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Abstract

We study a diffusion model of phase field type, consisting of a system of two partial differential equations of second order for the particle densities and the viscosity variable, coupled by a nonlocal drift term. We prove the existence of variational solutions in standard Hilbert spaces for the evolution system by a careful development of uniform estimates and applying finally a comparison principle .

Key words. Nonlocal phase separation models, viscous phase separation models, Cahn-Hilliard equation, integrodifferential equations, initial value problems, nonlinear evolution equations.

AMS subject classification. 80A22, 35B50, 45K05, 35K20, 35K45, 35K55, 35K65, 47J35

1 Introduction

In this article, we deal with an integrodifferential model for volume preserving isothermal phase transitions that takes into account long-range interactions between particles. The physical relevance of nonlocal interaction phenomena in phase separation and phase transition models was already described in the pioneering papers [15] and [1]; however, only recently both isothermal and nonisothermal models containing nonlocal terms have been analyzed in a more systematic way [7, 8]. Besides more slightly complicated models, which also take into account nonlocal viscosity effects has been suggested in [5]; these models are indeed generalizations of corresponding local viscous models, see [14].

Inspired by the nonlocal Cahn-Hilliard model studied by Gajewski in [7], we consider the following *nonlocal free energy* density

$$F(u) = f(u) + \frac{1}{2}uw, \tag{1}$$

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where u denotes the local concentration of a component occupying a spatial domain Ω, $f(u)$ is a convex function and

$$w(x) := \int \mathcal{K}(|x-y|)(1-2u(y))dy. \tag{2}$$

The kernel $\mathcal{K}$ of the integral term (2) describes *nonlocal or long-range interactions* [2, 9, 10, 11]. Hence, the difference between local and nonlocal models consists in a different choice of the particle interaction potential in the free energy. Moreover the local free energy can be obtained as a formal limit from the nonlocal one, see [12]. In [7] the above nonlocal free energy density has been used to derive a nonlocal Cahn-Hillard equation

$$u_t - \nabla \cdot (\mu \nabla (f'(u) + w)) = 0,$$

where in standard cases f is the convex (information) entropy function

$$f(u) = u\log(u) + (1-u)\log(1-u). \tag{3}$$

Consequently

$$f'(u) = \log\left(\frac{u}{1-u}\right) \quad \text{and} \quad u = f'^{-1}(v-w) = \frac{1}{1+\exp(v-w)},$$

where f'^{-1} is the Fermi-function, whose image is the interval $[0,1]$. Thus, the nonlocal model naturally satisfies the physical requirement

$$0 \le u(x) \le 1, \quad \forall t \ge 0.$$

and the maximum principle is available, which is not true for fourth order equations like in the case of the local Cahn-Hillard equations.

1.1 Nonlocal viscous model

As in [5] our aim is to formulate a general nonlocal model, which also takes into accout viscosity effects, see [14]. In the nonlocal philosophy these viscosity effects have also been formulated in a nonlocal manner, see [5], where we proposed two different models, namely:

model I:

$$-\gamma\Delta\psi_t + \psi = u_t, \quad \gamma > 0. \tag{4}$$

model II:

$$-\gamma\Delta\psi + \psi = u_t, \quad \gamma > 0.$$

In both cases γ is a model parameter, which is positive and guarantees the nonlocal structure of the additional term ψ in the chemical potential

$$v := \frac{\delta F(u)}{\delta u} + \psi. \tag{5}$$

Model I was analyzed in [5]. The mathematical anlaysis of model II is devoted to this paper. Taking into account (5) and (4) we end up with the *nonlocal viscous Cahn-Hillard equation*:

$$\begin{aligned} u_t - \nabla \cdot \mu \nabla v = 0, \quad v = f'(u) + w + \psi,\\ w(x) = \int_\Omega \mathcal{K}(|x-y|)(1-2u(y))dy,\\ -\gamma\Delta\psi + \psi = u_t, \quad \gamma > 0, \end{aligned} \tag{6}$$

which is complemented by suitable initial and boundary conditions.

In Section 2 we formulate the problem and general assumptions. Applying fixed-point arguments and comparison principles in Section 3 we prove the existence of variational solutions in standard Hilbert spaces for evolution systems.

