Equivariant Yamabe problem and
Hebey-Vaugon conjecture

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EQUIVARIANT YAMABE PROBLEM AND HEBEY–VAUGON
CONJECTURE

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Abstract. In their study of the Yamabe problem in the presence of isometry
group, E. Hebey and M. Vaugon announced a conjecture. This conjecture
generalizes T. Aubin’s conjecture, which has already been proven and is sufficient
to solve the Yamabe problem. In this paper, we generalize Aubin’s theorem
and we prove the Hebey–Vaugon conjecture in some new cases.

1. Introduction

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). Denote by
\(I(M, g)\), \(C(M, g)\) and \(R_g\) the isometry group, the conformal transformations group
and the scalar curvature, respectively. Let \(G\) be a subgroup of the isometry group
\(I(M, g)\). E. Hebey and M. Vaugon \(^3\) considered the following problem:

Hebey–Vaugon Problem. Is there some \(G\)-invariant metric \(g_0\) which minimizes
the functional

\[
J(g') = \frac{\int_M R_g^d\text{vol}(g')}{\left(\int_M \text{vol}(g')\right)^{\frac{n}{n-2}}}
\]

where \(g'\) belongs to the \(G\)-invariant conformal class of metrics \(g\) defined by:

\([g]^G := \{\tilde{g} = e^f g / f \in C^\infty(M), \ \sigma^* \tilde{g} = \tilde{g} \ \forall \sigma \in G\}\)

The positive answer would have two consequences. The first is that there exists an
\(I(M, g)\)-invariant metric \(g_0\) conformal to \(g\) such that the scalar curvature \(R_{g_0}\) is
constant. The second is that the A. Lichnerowicz’s conjecture \(^7\), stated below, is
true. By the works of J. Lelong-Ferrand \(^2\) and M. Obata \(^6\), we know that if \((M, g)\)
is not conformal to \((S_n, g_{can})\) (the unit sphere endowed with its standard metric
\(g_{can}\)), then \(C(M, g)\) is compact and there exists a conformal metric \(g'\) to \(g\) such
that \(I(M, g') = C(M, g)\). This implies that the first consequence is equivalent to the

A. Lichnerowicz Conjecture. For every compact Riemannian manifold \((M, g)\)
which is not conformal to the unit sphere \(S_n\) endowed with its standard metric, there
exists a metric \(\tilde{g}\) conformal to \(g\) for which \(I(M, \tilde{g}) = C(M, g)\), and the scalar
curvature \(R_{\tilde{g}}\) is constant.

To such metrics correspond functions which are necessarily solutions of the Yamabe
equation. In other words, if \(\tilde{g} = \psi^\frac{4}{n-2} g\), \(\psi\) is a \(G\)-invariant smooth positive function
then \(\psi\) satisfies

\[
\frac{4(n-1)}{n-2} \Delta_g \psi + R_g \psi = R_{\tilde{g}} \psi^{\frac{n+2}{n-2}}.
\]

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problem.
The classical Yamabe problem, which consists to find a conformal metric with constant scalar curvature on a compact Riemannian manifold, is the particular case of the problem above when $G = \{\text{id}\}$. Denote by $O_G(P)$ the orbit of $P \in M$ under $G$, $W_g$ the Weyl tensor associated to the manifold $(M, g)$ and $\omega$, the volume of the unit sphere $S_n$. We define the integer $\omega(P)$ at the point $P$ as

$$\omega(P) = \inf \{i \in \mathbb{N} / \|\nabla^i W_g(P)\| \neq 0 \} \quad (\omega(P) = +\infty \text{ if } \forall i \in \mathbb{N}, \|\nabla^i W_g(P)\| = 0)$$

**Hebey–Vaugon conjecture.** Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$ and $G$ be a subgroup of $I(M, g)$. If $(M, g)$ is not conformal to $(S_n, g_{can})$ or if the action of $G$ has no fixed point, then the following inequality holds

$$(1) \quad \inf_{g' \in [g]^G} J(g') < n(n - 1)\omega_n^{2/n}(\inf_{Q \in M} \text{card} O_G(Q))^{2/n}$$

**Remarks 1.1.**

1. This conjecture is the generalization of the former T. Aubin’s conjecture $[1]$ for the Yamabe problem corresponding to $G = \{\text{id}\}$, where the constant in the right side of the inequality is equal to $\inf_{g' \in [g_{can}]} J(g')$ for $S_n$. In this case, the conjecture is completely proved.

2. The inequality is obvious if $\inf_{g' \in [g]^G} J(g')$ is nonpositive, it is the case when there exists a Yamabe metric with nonpositive scalar curvature.

3. If for any $Q \in M$, $\text{card} O_G(Q) = +\infty$ then this conjecture is also obvious.

The only results known about this conjecture are given in the following theorem:

**Theorem 1.1** (E. Hebey and M. Vaugon). Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $G$ be a subgroup of $I(M, g)$. We always have:

$$\inf_{g' \in [g]^G} J(g') \leq n(n - 1)\omega_n^{2/n}(\inf_{Q \in M} \text{card} O_G(Q))^{2/n}$$

and inequality $[1]$ holds if one of the following items is satisfied.

