \in \Gamma^* \tag{2.3}$$

and

$$\Psi(q,w) \in \partial\Omega \quad \text{for all } q \in \partial\Gamma^*\,. \tag{2.4}$$

We also assume that for every (local) parametrization $q : D \to \Gamma^*$ with $D \subset \mathbb{R}^n$ open, the mapping $(y,w) \mapsto \Psi(q(y),w)$ is a locally invertible map from $\mathbb{R}^n$ to $\mathbb{R}^n$. At last, we choose a normal n^* of Γ^* and impose the condition that $\partial_w \Psi(q,0)\cdot n^*(q) \neq 0$ for $q \in \Gamma^*$, which means that there is some movement in normal direction. With a rescaling in the w-coordinate we can then even assume that

$$\partial_w \Psi(q,0)\cdot n^*(q) = 1 \quad \text{for } q \in \Gamma^*\,. \tag{2.5}$$

In [Vog00] there are some examples for situations when such a curvilinear coordinate system exists. Due to the angle condition at the boundary of Γ^*, we can conclude even more than (2.5) at the boundary $\partial\Gamma^*$.

Lemma 2.1. *For $q \in \partial\Gamma^*$, it holds that $\partial_w \Psi(q,0) = n^*(q)$.*

Proof. We see that for fixed $q \in \partial\Gamma^*$ the curve $c(w) := \Psi(q,w)$ lies on the boundary $\partial\Omega$, and with $c(0) = \Psi(q,0) = q$ it therefore holds $\partial_w\Psi(q,0) \in T_q(\partial\Omega)$. With the help of the angle condition we get $T_q\Gamma^* \perp T_q(\partial\Omega)$ and so we observe that $\partial_w\Psi(q,0)\cdot v = 0$ for all $v \in T_q\Gamma^*$. So $\partial_w\Psi(q,0)$ has just a normal part, that is $\partial_w\Psi(q,0) = (\partial_w\Psi(q,0)\cdot n^*(q))\, n^*(q)$. With the rescaling condition of the normal (2.5) the claim follows. $\square$

With the help of the mapping Ψ from (2.2) we define the hypersurfaces, that we want to consider. For a given smooth function

$$\rho : [0,T) \times \Gamma^* \longrightarrow (-d,d) \tag{2.6}$$

we introduce the mapping

$$\Phi^\rho : [0,T) \times \Gamma^* \longrightarrow \Omega\,, \qquad \Phi^\rho(t,q) := \Psi(q,\rho(t,q))\,. \tag{2.7}$$

Then we observe that for fixed t due to the assumptions on Ψ, the function

$$\Phi_t^\rho : \Gamma^* \longrightarrow \Omega\,, \qquad \Phi_t^\rho(q) := \Phi^\rho(t,q) \tag{2.8}$$

is a diffeomorphism onto its image. We denote this image by $\Gamma_\rho(t)$, that is

$$\Gamma_\rho(t) := \{\Phi_t^\rho(q) \mid q \in \Gamma^*\}\,. \tag{2.9}$$

In such a way we get an evolving hypersurface $\Gamma = \bigcup_{t\in[0,T)}\{t\} \times \Gamma_\rho(t)$ and we made sure that the hypersurfaces $\Gamma_\rho(t)$ always fulfill the conditions $\Gamma_\rho(t) \subset \Omega$ and $\partial\Gamma_\rho(t) \subset \partial\Omega$. We also observe that for $\rho \equiv 0$ it holds $\Gamma_{\rho\equiv 0}(t) = \Gamma^*$ for all $t \in [0,T)$.

At last we impose that the starting hypersurface Γ_0 is given with the help of a smooth function $\rho_0 : \Gamma^* \to \mathbb{R}$ through $\Gamma_0 = \{\Psi(q,\rho_0(q)) \,|\, q \in \Gamma^*\}$.

With the help of the diffeomorphisms Φ_t^ρ, we can finally formulate (2.1) over the fixed stationary hypersurface Γ^* as follows. Find ρ as in (2.6) as a solution to the problem

$$\left\{\begin{array}{rcllr} V(\Psi(q,\rho(t,q))) &=& -\Delta_{\Gamma_\rho(t)} H(\Psi(q,\rho(t,q))) & \text{in } \Gamma^* & \text{for all } t>0\,,\\ 0 &=& \left(\nabla_{\Gamma_\rho(t)} H \cdot n_{\partial\Gamma_\rho(t)}\right)(\Psi(q,\rho(t,q))) & \text{on } \partial\Gamma^* & \text{for all } t>0\,,\\ 0 &=& (n(t)\cdot\mu)\,(\Psi(q,\rho(t,q))) & \text{on } \partial\Gamma^* & \text{for all } t>0\,,\\ \rho(0,q) &=& \rho_0(q) & \text{in } \Gamma^*\,. & \end{array}\right. \tag{2.10}$$

3 Linearization

In this section we give the linearization of (2.10) around $\rho \equiv 0$, which corresponds to the linearization of (2.1) around the stationary state Γ^*. To get the linearization, we consider each term separately, write $\varepsilon\rho$ instead of ρ in (2.10), differentiate with respect to ε and set $\varepsilon = 0$.

Lemma 3.1. *The linearization of the surface diffusion equation from (2.10)*

$$V(\Psi(t,\rho(t,q))) = -\Delta_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q)))$$

around the stationary state represented through $\rho \equiv 0$ is given by

$$\partial_t\rho(t,q) = -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right),$$

where $q \in \Gamma^*$, $t > 0$, Δ_{Γ^*} is the Laplace-Beltrami operator on Γ^* and $|\sigma^*|^2$ is the squared norm of the second fundamental form of Γ^* with respect to n^*, which is given through the sum over the squared principal curvatures.

Proof. For the normal velocity we use the representation

$$V(t,\Psi(q,\rho(t,q))) = n(t,\Psi(q,\rho(t,q)))\cdot\frac{d}{dt}\Psi(q,\rho(t,q)) = \Big(n(t,\Psi(q,\rho(t,q)))\cdot\partial_w\Psi(q,\rho(t,q))\Big)\,\partial_t\rho(t,q)\,.$$

Therefore we can calculate

$$\begin{aligned} \frac{d}{d\varepsilon}V(t,\Psi(q,\varepsilon\rho(t,q)))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Big(n(t,\Psi(q,\varepsilon\rho(t,q)))\cdot\partial_w\Psi(q,\varepsilon\rho(t,q))\Big)\Big|_{\varepsilon=0}\underbrace{(\partial_t\varepsilon\rho(t,q))|_{\varepsilon=0}}_{=0}\\ &\quad+\Big(n(t,\Psi(q,\varepsilon\rho(t,q)))\cdot\partial_w\Psi(q,\varepsilon\rho(t,q))\Big)\Big|_{\varepsilon=0}\,\partial_t\rho(t,q)\\ &= \Big(n(t,\Psi(q,0))\cdot\partial_w\Psi(q,0)\Big)\,\partial_t\rho(t,q)\\ &\overset{(2.3)}{=} \Big(n^*(q)\cdot\partial_w\Psi(q,0)\Big)\,\partial_t\rho(t,q)\\ &= \partial_t\rho(t,q)\,, \end{aligned}$$

where we used (2.5) in the last line. To see $n(t,\Psi(q,0)) = n^*(q)$ in the line before, we observe the fact that $n(t,\Psi(q,\varepsilon\rho(t,q)))$ is the normal of $\Gamma_{\varepsilon\rho}(t)$ at $\Psi(q,\varepsilon\rho(t,q)) \in \Gamma_{\varepsilon\rho}(t)$, so that for $\varepsilon = 0$ the term $n(t,\Psi(q,0))$ is the normal of $\Gamma_{\rho\equiv 0}(t)$ at $\Psi(q,0) \in \Gamma_{\rho\equiv 0}$. With (2.3) and $\Gamma_{\rho\equiv 0}(t) = \Gamma^*$ for all t we find that $n(t,\Psi(q,0)) = n(t,q) = n^*(q)$ is the normal of Γ^* at $q \in \Gamma^*$.
For the Laplace-Beltrami operator of mean curvature we use the transformation rule

$$-\Delta_{\Gamma_\rho(t)} H(\Psi(t,\rho(t,q))) = -\Delta^{\rho}_{\Gamma^*}\left(\widetilde{H}_\rho(t,q)\right),$$

where $\widetilde{H}_\rho(t,q) = H(\Psi(t,\rho(t,q)))$ and $\Delta^\rho_{\Gamma^*}$ is the Laplace-Beltrami operator of Γ^* equipped with the pull-back metric $\left(\Phi^\rho_t\right)^*\eta$, where η is a symbol for the euclidian scalar product in $\mathbb{R}^{n+1}$.
Then we observe that for $\rho \equiv 0$ due to $\Phi^0_t = id|_{\Gamma^*}$ the identity $\Delta^0_{\Gamma^*} = \Delta_{\Gamma^*}$ holds, where Δ_{Γ^*} is the Laplace-Beltrami operator of Γ^* with respect to the restriction of the euclidian scalar product. We also have $\widetilde{H}_0 = H^*$, where H^* is the constant mean curvature of Γ^*. Therefore we get with a similar calculation as in the work of Escher, Mayer and Simonett [EMS98]

$$\frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\Big|_{\varepsilon=0}\widetilde{H}_0 = \frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\Big|_{\varepsilon=0} H^* = \frac{d}{d\varepsilon}\underbrace{(\Delta^{\varepsilon\rho}_{\Gamma^*}H^*)}_{=0}\Big|_{\varepsilon=0} = 0\,.$$

Finally, this gives for the right side of the surface diffusion equation

$$\begin{aligned}\frac{d}{d\varepsilon}\left(-\Delta^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\Big|_{\varepsilon=0} &= -\frac{d}{d\varepsilon}\Delta^{\varepsilon\rho}_{\Gamma^*}\Big|_{\varepsilon=0} H^* - \Delta_{\Gamma^*}\left(\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\rho}(t,q)\Big|_{\varepsilon=0}\right)\\ &= -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right),\end{aligned}$$

where we used the well-known linearization of mean curvature $\delta H = \Delta\rho + |\sigma^*|^2\rho$. A proof of this identity using the notion of normal-time derivative can be found in the work of the author [Dep10]. ☐

The next point is to linearize the first boundary condition in (2.10).

