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EQUIVARIANT YAMABE PROBLEM AND HEBEY–VAUGON CONJECTURE

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Abstract. In their study of the Yamabe problem in the presence of isometry group, E. Hebey and M. Vaugon announced a conjecture. This conjecture generalizes T. Aubin's conjecture, which has already been proven and is sufficient to solve the Yamabe problem. In this paper, we generalize Aubin's theorem and we prove the Hebey–Vaugon conjecture in some new cases.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Denote by $I(M, g)$, $C(M, g)$ and R_g the isometry group, the conformal transformations group and the scalar curvature, respectively. Let G be a subgroup of the isometry group $I(M, g)$. E. Hebey and M. Vaugon[5] considered the following problem:

Hebey–Vaugon problem. *Is there some $G-$invariant metric g_0 which minimizes the functional*

$$J(g') = \frac{\int_M R_{g'} \mathrm{d}v(g')}{(\int_M \mathrm{d}v(g'))^{\frac{n-2}{n}}}$$

where g' belongs to the $G-$invariant conformal class of metrics g defined by:

$$[g]^G := \{\tilde{g} = e^f g / f \in C^\infty(M),\ \sigma^* \tilde{g} = \tilde{g} \quad \forall \sigma \in G\}$$

The positive answer would have two consequences. The first is that there exists an $I(M, g)-$invariant metric g_0 conformal to g such that the scalar curvature R_{g_0} is constant. The second is that the A. Lichnerowicz's conjecture [7], stated below, is true. By the works of J. Lelong-Ferrand[6] and M. Obata[9], we know that if (M, g) is not conformal to (S_n, g_{can}) (the unit sphere endowed with its standard metric g_{can}), then $C(M, g)$ is compact and there exists a conformal metric g' to g such that $I(M, g') = C(M, g)$. This implies that the first consequence is equivalent to the

A. Lichnerowicz conjecture. *For every compact Riemannian manifold (M, g) which is not conformal to the unit sphere S_n endowed with its standard metric, there exists a metric $\tilde{g}$ conformal to g for which $I(M, \tilde{g}) = C(M, g)$, and the scalar curvature $R_{\tilde{g}}$ is constant.*

To such metrics correspond functions which are necessarily solutions of the Yamabe equation. In other words, if $\tilde{g} = \psi^{\frac{4}{n-2}} g$, ψ is a $G-$invariant smooth positive function then ψ satisfies

$$\frac{4(n-1)}{n-2} \Delta_g \psi + R_g \psi = R_{\tilde{g}} \psi^{\frac{n+2}{n-2}}.$$

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The classical Yamabe problem, which consists to find a conformal metric with constant scalar curvature on a compact Riemannian manifold, is the particular case of the problem above when $G = \{\mathrm{id}\}$. Denote by $O_G(P)$ the orbit of $P \in M$ under G, W_g the Weyl tensor associated to the manifold (M, g) and ω_n the volume of the unit sphere S_n. We define the integer $\omega(P)$ at the point P as

$$\omega(P) = \inf\{i \in \mathbb{N}/\|\nabla^i W_g(P)\| \neq 0\} \ (\omega(P) = +\infty \text{ if } \forall i \in \mathbb{N},\ \|\nabla^i W_g(P)\| = 0)$$

HEBEY–VAUGON CONJECTURE. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and G be a subgroup of $I(M, g)$. If (M, g) is not conformal to (S_n, g_{can}) or if the action of G has no fixed point, then the following inequality holds*

$$\inf_{g' \in [g]^G} J(g') < n(n-1)\omega_n^{2/n}(\inf_{Q \in M} \mathrm{card} O_G(Q))^{2/n} \tag{1}$$

Remarks 1.1.
(1) *This conjecture is the generalization of the former T. Aubin's conjecture* [1] *for the Yamabe problem corresponding to $G = \{\mathrm{id}\}$, where the constant in the right side of the inequality is equal to $\inf_{g' \in [g_{can}]} J(g')$ for S_n. In this case, the conjecture is completely proved.*
(2) *The inequality is obvious if $\inf_{g' \in [g]^G} J(g')$ is nonpositive, it is the case when there exists a Yamabe metric with nonpositive scalar curvature.*
(3) *If for any $Q \in M$, $\mathrm{card} O_G(Q) = +\infty$ then this conjecture is also obvious.*

The only results known about this conjecture are given in the following theorem:

Theorem 1.1 (E. Hebey and M. Vaugon)**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and G be a subgroup of $I(M, g)$. We always have :*

$$\inf_{g' \in [g]^G} J(g') \leq n(n-1)\omega_n^{2/n}(\inf_{Q \in M} \mathrm{card} O_G(Q))^{2/n}$$

and inequality (1) *holds if one of the following items is satisfied.*
(1) *The action of G on M is free*
(2) $3 \leq \dim M \leq 11$
(3) *There exists a point P with minimal orbit (finite) under G such that $\omega(P) > (n-6)/2$ or $\omega(P) \in \{0, 1, 2\}$.*

The case $\omega = 3$ was studied by A. Rauzy (private communications).