1. The action of $G$ on $M$ is free
2. $3 \leq \dim M \leq 11$
3. There exists a point $P$ with minimal orbit (finite) under $G$ such that $\omega(P) > (n - 6)/2$ or $\omega(P) \in \{0, 1, 2\}$.

The case $\omega = 3$ was studied by A. Rauzy (private communications).

In this we prove the following results:

**Main theorem.** The Hebey–Vaugon conjecture holds if there exists a point $P \in M$ with minimal orbit (finite) for which $\omega(P) \leq 15$ or if the degree of the leading part of $R_g$ is greater or equal to $\omega(P) + 1$, in the neighborhood of this point $P$.

**Corollary 1.1.** Hebey–Vaugon conjecture holds for every smooth compact Riemannian manifold $(M, g)$ of dimension $n \in [3, 37]$.

To prove the main theorem, we need to construct a $G$–invariant test function $\phi$ such that

$$I_g(\phi) < n(n - 1)\omega_n^{2/n}(\inf_{Q \in M} \text{card} O_G(Q))^{2/n}$$

Thus, all the difficulties are in the construction of such a function. For some cases, we can use the test functions constructed by T. Aubin $[1]$ and R. Schoen $[10]$ in the case of Yamabe problem. They have been already proven by E. Hebey and
M. Vaugon. But the item presented in Theorem 1 uses test functions different than T. Aubin and R. Schoen ones.

We multiply T. Aubin’s test function $u_{\varepsilon, P}$ by a function as follows:

$$
\varphi_{\varepsilon}(Q) = (1 - r^{m+2}f(\xi))u_{\varepsilon, P}(Q)
$$

(3) $u_{\varepsilon, P}(Q) = \begin{cases}
\left( \frac{\varepsilon}{r^2 + \varepsilon^2} \right)^{\frac{n-2}{2}} - \left( \frac{\varepsilon}{\delta^2 + \varepsilon^2} \right)^{\frac{n-2}{2}} & \text{if } Q \in B_P(\delta) \\
0 & \text{if } Q \in M - B_P(\delta)
\end{cases}
$

for all $Q \in M$, where $r = d(Q, P)$ is the distance between $P$ and $Q$. $(r, \xi')$ is a geodesic coordinates system in the neighborhood of $P$ and $B_P(\delta)$ is the geodesic ball of center $P$ with radius $\delta$ fixed sufficiently small. $f$ is a function depending only on $\xi$, chosen such that $\int_{S_{n-1}} f d\sigma = 0$. Without loss of generality, we suppose that in the coordinates system $(r, \xi)$ we have $\det g = 1 + o(r^m)$ for $m \gg 1$. In fact, E. Hebey and M. Vaugon proved that there exists $\tilde{g} \in [g]^\alpha$ for which $\det \tilde{g} = 1 + o(r^m)$ and $\inf_{g' \in [g]^\alpha} J(g')$ does not depend on the conformal $G$–invariant metric.

2. Computation of $\int_M R_g \varphi_{\varepsilon}^2 dv$

Let be

$$I_b^b(\varepsilon) = \int_0^{\varepsilon} \frac{b}{(1 + t^2)^\alpha} dt \text{ and } I_a^b = \lim_{\varepsilon \to 0} I_b^b(\varepsilon)
$$

then $I_a^{2a-1}(\varepsilon) = \log \varepsilon^{-1} + O(1)$. If $2a - b > 1$ then $I_a^b(\varepsilon) = I_a^b + O(\varepsilon^{2a-b-1})$ and by integration by parts, we establish the following relationships:

(4) $I_a^b = \frac{b - 1}{2a - b - 1} I_a^{b-2} = \frac{b - 1}{2a - 2} I_a^{b-2} = \frac{2a - b - 3}{2a - 2} I_a^{b-1}, \quad \frac{4(n-2)I_a^{n+1}}{(I_a^{n-2})^{n-2}} = n$

Using the inequality $(a - b)^\beta \geq a^\beta - \beta a^{\beta-1}b$ for $0 < b < a$, we have for $\beta \geq 2$, $0 \leq \alpha < (n-2)(\beta - 1) - n$

(5) $\int_M r^\alpha u_{\varepsilon, P}^2 dv = \omega_{n-1}I_a^{\alpha+n-1}e^{\alpha n/2} + O(\varepsilon^{n-2})$

This integral appears frequently in the forthcoming computations, and it allows us to neglect the constant term in the expression of $u_{\varepsilon, P}$, when we choose $\delta$ sufficiently small and $\varepsilon$ smaller than $\delta$.

Denote by $I_g$ the Yamabe functional defined for all $\psi \in H^1(M)$ by

(6) $I_g(\psi) = \left( \int_M |\nabla_g \psi|^2 dv + \frac{(n - 2)}{4(n - 1)} \int_M R_g \psi^2 dv \right) \|\psi\|_N^{-2}$

where $N = 2n/(n - 2)$ and $\nabla_g$ is the gradient of the metric $g$.