2 Statement of the problems and assumptions

Let be $\Omega \subset \mathbb{R}^3$ an open, bounded and smooth domain with boundary $\Gamma = \partial\Omega$ and ν the outer unit normal on Γ. In the sequel, $|\Omega|$ denotes the Lesbegue measure of Ω. We denote by $L^p(\Omega), W^{k,p}(\Omega)$ for $1 \le p \le \infty$ the Lesbegue spaces and Sobolev spaces of functions on Ω with the usual norms $\|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{W^{k,p}(\Omega)}$, and we write $H^k(\Omega) = W^{k,2}(\Omega)$, see [4]. For a Banach space X we denote its dual by X^*, the dual pairing between $f \in X^*$, $g \in X$ will be denoted by $\langle f, g\rangle$. If X is a Banach space with the norm $\|\cdot\|_X$, we denote for $T > 0$ by $L^p(0,T;X)$ $(1 \le p \le \infty)$ the Banach space of all (equivalence classes of) Bochner measurable functions $u : (0,T) \longrightarrow X$ such that $\|u(\cdot)\|_X \in L^p(0,T)$. We set $R^1_+ = (0,\infty)$ and, as already mentioned, $Q_T = (0,T)\times\Omega$, $\Gamma_T = (0,T)\times\Gamma$. "Generic" positive constants are denoted by C and for $u \in L^1(\Omega)$ we put

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

Furthermore we define following time dependent Sobolev spaces by

$$\begin{aligned} &W(0,T) := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)^*),\\ &\mathcal{V}^{2,\infty}(0,T) := \{f \in L^\infty(Q_T) \mid \nabla f \in L^\infty(Q_T), \Delta f \in L^\infty(Q_T)\}. \end{aligned}$$

We make the following general assumptions.
(A1) $f^\theta(u) = u^{1-\theta}\log u + (1-u)^{1-\theta}\log(1-u), \quad \theta \in (0,1/2).$

(A2) the potential operator P defined by

$$\rho \mapsto P\rho = \int_\Omega \mathcal{K}(|x-y|)\rho(y)dy$$

satisfies

$$\|P\rho\|_{W^{2,p}(\Omega)} \leq r_p \|\rho\|_{L^p}, \qquad 1 \leq p \leq \infty,$$

where the kernel $\mathcal{K} \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1)$ is such that

$$\int_\Omega \int_\Omega |\mathcal{K}(|x-y|)|dxdy = m_0 < \infty, \qquad \sup_{x\in\Omega} \int_\Omega |K(|x-y|)|dy = m_1 < \infty.$$

(A3) the mobility μ^θ has the form

$$\mu^\theta(u) = \frac{1}{(f^\theta)''(u)}, \qquad \theta \in (0, 1/2). \tag{7}$$

(A4) $u_0(x) \in [0,1]$ a.e. in Ω and $\overline{u}_0 \in (0,1)$,

Remark 1 In (A1) we have chosen a modified entropy function, which posses similar properties as (3) and does not mean any restriction in physical properties of the entropy function. In we choose $\theta = 0$ we would end up with (3). In our paper we only are able to prove existence in cases $\theta \neq 0$. We use a priori estimates which are not uniform in θ. The existence for the case $\theta = 0$ are led to future research.

Remark 2 The kernel $\mathcal{K}$ is chosen to be symmetric. Consequently the potential operator P is symmetric, too. Examples for kernels $\mathcal{K}$, see [7]

Remark 3 A concentration-dependent mobility appeared in the original derivation of the Cahn-Hillard equation, see [1], and a natural and thermodynamically reasonable choice is of the form (7) and were considered for $\theta = 0$ in [3].

Now we are going to formulate the nonlocal viscous Cahn-Hillard equation (6) with complemented initial and boundary values. So the initial-boundary value problem we want to discuss takes the form:

$$u_t - \nabla \cdot \overbrace{(\nabla u + \mu^\theta \nabla(w + \psi))}^{=\mu^\theta \nabla v} = 0 \quad \text{in } Q_T, \tag{8}$$

$$-\gamma\Delta\psi + \psi = u_t, \quad w = P(1-2u) \quad \text{in } Q_T, \tag{9}$$

$$\mu\nu \cdot \nabla v = \nu \cdot \nabla \psi = 0 \quad \text{on } \Gamma_T, \tag{10}$$

$$u(0,x) = u_0(x), \psi(0,x) = \psi_0(x) \quad x \in \Omega. \tag{11}$$

Theorem 1 *Suppose that the assumptions (A1)-(A4) hold. Then there exists a triple of functions* (u, w, ψ) *such that* $u(0) = u_0$, $\psi(0) = \psi_0$ *and*

$$(u, w, \psi) \in W(0,T) \times \mathcal{V}^{2,\infty}(0,T) \times L^2(0,T;H^1(\Omega))$$

with $0 \leq u(t,x) \leq 1$ *a.e. in* Q_T*, which satify equations (8)-(11) in the following sense:*

$$\int_0^T \langle u_t, \varphi \rangle \, dt + \int_0^T \int_\Omega \overbrace{(\nabla u + \mu^\theta \nabla (w + \psi))}^{=\mu^\theta \nabla v} \cdot \nabla \varphi dx dt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{12}$$