In this prove we prove the following results:
Main theorem. *The Hebey–Vaugon conjecture holds if there exists a point $P \in M$ with minimal orbit (finite) for which $\omega(P) \leq 15$ or if the degree of the leading part of R_g is greater or equal to $\omega(P) + 1$, in the neighborhood of this point P.*

Corollary 1.1. *Hebey–Vaugon conjecture holds for every smooth compact Riemannian manifold (M, g) of dimension $n \in [3, 37]$.*

To prove the main theorem, we need to construct a $G-$invariant test function ϕ such that

$$I_g(\phi) < n(n-1)\omega_n^{2/n}(\inf_{Q \in M} \mathrm{card} O_G(Q))^{2/n}$$

Thus, all the difficulties are in the construction of a such function. For some cases, we can use the test functions constructed by T. Aubin [1] and R. Schoen [10] in the case of Yamabe problem. They have been already proven by E. Hebey and

M. Vaugon [5]. But the item 3, presented in Theorem 1.1, uses test functions different than T. Aubin and R. Schoen ones.

We multiply T. Aubin's test function $u_{\varepsilon,P}$ by a function as follows:

$$\varphi_\varepsilon(Q) = (1 - r^{\omega+2} f(\xi)) u_{\varepsilon,P}(Q) \tag{2}$$

$$u_{\varepsilon,P}(Q) = \begin{cases} \left(\dfrac{\varepsilon}{r^2+\varepsilon^2}\right)^{\frac{n-2}{2}} - \left(\dfrac{\varepsilon}{\delta^2+\varepsilon^2}\right)^{\frac{n-2}{2}} & \text{if } Q \in B_P(\delta) \\ 0 & \text{if } Q \in M - B_P(\delta) \end{cases} \tag{3}$$

for all $Q \in M$, where $r = d(Q,P)$ is the distance between P and Q. (r,ξ^j) is a geodesic coordinates system in the neighborhood of P and $B_P(\delta)$ is the geodesic ball of center P with radius δ fixed sufficiently small. f is a function depending only on ξ, chosen such that $\int_{S_{n-1}} f d\sigma = 0$. Without loss of generality, we suppose that in the coordinates system (r,ξ^j) we have $\det g = 1 + o(r^m)$ for $m \gg 1$. In fact, E. Hebey and M. Vaugon proved that there exists $\tilde g \in [g]^G$ for which $\det \tilde g = 1 + o(r^m)$ and $\inf_{g' \in [g]^G} J(g')$ does not depend on the conformal $G-$invariant metric.

2. Computation of $\int_M R_g \varphi_\varepsilon^2 dv$

Let be

$$I_a^b(\varepsilon) = \int_0^{\delta/\varepsilon} \frac{t^b}{(1+t^2)^a} dt \text{ and } I_a^b = \lim_{\varepsilon \to 0} I_a^b(\varepsilon)$$

then $I_a^{2a-1}(\varepsilon) = \log \varepsilon^{-1} + O(1)$. If $2a - b > 1$ then $I_a^b(\varepsilon) = I_a^b + O(\varepsilon^{2a-b-1})$ and by integration by parts, we establish the following relationships :

$$I_a^b = \frac{b-1}{2a-b-1} I_a^{b-2} = \frac{b-1}{2a-2} I_{a-1}^{b-2} = \frac{2a-b-3}{2a-2} I_{a-1}^b, \quad \frac{4(n-2) I_n^{n+1}}{(I_n^{n-2})^{(n-2)/n}} = n \tag{4}$$

Using the inequality $(a-b)^\beta \geq a^\beta - \beta a^{\beta-1} b$ for $0 < b < a$, we have for $\beta \geq 2$, $0 \leq \alpha < (n-2)(\beta-1) - n$

$$\int_M r^\alpha u_{\varepsilon,P}^\beta \mathrm{d}v = \omega_{n-1} I_{(n-2)\beta/2}^{\alpha+n-1} \varepsilon^{\alpha+n-\beta(n-2)/2} + O(\varepsilon^{n-2}) \tag{5}$$

This integral appears frequently in the following computations, and it allows us to neglect the constant term in the expression of u_ε, when we choose δ sufficiently small and ε smaller than δ.
Denote by I_g the Yamabe functional defined for all $\psi \in H^1(M)$ by

$$I_g(\psi) = \left(\int_M |\nabla_g \psi|^2 \mathrm{d}v + \frac{(n-2)}{4(n-1)} \int_M R_g \psi^2 \mathrm{d}v \right) \|\psi\|_N^{-2} \tag{6}$$

where $N = 2n/(n-2)$ and ∇_g is the gradient of the metric g.

The second integral of the functional I_g with the scalar curvature term needs a special consideration. Let $\mu(P)$ be an integer defined as follows : $|\nabla_\beta R_g(P)| = 0$ for all $|\beta| < \mu(P)$ and there exists $\gamma \in \mathbb{N}^{\mu(P)}$ such that $|\nabla_\gamma R_g(P)| \neq 0$ then

$$R_g(Q) = \bar R + O(r^{\mu(P)+1})$$

where $\bar R = r^{\mu(P)} \sum_{|\beta| = \mu} \nabla_\beta R_g(P) \xi^\beta$ is a homogeneous polynomial of degree $\mu(P)$, the β are multi-indices.