Lemma 3.2. *The linearization of the boundary condition*

$$0 = \left(\nabla_{\Gamma_\rho(t)} H \cdot n_{\partial\Gamma_\rho(t)}\right)(\Psi(q,\rho(t,q)))$$

from (2.10) around the stationary state represented through $\rho \equiv 0$ is given by

$$0 = \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*|^2\rho(t,q)\right)\cdot\mu(q) = \partial_\mu\left(\Delta_{\Gamma^*}\rho(t,q) + |\sigma^*(q)|^2\rho(t,q)\right),$$

where $q \in \partial\Gamma^*$ and $t > 0$.

Proof. As for the Laplace-Beltrami operator we can correlate the surface gradient on $\Gamma_\rho(t)$ and on Γ^* equipped with the pull-back metric $\left(\Phi^\rho_t\right)^*\eta$ via

$$\nabla_{\Gamma_\rho(t)} H(\Psi(q,\rho(t,q))) = d_q\Phi^\rho_t\left(\nabla^\rho_{\Gamma^*}\widetilde{H}_\rho(t,q)\right),$$

where $p = \Phi^\rho_t(q) = \Psi(q,\rho(t,q)) \in \Gamma_\rho(t)$ and $d_q\Phi^\rho_t : T_q\Gamma^* \to T_{\Phi_t(q)}\Gamma_\rho(t)$ is the differential. With the same notation as in the previous lemma we get

$$\begin{aligned}\frac{d}{d\varepsilon}\left(d_q\Phi^{\varepsilon\rho}_t\left(\nabla^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\right)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left(d_q\Phi^{\varepsilon\rho}_t\right)\Big|_{\varepsilon=0}\underbrace{(\nabla_{\Gamma^*}H^*)}_{=0} + \underbrace{d_q\Phi^{\rho\equiv 0}_t}_{=Id}\left(\frac{d}{d\varepsilon}\left(\nabla^{\varepsilon\rho}_{\Gamma^*}\widetilde{H}_{\varepsilon\rho}(t,q)\right)\Big|_{\varepsilon=0}\right)\\ &= \frac{d}{d\varepsilon}\underbrace{\nabla^{\varepsilon\rho}_{\Gamma^*}H_0}_{=0}\Big|_{\varepsilon=0} + \nabla_{\Gamma^*}\left(\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\rho}(t,q)\Big|_{\varepsilon=0}\right)\\ &= \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)(t,q)\,.\end{aligned}$$

With the additional observation $n_{\partial\Gamma_{\rho\equiv 0}(t)} = n_{\partial\Gamma^*} = \mu$ due to the right angle condition for the fixed stationary hypersurface Γ^* we can show the assertion. $\square$

We proceed with the linearization of the boundary condition $n(t, \Psi(q, \rho(t, q))) \cdot \mu(\Psi(q, \rho(t, q))) = 0$ on $\partial\Gamma^*$ for $t > 0$ around $\rho \equiv 0$. To calculate this linearization at $q_0 \in \partial\Gamma^*$ and $t_0 > 0$, we choose a local parametrization of Γ^* around q_0 with nice properties. More precisely, let $U \subset \mathbb{R}^{n+1}$ be an open neighbourhood of q_0, $V \subset \mathbb{R}^{n+1}$ open and $\varphi : U \to V$ a diffeomorphism, such that

$$\varphi(U \cap \Gamma^*) = V \cap \left(\mathbb{R}^n_+ \times \{0\}\right) \quad \text{with} \quad \left(\varphi(q_0)\right)_n = 0\,.$$

We set $D \times \{0\} := V \cap \left(\mathbb{R}^n_+ \times \{0\}\right)$ and let $F = \left(\varphi^{-1}\right)\big|_D$, i.e.

$$F : D \longrightarrow \Gamma^* \subset \mathbb{R}^{n+1}\,, \qquad x \mapsto F(x)\,. \tag{3.1}$$

This is a local parametrization extended up to the boundary around q_0 with $F(x_0) = q_0$ for some $x_0 \in \partial D$. At the fixed point x_0, we can demand the following properties.

(A) $\partial_1 F(x_0), \ldots, \partial_n F(x_0)$ is an orthonormal basis of $T_{q_0}\Gamma^*$,

(B) $\partial_1 F(x_0) = n_{\partial\Gamma^*}(q_0)$, where $n_{\partial\Gamma^*}$ is the outer unit conormal of Γ^* at $\partial\Gamma^*$ and

(C) $\left(\partial_1 F \times \ldots \times \partial_n F\right)(x_0) = n^*(F(x_0))$, where we just fix the sign.

The third assumption (C) uses the cross product for n vectors in $\mathbb{R}^{n+1}$, which in this case due to the orthonormality of $\partial_1 F(x_0), \ldots, \partial_n F(x_0)$ lies by definition in normal direction and we just want to fix the sign.

With the parametrization F of Γ^* we also get a parametrization of $\Gamma_\rho(t)$ using the diffeomorphism $\Phi^\rho_t : \Gamma^* \to \Gamma_\rho(t)$ with $\Phi^\rho_{t_0}(q_0) = p_0$ for $p_0 \in \Gamma_\rho(t)$, which we denote by

$$G_t : D \longrightarrow \Gamma_\rho(t)\,, \qquad G_t(x) := \Phi^\rho_t(F(x)) = \Psi(F(x), \rho(t, F(x)))\,.$$

Locally around (t_0, p_0), the normal $n(t, p) = n(t, \Phi^\rho_t(q)) = n(t, \Phi^\rho_t(F(x)))$ of $\Gamma_\rho(t)$ is given with the help of the cross product of n vectors in $\mathbb{R}^{n+1}$ through

$$n(t, \Phi^\rho_t(F(x))) \quad = \quad \frac{\partial_1 G_t \times \ldots \times \partial_n G_t}{|\partial_1 G_t \times \ldots \times \partial_n G_t|}(x) = \frac{\partial_1 \Phi^\rho_t \times \ldots \times \partial_n \Phi^\rho_t}{|\partial_1 \Phi^\rho_t \times \ldots \times \partial_n \Phi^\rho_t|}(F(x))\,,$$

where ∂_i is the partial derivative with respect to x_i. To calculate the linearization of the right angle condition at the outer boundary, we need the following properties of Ψ at $w = 0$.

Lemma 3.3. *With the help of the parametrization F it holds for $F(x) = q \in \Gamma^*$*

(i) $\Psi(F(x), 0) = F(x)$, $\partial_i \Psi(F(x), 0) = \partial_i F(x)$,

and for $F(x) = q \in \partial\Gamma^*$ we have

(ii) $\partial_w \Psi(F(x), 0) = n^*(F(x))$, $\partial_i \partial_w \Psi(F(x), 0) \cdot n^*(F(x)) = 0$.

Additionally, for the fixed $F(x_0) = q_0 \in \partial\Gamma^*$ it holds

(iii) $\left(\partial_1 \Psi \times \ldots \times \partial_n \Psi\right)(F(x_0), 0) = n^*(F(x_0))$,

(iv) $\left(\partial_1 \Psi \times \ldots \times \overbrace{\partial_w \Psi}^{i\text{-th pos.}} \times \ldots \times \partial_n \Psi\right)(F(x_0), 0) = (-1)\partial_i F(x_0)$ *and*

$$(v)\quad \Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_i\partial_w\Psi}^{\textit{i-th pos.}}\times\ldots\times\partial_n\Psi\Big)(F(x_0),0) = \Big(\partial_i\partial_w\Psi(F(x_0),0)\cdot\partial_i F(x_0)\Big)\, n^*(F(x_0)),$$

where $i=1,\ldots,n$ in each case.

Proof. This is a direct calculation using the properties of the vector product and the parametrization F from (3.1) and will be omitted here. ☐

Now we can show the following linearization of the right angle condition.