$$\gamma \int_0^T \langle \nabla \psi, \nabla \phi \rangle \, dt + \int_0^T \int_\Omega \psi \phi dx dt = \int_0^T \langle u_t, \phi \rangle \, dt, \quad \forall \phi \in L^2(0,T;H^1(\Omega)), \tag{13}$$

$$w = P(1 - 2u) \text{ a.e. in } Q_T. \tag{14}$$

3 Proof of Theorem 1

The idea of the existence proof is as follows: we construct regularized problems with truncated nonlinearities. After proving the existence result for such problems we establish the existence result for the original problem by giving a priori estimates.
To do so, for $c \in \mathbb{R}$ we define the truncation

$$c^\diamond := \min\{\max\{c, \varepsilon\}, 1 - \varepsilon\}, \tag{15}$$

and we carry over this setting in the usual way to the concept of truncated functions. Thus we define the regurized entropy function in the following manner:

$$f_\varepsilon^\theta(u) := f^\theta(u^\diamond) \tag{16}$$

Remark 4 We have by (A1) for $u \geq 1/2$

$$\begin{aligned}
(f_\varepsilon^\theta)''(u) &= u^{-(1+\theta)}[-\theta(1-\theta)\log u + (1-2\theta)] + (1-u)^{-(1+\theta)}[-\theta(1-\theta)\log(1-u) + (1-2\theta)] \\
&\geq (1-2\theta)[u^{-(1+\theta)} + (1-u)^{-(1+\theta)}] \\
&\geq (1-2\theta)(1-u)^{-(1+\theta)}
\end{aligned}$$

Remark 5 $\exists \varepsilon_0 := \varepsilon_0(w)$ *so that* $\forall \varepsilon \in (0, \varepsilon_0]$*:*

$$F_{NL,\varepsilon}^\theta(u) := \int_\Omega \left(f_\varepsilon^\theta(u) + \frac{1}{2} u w \right) dx \geq -C_F,$$

where $C_F > 0$.

Proof of Remark 5. Using (A1), (3.3) and (19) we see that it depends on the choice of ε to ensure that $f_\varepsilon(u)$ dominates $\frac{1}{2}uw$. Thus, there exists an $\varepsilon_0 = \varepsilon_0(w)$ so that $\forall \varepsilon \in (0, \varepsilon_0]$ this is true. □

3.1 Regularized problems

For the system (12)-(14) we get by (15) and (3.3) the *regularized system:*

$$\int_0^T \langle u_t, \varphi\rangle\, dt + \int_0^T \int_\Omega \big(\nabla u + \mu_\varepsilon^\theta \nabla(w+\psi)\big)\cdot \nabla\varphi dxdt = 0, \quad \forall \varphi \in L^2(0,T;H^1(\Omega)), \tag{17}$$

$$\gamma \int_0^T \langle \nabla\psi, \nabla\phi\rangle\, dt + \int_0^T \int_\Omega \psi\phi dxdt = \int_0^T \langle u_t, \phi\rangle\, dt, \quad \forall \phi \in L^2(0,T;H^1(\Omega)), \tag{18}$$

$$w_\varepsilon(u) = P(1-2u^\diamond) \text{ a.e. in } Q_T. \tag{19}$$

Lemma 1 *There exists* $\varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0]$ *there exist* $(u_\varepsilon, w_\varepsilon, \psi_\varepsilon)$ *such that* $u_\varepsilon(0) = u_0$, $\psi_\varepsilon(0) = \psi_0$ *and*

$$(u_\varepsilon, w_\varepsilon, \psi_\varepsilon) \in W(0,T) \times \mathcal{V}^{2,\infty}(0,T) \times L^2(0,T;H^1(\Omega)),$$

which satify (17)-(19).

Proof of Lemma 1 This proof is similar to the proof established in [5]. Hence, we skip here the details. For the proof we replace the regularized problem (17)-(19) by a semi-discrete approximation, which we solve by Schauder's fixed-point principle. After constructing suitable a priori estimates and compactness we can converge from the semi-discrete approximation to the regularized problem. □

To get the solution for $\varepsilon \searrow 0$ one usually needs a-priori estimates which guaratee compactness and finally the convergence to (u, w, ψ). But we will see that here for our problem this is not necessary, if we are able to show that u_ε lives on some smaller sub-intervall of $[0, 1]$. So we will investigate that the regularization is "effectless" and that we can "skip" it. Do do so we need in the following some estimates.