The second integral of the functional $I_g$ with the scalar curvature term needs a special consideration. Let $\mu(P)$ be an integer defined as follows: $|\nabla_{\gamma} R_g(P)| = 0$ for all $|\beta| < \mu(P)$ and there exists $\gamma \in \mathbb{N}^\mu(P)$ such that $|\nabla_{\gamma} R_g(P)| \neq 0$ then

$$R_g(Q) = \bar{R} + O(\varepsilon^{\mu(P)+1})$$

where $\bar{R} = r^\mu(P)\sum_{|\beta| = \mu} \nabla_{\beta} R_g(P) \xi^\beta$ is a homogeneous polynomial of degree $\mu(P)$, the $\beta$ are multi-indices.

For simplicity, we drop the letter $P$ in $\omega(P)$ and $\mu(P)$.

By E. Hebey and M. Vaugon results:
Lemma 2.1. \( \mu \geq \omega , \ g_{ij} = \delta_{ij} + O(r^{\omega+2}) \) and \( \int_{S(r)} Rg = O(r^{2\omega+2}) \) which implies that \( \int_{S(r)} \tilde{R}\sigma = 0 \) when \( \mu < 2\omega + 2 \).

\( \tilde{g} \) denotes the average.\( \int_{M} R_{g} \varphi_{\varepsilon}^{2} dv = \int_{M} R_{g} u_{\varepsilon,p}^{2} dv - 2 \int_{M} f u_{\varepsilon,p}^{2} R_{g} r^{\omega+2} dv + \int_{M} f^{2} u_{\varepsilon,p}^{2} R_{g} r^{2\omega+4} dv \\
= \varepsilon^{2\omega+4} \omega_{n-1} \int_{S(r)} r^{-2\omega-2} R_{g} d\sigma r^{n+2\omega+1}(\varepsilon) - 2\varepsilon^{\omega+\mu+4} I_{n-2}^{\omega+\mu+n+1}(\varepsilon) \omega_{n-1} \int_{S(r)} r^{-\mu} f(\xi) \tilde{R} d\sigma(\xi) + O(\varepsilon^{-2}) \)

Moreover T. Aubin [2] proved that:

Theorem 2.1. If \( \mu \geq \omega + 1 \) then there exists \( C(n, \omega) > 0 \) such that

\( \int_{S_{n-1}(r)} \tilde{R} d\sigma = C(n, \omega) (-\Delta_{g})^{\omega+1} R(P) r^{2\omega+2} + o(r^{2\omega+2}) \)

\((-\Delta_{g})^{\omega+1} R(P) \) is negative. Then \( I_{g}(u_{\varepsilon, p}) < \frac{n(n-2)}{4} \omega_{n-1}^{2/n} \).

From now until the end of this section, we make the assumption that \( \mu = \omega \). Now, we recall some results obtained by T. Aubin in his papers [3][4]:

\( \tilde{R} \) is homogeneous polynomial of degree \( \omega \) then \( \Delta_{g} \tilde{R} \) is homogeneous of degree \( \omega-2 \) and

\( \Delta_{\varepsilon} \tilde{R} = r^{-2}(\Delta_{S} \tilde{R} - \omega(n + \omega - 2) \tilde{R}) \)

where \( \Delta_{g} \) is the Euclidean Laplacian and \( \Delta_{s} \) is the Laplacian on the sphere \( S_{n-1} \).

\( \Delta_{g}^{k-1} \tilde{R} \) is homogeneous of degree \( \omega - 2k + 2 \) and

\( \Delta_{g}^{k} \tilde{R} = r^{-2}(\Delta_{S} - \nu_{k} \text{id}) \Delta_{g}^{k-1} \tilde{R} = r^{-2k} \prod_{p=1}^{k}(\Delta_{S} - \nu_{p} \text{id}) \tilde{R} \)

with

\( \nu_{k} = (\omega - 2k + 2)(n + \omega - 2k) \)

The sequence of integers \( \{\nu_{k}\}_{1 \leq k \leq [\omega/2]} \) is decreasing. It will play the role of the eigenvalues of the Laplacian on the sphere \( S_{n-1} \). It is known that the eigenvalues of the geometric Laplacian are non-negative and increasing. Our \( \nu_{k} \) are in the opposite order.

