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Abstract

A nonlocal viscous model of phase separation is presented. It is derived from a minimization of free energy containing a nonlocal part due to particle interaction. In contrast to the classical Cahn-Hilliard theory with higher order terms this leads to an evolution system of second order parabolic equations for the particle densities, coupled by nonlocal drift and viscosity terms, which allow reasonable bounds for the concentrations. Applying fixed-point arguments and compactness results we prove the existence of variational solutions in standard Hilbert spaces for evolution systems. Using the free energy as Lyapunov functional the asymptotic state of the system is investigated and characterized by a variational principle.

Key words. Nonlocal phase separation models; viscous phase separation models, Cahn-Hilliard equation; Integrodifferential equations ; Initial value problems; Nonlinear evolution equations.

AMS subject classification. 80A22, 35B40, 35B50, 45K05, 35K20, 35K45, 35K55, 35K65, 47J35

1 Introduction

Phase separation phenomena in material sciences are modeled usually by Cahn-Hillard equation, see [5] and references therein, which is derived from a free energy functional. Often the classical Ginzburg-Landau free energy which contains gradient terms is used as the free energy functional. These models have been extensively analyzed, see [21] and references therein. But inspecting Van der Waals early works, see [20], and later Cahn and Hillard paper [4] it seems to be reasonable and even more adequate, see [10], to choose an alternative expression for the *free energy* functional like

$$F_{NL}(u) = \int_{\Omega} F(u)dx, \tag{1.1}$$

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where u denotes the local concentration of a component occupying a spatial domain Ω and $F(u) = f(u) + \frac{1}{2}uw$. Here $f(u)$ is a convex function and

$$w(x) := \int \mathcal{K}(|x-y|)(1-2u(y))dy. \tag{1.2}$$

The kernel $\mathcal{K}$ of the integral term (1.2) describes *nonlocal or long-range interactions* [5, 11, 12, 14]. Hence, the difference between local and nonlocal models consists in a different choice of the particle interaction potential in the free energy functional. Moreover the local free energy can be obtained as a formal limit from the nonlocal one, see [17]. In [10] the above nonlocal free energy functional has been used to derive a nonlocal Cahn-Hillard equation

$$u_t - \nabla \cdot (\mu \nabla (f'(u) + w)) = 0,$$

where f is the convex (Information) entropy function

$$f(u) = u \log(u) + (1-u)\log(1-u).$$

Consequently

$$f'(u) = \log\left(\frac{u}{1-u}\right) \quad \text{and} \quad u = f'^{-1}(v-w) = \frac{1}{1+\exp(v-w)},$$

where f'^{-1} is the Fermi-function, whose image is the interval $[0,1]$. Thus, the nonlocal model naturally satisfies the physical requirement

$$0 \leq u(x) \leq 1, \quad \forall t \geq 0.$$

and the maximum principle is available, which is not true for fourth order equations like in the case of the local Cahn-Hillard equations.

1.1 Nonlocal viscous model

Following [10] our aim is to formulate a general nonlocal model, which also takes into accout viscosity effects, see [19]. In the local theory this was done by adding a rate term to the chemical potential. Now we are going to formulate this additional term in the nonlocal philosophy, so we not only want to get nonlocality in space (1.2) but also nonlocality in time. Hence, the chemical potential in our case is given by

$$v := \frac{\delta F(u)}{\delta u} + \psi, \tag{1.3}$$

We propose two models:

Model I:

$$-\gamma \Delta \psi_t + \psi = u_t, \quad \gamma > 0. \tag{1.4}$$

Model II:

$$-\gamma\Delta\psi + \psi = u_t, \quad \gamma > 0.$$

In both cases γ is a model parameter, which is positive and guarantees the nonlocal structure of the additional term ψ in the chemical potential (1.3). This means in Gurtin's language that the influence of microforces is nonlocal, but we are not able to postulate a generalized nonlocal balance law for nonlocal microforces similar to the balance law in Gurtin [16]. Setting $\gamma = 0$ we recover the local viscous model, see [19]. Hence, our model is a real expansion of previous existing models. From mathematical point of view the terms $-\gamma\Delta\psi_t$ respectively $-\gamma\Delta\psi$ have regularizing effects. Model I will be analysed in this paper. The Anlayis of Model II are left to a forthcoming paper, see [8]. Taking into account (1.3) and (1.4) we end up with the *nonlocal viscous Cahn-Hillard equation*:

$$\begin{aligned} u_t - \nabla\cdot\mu\nabla v = 0, \quad v = f'(u) + w + \psi, \\ w(x) = \int\limits_{\Omega} \mathcal{K}(|x-y|)(1-2u(y))dy, \\ -\gamma\Delta\psi_t + \psi = u_t, \quad \gamma > 0, \end{aligned} \tag{1.5}$$

which is complementedby suitable initial and boundary conditions.

In Section 2 we formulate the problem, general assumptions and the main theorems. The rest of the Sections are devoted to the proof of the corresponding theorems.

2 Assumptions and main results

2.1 Statement of the problems and assumptions

Let be $\Omega \subset \mathbb{R}^3$ an open, bounded and smooth domain with boundary $\Gamma = \partial\Omega$ and ν the outer unit normal on Γ. In the sequel, $|\Omega|$ denotes the Lebesgue measure of Ω. We denote by $L^p(\Omega), W^{k,p}(\Omega)$ for $1 \le p \le \infty$ the Lesbegue spaces and Sobolev spaces of functions on Ω with the usual norms $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{W^{k,p}(\Omega)}$, and we write $H^k(\Omega) = W^{k,2}(\Omega)$, see [7]. For a Banach space X we denote its dual by X^*, the dual pairing between $f \in X^*$, $g \in X$ will be denoted by $\langle f, g\rangle$. If X is a Banach space with the norm $\|\cdot\|_X$, we denote for $T > 0$ by $L^p(0,T;X)$ ($1 \le p \le \infty$) the Banach space of all (equivalence classes of) Bochner measurable functions $u : (0,T) \longrightarrow X$ such that $\|u(\cdot)\|_X \in L^p(0,T)$. We set $R^1_+ = (0,\infty)$ and, as already mentioned, $Q_T = (0,T)\times\Omega$, $\Gamma_T = (0,T)\times\Gamma$. "Generic" positive constants are denoted by C and for $u \in L^1(\Omega)$ we put

$$\overline{u} = \frac{1}{|\Omega|}\int_{\Omega} u(x)dx.$$

Furthermore we define following time dependent Sobolev spaces by

$$\mathcal{V}^{1,\infty}(0,T) := \{f \in L^\infty(Q_T) \mid \nabla f \in L^\infty(Q_T)\},$$
$$\mathcal{V}^{2,\infty}(0,T) := \{f \in L^\infty(Q_T) \mid \nabla f \in L^\infty(Q_T), \Delta f \in L^\infty(Q_T)\}.$$

So the initial-boundary value problem we want to discuss takes the form:

$$u_t - \nabla \cdot \overbrace{(\nabla u + \mu\nabla(w+\psi))}^{=\mu\nabla v} = 0 \quad \text{in } Q_T, \tag{2.1}$$
$$-\gamma\Delta\psi_t + \psi = u_t, \quad w = P(1-2u) \quad \text{in } Q_T, \tag{2.2}$$
$$\mu\nu \cdot \nabla v = \nu \cdot \nabla\psi = 0 \quad \text{on } \Gamma_T, \tag{2.3}$$
$$\nu \cdot \nabla\psi_0 = 0, u(0,x) = u_0(x), \psi(0,x) = \psi_0(x) \quad x \in \Omega. \tag{2.4}$$

We make the following general assumptions.
(A1) $f(u) = u\log u + (1-u)\log(1-u)$.
(A2) The potential operator P defined by

$$\rho \mapsto P\rho = \int_\Omega \mathcal{K}(|x-y|)\rho(y)dy$$

satisfies

$$\|P\rho\|_Y \le r_p\|\rho\|_{L^p}, \qquad 1 \le p \le \infty,$$

where the kernel $\mathcal{K} \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1)$ is such that

$$\int_\Omega\int_\Omega |\mathcal{K}(|x-y|)|dxdy = m_0 < \infty, \qquad \sup_{x\in\Omega}\int_\Omega |K(|x-y|)|dy = m_1 < \infty.$$

(A3) The mobility μ has the form

$$\mu(u) = \frac{1}{f''(u)} = u(1-u). \tag{2.5}$$

(A4) $u_0(x) \in [0,1]$ a.e. in Ω and $\overline{u}_0 \in (0,1)$.

The next assumptions concern different regularity assumptions on the data.

(B1) $u_0 \in L^\infty(\Omega)$	or	**(B1')** $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$,
(B2) $\psi_0 \in H^2(\Omega)$	or	**(B2')** $\psi_0 \in H^3(\Omega)$,
(B3) $Y := W^{1,p}(\Omega)$	or	**(B3')** $Y := W^{2,p}(\Omega)$.

Remark 1. The kernel $\mathcal{K}$ is chosen to be symmetric. Hence, the potential operator P is symmetric, too. Examples for kernels $\mathcal{K}$ satisfying (A2) are Newton potentials, Gauss functions and usual mollifiers, see [10].

Remark 2. A concentration-dependent mobility appeared in the original derivation of the Cahn-Hillard equation, see [4], and a natural and thermodynamically reasonable choice is of the form (2.5) and were considered in [6].

2.2 Main results

Due to different regularity assumptions on the initial data we formulate two different Theorems, which will be proven separately in the next two chapters.

Theorem 1. *Suppose that the assumptions (A1)-(A4) and (B1)-(B3) hold. Then there exists a unique triple of functions* (u, w, ψ) *such that* $u(0) = u_0$, $\psi(0) = \psi_0$ *and*

1. $u \in L^2(0,T;H^1(\Omega)), \quad 0 \le u(t,x) \le 1$ *a.e. in* Q_T,
2. $u_t \in L^2(0,T;H^1(\Omega)^*)$,
3. $w \in \mathcal{V}^{1,\infty}(0,T)$,
4. $\psi \in L^2(0,T;L^2(\Omega))$,
5. $\nabla\psi \in L^\infty(0,T;H^1(\Omega))$,
6. $\nabla\psi_t \in L^2(0,T;H^1(\Omega)^*)$,

which satify (2.1)-(2.4) in the following sense:

$$\int_0^T \langle u_t, \varphi\rangle\, dt + \int_0^T\int_\Omega (\nabla u + \mu\nabla(w+\psi))\nabla\varphi dxdt = 0, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)), \tag{2.6}$$

$$\gamma\int_0^T \langle \nabla\psi_t, \nabla\phi\rangle\, dt + \int_0^T\int_\Omega \psi\phi dxdt = \int_0^T \langle u_t, \phi\rangle\, dt, \quad \forall\phi \in L^2(0,T;H^1(\Omega)), \tag{2.7}$$

$$w = P(1-2u) \text{ a.e. in } Q_T. \tag{2.8}$$

Theorem 2. *Suppose that the assumptions (A1)-(A4) and (B1')-(B3') hold. Then there exists a unique triple of functions* (u, w, ψ) *such that* $u(0) = u_0$, $\psi(0) = \psi_0$ *and*

1. $u \in L^2(0,T;H^2(\Omega)), \quad 0 \le u(t,x) \le 1$ *a.e. in* Q_T,
2. $u_t \in L^2(0,T;L^2(\Omega))$,

3. $w \in \mathcal{V}^{2,\infty}(0,T)$,

4. $\psi \in L^2(0,T;L^2(\Omega))$,

5. $\nabla\psi \in L^\infty(0,T;H^2(\Omega))$,

6. $\nabla\psi_t \in L^2(0,T;L^2(\Omega))$,

which satisfy (2.1)-(2.4) in the following sense:

$$\int_0^T\int_\Omega u_t\varphi dxdt + \int_0^T\int_\Omega (\nabla u + \mu\nabla(w+\psi))\,\nabla\varphi dxdt = 0, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)), \tag{2.9}$$

$$\gamma\int_0^T\int_\Omega \nabla\psi_t\cdot\nabla\phi dt + \int_0^T\int_\Omega \psi\phi dxdt = \int_0^T\int_\Omega u_t\phi dxdt,, \quad \forall\phi \in L^2(0,T;H^1(\Omega)), \tag{2.10}$$

$$w = P(1-2u) \text{ a.e. in } Q_T. \tag{2.11}$$

Remark 3. Note that using the testfunctions $\varphi = 1$ and $\phi = 1$ in (2.6)-(2.7) we get

$$\overline{u(t,x)} = \overline{u_0} \text{ a.e. in } [0,T], \qquad \int_0^T\int_\Omega \psi(t,x)dxdt = 0. \tag{2.12}$$

Under the assumptions of Theorem 2 we can state the following

Theorem 3. *Suppose* $f'(u_0) \in L^\infty(\Omega)$*. Then* $f'(u) \in L^\infty(Q_T)$*.*

Remark 4. *We get from Theorem 3 that* $0 < u(t,x) < 1$ *a.e. in* Q_T*, provided* $0 < u_0(x) < 1$ *a.e. in* Ω*.*

The main tool for studing the global behaviour of the solution (2.9)-(2.11) for $T \to \infty$ is the energy estimate. Because of Theorem 3 and (A2) the function $f'(u) + w + \psi =: v$ is in $L^\infty(Q_T)$ and an admissible testfunction in (2.9) and gives the global energy estimate

$$\frac{\gamma}{2}\sup_{t\geq 0}\int_\Omega |\nabla\psi(t)|^2dx + \int_0^\infty\int_\Omega |\psi|^2dxdt + \int_0^\infty\int_\Omega \mu(u)|\nabla v|^2dxdt \leq C_{55} < \infty. \tag{2.13}$$

Thus we can state the following Theorem which can be proven exploiting (2.13), see [9].