For simplicity, we drop the letter P in $\omega(P)$ and $\mu(P)$.

By E. Hebey and M. Vaugon [5] results:

Lemma 2.1. *$\mu \geq \omega$, $g_{ij} = \delta_{ij} + O(r^{\omega+2})$ and $\bar{\int}_{S(r)} R_g = O(r^{2\omega+2})$ which implies that $\int_{S(r)} \bar{R} d\sigma = 0$ when $\mu < 2\omega + 2$*

$\bar{\int}$ denotes the average. Then

$$\begin{aligned}\int_M R_g \varphi_\varepsilon^2 \mathrm{d}v &= \int_M R_g u_{\varepsilon,P}^2 \mathrm{d}v - 2\int_M f u_{\varepsilon,P}^2 R_g r^{\omega+2} \mathrm{d}v + \int_M f^2 u_{\varepsilon,P}^2 R_g r^{2\omega+4} \mathrm{d}v \\ &= \varepsilon^{2\omega+4} \omega_{n-1} \bar{\int}_{S(r)} r^{-2\omega-2} R_g \mathrm{d}\sigma I_{n-2}^{n+2\omega+1}(\varepsilon) - \\ & 2\varepsilon^{\omega+\mu+4} I_{n-2}^{\omega+\mu+n+1}(\varepsilon) \omega_{n-1} \bar{\int}_{S(r)} r^{-\mu} f(\xi) \bar{R} \mathrm{d}\sigma(\xi) + O(\varepsilon^{n-2})\end{aligned} \tag{7}$$

Moreover T. Aubin [2] proved that:

Theorem 2.1. *If $\mu \geq \omega + 1$ then there exists $C(n,\omega) > 0$ such that*

$$\bar{\int}_{S_{n-1}(r)} R \mathrm{d}\sigma = C(n,\omega)(-\Delta_g)^{\omega+1} R(P) r^{2\omega+2} + o(r^{2\omega+2})$$

$(-\Delta_g)^{\omega+1} R(P)$ is negative. Then $I_g(u_{\varepsilon,P}) < \frac{n(n-2)}{4} \omega_{n-1}^{2/n}$.

From now until the end of this section, we make the assumption that $\mu = \omega$. Now, we recall some results obtained by T. Aubin in his papers [3, 4]:

$\bar{R}$ is homogeneous polynomial of degree ω then $\Delta_{\mathcal{E}} \bar{R}$ is homogeneous of degree $\omega - 2$ and

$$\Delta_{\mathcal{E}} \bar{R} = r^{-2}(\Delta_s \bar{R} - \omega(n+\omega-2)\bar{R})$$

where $\Delta_{\mathcal{E}}$ is the Euclidean Laplacian and Δ_s is the Laplacian on the sphere S_{n-1}. $\Delta_{\mathcal{E}}^{k-1} \bar{R}$ is homogeneous of degree $\omega - 2k + 2$ and

$$\Delta_{\mathcal{E}}^k \bar{R} = r^{-2}(\Delta_s - \nu_k \mathrm{id}) \Delta_{\mathcal{E}}^{k-1} \bar{R} = r^{-2k} \prod_{p=1}^{k} (\Delta_S - \nu_p \mathrm{id}) \bar{R}$$

with

$$\nu_k = (\omega - 2k + 2)(n + \omega - 2k) \tag{8}$$

The sequence of integers $(\nu_k)_{\{1 \leq k \leq [\omega/2]\}}$ is decreasing. It will play the role of the eigenvalues of the Laplacian on the sphere S_{n-1}. It is known that the eigenvalues of the geometric Laplacian are non-negative and increasing. Our ν_k are in the opposite order.
We know by T. Aubin's paper [2] that $\Delta_{\mathcal{E}}^{[\omega/2]} \bar{R} = 0$ and $\int_{S(r)} \bar{R} \mathrm{d}\sigma = 0$, then

$$q = \min\{k \in \mathbb{N} / \Delta_{\mathcal{E}}^k \bar{R} = 0\}$$

is well defined and $r^{-\omega} \bar{R} \in \bigoplus_{k=1}^q E_k$, with E_k the eigenspace associated to the positive eigenvalues ν_k of the Laplacian Δ_s on the sphere S_{n-1}. If $j \neq k$, then E_k is orthogonal to E_j, for the standard scalar product in $H_1^2(S_{n-1})$. Moreover, since $\int \bar{R} d\sigma = 0$ there exist $\varphi_k \in E_k$ (eigenfunctions of Δ_s) such that

$$\bar{R} = r^\omega \Delta_s \sum_{k=1}^q \varphi_k = r^\omega \sum_{k=1}^q \nu_k \varphi_k \tag{9}$$