We know by T. Aubin’s paper [2] that \( \Delta_{g}^{[\omega/2]} \tilde{R} = 0 \) and \( \int_{S(r)} \tilde{R} d\sigma = 0 \), then

\( q = \min\{k \in \mathbb{N} / \Delta_{g}^{k} \tilde{R} = 0\} \)

is well defined and \( r^{-\omega} \tilde{R} \in \bigoplus_{k=1}^{q} E_{k} \), with \( E_{k} \) the eigenspace associated to the positive eigenvalues \( \nu_{k} \) of the Laplacian \( \Delta_{g} \) on the sphere \( S_{n-1} \). If \( j \neq k \), then \( E_{k} \) is orthogonal to \( E_{j} \), for the standard scalar product in \( H_{1}^{2}(S_{n-1}) \). Moreover, since \( \int \tilde{R} d\sigma = 0 \) there exist \( \varphi_{k} \in E_{k} \) (eigenfunctions of \( \Delta_{g} \)) such that

\( \tilde{R} = r^{\omega} \sum_{k=1}^{q} \varphi_{k} = r^{\omega} \sum_{k=1}^{q} \nu_{k} \varphi_{k} \)

According to Lemma 2.1 we can split the metric \( g \) in the following way:

\( g = \mathcal{E} + h \)
where $E$ is the Euclidean metric and $h$ is a symmetric 2-tensor defined in our geodesic coordinates system by

$$h_{ij} = r^{p+2} g_{ij} + r^{2(\omega+2)} \tilde{g}_{ij} + \tilde{h}_{ij}$$

and $h_{rr} = h_{r\theta} = 0$.

where $\tilde{g}$, $\tilde{g}$ and $\tilde{h}$ are symmetric 2-tensors defined on the sphere $S_{n-1}$. We denote by $s$ the standard metric on the sphere, $\nabla$, $\Delta$ are the associated gradient and Laplacian on $S_{n-1}$. By straightforward computations, Aubin [29] proved that:

Lemma 2.2.

$$\tilde{R} = \nabla^i \tilde{g}_{ij} r^\omega$$

$$\int_{S_{n-1}(r)} \tilde{R} d\sigma = [B/2 - C/4 - (1 + \omega/2)^2 Q] r^{2(\omega+1)} + o(r^{2(\omega+1)})$$

where $B = \int \nabla^i \tilde{g}^j \nabla_i \tilde{g}^k d\sigma$, $C = \int \nabla^i \tilde{g}^j \nabla_i \tilde{g}^k d\sigma$ and $Q = \int \tilde{R}_{ij} \tilde{g}^{ij} d\sigma$

For further details refer to [3].

The integrals $Q$, $B$ and $C$ are given in terms of the tensor $\tilde{g}$. Our goal is to compute them using the eigenfunctions $\varphi_k$ above. Let us define

$$b_{ij} = \sum_{k=1}^{q} \frac{1}{n-2(n_k + 1 - n)} [(n - 1) \nabla_{ij} \varphi_k + \nu_k \varphi_k s_{ij}]$$

and $a_{ij}$ such that $\tilde{g}_{ij} = a_{ij} + b_{ij}$ then, according to [2], we check that

$$\tilde{R} = \tilde{R}_a = \nabla^i b_{ij} r^\omega$$

and $\tilde{R}_a = \nabla^i a_{ij} r^\omega = 0$

If $\bar{g}_{ij} = a_{ij}$ then $\bar{R} = \bar{R}_a = 0$ and $\mu \geq \omega + 1$. By Theorem 2.1

$$\int_{S_{n-1}(r)} \tilde{R} d\sigma = 0$$

If $\bar{g}_{ij} = b_{ij}$ then

$$\int_{S_{n-1}(r)} \tilde{R} d\sigma = \int_{S_{n-1}(r)} \tilde{R}_a d\sigma = [B_b/2 - C_b/4 - (1 + \omega/2)^2 Q_b] r^{2(\omega+1)} + o(r^{2(\omega+1)})$$

where $B_b$, $C_b$ and $Q_b$ are the same integrals defined in Lemma 2.2 when the considered tensor $\bar{g}_{ij} = b_{ij}$. We compute them in terms of $\varphi_k$

$$Q_b = \int_{S_{n-1}} \bar{b}_{ij} \bar{b}^{ij} d\sigma = \frac{n-1}{n-2} \sum_{k=1}^{q} \frac{\nu_k}{\nu_k + n - 1} \int_{S_{n-1}} \varphi_k^2 d\sigma$$

$$B_b = -(n - 1) Q_b + \sum_{k=1}^{q} \nu_k \int_{S_{n-1}} \varphi_k^2 d\sigma$$

$$C_b = -(n - 1) Q_b + \frac{n-1}{n-2} \sum_{k=1}^{q} \nu_k \int_{S_{n-1}} \varphi_k^2 d\sigma$$

To find these expressions, we used several times the identity $\nabla^i b_{ij} = -\sum_{k=1}^{q} \nabla_j \varphi_k$ and Stokes formula (more details are given in [34] and [35]). In the general case, we deduce that

Lemma 2.3. If $\mu = \omega$ and $\bar{g}_{ij} = a_{ij} + b_{ij}$, where $b_{ij}$ is defined above,

$$\int_{S_{n-1}(r)} \tilde{R} d\sigma = \int_{S_{n-1}(r)} \tilde{R}_a + \tilde{R}_b d\sigma \leq [B_b/2 - C_b/4 - (1 + \omega/2)^2 Q_b] r^{2(\omega+1)} + o(r^{2(\omega+1)})$$

and

$$B_b/2 - C_b/4 - (1 + \omega/2)^2 Q_b = \sum_{k=1}^{q} \nu_k \int_{S_{n-1}} \varphi_k^2 d\sigma$$
with
\[
\begin{align*}
  u_k &= \left( \frac{n-3}{(n-2)^2} + \frac{(n-1)^2 + (n-1)(\omega + 2)^2}{(n-2)(\nu_k - n + 1)} \right)^{\frac{1}{2}} 
\end{align*}
\]

\(u_k\) is obtained using the expressions of \(Q_b, B_b\) and \(C_b\) above.