3.2 A priori estimates

Estimate 1 *There exists a constant* ε_1 *such that for all* $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$ *the following estimate holds:*

$$\|\psi_\varepsilon\|^2_{L^2(0,t;H^1(\Omega))} \leq C(\gamma, t, \Omega)(1 + \|u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}) \tag{20}$$

Remark 6 *The existence of* ε_1 *will be given by Lemma 3.*

Proof. 1. We apply the admissible testfunctions $\psi_\varepsilon \in L^2(\Omega)$ in (17) and in (18), $-u_\varepsilon/\gamma$ in (18) and get We obtain by using $-u_\varepsilon/\gamma$ as a testfunction

$$\gamma \int_0^t \int_\Omega |\nabla \psi_\varepsilon|^2 dxds + \int_0^t \int_\Omega |\psi_\varepsilon|^2 dxds - \frac{1}{\gamma} \int_0^t \int_\Omega \psi_\varepsilon u_\varepsilon dxds$$
$$+ \frac{1}{\gamma} \int_0^t \frac{1}{2} \frac{d}{dt} \int_\Omega |u_\varepsilon|^2 dxds + \int_0^t \int_\Omega \mu_\varepsilon^\theta \nabla w_\varepsilon \cdot \nabla \psi_\varepsilon dxds + \int_0^t \int_\Omega \mu_\varepsilon^\theta |\nabla \psi_\varepsilon|^2 dxds = 0$$

for all $t \in [0, T]$. Using Young's inequality we find after standard calculations

$$\gamma \|\nabla \psi_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} + \|\psi_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} \le \frac{C(r_2, \Omega)}{\gamma} \|\nabla w_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{\gamma^2} \|u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}$$

Using (A2), (15) and (19) we obtain (20). □

Estimate 2 *There exists a constant* ε_1 *such that for all* $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$ *the following estimate holds:*

$$0 \le u_\varepsilon(t, x) \le 1, \qquad a.e.\ in\ Q_T.$$

Proof. Using in (12) the admissible testfunctions $u_\varepsilon^\odot := \min(u_\varepsilon, 0)$ and $u_\varepsilon^\circledast := \min(1 - u_\varepsilon, 0)$ we get

$$\frac{1}{2} \int_\Omega |u_\varepsilon^\circ(t)|^2 dx + \int_0^T \int_\Omega |\nabla u_\varepsilon^\circ|^2 dxdt + \int_0^T \int_\Omega \mu_\varepsilon^\theta \nabla (w_\varepsilon + \psi_\varepsilon) \cdot \nabla u_\varepsilon^\circ dxdt = 0.$$

where $\circ \in \{\odot, \circledast\}$. Because of $\mu_\varepsilon^\theta \nabla u_\varepsilon^\circ = 0$ for $\circ \in \{\odot, \circledast\}$ the last term vanishes and we get

$$0 = \frac{1}{2} \int_\Omega |u_\varepsilon^\circ(t)|^2 dx + \int_0^T \int_\Omega |\nabla u_\varepsilon^\circ|^2 dxdt \ge \frac{1}{2} \int_\Omega |u_\varepsilon^\circ(t)|^2 dx,$$

that means $u_\varepsilon^\circ(t, x) = 0$ a.e. in Q_T, hence $1 \ge u_\varepsilon(t, x) \ge 0$ a.e. in Q_T. □

We introduce following notations

$$\tilde{u} := \max(0, u - k) \tag{21}$$
$$M(k, t) := \{x \in \Omega \mid \tilde{u}(t, x) > 0\} \tag{22}$$

Estimate 3 *There exists a constant ε_1 such that for all $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$ the following estimate holds with a constant ϑ independent of ε:*

$$\int_0^t \|\tilde{u}_\varepsilon(s)\|^2_{L^2(\Omega)} ds \leq \frac{\vartheta^2}{(f_\varepsilon^\theta)''^2(k)} \left(\int_0^t M(k,\tau) d\tau \right)^{2/p'} \tag{23}$$

Proof. We only will show the proof for one side, the other side can be proven analogously.
1. Let be $k \in [k_0, 1), k_0 \in [1/2, 1)$, $\tilde{u}_0 = 0$ and $2 \leq p \leq \frac{2(N+1)}{N}$. The function $\varphi = \max(0, (f_\varepsilon^\theta)'(u_\varepsilon) - (f_\varepsilon^\theta)'(k)) \in L^2(0,T; H^1(\Omega))$ is a valid testfunction in (17). Therefore we obtain

$$\int_0^t \langle \partial_t u_\varepsilon, \max(0, (f_\varepsilon^\theta)'(u_\varepsilon) - (f_\varepsilon^\theta)'(k)) \rangle ds \tag{24}$$

$$+ \int_0^t \int_\Omega \mu_\varepsilon^\theta(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \max(0, (f_\varepsilon^\theta)'(u_\varepsilon) - (f_\varepsilon^\theta)'(k)) dx ds := J_1 + J_2 = 0 \tag{25}$$

for a.e. in $[0, T]$.
We first treat the first term J_1: We define *steklov averaged* functions

$$u_{\varepsilon h}(t,x) := \frac{1}{h} \int_{t-h}^t u_\varepsilon(\tau, x) d\tau, \tag{26}$$

where we set $u_\varepsilon(t,x) = u_0(x)$ when $t \leq 0$. From [13] we have

$$u_{\varepsilon h} \longrightarrow u_\varepsilon \quad \text{strongly} \quad \text{in} \quad L^2(0,T; H^1(\Omega)) \quad \text{as } h \searrow 0.$$