Theorem 4. *Let* (u,w,ψ) *be a solution of (2.9)-(2.11). Then there exist a sequence* $\{t_k : k = 1,2,\cdots\}$ *with* $t_k \to \infty$ *for* $k \to \infty$ *and a triplet* (u^*,w^*,ψ^*) *such that* $u_k = u(t_k), w_k = w(t_k), \psi_k = \psi(t_k)$ *satisfy*

$$\begin{array}{llll} u_k & \to & u^* & \text{strongly in } L^2 \text{ and weakly in } H^1, \\ w_k & \to & w^* & \text{strongly in } H^1, \\ \psi_k & \to & 0 & \text{strongly in } L^2, \\ \nabla\psi_k & \to & \nabla\psi^* & \text{strongly in } L^2, \end{array} \tag{2.14}$$

and

$$\arctan(e^{-v_k/2}) \to \arctan(e^{-v^*/2}) \textit{ strongly in } H^1, v^* = \textit{ const. } . \tag{2.15}$$

Moreover, the following relations hold:

$$w^* = \int\limits_{\Omega} \mathcal{K}(|x-y|)(1-2u^*(y))dy, \quad \overline{u^*} = \overline{u_0}, \tag{2.16}$$

$$u^* = \frac{1}{1+\exp(w^*-v^*)}, \quad v^* = \textit{const.} \tag{2.17}$$

3 Proof of Theorem 1

3.1 Existence

The idea of existence proof is as follows: we construct regularized problems with non-degenerate mobility functions. These regularized problems then are approximated by semi-discrete problems, which we solve by applying the Schauder's fixed-point principle. After constructing suitable a priori estimates and compactness we can converge from the semi-discrete approximation to the regularized problem. The similar procedure we repeat for regularized problem to get uniform a priori estimates and compactness results, which finaly give convergence to the original problem. We devide our existence proof into a sequence of steps.

3.1.1 Regularized problems

At first we modify the moblity. We introduce a non-degenerate positive mobility μ_ε as

$$\mu_\varepsilon(u) := \begin{cases} \mu(\varepsilon) & \text{for} \quad u \le \varepsilon, \\ \mu(u) & \text{for} \quad \varepsilon < u \le 1-\varepsilon, \\ \mu(1-\varepsilon) & \text{for} \quad u > 1-\varepsilon. \end{cases} \tag{3.1}$$

This means that we symmetrically cut the mobility and constantly extend it to whole $\mathbb{R}$. Simillarly we regularize $f''(u)$ and $f'(u)$, see [9]. Furthermore we introduce the truncation

$$\Pi u := \begin{cases} 1 & \text{for} \quad u \ge 1, \\ u & \text{for} \quad 0 < u < 1, \\ 0 & \text{for} \quad u \le 0, \end{cases} \tag{3.2}$$

which is necessary to be able to apply the Schauder's fixed-point principle. Hence, we get the *truncated regularized system:*

$$\int\limits_0^T \langle u_t, \varphi \rangle \, dt + \int\limits_0^T \int\limits_\Omega (\nabla u + \mu_\varepsilon \nabla(w+\psi)) \cdot \nabla \varphi dx dt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{3.3}$$

$$\gamma \int\limits_0^T \langle \nabla \psi_t, \nabla \phi \rangle \, dt + \int\limits_0^T \int\limits_\Omega \psi \phi dx dt = \int\limits_0^T \langle u_t, \phi \rangle \, dt, \quad \forall \phi \in L^2(0,T;H^1(\Omega)), \tag{3.4}$$

$$w = P(1 - 2\Pi u) \text{ a.e. in } Q_T. \tag{3.5}$$

Remark 5. We have by (A2) and (A4)

$$\|w\|^2_{H^1(\Omega)} \le r_2^2 \|1 - 2\Pi u\|_{L^2(\Omega)} \le r_2^2 |\Omega|. \tag{3.6}$$

Remark 6. $\exists \varepsilon_0 := \varepsilon_0(w)$ so that $\forall \varepsilon \in (0, \varepsilon_0]$

$$F_{NL,\varepsilon}(u) := \int\limits_\Omega \left(f_\varepsilon(u) + \frac{1}{2} uw \right) dx \ge -C_F,$$

where $C_F > 0$.

Proof. Using (A1), (3.1) and (3.5) we see that it depends on the choice of ε to ensure that $f_\varepsilon(u)$ dominates $\frac{1}{2}uw$. Thus, there exists an $\varepsilon_0 = \varepsilon_0(w)$ so that $\forall \varepsilon \in (0, \varepsilon_0]$ this is true. □

Existence result for the regularized problems We will denote the solution to the regularized system (3.3)-(3.5) by $(u_\varepsilon, w_\varepsilon, \psi_\varepsilon)$. Let $\forall \varepsilon \in (0, \varepsilon_0]$ be fixed but arbitrary. The strategy of constructing solutions to (3.3)-(3.5) is to employ a semi-discrete approximation. To this end, let $M \in N$ be given and $h := T/M$. In the sequel, we will denote by $C_i, i \in \mathbb{N}$, positive constants that may depend on Ω, T and the initial data, but not on M or $m \in \{1, \dots, M\}$. For $1 \le m \le M$, we consider the *semi-discrete problem* on the time level $t := mh$ for the unknown functions $u^m, w^m, \psi^m : \Omega \to \mathbb{R}$ given by

$$\frac{1}{h} \int\limits_\Omega (u^m - u^{m-1}) \varphi dx +$$

$$+ \int\limits_\Omega \left[\nabla u^m + \mu_\varepsilon(u^m) \nabla \left(\frac{w^m + w^{m-1}}{2} + \psi^m \right) \right] \cdot \nabla \varphi dx = 0, \quad \forall \varphi \in H^1(\Omega), \tag{3.7}$$

$$\frac{\gamma}{h} \int\limits_\Omega \nabla(\psi^m - \psi^{m-1}) \cdot \nabla \phi dx + \int\limits_\Omega \psi^m \phi dx = \frac{1}{h} \int\limits_\Omega (u^m - u^{m-1}) \phi dx, \quad \forall \phi \in H^1(\Omega), \tag{3.8}$$

$$w^m = P(1 - 2\Pi u^m) \text{ a.e. in } \Omega. \tag{3.9}$$

For $1 \le m \le M$ the system (3.7)-(3.9) is a nonlinear elliptic system. Note that $u^0 = u_0, \psi^0 = \psi_0$.

Remark 7. For $\varphi = 1$ and $\phi = 1$ we get from (3.7)-(3.9)

$$\overline{u^m} = \overline{u_0} \quad \text{and} \quad \int\limits_\Omega \psi^m = 0, \quad \forall m \in \{1, \dots, M\}.$$

We prove existence of approximate solutions step by step via Schauder's fixed-point principle.

Lemma 1. *Suppose that the assumptions (A1)-(A4) and (B1)-(B3) hold. Then for every* $m \in \{1, ..., M\}$ *there exists a triple of functions* $(u^m, w^m, \psi^m) \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ *satifying (3.7)-(3.9).*

Proof. 1. Our proof is based on the application of Schauder's fixed-point principle. Let $m \in \{1, ..., M\}$ be fixed but arbitrary, and assume that the data (u^{m-1}, ψ^{m-1}) are known and given. Now for a given $u^m \in L^2$ we consider the *auxiliary linear problems*

$$\int_\Omega \nabla(\mathcal{T}_m u^m) \cdot \nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_m u^m \varphi dx = \int_\Omega g_2 \varphi dx, \quad \forall \varphi \in H^1(\Omega), \tag{3.10}$$

$$\int_\Omega \nabla\psi^m \cdot \nabla\phi dx + \frac{h}{\gamma}\int_\Omega \psi^m \phi dx = \int_\Omega g_1 \phi dx, \quad \forall \phi \in H^1(\Omega), \tag{3.11}$$

where

$$\int_\Omega g_1 \phi dx := \frac{1}{\gamma}\int_\Omega (u^m - u^{m-1})\phi dx + \int_\Omega \nabla\psi^{m-1} \cdot \nabla\phi dx, \quad \forall \phi \in H^1(\Omega), \tag{3.12}$$

$$\int_\Omega g_2 \varphi dx :=$$

$$\int_\Omega \mu_\varepsilon(u^m)\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right) \cdot \nabla\varphi dx + \frac{1}{h}\int_\Omega \mathcal{T}_{m-1} u^{m-1}\varphi dx, \forall \varphi \in H^1(\Omega). \tag{3.13}$$

The existence and uniqueness theory of (3.10)-(3.11) is standard and can be found in [7]. The strategy is to convert the integral expression in (3.10)-(3.13) into linear- and bilinearforms and use the Lax-Milgram Theorem. From [15], Corollary 2.2.2.4, respectively, we find that for a given $u^m \in L^2$ and consequently a given $g_1 \in L^2(\Omega)$ in (3.12) the linear equation (3.11) admits a unique solution $\psi^m \in H^2(\Omega)$. Setting $w^m = P(1 - 2\Pi u^m) \in H^1(\Omega)$, we find (3.9). Finally again from [15], Corollary 2.2.2.4, respectively, we conclude that for a given $g_2 \in L^2(\Omega)$ (3.10) admits a unique solution $\mathcal{T}_m u^m \in H^1(\Omega)$.

2. Thus, we have properly defined a fixed-point operator $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$. We can apply Schauder's theorem, if we are able to prove, that $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$ is completely continuous and $\mathcal{T}_m[\mathcal{B}] \subset \mathcal{B}$ hold true for a closed ball $\mathcal{B} \subset L^2(\Omega)$ with a radius depending only on the data of the problem.