### 3. Generalization of T. Aubin’s theorem

**Theorem 3.1.** If there exists \(P \in M\) such that \(\omega(P) \leq (n-6)/2\) then there exists \(f \in C^\infty(S_{n-1})\) with vanishing mean integral such that

\[
I_g(\varphi_\epsilon) < \frac{n(n-2)}{4} \omega_{n-1}^{2/n}
\]

The case \(\omega = 0\) of the this theorem has already been proven by T. Aubin [1]. He also proved the theorem when \(\mu \geq \omega + 1\) (see Theorem 24).

From now until the end of this paper, we drop the letter \(P\) in \(\omega(P)\) and \(\mu(P)\).

**Proof.** If \(\mu \geq \omega + 1\) then the inequality holds by Theorem 24. So we suppose that \(\mu = \omega\) until the end of the proof. We start by computing the first integral of the Yamabe functional (23) with \(\psi = \varphi_\epsilon\). Using formula \(|\nabla_g \varphi_\epsilon|^2 = (\partial_r \varphi_\epsilon)^2 + r^{-2} |\nabla_\epsilon \varphi_\epsilon|^2\), we obtain:

\[
\int_M |\nabla_g \varphi_\epsilon|^2 d\nu = \int_M |\nabla_g u_\epsilon, P|^2 d\nu + \int_0^\delta [\partial_r (r^{\omega + 2} u_\epsilon, P)]^2 r^{n-1} dr
\]

The substitution \(t = \frac{r}{\epsilon}\) gives

\[
\int_M |\nabla_g \varphi_\epsilon|^2 d\nu = (n-2)^2 \omega_{n-1} I_{n-1}^{n+1}(\epsilon) + \epsilon^{2\omega + 4} \int_{S_{n-1}} |\nabla f|^2 d\sigma I_{n-1}^{2\omega + n+1}(\epsilon) + 
\]

\[
\int_{S_{n-1}} f^2 d\sigma [\omega(n+4)^2 I_{n+1}^{2\omega+n+5}(\epsilon) + 2(\omega+2)(\omega-n+4) I_{n+1}^{2\omega+n+3}(\epsilon) + (\omega+2)^2 I_{n+1}^{2\omega+n+1}(\epsilon)]
\]

For \(|\varphi_\epsilon|_N^2\), we need to compute the Taylor expansion of:

\[
\varphi_\epsilon^N(Q) = [1 - N \epsilon^{\omega + 2} f(\xi) + \frac{N(N-1)}{2} \epsilon^{2\omega + 4} f^2(\xi) + o(\epsilon^{2\omega + 4})] u_\epsilon, P
\]

Using the fact that \(f d\sigma(\xi) = 0\) and formula (4), we conclude that

\[
|\varphi_\epsilon|_N^2 = \int_0^\delta \int_{S_{n-1}} [1 + \frac{N(N-1)}{2} \epsilon^{2(\omega + 2)} f^2(\xi) + o(\epsilon^{2\omega + 4})] u_\epsilon, P^2 d\sigma(\xi)
\]

\[
= \omega_{n-1} I_{n-1}^{n+1} + \frac{N(N-1)}{2} \epsilon^{2(\omega + 2)} \int_{S_{n-1}} f^2 d\sigma I_{n}^{2\omega+n+3} + o(\epsilon^{2\omega + 4})
\]

then

\[
|\varphi_\epsilon|_N^2 = (\omega_{n-1} I_{n-1}^{n+1})^{-2/N} \{1
\]

\[
- (N-1) \epsilon^{2(\omega + 2)} \int_{S_{n-1}} f^2 d\sigma I_{n}^{2\omega+n+3} / (\omega_{n-1} I_{n-1}^{n+1}) + o(\epsilon^{2\omega + 4})
\]

By Eqs (10), (14), (7) and the relationship (4), if \(n > 2\omega + 6\) then:
\[ I_g(\varphi \varepsilon) = \frac{n(n - 2)}{4} \omega_n + (\omega_n - 1 \mathcal{L}_n)^{-2/N} I_{n-2}^{n+2\omega+4} \times \]

\[
\left\{ \frac{(n - 2)\omega_n - 1}{4(n - 1)} \int_{S(r)} r^{-2\omega - 2} R_g d\sigma - \frac{n - 2}{2(n - 1)} \int_{S_{n-1}} f(\xi) \vec{R} d\sigma + \int_{S_{n-1}} |\nabla f|^2 d\sigma + \int_{S_{n-1}} f^2 d\sigma \right\} + o(\varepsilon^{2\omega + 4})
\]