Lemma 3.4. *The linearization of the right angle condition at the outer boundary for $t>0$ and $q\in\partial\Gamma^*$ is given by*

$$\frac{d}{d\varepsilon}\Big(n(t,\Psi(q,\varepsilon\rho(t,q)))\cdot\mu(\Psi(q,\varepsilon\rho(t,q)))\Big)\Big|_{\varepsilon=0} = -\,\nabla_{\Gamma^*}\rho(t,q)\cdot\mu(q) + S_q(n^*(q),n^*(q))\rho(t,q)\,, \tag{3.2}$$

where S is the second fundamental form of $\partial\Omega$ with respect to $-\mu$. Note that $n^*(q)\in T_q\partial\Omega$ because due to the angle condition for the stationary state Γ^* the relation $n^*(q)\cdot\mu(q)=0$ for $q\in\partial\Gamma^*$ holds true.

Proof. We calculate the linearization at a fixed point $q_0\in\partial\Gamma^*$ and $t_0>0$. Using the above notation for the parametrization F and Φ_t we have to calculate

$$\frac{d}{d\varepsilon}\Big[\Big(\partial_1\Phi_t^{\varepsilon\rho}\times\ldots\times\partial_n\Phi_t^{\varepsilon\rho}\Big)\cdot\Big(\mu\circ\Phi_t^{\varepsilon\rho}\Big)(F(x))\Big]\Big|_{\varepsilon=0} \tag{3.3}$$

at the fixed point (t_0,x_0).
For the vector product in the above formula we do firstly some calculations without ε to get

$$\partial_i\Phi_t^{\rho}(F(x)) = \partial_i\Big(\Psi(F(x),\rho(t,F(x)))\Big) = \partial_i\Psi + \partial_w\Psi\,\partial_i\rho\,, \tag{3.4}$$

where we used some short notation without variables. Furthermore we observe

$$\begin{aligned}
&\Big(\partial_1\Phi_t^{\rho}\times\ldots\times\partial_n\Phi_t^{\rho}\Big)\\
={}&\Big((\partial_1\Psi+\partial_i\rho\,\partial_w\Psi)\times\ldots\times(\partial_n\Psi+\partial_i\rho\,\partial_w\Psi)\Big)\\
={}&\Big(\partial_1\Psi\times\ldots\times\partial_n\Psi\Big)+\sum_{i=1}^n\partial_i\rho\Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\ldots\times\partial_n\Psi\Big)\\
&+\sum_{\substack{i,j=1\\ i\neq j}}^n\partial_i\rho\,\partial_j\rho\underbrace{\Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\ldots\times\overbrace{\partial_w\Psi}^{\text{j-th pos.}}\times\ldots\times\partial_n\Psi\Big)}_{=0}\\
&+\text{h.o.t.}\\
={}&\Big(\partial_1\Psi\times\ldots\times\partial_n\Psi\Big)+\sum_{i=1}^n\partial_i\rho\Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\ldots\times\partial_n\Psi\Big)\,.
\end{aligned}$$

Herein the terms h.o.t. contain more than two $\partial_w\Psi$ in the cross product and therefore they also vanish. Inserting the last identity into (3.3) for the fixed (t_0,x_0) with $F(x_0)=q_0$, we can do the following calculation

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big[\Big(\partial_1\Phi_{t_0}^{\varepsilon\rho}\times\ldots\times\partial_n\Phi_{t_0}^{\varepsilon\rho}\Big)\cdot\Big(\mu\circ\Phi_{t_0}^{\varepsilon\rho}\Big)(F(x_0))\Big]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Big\{\Big[\Big(\partial_1\Psi\times\ldots\times\partial_n\Psi\Big)(q_0,\varepsilon\rho(t_0,q_0))\\
&+\sum_{i=1}^n\partial_i\varepsilon\rho(t_0,q_0)\Big(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\ldots\times\partial_n\Psi\Big)(q_0,\varepsilon\rho(t_0,q_0))\Big]\cdot\mu(\Psi(q_0,\varepsilon\rho(t_0,q_0)))\Big\}\Big|_{\varepsilon=0}
\end{aligned}$$

$$
\begin{aligned}
= &\frac{d}{d\varepsilon}\Bigg[\Bigg[\underbrace{\Big(\partial_1\Psi\times\ldots\times\partial_n\Psi\Big)(q_0,\varepsilon\rho(t_0,q_0))}_{(1)} \\
&+\underbrace{\sum_{i=1}^n \partial_i\varepsilon\rho(t_0,q_0)\left(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\ldots\times\partial_n\Psi\right)(q_0,\varepsilon\rho(t_0,q_0))}_{(2)}\Bigg]\Bigg|_{\varepsilon=0}\cdot\mu(\Psi(q_0,0)) \\
&+\Bigg[\underbrace{\Big(\partial_1\Psi\times\ldots\times\partial_n\Psi\Big)(q_0,0)}_{(3)}+0\Bigg]\cdot\frac{d}{d\varepsilon}\underbrace{\mu(\Psi(q_0,\varepsilon\rho(t_0,q_0)))}_{(4)}\Bigg|_{\varepsilon=0}.
\end{aligned}
$$

We will consider the above numbered terms separately. For the first one, we calculate

$$
\begin{aligned}
\frac{d}{d\varepsilon}(1)\Big|_{\varepsilon=0} &= \sum_{k=1}^n\left(\partial_1\Psi\times\ldots\times\overbrace{\partial_w\partial_k\Psi}^{\text{k-th pos.}}\times\ldots\times\partial_n\Psi\right)(q_0,0)\,\rho(t_0,q_0) \\
&\overset{3.3,(v)}{=} \sum_{k=1}^n n^*(q_0)\Big(\partial_k\partial_w\Psi(F(x_0),0)\cdot\partial_kF(x_0)\Big)\,\rho(t_0,q_0)\,.
\end{aligned}
$$

Therefore we get

$$
\frac{d}{d\varepsilon}(1)\Big|_{\varepsilon=0}\cdot\mu(q_0)=\sum_{k=1}^n\Big(\partial_k\partial_w\Psi(F(x_0),0)\cdot\partial_kF(x_0)\Big)\;\rho(t_0,q_0)\;\underbrace{(n^*(q_0)\cdot\mu(q_0))}_{=0}=0\,,
$$

where we used $\mu(\Psi(q_0,0))=\mu(q_0)$ due to (2.3) and the angle condition for Γ^* to conclude $n^*\cdot\mu=0$.
For the second term, we observe

$$
\begin{aligned}
\frac{d}{d\varepsilon}(2)\Big|_{\varepsilon=0} &= \sum_{i=1}^n\partial_i\rho(t_0,q_0)\left(\partial_1\Psi\times\cdots\times\overbrace{\partial_w\Psi}^{\text{i-th pos.}}\times\cdots\times\partial_n\Psi\right)(F(x_0),0) \\
&\overset{3.3,(iv)}{=} -\sum_{i=1}^n\partial_i\rho(t_0,q_0)\partial_iF(x_0)\;=\;-\nabla_{\Gamma^*}\rho(t_0,q_0)\,,
\end{aligned}
$$

where the last identity can be seen with the representation of the surface gradient in local coordinates due to assumption (A) for F at the fixed x_0. Taking the scalar product with the normal yields

$$
\frac{d}{d\varepsilon}(2)\Big|_{\varepsilon=0}\cdot\mu(q_0)=-\nabla_{\Gamma^*}\rho(t_0,q_0)\cdot\mu(q_0)\,,
$$

which is the directional derivative $-\partial_\mu\rho(t_0,q_0)$ of ρ in direction of the outer unit conormal μ of Γ^* at $\partial\Gamma^*$. Here we used the fact $\mu(q)=n_{\partial\Gamma^*}(q)$ on $\partial\Gamma^*$, that is the outer unit normal of Ω equals the outer unit conormal of Γ^* at $\partial\Gamma^*$ due to the angle condition.
For the remaining terms we observe

$$
\begin{aligned}
(3)\cdot\frac{d}{d\varepsilon}(4)\Big|_{\varepsilon=0} &= (\partial_1\Psi\times\cdots\partial_n\Psi)(F(x_0),0)\cdot\frac{d}{d\varepsilon}\mu(\Psi(q_0,\varepsilon\rho(t_0,q_0)))\Big|_{\varepsilon=0} \\
&\overset{3.3,(iii)}{=} n^*(q_0)\cdot\partial_{(n^*(q_0)\,\rho(t_0,\,q_0))}\mu\,,
\end{aligned}
$$

where the directional derivative appears by definition with the help of the curve $c(\varepsilon) = \Psi(q_0, \varepsilon\rho(t_0, q_0))$, which fulfills

$$c(\varepsilon) \in \partial\Omega\,,\ c(0) = \Psi(q_0, 0) = q_0 \ \text{ and}$$
$$c'(0) = \partial_w \Psi(q_0, 0)\,\rho(t_0, q_0) \overset{3.3,(ii)}{=} n^*(q_0)\rho(t_0, q_0)\,.$$