3. Let $\psi^m \in H^2(\Omega)$ be a solution of (3.11). We obtain using ψ^m as a testfunction in (3.11)

$$\int_\Omega |\nabla\psi^m|^2 dx + \frac{h}{\gamma}\int_\Omega |\psi^m|^2 dx \le \frac{1}{\gamma}\int_\Omega (u^m - u^{m-1})\psi^m dx + \int_\Omega \nabla\psi^{m-1} \cdot \nabla\psi^m dx.$$

Applying Young's inequality in the form

$$\frac{1}{\gamma}\int\limits_{\Omega}(u^m-u^{m-1})\psi^m dx \le \frac{\epsilon}{2\gamma}\int\limits_{\Omega}|u^m-u^{m-1}|^2dx+\frac{1}{2\gamma\epsilon}\int\limits_{\Omega}|\psi^m|^2dx, \tag{3.14}$$

and using the Poincaré inequality for the last term in (3.14) by choosing $\epsilon = 2c_p/\gamma$, where c_p is the Poincaré constant, we finally conclude

$$\int\limits_{\Omega}|\nabla\psi^m|^2dx+\frac{4h}{\gamma}\int\limits_{\Omega}|\psi^m|^2dx \le \frac{4c_p}{\gamma^2}\int\limits_{\Omega}|u^m-u^{m-1}|^2dx+2\int\limits_{\Omega}|\nabla\psi^{m-1}|^2dx. \tag{3.15}$$

4. Let $\mathcal{T}_m u^m \in H^1(\Omega)$ be a solution of (3.10). Applying the admissible test function $\varphi = \mathcal{T}_m u^m \in H^1(\Omega)$ in (3.10) and Young's inequality we get

$$\begin{aligned}\frac{1}{2h}\int\limits_{\Omega}|\mathcal{T}_m u^m|^2dx &\le \frac{1}{4}\int\limits_{\Omega}\left|\mu_\varepsilon(u^m)\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\right|^2dx+\frac{1}{2h}\int\limits_{\Omega}|\mathcal{T}_{m-1}u^{m-1}|^2dx\\ &\le \frac{2}{4^3}\int\limits_{\Omega}\left|\nabla\left(\frac{w^m+w^{m-1}}{2}\right)\right|^2dx+\frac{2}{4^3}\int\limits_{\Omega}|\nabla\psi^m|^2\,dx\\ &\quad+\frac{1}{2h}\int\limits_{\Omega}|\mathcal{T}_{m-1}u^{m-1}|^2dx. \end{aligned}\tag{3.16}$$

Using (3.6) we obtain by the estimates (3.15), (3.16)

$$\begin{aligned}\|\mathcal{T}_m u^m\|^2_{L^2(\Omega)} \le &\frac{hr_2^2|\Omega|}{4^2}+\frac{2h}{4^2}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)}+\|\mathcal{T}_{m-1}u^{m-1}\|^2_{L^2(\Omega)}\\ &+\frac{2h}{4^2}\|u^{m-1}\|^2_{L^2(\Omega)}+\frac{2h}{4^2}\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

That means, we have $\|\mathcal{T}_m u^m\|^2_{L^2(\Omega)} \le \lambda^2$ for all $u^m \in L^2(\Omega)$, if we choose h so that $1-h/8 =: 1/\beta > 0$ and fix radius $\lambda > 0$ by

$$\lambda^2 \equiv \frac{h\beta r_2^2|\Omega|}{4^2}+\frac{2h\beta}{4^2}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)}+\beta\|\mathcal{T}_{m-1}u^{m-1}\|^2_{L^2(\Omega)}+\frac{2h\beta}{4^2}\|u^{m-1}\|^2_{L^2(\Omega)}.$$

Hence, we get $\mathcal{T}_m[\mathcal{B}] \subset \mathcal{B}$ for a closed ball $\mathcal{B} := \{u^m \in L^2(\Omega) : \|u^m\|_{L^2(\Omega)} \le \lambda\}$.
5. To show the continuity of $\mathcal{T}_m$, let $\{u_i^m\}_{i\in\mathbb{N}} \subset L^2(\Omega)$ be a sequence such that $\lim_{i\to\infty}\|u_i^m - u^m\|_{L^2(\Omega)} = 0$. For every $i \in \mathbb{N}$ there exists a uniquely determined solution $\mathcal{T}_m u_i^m \in H^1(\Omega)$ of the problem (3.10)-(3.13). Because $\mathcal{T}_m u^m \in H^1(\Omega)$ is a solution of

the problem (3.10), for every $i \in \mathbb{N}$ it follows

$$\int_\Omega \nabla(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)\cdot\nabla\varphi dx + \\ + \frac{1}{h}\int_\Omega (\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)\varphi dx = \int_\Omega (g_{i,2}-g_2)\varphi dx, \quad \forall\varphi \in H^1(\Omega), \tag{3.17}$$

$$\int_\Omega \nabla(\psi_i^m - \psi^m)\cdot\nabla\phi dx + \frac{h}{\gamma}\int_\Omega (\psi_i^m - \psi^m)\phi dx = \int_\Omega (g_{1,i}-g_1)\phi dx, \quad \forall\phi \in H^1(\Omega), \tag{3.18}$$

where

$$\int_\Omega (g_{i,1}-g_1)\phi dx := \frac{1}{\gamma}\int_\Omega (u_i^m - u^m)\phi dx - \frac{1}{\gamma}\int_\Omega (u_i^{m-1}-u^{m-1})\phi dx \\ + \int_\Omega \nabla(\psi_i^{m-1}-\psi^{m-1})\cdot\nabla\phi dx, \quad \forall\varphi\in H^1(\Omega), \tag{3.19}$$

and

$$\int_\Omega (g_{i,2}-g_2)\varphi dx := \int_\Omega (\mu_\varepsilon(u^m)-\mu_\varepsilon(u_i^m))\nabla\left(\frac{w^m+w^{m-1}}{2}+\psi^m\right)\cdot\nabla\varphi dx \\ + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(w_i^m - w^m)\cdot\nabla\varphi dx \\ + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(w_i^{m-1}-w^{m-1})\cdot\nabla\varphi dx \\ + \int_\Omega \mu_\varepsilon(u_i^m)\nabla(\psi_i^m - \psi^m)\cdot\nabla\varphi dx \\ + \frac{1}{h}\int_\Omega (\mathcal{T}_{m-1}u_i^{m-1} - \mathcal{T}_{m-1}u^{m-1})\varphi dx, \quad \forall\varphi\in H^1(\Omega). \tag{3.20}$$

Using in (3.18) the testfunction $\phi = (\psi_i^m - \psi^m) \in H^2(\Omega)$ we find that

$$\int_\Omega |\nabla(\psi_i^m-\psi^m)|^2 dx + \frac{h}{\gamma}\int_\Omega |\psi_i^m - \psi^m|^2 dx = \int_\Omega (g_{i,1}-g_1)(\psi_i^m-\psi^m)dx.$$

Similar calculations like in (3.15) give

$$\|\nabla(\psi_i^m - \psi^m)\|^2_{L^2(\Omega)} \le \frac{8c_p}{\gamma^2}\|u_i^m - u^m\|^2_{L^2(\Omega)} + \frac{8c_p}{\gamma^2}\|u_i^{m-1}-u^{m-1}\|^2_{L^2(\Omega)} \tag{3.21}$$

$$+ 2\|\nabla(\psi_i^{m-1}-\psi^{m-1})\|^2_{L^2(\Omega)}. \tag{3.22}$$

where c_p is the Poincaré constant.
Applying $\varphi = \mathcal{T}_m u_i^m - \mathcal{T}_m u^m \in H^1(\Omega)$ as a testfunction in (3.17) we get

$$\int_\Omega |\nabla(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m)|^2 dx + \frac{1}{h}\int_\Omega |\mathcal{T}_m u_i^m - \mathcal{T}_m u^m|^2 dx = \int_\Omega (g_{i,2} - g_2)(\mathcal{T}_m u_i^m - \mathcal{T}_m u^m) dx.$$

Young's inequality gives

$$\begin{aligned}\frac{1}{h}\int_\Omega |\mathcal{T}_m u_i^m - \mathcal{T}_m u^m|^2 dx \leq & 4\int_\Omega \left|(\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m))\nabla\left(\frac{w^m + w^{m-1}}{2} + \psi^m\right)\right|^2 dx \\ & + 8\int_\Omega |\mu_\varepsilon(u_i^m)\nabla(w_i^m - w^m)|^2 dx \\ & + 4^2\int_\Omega |\mu_\varepsilon(u_i^m)\nabla(w_i^{m-1} - w^{m-1})|^2 dx \\ & + 4^2\int_\Omega |\mu_\varepsilon(u_i^m)\nabla(\psi_i^m - \psi^m)|^2 dx \\ & =: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Each summand will be analyzed separately: Because of the continuity of Π we get

$$I_2 \leq 8\|w_i^m - w^m\|^2_{H^1(\Omega)} \leq 2\|P(\Pi u_i^m - \Pi u^m)\|^2_{H^1(\Omega)} \leq 2Cr_2^2\|u_i^m - u^m\|^2_{L^2(\Omega)}.$$

The summand I_3 can be trated in a similar way like I_2. For I_4 we use the estimate (3.22). In the limit process $i \longrightarrow \infty$ the expression $(\mu_\varepsilon(u^m) - \mu_\varepsilon(u_i^m))$ tends pointwise to zero, because of the Lipschitz continuity of $u^m \mapsto \mu_\varepsilon(u^m)$ and the convergence $\lim_{i\to\infty}\|u_i^m - u^m\|_{L^2(\Omega)} = 0$. Hence applying Lebesgue's theorem I_1 tends to zero. The convergence of I_2-I_4 follows from the boundedness of $\mu_\varepsilon(u_i^m)$ and the convergence $\lim_{i\to\infty}\|u_i^m - u^m\|_{L^2(\Omega)} = 0$.

6. Bacause of $\mathcal{T}_m[L^2(\Omega)] \in H^1(\Omega)$ and the completely continuous embedding of $H^1(\Omega)$ into $L^2(\Omega)$, the fixed-point map $\mathcal{T}_m : L^2(\Omega) \longrightarrow L^2(\Omega)$ is completely continuous. Having in mind the first step of the proof, Schauder's fixed-point theorem yields a solution $u^m \in H^1(\Omega) \cap \mathcal{B}$ of the equation $\mathcal{T}u^m = u^m$. Setting $w^m = P(1 - \Pi u^m) \in H^1(\Omega)$, we have found a solution $(u^m, w^m, \psi^m) \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ of the problem (3.7)-(3.9). □

In order to prove convergence of the semi-discrete problem (3.7)-(3.9) to the regularized problems (3.3)-(3.5) we need to derive (uniformly in m) a priori estimates. The key estimate is the following energy estimate in its discrete form

Lemma 2. *(Discrete energy estimate) Let (u^m, w^m, ψ^m) be solution of (3.7)-(3.9) for every*

$m \in \{1, ..., M\}$. Then

$$\frac{\gamma}{2}\sum_{m=1}^{M}\|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\max_{1\le k\le M}\|\nabla\psi^k\|^2_{L^2(\Omega)} + \sum_{m=1}^{M} h\|\psi^m\|^2_{L^2(\Omega)} \tag{3.23}$$

$$+\sum_{m=1}^{M} h\int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2 dx \le C_1 \tag{3.24}$$

and

$$\sum_{m=1}^{M} h\int_\Omega |\nabla u^m|^2 dx \le C_2(T). \tag{3.25}$$

Proof. 1. Because of Lemma 1 the testfunction $\varphi = v^m = f'_\varepsilon(u^m) + \frac{w^m + w^{m-1}}{2} + \psi^m \in H^1(\Omega)$ is admissible in (3.7) and we find

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\left(f'_\varepsilon(u^m) + \frac{w^m + w^{m-1}}{2} + \psi^m\right)dx + \int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2 dx = 0.$$

We will estimate the first summand term by term.
2. The first term can be estimated as follows

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})f'_\varepsilon(u^m)dx \ge \frac{1}{h}\int_\Omega f_\varepsilon(u^m) - f_\varepsilon(u^{m-1})dx,$$

where we have used the convexity of $f_\varepsilon(u)$, see (A1).
3. In order to estimate the second term we use the symmetry of P, see Remark 1.