Because of (A2) and the continuity of f'_ε it is easily proven that as $h \searrow 0$

$$\begin{array}{llll} w_{\varepsilon h} \longrightarrow w_\varepsilon & \text{strongly} & \text{in} & L^2(0,T; H^1(\Omega)), \\ f'_\varepsilon(u_{\varepsilon h}) \longrightarrow f'_\varepsilon(u_\varepsilon) & \text{strongly} & \text{in} & L^2(0,T; H^1(\Omega)). \end{array} \tag{27}$$

We define $g_{\varepsilon h} := (f_\varepsilon^\theta)'(u_{\varepsilon h}) + w_{\varepsilon h}$, and $v_{\varepsilon h} := g_{\varepsilon h} + \psi_{\varepsilon h}$. It follows from (20) that

$$\psi_{\varepsilon h} \longrightarrow \psi_\varepsilon \text{ strongly} \quad \text{in} \quad L^2(0,T; H^1(\Omega)) \quad \text{as } h \searrow 0. \tag{28}$$

Furthermore, we can show $\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon$ strongly in $L^2(0,T;H^1(\Omega)^*)$. For any $\varphi \in L^2(0,T;H^1(\Omega))$ we have

$$\begin{aligned}
|\langle \partial_t u_{\varepsilon h} - \partial_t u_\varepsilon, \varphi\rangle| &= \frac{1}{h}\left|\int_0^T \left\langle \int_{t-h}^t (\partial_t u_\varepsilon(\tau) - \partial_t u_\varepsilon(t))d\tau, \varphi\right\rangle dt\right| \\
&= \frac{1}{h}\left|\int_0^T \left\langle \int_{-h}^0 (\partial_t u_\varepsilon(t+s) - \partial_t u_\varepsilon(t))ds, \varphi\right\rangle dt\right| \\
&\le \frac{1}{h}\int_{-h}^0 \left|\int_0^T \int_\Omega \left(\mu_\varepsilon^\theta(u_\varepsilon(t+s))\nabla v_\varepsilon - \mu_\varepsilon^\theta(u_\varepsilon(t))\nabla v_\varepsilon\right) \nabla\varphi dx dt\right| ds \\
&\le \max_{-h\le s\le 0} \|(\mu_\varepsilon^\theta(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s) - \mu_\varepsilon^\theta(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)} \|\nabla\varphi\|_{L^2(Q_T)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\max_{-h\le s\le 0}\|\mu_\varepsilon^\theta(u_\varepsilon(t+s))\nabla v_\varepsilon(t+s) - \mu_\varepsilon^\theta(u_\varepsilon(t))\nabla v_\varepsilon(t)\|_{L^2(Q_T)} \\
&\qquad \le \max_{-h\le s\le 0} \|[\mu_\varepsilon^\theta(u_\varepsilon(t+s)) - \mu_\varepsilon^\theta(u_\varepsilon(t))]\nabla v_\varepsilon(t+s)\|_{L^2(Q_T)} \\
&\qquad\quad + C \max_{-h\le s\le 0} \|g_\varepsilon(t+s) - g_\varepsilon(t)\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad\quad + C \max_{-h\le s\le 0} \|\nabla\psi_\varepsilon(t+s) - \nabla\psi_\varepsilon(t)\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

The first part of the right hand side tends as $h \to 0$ pointwise to zero, because of the Lipschitz continuity of $u_\varepsilon \mapsto \mu_\varepsilon^\theta(u_\varepsilon)$ and the convergence

$$\max_{-h\le s\le 0} \|u_\varepsilon(t+s) - u_\varepsilon(t)\|_{L^2(Q_T)} \to 0 \quad \text{as} \quad h \to 0.$$

The second and the third part follow from (27) and (28). It follows that

$$\partial_t u_{\varepsilon h} \longrightarrow \partial_t u_\varepsilon \quad \text{strongly} \quad \text{in} \quad L^2(0,T;H^1(\Omega)^*) \quad \text{as } h \searrow 0.$$