According to Lemma 2.1, we can split the metric g in the following way:

$$g = \mathcal{E} + h \tag{10}$$

where $\mathcal{E}$ is the Euclidean metric and h is a symmetric 2-tensor defined in our geodesic coordinates system by

$$h_{ij} = r^{\omega+2}\bar{g}_{ij} + r^{2(\omega+2)}\hat{g}_{ij} + \tilde{h}_{ij} \text{ and } h_{ir} = h_{rr} = 0 \tag{11}$$

where $\bar{g}$, $\hat{g}$ and $\tilde{h}$ are symmetric 2-tensors defined on the sphere S_{n-1}. We denote by s the standard metric on the sphere, ∇, Δ are the associated gradient and Laplacian on S_{n-1}. By straightforward computations, Aubin [3] proved that:

Lemma 2.2.

$$\bar{R} = \nabla^{ij}\bar{g}_{ij}r^{\omega} \textit{ and}$$

$$\fint_{S_{n-1}(r)} R\mathrm{d}\sigma = [B/2 - C/4 - (1+\omega/2)^2 Q]r^{2(\omega+1)} + o(r^{2(\omega+1)})$$

where $B = \fint_{S_{n-1}} \nabla^i \bar{g}^{jk}\nabla_j \bar{g}_{ik}\mathrm{d}\sigma$, $C = \fint_{S_{n-1}} \nabla^i \bar{g}^{jk}\nabla_i \bar{g}_{jk}\mathrm{d}\sigma$ *and* $Q = \fint_{S_{n-1}} \bar{g}_{ij}\bar{g}^{ij}\mathrm{d}\sigma$

For further details refer to [8].
The integrals Q, B and C are given in terms of the tensor $\bar{g}$. Our goal is to compute them using the eigenfunctions φ_k above. Let us define

$$b_{ij} = \sum_{k=1}^{q} \frac{1}{(n-2)(\nu_k + 1 - n)}[(n-1)\nabla_{ij}\varphi_k + \nu_k\varphi_k s_{ij}]$$

and a_{ij} such that $\bar{g}_{ij} = a_{ij} + b_{ij}$ then, according to (9), we check that

$$\bar{R} = \bar{R}_b = \nabla^{ij}b_{ij}r^{\omega} \quad \text{and } \bar{R}_a = \nabla^{ij}a_{ij}r^{\omega} = 0 \tag{12}$$

If $\bar{g}_{ij} = a_{ij}$ then $\bar{R} = \bar{R}_a = 0$ and $\mu \geq \omega + 1$. By Theorem 2.1

$$\fint_{S_{n-1}(r)} R\mathrm{d}\sigma = \fint_{S_{n-1}(r)} R_a\mathrm{d}\sigma < 0$$

If $\bar{g}_{ij} = b_{ij}$ then

$$\fint_{S_{n-1}(r)} R\mathrm{d}\sigma = \fint_{S_{n-1}(r)} R_b\mathrm{d}\sigma = [B_b/2 - C_b/4 - (1+\omega/2)^2 Q_b]r^{2(\omega+1)} + o(r^{2(\omega+1)})$$

where B_b, C_b and Q_b are the same integrals defined in Lemma 2.2 when the considered tensor $\bar{g}_{ij} = b_{ij}$. We compute them in terms of φ_k

$$Q_b = \fint_{S_{n-1}} \bar{b}_{ij}\bar{b}^{ij}\mathrm{d}\sigma = \frac{n-1}{n-2}\sum_{k=1}^{q}\frac{\nu_k}{\nu_k - n + 1}\fint_{S_{n-1}} \varphi_k^2\mathrm{d}\sigma$$

$$B_b = -(n-1)Q_b + \sum_{k=1}^{q}\nu_k \fint_{S_{n-1}} \varphi_k^2\mathrm{d}\sigma$$

$$C_b = -(n-1)Q_b + \frac{n-1}{n-2}\sum_{k=1}^{q}\nu_k \fint_{S_{n-1}} \varphi_k^2\mathrm{d}\sigma$$

To find these expressions, we used several times the identity $\nabla^i b_{ij} = -\sum_{k=1}^{q}\nabla_j\varphi_k$ and Stokes formula (more details are given in [3, 4] and [8]). In the general case, we deduce that

Lemma 2.3. *If* $\mu = \omega$ *and* $\bar{g}_{ij} = a_{ij} + b_{ij}$, *where* b_{ij} *is defined above,*

$$\fint_{S_{n-1}(r)} R\mathrm{d}\sigma = \fint_{S_{n-1}(r)} R_a + R_b\mathrm{d}\sigma \leq [B_b/2 - C_b/4 - (1+\omega/2)^2 Q_b]r^{2(\omega+1)} + o(r^{2(\omega+1)}) \tag{13}$$

and

$$B_b/2 - C_b/4 - (1+\omega/2)^2 Q_b = \sum_{k=1}^{q} u_k \fint_{S_{n-1}} \varphi_k^2\mathrm{d}\sigma \tag{14}$$

with

$$u_k = \left(\frac{n-3}{4(n-2)} - \frac{(n-1)^2+(n-1)(\omega+2)^2}{4(n-2)(\nu_k-n+1)}\right)\nu_k \tag{15}$$

u_k is obtained using the expressions of Q_b, B_b and C_b above.