Due to linearity of the directional derivative, we finally get

$$(3) \cdot \frac{d}{d\varepsilon}(4)\Big|_{\varepsilon=0} = \Big(n^*(q_0) \cdot \partial_{n^*(q_0)}\mu\Big)\ \rho(t_0, q_0) = S_{q_0}(n^*(q_0), n^*(q_0))\,\rho(t_0, q_0)\,,$$

where S is the second fundamental form of $\partial\Omega$ equipped with normal $-\mu$. Note that $n^*(q_0) \in T_{q_0}\partial\Omega$ due to the angle condition for the stationary state Γ^*.
Altogether, the linearization of the boundary condition

$$n(t, \Psi(q, \rho(t, q))) \cdot \mu(\Psi(q, \rho(t, q)) = 0$$

at the fixed point (t_0, q_0) yields

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}(1)\Big|_{\varepsilon=0} \cdot \mu(q_0) + \frac{d}{d\varepsilon}(2)\Big|_{\varepsilon=0} \cdot \mu(q_0) + (3) \cdot \frac{d}{d\varepsilon}(4)\Big|_{\varepsilon=0} \\ &= 0 - \nabla_{\Gamma^*}\rho(t_0, q_0) \cdot \mu(q_0) + S_{q_0}(n^*(q_0), n^*(q_0))\,\rho(t_0, q_0)\,, \end{aligned}$$

Since the fixed point (t_0, q_0) was arbitrary, we can conclude the above linearization for every $q \in \partial\Gamma^*$ and $t > 0$, which completes the proof of Lemma 3.4. □

Putting the last lemmata together, we get the following linearization of (2.10) around $\rho \equiv 0$.

$$\left\{ \begin{array}{rcllr} \partial_t\rho &=& -\Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) & \text{in } \Gamma^* & \text{for all } t > 0\,, \\ 0 &=& \Big(\partial_\mu - S(n^*, n^*)\Big)\,\rho & \text{on } \partial\Gamma^* & \text{for all } t > 0\,, \\ 0 &=& \partial_\mu\Big(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\Big) & \text{on } \partial\Gamma^* & \text{for all } t > 0\,, \\ \rho(0, q) &=& 0 & \text{in } \Gamma^*\,. & \end{array} \right. \tag{3.5}$$

4 Stability analysis

In this section we derive conditions for the asymptotic stability of the zero solution of the linearized problem (3.5). We first show that (3.5) can be interpreted as a gradient flow with respect to an energy E given by a bilinear form I. Then we can show that the solution operator $\mathcal{A}$ of (3.5) is self-adjoint and we will study its spectrum. Finally, we describe asymptotic stability through the condition that I is positive definite.

We generalize the work of Garcke, Ito and Kohsaka [GIK05] from curves to higher dimensions, which is a non-trivial task as the geometry becomes much more involved. Since the problem (3.5) will be a gradient flow with respect to the H^{-1}-inner product, we give its definition. We denote by $\langle .\,, . \rangle$ the duality pairing between the dual space $\left(H^1(\Gamma^*)\right)'$ and $H^1(\Gamma^*)$ and we define the space $H^{-1}(\Gamma^*)$ by

$$H^{-1}(\Gamma^*) := \left\{\rho \in \left(H^1(\Gamma^*)\right)' \mid \langle \rho, 1\rangle = 0\right\}\,. \tag{4.1}$$

Definition 4.1. *We say that $u_v \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} u_v = 0$ for a given $v \in H^{-1}(\Gamma^*)$ is a weak solution of*

$$\left\{ \begin{array}{rcll} -\Delta_{\Gamma^*}u_v &=& v & \textit{in } \Gamma^*\,, \\ \nabla_{\Gamma^*}u_v \cdot n_{\partial\Gamma^*} &=& 0 & \textit{on } \partial\Gamma^*\,, \end{array} \right. \tag{4.2}$$

if and only if u_v satisfies $\langle v, \xi\rangle = \int_{\Gamma^*} \nabla_{\Gamma^*}u_v \cdot \nabla_{\Gamma^*}\xi$ for all $\xi \in H^1(\Gamma^*)$.

For $\rho_i \in H^{-1}(\Gamma^*)$, $i = 1, 2$, we introduce the inner product $(\rho_1, \rho_2)_{-1} := \int_{\Gamma^*} \nabla_{\Gamma^*} u_{\rho_1} \cdot \nabla_{\Gamma^*} u_{\rho_2}$, called the H^{-1}-inner product, where u_{ρ_i} is defined as the weak solution of (4.2) with respect to ρ_i. By definition, we have the identity

$$(\rho_1, \rho_2)_{-1} = \langle \rho_1, u_{\rho_2} \rangle \tag{4.3}$$

for $\rho_i \in H^{-1}(\Gamma^*)$. For further use we also introduce the notation $V := \{\rho \in H^1(\Gamma^*) \mid \int_{\Gamma^*} \rho = 0\}$.

Definition 4.2. *For $\rho_1, \rho_2 \in H^1(\Gamma^*)$ we define*

$$I(\rho_1, \rho_2) := \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho_1 \cdot \nabla_{\Gamma^*} \rho_2 - |\sigma^*|^2 \rho_1 \rho_2 \right) - \int_{\partial\Gamma^*} S(n^*, n^*) \rho_1 \rho_2 \tag{4.4}$$

and the associated energy for $\rho \in H^1(\Gamma^*)$ by $E(\rho) := \frac{1}{2} I(\rho, \rho)$.

The next point is to show that the linearized problem (3.5) is the gradient flow of E with respect to the H^{-1}-inner product $(.\,,.)_{-1}$. This means that a solution ρ of (3.5) fulfils

$$(\partial_t \rho, \xi)_{-1} = -\partial E(\rho(t))(\xi)$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. Here, $\partial E(\rho(t))(\xi)$ denotes the derivative of E at $\rho(t)$ in direction of ξ. Because of the definition of E via the bilinear form I, this derivative is given by

$$\partial E(\rho(t))(\xi) = I(\rho(t), \xi)\,.$$

To simplify notation, we introduce the following time independent problem.

Definition 4.3. *For a given $v \in H^{-1}(\Gamma^*)$ we say that $\rho \in H^3(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ is a weak solution of the boundary value problem*

$$\left\{ \begin{array}{rcll} v &=& -\Delta_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) & \text{in } \Gamma^*\,, \\ 0 &=& \partial_\mu \rho - S(n^*, n^*) \rho & \text{on } \partial\Gamma^*\,, \\ 0 &=& \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot n_{\partial\Gamma^*} & \text{on } \partial\Gamma^*\,, \end{array} \right. \tag{4.5}$$

if and only if ρ satisfies

$$\langle v, \xi \rangle = \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} \xi$$

for all $\xi \in H^1(\Gamma^*)$ and $0 = \partial_\mu \rho - S(n^*, n^*) \rho$ on $\partial\Gamma^*$.

In the case that $v \in L^2(\Gamma^*)$ with $\int_{\Gamma^*} v = 0$, we obtain from elliptic regularity theory on manifolds that $v = -\Delta_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right)$ is fulfilled almost everywhere in Γ^* and $\nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot n_{\partial\Gamma^*} = 0$ is fulfilled almost everywhere on $\partial\Gamma^*$. The fact that the linearized problem is the gradient flow of E with respect to the H^{-1}-inner product follows from the next lemma.

Lemma 4.4. *Let $v \in H^{-1}(\Gamma^*)$ and $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ be given. Then ρ is a weak solution of (4.5) if and only if*

$$(v, \xi)_{-1} = -I(\rho, \xi)$$

holds for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$.

Proof. Let $\rho \in H^3(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ be a weak solution of (4.5). By (4.3) and Definition 4.3, we deduce for $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$ the identities

$$(v, \xi)_{-1} = \langle v, u_\xi \rangle = \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} u_\xi \, .$$

Here, $u_\xi \in H^1(\Gamma^*)$ is the weak solution of (4.2) for the given $\xi \in H^1(\Gamma^*)$. Then, by virtue of $\left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \in H^1(\Gamma^*)$ we see from the definition of the weak solution u_ξ with $\left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right)$ as testfunction

$$\int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} u_\xi = \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \xi \, .$$

Now we conclude with integration by parts.

$$\begin{aligned}
(v, \xi)_{-1} &= \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \xi = - \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} \nabla_{\Gamma^*} \rho \cdot n_{\partial \Gamma^*} \, \xi \\
&= - \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \, \rho \, \xi = -I(\rho, \xi) \, ,
\end{aligned}$$

where we used the boundary condition $\nabla_{\Gamma^*} \rho \cdot n_{\partial \Gamma^*} = \partial_\mu \rho = S(n^*, n^*) \rho$ on $\partial \Gamma^*$ for ρ.