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\left(\frac{w^m + w^{m-1}}{2}\right)dx = \frac{1}{h}\int_\Omega \frac{1}{4}\{(u^m + u^{m-1})(w^m - w^{m-1})$$
$$+ (u^m - u^{m-1})(w^m + w^{m-1})\}dx = \frac{1}{h}\int_\Omega \frac{1}{2}\{u^m w^m - u^{m-1}w^{m-1}\}dx.$$

4. For the third term we use the testfunction $\phi = \psi^m \in H^2(\Omega)$ in (3.8) to get

$$\frac{1}{h}\int_\Omega (u^m - u^{m-1})\psi^m dx = \frac{\gamma}{h}\int_\Omega \nabla(\psi^m - \psi^{m-1})\cdot\nabla\psi^m dx + \int_\Omega |\psi^m|^2 dx.$$

The above estimates give

$$\frac{\gamma}{2h}\|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2h}\|\nabla\psi^m\|^2_{L^2(\Omega)} - \frac{\gamma}{2h}\|\nabla\psi^{m-1}\|^2_{L^2(\Omega)} + \|\psi^m\|^2_{L^2(\Omega)}$$
$$+\int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2 dx + \frac{1}{h}\left[F_{NL,\varepsilon}(u^m) - F_{NL,\varepsilon}(u^{m-1})\right] \le 0. \tag{3.26}$$

We multiply (3.26) by h and sum both sides from $m=1$ to $m=k$, where $1 \le k \le M$. Using Remark 6 we conclude

$$\frac{\gamma}{2}\sum_{m=1}^{M}\|\nabla(\psi^m-\psi^{m-1})\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\max_{1\le k\le M}\|\nabla\psi^k\|^2_{L^2(\Omega)}+\sum_{m=1}^{M}h\|\psi^m\|^2_{L^2(\Omega)}$$
$$+\sum_{m=1}^{M}h\int_\Omega \mu_\varepsilon(u^m)|\nabla v^m|^2dx \le C_1,$$

where $C_1 := F_{NL}(u_0) - F_{NL,\varepsilon}(u^M) + \frac{\gamma}{2}\|\nabla\psi^0\|^2_{L^2(\Omega)}$.
Defining $\tilde{w}^m := \frac{w^m+w^{m-1}}{2} + \psi^m$ we have the following estimate

$$\begin{aligned}\int_\Omega \mu_\varepsilon(u^m)|\nabla(f'_\varepsilon(u^m)+\tilde{w}^m)|^2dx &= \int_\Omega \left(f''_\varepsilon(u^m)|\nabla u^m|^2 + 2\nabla u^m\cdot\nabla\tilde{w}^m + \frac{|\nabla\tilde{w}^m|^2}{f''_\varepsilon(u^m)}\right)dx\\ &\ge \int_\Omega \left(\frac{f''_\varepsilon(u^m)}{2}|\nabla u^m|^2 - \frac{|\nabla\tilde{w}^m|^2}{f''_\varepsilon(u^m)}\right)dx \qquad (3.27)\\ &\ge \int_\Omega \left(2|\nabla u^m|^2 - \frac{1}{4}|\nabla\tilde{w}^m|^2\right)dx,\end{aligned}$$

where we have used Young's inequality and the fact that $f''_\varepsilon(u^m) \ge 4$. We multiply (3.27) by h and sum both sides from $m=1$ to $m=k$, where $1\le k\le M$, to get

$$C_1 \ge \sum_{m=1}^{k}h\int_\Omega\left(2|\nabla u^m|^2-\frac{1}{4}|\nabla\tilde{w}^m|^2\right)dx,$$

where C_1 is the constant in (3.23). The definition of $\tilde{w}^m$, (A2), (B4) and (3.23), (3.6) give

$$\sum_{m=1}^{M}h\int_\Omega|\nabla u^m|^2dx \le C_2(T),$$

where $C_2(T) := T\left\{r_2^2|\Omega| + 4r_2^2 + \frac{1}{4}\max_{1\le k\le M}\|\nabla\psi^k\|^2_{L^2(\Omega)}\right\} + \frac{C_1}{2}$. □

Lemma 3. *Let (u^m, w^m, ψ^m) be solution of (3.7)-(3.9) for every $m \in \{1,...,M\}$. Then*

$$\max_{1\le k\le M}\|\Delta\psi^k\|^2_{L^2(\Omega)} \le C_3. \qquad (3.28)$$

Proof. 1. Because of Lemma 1 $\Delta\psi^m \in L^2(\Omega)$, thus an admissible testfunction in (3.8)

$$\frac{\gamma}{h}\int_\Omega\Delta(\psi^m-\psi^{m-1})\Delta\psi^m dx + \int_\Omega|\nabla\psi^m|^2dx + \frac{1}{h}\int_\Omega(u^m-u^{m-1})\Delta\psi^m dx = 0.$$

2. Applying the testfunction $-u^m/\gamma$ in (3.8) we have

$$\frac{1}{h}\int_\Omega \Delta(\psi^m - \psi^{m-1})u^m dx - \frac{1}{\gamma}\int_\Omega \psi^m u^m dx + \frac{1}{\gamma h}\int_\Omega (u^m - u^{m-1})u^m dx = 0.$$

3. We use the identity

$$\int_\Omega \left((u^m - u^{m-1})\Delta\psi^m + (\Delta\psi^m - \Delta\psi^{m-1})u^m\right) dx$$
$$= \int_\Omega u^m \Delta\psi^m dx - \int_\Omega u^{m-1}\Delta\psi^{m-1} dx + \int_\Omega \left((u^m - u^{m-1})(\Delta\psi^m - \Delta\psi^{m-1})\right) dx,$$

and Young's inequality in the following way

$$\frac{2}{\gamma}\int_\Omega (u^m - u^{m-1})(\Delta\psi^m - \Delta\psi^{m-1})dx \leq \frac{1}{\gamma^2}\|u^m - u^{m-1}\|^2_{L^2(\Omega)} + \|\Delta\psi^m - \Delta\psi^{m-1}\|^2_{L^2(\Omega)},$$
$$\frac{2}{\gamma}\int_\Omega u^m \Delta\psi^m dx \leq \frac{2}{\gamma^2}\|u^m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta\psi^m\|^2_{L^2(\Omega)},$$

and get

$$\begin{aligned}\|\Delta\psi^k\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\sum_{m=1}^k h\|\nabla\psi^m\|^2_{L^2(\Omega)} =& 3\|\Delta\psi^0\|^2_{L^2(\Omega)} + \frac{6}{\gamma^2}\|u^0\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\|u^k\|^2_{L^2(\Omega)} \\ &+ \frac{1}{\gamma^3}\sum_{m=1}^k h\|\nabla u^m\|^2_{L^2(\Omega)} + \frac{2}{\gamma^2}\sum_{m=1}^k h\|\psi^m\|^2_{L^2(\Omega)} \\ &+ \frac{2}{\gamma^2}\sum_{m=1}^k h\|u^m\|^2_{L^2(\Omega)}.\end{aligned}$$

where we have summed both sides from $m = 1$ to $m = k$, $(1 \leq k \leq M)$. The energy estimate (3.23) and (3.25) give (3.28). □

To indicate the dependence on M, we denote for any $M \in \mathbb{N}$ the solutions of (3.7)-(3.9) by (u^m_M, w^m_M, ψ^m_M). We define the piecewise linear

$$\hat{u}_M(x,t) = u^m + \frac{t - mh}{h}(u^m - u^{m-1}) \text{ for } t \in [(m-1)h, mh], \tag{3.29}$$
$$\hat{\psi}_M(x,t) = \psi^m + \frac{t - mh}{h}(\psi^m - \psi^{m-1}) \text{ for } t \in [(m-1)h, mh], \tag{3.30}$$

as well as the constant interpolates

$$\check{u}_M(x,t) = u^m \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.31}$$
$$\check{w}_M(x,t) = \frac{w^m + w^{m-1}}{2} \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.32}$$
$$\check{\psi}_M(x,t) = \psi^m \quad \text{for} \quad t \in [(m-1)h, mh], \tag{3.33}$$

for $1 \le m \le M$. With these notations we obtain

$$\int_0^T \int_\Omega \hat{u}_{M,t}\varphi \, dxdt +$$
$$+ \int_0^T \int_\Omega (\nabla \check{u}_M + \mu_\varepsilon(\check{u}_M)\nabla(\check{w}_M + \check{\psi}_M)) \cdot \nabla\varphi \, dxdt = 0, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)) \tag{3.34}$$
$$\gamma \int_0^T \int_\Omega \nabla\hat{\psi}_{M,t} \cdot \nabla\phi dx\, dt + \int_0^T \int_\Omega \check{\psi}_M \phi \, dxdt = \int_0^T \int_\Omega \hat{u}_{M,t}\phi dxdt, \quad \forall\phi \in L^2(0,T;H^1(\Omega)).$$

We again get like in Remark 3 using $\varphi = 1$ and $\phi = 1$ in (3.34)

$$\overline{\check{u}(t)} = \overline{u_0}, \quad \int_0^T \int_\Omega \check{\psi}(x) dxdt = 0. \tag{3.35}$$

By virtue of the energy estimate (3.23),

$$\frac{\gamma}{2}\sum_{m=1}^{M} \|\nabla(\psi^m - \psi^{m-1})\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\sup_{0\le t\le T} \|\nabla\check{\psi}_M(t)\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega |\check{\psi}_M|^2 dxdt$$
$$+ \int_0^T \int_\Omega \mu_\varepsilon(\check{u}_M)|\nabla\check{v}_M|^2 \, dxdt \le C_1, \tag{3.36}$$

where $\check{v}_M := f'_\varepsilon(\check{u}_M) + \check{w}_M + \check{\psi}_M$. Using (3.35) and the generalized Poincaré inequality we find from (3.25) that

$$\|\check{u}_M\|_{L^2(0,T;H^1(\Omega))} \le C_4(\sqrt{T}).$$

Moreover we find from (3.34) and (3.36)

$$\left| \int_0^T \int_\Omega \hat{u}_{M,t}\varphi dxdt \right| \le \left| \int_0^T \int_\Omega \mu_\varepsilon(\check{u}_M)\nabla\check{v}_M \cdot \nabla\varphi \, dxdt \right|$$
$$\le \frac{1}{2}\left(\int_0^T \int_\Omega \mu_\varepsilon(\check{u}_M)\, |\nabla\check{v}_M|^2 \, dxdt \right)^{1/2} \left(\int_0^T \int_\Omega |\nabla\varphi|^2 dxdt \right)^{1/2} \tag{3.37}$$
$$\le \frac{C_1}{2}\left(\int_0^T \int_\Omega |\nabla\varphi|^2 dxdt \right)^{1/2}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$. We get

$$\|\hat{u}_{M,t}\|_{L^2(0,T;H^1(\Omega)^*)} = \sup_{\varphi\in L^2(0,T;H^1(\Omega))} \frac{|\int_0^T \int_\Omega \hat{u}_{M,t}(x,t)\varphi dxdt|}{\|\varphi\|_{L^2(0,T;H^1(\Omega))}} \le C_5.$$

Thus, we find

$$\begin{aligned} \left|\int_0^T\int_\Omega \nabla\hat{\psi}_{M,t}\cdot\nabla\phi dxdt\right| \le & \left|\int_0^T\int_\Omega \hat{u}_{M,t}\phi dxdt\right| \\ & + \left(\int_0^T\int_\Omega |\check{\psi}_M|^2 dxdt\right)^{1/2}\left(\int_0^T\int_\Omega |\phi|^2 dxdt\right)^{1/2} \\ \le & C_6\|\phi\|^2_{L^2(0,T;H^1(\Omega))}, \end{aligned} \tag{3.38}$$

and we find

$$\|\nabla\hat{\psi}_{M,t}\|_{L^2(0,T;H^1(\Omega)^*)} \le C_6, \quad \sup_{0\le t\le T}\|\Delta\check{\psi}_M(t)\|^2_{L^2(\Omega)} \le C_3(T).$$

In addition, (3.25), (3.29) and (3.31) imply that

$$\|\nabla\check{u}_M - \nabla\hat{u}_M\|^2_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^M \|\nabla u_M^m - \nabla u_M^{m-1}\|^2_{L^2(\Omega)} \to 0 \text{ as } M\to\infty.$$

We also get from (3.29), (3.31) and Remark 7 that

$$\|\check{u}_M - \hat{u}_M\|^2_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^M \|u_M^m - u_M^{m-1}\|^2_{L^2(\Omega)} \to 0 \text{ as } M\to\infty.$$

We obtain using the generalized Poincaré inequality the following convergence

$$\|\check{u}_M - \hat{u}_M\|_{L^2(0,T;H^1(\Omega))} \to 0 \text{ as } M\to\infty. \tag{3.39}$$

Moreover we have with Remark 7

$$\|\check{\psi}_M - \hat{\psi}_M\|^2_{L^2(0,T;L^2(\Omega))} = \frac{T}{3M}\sum_{m=1}^M \|\psi_M^m - \psi_M^{m-1}\|^2_{L^2(\Omega)} = 0,$$

and by (3.30), (3.33) and (3.23), as $M\to\infty$

$$\|\nabla\check{\psi}_M - \nabla\hat{\psi}_M\|_{L^\infty(0,T;L^2(\Omega))} = \max_{0\le t\le T}\|\nabla\psi_M^m - \nabla\psi_M^{m-1}\|_{L^2(\Omega)} \to 0. \tag{3.40}$$

In conclusion, there are functions $\check{u}, \hat{u}_t, \check{\psi}, \hat{\psi}_t$, such that for $M \to \infty$, possibly after selecting subsequences,

$$\begin{array}{llllll}
\check{u}_M & \longrightarrow & \check{u} & \text{weakly} & \text{in} & L^2(0,T;H^1(\Omega)), \\
\hat{u}_{M,t} & \longrightarrow & \hat{u}_t & \text{weakly} & \text{in} & L^2(0,T;H^1(\Omega)^*), \\
\check{\psi}_M & \longrightarrow & \check{\psi} & \text{weakly} & \text{in} & L^2(0,T;L^2(\Omega)), \\
\nabla\check{\psi}_M & \longrightarrow & \nabla\check{\psi} & \text{weakly-star} & \text{in} & L^\infty(0,T;H^1(\Omega)), \\
\nabla\hat{\psi}_{M,t} & \longrightarrow & \nabla\hat{\psi}_t & \text{weakly} & \text{in} & L^2(0,T;H^1(\Omega)^*).
\end{array} \tag{3.41}$$

Taking (3.39) and (3.40) into account, we see that $\check{u} = \hat{u}$ and $\check{\psi} = \hat{\psi}$. It follows from (3.41) that we may pass to the limit as $M \to \infty$ in (3.34). The convergence of the linear terms in (3.34) are standard. We take a closer look on the convergence of the nonlinear term

$$\begin{aligned}
\int_0^T \int_\Omega (\mu_\varepsilon(u)\nabla(w+\psi) - \mu_\varepsilon(u_M)\nabla(w_M+\psi_M)) \cdot \nabla\varphi \, dxdt \\
= \int_0^T \int_\Omega (\mu_\varepsilon(u) - \mu_\varepsilon(u_M))\nabla(w+\psi) \cdot \nabla\varphi \, dxdt \qquad (3.42) \\
+ \int_0^T \int_\Omega \mu_\varepsilon(u_M)\nabla\left[(w-w_M)+(\psi-\psi_M)\right] \cdot \nabla\varphi \, dxdt.
\end{aligned}$$

Because of the Lipschitz continuity of μ_ε and the compactness results in (3.41) the first term on the right hand side converges to zero. The second term converges to zero again by taking into account (3.41). Now we have proved the existence of solutions to the *regularized problems* (3.3)-(3.5).