3. Generalization of T. Aubin's theorem

Theorem 3.1. *If there exists $P\in M$ such that $\omega(P)\leq (n-6)/2$ then there exists $f\in C^\infty(S_{n-1})$ with vanishing mean integral such that*

$$I_g(\varphi_\varepsilon) < \frac{n(n-2)}{4}\omega_{n-1}^{2/n}$$

The case $\omega=0$ of the this theorem has already been proven by T. Aubin [1]. He also proved the theorem when $\mu\geq\omega+1$ (see Theorem 2.1).

From now until the end of this paper, we drop the letter P in $\omega(P)$ and $\mu(P)$.

Proof. If $\mu\geq\omega+1$ then the inequality holds by Theorem 2.1. So we suppose that $\mu=\omega$ until the end of the proof. We start by computing the first integral of the Yamabe functional (6) with $\psi=\varphi_\varepsilon$. Using formula $|\nabla_g\varphi_\varepsilon|^2=(\partial_r\varphi_\varepsilon)^2+r^{-2}|\nabla_s\varphi_\varepsilon|^2$, we obtain:

$$\int_M|\nabla_g\varphi_\varepsilon|^2\mathrm{d}v = \int_M|\nabla_g u_{\varepsilon,P}|^2\mathrm{d}v + \int_0^\delta[\partial_r(r^{(\omega+2)}u_{\varepsilon,P})]^2r^{n-1}\mathrm{d}r\int_{S_{n-1}}f^2\mathrm{d}\sigma + \int_0^\delta u_{\varepsilon,P}^2r^{n+2\omega+1}\mathrm{d}r\int_{S_{n-1}}|\nabla f|^2\mathrm{d}\sigma$$

The substitution $t=r/\varepsilon$ gives

$$\int_M|\nabla_g\varphi_\varepsilon|^2\mathrm{d}v = (n-2)^2\omega_{n-1}I_n^{n+1}(\varepsilon)+\varepsilon^{2\omega+4}\Big\{\int_{S_{n-1}}|\nabla f|^2\mathrm{d}\sigma I_{n-2}^{2\omega+n+1}(\varepsilon)+ \int_{S_{n-1}}f^2\mathrm{d}\sigma[(\omega-n+4)^2I_n^{2\omega+n+5}(\varepsilon)+2(\omega+2)(\omega-n+4)I_n^{2\omega+n+3}(\varepsilon)+(\omega+2)^2I_n^{2\omega+n+1}(\varepsilon)]\Big\} \tag{16}$$

For $\|\varphi_\varepsilon\|_N^{-2}$, we need to compute the Taylor expansion of :

$$\varphi_\varepsilon^N(Q)=[1-Nr^{\omega+2}f(\xi)+\frac{N(N-1)}{2}r^{2\omega+4}f^2(\xi)+o(r^{2\omega+4})]u_{\varepsilon,P}^N$$

Using the fact that $\int_{S_{n-1}}f\mathrm{d}\sigma(\xi)=0$ and formula (5), we conclude that

$$\begin{aligned}\|\varphi_\varepsilon\|_N^N &= \int_0^\delta\int_{S_{n-1}}[1+\frac{N(N-1)}{2}r^{2(\omega+2)}f^2(\xi)+o(r^{2\omega+4})]r^{n-1}u_{\varepsilon,P}^N\mathrm{d}r\mathrm{d}\sigma(\xi)\\ &= \omega_{n-1}I_n^{n-1}+\frac{N(N-1)}{2}\varepsilon^{2(\omega+2)}\int_{S_{n-1}}f^2\mathrm{d}\sigma I_n^{2\omega+n+3}+o(\varepsilon^{2\omega+4})\end{aligned}$$

then

$$\|\varphi_\varepsilon\|_N^{-2}=(\omega_{n-1}I_n^{n-1})^{-2/N}\{1 - (N-1)\varepsilon^{2(\omega+2)}\int_{S_{n-1}}f^2\mathrm{d}\sigma I_n^{2\omega+n+3}/(\omega_{n-1}I_n^{n-1})\}+o(\varepsilon^{2\omega+4}) \tag{17}$$

By Eqs (16), (17), (7) and the relationship (4), if $n>2\omega+6$ then :

$$I_g(\varphi_\varepsilon) = \frac{n(n-2)}{4}\omega_{n-1}^{2/n} + (\omega_{n-1}I_n^{n-1})^{-2/N} I_{n-2}^{n+2\omega+1}\varepsilon^{2\omega+4}\times \left\{\frac{(n-2)\omega_{n-1}}{4(n-1)}\fint_{S(r)} r^{-2\omega-2}R_g \mathrm{d}\sigma - \frac{n-2}{2(n-1)}\int_{S_{n-1}} f(\xi)\bar{R}\mathrm{d}\sigma + \int_{S_{n-1}}|\nabla f|^2\mathrm{d}\sigma + \right. \left. - \frac{n(n-2)^2-(\omega+2)^2(n^2+n+2)}{(n-1)(n-2)}\int_{S_{n-1}} f^2\mathrm{d}\sigma\right\} + o(\varepsilon^{2\omega+4})$$