Conversely, assume that $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \rho = 0$ satisfies $(v, \xi)_{-1} = -I(\rho, \xi)$ for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. Now we choose $\xi = -\Delta_{\Gamma^*} \eta$ for a given function $\eta \in H^3(\Gamma^*)$ with $\nabla_{\Gamma^*} \eta \cdot n_{\partial \Gamma^*} = 0$ on $\partial \Gamma^*$. From Definition 4.1 we can write $\eta = u_\xi$ and with (4.3) it holds

$$\begin{aligned}
\langle v, \eta \rangle = (v, \xi)_{-1} = -I(\rho, \xi) &= - \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} \xi - |\sigma^*|^2 \rho \xi \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \rho \xi \\
&= \int_{\Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot \nabla_{\Gamma^*} (\Delta_{\Gamma^*} \eta) - |\sigma^*|^2 \rho \, (\Delta_{\Gamma^*} \eta) \right) + \int_{\partial \Gamma^*} S(n^*, n^*) \rho \, (\Delta_{\Gamma^*} \eta) \, .
\end{aligned}$$

Since $v \in \left(H^1(\Gamma^*) \right)'$ we deduce from the above identity and elliptic regularity theory that $\rho \in H^3(\Gamma^*)$. Integration by parts gives then

$$\begin{aligned}
\langle v, \eta \rangle &= - \int_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho \, \Delta_{\Gamma^*} \eta - \nabla_{\Gamma^*} (|\sigma^*|^2 \rho) \cdot \nabla_{\Gamma^*} \eta \right) \\
&\quad + \int_{\partial \Gamma^*} \left(\nabla_{\Gamma^*} \rho \cdot n_{\partial \Gamma^*} \, \Delta_{\Gamma^*} \eta - |\sigma^*|^2 \rho \underbrace{\nabla_{\Gamma^*} \eta \cdot n_{\partial \Gamma^*}}_{=0} - S(n^*, n^*) \rho \, \Delta_{\Gamma^*} \eta \right) \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} \eta - \int_{\partial \Gamma^*} \Delta_{\Gamma^*} \rho \underbrace{\nabla_{\Gamma^*} \eta \cdot n_{\partial \Gamma^*}}_{=0} \\
&\quad + \int_{\partial \Gamma^*} \left(\partial_\mu \rho - S(n^*, n^*) \rho \right) \Delta_{\Gamma^*} \eta \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} \eta + \int_{\partial \Gamma^*} \left(\partial_\mu \rho - S(n^*, n^*) \rho \right) \Delta_{\Gamma^*} \eta \, .
\end{aligned}$$

To show that ρ is a weak solution of (4.5), we choose a sequence $g_n \in C^\infty(\overline{\Gamma^*})$ with given boundary data $g_n|_{\partial \Gamma^*} = g$ with $\int_{\Gamma^*} g_n = 0$ and which fulfills $\|g_n\|_{L^2(\Gamma^*)} \to 0$ for $n \to \infty$. Then we solve the problem

$$\begin{aligned}
\Delta_{\Gamma^*} \eta_n &= g_n \quad \text{in } \Gamma^* \, , \\
\nabla_{\Gamma^*} \eta_n \cdot n_{\partial \Gamma^*} &= 0 \quad \text{on } \partial \Gamma^*
\end{aligned}$$

with additional condition $\int_{\Gamma^*} \eta_n = 0$. A solution fulfills $\|\eta_n\|_{H^1} \to 0$, which leads to

$$0 = \int_{\partial\Gamma^*} (\partial_\mu \rho - S(n^*, n^*)\rho)\, g$$

for arbitrary boundary data $g \in L^2(\partial\Gamma^*)$. Therefore we conclude with the fundamental lemma that $\partial_\mu \rho - S(n^*, n^*)\rho = 0$ on $\partial\Gamma^*$ and we are led to the identity

$$\langle v, \eta \rangle = \int_{\Gamma^*} \nabla_{\Gamma^*} \left(\Delta_{\Gamma^*} \rho + |\sigma^*|^2 \rho \right) \cdot \nabla_{\Gamma^*} \eta \tag{4.6}$$

for $\eta \in H^3(\Gamma^*)$ with $\nabla_{\Gamma^*} \eta \cdot n_{\partial\Gamma^*} = 0$ on $\partial\Gamma^*$. We can approximate an arbitrary function $\varphi \in H^1(\Gamma^*)$ with such testfunctions η in the H^1-norm. Therefore let w.l.o.g. $\varphi \in C^\infty(\overline{\Gamma^*})$ (otherwise we use an additional approximation $\varphi_n \to \varphi$ in $H^1(\Gamma^*)$ for smooth functions $\varphi_n \in C^\infty(\overline{\Gamma^*})$). In a small neighbourhood around $\partial\Gamma^*$ we choose an extension $u \in H^3(\Gamma^*)$ of $\varphi|_{\partial\Gamma^*}$ which is extended constantly in normal direction and fulfills

$$\begin{aligned} u &= \varphi \quad \text{on } \partial\Gamma^*\,, \\ \nabla_{\Gamma^*} u \cdot n_{\partial\Gamma^*} &= 0 \quad \text{on } \partial\Gamma^*\,. \end{aligned}$$

With the notation $\Gamma^*_\varepsilon := \{p \in \Gamma^* \,|\, \mathrm{dist}(p, \partial\Gamma^* < \varepsilon\}$, where dist is built with the usual metric on a hypersurface given by the infimum over all length of connecting curves, we choose additionally smooth cut-off functions $\zeta_n \in C^\infty(\overline{\Gamma^*})$ with

$$\zeta_n = 1 \ \text{ in } \Gamma^*_{\frac{2}{n}}\,, \quad \zeta_n = 0 \ \text{ in } \Gamma^*_{\frac{1}{n}} \ \text{ and } \ \|\nabla_{\Gamma^*} \zeta_n\|_{L^\infty} \le n\,.$$

Then we set $\eta_n := \varphi\zeta_n + u(1-\zeta_n)$, which by definition fulfills $\eta_n \in H^3(\Gamma^*)$ and $\nabla_{\Gamma^*}\eta_n \cdot n_{\partial\Gamma^*} = 0$ on $\partial\Gamma^*$. Finally, it holds that $\eta_n \to \varphi$ in $H^1(\Gamma^*)$, since on the one hand

$$\|\eta_n - \varphi\|_{L^2} \le \|\varphi(\zeta_n - 1)\|_{L^2} + \|u(1-\zeta_n)\|_{L^2} \longrightarrow 0$$

and on the other hand

$$\|\nabla_{\Gamma^*}\eta_n - \nabla_{\Gamma^*}\varphi\|_{L^2} = \|\nabla_{\Gamma^*}\left((\varphi - u)(\zeta_n - 1)\right)\|_{L^2} \le \|\nabla_{\Gamma^*}(\varphi - u)\,(\zeta_n - 1)\|_{L^2} + \|(\varphi - u)\,\nabla_{\Gamma^*}(\zeta_n - 1)\|_{L^2}\,.$$

The first term tends to 0 and for the second one we observe with $\Sigma_n := \left(\Gamma^*_{\frac{1}{n}} \backslash \Gamma^*_{\frac{2}{n}}\right)$ that

$$\int_{\Gamma^*} |(\varphi - u)\nabla_{\Gamma^*}(\zeta_n - 1)|^2 = \int_{\Sigma_n} |(\varphi - u)\,\nabla_{\Gamma^*}\zeta_n|^2 \le \int_{\Sigma_n} |\varphi - u|^2 \cdot \int_{\Sigma_n} |\nabla_{\Gamma^*}\zeta_n|^2\,.$$

Now we use that u emerges from φ by an extension constant in normal direction and the fact that φ is locally lipschitz continuous to get for $q \in \Sigma_n$ and some $q^* \in \partial\Gamma^*$ the inequality

$$|\varphi(q) - u(q)|^2 = |\varphi(q) - \varphi(q^*)|^2 \le L\, d(q, q^*)^2 \le L \left(\frac{2}{n}\right)^2\,.$$

Together with $|\nabla_{\Gamma^*}\zeta_n|^2 \le n^2$ we get finally

$$\int_{\Gamma^*} |(\varphi - u)\nabla_{\Gamma^*}(\zeta_n - 1)|^2 \le C\frac{1}{n^2} n^2 \left(\int_{\Sigma_n} 1\right)^2 \longrightarrow 0\,.$$

With this approximation we can write (4.6) with arbitrary testfunctions $\varphi \in H^1(\Gamma^*)$, which yields that ρ is a weak solution of (4.5). We remark that this part of the proof strongly differs from the curve case in [GIK05]. $\square$

The next steps consist in showing that the linearized operator is self-adjoint and to study its spectrum. This linearized operator corresponding to (3.5) is given by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H\,,$$

with

$$\begin{cases} \mathcal{D}(\mathcal{A}) &= \{\rho \in H^3(\Gamma^*)\,|\, (\partial_\mu - S(n^*, n^*))\,\rho = 0 \text{ on } \partial\Gamma^* \text{ and } \int_{\Gamma^*} \rho = 0\}\,, \\ H &= \{\rho \in \left(H^1(\Gamma^*)\right)' \,|\, \langle \rho, 1\rangle = 0\} \end{cases} \tag{4.7}$$

by

$$\langle \mathcal{A}\rho, \xi\rangle \coloneqq \int_{\Gamma^*} \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) \cdot \nabla_{\Gamma^*}\xi\,. \tag{4.8}$$

Then we can relate the boundary value problem (4.5) to the problem of finding a $\rho \in \mathcal{D}(\mathcal{A})$ with $\mathcal{A}\rho = v$. By Lemma 4.4 we also have for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$ the identity $(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)$.