If \( n = 2\omega + 6 \) then

\[ I_g(\varphi \varepsilon) = \frac{n(n - 2)}{4} \omega_n + (\omega_n - 1 \mathcal{L}_n)^{-2/N} \varepsilon^{2\omega + 4} \times \]

\[
\left\{ \frac{(n - 2)\omega_n - 1}{4(n - 1)} \int_{S(r)} r^{-2\omega - 2} R_g d\sigma - \frac{n - 2}{2(n - 1)} \int_{S_{n-1}} f(\xi) \vec{R} d\sigma + \int_{S_{n-1}} |\nabla f|^2 d\sigma + (\omega + 2)^2 \int_{S_{n-1}} f^2 d\sigma \right\} + O(\varepsilon^{2\omega + 4})
\]

For further details refer to [3].

Let \( I_S \) be the functional defined for a function \( f \) on the sphere \( S_{n-1} \), with zero mean integral, by

\[ I_S(f) = \int_{S_{n-1}} 4(n - 1)(n - 2) |\nabla f|^2 - [4n(n - 2)^2 - 4(\omega + 2)^2(n^2 + n + 2)] f^2 +
\]

\[ - 2(n - 2)^2 f \vec{R} d\sigma \]

This implies that if \( n > 2\omega + 6 \)

\[ I_g(\varphi \varepsilon) = \frac{n(n - 2)}{4} \omega_n + \frac{\omega_n^{2/n} I_n^{n+2\omega+1} \varepsilon^{2\omega + 4}}{4(n - 1)(n - 2)(\mathcal{L}_n)^{2/N}} \times \]

\[
\left\{ (n - 2)^2 \int_{S(r)} r^{-2\omega - 2} R_g d\sigma + I_S(f) \right\} + o(\varepsilon^{2\omega + 4})
\]

and if \( n = 2\omega + 6 \)

\[ I_g(\varphi \varepsilon) = \frac{n(n - 2)}{4} \omega_n + \frac{\omega_n^{2/n} I_n^{n+2\omega+1} \varepsilon^{2\omega + 4} \log \varepsilon^{-1}}{4(n - 1)(n - 2)(\mathcal{L}_n)^{2/N}} \times \]

\[
\left\{ (n - 2)^2 \int_{S(r)} r^{-2\omega - 2} R_g d\sigma + I_S(f) \right\} + O(\varepsilon^{2\omega + 4})
\]

Notice that if \( k \neq j \) then \( I_g(\varphi_k + \varphi_j) = I_g(\varphi_k) + I_g(\varphi_j) \). Indeed, \( \varphi_k \) and \( \varphi_j \) are orthogonal for the standard scalar product in \( H_k^2(S_{n-1}) \).

\[ I_S(c_k \varphi_k) = \left\{ d_k \varphi_k^2 - 2(n - 2)^2 c_k \right\} \varphi_k^2 \int_{S_{n-1}} \varphi_k^2 d\sigma \]

\[ = \frac{(n - 2)^4}{d_k} \varphi_k^2 \int_{S_{n-1}} \varphi_k^2 d\sigma \]

where

\[ d_k = 4[(n - 1)(n - 2) + (\omega + 2)^2(n^2 + n + 2)] \text{ and } c_k = \frac{(n - 2)^2}{d_k} \]
Using (3) we can check easily that $d_k$ is positive for any $1 \leq k \leq \lfloor \omega/2 \rfloor$. Now, let us consider $f = \sum_{k=1}^{q} c_k v_k \varphi_k$. Then

$$I_S(f) = -\sum_{k=1}^{q} \frac{(n-2)^4}{d_k} v_k^2 \int_{S_{n-1}} \varphi_k^2 d\sigma$$

and by Lemma 2.8

$$(n-2)^2 \int_{S_{\omega/2}} R_g d\sigma + I_S(f) \leq \sum_{k=1}^{q} (u_k(n-2)^2 - \frac{(n-2)^4}{d_k} v_k^2) \int_{S_{n-1}} \varphi_k^2 d\sigma + o(1)$$

The following lemma implies that $I_g(\varphi) < \frac{n(n-2)}{4} \omega^{2/n}_{n-1}$

Lemma 3.1. For any $k \leq q \leq \lfloor \omega/2 \rfloor$ the following inequality holds

$$u_k - \frac{(n-2)^2}{d_k} v_k^2 < 0$$

Proof. Recall the expression of $v_k$ given in (3). The sequence $(U_k)$ defined by

$$U_k := (v_k - n + 1) d_k \{(n-2) \frac{u_k}{v_k} - \frac{(n-2)^3}{d_k} v_k\}$$

is polynomial decreasing in $v_k$ when $v_k \geq 0$. In fact, $U_k = P(v_k)$ with $P$ the decreasing polynomial in $\mathbb{R}_+$, defined by

$$P(x) = [(n-1)(n-2)x - n(n-2)^2 + (\omega + 2)^2(n^2 + n + 2)] \times [(n-3)(x-n+1) - (n-1)^2 - (n-1)(\omega + 2)^2] - (n-2)^3(x^2 - (n-1)x)$$

The derivative of $P$ is

$$P'(x) = -2(n-2)x - 2n(n-2)^3 + 2(n^2 - 3n - 2)(\omega + 2)^2$$

By assumption $\omega + 2 \leq (n-2)/2$ then $P$ is decreasing in $\mathbb{R}_+$. Hence

$$U_k = P(v_k) \leq P(u_{\omega/2}) = U_{\omega/2}$$

for all $k \leq \omega/2$. It easy to check that $u_{\omega/2}$ is negative so $U_k \leq U_{\omega/2} < 0$.