If $n = 2\omega + 6$ then

$$I_g(\varphi_\varepsilon) = \frac{n(n-2)}{4}\omega_{n-1}^{2/n} + (\omega_{n-1}I_n^{n-1})^{-2/N}\varepsilon^{2\omega+4}\log\varepsilon^{-1}\times \left\{\frac{(n-2)\omega_{n-1}}{4(n-1)}\fint_{S(r)} r^{-2\omega-2}R_g\mathrm{d}\sigma - \frac{n-2}{2(n-1)}\int_{S_{n-1}} f(\xi)\bar{R}\mathrm{d}\sigma + \right. \left. \int_{S_{n-1}}|\nabla f|^2\mathrm{d}\sigma + (\omega+2)^2\int_{S_{n-1}} f^2\mathrm{d}\sigma\right\} + O(\varepsilon^{2\omega+4})$$

For further details refer to [8].

Let I_S be the functional defined for a function f on the sphere S_{n-1}, with zero mean integral , by

$$I_S(f) = \fint_{S_{n-1}} 4(n-1)(n-2)|\nabla f|^2 - [4n(n-2)^2 - 4(\omega+2)^2(n^2+n+2)]f^2 + - 2(n-2)^2 f\bar{R}\mathrm{d}\sigma$$

This implies that if $n > 2\omega + 6$

$$(18)\quad I_g(\varphi_\varepsilon) = \frac{n(n-2)}{4}\omega_{n-1}^{2/n} + \frac{\omega_{n-1}^{2/n} I_{n-2}^{n+2\omega+1}\varepsilon^{2\omega+4}}{4(n-1)(n-2)(I_n^{n-1})^{2/N}}\times \{(n-2)^2\fint_{S(r)} r^{-2\omega-2}R_g\mathrm{d}\sigma + I_S(f)\} + o(\varepsilon^{2\omega+4})$$

and if $n = 2\omega + 6$

$$(19)\quad I_g(\varphi_\varepsilon) = \frac{n(n-2)}{4}\omega_{n-1}^{2/n} + \frac{\omega_{n-1}^{2/n} I_{n-2}^{n+2\omega+1}\varepsilon^{2\omega+4}\log\varepsilon^{-1}}{4(n-1)(n-2)(I_n^{n-1})^{2/N}}\times \{(n-2)^2\fint_{S(r)} r^{-2\omega-2}R_g\mathrm{d}\sigma + I_S(f)\} + O(\varepsilon^{2\omega+4})$$

Notice that if $k \neq j$ then $I_S(\varphi_k + \varphi_j) = I_S(\varphi_k) + I_S(\varphi_j)$. Indeed, φ_k and φ_j are orthogonal for the standard scalar product in $H_1^2(S_{n-1})$.

$$\begin{aligned} I_S(c_k\nu_k\varphi_k) &= \{d_k c_k^2 - 2(n-2)^2 c_k\}\nu_k^2 \fint_{S_{n-1}} \varphi_k^2\mathrm{d}\sigma \\ &= -\frac{(n-2)^4}{d_k}\nu_k^2\fint_{S_{n-1}}\varphi_k^2\mathrm{d}\sigma \end{aligned}$$

where

$$d_k = 4[(n-1)(n-2)\nu_k - n(n-2)^2 + (\omega+2)^2(n^2+n+2)] \text{ and } c_k = \frac{(n-2)^2}{d_k}$$

Using (8), we can check easily that d_k is positive for any $1 \leq k \leq [\omega/2]$. Now, let us consider $f = \sum_1^q c_k \nu_k \varphi_k$. Then

$$I_S(f) = -\sum_1^q \frac{(n-2)^4}{d_k} \nu_k^2 \int_{S_{n-1}} \varphi_k^2 \mathrm{d}\sigma$$

and by Lemma 2.3

$$(n-2)^2 \int_{S(r)} r^{-2\omega-2} R_g \mathrm{d}\sigma + I_S(f) \leq \sum_1^q (u_k(n-2)^2 - \frac{(n-2)^4}{d_k}\nu_k^2) \int_{S_{n-1}} \varphi_k^2 \mathrm{d}\sigma + o(1)$$

The following lemma implies that $I_g(\varphi_\varepsilon) < \frac{n(n-2)}{4}\omega_{n-1}^{2/n}$ □

Lemma 3.1. *For any $k \leq q \leq [\omega/2]$ the following inequality holds*

$$u_k - \frac{(n-2)^2}{d_k}\nu_k^2 < 0$$

Proof. Recall the expression of ν_k given in (8). The sequence (U_k) defined by

$$U_k := (\nu_k - n + 1) d_k \{(n-2)\frac{u_k}{\nu_k} - \frac{(n-2)^3}{d_k}\nu_k\}$$

is polynomial decreasing in ν_k when $\nu_k \geq 0$. In fact, $U_k = P(\nu_k)$ with P the decreasing polynomial in $\mathbb{R}_+$, defined by

$$P(x) = [(n-1)(n-2)x - n(n-2)^2 + (\omega+2)^2(n^2+n+2)]\times$$
$$[(n-3)(x-n+1) - (n-1)^2 - (n-1)(\omega+2)^2] - (n-2)^3(x^2 - (n-1)x)$$