Lemma 4.5. *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(\,.\,,.)_{-1}$.*

Proof. For $\rho, \xi \in \mathcal{D}(\mathcal{A})$ we have

$$(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi) = -I(\xi, \rho) = (\mathcal{A}\xi, \rho)_{-1} = (\rho, \mathcal{A}\xi)_{-1}\ ,$$

so that $\mathcal{A}$ is symmetric. $\square$

The spectrum of $\mathcal{A}$ is related to the functional I with the help of the inner product $(\,.\,,.)_{-1}$. In fact, for an eigenfunction $\rho \in \mathcal{D}(\mathcal{A})$ to the eigenvalue λ of $\mathcal{A}$, it holds

$$\lambda\,(\rho, \xi)_{-1} = (\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)$$

for all $\xi \in H^1(\Gamma^*)$ with $\int_{\Gamma^*} \xi = 0$. The next point is to show boundedness of eigenvalues of $\mathcal{A}$ from above. Therefore we need the following two lemmata.

Lemma 4.6. *For all $\delta > 0$ there exists a $C_\delta > 0$, such that for all functions $\rho \in V$ the inequality*

$$\|\rho\|^2_{L^2(\partial\Gamma^*)} \le \delta\,\|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} + C_\delta\,\|\rho\|^2_{-1}$$

holds.

Proof. Assume by contradiction that there exists $\delta > 0$ such that we can find a sequence $(\widetilde{\rho}_n)_{n\in\mathbb{N}} \subset V$ such that

$$\|\widetilde{\rho}_n\|^2_{L^2(\partial\Gamma^*)} > \delta\,\|\nabla_{\Gamma^*}\widetilde{\rho}_n\|^2_{L^2(\Gamma^*)} + n\,\|\widetilde{\rho}_n\|^2_{-1}\,.$$

In particular we observe $\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)} > 0$ for all $n \in \mathbb{N}$. Therefore, we get for the scaled functions $\rho_n = \widetilde{\rho_n}\,\left(\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)}\right)^{-1}$ by multiplying with $\left(\|\widetilde{\rho_n}\|_{L^2(\partial\Gamma^*)}\right)^{-2}$ the inequality

$$1 > \delta\,\|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} + n\,\|\rho_n\|^2_{-1}\,.$$

This implies

$$\|\rho_n\|^2_{-1} < \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty \qquad \text{and} \qquad \|\nabla_{\Gamma^*}\rho_n\|^2_{L^2(\Gamma^*)} < \frac{1}{\delta}\,.$$

Since $\int_{\Gamma^*} \rho_n = 0$, we conclude from Poincaré's inequality that ρ_n is bounded uniformly in $H^1(\Gamma^*)$. Therefore it converges weakly for a subsequence $\rho_n \rightharpoonup \overline{\rho}$ in $H^1(\Gamma^*)$ to some $\overline{\rho} \in H^1(\Gamma^*)$. Due to $0 = (\rho_n, 1)_{L^2} \to (\overline{\rho}, 1)_{L^2} = \int_{\Gamma^*} \overline{\rho}$ we observe $\int_{\Gamma^*} \overline{\rho} = 0$. Furthermore from the compact embedding $\left\{\rho \in H^1(\Gamma^*) \mid \int_{\Gamma^*} \rho = 0\right\} \hookrightarrow H^{-1}(\Gamma^*)$ we see the strong convergence $\rho_n \to \overline{\rho}$ in $H^{-1}(\Gamma^*)$. By uniqueness of the limit and $\|\rho_n\|_{H^{-1}} \to 0$ we get finally $\overline{\rho} = 0$. So we have $\rho_n \rightharpoonup 0$ in $H^1(\Gamma^*)$ By another compact embedding $H^1(\Gamma^*) \hookrightarrow L^2(\partial\Gamma^*)$ we see $\rho_n \to 0$ in $L^2(\partial\Gamma^*)$, which at last contradicts the fact $\|\rho_n\|_{L^2(\partial\Gamma^*)} = 1$ for all $n \in \mathbb{N}$. □

Lemma 4.7. *There exist positive constants C_1 and C_2, such that*

$$\|\rho\|^2_{H^1(\Gamma^*)} \leq C_1 \|\rho\|^2_{-1} + C_2\, I(\rho, \rho)$$

for all $\rho \in V$.

Proof. With an analogue argumentation as in the previous lemma we get the following inequality. For all $\delta > 0$ there exists a $C_\delta > 0$, such that

$$\|\rho\|^2_{L^2(\Gamma^*)} \leq \delta \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} + C_\delta \|\rho\|^2_{-1}$$

holds for all $\rho \in V$. To this end, we just need the compact embedding $H^1(\Gamma^*) \hookrightarrow L^2(\Gamma^*)$ instead of $H^1(\Gamma^*) \hookrightarrow L^2(\partial\Gamma^*)$. Now we obtain with the help of the above inequality and Lemma 4.6

$$\begin{aligned}
I(\rho, \rho) &= \int_{\Gamma^*} |\nabla_{\Gamma^*}\rho|^2 - \int_{\Gamma^*} |\sigma^*|^2 \rho^2 - \int_{\partial\Gamma^*} S(n^*, n^*)\, \rho^2 \\
&\geq \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \cdot \|\rho\|^2_{L^2(\Gamma^*)} - \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} \cdot \|\rho\|^2_{L^2(\partial\Gamma^*)} \\
&\geq \Big(1 - \delta_1 \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)}\Big) \cdot \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} - \||\sigma^*|^2\|_{L^\infty(\Gamma^*)} \cdot \|\rho\|^2_{L^2(\Gamma^*)} \\
&\quad - \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} \cdot C_{\delta_1} \|\rho\|^2_{-1} \\
&\geq \Big(1 - \delta_1 \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)} - \delta_2 \||\sigma^*|^2\|_{L^\infty(\Gamma^*)}\Big) \cdot \|\nabla_{\Gamma^*}\rho\|^2_{L^2(\Gamma^*)} \\
&\quad - \Big(\||\sigma^*|^2\|_{L^\infty(\Gamma^*)}\, C_{\delta_2} + \|S(n^*, n^*)\|_{L^\infty(\partial\Gamma^*)}\, C_{\delta_1}\Big) \cdot \|\rho\|^2_{-1}\,.
\end{aligned}$$

With the help of the Poincaré inequality on V and by choosing δ_1 and δ_2 small enough, we get the assertion. □

With the previous two lemmata we show boundedness from above for the eigenvalues of $\mathcal{A}$.

Lemma 4.8. *Let λ be an eigenvalue of $\mathcal{A}$. Then the following inequality holds*

$$\lambda \leq \frac{C_1}{C_2}\,,$$

where C_1 and C_2 are the positive constants of the above Lemma 4.7.

Proof. Let $\rho \in \mathcal{D}(\mathcal{A})$ be an eigenvector to the eigenvalue λ, which in particular means $\rho \neq 0$. It holds $\lambda\, (\rho, \rho)_{-1} = (\mathcal{A}\rho, \rho)_{-1} = -I(\rho, \rho)$. Assuming that $\lambda > \frac{C_1}{C_2}$, we would have

$$0 = I(\rho, \rho) + \lambda\, (\rho, \rho)_{-1} > I(\rho, \rho) + \frac{C_1}{C_2}\, (\rho, \rho)_{-1} \geq \frac{1}{C_2} \|\rho\|^2_{H^1(\Gamma^*)} > 0\,,$$

which is a contradiction. □

Now we are able to show that $\mathcal{A}$ is self-adjoint with respect to the $(\,.\,,.)_{-1}$ inner product. Therefore we use a property that implies the equivalence of symmetry and self-adjointness from [Weid76].

Lemma 4.9. *The operator $\mathcal{A}$ is self-adjoint with respect to the $(.\,,.)_{-1}$ inner product.*

Proof. We use the following theorem of operator theory. If there exists an $\omega \in \mathbb{R}$, such that

$$\operatorname{im}(\omega\, Id - \mathcal{A}) = H^{-1}(\Gamma)\,,$$

the properties symmetry and self-adjointness of $\mathcal{A}$ are equivalent, see for example [Weid76].