Using $\partial_t u_{\varepsilon h} \in L^2(0,T;L^2(\Omega))$, we have for almost all $t \in [0,T]$

$$
\begin{aligned}
\int_0^t \int_\Omega \partial_t u_{\varepsilon h}, \max(0,(f_\varepsilon^\theta)'(u_{\varepsilon h}) - (f_\varepsilon^\theta)'(k)) dxds &= \int_0^t \int_{M(k,s)} \partial_t u_{\varepsilon h}[(f_\varepsilon^\theta)'(u_{\varepsilon h}) - (f_\varepsilon^\theta)'(k)] dxds \\
&= \int_0^t \partial_s \int_{M(k,s)} \left[f_\varepsilon^\theta(u_{\varepsilon h}(s)) - (f_\varepsilon^\theta)'(k)\tilde{u} \right] dxds \\
&= \int_{M(k,t)} \left[f_\varepsilon^\theta(u_{\varepsilon h}(t)) - f_\varepsilon^\theta(k) - (f_\varepsilon^\theta)'(k)\tilde{u}(t) \right] dx \\
&\geq \frac{1}{2} \int_{M(k,t)} (f_\varepsilon^\theta)''(k) |\tilde{u}_\varepsilon(t)|^2 dx,
\end{aligned}
$$

where we used for the last inequality the Taylor expansion of f_ε^θ and $(f_\varepsilon^\theta)'''(u_\varepsilon) \geq 0$ for $u_\varepsilon \geq 1/2$. Passing to the limit ($h \searrow 0$) in this equation, where we apply the convergence properties of $u_{\varepsilon h}$ proved above and using Remark 5, we obtain a.e. in $[0,T]$

$$
J_1 \geq \frac{(f_\varepsilon^\theta)''(k)}{2} \|\tilde{u}_\varepsilon(t)\|^2_{L^2(\Omega)}.
$$

Moreover we have

$$
J_2 = \int_0^t \int_\Omega \nabla v_\varepsilon \cdot \nabla \tilde{u}_\varepsilon dxds = \int_0^t \int_\Omega \left[(f_\varepsilon^\theta)''(u_\varepsilon) |\nabla \tilde{u}_\varepsilon|^2 + \nabla \psi_\varepsilon \cdot \nabla \tilde{u}_\varepsilon - \Delta w_\varepsilon \tilde{u}_\varepsilon \right] dxds.
$$

Testing (17) by the admissible testfunction $\tilde{u}_\varepsilon$ and using (20) we have

$$
\begin{aligned}
\int_0^t \int_\Omega \nabla \psi_\varepsilon \cdot \nabla \tilde{u}_\varepsilon dxds &= \frac{1}{2\gamma} \int_\Omega |\tilde{u}_\varepsilon(t)|^2 dx - \frac{1}{\gamma} \int_0^t \int_\Omega \psi_\varepsilon \tilde{u}_\varepsilon dxds \\
&\geq -\frac{1}{2\gamma^2} \|\psi_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} - \frac{1}{2} \|u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} \\
&\geq -C(\gamma,t,\Omega) \|u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))}.
\end{aligned}
$$

Hence, applying Estimate 2, (A2), Hölder's and Young's inequalities, we find

$$\begin{aligned}
J_2 &\geq \int_0^t \int_\Omega (f_\varepsilon^\theta)''(k)|\nabla \tilde{u}_\varepsilon|^2 dx ds - C(r_\infty,\gamma,t,\Omega)\int_0^t M(k,s)^{1/p'}\|\tilde{u}_\varepsilon\|_{L^p(\Omega)} ds \\
&\geq \int_0^t \int_\Omega (f_\varepsilon^\theta)''(k)|\nabla \tilde{u}_\varepsilon|^2 dx ds - C(r_\infty,\gamma,t,\Omega)\left(\int_0^t M(k,s) ds\right)^{1/p'} \left(\int_0^t \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} ds\right)^{1/p} \\
&\geq (f_\varepsilon^\theta)''(k)\int_0^t \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 dx ds - \frac{C(r_\infty,\gamma,t,\Omega)^2}{2\delta (f_\varepsilon^\theta)''(k)}\left(\int_0^t M(k,s) ds\right)^{2/p'} \\
&\quad - \frac{\delta}{2}(f_\varepsilon^\theta)''(k)\left(\int_0^t \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} ds\right)^{2/p},
\end{aligned}$$

where $1/p + 1/p' = 1$.
Using the Gagliardo-Nierenberg-inequality (with the constant C_g)

$$\|\tilde{u}_\varepsilon\|_{L^p(\Omega)} \leq C_g \left(\|\tilde{u}_\varepsilon\|^{1-\beta}_{L^2(\Omega)}\|\nabla\tilde{u}_\varepsilon\|^{\beta}_{L^2(\Omega)}\right)$$

with $\beta = 1/p$, we find for the last term applying the Hölder inequality

$$\begin{aligned}
\left(\int_0^t \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} ds\right)^{2/p} &\leq C_g \left(\int_0^t \|\tilde{u}_\varepsilon\|^{p(1-\beta)}_{L^2(\Omega)}\|\nabla\tilde{u}_\varepsilon\|^{p\beta}_{L^2(\Omega)} ds\right)^{2/p} \\
&\leq C_g \left(\int_0^t \|\tilde{u}_\varepsilon\|^{p-1}_{L^2(\Omega)}\|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)} ds\right)^{2/p} \\
&\leq C_g \left(\int_0^t \|\tilde{u}_\varepsilon\|^{2(p-1)}_{L^2(\Omega)} ds\right)^{1/p} \left(\int_0^t \|\nabla\tilde{u}_\varepsilon\|_{L^2(\Omega)} ds\right)^{1/p}.
\end{aligned}$$