3.1.2 Existence result for the original problem

Our aim is now to show the existence for the original problem (2.6)-(2.8) by showing the convergence of $\varepsilon \to 0$. To do this we need a priori estimates uniformly in the regularization parameter ε. Our starting point will again be the energy estimate.
We denote the solutions of the regularized problem by $(u_\varepsilon, w_\varepsilon, \psi_\varepsilon)$.

Lemma 4. *(energy estimate) There exists an ε_0, see Remark 6, such that for all $0 < \varepsilon \le \varepsilon_0$ the following estimate holds with constants C_7, C_8 independent of ε:*

$$\frac{\gamma}{2} \max_{0\le t\le T} \int_\Omega |\nabla\psi_\varepsilon(t)|^2 dx + \int_0^T \int_\Omega |\psi_\varepsilon|^2 dxdt + \int_0^T \int_\Omega \mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 dxdt \le C_7, \tag{3.43}$$

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 dxdt \le C_8. \tag{3.44}$$

Proof. 1. The function $v_\varepsilon = f'_\varepsilon(u_\varepsilon) + w_\varepsilon + \psi_\varepsilon \in L^2(0,T;H^1(\Omega))$ is a valid testfunction in (3.3). Therefore we obtain

$$\int_0^t \langle \partial_t u_\varepsilon, f'_\varepsilon(u_\varepsilon) + w_\varepsilon + \psi_\varepsilon \rangle dt = -\int_0^t \int_\Omega \mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 dxdt \tag{3.45}$$

for almost all $t \in [0,T]$. To prove this we define *steklov averaged* functions

$$u_{\varepsilon h}(t,x) := \frac{1}{h}\int_{t-h}^t u_\varepsilon(\tau,x)d\tau, \tag{3.46}$$

where we set $u_\varepsilon(t,x) = u_0(x)$ when $t \le 0$. From [18] it follows that $u_{\varepsilon h}$ converge strongly to u_ε in $L^2(0,T;H^1(\Omega))$. Because of (A2), (B3) and the continuity of f'_ε it is easily proven that

$$\begin{array}{lllll} w_{\varepsilon h} & \longrightarrow & w_\varepsilon & \text{strongly} & \text{in} \quad L^2(0,T;H^1(\Omega)), \\ f'_{\varepsilon h}(u_{\varepsilon h}) & \longrightarrow & f'_\varepsilon(u_\varepsilon) & \text{strongly} & \text{in} \quad L^2(0,T;H^1(\Omega)). \end{array} \tag{3.47}$$

We define $g_{\varepsilon h} := f'_{\varepsilon h}(u_{\varepsilon h}) + w_{\varepsilon h}$, and $v_{\varepsilon h} := g_{\varepsilon h} + \psi_{\varepsilon h}$. By Lemma 5 we have

$$\nabla\psi_{\varepsilon h} \longrightarrow \nabla\psi_\varepsilon \text{ strongly} \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{3.48}$$

Furthermore, we can show $\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon$ strongly in $L^2(0,T;H^1(\Omega)^*)$. For any $\varphi \in L^2(0,T;H^1(\Omega))$ we have

$$\begin{aligned} |\langle \partial_t u_{\varepsilon h} - \partial_t u_\varepsilon, \varphi\rangle| &= \frac{1}{h}\left|\int_0^T \left\langle \int_{t-h}^t (\partial_t u_\varepsilon(\tau) - \partial_t u_\varepsilon(t))d\tau, \varphi \right\rangle dt\right| \\ &= \frac{1}{h}\left|\int_0^T \left\langle \int_{-h}^0 (\partial_t u_\varepsilon(t+s) - \partial_t u_\varepsilon(t))ds, \varphi \right\rangle dt\right| \\ &\le \frac{1}{h}\int_{-h}^0 \left|\int_0^T \int_\Omega (\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon - \mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon)\,\nabla\varphi dxdt\right| ds \\ &\le \max_{-h\le s\le 0} \|(\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s) - \mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)}\,\|\nabla\varphi\|_{L^2(Q_T)}. \end{aligned}$$

We have

$$\begin{aligned} &\max_{-h\le s\le 0}\|\mu_\varepsilon(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s) - \mu_\varepsilon(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)} \\ &\qquad \le \max_{-h\le s\le 0}\|[\mu_\varepsilon(u_\varepsilon(t+s)) - \mu_\varepsilon(u_\varepsilon(t))]\nabla v_\varepsilon(t+s)\|_{L^2(Q_T)} \\ &\qquad\quad + C\max_{-h\le s\le 0}\|g_\varepsilon(t+s) - g_\varepsilon(t)\|_{L^2(0,T;H^1(\Omega))} \\ &\qquad\quad + C\max_{-h\le s\le 0}\|\nabla\psi_\varepsilon(t+s) - \nabla\psi_\varepsilon(t)\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

The first part of the right hand side tends as $h \to 0$ pointwise to zero, because of the Lipschitz continuity of $u_\varepsilon \mapsto \mu_\varepsilon(u_\varepsilon)$ and the convergence

$$\max_{-h\le s\le 0} \|u_\varepsilon(t+s) - u_\varepsilon(t)\|_{L^2(Q_T)} \to 0 \quad \text{as} \quad h \to 0.$$

The second and the third part follow from (3.47) and (3.48). It follows that

$$\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon \quad \text{strongly in } L^2(0,T;H^1(\Omega)^*).$$

Using $\partial_t u_{\varepsilon h} \in L^2(0,T;L^2(\Omega))$, we have for almost all $t \in [0,T]$

$$\begin{aligned}
\int_0^t \langle \partial_t u_{\varepsilon h}, f_\varepsilon'(u_{\varepsilon h}) + w_{\varepsilon h} + \psi_{\varepsilon h}\rangle dt &= \int_0^t \int_\Omega \partial_t u_{\varepsilon h}(f_\varepsilon'(u_{\varepsilon h}) + w_{\varepsilon h} + \psi_{\varepsilon h}) dxdt \\
&= \partial_t \int_0^t \int_\Omega \left(f_\varepsilon(u_{\varepsilon h}) + \frac{1}{2} u_{\varepsilon h} w_{\varepsilon h} + \frac{1}{2}|\nabla \psi_{\varepsilon h}|^2 \right) dxdt \\
&\quad + \int_0^t \int_\Omega |\psi_{\varepsilon h}|^2 dxdt \\
&= \int_\Omega \left(f_\varepsilon(u_{\varepsilon h}(t)) + \frac{1}{2} u_{\varepsilon h}(t) w_{\varepsilon h}(t) + \frac{1}{2}|\nabla \psi_{\varepsilon h}(t)|^2 \right) dxdt \\
&+ \int_\Omega \left(f_\varepsilon(u_0) + \frac{1}{2} u_0 w_0 + \frac{1}{2}|\nabla \psi_0|^2 \right) dxdt + \int_0^t \int_\Omega |\psi_{\varepsilon h}|^2 dxdt.
\end{aligned}$$

Passing to the limit ($h \searrow 0$) in this equation, where we apply the convergence properties of $u_{\varepsilon h}$ proved above, and using Remark 6, (3.45) we obtain for almost all t

$$\begin{aligned}
\int_\Omega \frac{1}{2}|\nabla \psi_{\varepsilon h}(t)|^2 + \int_0^t \int_\Omega |\psi_{\varepsilon h}|^2 dxdt + \int_0^t \int_\Omega \mu_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 dxdt \\
\le F_{NL,\varepsilon}(u_0) + \int_\Omega \frac{1}{2}|\nabla \psi_0|^2 dxdt \le C_7.
\end{aligned}$$

The proof of (3.44) is similar to the proof in the discrete case (3.25). □

We get further a priori estimates for $\partial_t u_\varepsilon$ and $\nabla \partial_t \psi_\varepsilon$ in a similar way to the discrete case (3.37) and (3.38). Moreover we have

Lemma 5. *There exists an ε_0, see Remark 6, such that for all $0 < \varepsilon \le \varepsilon_0$ the following estimate holds with a constant C_{11} independent of ε:*

$$\max_{0\le t\le T} \|\Delta \psi_\varepsilon\|^2_{L^2(\Omega)} \le C_{11}. \tag{3.49}$$

Proof. 1. We again make use of (3.46). We apply the admissible testfunction $-\Delta\psi_{\varepsilon h} \in L^2(\Omega)$ in (3.4) and get

$$\gamma \int_0^t \frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\psi_{\varepsilon h}|^2 dxdt + \int_0^t \int_\Omega |\nabla\psi_{\varepsilon h}|^2 dxdt + \int_0^t \int_\Omega \partial_t u_{\varepsilon h}\Delta\psi_{\varepsilon h} dxdt = 0,$$

for almost all $t \in [0,T]$.
2. We obtain by using $-u_{\varepsilon h}/\gamma$ as a testfunction in (3.4)

$$-\int_0^t \int_\Omega \partial_t \nabla\psi_{\varepsilon h} \cdot \nabla u_{\varepsilon h} dxdt - \frac{1}{\gamma}\int_0^t \int_\Omega \psi_{\varepsilon h} u_{\varepsilon h} dxdt + \frac{1}{\gamma}\int_0^t \frac{1}{2}\frac{d}{dt}\int_\Omega |u_{\varepsilon h}|^2 dxdt = 0$$

for almost all $t \in [0,T]$. We find after standard calculations similar to the discrete case by passing to the limit ($h \searrow 0$)

$$\begin{aligned} \max_{0\le t\le T} \|\Delta\psi_\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^T \|\nabla\psi_\varepsilon\|^2_{L^2(\Omega)} dt =& 3\|\Delta\psi_\varepsilon(0)\|^2_{L^2(\Omega)} + \frac{8}{\gamma^2}\|u_\varepsilon(0)\|^2_{L^2(\Omega)} \\ &+ \frac{2}{\gamma^2}\int_0^T \|\psi_\varepsilon\|^2_{L^2(\Omega)} dt + \frac{2c_p}{\gamma^2}\int_0^T \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} dt. \end{aligned}$$