4. Proof of the main theorem

By Remark 14 we consider only the positive case (i.e., $\inf_{g \in [g]} J(g') > 0$) and the case when there exists $P \in M$ such that

$$O_G(P) = \{P_i\}_{i \leq i \leq m}, \quad m = \text{card} O_G(P) = \inf_{Q \in M} \text{card} O_G(Q), \quad \omega \leq \frac{n-6}{2} \quad \text{and} \quad P_1 = P$$

Let $\bar{\varphi}_g$ be a function defined as follows:

$$\bar{\varphi}_{g, i}(Q) = (1 - r_i^{\omega+2} f_i(\xi)) u_{e, P_i}(Q)$$

where $r_i = d(Q, P_i)$, the function $u_{e, P_i}$ is defined as in (3) and $f_i$ is defined by:

$$f_i(Q) = e^{r_i^{-\omega} R(P_i)}(\exp_{P_i}^{-1} Q, \cdots, \exp_{P_i}^{-1} Q)$$

$\exp_{P_i}$ is the exponential map. In a geodesic coordinates system $\{r, \xi^I\}$ with origin $P$, induced by the exponential map

$$f_1 = e^{r^{-\omega}} \tilde{R} = c \sum_{k=1}^{q} v_k \varphi_k$$

where $\tilde{R}$, $\varphi_k$ and $v_k$ are defined in Section 2. Thus the functions $f_i$ are defined on the sphere $S_{n-1}$. The choice of the constant $c$ is important.
Lemma 4.1. Suppose that \( \omega \leq (n - 6)/2 \). If \( \omega \in [3, 15] \) or if \( \deg \hat{R} \geq \omega + 1 \) then there exists \( c \in \mathbb{R} \) such that the corresponding functions \( \hat{\varphi}_{c,i} \) satisfy:

\[
I_g(\varphi_{c,i}) < \frac{1}{4}n(n - 2)\omega^{2/n}
\]

Remarks 4.1. (1) We proved inequality of this lemma for any \( \omega \leq (n - 6)/2 \), using test function \( \varphi_{c} \) (see Theorem 2.1). We notice that the difference between \( \varphi_{c} \) and \( \hat{\varphi}_{c,i} \) is on the construction of the corresponding functions \( f_{\omega} \) and \( f_{i} \) respectively. From \( \hat{\varphi}_{c,i} \) we define a \( G \)-invariant function (see proof of the main theorem below), this property is not possible with the function \( \varphi_{c} \).

(2) For \( \omega = 16 \) and \( n \) sufficiently big, we can check that for any \( c \in \mathbb{R} \), inequality (22) is false.

Proof. 1. If \( \deg \hat{R} \geq \omega + 1 \), then by Theorem 2.1

\[
I_g(u_{c, p_i}) < \frac{n(n - 2)}{4} \omega^{2/n}
\]

It is sufficient to take \( c = 0 \), hence \( \hat{\varphi}_{c,i} = u_{c, p_i} \).

2. If \( \deg \hat{R} = \omega \). Using estimates given in the proof of Theorem 3.1 (see [18, 19]), it is sufficient to show that there exists \( c \in \mathbb{R} \) such that

\[
I_g(f_1) + (n - 2)^2 \int_{S(r)} r^{-2\omega-2} R_g d\sigma_r < 0
\]

We keep the notations used in the proof of Theorem 3.1. Thus

\[
I_g(f_1) = \sum_{k=1}^{q} I_g(cv_k \varphi_k) = \left\{ d_k c^2 - 2(n - 2)^2 c \right\} \nu_k^2 \int_{S_{n-1}} \varphi_k^2 d\sigma
\]

and

\[
\int_{S(r)} r^{-2\omega-2} R_g d\sigma_r = \sum_{k=1}^{q} u_k \int_{S_{n-1}} \varphi_k^2 d\sigma
\]

To prove inequality (23), it is sufficient to prove that

\[
\forall k \leq q \quad \frac{d_k}{2(n - 2)} c^2 - (n - 2)c + (n - 2) \frac{u_k}{2\nu_k^2} < 0
\]

The left side of the inequality above is a second degree polynomial with variable \( c \), its discriminant is:

\[
\Delta_k = (n - 2)^2 - \frac{d_k u_k}{\nu_k^2}
\]