Furthermore using the Young inequality we get

$$\left(\int_0^t \|\tilde{u}_\varepsilon\|^p_{L^p(\Omega)} ds\right)^{2/p} \leq C_g \left[\frac{1}{p'}\left(\int_0^t \|\tilde{u}_\varepsilon\|^{2(p-1)}_{L^2(\Omega)} ds\right)^{\frac{1}{p-1}} + \frac{1}{p}\left(\int_0^t \|\nabla\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} ds\right)\right].$$

Standard calculations give

$$\left(\int_0^t \|\tilde u_\varepsilon\|^p_{L^p(\Omega)}ds\right)^{2/p} \le C_g\left[\frac{1}{p'}\left(\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^{\frac{2(p-2)}{p-1}}_{L^2(\Omega)}\right)\left(\int_0^t \|\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds\right)^{\frac{1}{p-1}} + \frac{1}{p}\left(\int_0^t \|\nabla\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds\right)\right]$$

$$\le \frac{C_g}{p'}\left(\frac{p-2}{p-1}\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^2_{L^2(\Omega)} + \frac{C_g}{p-1}\int_0^t \|\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds\right)$$

$$+\frac{1}{p}\left(\int_0^t \|\nabla\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds\right)$$

$$\le C_g\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^2_{L^2(\Omega)} + C_g\left(1+\frac{1}{C_P}\right)\int_0^t \|\nabla\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds,$$

where we have used the Poincaré inequality with the Poincaré constant C_P for the last step. Choosing $\delta = \frac{1}{C_g\left(1+\frac{1}{C_P}\right)}$ we obtain

$$J_2 \ge \frac{(f^\theta_\varepsilon)''(k)}{2}\int_0^t\int_\Omega |\nabla\tilde u_\varepsilon|^2 - \frac{C(r_\infty,\gamma,t,\Omega)^2C_g(1+1/C_p)}{2(f^\theta_\varepsilon)''(k)}\left(\int_0^t M(k,s)ds\right)^{2/p'}$$
$$-\frac{(f^\theta_\varepsilon)''(k)C_p}{2(1+C_p)}\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^2_{L^2(\Omega)}.$$

We finally obtain for J_1+J_2

$$\|\tilde u_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\tilde u_\varepsilon\|^2_{L^2(\Omega)}ds \le \frac{C(r_\infty,\gamma,t,\Omega)^2C_g(1+1/C_p)}{(f^\theta_\varepsilon)''^2(k)}\left(\int_0^t M(k,s)ds\right)^{2/p'} + \frac{C_p}{2(1+C_p)}\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^2. \tag{29}$$

This implies

$$\sup_{0\le s\le t}\|\tilde u_\varepsilon(s)\|^2_{L^2(\Omega)} \le \frac{\vartheta^2}{(f^\theta_\varepsilon)''^2(k)}\left(\int_0^t M(k,s)ds\right)^{2/p'},$$

where $\vartheta^2 = C(r_\infty,\gamma,t,\Omega)^2C_g(1+1/C_p)(1+C_p)$ and hence with $Y(t) := \int_0^t \|\tilde u_\varepsilon(s)\|^2_{L^2(\Omega)}ds$ and the Poincaré inequality (30) becomes

$$Y'(t) + (1+C_p)C_pY(t) \le \frac{\vartheta^2}{(f^\theta_\varepsilon)''^2(k)}\left(\int_0^t M(k,s)ds\right)^{2/p'}. \tag{30}$$

We get by integration with respect to time

$$\int_0^t \|\tilde{u}_\varepsilon(s)\|^2_{L^2(\Omega)} ds \leq \frac{\vartheta^2}{(f^\theta_\varepsilon)''^2(k)} \int_0^t \exp(1+C_p)(s-t) \left(\int_0^s M(k,\tau) d\tau \right)^{2/p'} ds$$
$$\leq \frac{\vartheta^2}{(f^\theta_\varepsilon)''^2(k)} \left(\int_0^t M(k,\tau) d\tau \right)^{2/p'}.$$