By (3.43) and (3.44) we obtain (3.49). Thus we have as $\varepsilon \to 0$

$$\begin{array}{lllll} u_\varepsilon & \longrightarrow & u & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)), \\ \partial_t u_\varepsilon & \longrightarrow & \partial_t u & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*), \\ \psi_\varepsilon & \longrightarrow & \psi & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\ \nabla\psi_\varepsilon & \longrightarrow & \nabla\psi & \text{weakly-star} & \text{in } L^\infty(0,T;H^1(\Omega)), \\ \nabla\partial_t\psi_\varepsilon & \longrightarrow & \nabla\psi_t & \text{weakly} & \text{in } L^2(0,T;H^1(\Omega)^*), \end{array} \tag{3.50}$$

and by (A2) $w_\varepsilon \longrightarrow w$ in $L^2(0,T;H^1(\Omega))$, so that as $\varepsilon \longrightarrow 0$ we may pass to the limit in (3.3)-(3.5). The convergence of the linear terms in (3.3)-(3.5) are standard. We take a closer look on the convergence of the nonlinear term

$$\begin{aligned} &\int_0^T \int_\Omega (\mu(u)\nabla(w+\psi) - \mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon+\psi_\varepsilon)) \cdot \nabla\varphi \, dxdt \\ &\qquad = \int_0^T \int_\Omega (\mu(u) - \mu_\varepsilon(u_\varepsilon))\nabla(w+\psi) \cdot \nabla\varphi \, dxdt \\ &\qquad + \int_0^T \int_\Omega \mu_\varepsilon(u_\varepsilon)\nabla\left[(w-w_\varepsilon) + (\psi-\psi_\varepsilon)\right] \cdot \nabla\varphi \, dxdt. \end{aligned} \tag{3.51}$$

Using the fact that for all $z \in \mathbb{R}$

$$|\mu(z) - \mu_\varepsilon(z)| \leq \sup_{\substack{0\leq z\leq \varepsilon \\ 1-\varepsilon\leq z\leq 1}} |\mu(z)| \longrightarrow 0, \quad \text{as} \quad \varepsilon \longrightarrow 0,$$

it follows that $\mu_\varepsilon \longrightarrow \mu$ uniformly and the first term on the right hand side tends to zero. The convergence of the second term is a standard consequence of the compactness result (3.50). □

Now we have shown that the problem (2.6), (2.7) and (3.5) has a solution. The next step is to overcome the truncation in (3.5) and to show that that the solution to the truncated problem is also a solution to thze original problem. The rest of the proof is formulated as

Proposition 1. *Let (u, w, ψ) be solution of the problem (2.6)-(2.8) then*

$$0 \leq u(t, x) \leq 1, \qquad a.e. \ in \ Q_T. \tag{3.52}$$

Proof. Using in (2.6) the admissible testfunctions $u^\bullet := \min(u, 0)$ and $u^\diamond := \min(1 - u, 0)$ we get

$$\frac{1}{2}\int_\Omega |u^\circ(t)|^2 dx + \int_0^T\int_\Omega |\nabla u^\circ|^2 dxdt + \int_0^T\int_\Omega \mu(u)\nabla(w + \psi)\cdot\nabla u^\circ dxdt = 0.$$

where $\circ \in \{\bullet, \diamond\}$. Because of $\mu(u)\nabla u^\circ = 0$ for $\circ \in \{\bullet, \diamond\}$ the last term vanishes and we get

$$0 = \frac{1}{2}\int_\Omega |u^\circ(t)|^2 dx + \int_0^T\int_\Omega |\nabla u^\circ|^2 dxdt \geq \frac{1}{2}\int_\Omega |u^\circ(t)|^2 dx,$$

that means $u^\circ(t, x) = 0$ a.e. in Q_T, hence $1 \geq u(t, x) \geq 0$ a.e. in Q_T. □
Hence, by Proposition 1 we have $u \in L^\infty(Q_T)$ and (A2) provides $w \in \mathcal{V}^{1,\infty}(0, T)$.

3.2 Uniqueness

Let (u_i, w_i, ψ_i), $i \in \{1, 2\}$ be solutions to (2.6)-(2.8). We define $u := u_1 - u_2, w := w_1 - w_2$ and $\psi := \psi_1 - \psi_2$. Our aim is to derive a Gronwall type inequality for u, w and ψ to prove for the uniqueness. Now (u, w, ψ) fulfill

$$\int_0^T \langle u_t, \varphi\rangle\, dt + \int_0^T\int_\Omega \nabla u\cdot\nabla\varphi dxdt \tag{3.53}$$

$$+ \int_0^T\int_\Omega (\mu(u_1)\nabla(w_1 + \psi_1) - \mu(u_2)\nabla(w_2 + \psi_2))\cdot\nabla\varphi dxdt = 0, \forall\varphi \in L^2(0, T; H^1(\Omega)),$$

$$\gamma\int_0^T \langle\nabla\psi_t, \nabla\phi\rangle\, dt + \int_0^T\int_\Omega \psi\phi dxdt = \int_0^T \langle u_t, \phi\rangle\, dt, \quad \forall\phi \in L^2(0, T; H^1(\Omega)), \tag{3.54}$$

$$w = P(-2u) \text{ a.e. in } Q_T. \tag{3.55}$$

Testing (3.53) by u and ψ and (3.54) by ψ we find

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u\|^2_{L^2(\Omega)} dt &= -\int_0^t \int_\Omega (\mu(u_1) - \mu(u_2))\nabla(w_1 + \psi_1) \cdot \nabla u \, dxdt \\ &\quad - \int_0^t \int_\Omega \mu(u_2)\nabla w \cdot \nabla u \, dx - \int_0^t \int_\Omega \mu(u_2)\nabla\psi \cdot \nabla u \, dxdt \\ &=: \int_0^t I_5 dt + \int_0^t I_6 dt + \int_0^t I_7 dt.\end{aligned}$$

and

$$\begin{aligned}&\frac{\gamma}{2}\|\nabla\psi(t)\|^2_{L^2(\Omega)} + \int_0^t \|\psi\|^2_{L^2(\Omega)} dt + \int_0^t \int_\Omega \mu(u_2)|\nabla\psi|^2 dxdt \\ &\qquad = -\int_0^t \int_\Omega (\mu(u_1) - \mu(u_2))\nabla(w_1 + \psi_1) \cdot \nabla\psi \, dx \\ &\qquad\quad - \int_0^t \int_\Omega \nabla u \cdot \nabla\psi \, dx - \int_0^t \int_\Omega \mu(u_2)\nabla w \cdot \nabla\psi \, dx \\ &\qquad =: \int_0^t I_8 dt + \int_0^t I_9 dt + \int_0^t I_{10} dt.\end{aligned}$$

We will estimate the $I_i, i \in \{1, ..., 6\}$ separately. Because of the Lipschitz continuity of μ we get by using Hölder's inequality

$$\begin{aligned}|I_5| &\leq C_{12} \int_\Omega |u| \, |\nabla w_1| \, |\nabla u| \, dx + C_{12} \int_\Omega |u| \, |\nabla\psi_1| \, |\nabla u| \, dx \\ &\leq C_{12}\|u\|_{L^2(\Omega)} \, \|\nabla w_1\|_{L^\infty(\Omega)} \, \|\nabla u\|_{L^2(\Omega)} + C_{12}\|u\|_{L^3(\Omega)} \, \|\nabla\psi_1\|_{L^6(\Omega)} \, \|\nabla u\|_{L^2(\Omega)},\end{aligned} \tag{3.56}$$

where C_{12} is the Lipschitz constant. For the first term on the right hand side we get using (A2) and (B3)

$$\begin{aligned}C_{12}\|u\|_{L^2(\Omega)} \, \|\nabla w_1\|_{L^\infty(\Omega)} \, \|\nabla u\|_{L^2(\Omega)} &\leq C_{13}\|u\|_{L^2(\Omega)} \, \|\nabla u\|_{L^2(\Omega)} \\ &\leq \frac{C_{13}}{2\epsilon_1}\|u\|^2_{L^2(\Omega)} + \frac{C_{13}\epsilon_1}{2}\|\nabla u\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used for the last operation Young's inequality. The Gagliardo-Nirenberg inequality for $\dim(\Omega) = 3$ and Sobolev embedding theorem give for the second term in (3.56)

$$\begin{aligned} C_{12}\|u\|_{L^3(\Omega)}\,\|\nabla\psi_1\|_{L^6(\Omega)}\,\|\nabla u\|_2 &\le C_{14}\|u\|^{1/2}_{L^2(\Omega)}\,\|\nabla\psi_1\|_{H^2(\Omega)}\,\|\nabla u\|^{3/2}_{L^2(\Omega)} \\ &\le \frac{C_{15}}{4\epsilon_2}\|u\|^2_{L^2(\Omega)} + \frac{3C_{15}\epsilon_2}{4}\|\nabla u\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used Young's inequality for the last row. Finally we find

$$|I_5| \le \left(\frac{C_{13}}{2\epsilon_1} + \frac{C_{15}}{4\epsilon_2}\right)\|u\|^2_{L^2(\Omega)} + \left(\frac{C_{13}}{2}\epsilon_1 + \frac{3C_{15}\epsilon_2}{4}\right)\|\nabla u\|^2_{L^2(\Omega)}.$$

Young's inequality and (3.55) together with (A2) and (B3) give

$$|I_6| \le r_2^2\|u\|^2_{L^2(\Omega)} + \frac{1}{4^2}\|\nabla u\|^2_{L^2(\Omega)},$$

$$|I_7| \le \|\nabla\psi\|^2_{L^2(\Omega)} + \frac{1}{8^2}\|\nabla u\|^2_{L^2(\Omega)}.$$

Furthermore we get

$$\begin{aligned} |I_8| &\le C_{16}\int_\Omega |u|\,|\nabla w_1|\,|\nabla\psi|\,dx + C_{16}\int_\Omega |u|\,|\nabla\psi_1|\,|\nabla\psi|\,dx \\ &\le C_{16}\|u\|_{L^2(\Omega)}\|\nabla w_1\|_{L^\infty(\Omega)}\|\nabla\psi\|_{L^2(\Omega)} + C_{16}\|u\|_{L^3(\Omega)}\|\nabla\psi_1\|_{L^6(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}. \end{aligned}$$

where C_{16} is the Lipschitz constant. For the first term we get by using (A2), (B3) and Young's inequality

$$C_{16}\|u\|_2\|\nabla w_1\|_{L^\infty(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)} \le \frac{C_{17}}{2}\|u\|^2_{L^2(\Omega)} + \frac{C_{17}}{2}\|\nabla\psi\|^2_{L^2(\Omega)}.$$

The Gagliardo-Nirenberg inequality for $\dim(\Omega) = 3$ and Sobolev embedding theorem give for the second term

$$\begin{aligned} C_{16}\|u\|_{L^3(\Omega)}\,\|\nabla\psi_1\|_{L^6(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)} &\le C_{18}\|u\|^{1/2}_{L^2(\Omega)}\,\|\nabla u\|^{1/2}_{L^2(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)} \\ &\le \frac{C_{18}}{4\epsilon_3}\|u\|^2_{L^2(\Omega)} + \frac{C_{18}\epsilon_3}{4}\|\nabla u\|^2_{L^2(\Omega)} + \frac{C_{18}}{2}\|\nabla\psi\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have applied Young's inequality for the last estimates. Thus, we get

$$|I_8| \le \left(\frac{C_{17}}{2} + \frac{C_{18}}{4\epsilon_3}\right)\|u\|^2_{L^2(\Omega)} + \frac{C_{18}\epsilon_3}{4}\|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{C_{17}}{2} + \frac{C_{18}}{2}\right)\|\nabla\psi\|^2_{L^2(\Omega)}. \quad (3.57)$$

Using Hölder's inequality and (A2) and (B3) we get

$$|I_9| \leq 2\|\nabla\psi\|^2_{L^2(\Omega)} + \frac{1}{8}\|\nabla u\|^2_{L^2(\Omega)},$$
$$|I_{10}| \leq \frac{r_2^2}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{8}\|\nabla\psi\|^2_{L^2(\Omega)},$$

Finally we conclude

$$\frac{1}{2}\left[\|u(t)\|^2_{L^2(\Omega)} + \gamma\|\nabla\psi(t)\|^2_{L^2(\Omega)}\right] + \nu(\epsilon_1,\epsilon_2,\epsilon_3)\int_0^t \|\nabla u\|^2_{L^2(\Omega)}dt + \int_0^t \|\psi\|^2_{L^2(\Omega)}dt$$
$$+ \|\psi\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \mu(u_2)|\nabla\psi|^2 dxdt \leq C_{19}(\epsilon_1,\epsilon_2,\epsilon_3)\int_0^t \|u\|^2_{L^2(\Omega)}dt + C_{20}(\epsilon_1,\epsilon_2,\epsilon_3)\int_0^t \|\nabla\psi\|^2_{L^2(\Omega)}dt,$$

where $\nu(\epsilon_1,\epsilon_2,\epsilon_3) := 1 - C_{13}\epsilon_1/2 - 3C_{15}\epsilon_2/4 - C_{18}\epsilon_3/4 - 13/4^3$. Choose $\epsilon_i, i = 1,2,3$, so that $\nu > 0$ Gronwall's Lemma gives the uniqueness.