So we have to show that there exists an $\omega \in \mathbb{R}$, such that for given $f \in H^{-1}(\Gamma^*)$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with $\omega\rho - \mathcal{A}\rho = f$. This means that $\rho \in H^3(\Gamma^*)$ is a weak solution of the boundary value problem

$$\begin{cases} \Delta_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right) + \omega\rho &= f \quad \text{in } \Gamma^*\,, \\ \partial_\mu\rho - S(n^*, n^*)\rho &= 0 \quad \text{on } \partial\Gamma^*\,, \\ \nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot n_{\partial\Gamma^*} &= 0 \quad \text{on } \partial\Gamma^*\,. \end{cases} \tag{4.9}$$

The weak formulation consists in finding a $\rho \in H^3(\Gamma^*)$ with $\partial_\mu\rho - S(n^*, n^*)\rho = 0$ on $\partial\Gamma^*$ and

$$\int_{\Gamma^*} -\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\rho + |\sigma^*|^2\rho\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\rho\,\xi = \langle f, \xi\rangle$$

for all $\xi \in H^1(\Gamma^*)$. Due to $\langle f, 1\rangle = 0$, inserting $\xi \equiv 1$ in this equation yields $\int_{\Gamma^*}\rho = 0$, so that a solution ρ really belongs to $\mathcal{D}(\mathcal{A})$. To obtain such a solution ρ, we use the minimization problem

$$F(\rho) := \frac{1}{2}\int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2 - |\sigma^*|^2\rho^2\right) - \int_{\partial\Gamma^*} S(n^*, n^*)\,\rho^2 + \frac{\omega}{2}\,\|\rho\|_{-1}^2 - \int_{\Gamma^*} u_f\,\rho \;\to \min$$

under all $\rho \in H^1(\Gamma^*)$ with $\int_{\Gamma^*}\rho = 0$. Here, $u_f \in H^1(\Gamma^*)$ is the weak solution of (4.2) with respect to $f \in H^{-1}(\Gamma^*)$. With the help of Lemma 4.7 we can show that F is coercive for large ω and therefore there exists a unique minimizer $\overline{\rho} \in V$, which is characterized by the first variation of F through

$$0 = \frac{d}{d\varepsilon}F(\overline{\rho} + \varepsilon v)\Big|_{\varepsilon=0} = \int_{\Gamma^*}\left(\nabla_{\Gamma^*}\overline{\rho}\cdot\nabla_{\Gamma^*}v - |\sigma^*|^2\overline{\rho}\,v\right) - \int_{\partial\Gamma^*} S(n^*, n^*)\overline{\rho}\,v + \omega\,(\overline{\rho}, v)_{-1} - \int_{\Gamma^*} u_f\,v\,,$$

where $v \in V$ is arbitrary. By the Definition of $u_{\overline{\rho}}$ in (4.2) and the identity (4.3), we observe that $\omega\,(\overline{\rho}, v)_{-1} = \omega\,\langle v, u_{\overline{\rho}}\rangle = \omega\int_{\Gamma^*} u_{\overline{\rho}}\,v$. Since in the above equation the testfunctions v have to fulfill the constraint $\int_{\Gamma^*} v = 0$, the identity is the weak version of the boundary value problem

$$\begin{cases} -\left(\Delta_{\Gamma^*}\overline{\rho} + |\sigma^*|^2\,\overline{\rho}\right) + \omega u_{\overline{\rho}} + \lambda &= u_f \quad \text{in } \Gamma^*\,, \\ \partial_\mu\overline{\rho} - S(n^*, n^*)\overline{\rho} &= 0 \quad \text{on } \partial\Gamma^*\,. \end{cases} \tag{4.10}$$

Here the Lagrange-multiplier λ is given through

$$\lambda = \frac{1}{|\Gamma^*|}\left(\int_{\Gamma^*}\left(|\sigma^*|^2\,\overline{\rho} - \omega\,u_{\overline{\rho}} + u_f\right) + \int_{\partial\Gamma^*} S(n^*, n^*)\overline{\rho}\right).$$

Since $u_{\overline{\rho}}$ and u_f are in $H^1(\Gamma^*)$, we obtain from elliptic regularity theory that $\overline{\rho} \in H^3(\Gamma^*)$. Therefore we can differentiate the first line in (4.10) and take the L^2-inner product with $\nabla_{\Gamma^*}\xi$ for some arbitrary $\xi \in H^1(\Gamma^*)$ to obtain

$$-\int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\overline{\rho} + |\sigma^*|^2\,\overline{\rho}\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\nabla_{\Gamma^*}u_{\overline{\rho}}\cdot\nabla_{\Gamma^*}\xi = \int_{\Gamma^*}\nabla_{\Gamma^*}u_f\cdot\nabla_{\Gamma^*}\xi\,.$$

With the Definition of the weak solutions $u_{\overline{\rho}}$ and u_f from (4.2) we finally get

$$-\int_{\Gamma^*}\nabla_{\Gamma^*}\left(\Delta_{\Gamma^*}\overline{\rho} + |\sigma^*|^2\,\overline{\rho}\right)\cdot\nabla_{\Gamma^*}\xi + \omega\int_{\Gamma^*}\overline{\rho}\,\xi = \int_{\Gamma^*}\langle f, \xi\rangle$$

for all $\xi \in H^1(\Gamma^*)$. So together with the boundary condition from (4.10), we found a $\rho \in \mathcal{D}(\mathcal{A})$ with $\omega\rho - \mathcal{A}\rho = f$, provided $\omega > \frac{C_1}{C_2}$, where C_1 and C_2 are the positive constants from Lemma 4.7. $\square$

In the following theorem we give a stability criterion for the zero solution of the linearized operator $\mathcal{A}$.

Theorem 4.10.

(*i*) *The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.*

(*ii*) *The initial value problem (3.5) is solvable for initial data in $H^{-1}(\Gamma^*)$.*

(*iii*) *The zero solution of (3.5) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative, in short notation $\sigma(\mathcal{A}) < 0$.*

Proof. ad (i). We want to show that for some $\lambda \in \mathbb{R}$, the operator $(\lambda I - \mathcal{A})^{-1} : H \to H$ exists and is compact. For $\lambda > \frac{C_1}{C_2}$, where C_1 and C_2 the positive constants from Lemma 4.7 we showed surjectivity of $\lambda I - \mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow H$ in the last Lemma 4.9. Since every eigenvalue $\mu \in \sigma(\mathcal{A})$ fulfills $\mu \leq \frac{C_1}{C_2}$ from Lemma 4.8, we see from the identity $\sigma(\lambda I - \mathcal{A}) = \lambda - \sigma(\mathcal{A})$ for the spectrum that there exists no eigenvalue zero of $\lambda I - \mathcal{A}$ provided $\lambda > \frac{C_1}{C_2}$. For a linear operator this means in particular that it is injective.
Continuity of the resolvent

$$(\lambda I - \mathcal{A})^{-1} : H \longrightarrow \mathcal{D}(\mathcal{A})$$

for $\lambda > \frac{C_1}{C_2}$ can be seen by observing that

$$(\lambda I - \mathcal{A})^{-1}(f) = \rho \quad \Leftrightarrow \quad (\lambda I - \mathcal{A})(\rho) = f\,,$$

which means that $\rho \in \mathcal{D}(\mathcal{A})$ is a weak solution for the boundary value problem (4.9) with $\omega = \lambda$. Solutions of this problem fulfill an inequality

$$\|\rho\|_{H^3(\Gamma^*)} \leq C\,\|f\|_{H^{-1}(\Gamma^*)}\,,$$

which gives continuity of the resolvent. Since the embedding $\mathcal{D}(\mathcal{A}) \hookrightarrow H^{-1}(\Gamma^*)$ is compact, we get by composition a compact operator $(\lambda I - \mathcal{A})^{-1} : H \longrightarrow H$, provided $\lambda > \frac{C_1}{C_2}$. Together with the self-adjointness of $\mathcal{A}$ from Lemma 4.9, we get the claim (i) with the help of an abstract operator theorem from the book of Kato [Kat95].
ad (ii) and (iii). Existence and stability of the problem

$$\text{Find } \rho(t) \in \mathcal{D}(\mathcal{A})\,, \text{ such that } \partial_t\rho(t) = \mathcal{A}(t)$$

can be treated with the theory of analytic semigroups as in the book of Lunardi [Lun95]. $\square$

The next lemma, which follows with classical arguments from Courant and Hilbert [CH68], gets together eigenvalues of $\mathcal{A}$ and properties of the bilinear form I.

Lemma 4.11. *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$$

be the eigenvalues of $\mathcal{A}$ (taken multiplicity into account).

(i) For all $n \in \mathbb{N}$, the following description of the eigenvalues holds

$$\lambda_n = \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{-1}},$$
$$-\lambda_n = \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^\perp \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}},$$

where Σ_n is the collection of n-dimensional subspaces of V and $W^\perp$ is the orthogonal complement with respect to the $(.\,,.)_{-1}$ inner product.

(ii) The eigenvalues λ_n depend continuously on $S(n^*, n^*)$ and $|\sigma^*|$ in the L^∞-norm.

Proof. The first part follows with the help Courant's maximum-minimum principle from [CH68] and the second part follows due to the structure of I,

$$I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*} \rho|^2 - |\sigma^*|^2 \rho^2 \right) d\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*, n^*) \rho^2 \, d\mathcal{H}^{n-1},$$

from which the continuous dependence can be seen directly. □

Now we can describe the eigenvalue λ_1 in the above lemma more explicitly.

Remark 4.12. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 = \min_{\rho \in V \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}. \tag{4.11}$$

From Theorem 4.10 we have asymptotic stability of the zero solution of the linearized equation (3.5) if and only if $\lambda_1 < 0$. This leads to the following main conclusion.

Theorem 4.13. *The zero solution of the linearized equation (3.5) is asymptotically stable if and only if*

$$I(\rho,\rho) \quad > \quad 0$$

for all $\rho \in V \setminus \{0\}$, where $I(\rho,\rho) = \int_{\Gamma^*} \left(|\nabla_{\Gamma^*} \rho|^2 - |\sigma^*|^2 \rho^2 \right) \mathrm{d}\mathcal{H}^n - \int_{\partial\Gamma^*} S(n^*, n^*) \rho^2 \, \mathrm{d}\mathcal{H}^{n-1}$.