The derivative of P is

$$P'(x) = -2(n-2)x - 2n(n-2)^3 + 2(n^2 - 3n - 2)(\omega+2)^2$$

By assumption $\omega + 2 \leq (n-2)/2$ then P is decreasing in $\mathbb{R}_+$. Hence

$$U_k = P(\nu_k) \leq P(\nu_{\omega/2}) = U_{\omega/2}$$

for all $k \leq \omega/2$. It easy to check that $u_{\omega/2}$ is negative so $U_k \leq U_{\omega/2} < 0$. □

4. Proof of the main theorem

By Remarks 1.1, we consider only the positive case (i.e., $\inf_{g' \in [g]^G} J(g') > 0$) and the case when there exists $P \in M$ such that

$$O_G(P) = \{P_i\}_{1 \leq i \leq m}, \;\; m = \mathrm{card} O_G(P) = \inf_{Q \in M} \mathrm{card} O_G(Q), \; \omega \leq \frac{n-6}{2} \text{ and } P_1 = P$$

Let $\tilde{\varphi}_{\varepsilon,i}$ be a function defined as follows:

$$\tilde{\varphi}_{\varepsilon,i}(Q) = (1 - r_i^{\omega+2} f_i(\xi)) u_{\varepsilon,P_i}(Q) \tag{20}$$

where $r_i = d(Q, P_i)$, the function u_{ε,P_i} is defined as in (3) and f_i is defined by:

$$f_i(Q) = c r_i^{-\omega} \nabla_g^\omega R_{(P_i)}(\exp_{P_i}^{-1} Q, \cdots, \exp_{P_i}^{-1} Q) \tag{21}$$

$\exp_{P_i}$ is the exponential map. In a geodesic coordinates system $\{r, \xi^j\}$ with origin P, induced by the exponential map

$$f_1 = c r^{-\omega} \bar{R} = c \sum_{k=1}^q \nu_k \varphi_k$$

where $\bar{R}$, φ_k and ν_k are defined in Section 2. Thus the functions f_i are defined on the sphere S_{n-1}. The choice of the constant c is important.

Lemma 4.1. *Suppose that $\omega \le (n-6)/2$. If $\omega \in [3,15]$ or if $\deg \bar{R} \ge \omega + 1$ then there exists $c \in \mathbb{R}$ such that the corresponding functions $\tilde{\varphi}_{\varepsilon,i}$ satisfy :*

$$I_g(\tilde{\varphi}_{\varepsilon,i}) < \frac{1}{4} n(n-2)\omega_n^{2/n} \tag{22}$$

Remarks 4.1. (1) *We proved inequality of this lemma for any $\omega \le (n-6)/2$, using test function φ_ε (see Theorem 3.1). We notice that the difference between φ_ε and $\tilde{\varphi}_{\varepsilon,i}$ is on the construction of the corresponding functions f and f_i respectively. From $\tilde{\varphi}_{\varepsilon,i}$ we define a $G-$invariant function (see proof of the main theorem below), this property is not possible with the function φ_ε.*

(2) *For $\omega = 16$ and n sufficiently big, we can check that for any $c \in \mathbb{R}$, inequality (22) is false.*

Proof. 1. If $\deg \bar{R} \ge \omega + 1$, then by Theorem 2.1

$$I_g(u_{\varepsilon,P_i}) < \frac{n(n-2)}{4}\omega_n^{2/n}$$

It is sufficient to take $c = 0$, hence $\tilde{\varphi}_{\varepsilon,i} = u_{\varepsilon,P_i}$.

2. If $\deg \bar{R} = \omega$. Using estimates given in the proof of Theorem 3.1 (see (18), (19)), it is sufficient to show that there exists $c \in \mathbb{R}$ such that

$$I_S(f_1) + (n-2)^2 \bar{\int}_{S(r)} r^{-2\omega-2} R_g \mathrm{d}\sigma_r < 0 \tag{23}$$

We keep the notations used in the proof of Theorem 3.1. Thus

$$I_S(f_1) = \sum_{k=1}^{q} I_S(c\nu_k\varphi_k) = \left\{d_k c^2 - 2(n-2)^2 c\right\}\nu_k^2 \bar{\int}_{S_{n-1}} \varphi_k^2 \mathrm{d}\sigma$$

$$\text{and } \bar{\int}_{S(r)} r^{-2\omega-2} R_g \mathrm{d}\sigma_r = \sum_{k=1}^{q} u_k \bar{\int}_{S_{n-1}} \varphi_k^2 \mathrm{d}\sigma$$

To prove inequality (23), it is sufficient to prove that

$$\forall k \le q \quad \frac{d_k}{2(n-2)} c^2 - (n-2)c + (n-2)\frac{u_k}{2\nu_k^2} < 0 \tag{24}$$