4 Proof of Theorem 2

4.1 Existence

Unlike the proof of Theorem 1 we here will not only apply the regularization and truncation (3.1) and (3.2), but also we will use a *biharmonic regularization* of the ψ-equation.

4.1.1 Regularized problems

For the system (2.9)-(2.11) we consider for $(\varepsilon,\delta > 0)$ the *regularized system*

$$\int_0^T\int_\Omega u_t\varphi dxdt + \int_0^T\int_\Omega (\nabla u + \mu_\varepsilon\nabla(w+\psi))\cdot\nabla\varphi dxdt = 0, \quad \forall\varphi \in L^2(0,T;H^1(\Omega)), \tag{4.1}$$

$$\delta\int_0^T\int_\Omega \Delta\psi\Delta\phi dxdt + \gamma\int_0^T\int_\Omega \nabla\psi_t\cdot\nabla\phi dt + \int_0^T\int_\Omega \psi\phi dxdt =$$
$$= \int_0^T\int_\Omega u_t\phi dxdt \quad \forall\phi \in L^2(0,T;H^2_\bullet(\Omega)), \tag{4.2}$$

$$w = P(1 - 2\Pi u) \text{ a.e. in } Q_T. \tag{4.3}$$

where $H^2_\bullet(\Omega) := \{\phi \in H^2(\Omega)| \quad \nu\cdot\nabla\phi = 0 \text{ on } \partial\Omega\}$ a dense subset of $H^1(\Omega)$, so that the choice of the testfunction space is consistent if we take $\delta \searrow 0$.
By analogous arguments similar to the previous section we can prove following

Lemma 6. *(existence) There exists an* ε_0*, see Remark 6, such that for all* $0 < \varepsilon \leq \varepsilon_0$ *and for all* $\delta > 0$ *there exist*

1) $u_{\delta,\varepsilon} \in L^2(0,T;H^2(\Omega))$	2) $\partial_t u_{\delta,\varepsilon} \in L^2(0,T;L^2(\Omega))$,
3) $w_{\delta,\varepsilon} \in L^2(0,T;H^2(\Omega))$	4) $\psi_{\delta,\varepsilon} \in L^2(0,T;L^2(\Omega))$,
5) $\nabla\psi_{\delta,\varepsilon} \in L^\infty(0,T;H^2(\Omega))$	6) $\delta_t \nabla\psi_{\delta,\varepsilon} \in L^2(0,T;L^2(\Omega))$,

which satisfy (4.1)-(4.3)

For Proof, see [9].

4.1.2 Existence result of the original problem

To get rid of the regularizations we need a priori estimates, which will be proven as the next step.

Lemma 7. *(energy estimate) There exists an* ε_0*, see Remark 6, such that for all* $0 < \varepsilon \leq \varepsilon_0$ *and for all* $\delta > 0$ *the following estimates hold with constants* C_{38}, C_{39} *independent of* ε *and* δ*:*

$$\delta \int_0^T \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt + \frac{\gamma}{2} \max_{0\leq t\leq T} \|\nabla\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} + \int_0^T \|\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt + \int_0^T \int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon}) |\nabla v_{\delta,\varepsilon}|^2 dx dt \leq C_{38}, \tag{4.4}$$

and

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 dx dt \leq C_{39}, \tag{4.5}$$

where $v_{\delta,\varepsilon} := f'_\varepsilon(u_{\delta,\varepsilon}) + w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon}$.

Proof. The proof is similar to the proof of Lemma 4. Because of sufficient regularity of the function $v_{\delta,\varepsilon} \in L^2(0,T;H^2(\Omega))$ we here don't make use of steklov averaging.

Lemma 8. *(A priori estimates) There exists an* ε_0*, see Remark 6, such that for all* $0 < \varepsilon \leq \varepsilon_0$ *and for all* $\delta > 0$ *the following estimate holds with positive constant* $C_{40}, C_{46}, C_{51}, C_{52}, C_{53}$ *independent of* ε *and* δ*:*

a) $\max_{0\leq t\leq T} \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} \leq C_{40}$	*b)* $\int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \leq C_{46}$,
c) $\int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \leq C_{51}$	*d)* $\int_0^T \|\nabla\partial_t\psi_{\delta,\varepsilon}\|^2 dt \leq C_{52}$,
e) $\max_{0\leq t\leq T} \|\nabla\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} \leq C_{53}$,	

Proof. of a) Similar to the proof of Lemma 5 without steklov averaging.
Proof. of b) 1. Because of Lemma (6) $-\Delta u_\varepsilon$ is an admissible testfunction in (4.1). Using the chain rule we get after partial integration in (4.1)

$$\begin{aligned}\|\nabla u_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \\ = -\int_0^T\int_\Omega \mu'_\varepsilon(u_{\delta,\varepsilon})\nabla u_{\delta,\varepsilon}\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\Delta u_{\delta,\varepsilon}dxdt \\ -\int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\Delta(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\Delta u_{\delta,\varepsilon}dxdt \\ \equiv I_{15}+I_{16}.\end{aligned} \tag{4.6}$$

2. Using Hölder's inequality we get

$$\begin{aligned}I_{15} &\le \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\|_{L^6(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt \\ &\le C_{41}\int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt,\end{aligned}$$

where we have used the embedding $H^2(\Omega)\subseteq H^{1,6}(\Omega)$ in the last step. The Gagliardo-Nirenberg inequality for $\dim(\Omega)=3$

$$\|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)} \le C_g\|\nabla u_{\delta,\varepsilon}\|^{1/2}_{L^2(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|^{1/2}_{L^2(\Omega)}$$

gives

$$I_{15} \le C_{42}\int_0^T \|\nabla u_{\delta,\varepsilon}\|^{1/2}_{L^2(\Omega)}\ \|w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|^{3/2}_{L^2(\Omega)}dt.$$

We obtain using (A2), (B3') and *a)*

$$I_{15} \le C_{43}\int_0^T \|\nabla u_{\delta,\varepsilon}\|^{1/2}_{L^2(\Omega)}\ \|\Delta u_{\delta,\varepsilon}\|^{3/2}_{L^2(\Omega)}dt.$$

Young's inequality together with (4.5) gives

$$I_{15} \le C_{44} + \frac{1}{4}\int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_2\,dt.$$

3. For the second term in (4.6) we get

$$I_{16} \leq \frac{1}{8}\int_0^T \|\Delta(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\|^2_{L^2(\Omega)} + \frac{1}{8}\int_0^T \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt$$
$$\leq C_{45} + \frac{1}{8}\|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)},$$

where we have used (A2), (B3') and *a)*.
Proof. of c) 1. Because of Lemma (6) $\partial_t u_{\delta,\varepsilon} \in L^2(0,T;L^2(\Omega))$, thus an admissible testfuction in (4.1). Again using the chain rule we get after partial integration

$$\begin{aligned}\int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt + &\|\nabla u_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)}\\ &= \int_0^T\int_\Omega \mu'_\varepsilon(u_{\delta,\varepsilon})\nabla u_{\delta,\varepsilon}\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\partial_t u_{\delta,\varepsilon}dxdt\\ &\quad+\int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\Delta(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\partial_t u_{\delta,\varepsilon}dxdt\\ &\equiv I_{17}+I_{18}.\end{aligned} \tag{4.7}$$

2. Using Hölder's inequality and again the embedding $H^2(\Omega) \subseteq H^{1,p}(\Omega)$, $p \in [1,6]$, we get

$$I_{17} \leq \int_0^T \|\nabla u_{\delta,\varepsilon}\|_{L^3(\Omega)}\ \|\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon})\|_{L^6(\Omega)}\ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt$$
$$C_{47}\int_0^T \|u_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)}dt.$$

Assumption (A2), (B3') and (4.6) gives

$$I_{17} \leq C_{48}\int_0^T \|u_{\delta,\varepsilon}\|_{H^2(\Omega)}\ \|\partial_t u_{\delta,\varepsilon}\|_{L^2(\Omega)}\,dt.$$

Applying Young's inequality together with *b)* we find that

$$I_{17} \leq C_{49} + \frac{1}{4}\int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt.$$

3. Young's inequality gives for the second right term of (4.7)

$$I_{18} \leq \int_0^T \|(w_{\delta,\varepsilon} + \psi_{\delta,\varepsilon})\|^2_{H^2(\Omega)} dt + \frac{1}{4} \int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt$$

By (A2), (B3') and *a)* we get

$$I_{18} \leq C_{50} + \frac{1}{4} \int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt.$$

Proof. of d) 1. Testing (4.2) by the admissible testfunction $\partial_t \psi_{\delta,\varepsilon}$ we find

$$\begin{aligned} \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \gamma \int_0^T \|\nabla\partial_t\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \\ + \frac{1}{2}\|\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} = \int_0^T \int_\Omega \partial_t u_{\delta,\varepsilon} \partial_t \psi_{\delta,\varepsilon} dx dt. \end{aligned} \tag{4.8}$$

2. Using Young's inequality together with Poincaré inequality for the right hand side of (4.8) we get

$$\begin{aligned} \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\|\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \frac{\gamma}{2} \int_0^T \|\nabla\partial_t\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt \\ + \frac{1}{2}\|\psi_{\delta,\varepsilon}(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} \leq \frac{1}{2} \int_0^T \|\partial_t u_{\delta,\varepsilon}\|^2_{L^2(\Omega)} dt. \end{aligned}$$

Using *c)* we get *d)*
Proof. of e) 1. To prove this we use again the steklov averaging technique (3.46). We use the admissible testfunction $-\Delta^2\psi_{\delta,\varepsilon h}$ in (4.2) and get after partial integration:

$$\begin{aligned} \delta \int_0^t \|\Delta^2\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} dt \\ = \int_0^t \int_\Omega \partial_t u_{\delta,\varepsilon h} \Delta^2 \psi_{\delta,\varepsilon h} dx dt + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.9}$$

for almost all $t \in [0, T]$.

2. We obtain applying the testfunction $\Delta u_{\delta,\varepsilon h}/\gamma$ in (4.2)

$$\frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt - \int_0^t\int_\Omega \Delta\partial_t\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt + \frac{1}{\gamma}\int_0^t\int_\Omega \psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt = \int_0^t\int_\Omega \partial_t u_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt, \tag{4.10}$$

for almost all $t \in [0,T]$.
3. Because of the steklov averaging $\Delta^2\partial_t\psi_{\delta,\varepsilon h} \in L^2(0,T;L^2(\Omega))$ and by partial integration we get for (4.10)

$$\frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt - \int_0^t\int_\Omega \Delta^2\partial_t\psi_{\delta,\varepsilon h}u_{\delta,\varepsilon h}dxdt + \frac{1}{\gamma}\int_0^t\int_\Omega \nabla\psi_{\delta,\varepsilon h}\cdot\nabla u_{\delta,\varepsilon h}dxdt = \int_0^t\int_\Omega \partial_t u_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt, \tag{4.11}$$

for almost all $t \in [0,T]$. Using the formula for partial integration in time

$$\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}(t)u_{\delta,\varepsilon h}(t)dx - \int_\Omega \Delta^2\psi_{\delta,\varepsilon h}(0)u_{\delta,\varepsilon h}(0)dx = \int_0^t\int_\Omega \Delta^2\partial_t\psi_{\delta,\varepsilon h}u_{\delta,\varepsilon h}dxdt + \int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\partial_t u_{\delta,\varepsilon h}dxdt,$$

for almost all $t \in [0,T]$, we find

$$\begin{aligned}
&\delta\int_0^t \|\Delta^2\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt + \frac{1}{\gamma}\|\nabla u_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} \\
&= + \frac{\gamma}{2}\|\nabla\Delta\psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|\nabla u_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} - \int_\Omega \nabla\Delta\psi_{\delta,\varepsilon h}(t)\cdot\nabla u_{\delta,\varepsilon h}(t)dx \\
&\quad + \int_\Omega \nabla\Delta\psi_{\delta,\varepsilon h}(0)\cdot\nabla u_{\delta,\varepsilon h}(0)dx - \frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt + \frac{1}{\gamma}\int_0^t\int_\Omega \nabla\psi_{\delta,\varepsilon h}\cdot\nabla u_{\delta,\varepsilon h}dxdt,
\end{aligned} \tag{4.12}$$