5 Example

In this section, we consider an explicit given geometry. This means we will specify a region Ω together with a hypersurface Γ^* lying inside Ω and touching the boundary at a right angle. Γ^* will be a stationary solution of (2.1) and we want to determine a characteristic behaviour concerning the linearized stability of Γ^*. For $a, c > 0$ we let $\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} < 1\}$ be surrounded by the ellipsoid

$$\partial\Omega = E = \{(x,y,z) \in \mathbb{R}^3 \,|\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1\}.$$

A parametrization of E is given by $f : [0,\pi] \times [0, 2\pi] \longrightarrow E,\ f(u,v) = (a \sin u \cos v, a \sin u \sin v, c \cos u)$. We consider a stationary solution Γ^* of the surface diffusion equation (2.1) given by

$$\Gamma^* = \{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 \le a^2\}.$$

Γ^* is a circle in the (x,y)-plane lying inside the ellipsoid E with boundary

$$\partial\Gamma^* = \{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = a^2\} = \{f(\frac{\pi}{2}, v) \,|\, v \in [0, 2\pi]\},$$

that touches E at a right angle.
To decide on linearized stability of Γ^*, we have to examine due to Theorem 4.13 the positivity of

$$I(\rho,\rho)=\int_{\Gamma^*}\left(|\nabla_{\Gamma^*}\rho|^2-|\sigma^*|^2\rho^2\right)-\int_{\partial\Gamma^*}S(n^*,n^*)\rho^2$$

for all $\rho\in H^1(\Gamma^*)\backslash\{0\}$ with $\int_{\Gamma^*}\rho=0$.
A straightforward calculation gives $|\sigma^*|^2=0$ for the squared norm of the second fundamental form and $S(n^*,n^*)=\frac{a}{c^2}$ for the second fundamental form of $\partial\Omega$ with respect to the inwards pointing unit normal $(-\mu)$ of Ω. With this results the bilinear form from Theorem 4.13 reduces to

$$I(\rho,\rho)=\int_{\Gamma^*}|\nabla_{\Gamma^*}\rho|^2-\frac{a}{c^2}\int_{\partial\Gamma^*}\rho^2\,. \tag{5.1}$$

To determine the minimum of I we proceed in an analogue manner as in Courant and Hilbert [CH68]. By using the fact that Γ^* is a flat disc in $\mathbb{R}^3$ with radius $a>0$, we can replace the bilinear form (5.1) by the following one.

$$I(\rho,\rho)=\int_{B_a(0)}|\nabla\rho|^2-\frac{a}{c^2}\int_{\partial B_a(0)}\rho^2\,, \tag{5.2}$$

where $B_a(0)$ is the ball in $\mathbb{R}^2$ with center 0 and radius $a>0$, and $\rho\in H^1(B_a(0))$ with $\int_{B_a(0)}\rho=0$. Note that ∇ is the usual gradient in $\mathbb{R}^2$. We can simplify the bilinear form (5.2) further by introducing polar coordinates (r,ϑ) to get

$$I(\varphi,\varphi)=\int_0^{2\pi}\int_0^a\left((\partial_r\varphi)^2+\frac{1}{r^2}(\partial_\vartheta\varphi)^2\right)r\,dr\,d\vartheta-\frac{a}{c^2}\int_0^{2\pi}\left(\varphi(a,\theta)\right)^2\,a\,d\vartheta\,, \tag{5.3}$$

where $\varphi=\rho\circ\Pi$ for polar coordinates $\Pi(r,\vartheta)$ with $\varphi\in H^1((0,2\pi)\times(0,a))$ and $\int_0^{2\pi}\int_0^a\varphi\,r=0$. Here we used the transformation rule $|\nabla\rho|^2=(\partial_r\varphi)^2+\frac{1}{r^2}(\partial_\vartheta\varphi)^2$.
If we now want to solve the minimization problem

$$I(\varphi,\varphi)\longrightarrow\min\,,\quad\varphi\in H^1((0,2\pi)\times(0,a))\ \text{ and }\ \int_0^{2\pi}\int_0^a\varphi\,r=0\,, \tag{5.4}$$

we can assume for φ a Fourier series expansion as

$$\varphi(r,\vartheta)=\frac{1}{2}f_0(r)+\sum_{n=1}^{\infty}\left(f_n(r)\cos(n\vartheta)+g_n(r)\sin(n\vartheta)\right)\,, \tag{5.5}$$

for functions f_0, f_n and g_n. Due to the volume constraint we observe that $\int_0^a f_0(r)\,r\,dr=0$. At the boundary of $B_a(0)$, formula (5.5) gives for $r=a$

$$\varphi(a,\vartheta)=\frac{1}{2}f_0(a)+\sum_{n=1}^{\infty}\left(f_n(a)\cos(n\vartheta)+g_n(a)\sin(n\vartheta)\right)\,.$$

Differentiating (5.5) with respect to r and ϑ, inserting it into formula (5.3) for $I(\varphi,\varphi)$ and using the orthogonality of the trigonometric functions, we deduce the following expression for I.

$$\begin{aligned}I(\varphi,\varphi)=&\pi\int_0^a(f_0'(r))^2\,r\,dr+\pi\sum_{n=1}^{\infty}\int_0^a\left((f_n'(r))^2+\frac{n^2}{r^2}(f_n(r))^2\right)r\,dr\\&+\pi\sum_{n=1}^{\infty}\int_0^a\left((g_n'(r))^2+\frac{n^2}{r^2}(g_n(r))^2\right)r\,dr-\frac{a^2}{c^2}\pi(f_0(a))^2-\frac{a^2}{c^2}\pi\sum_{n=1}^{\infty}(f_n(a))^2-\frac{a^2}{c^2}\pi\sum_{n=1}^{\infty}(g_n(a))^2\,.\end{aligned} \tag{5.6}$$

Due to this structure we can minimize instead of I also the series of problems given by

$$\int_0^a (f_0'(r))^2\, r\, dr - \frac{a^2}{c^2}(f_0(a))^2 \longrightarrow \min\,, \tag{5.7}$$

$$\int_0^a \left((f_n'(r))^2 + \frac{n^2}{r^2}(f_n(r))^2\right) r\, dr - \frac{a^2}{c^2}(f_n(a))^2 \longrightarrow \min \;\text{ for }\; n \in \mathbb{N}\,, \tag{5.8}$$

$$\int_0^a \left((g_n'(r))^2 + \frac{n^2}{r^2}(g_n(r))^2\right) r\, dr - \frac{a^2}{c^2}(g_n(a))^2 \longrightarrow \min \;\text{ for }\; n \in \mathbb{N}\,. \tag{5.9}$$

The first line (5.7) can be minimized at once by $f_0' = 0$, and therefore $f_0(r) \equiv c$ for some constant c. Due to the constraint $\int_0^a f_0(r)\, r\, dr = 0$, we observe $f_0(r) \equiv 0$ and in particular $f_0(a) = 0$. So the first line will yield the minimal value 0.
For $n > 0$, we must have $f_n(0) = 0$, otherwise the function $\frac{n^2}{r^2}(f_n(r))^2 r = \frac{n^2}{r}(f_n(r))^2$ from (5.6) would have a pole at $r = 0$ that is not integrable. Therefore we can rewrite the integral in (5.8) as follows.

$$\int_0^a \left((f_n')^2 + \frac{n^2}{r^2}(f_n)^2\right) r\, dr = \int_0^a \left(f_n' - \frac{n}{r}f_n\right)^2 r + 2n f_n\, f_n'\, dr = \int_0^a \left(f_n' - \frac{n}{r}f_n\right)^2 r\, dr + n(f_n(a))^2\,,$$

so that the above minimization problem for f_n reads as

$$\int_0^a \left(f_n' - \frac{n}{r}f_n\right)^2 r\, dr + \left(n - \frac{a^2}{c^2}\right)(f_n(a))^2 \longrightarrow \min \;\text{ for }\; n \in \mathbb{N}\,.$$

The minimum is attained if $f_n' - \frac{n}{r}f_n = 0$, which gives $f_n(r) = c_n r^n$ for some constant c_n. The minimal value is then given by

$$\left(n - \frac{a^2}{c^2}\right)(f_n(a))^2\,.$$

Analogous calculations for g_n yield finally the minimal value of I given by

$$\pi \sum_{n=1}^{\infty} \left(n - \frac{a^2}{c^2}\right)\left((f_n(a))^2 + (g_n(a))^2\right)\,. \tag{5.10}$$

With this minimal value we can give the following result about linear stability of Γ^*.

Lemma 5.1. *With the above notations we get the following result for Γ^*.*

(i) If $c > a$, Γ^* is linearly asymptotically stable.

(ii) If $c < a$, Γ^* is linearly asymptotically instable.

Proof. If $c > a$, we see that $\left(n - \frac{a^2}{c^2}\right) \geq \left(1 - \frac{a^2}{c^2}\right) > 0$ and the above minimal value is positive.
If on the other hand $c < a$, we choose $f_1(a) = g_1(a) = \frac{1}{2}$ and $f_n(a) = g_n(a) = 0$ for $n > 1$, so that the above minimal value simplifies to

$$\pi\left(1 - \frac{a^2}{c^2}\right) < 0\,.$$

Using Theorem 4.13 yields the proof. □

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