□

3.3 Existence to the original problem

Lemma 2 *(Auxilliary Lemma) Let $\Phi(\xi)$ be a function defined for $\xi \geq M$, nonnegative and nondecreasing such that for $h > k \geq M$ the estimate*

$$\Phi(h) \leq \frac{\beta k^\varsigma}{(h-k)^\alpha} \Phi(k)^{1+\chi} \tag{31}$$

holds. Here α, β and χ are positive constants. Moreover $\varsigma < \alpha(1+\chi)$. Then $\Phi(2d) = 0$ where $d > M$ is the root of the equation

$$d = M + \lambda M^{\varsigma/\alpha} d^{\frac{\varsigma-\alpha}{\chi\alpha}} \tag{32}$$

and

$$\lambda^\alpha = 2^{\frac{\alpha+\varsigma}{\chi} + \frac{\alpha}{\chi^2}} \beta^{1+\frac{1}{\chi}} \Phi(M)^{1+\chi}. \tag{33}$$

Proof. Set $k_j = d(2 - 2^{-j})$, for $j = 0, 1, 2, ...$ We want to show that

$$\Phi(k_j) \leq \left[\frac{d^{\alpha-\varsigma}}{2^{\alpha(j+1+1/\chi)+\varsigma}\beta} \right]^{1/\chi}. \tag{34}$$

This proves that $\Phi(2d) = 0$, since $\lim_{j\to+\infty} k_j = 2d$. Equation (31) for $h = k_0$ and $k = M$ shows that

$$\Phi(k_0) \leq \frac{\beta M^\varsigma}{(d-M)^\alpha} \Phi(M)^{1+\chi}. \tag{35}$$

By replacing $(d-M)^\alpha$ by the value obtained from equation (32) it readily follows that the right hand side of (35) is equal to the right hand side of (34) for $j = 0$. Next, by supposing that (34) holds for some $j \geq 0$ and by using (31), we prove that

$$\Phi(k_{j+1}) \leq \frac{\beta 2^{\varsigma+(j+1)\alpha}}{d^{\alpha-\varsigma}} \left[\frac{d^{\alpha-\varsigma}}{2^{\alpha(j+1+1/\chi)+\varsigma}\beta} \right]^{1+1/\chi}. \tag{36}$$

Straightforward calculations show that the right hand side of (36) is equal to the right hand side of (34) if here we replace j by $j+1$. □

Lemma 3 *There exists a constant $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1)]$*

$$\varepsilon_1 \le u_\varepsilon(t,x) \le 1-\varepsilon_1 \;\; a.e. \;\; in \;\; Q_T. \tag{37}$$

Proof . Let now $1 > h > k \ge 1/2$. We have

$$\int_0^t \|\tilde{u}_\varepsilon(s)\|^2_{L^2(\Omega)} ds \ge (h-k)^2 \int_0^t M(h,\tau)d\tau \tag{38}$$

and consequently with $1+\chi := 2/p'$ and by Estimate 3 we obtain

$$\left(\int_0^t M(h,\tau)d\tau\right)^{1/2} \le \frac{\vartheta}{(f_\varepsilon^\theta)''(k)(h-k)} \left[\left(\int_0^t M(k,\tau)d\tau\right)^{1/2}\right]^{1+\chi}. \tag{39}$$

Defining $\pi(\xi) := \left(\int_0^t M(\xi,\tau)d\tau\right)^{1/2}$ and using Remark 4 we get

$$\pi(h) \le \frac{\vartheta(1-k)^{1+\theta}}{(1-2\theta)(h-k)} \pi(k)^{1+\chi}. \tag{40}$$

By $\xi := 1 - 1/\Xi$ and $\Pi := \pi \circ \xi$ we find

$$\Pi(H) \le \frac{\vartheta K^{-\theta} H}{(1-2\theta)(H-K)} \Pi(K)^{1+\chi} \tag{41}$$

Defining $\Phi(\Xi) := \Pi(\Xi)/\Xi$, $\beta_\theta := \vartheta/(1-2\theta)$ and $\varsigma := 1+\chi-\theta$ we end up with (31). So by Lemma 2 there exists a D which is characterized by (32) and (33) for which we have $\Phi(2D) = 0$. That means that there exists a value $(1/2, 1) \ni u_r := 1 - \frac{1}{2D}$ for which we have $\pi(d) = 0$. Hence the solution $u_\varepsilon(t,x) \le u_r$ a.e. in Q_T. Analogously we can prove that there exists a $(0, 1/2) \ni u_l$ such that $u_\varepsilon(t,x) \ge u_l$ a.e. in Q_T. So defining $\varepsilon_1 := \min(u_l, u_r)$ we end up with (37). □

Remark 7 *The constant ε_1 depends on θ.*

Hence, by the definition of the truncation (15) we have $f^\theta(u) = f^\theta(u^\diamond)$, that means that the solution to (17)-(19) is a solution to the original problem (12)-(14), too.

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