for almost all $t \in [0,T]$. Using Young's inequality in the form

$$\frac{\delta}{\gamma}\int_0^t\int_\Omega \Delta^2\psi_{\delta,\varepsilon h}\Delta u_{\delta,\varepsilon h}dxdt \le \delta\int_0^t \|\Delta^2\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt + \frac{\delta}{\gamma^2}\int_0^t \|\Delta u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt,$$

$$\int_\Omega \nabla\Delta\psi_{\delta,\varepsilon h}(t)\cdot\nabla u_{\delta,\varepsilon h}(t)dx \le \frac{\gamma}{4}\|\nabla\Delta\psi_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|\nabla u_{\delta,\varepsilon h}(t)\|^2_{L^2(\Omega)},$$

$$\frac{1}{\gamma}\int_0^t\int_\Omega \nabla\psi_{\delta,\varepsilon h}\cdot\nabla u_{\delta,\varepsilon h}dxdt \le \frac{1}{2\gamma}\int_0^t \|\nabla\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} + \frac{1}{2\gamma}\int_0^t \|\nabla u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)},$$

for almost all $t \in [0,T]$, we get from (4.12)

$$\begin{aligned}\|\nabla\Delta\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^t \|\Delta\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt =& 3\|\nabla\Delta\psi_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|\nabla u_{\delta,\varepsilon h}(0)\|^2_{L^2(\Omega)}\\ &+ \frac{4\delta}{\gamma^3}\int_0^t \|\Delta u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt\\ &+ \frac{2}{\gamma^2}\int_0^t \|\nabla\psi_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt + \frac{2}{\gamma^2}\int_0^t \|\nabla u_{\delta,\varepsilon h}\|^2_{L^2(\Omega)}dt,\end{aligned}$$

for almost all $t \in [0,T]$. Passing to the limit ($h \searrow 0$) we obtain

$$\begin{aligned}\|\nabla\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)} + \frac{4}{\gamma}\int_0^t \|\Delta\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt =& 3\|\nabla\Delta\psi_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|\nabla u_{\delta,\varepsilon}(0)\|^2_{L^2(\Omega)}\\ &+ \frac{4\delta}{\gamma^3}\int_0^t \|\Delta u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt\\ &+ \frac{2}{\gamma^2}\int_0^t \|\nabla\psi_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt + \frac{2}{\gamma^2}\int_0^t \|\nabla u_{\delta,\varepsilon}\|^2_{L^2(\Omega)}dt,\end{aligned}$$

for almost all $t \in [0,T]$. The estimates (4.4), (4.5) and *a)* give *e)*.

Remark 8. Because of Lemma 8 we have

$$\max_{0\le t\le T}\|\nabla\psi_{\delta,\varepsilon}\|^2_{H^2(\Omega)} \le C_{53}.$$

and by the Sobolev embedding Theorem for $\dim(\Omega) = 3$ we get

$$\max_{0\le t\le T}\|\nabla\psi_{\delta,\varepsilon}\|_{L^\infty(\Omega)} \le \tilde{C}_{53}. \tag{4.13}$$

By Lemma 8 We have as $\delta, \varepsilon \longrightarrow 0$

$$
\begin{array}{lllll}
u_{\delta,\varepsilon} & \longrightarrow & u & \text{weakly} & \text{in } L^2(0,T;H^2(\Omega)), \\
\partial_t u_{\delta,\varepsilon} & \longrightarrow & \partial_t u & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\
\psi_{\delta,\varepsilon} & \longrightarrow & \psi & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)), \\
\nabla\psi_{\delta,\varepsilon} & \longrightarrow & \nabla\psi & \text{weakly-star} & \text{in } L^\infty(0,T;H^2(\Omega)), \\
\nabla\partial_t\psi_{\delta,\varepsilon} & \longrightarrow & \nabla\psi_t & \text{weakly} & \text{in } L^2(0,T;L^2(\Omega)),
\end{array}
\tag{4.14}
$$

and by (A2) $w_{\delta,\varepsilon} \longrightarrow w$ in $L^2(0,T;H^2(\Omega))$, so that as $\delta, \varepsilon \longrightarrow 0$ we may pass to the limit in (4.1)-(4.3). The convergence of the linear terms in (4.1)-(4.3) are standard. We take a closer look on the convergence of the nonlinear term. First we prove as $\delta \longrightarrow 0$ the passage to the limit of the nonlinear term

$$
\begin{aligned}
\int_0^T\int_\Omega &(\mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon+\psi_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon})\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}))\cdot\nabla\varphi\,dxdt \\
&= \int_0^T\int_\Omega (\mu_\varepsilon(u_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon}))\nabla(w_\varepsilon+\psi_\varepsilon)\cdot\nabla\varphi\,dxdt \\
&\quad + \int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\nabla\left[(w_\varepsilon - w_{\delta,\varepsilon}) + (\psi_\varepsilon - \psi_{\delta,\varepsilon})\right]\cdot\nabla\varphi\,dxdt.
\end{aligned}
$$

We follow the same argument as in (3.42) and skip here the details of the proof. Now we are able to prove the passage to the limit as $\varepsilon \to 0$

$$
\begin{aligned}
\int_0^T\int_\Omega &(\mu_\varepsilon(u_\varepsilon)\nabla(w_\varepsilon+\psi_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon})\nabla(w_{\delta,\varepsilon}+\psi_{\delta,\varepsilon}))\cdot\nabla\varphi\,dxdt \\
&= \int_0^T\int_\Omega (\mu_\varepsilon(u_\varepsilon) - \mu_\varepsilon(u_{\delta,\varepsilon}))\nabla(w_\varepsilon+\psi_\varepsilon)\cdot\nabla\varphi\,dxdt \\
&\quad + \int_0^T\int_\Omega \mu_\varepsilon(u_{\delta,\varepsilon})\nabla\left[(w_\varepsilon - w_{\delta,\varepsilon}) + (\psi_\varepsilon - \psi_{\delta,\varepsilon})\right]\cdot\nabla\varphi\,dxdt.
\end{aligned}
$$

Here we use arguments as in (3.51) to justify this passage. Furthermore we get by Proposition 1 that $u \in L^\infty(Q_T)$ and from (A2) we end up with $w \in \mathcal{V}^{2,\infty}(0,T)$. We also skip here the proof of the uniqueness, which is similar to the corresponding proof in Chapter 3.

5 Proof of Theorem 3

In this chapter we will prove a separation result for u in Theorem 2. For this we will use the Moser iteration technique in the form of Alikakos [1] to establish the separation result. The key point is a proper choice of testfunctions.

Proof. We use here ideas close to [1] and [10]. Denote by

$$z := f'(u) = v - (w + \psi).$$

We introduce

$$\sigma(z) := u = \frac{1}{1 + \exp(-z)},$$

and have

$$\sigma'(z) = u(1-u) = \frac{\exp(-z)}{1 + \exp(-z))^2} = \frac{1}{f''(u)},$$
$$\sigma''(z) = \frac{(\exp(-z) - 1)\exp(-z)}{(1 + \exp(-z))^3}.$$

Becauce of (A2), (B3') we have

$$\sigma''(z) \leq 0 \quad \text{if } z \geq 0 \tag{5.1}$$
$$\sigma''(z) \geq 0 \quad \text{if } z \leq 0 \tag{5.2}$$

Using (5.1) and testing (2.9) with (see [10])

$$\varphi = \frac{z_+^{2^k - 1}}{\sigma'(z)}, \quad k \geq 1, \quad z_+ = \max(0, z),$$

and taking into account

$$\nabla\varphi = \frac{(2^k - 1) z_+^{2^k} \nabla z}{\sigma'} - \frac{\sigma''}{\sigma'^{\,2}} \nabla z z_+^r = \frac{1}{\sigma'} \left\{ (2^k - 1) z_+^{2^k - 2} \nabla z - \sigma'' \varphi \nabla z \right\},$$
$$\frac{\partial \sigma(z)}{\partial t} \varphi = z_t z_+^{2^k - 1} = \frac{1}{2^k} \frac{d}{dt} z_+^{2^k},$$

we get

$$\frac{1}{2^k} \frac{d}{dt} \int\limits_{\{z \geq 0\}} z_+^{2^k} dx + \int\limits_{\{z \geq 0\}} \nabla v \cdot \{(2^k - 1) z_+^{2^k - 2} \nabla z - \sigma'' \varphi \nabla z\} dx = 0. \tag{5.3}$$

We expand the integrand of the second integral in the form

$$\begin{aligned} S =& [\nabla z + \nabla(w + \psi)] \cdot \left\{ (2^k - 1) z_+^{2^k - 2} \nabla z - \sigma'' \varphi \nabla z \right\} \\ =& (2^k - 1) z_+^{2^k - 2} \left\{ |\nabla z|^2 + \nabla(w + \psi) \cdot \nabla z \right\} - \sigma''(z) \varphi \left\{ |\nabla z|^2 + \nabla(w + \psi) \cdot \nabla z \right\}. \end{aligned}$$

Because of (5.1) we can estimate using Young's inequality

$$\begin{aligned} S \geq& (2^k - 1) z_+^{2^k - 2} \left\{ |\nabla z|^2 - \frac{1}{2}(|\nabla(w + \psi)|^2 + |\nabla z|^2) \right\} \\ &- \sigma''(z) \varphi \left\{ |\nabla z|^2 - \frac{1}{2}(k|\nabla(w + \psi)|^2 + \frac{1}{k}|\nabla z|^2) \right\}. \end{aligned}$$

We find with the choise $k = 1/2$

$$\begin{aligned} S \geq & \frac{1}{2}(2^k - 1) z_+^{2^k-2} |\nabla z|^2 - \frac{1}{2}(2^k - 1) z_+^{2^k-2} |\nabla (w+\psi)|^2 \\ & + \frac{1}{4}\frac{\sigma''(z)}{\sigma'(z)} z_+^{2^k-1} |\nabla (w+\psi)|^2. \end{aligned}$$

Because of

$$-1 \leq \frac{\sigma''(z)}{\sigma'(z)} \leq 1,$$

we obtain

$$\begin{aligned} S \geq & \frac{1}{2}(2^k - 1) z_+^{2^k-2} |\nabla z|^2 - \frac{1}{2}(2^k - 1) z_+^{2^k-2} |\nabla (w+\psi)|^2 \\ & - \frac{1}{4} z_+^{2^k-1} |\nabla (w+\psi)|^2. \end{aligned}$$

Because of assumption (A2), (B3') and (4.13) we obtain

$$S \geq \frac{1}{2}(2^k - 1) z_+^{2^k-2} |\nabla z|^2 - \frac{C_{54}}{2}(2^k - 1) z_+^{2^k-2} - \frac{C_{54}}{4} z_+^{2^k-1}.$$

Taking into account

$$z_+^{2^k-2} |\nabla z_+|^2 = \frac{4|\nabla (z_+^{2^{k-1}})|^2}{(2^k)^2},$$

we finally get from the identity (5.3)

$$\begin{aligned} \frac{1}{2^k}\frac{d}{dt}\int_\Omega z_+^{2^k} dx \leq & -\frac{2(2^k-1)}{(2^k)^2}\int_\Omega |\nabla (z_+^{2^{k-1}})|^2 dx \\ & \frac{C_{54}}{4}\int_\Omega \{2(2^k-1) z_+^{2^k-2} + z_+^{2^k-1}\} dx. \end{aligned} \tag{5.4}$$

For $k = 1$ we obtain from (5.4), the embedding $L^2 \subset L^1$ and by integration with respect to t

$$\frac{1}{2}\int_\Omega z_+(t)^2 dx + \frac{1}{2}\int_\Omega |\nabla z_+(t)|^2 dx \leq \frac{1}{2}\int_\Omega z_+(0)^2 dx + \frac{C_{54}}{4}\left\{ 2|Q_t| + \int_0^t \int_\Omega z_+^2 dx \right\}.$$

Recalling that $z_+(0) = \max(0, f'(u_0)) \in L^\infty(\Omega)$ we conclude from Gronwall's Lemma

$$\|z_+\|_{L^\infty(0,T;L^2)} \leq K,$$

where K is a positive constant, which depends on T. Consequently we find

$$\|z_+\|_{L^\infty(0,T;L^1)} \leq K. \tag{5.5}$$

Applying now the Thorem 3.1 in [1] we obtain the L^∞ estimate for z_+. Analogously, from (5.2) we get an L^∞ estimate for z_- by using the testfunction

$$\varphi = \frac{z_-^{2^k-1}}{\sigma'(z)}, \quad k \geq 1, \quad z_- = -\min(0, z).$$

□

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