Using Lemma 3.1 we deduce that for any \( k \leq q \), \( \Delta_k > 0 \). Hence, the polynomial above admits two different roots denoted \( x_k < y_k \) and given by

\[
x_k = \frac{(n - 2)^2 - (n - 2)\sqrt{\Delta_k}}{d_k}, \quad y_k = \frac{(n - 2)^2 + (n - 2)\sqrt{\Delta_k}}{d_k}
\]

Inequality (24) holds if and only if

\[
\forall k \leq q \quad (x_k, y_k) \neq \emptyset
\]

The sequence \( (d_k)_{k \leq \lfloor \omega/2 \rfloor} \) decreases. It is easy to check that

\[
\forall k < j \leq \lfloor \omega/2 \rfloor \quad x_k < y_j
\]

Hence intersection (26) is not empty if

\[
\forall k < j \leq \lfloor \omega/2 \rfloor \quad x_j < y_k
\]
We also check that if $\omega$ is even, $u_{\omega/2} < 0$, which implies $x_{\omega/2} < 0$.

i. If $\omega = 3$ then $q = 1$, intersection above is not empty. It is sufficient to take $c = (x_1 + y_2)/2$.

ii. If $\omega = 4$ then $k \in \{1, 2\}$, $x_2 < 0$ (because $u_2 < 0$) and $0 < x_1 < y_2$. Hence intersection $[x_1, y_1]\cap[x_2, y_2]$ is not empty.

iii. If $5 \leq \omega \leq 15$, it is sufficient to prove (28) which is equivalent to prove that

$$\forall k < j \leq \left[\frac{\omega}{2}\right] \quad (n - 2)(d_j - d_k) + d_k\sqrt{\Delta_j} + d_j\sqrt{\Delta_k} > 0$$

Notice that $\Delta_k$ given by (28) is a rational fraction in $n$. By straightforward computations, we check that there exists reel numbers $a_k$, $b_k$, $c_k$, $h_k$ and $s_k$ which depend on $k$ and $\omega$ such that

$$\Delta_k = a_k n^2 + b_k n + c_k + \frac{h_k}{n - 2} + \frac{s_k}{\nu_k + 1 - n}$$

$$\sqrt{\Delta_k} > \sqrt{a_k(n + \frac{b_k}{2\nu_k})}$$

Inequality (29) holds if we use (31).

The expressions of the reel numbers above are known explicitly (we used the software Maple to compute them, see [5]). For simplicity, we omit to give these expressions.

Proof of the main theorem. The orbit of $P$ under the action of $G$ is supposed to be minimal (i.e. $\text{card}O_G(P) = \inf_{Q \in M} \text{card}O_G(Q)$). Without loss of generality, we suppose that $3 \leq \omega \leq (n - 6)/2$, because if $\omega > (n - 6)/2$ or $\omega \leq 2$, we conclude using Theorem 1.4.

From functions $\tilde{\varphi}_{\varepsilon_i}^{\omega}$ defined by (20), we define the function $\phi_{\varepsilon}$ as follows:

$$\phi_{\varepsilon} = \sum_{k=1}^{m} \tilde{\varphi}_{\varepsilon, i}$$

$\phi_{\varepsilon}$ is $G$-invariant. In fact, for any $\sigma \in G$, such that $\sigma(P_i) = P_j$

$$u_{\varepsilon, P_i} = u_{\varepsilon, P_j} \circ \sigma$$

and $f_i = f_j \circ \sigma$

$f_i$ are defined by (24), we deduce that

$$\tilde{\varphi}_{\varepsilon, i} = \tilde{\varphi}_{\varepsilon, j} \circ \sigma$$

The support of $\tilde{\varphi}_{\varepsilon, i}$ is included in the ball $B_{P_i}(\delta)$. We choose $\delta$ sufficiently small such that for all integers $i \neq j$ in $[1, m]$, intersection $B_{P_i}(\delta) \cap B_{P_j}(\delta) = \emptyset$. Thus

$$I_{\phi}(\varphi_{\varepsilon}) = (\text{card}O_G(P))^{2/n}I_{\phi}(\varphi_{\varepsilon})$$

By Lemma 4.1, we conclude that

$$I_{\phi}(\varphi_{\varepsilon}) < \frac{n(n - 2)}{4} \omega_{n-1}^{2/n}(\text{card}O_G(P))^{2/n}$$

It remains to notice that if $\tilde{g} = \phi_{\varepsilon}(n-2)/g$ then

$$J(\tilde{g}) = \frac{4n - 1}{n - 2} I_{\phi}(\varphi_{\varepsilon}) < n(n - 1)\omega_{n-1}^{2/n}(\text{card}O_G(P))^{2/n}$$

where $\varepsilon$ is sufficiently smaller than $\delta$.

Proof of the Corollary 1.4. Suppose that the orbit of $P$ under the action of $G$ is minimal (otherwise the conjecture is obvious).

If $\omega = \omega(P) > (n - 6)/2$, we conclude using Theorem 1.4.

If $\omega \leq (n - 6)/2 \leq 15$, we conclude using main theorem.
REFERENCES


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