The left side of the inequality above is a second degree polynomial with variable c, his discriminant is:

$$\Delta_k = (n-2)^2 - \frac{d_k u_k}{\nu_k^2} \tag{25}$$

Using Lemma 3.1, we deduce that for any $k \le q$, $\Delta_k > 0$. Hence, the polynomial above admits two different roots denoted $x_k < y_k$ and given by

$$x_k = \frac{(n-2)^2 - (n-2)\sqrt{\Delta_k}}{d_k}, \qquad y_k = \frac{(n-2)^2 + (n-2)\sqrt{\Delta_k}}{d_k}$$

Inequality (24) holds if and only if

$$\bigcap_{k=1}^{q} (x_k, y_k) \ne \varnothing \tag{26}$$

The sequence $(d_k)_{k \le [\omega/2]}$ decreases. It is easy to check that

$$\forall k < j \le [\frac{\omega}{2}] \qquad x_k < y_j \tag{27}$$

Hence intersection (26) is not empty if

$$\forall k < j \le [\frac{\omega}{2}] \qquad x_j < y_k \tag{28}$$

We also check that if ω is even, $u_{\omega/2} < 0$, which implies $x_{\omega/2} < 0$.

i. If $\omega = 3$ then $q = 1$, intersection above is not empty. It is sufficient to take $c = (x_1 + y_2)/2$.

ii. If $\omega = 4$ then $k \in \{1, 2\}$, $x_2 < 0$ (because $u_2 < 0$) and $0 < x_1 < y_2$. Hence intersection $]x_1, y_1[\cap]x_2, y_2[$ is not empty.

iii. If $5 \leq \omega \leq 15$, it is sufficient to prove (28) which is equivalent to prove that

$$\forall k < j \leq [\frac{\omega}{2}] \quad (n-2)(d_j - d_k) + d_k\sqrt{\Delta_j} + d_j\sqrt{\Delta_k} > 0 \tag{29}$$

Notice that Δ_k given by (25) is a rational fraction in n. By straightforward computations, we check that there exists reel numbers a_k, b_k, e_k, h_k and s_k which depend on k and ω such that

$$\Delta_k = a_k n^2 + b_k n + e_k + \frac{h_k}{n-2} + \frac{s_k}{\nu_k + 1 - n} \tag{30}$$

$$\sqrt{\Delta_k} > \sqrt{a_k}(n + \frac{b_k}{2a_k}) \tag{31}$$

Inequality (29) holds if we use (31).

The expressions of the reel numbers above are known explicitly (we used the software Maple to compute them, see [8]). For simplicity, we omit to give these expressions.

□

Proof of the main theorem. The orbit of P under the action of G is supposed to be minimal (i.e. $\mathrm{card}O_G(P) = \inf_{Q\in M} \mathrm{card}O_G(Q)$). Without loss of generality, we suppose that $3 \leq \omega \leq (n-6)/2$, because if $\omega > (n-6)/2$ or $\omega \leq 2$, we conclude using Theorem 1.1. From functions $\tilde{\varphi}_{\varepsilon,i}$ defined by (20), we define the function ϕ_ε as follows:

$$\phi_\varepsilon = \sum_{k=1}^{m} \tilde{\varphi}_{\varepsilon,i}$$

ϕ_ε is $G-$invariant. In fact, for any $\sigma \in G$, such that $\sigma(P_i) = P_j$

$$u_{\varepsilon,P_i} = u_{\varepsilon,P_j} \circ \sigma \text{ and } f_i = f_j \circ \sigma$$

f_i are defined by (21), we deduce that

$$\tilde{\varphi}_{\varepsilon,i} = \tilde{\varphi}_{\varepsilon,j} \circ \sigma$$

The support of $\tilde{\varphi}_{\varepsilon,i}$ is included in the ball $B_{P_i}(\delta)$. We choose δ sufficiently small such that for all integers $i \neq j$ in $[1, m]$, intersection $B_{P_j}(\delta) \cap B_{P_i}(\delta) = \varnothing$. Thus

$$I_g(\phi_\varepsilon) = (\mathrm{card}O_G(P))^{2/n} I_g(\varphi_\varepsilon)$$

By Lemma 4.1, we conclude that

$$I_g(\phi_\varepsilon) < \frac{n(n-2)}{4}\omega_{n-1}^{2/n}(\mathrm{card}O_G(P))^{2/n}$$

It remains to notice that if $\tilde{g} = \phi_\varepsilon^{4/(n-2)} g$ then

$$J(\tilde{g}) = 4\frac{n-1}{n-2} I_g(\phi_\varepsilon) < n(n-1)\omega_{n-1}^{2/n}(\mathrm{card}O_G(P))^{2/n}$$

where ε is sufficiently smaller than δ. □

Proof of the Corollary 1.1. Suppose that the orbit of P under the action of G is minimal (otherwise the conjecture is obvious).
If $\omega = \omega(P) > [(n-6)/2]$, we conclude using Theorem 1.1.
If $\omega \leq [(n-6)/2] \leq 15$, we conclude using main theorem. □

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